Lecture 1: Static Games of Complete Information

Recap

- Centralized versus decentralized view

- Ice cream stands example and prisoners' dilemma

- The idea of "equilibrium"? No one can be better-off by a unilateral change in his/her decision.

Concept – static games of complete information

- Complete information: each player's payoff function is ***common knowledge*** among all the players.
- Incomplete information: games in which some player is ***uncertain*** about another player's payoff function.
  - Ex. In an auction where each bidder's willingness to pay for the good being sold is unknown to the other bidders.
- Static game: a static game is one in which all players make decisions simultaneously, without knowledge of the strategies that are being chosen by other players.
- Dynamic game: when players interact by playing a similar stage game numerous times, the game is called a dynamic, or repeated game. Unlike simultaneous games, players have at least some information about the strategies chosen by others in the **past**.

Normal (or strategic) form game: Notation

- A set of N players $I = \{1, \cdots, N\}$
- Each player $i \in I$ has an action set $A_i = \left\{a_1^i, a_2^i, \cdots, a_{k_i}^i\right\}$
- Each player i has a payoff function $\pi^i(a)$ where $a = \{a^1, a^2, ..., a^N\}$ is game outcome
- $a^{-i} = \left\{a^1, a^2, ..., a^{i-1}, a^{i+1}, ..., a^N\right\}$ denotes the actions of all players except i in the outcome.

Normal form

The normal form is a matrix representation of a simultaneous game. For two players, one is the "row" player, and the other is the "column" player. Each row or column represents a strategy and each box represents the payoffs to each player for every combination of strategies. Generally, such games are solved using the concept of a Nash equilibrium.

Example:

Recap – Prisoners' dilemma

Solution Approach

		Prisoner 2	
		C (cooperate)	D (defect)
Prisoner 1	C	0, 0	-10, 2
	D	2, -10	-5, -5

Equilibrium Solution (Stable solution)

Other equilibrium solution?

Example: Grade game[1]

Without showing your neighbor what you are doing, write down on a form either the letter α or the letter β. Think of this as a 'grade bid'. We will randomly pair your form with one other form. Neither you nor your pair will ever know with whom you were paired. Here is how grades may be assigned for this course.

- If you put α and your pair puts β, then you will get grade A, and your pair grade C.
- If both you and your pair put α, then you both will get grade B-.
- If you put β and your pair puts α, then you will get grade C, and your pair grade A.
- If both you and your pair put β, then you will both get grade B+.

What is your choice?

α β

[1] The example is available at Yale open course, game theory.

Outcome matrix

		pair	
		α	β
me	α	(B-,B-)	(A,C)
	β	(C,A)	(B+,B+)

- Possible payoffs: A (3), B+(1), B-(0), C(-1)

		pair	
		α	β
me	α		
	β		

Definition: Dominant actions

- (α, α) is an *equilibrium in dominant actions*.
- Dominant actions may not always exist

Other possible payoffs for the grade game:

- In contrast to the case where all players are evil, suppose that each person cares not only about her own grade but also about the grade of the person with whom she is paired. For example, each player might be an "angel": she likes getting an A but she feels guilty that this is at the expense of her pair getting a C. The guilt lowers her payoff from 3 to -1.
- If she gets a C because her pair gets an A, it reduces the payoff from -1 to -3.

		pair	
		α	β
me	α		
	β		

- Equilibrium in dominant actions?

The Devil versus Angel

- What if I am a 'Devil' but I know my opponent is an 'Angel?'

		pair	
		α	β
me	α		
	β		

- "Choosing α" is a *dominant action* for me.
- What if I am an Angel but I know my opponent is a Devil?

		pair	
		α	β
me	α		
	β		

- Neither of my strategies dominates the other. But, my pair's strategy α strictly dominates her strategy β. If I know she is rational then I know she will play α. Hence, I should play α.

- Lesson: If you do not have a dominant strategy, put yourself in your opponents' shoes to try to predict what they will do.
- In larger experiments with 'normal people', about 30% chose β. What is your choice?

Definition: Dominated actions

- Do not play a strictly dominated strategy
- The action $a_1{}^i$ is ***weakly dominated*** if there exists a $a_2{}^i$ such that inequality (1) holds with weak inequality, and the inequality is strict for at least one a^{-i}.

