Review of Linear Programming

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Agenda

- Linear Programming Problem
- Graphical Solution
- The Simplex Algorithm
- The Two-phase Simplex Method
- Duality

Linear Programming

- Linear Programming (LP) problems are optimization problems involving only **linear functions** representing a decision, given an objective and resource constraints.
- The model includes
  - Decision variables
  - Objective functions
  - Model constraints
  - Parameters

Assumptions

- Deterministic: all parameters are all known.
- Divisibility: non-integer values for the decision variables are permitted.
- Proportionality: increase x_j which also proportionally contributes to the objectives and associated constraints.
- Additivity: total cost is the sum of the individual costs

The Format of Linear Programming Model

$$\begin{aligned} \text{maximize}\; z &= c^t x \\ \text{subject to}\quad Ax &\le b, \\ x &\ge 0 \end{aligned}$$

The Format of Linear Programming Model

$$\max Z = c_1x_1 + c_2x_2 + \ldots. + c_nx_n$$

$$\begin{aligned} s.t.\quad & a_{11}x_1 + a_{12}x_2 + \ldots. + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots. + a_{2n}x_n \leq b_2 \\ & \ldots. \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots. + a_{mn}x_n \leq b_m \\ & x_j \geq 0 \qquad \text{for } 1 \leq j \leq n \end{aligned}$$

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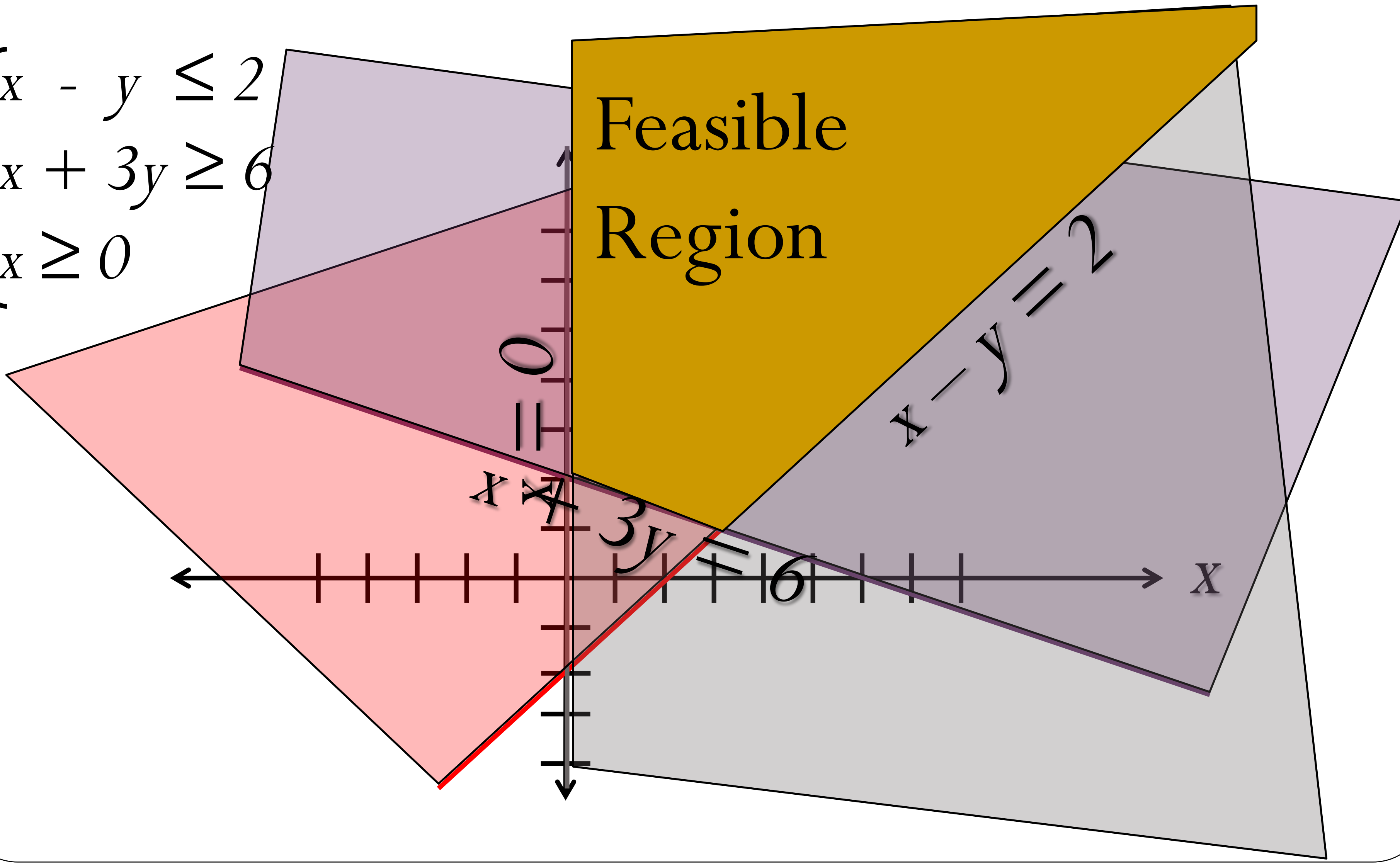
Graphic Solution

