Dual Fractional Mixed Integer Programming

Instructor: Kwei-Long Huang

Agenda

- Basic Procedures
- Form of the Cut
- Derivation of the Cut
- Mixed Cut to IP

Basic Procedures

The there main steps are:

- Solve the MIP by LP. If the problem is an ILP, start with an all-integer tableau (or with tableau of rational numbers). If the problem is infeasible or has an integer solution, stop. Otherwise, go to step 2.
- For an integer constrained variable without an integral value, derive a new inequality constraint. Add a new constraint to the tableau, which will produce primal infeasibility.
- Re-optimize with the lexicographic dual simplex (LDS) method. If the new problem is infeasible or has an integer solution for integer constrained variables, stop. Otherwise, go to 2.

MILP

- It is a cutting plane method for MILP problems using the LDS method.

(MILP) Maximize $z = C^T x$,

subject to $Ax \leq b$

$x_j \geq 0$, integer , $j = 1, \ldots, I$,

$x_j \geq 0$ $j = I+1, \ldots, n$.

- The initial tableau is not required to be integer. However, to ensure convergence the objective function value $z\ (x_o)$ is required to be integer.

Generate a Cut

- The LDS method will produce an optimal tableau such that

$$\alpha_j \overset{\ell}{\succ} 0,\ j=1,\ \ldots, n \qquad \Rightarrow \qquad a_{0,j} \geq 0,\ j=1,\ \ldots, n,$$
$$a_{i0} \geq 0,\ i=1,\ \ldots, n+m.$$

- In addition, if a_{j0} is integer , $i=1$,… ,I ,the MILP is solved:
  - $z^* = a_{0\ ,0}$,
  - $x^* = a_{\mathrm{i}\ ,0}$, $i = i$,… , $n+m$.
  - Otherwise , we need to generate a "Gomory" cut .

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Form of the Cut

There is a row v $\left(1 \le v \le I\right)$, $\quad x_v = a_{v,0} + \sum_{j=1}^{n} a_{v,j}\left(-x_{J(j)}\right)$, $\quad$ with $a_{v,0}$ fractional.

k^{th} Gomory cut: $\quad x_{n+m+k} = -f_{v,0} + \sum_{j=1}^{n}\left(\text{-}g_{v,j}\right)\left(-x_{J(j)}\right) \ge 0,$ $\quad \left(\text{Gomory, 1960}\right)$

$$
\text{where } g_{v,j} = \begin{cases} a_{v,j}, & \text{if } a_{v,j} \ge 0 \text{ and } x_{J(j)} \text{ is a continuous variable,} \\ \dfrac{f_{v,0}}{f_{v,0}-1} a_{v,j}, & \text{if } a_{v,j} < 0 \text{ and } x_{J(j)} \text{ is a continuous variable,} \\ f_{v,j}, & \text{if } f_{v,j} \le f_{v,0} \text{ and } x_{J(j)} \text{ is an integer variable,} \\ \dfrac{f_{v,0}}{1-f_{v,0}}\left(1-f_{v,j}\right), & \text{if } f_{v,j} > f_{v,0} \text{ and } x_{J(j)} \text{ is an integer variable,} \end{cases}
$$

$$f_{v,j} = a_{v,j} - \left\lfloor a_{v,j} \right\rfloor,\ j = 0,\ \ldots,\ I,$$

and $\quad x_{n+m+k}$ is the gomory slack varible.

Note that $\quad 0 \le f_{v,j} < 1,\ j = 1,\ \ldots,\ I, \quad 0 < f_{v,0} < 1.$

Example (1/7)

Maximize $z = -4x_1 - 5x_2,$

subject to

$$-x_1 - 4x_2 + x_3 = -5, \qquad (*)$$

$$-3x_1 - 2x_2 + x_4 = -7, \qquad (**)$$

$x_1 \geq 0$, integer,

$x_2, x_3, x_4 \geq 0.$

Example (2/7)