Weakly dominated by

Example: Strictly and weakly dominated actions

		Player 2			
Player 1		cc	cw	wc	ww
	w	9, 1	9, 1	0, 0	0, 0
	c	5, 3	4, 4	5, 3	4, 4

Strictly dominated by

Assumptions

- Payoffs are known and fixed.
- Players are risk <u>neutral</u>, i.e., maximize expected payoffs.
  - Ex: a risk neutral person is **indifferent** between
    - Scenario A: \$70 for certain or
    - Scenario B: a 70% chance of earning \$100 and a 30% chance of earning \$0.
      $0.7\times100+0.3\times0=70$
  - A risk averse person prefers Scenario A to B.
  - A risk preferred person likes Scenario B.
- All players behave <u>rationally</u>. They understand and seek to maximize their own payoffs.
- The rules of the game are <u>common knowledge</u>

Assumptions – common knowledge

- It is helpful to think of players' strategies as corresponding to various "buttons" on a computer keyboard. The players are thought of as being in separate rooms, and being asked to choose a button without communicating with each other.
- Each player knows the set of players, strategies and payoffs from all possible combinations of strategies: call this information set "X".
- Common knowledge means that each player knows X,
  - each player knows that all players know X,
    - each player knows that all players know that all players know X,
      - each player knows that all players know that all players know that all players know X,
        - and so forth *ad infinitum*.

- This means that everyone knows X, and everyone knows everyone knows X, and so on *ad infinitum*.

Assumptions – rationality is common knowledge

- Most game-theoretic analysis makes the further assumption that players' rationality is common knowledge
  - Each player is rational,
    - each player *knows* that each player is rational,
      - each player *knows that each player knows* that each player is rational,
        - …,
          - and so on, *ad infinitum*.

Number game[2]

- Pick a number (integer) between 1 and 100 without showing your neighbor what you are doing. We will calculate the average number chosen in the class. The winner in this game is the person whose number is closest to two-thirds times the average in the class.
- Example: 3 people: 25, 5, and 60. The total is 90. The average =30 2/3(avearge)=20. 25 wins.
- What is your answer?
- Analysis:

- "1" is the model answer
- Why wasn't "1" the winner answer?
- Now play again.

[2] The example is available at Yale open course, game theory.

Definition: Best response

		Player 2	
		x	y
Player 1	x	20, 10	0, 0
	y	0, 0	10, 20

- If player 1 chooses x, player 2's best response: $R^2(x)=x$
- If player 1 chooses y, player 2's best response: $R^2(y)=y$
- If player 2 chooses x, player 1's best response: $R^1(x)=x$
- If player 2 chooses y, player 1's best response: $R^1(y)=y$

What is the likely outcome of this game?

		Player 2	
		x	y
Player 1	x	20, 10	0, 0
	y	0, 0	10, 20

- There is no equilibrium in *dominant actions*.

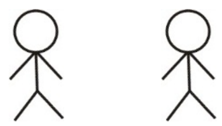
Nash equilibrium (NE)

- Each player's predicted action must be that player's best response to the predicted actions of the other players. Such a prediction can be called *strategically stable* or *self-enforcing*, because no single player wants to deviate from his or her predicted action.

- The action $a^* = \{a^1, a^2, ..., a^N\}$ is a Nash equilibrium if no player has incentive to deviate from his action given that the other players do not deviate. In other words, in a NE solution, no one can be better off by a unilateral change in its solution.
- For example: Two players and each player with three actions.

Suppose (a_2^{1*}, a_3^{2*}) is a NE

What are the sufficient conditions?



