(3rd Assignment is due on March 22. The paper work must be submitted on the class.)

To estimate $I = \int_a^b f(x)dx$, assume that $X_1, X_2, ..., X_n$ are independently sampled from some distribution wiht the p.d.f $p(x) : [a, b] \to R^+$, and monte carlo integration is defined by $\hat{I}_M = \frac{1}{n}\sum_{i=1}^n \frac{f(X_i)}{p(X_i)}$. Please show

(1) $E(\hat{I}_M) = I$;

(2) Please show that $\hat{I}_M \to^p I$ when $E(\frac{f(X)}{p(X)})^2 < \infty$, where X is a random variable with p.d.f $p(x)$. (Hint: Chebyshev's inequality);

(3) Please apply monte carlo integration for $\int_0^1 \exp(-x^2)dx$ with the uniform samples (Use: "runif: in R to generate data from U(a,b))

(4) Suppose $f(x) = x^{-1/3} + \frac{x}{10}, 0 < x \leq 1$. When $p_1(x) = 1, 0 < x < 1$, and $p_2(x) = \frac{2}{3}x^{-1/3}, 0 < x \leq 1$, please compare their variances of monte carlo integrations, and interpret what we earn.

(4th Assignment is due on March 29. The paper work must be submitted on the class.)

(5) Please verify your answer in (4) through the numerical evidence.

(6) Please review the concept of the rejection sampling before you answer this question.

In the rejection sampling method, for the selected density $g(x)$, which is easily sampling, we need to choose a postive constant α such that $e(x) \geq f(x)$, where $e(x) = \alpha g(x)$.

(a) N: The number of the required iteration to successfully generate X. Please show N follow the geometric distribution.

(b) Verify $E(N) = \alpha$.

(7) Assume that $X \sim N(0, 1)$. Please compute $E(X^4)$ through monte carlo integration + sampling method.