#3	1	$(-x_4)$	$(-x_3)$
x_0	$-{}^{112}/_{10}$	${}^{11}/_{10}$	${}^{7}/_{10}$
x_1	${}^{18}/_{10}$	$-{}^{4}/_{10}$	${}^{2}/_{10}$
x_2	${}^{8}/_{10}$	$1/10$	$-{}^{3}/_{10}$
x_3	0	0	-1
x_4	0	-1	0

Example (3/7)

#4	1	$(-x_5)$	$(-x_3)$
x_0	$-{}^{188}/_{16}$	${}^{11}/_{16}$	${}^{9}/_{16}$
x_1	2	$-{}^{4}/_{16}$	${}^{4}/_{16}$
x_2	${}^{12}/_{16}$	${}^{1}/_{16}$	$-{}^{5}/_{16}$
x_3	0	0	-1
x_4	${}^{8}/_{16}$	$-{}^{10}/_{16}$	${}^{2}/_{16}$
x_5	0	-1	0

Example (4/7)

#5	1	$(-x_6)$	$(-x_3)$
x_0	-12	1	0
x_1	$^{23}/_{11}$	$-^{4}/_{11}$	$^{5}/_{11}$
x_2	$^{8}/_{11}$	$^{1}/_{11}$	$-^{4}/_{11}$
x_3	0	0	-1
x_4	$^{8}/_{11}$	$-^{10}/_{11}$	$^{7}/_{11}$
x_5	$^{4}/_{11}$	$-^{16}/_{11}$	$^{9}/_{11}$
x_6	0	-1	0
x_7	$-^{1}/_{11}$	$-^{4}/_{110}$	$-^{5}/_{11}$

Cut derivation:

$$f_{10} = 23/11 - [23/11] = 1/11$$

$$g_{11} = \tfrac{1/11}{1/11-1}(-4/11) = 4/110$$

$$g_{12} = 5/11$$

$$x_7 = -\frac{1}{11} - \frac{4}{110}(-x_6) - \frac{5}{11}(-x_3)$$

Example (5/7)