$a_1^1 \; a_2^1 \; a_3^1 \qquad a_1^2 \; a_2^2 \; a_3^2$

Applications of Static Games: Duopoly models

- So far, we have discussed how to find Nash equilibrium when there are few players, each of whom has <u>few</u> strategies, but not <u>many</u> strategies.
- There are two competing firms selling a homogeneous good.
- The marginal cost of producing each unit of the good: c_1 and c_2.
- The market price, P is determined by (inverse) market demand $P = a - Q$ if $a > Q$, P=0 otherwise.
- Both firms seek to maximize profits
- Cournot model: Firms set quantities simultaneously (this game lies between the two extreme cases of perfect competition and monopoly.)
- Bertrand model: Firms set prices simultaneously

Case of Duopoly

- Pepsi vs. Coca-Cola in Beverages market
- Airbus vs. Boeing in Commercial Aircraft market
- CPC and Formosa in petroleum market in Taiwan

Cournot Competition

- The market price, P is determined by (inverse) market demand: $P = a - Q$ if $a > Q$, P=0 otherwise.
- Each firm decides on the quantity to sell (market share): q_1 and q_2
- $Q = q_1 + q_2$ total market demand
- Both firms seek to maximize profits

Cournot Competition: Best response of firm 1

Graph

Consider a symmetric case where $c = c_1 = c_2$

- Delete the strategies that are never a best response, and then delete strategies that are never best response to anything that is a best response and so on and so forth.

Cournot Equilibrium

Some Implications

- Each firm would of course like to be a monopolist in the market, in which case it would choose q_i to maximize $\prod_i (q_i, 0)$.

- Given that there are two firms, aggregate profits for the duopoly would be maximized by setting the aggregate quantity $q_1 + q_2$ equal to the monopoly quantity q_m, as would occur if $q_i = q_m / 2$ for each i.

- The problem with this arrangement is that each firm has an incentive to deviate: because the monopoly quantity is low, the associated price $P(q_m)$ is high, and at this price each firm would like to increase its quantity, in spite of the fact that such an increase in production drives down the market-clearing price. (How to check this?)

- Another concern of collusion: new entrants

  - OPEC, central American countries, Russia

  - Pepsi, Coca-Cola, Dr. Pepper,…

- Total market quantity (symmetric case)

Perfect competition		Cournot		Monopoly
$a-c$	>	$\frac{2}{3}(a-c)$	>	$\frac{1}{2}(a-c)$

Example: Cournot Competition

- $P = 130-(q_1 + q_2)$, so $a=130$, $c_1 = c_2 = c = 10$

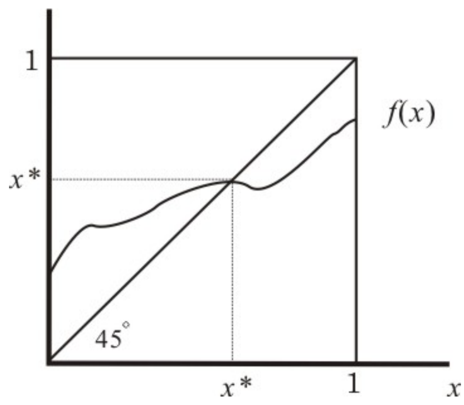
Cournot Competition

$$R^1(q_2) = 60 - q_2/2 \qquad R^2(q_1) = 60 - q_1/2$$

The existence of the Cournot equilibrium: Fixed point theorem

- Nash equilibrium solutions are the "fixed points" of the composite functions $R^1\left(R^2(\cdot)\right)$ and $R^2\left(R^1(\cdot)\right)$.

- Fixed-point theorem: suppose *f*(*x*) is a continuous function with domain [0,1] and range [0,1]. Then, fixed-point theorem guarantees that there exists at least one fixed point – that is, there exists at least one value x^* in [0,1] such that $f\left(x^*\right) = x^*$.



Cournot Equilibrium with *N* firms

Bertrand Equilibrium Model

- Firms set prices rather than quantities. $P = a - Q$
- Customers buy from the firm with the cheapest price.
- The market is split evenly if firms offer the same price.

Best response

- Firm 1's profit function
- To ensure q_1>0 (recall: $P = a - Q$ and Q=a-P): $P_1 < a$
- To ensure nonnegative profits: $P_1 \geq c_1$
- Firm 1 should choose $c_1 \leq P_1 \leq a$
- Similarly, firm 2 should choose $c_2 \leq P_2 \leq a$
- Firm i's demand depends on the relationship between P_1 and P_2.