$$\begin{cases} x - y \le 2 \\ x + 3y \ge 6 \\ x \ge 0 \end{cases}$$



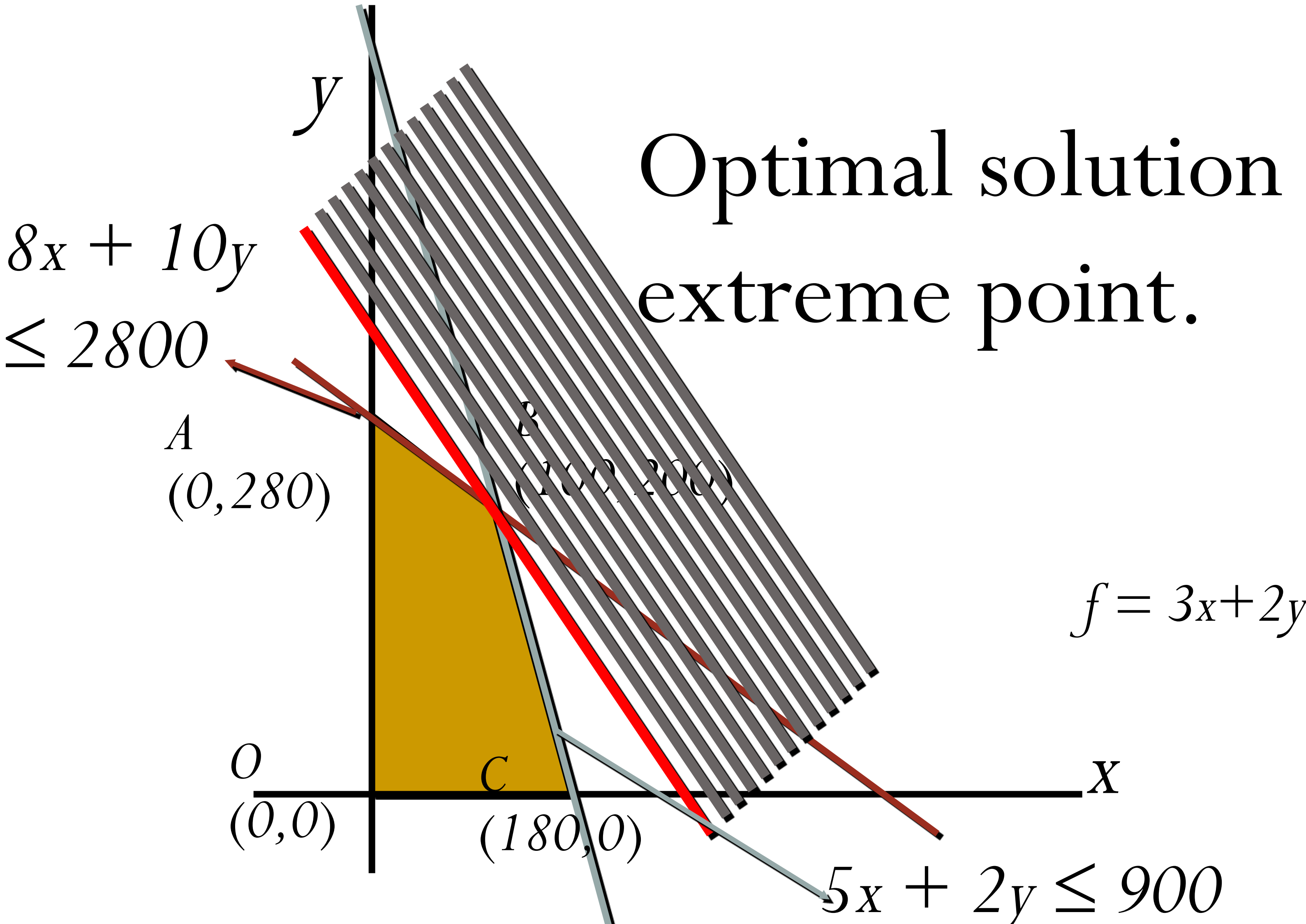
Some Theorems

- The feasible region of a LP problem is a convex set.
- If the feasible region is bounded, at least one optimal solution occurs at one of its extreme points.