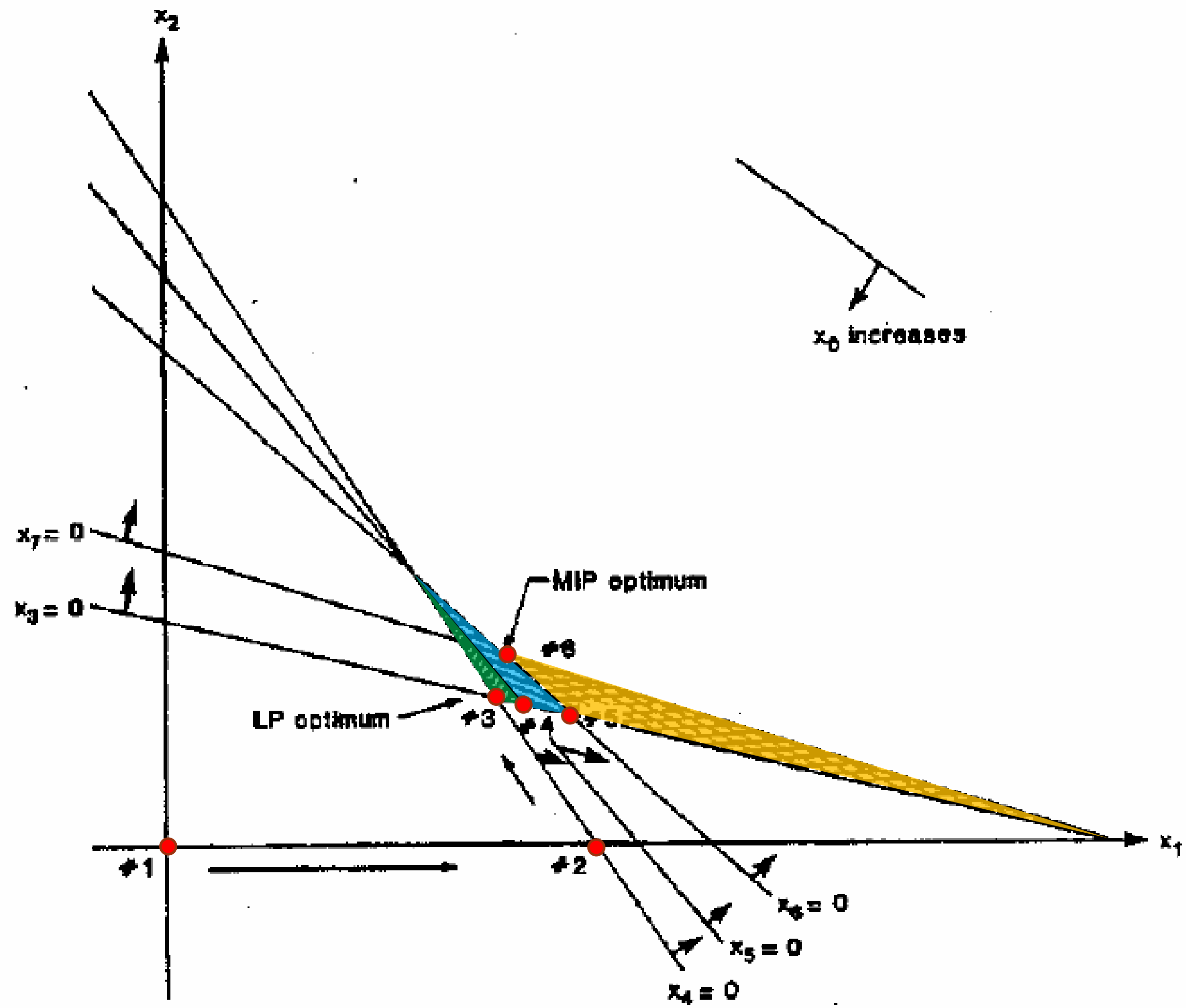
#6	1	$(-x_6)$	$(-x_7)$
x_0	-12	1	0
x_1	2	$-{}^{2}/_{5}$	1
x_2	${}^{4}/_{5}$	${}^{6}/_{50}$	$-{}^{4}/_{5}$
x_3	${}^{1}/_{5}$	${}^{4}/_{50}$	$-{}^{11}/_{5}$
x_4	${}^{3}/_{5}$	$-{}^{48}/_{50}$	${}^{7}/_{5}$
x_5	${}^{1}/_{5}$	$-{}^{76}/_{50}$	${}^{9}/_{5}$
x_6	0	-1	0
x_7	0	0	-1

Example (6/7)

- The cuts in terms of original non-basic variables

$$
\begin{aligned}
x_5 &= -13 + 5x_1 + 4x_2 \geq 0, \\
x_6 &= -12 + 4x_1 + 5x_2 \geq 0, \\
x_7 &= -14/5 + (3/5)x_1 + 2x_2 \geq 0,
\end{aligned}
$$

Example (7/7)



x2
x0 increases
x7 = 0
x3 = 0
MIP optimum
#6
LP optimum
#3
#4
x1
#1
#2
x6 = 0
x5 = 0
x4 = 0

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Derivation of the Cut (1/7)

$$x = a_0 + \sum_{j=1}^{n} a_j \left(-x_{J(j)}\right),$$

$$0 \equiv a_0 + \sum_{j=1}^{n} a_j \left(-x_{J(j)}\right) \pmod{1}$$

$$0 \equiv f_0 + \sum_{j=1}^{n} a_j \left(-x_{J(j)}\right) \pmod{1}. \qquad f_0 = a_0 - \lfloor a_0 \rfloor$$

$$\sum_{j=1}^{n} a_j x_{J(j)} \equiv f_0 \pmod{1}.$$

Derivation of the Cut (2/7)

- Assume the left-hand side ($\sum_{j=1}^{n} a_j \left(x_{J(j)}\right)$) is positive:

$$\sum_{j=1}^{n} a_j x_{J(j)} \ge f_0$$

$$\sum_{j \in P} a_j x_{J(j)} \ge \sum_{j=1}^{n} a_j x_{J(j)} \ge f_0, \qquad \text{where } P = \left\{ j \mid a_j \ge 0,\ \ j = 1, \ldots, n \right\}.$$

Derivation of the Cut (3/7)