Bertrand equilibrium

Cournot behavior with general cost and demand functions

- We will start by generalizing the Cournot Game from linear demand and constant costs to a wider class of functions. The two players are firm 1 and firm 2, and their strategies are their choices of the quantities q_1 and q_2.
- The payoffs are based on the total cost functions, $c(q_1)$ and $c(q_2)$, and the demand function, $p(q)$, where $q= q_1+ q_2$. (here we still assume that both firms employ the same functional form of the production cost.)
- Recall
  - Product rule: $(fg)' = f'q + fg'$
  - Chain rule: $\begin{aligned} f(x) &= h(g(x)) \\ f'(x) &= h'(x)g'(x) \end{aligned}$
  - Quotient rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

The general model faces two problems: nonuniqueness and nonexistence.

- If demand is concave and costs are convex, which implies that $p'' < 0$ and $c'' > 0$, then all is well as far as existence goes. Since price is greater than marginal cost ($p > c'$), (7) tells us that the reaction functions are downward sloping, because $2p'^2 - c''p' - \left(p - c'\right)p''$ is positive and both $\left(p - c'\right)p''$ and $-p'^2$ are negative. If the reaction curves are downward sloping, they cross and an equilibrium exists.

- We usually assume that costs are at least weakly convex, since that is the result of diminishing or constant returns, but there is no reason to believe that demand is either concave or convex. If the demand curves are not linear, the contorted reaction functions of (7) might give rise to multiple Cournot equilibria as in figure.

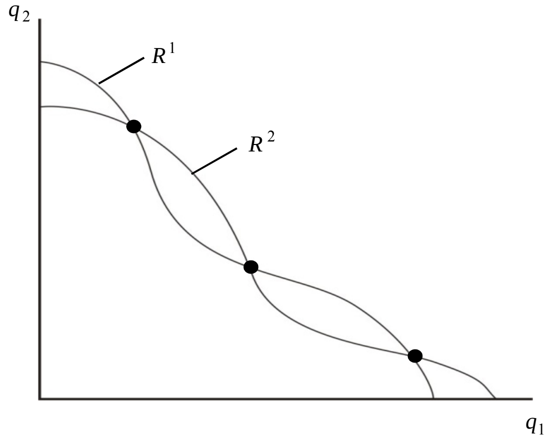


Figure. Multiple Cournot-Nash equilibria

■ If demand is convex or costs are concave, so $p'' > 0$ or $c'' < 0$, the reaction functions can be upward sloping, in which case they might never cross and no equilibrium would exist. The problem can also be seen from firm 1's payoff function, equation (2). If $p(q)$ is convex, the payoff function might not be concave, in which case standard maximization techniques break down.

Mixed-strategy

- So far the strategies (or actions) available to players are "pure strategy". The new type of strategy is to randomize over the existing strategies. How to choose randomly form one of their pure strategies?
- Mixed-strategy: A randomization over your pure strategies
- Rock, Paper, and Scissors (RPS) game:

	Player 2			
Player 1		Rock (R)	Paper (P)	Scissor (S)
	Rock (R)	0, 0	-1, 1	1, -1
	Paper (P)	1, -1	0, 0	-1, 1
	Scissor (S)	-1, 1	1, -1	0, 0

Obviously there is no pure strategy Nash equilibrium. Any attempt to look for best responses that are best responses to each other can lead to a cycle. (no pure strategies that are best responses to each other)

What is the Nash equilibrium for the RPS game?

- A pure strategy is a special case of mixed-strategies in which some strategy's probability is 1.
- The expected payoff? The weighted average of the expected payoffs of each of the pure strategies in the mix.

- Example: (*Matching Pennies*) Player 1 and 2 both toss a penny on a table simultaneously. If the two pennies come up the same (heads or tails) then player 2 gets both; if the pennies do not match then 1 wins 2's penny.

- If player 2 plays H with probability q and T with probability 1-q and player 1 plays H with probability p and T with probability 1-p,…

	Player 2		
		H (q)	T (1-q)
Player 1	H (p)	-1, $\underline{1}$	$\underline{1}$, -1
	T (1-p)	$\underline{1}$, -1	-1, $\underline{1}$

- A mixed strategy for player 1 is the probability distribution (p, 1-p), where p is the probability of playing Head, 1-p is the probability of playing Tail, and $0 \le p \le 1$.

- The mixed strategy (0, 1) is simply the pure strategy Tail; likewise, the mixed strategy (1, 0) is the pure strategy Head.

- **Definition**: In the two-player normal-form game, the mixed strategy (p_1^*, p_2^*) is a Nash equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy.