- The feasible region has a finite number of extreme points.

Example:

$$\begin{aligned} \text{Max } & 3x + 2y \\ \text{s.t. } & 5x + 2y \leq 900 \\ & 8x + 10y \leq 2800 \\ & x \geq 0, y \geq 0 \end{aligned}$$

Solve the Problem



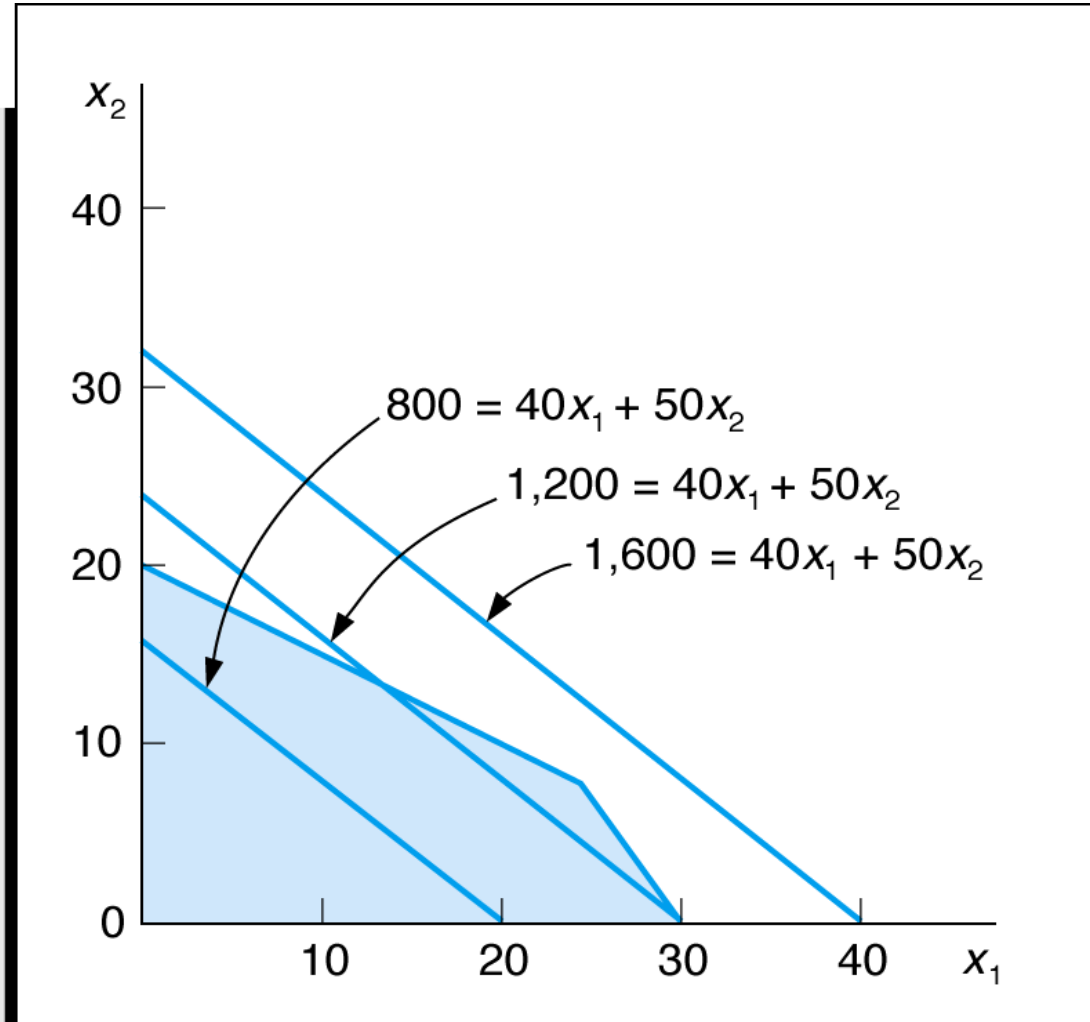
Optimal solution is at extreme point.