- Assume the left-hand side ($\sum_{j=1}^{n} a_j \left(x_{J(j)}\right)$) is negative:

$$\sum_{j=1}^{n} a_j x_{J(j)} \le -1 + f_0$$

$$\sum_{j \in N} a_j x_{J(j)} \le \sum_{j=1}^{n} a_j x_{J(j)} \le -1 + f_0, \quad \text{where } N = \left\{ j \mid a_j < 0,\ j = 1, \ldots, n \right\}.$$

- Multiplying the negative number ${f_0}/{-1+f_0}$ gives

$$\sum_{j \in N} \left(\frac{f_0}{-1 + f_0} \right) a_j x_{J(j)} \ge f_0$$

Derivation of the Cut (4/7)

- Combine the two inequalities:

$$\sum_{j\in P} a_j x_{J(j)} + \sum_{j\in N}\left(\frac{f_0}{-1+f_0}\right) a_j x_{J(j)} \geq f_0$$

- Thus, a non-negative Gomory slack variable x' is

$$x' = -f_0 + \sum_{j\in P} a_j x_{J(j)} + \sum_{j\in N}\left(\frac{f_0}{-1+f_0}\right) a_j x_{J(j)} \geq 0$$

Derivation of the Cut (5/7)

- Consider any nonbasic integer constrained variable x_t with $a_t \neq 0$ in the following congruence relation

$$0 \equiv f_0 + \sum_{j=1}^{n} a_j(-x_{J(j)}) \pmod 1.$$

- Add or subtract integer multiples of to or from the relation will not invalidate it.

Derivation of the Cut (6/7)

- If t is placed in P set, the smallest coefficient is .
- If t is place in N set, the smallest coefficient is .
- Thus, when $f_t \leq \frac{f_0}{-1+f_0}(f_t - 1)$, it is better to put t in set.
- In addition, the equation is true when $f_t \leq f_0$.

Derivation of the Cut (7/7)

$$x' = -f_0 + \sum_{j=1}^{n} g_j x_{J(j)} \geq 0,$$

where

$$g_{v,j} = \begin{cases} a_j & \text{if } a_j \geq 0 \text{ and } x_{J(j)} \text{ is a continuous variable.} \\ \dfrac{f_0}{f_0 - 1} a_j & \text{if } a_j < 0 \text{ and } x_{J(j)} \text{ is a continuous variable.} \\ f_j & \text{if } f_j \leq f_0 \text{ and } x_{J(j)} \text{ is an integer variable.} \\ \dfrac{f_0}{1 - f_0}\left(1 - f_j\right), & \text{if } f_j > f_0 \text{ and } x_{J(j)} \text{ is an integer variable.} \end{cases}$$

Example

- Suppose the source row is (x_1 and x_2 are integer):

$$x = {}^{23}\!/_{10} + ({}^{14}\!/_{10})(-x_1) + ({}^{12}\!/_{10})(-x_2) - ({}^{11}\!/_{10})(-x_3) + ({}^{18}\!/_{10})(-x_4)$$

$$0 \equiv {}^{3}\!/_{10} + ({}^{14}\!/_{10})(-x_1) + ({}^{12}\!/_{10})(-x_2) - ({}^{11}\!/_{10})(-x_3) + ({}^{18}\!/_{10})(-x_4)$$

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MILP Gomory Cut

- Assume all non-basic variables in the current optimal are integer constrained, the MILP Gomory cut is

$$x^M = -f_0 + \sum_{j \in J^1} f_j x_{J(j)} + \sum_{j \in J^2} \left(\frac{f_0}{1 - f_0} \right) \left(1 - f_j\right) x_{J(j)} \geq 0,$$

where $J^1 = \left\{ j : f_j \leq f_0,\ j \in J \right\}$, and $J^2 = \left\{ j : f_j > f_0,\ j \in J \right\}$.

and the ILP Gomory cut is

$$x^I = -f_0 + \sum_{j=1}^{n} f_j x_{J(j)} \geq 0.$$

Stronger Cut

Theorem: If $x_{J(j)}$, $j \in J$, are integer variables and $J^2 \neq \varnothing$, then the MILP cut is stronger than the ILP cut.