- If player 2 plays H with probability q and T with probability 1-q,…

	Player 2		
		H (q)	T (1-q)
Player 1	H	-1, $\underline{1}$	$\underline{1}$, -1
	T	$\underline{1}$, -1	-1, $\underline{1}$

- If player 1 plays (p, 1-p), what is player 2's best response?

- Graphical representation of players 1 and 2's best response.

Concept of the Mixed-strategy

	Player 2		
Player 1		H (1/3)	T (2/3)
	H	-1, 1	1, -1
	T	1, -1	-1, 1

- If player 2 plays H with probability 1/3 and T with probability 2/3. (denoted by $\left(\frac{1}{3},\frac{2}{3}\right)$)

- Under the scheme of mixed strategies (not talking about the pure strategy case), players are able to assign any probabilities to strategies. If a mixed strategy is a best response, each of the pure strategies in the mix must themselves be best responses. In particular, each must yield the same expected payoff.

- Definition: A mixed strategy profile $\left(p_1^*, p_2^*, \ldots, p_n^*\right)$ is a mixed strategy Nash equilibrium if for each player i, p_i^* is a best response to $p_{-i}^* = \left(p_1^*, \ldots, p_{i-1}^*, p_{i+1}^*, \ldots, p_n^*,\right)$.

- Revisiting the matching pennies example

	Player 2		
Player 1		H (q)	T (1-q)
	H (p)	-1, 1	1, -1
	T (1-p)	1, -1	-1, 1

- The *Battle of the Sexes*: Bob and Alice.

	Bob		
Alice		Ballet (q)	Football (1-q)
	Ballet (p)	$\underline{2}$, $\underline{1}$	0, 0
	Football (1-p)	0, 0	$\underline{1}$, $\underline{2}$

- Both (Ballet, Ballet) and (Football, Football) are Nash equilibria.

- Graphical representation of Alice and Bob's best response.

- What is the difference between Matching Pennies game and the Battle of Sexes game?
- There is only one intersection of the players' best-response functions in Matching Pennies game.
- There are three intersections of the players' best-response functions in the Battle of Sexes game. (Two pure strategies and one mixed strategy equilibrium.
- All the examples we looked at so far had discrete action sets.

Tennis Game

- Player V (server) aims to Player S's (receiver) left (L) or right (R).
- Player S (receiver) prepares her grip and starts to move toward that side in anticipation (l or r).

	Player S (receiver's move)		
Player V (server's aim)		l (q)	r (1-q)
	L (p)	50, <u>50</u>	<u>80</u>, 20
	R (1-p)	<u>90</u>, 10	20, <u>80</u>

The receiver's forehand is somewhat stronger. If she anticipates correctly, her forehand return will be successful 80 percent of the time, while an anticipated backhand return will be successful only 50 percent of the time.

If receiver starts to move to left and the service goes to the right, the receiver returns successfully only 10 percent. The other way around, the chances are 20 percent.

- There is no pure-strategy NE. Let's find a mixed-strategy NE.

- Suppose player S leant to the left more than .6 of the time. If you were player V's coach, what would you advise player to do? Player V should shoot the ball to the right all the time.
- Suppose player S leant to the left less than .6 of the time. If you were player V's coach, what would you advise player to do? Player V should shoot the ball to the left all the time.

  If player S doesn't choose exactly this mix, then player V's best response is actually a pure strategy.
- Similarly, if player V is hitting the ball to player S's left more than 0.7 of the time, S should just always go to her left, and if V is hitting to the left less than .7 of the time, then S should always go to the right.

■ Player S hires a new coach and player S's backhand has been improved….

	Player S (receiver's move)		
Player V (server's aim)		*l* (*q*)	*r* (1-*q*)
	L (*p*)	30, 70	80, 20
	R (1-*p*)	90, 10	20, 80

■ Interpretations

Before we made any changes, player V was indifferent between shooting to the left and shooting to the right. Then we improved player S's ability to return the ball to her left. If we had not changed the way player S played then what would player V have done?

Suppose in fact S's q had not changed. V would never, ever have shot to the left anymore.