Alternative Objective Function Solution Lines

Maximize $Z = \$40x_1 + \$50x_2$

subject to:

$$1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$



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Basic Feasible Solution

- Non-Basic variables
  - $x_n = 0$
- Basic variables
  - $x_B = B^{-1}b$

Basic Feasible Solution

- Partition A matrix into 2 sub-matrices $A = (B, N)$, where B is invertible
- In addition, if $x_B \geq 0$, then it is called a basic feasible solution or BFS
- If $x_B > 0$ then it is called a non-degenerate BFS

Example

$$\max \ 3x_1 + 5x_2$$

S.t.

$$x_1 \leq 4$$

$$x_2 \leq 6$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1, x_2 \geq 0$$

Basic Feasible Solution

- Consider BFS, is this solution optimal?
- Examine the canonical form, that is solve for z and x_B in terms of x_N:

$$cx = c_B x_B + c_N x_N$$

$$Ax = b$$

$$Bx_B + Nx_N = b$$

Simplex Method

- Finding a starting basic feasible solution
- The current basic feasible solution
- Test for optimality
- Improving the current basic feasible solution
- Pivot step

Example

$$\max \; 4x_1 + 3x_2$$

S.t.

$$2x_1 + 1x_2 \le 12$$

$$-x_1 + 2x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Some observations

- Unbounded objective
- Unique optimal solution
- Alternative optimal solution

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Two-phase Simplex

- A convenient starting solution is not available

$$\max \ z = cx$$

S.t.

$$Ax = b$$
$$x \geq 0$$

- Artificial variable technique
  - Creating a convenient starting basis by adding a set of artificial variables

$$Ax + Dx_a = b$$
$$x \geq 0$$

Example

$$\max \;\; 2x_1 + x_2$$

S.t.

$$x_1 + x_2 \le 5$$

$$x_1 \ge 1$$

$$-x_1 + x_2 = 1$$

$$x_2 \ge 0$$

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Duality

- Primal Problem

$$\max \; cx$$

S.t.

$$Ax \le b$$

$$x \ge 0$$

- Dual Problem

$$\min \; wb$$

S.t.

$$wA \ge c$$

$$w \ge 0$$

Economic Interpretation

- A manufacturer produce a set of products $(x_1,\ldots,x_n)$ and sell them at prices $c_1,\ldots,c_n$.
- It requires a set of raw material ($\boldsymbol{b}$) to produce these products.
- The resource consumption is modeled as $AX \leq \boldsymbol{b}$.
- A resource collector would like to buy these raw material with min cost.
- Assume the value of raw material i is w_i.
- Total value of raw material i is equal to $b_i\, w_i$.
- In what conditions, the manufacturer is willing to sell its resource?

Weak Duality

- Let *x'* be a feasible solution to (P), and *w'* be a feasible to (D)
- Then, $cx' \leq w'b$

- Corollaries:
  - Each feasible solution to (D) provides _______________ for the objective of (P).
  - If the objective of (D) is unbounded, (P) is__________ .
  - If (P) is infeasible, (D) is either infeasible or has ______________________ .