Proof:

Note that $\{1,\ 2,\ \ldots,\ n\} = J^1 \cup J^2$. Then,

$$x^I = -f_0 + \sum_{j \in J^1} f_j x_{J(j)} + \sum_{j \in J^2} f_j x_{J(j)} \geq 0,$$

and $$x^M = -f_0 + \sum_{j \in J^1} f_j x_{J(j)} + \sum_{j \in J^2} \left(\frac{f_0}{1 - f_0} \right) \left(1 - f_j\right) x_{J(j)} \geq 0.$$

In addition, note that

$j \in J^2 \quad \Rightarrow \quad \Rightarrow \quad x^M \leq x^I$

Thus, x^M produces a stronger inequality (cut) than x^I.

Example (1/6)

$$
\begin{array}{llll}
\text{Maximize} & x_1 + x_2 = x_0 & & \\
\text{subject to} & -4x_1 + x_2 \le -1 & & (x_3) \\
 & 4x_1 + x_2 \le \ 3 & & (x_4) \\
\text{and} & x_1,\ \ x_2 \ \ge 0, & \text{integer.} &
\end{array}
$$

Example - MILP (2/6)

#4	1	$(-x_3)$	$(-x_4)$
x_0	$12/8$	$3/8$	$5/8$
x_1	$4/8$	$-{}^{1}/_{8}$	${}^{1}/_{8}$
x_2	1	${}^{4}/_{8}$	${}^{4}/_{8}$
x_3	0	-1	0
x_4	0	0	-1
x^M	$-{}^{4}/_{8}$	$-{}^{1}/_{8}$	$-{}^{1}/_{8}$

$$f_{10} = \frac{4}{8}$$

$$f_{11} = \frac{7}{8}$$

$$f_{12} = \frac{1}{8}$$

#5	1	$(-x^M)$	$(-x_4)$
x_0	0	3	$1/4$
x_1	1	-1	$1/4$
x_2	-1	4	0
x_3	4	-8	1
x_4	0	0	-1
x^M	0	-1	0

Example - ILP (3/6)

#4	1	$(-x_3)$	$(-x_4)$
$\rightarrow x_0$	$12/8$	$3/8$	$5/8$
x_1	$4/8$	$-1/8$	$1/8$
x_2	1	$4/8$	$4/8$
x_3	0	-1	0
x_4	0	0	-1
x_5	$-4/8$	$-3/8$	$-5/8$

$(x_1 = 4/8, x_2 = 1)$

Minimun $\{\frac{3/8}{|-3/8|}\ ,\ \frac{5/8}{|-5/8|}\}$ tie;

to maintain lexicographically positive columns α_1 and α_2, compute

Minimun $\{\frac{-1/8}{|-3/8|}\ ,\ \frac{1/8}{|-5/8|}\} = -\frac{1}{3}$;

x_3 becomes basic.

Example - ILP (4/6)

#5	1	$(-x_5)$	$(-x_4)$
x_0	1	1	0
→ x_1	$2/3$	$-1/3$	$1/3$
x_2	$1/3$	$4/3$	$-1/3$
x_3	$4/3$	$-8/3$	$5/3$
x_4	0	0	-1
x_5	0	-1	0
x_6	$-2/3$	$-2/3$	$-1/3$

$(x_1 = 2/3, x_2 = 1/3)$

#6	1	$(-x_5)$	$(-x_6)$
x_0	1	1	0
x_1	0	-1	1
x_2	1	2	-1
x_3	-2	-6	5
x_4	2	2	-3
x_5	0	-1	0
x_6	0	0	-1

$(x_1 = 0, x_2 = 1)$

Example - ILP (5/6)

#7	1	$(-x_3)$	$(-x_6)$
x_0	$4/6$	$1/6$	$5/6$
x_1	$2/6$	$-1/6$	$1/6$
x_2	$2/6$	$2/6$	$4/6$
x_3	0	-1	0
x_4	$8/6$	$2/6$	$-8/6$
x_5	$2/6$	$-1/6$	$-5/6$
x_6	0	0	-1
x_7	$-4/6$	$-1/6$	$-5/6$