S would like to change q to bring V back into equilibrium (it was S moving to the left less often and moving to the right more often)

Conversely

If V hadn't changed her behavior, S would have only gone to the left. So it must be something about V's play that brings S back into equilibrium. (V starts shooting to the right more often.)

Interpretations of Mixed-strategy

- Players really randomize over their strategies
- Players' behaviors of chosen actions (percent of time)
- Proportion of players

Tax game[3]

	Taxpayer		
Auditor		H (q)	C(1-q)
	A (p)	2, <u>0</u>	<u>4</u>, -10
	N (1-p)	<u>4</u>, 0	0, <u>4</u>

No pure strategy NE!

Find mixed-strategy NE

[3] The example is available at Yale open course, game theory.

■ New tax policy : raise the fine to -20

	Taxpayer		
Auditor		H (q)	C(1-q)
	A (p)	2, $\underline{0}$	$\underline{4}$, **-20**
	N (1-p)	$\underline{4}$, 0	0, $\underline{4}$

What happens to the new q? (compliance rate)

To get a higher compliance rate:

- ➢ Change payoffs to auditor
  - ✓ Make it less costly to do an audit
  - ✓ Give a bigger gain for catching a cheater
- ➢ Set audit rates higher by lawmakers

Application: Final-Offer Arbitration (Farber 1980)

- A firm and a union dispute the wage. First, the firm and the union simultaneously make offers, denoted by w_f and w_u, respectively. Second, the arbitrator chooses one of the two offers as the settlement. Assume that the arbitrator has an ideal settlement she would like to impose, denoted by *x*. After observing the parties' offers, w_f and w_u, the arbitrator simply chooses the offer that is closer to *x*.
- The arbitrator chooses w_f if $x < (w_f + w_u)/2$
- The arbitrator chooses w_u if $x > (w_f + w_u)/2$
- The arbitrator knows *x* but the parties do not. The parties believe that *x* is randomly distributed according to a cumulative probability distribution denoted by *F*(*x*), with associated probability density function denoted by *f*(*x*).
  (Recall: $F(x^*)$: the probability that the random variable *X* is less than an arbitrary value x^*. The derivative of this probability (cdf) with respect to x^* is denoted by $f(x^*)$ (pdf).)
- Nash equilibrium wage w_f and w_u?
- Analysis:

Existence of a Mixed-Strategy Equilibrium

- Theorem (Nash 1950) Every finite normal form game has a mixed-strategy equilibrium.

Preliminary

- If *A* and *B* are non-empty sets, then the Cartesian Product, $A \times B$, is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.

- A convergent sequence: Let $(\underline{x}_n)$ be a sequence in $\mathbf{R}^p$ and $\underline{x} \in \mathbf{R}^p$. Then $\underline{x}$ is a limit of $(\underline{x}_n)$ iff for each $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbf{N}$ such that

$$n \geq K(\varepsilon) \Rightarrow \|\underline{x}_n - \underline{x}\| < \varepsilon .$$

- A **compact** set must be both **closed** and **bounded**. E.g. [0,1] is compact.

- A set *X* is convex if for any *x* and x' belonging to *X* and any $\lambda \in [0,1]$, $\lambda x + (1-\lambda)x' \in X$.

- A set *S* is closed *iff* for any convergent sequence of points $\{x_k\}$ contained in *S* with limit point $\bar{x}$, we also have that $\bar{x} \in S$.

- Suppose that $F : \mathbf{R}^m \to \mathbf{R}^n$ is a correspondence. Then the **graph** of *F* is defined to be

$$\begin{aligned} G_F &= \{(x_1,...,x_m,y_1,...,y_n) \in \mathbf{R}^{m+n} : (y_1,...,y_n) \in F(x_1,...,x_m)\} \\ &= \{(x,y) \in \mathbf{R}^m \times \mathbf{R}^n : y \in F(x)\} \end{aligned}$$

- A correspondence *C* is called **upper semi-continuous** (usc), if the **graph** of the correspondence $\{(x,y) : y \in C(x)\}$ is closed.