Strong Duality Theorem

- If (P) and (D) are both feasible, they both have finite optimal solution with the same objective value

$$cx^* = (w^*)Ax^* = w^*b$$

Complementary Slackness

- Let v_j be the reduced cost for x_j, w_i is the reduced cost for x_{si}
- $x_j > 0 \Rightarrow x_j$ is basic $\Rightarrow z_j - c_j = 0 = v_j$
- $v_j \neq 0 \Rightarrow x_j$ is nonbasic $\Rightarrow x_j = 0$
- Then, $x_j v_j = 0$ for all j; similarly, $w_i x_{si} = 0$ for all i

Example

$$\max \ 2x_1 + x_2$$

S.t.

$$x_1 - x_2 \le 1$$

$$x_1 + x_2 \le 3$$

$$x_1, x_2 \ge 0$$

$$\min \ 1w_1 + 3w_2$$

S.t.

$$w_1 + w_2 \ge 2$$

$$-w_1 + w_2 \ge 1$$

$$w_1, w_2 \ge 0$$

Primal-Dual Forms

	Primal		Dual	
Obj.	Max Z	$\Leftrightarrow$	Min W	
St.	$\le$	$\Rightarrow$	$y_i \ge 0$	
	$=$	$\Rightarrow$	y_i unrestricted	
	$\ge$	$\Rightarrow$	$y_i \le 0$	
	$x_i \ge 0$	$\Rightarrow$	St.	$\ge$
	x_i unrestricted	$\Rightarrow$		$=$
	$x_i \le 0$	$\Rightarrow$		$\le$

Example 1: The Dual

$$\text{Max } Z = 10x_1 + 20x_2 + 30x_3$$
$$\text{s.t. } 2x_1 + 3x_2 + 4x_3 = 100$$
$$5x_1 + 6x_2 + 7x_2 \geq 200$$
$$8x_1 + 9x_2 + 10x_2 \leq 300,$$
$$x_1 \leq 0,\ x2 \text{ unrestricted},\ x3 \geq 0$$

$$\text{Min } W = 100y_1 + 200y_2 + 300y_3$$
$$\text{s.t. } 2y_1 + 5y_2 + 8y_3 \leq 10$$
$$3y_1 + 6y_2 + 9y_3 = 20$$
$$4y_1 + 7y_2 + 10y_3 \geq 30$$
$$y_1 \text{ unrestricted},\ y_2 \leq 0\ ,\ y_3 \geq 0$$

Example 2: The Dual

$$\begin{aligned}
&Min\ Z = 0.4x_1 + 0.5x_2 \\
&s.t.\ 0.3x_1 + 0.1x_2 \le 2.7 \\
&\quad 0.5x_1 + 0.5x_2 = 6 \\
&\quad 0.6x_1 + 0.4x_2 \ge 6, \quad x_1 \text{ and } x_2 \ge 0
\end{aligned}$$

$$\begin{aligned}
&Max\ W = 2.7y_1 + 6y_2 + 6y_3 \\
&s.t.\ 0.3y_1 + 0.5y_2 + 0.6y_3 \le 0.4 \\
&\quad 0.1y_1 + 0.5y_2 + 0.4y_3 \le 0.5 \\
&\quad y_1 \le 0,\ y_2 \text{ unrestricted},\ y_3 \ge 0
\end{aligned}$$

Dual variables

- What do the dual variables mean in an economic sense?

$$z = wb = C_B B^{-1} b$$

$$\frac{\partial z}{\partial b_i} = w_i$$

- The dual variables are also called dual prices, shadow prices or marginal prices.
- Consider constraints and increase RHS values:

$$A_{i.} x \leq b_i$$

$$A_{i.} x \geq b_i$$

Dual Simplex

- Choose the leaving variables
  - Choose the variable x_r with the most negative b_i
- Find the entering variables (maintain dual feasibility)
  - Use min ratio: $z_k - c_k / -y_r^k = \min_{y_r^j < 0}(z_j - c_j / -y_r^j)$
- Pivot on $y^k{}_r$

Example

$$\min \; 4x_1 + 10x_2$$

S.t.

$$x_1 + 3x_2 \geq 4$$

$$2x_1 + 4x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

$$\max \; -4x_1 - 10x_2$$

S.t.

$$x_1 + 3x_2 - x_3 = 4$$

$$2x_1 + 4x_2 - x_4 = 6$$

$$x_i \geq 0$$

Questions?

- Homework 1 can be downloaded from Ceiba and due on 3/16.