■ Definition: Mix-Strategy NE (2 Players)

In the two-player normal-form game, the mixed strategy (p_1^*, p_2^*) is a Nash Equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy.

■ Definition: Mix-Strategy NE (n Players)

A mixed strategy profile $(p_1^*, p_2^*, \cdots, p_n^*)$ is a Mixed-Strategy Nash Equilibrium if for each player i, p_i^* is a best response to $p_{-i}^* = (p_1^*, \cdots, p_{i-1}^*, p_{i+1}^*, \cdots, p_n^*)$.

■ Existence of a Mixed-Strategy Equilibrium

Theorem (Nash, 1950) Every finite normal form game has a mixed-strategy equilibrium.

■ Preliminary

- ✓ If A and B are non-empty sets, then the **Cartesian Product** $A \times B$ is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.
- ✓ Let $\{a_n\}$ be a convergent sequence in $\mathbb{R}^p$ and $a \in \mathbb{R}^p$. Then L is a limit of $\{a_n\}$ if and only if for each $\epsilon > 0$, there exist $K(\epsilon) \in \mathbb{N}$ such that

$$n \geq K(\epsilon) \implies |a_n - L| < \epsilon$$

- ✓ A **compact** set must be both **closed** and **bounded**.
- ✓ A set C is **convex** if the line segment between any two points in C lies in C, i.e. $\forall x_1, x_2 \in C$, $\forall \lambda \in [0, 1]$:

$$\lambda x_1 + (1 - \lambda) x_2 \in C$$

- ✓ A set S is **closed** if and only if for any convergent sequence of points $\{a_n\}$ contained in S with limit point $L \in S$.
- ✓ Suppose that $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a correspondence. Then the **graph** of F is defined to be

$$G_F = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : y \in F(x)\}.$$

- ✓ A correspondence C is called **upper semi-continuous (usc)**, if the graph of the correspondence $\{(x, y) : y \in C(x)\}$ is closed.

■ Kakutani's Fixed-Point Theorem

Let $S \subset \mathbb{R}^n$ be a compact and convex set. Let C be a correspondence from S into itself that is upper semi-continuous and convex valued. Then, there is an $x^* \in S$ such that $x^* \in C(x^*)$.

此固定點定理的用處在於，上述 $C(x^*)$ 即為最佳反應(Best Response)：

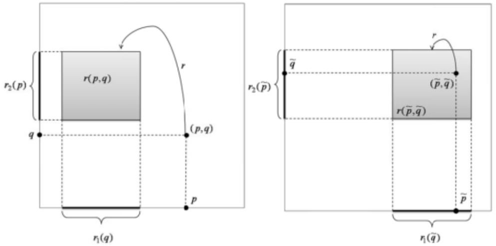
$$x^* = r(p^*, q^*) \text{ if } (p^*, q^*) \text{ is a Nash Equilibrium.}$$

$$\Longleftrightarrow$$

q^* is one of Best Response to p^*.
p^* is one of Best Response to q^*.

■ Notations

- ✓ Player i's Best Response Correspondence, r_i, maps each strategy profile σ to the set of mixed strategies that maximize player i's payoff when his opponents play σ_{-i}. (Although r_i depends only on σ_{-i} and not on σ_i, we write it as a function of the strategies of all players, because later we will look for a fixed point in the space Σ of strategy profiles.)
- ✓ Player i's payoff function: $u_i(\sigma)$.
- ✓ **Theorem (Nash, 1950)** Every finite normal form game has a mixed-strategy equilibrium.
- ✓ The correspondence in Nash's Theorem:



■ Stackelberg Model of Duopoly

A dominant firm moves first and a subordinate firm moves second:

(1) Firm 1 chooses a quantity $q_1 \geq 0$
(2) Firm 2 observes q_1 and then chooses a quantity $q_2 \geq 0$
(3) Payoff to firm i is given by the profit function:

$$\pi_i(q_i, q_j) = q_i[P(Q) - c] = q_i[(a - Q) - c]$$

<u>Compute 2's Best Response to 1</u>

$$\max_{q_2 \geq 0} \pi_2(q_1, q_2) = \max_{q_2 \geq 0} q_2[(a - q_2 - q_2) - c]$$

$\boxed{\text{F.O.C}}$ $\frac{d\pi_2(q_1, q_2)}{dq_2} = 0 \implies R_2(q_1) = \frac{a - q_1 - c}{2}$

<u>Compute 1's Best Response to 2 (According to q₂)</u>

$$\max_{q_1 \geq 0} \pi_1(q_1, (R_2(q_1))) = \max_{q_1 \geq 0} q_1 \times \frac{a - q_1 - c}{2}$$

$\boxed{\text{F.O.C}}$ $\frac{d\pi_1(q_1, R_2(q_1))}{dq_1} = 0 \implies q_1^* = \frac{a - c}{2}$

Then we have $q_1^* = \frac{a-c}{2}$ and $q_2^* = R_2(q_1^*) = \frac{a-c}{4}$

Homework 03-01

Let $x_n = 1/n,\ n \geq 1$. By the definition of sequence convergence, prove the sequence x_n converges to 0.