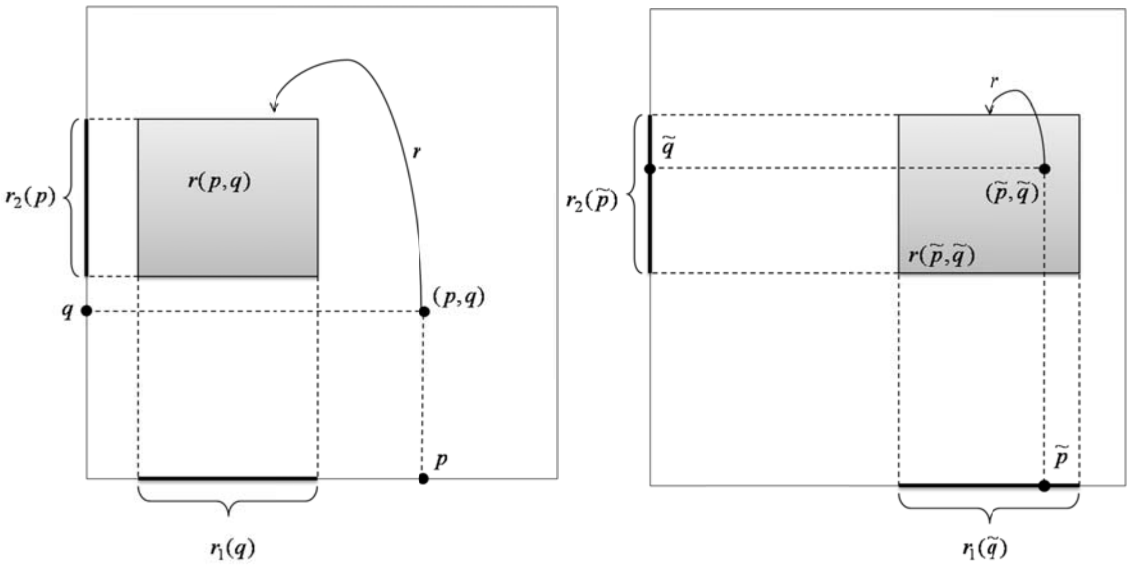
- **Kakutani's fixed point theorem**

  Let $S \subset \mathbf{R}^n$ be a compact and convex set. Let *C* be a correspondence from *S* into itself that is upper semi-continuous and convex valued. Then, there is an $x^* \in S$ such that $x^* \in C(x^*)$.

Notations

- Player *i*'s *best response correspondence*, r_i, maps each strategy profile σ to the set of mixed strategies that maximize player *i*'s payoff when his opponents play σ_{-i}. (Although r_i depends only on σ_{-i} and not on σ_i, we write it as a function of the strategies of all players, because later we will look for a fixed point in the space Σ of strategy profiles.)

- Player *i*'s payoff function: $u_i(\sigma)$.
- Theorem (Nash 1950) Every finite normal form game has a mixed-strategy equilibrium.
- Two-player case (graph illustration)



(a) (b)

The correspondence *r* in Nash's theorem.

Proof:

The idea of the proof is to apply Kakutani's fixed-point theorem to the players' "best response correspondences." Define the correspondence $r:\Sigma\to\Sigma$ to be the Cartesian product of the r_i.

A *fixed point* of *r* is a σ such that $\sigma\in r(\sigma)$, so that, for each player, $\sigma_i\in r_i(\sigma)$. Thus, a fixed point of *r* is a Nash equilibrium. (That is, for each *i*, σ_i must be (one of) player i's best response(s) to σ_{-i}, but this is precisely the statement that σ is a Nash equilibrium.)

From Kakutani's theorem, the following are sufficient conditions for $r:\Sigma\to\Sigma$ to have a fixed point:

(1) Σ is a compact, convex, nonempty subset of a (finite-dimensional) Euclidean space.

(2) $r(\sigma)$ is nonempty for all σ.

(3) $r(\sigma)$ is convex for all σ.

(4) $r(\cdot)$ has a closed graph: If the sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in r(\sigma^n)$, then $\hat{\sigma} \in r(\sigma)$.

Let us check that these conditions are satisfied.

Once existence has been established, it is natural to consider the characterization of the equilibrium set. Ideally one would prefer there to be a unique equilibrium, but this is true only under very strong conditions. When several equilibria exist, one must see which, if any, seem to be reasonableness of one equilibrium may depend on whether there are others with competing claims. Unfortunately, in many interesting games the set of equilibria is difficult to characterize.