$(x_1 = x_2 = 2/6)$

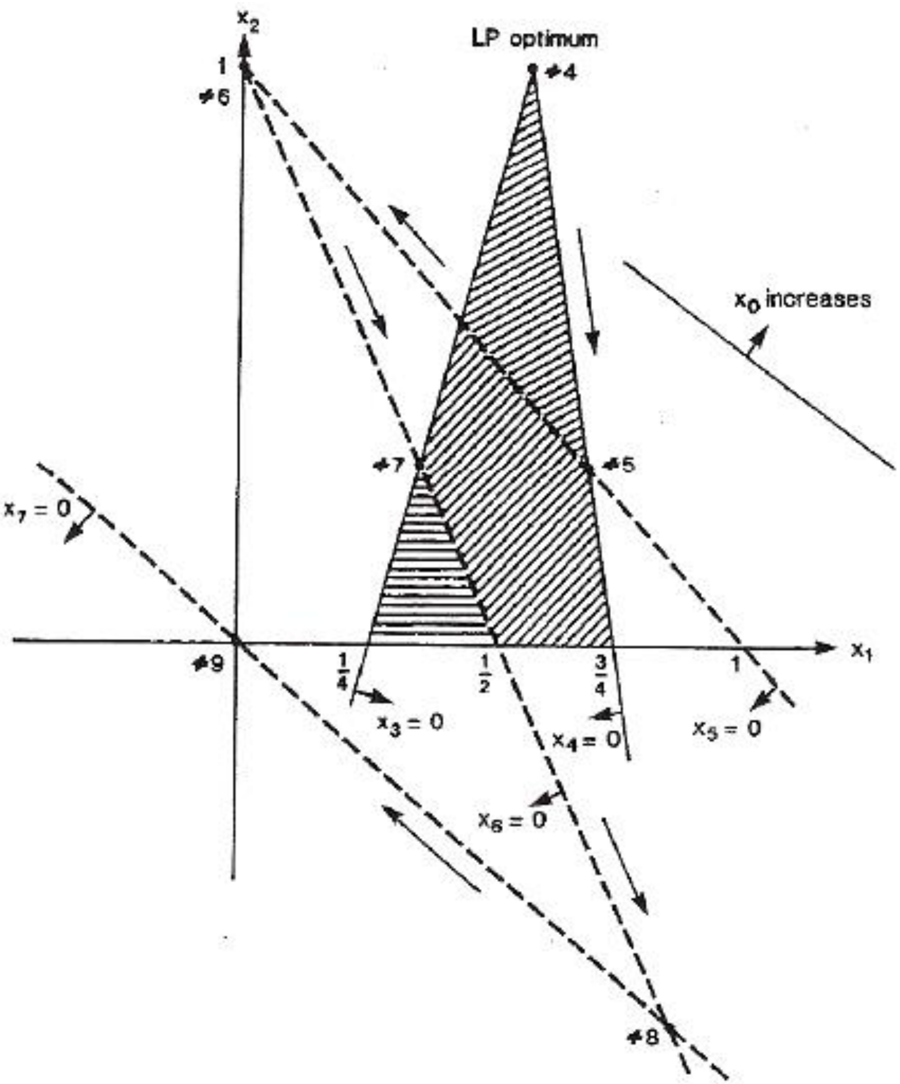
#8	1	$(-x_7)$	$(-x_6)$
x_0	0	1	0
x_1	1	-1	1
x_2	-1	2	-1
x_3	4	-6	5
x_4	0	2	-3
x_5	1	-1	0
x_6	0	0	-1
x_7	0	-1	0

$(x_1 = 1, x_2 = -1)$

#9	1	$(-x_7)$	$(-x_2)$
x_0	0	1	0
x_1	0	1	1
x_2	0	0	-1
x_3	-1	4	5
x_4	3	-4	-3
x_5	1	-1	0
x_6	1	-2	-1
x_7	0	-1	0

$(x_1 = x_2 = 0)$

Example- ILP (6/6)



Questions?

Reminder:

- No class on 4/6
- HW3 due on 11:59 AM 4/17(Monday)
- Midterm on April 20