Calculate the limit

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(1/n) = 0.$$

$\forall \epsilon > 0$, there is a corresponding positive integer $N \in \mathbb{N}$ such that $(1/N) < \epsilon$ (by the Archimedean Property)

Whenever $n \geq N$, we have that

$$|x_n - 0| = |\tfrac{1}{n} - 0| = \tfrac{1}{n} \leq \tfrac{1}{N} < \epsilon$$

Q.E.D

Homework 03-02

Let X and Y be sequences in $\mathbb{R}^p$ that converge to x and y respectively. Prove that $X+Y$ converges to $x+y$.

Since the sequences X and Y are convergent. $\forall \epsilon > 0$, there are corresponding positive integers $N_1, N_2 \in \mathbb{N}$ so that

$$\text{If } n > N_1 \text{ then } |X - x| < \tfrac{\epsilon}{2}, \text{ and}$$

$$\text{If } n > N_2 \text{ then } |Y - y| < \tfrac{\epsilon}{2}$$

Thus whenever we take $n > \max\{N_1, N_2\} = N$, then

$$|(X+Y) - (x+y)| \leq |X - x| + |Y - y| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

Q.E.D

Homework 03-03

(a) $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$ is closed in $\mathbb{R}$. True

(b) $\{x \in \mathbb{R} : x \geq 0\}$ is closed in $\mathbb{R}$. True

(c) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is closed in $\mathbb{R}^2$. True

(d) $\{x \in \mathbb{R} : 0 \leq x < 1\}$ is closed in $\mathbb{R}$. False

Closed Set, True or False?

Definition A subset S of a metric space (X, d) is **closed** if it is the complement of an open set.

Definition A subset S of a metric space (X, d) is **open** if it contains an open ball about each of its points – i.e., if

$$\forall x \in S : \exists \epsilon > 0 : B(x, \epsilon) \subseteq S$$

Homework 03-04

Please give an example of the sequence and closed set to demonstrate that a set S is closed if and only if for any convergent sequence of points $\{x_k\}$ contained in S with limit point $\bar{x}$, we also have that $\bar{x} \in S$.

We can choose the sequence $\{x_k\} = 1/k^2$ and the closed set $[0, \infty)$. So that the sequence converges to $0 \in [0, \infty)$.

In fact, $[0, \infty)$ is closed, since every sequence of positive numbers converging to a limit would have an non-negative limit which is in $[0, \infty)$.

Homework 03-05

Disprove that $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i{}^2 = 1\}$ is convex.

Definition A set C is **convex** if the line segment between any two points in C lies in C, $\forall x_1, x_2 \in C, \forall \theta \in [0,1]$:

$$\theta x_1 + (1-\theta) x_2 \in C$$

By the definition above. Let arbitrary two point $x_1, x_2 \in \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i{}^2 = 1\},\ 0 \leq \theta \leq 1$.

$$|\theta x_1 + (1-\theta)x_2| \leq \theta|x_1| + (1-\theta)|x_2|$$

Since that

$$|\theta x_1 + (1-\theta)x_2| \leq \theta|x_1| + (1-\theta)|x_2| \leq \theta + (1-\theta)$$

only hold when $x_1, x_2 \in \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i{}^2 \leq 1\}$. The set $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i{}^2 = 1\}$ is not convex.

Homework 04

	Head (q)	Tail (1-q)
Head (p)	-1,1	1,-1
Tail (1-p)	1,-1	-1,1

Notations:

- The strategy profile: $\sigma = (p, q)$
- The strategy profile except player i's action: σ_{-i}
- The space of strategy profile: Σ
- Player i's payoff function: $u_i(\sigma)$
- Player i's best response correspondence, r_i, maps each strategy profile σ to the set of mixed strategies that maximize player i's payoff when his opponents play σ_{-i}
- The correspondence $r : \Sigma \to \Sigma$ to be the Cartesian product of the r_i
- The graph G_R of the correspondence r: $G_R = \{(p, q, \hat{p}, \hat{q} : (\hat{p}, \hat{q}) \in r(p,q)\}$

Homework 04-01

Please explicitly write the following terms in this example.

(a) The space of action (or strategy) profile Σ

$$\Sigma = [0,1] \times [0,1]$$

(b) Player 1's expected payoff function $u_1(\sigma)$

$$u_1(\sigma) = p[(-q) + (1-q)] + (1-p)[q + (1-q)(-1)]$$

$$= 2p + 2q - 4pq - 1$$

Homework 04-02

Suppose $\sigma' = (p', q') \in r(p,q)$ and $\sigma'' = (p'', q'') \in r(p,q)$ where $\sigma = (p,q)$. Show that $\lambda p' + (1-\lambda)p''$ is player 1 's best response to q for $\lambda \in (0,1)$.

I. $q < 1/2$

$p' = p'' = \lambda p' + (1-\lambda)p'' \implies p = 1$

II. $q = 1/2$

$p', p'' \in [0,1]$, can be arbitrary number between 0 and 1

$p' = p'' = \lambda p' + (1-\lambda)p'' \in [p', p'']$ or $[p'', p']$

III. $q > 1/2$

$p' = p'' = \lambda p' + (1-\lambda)p'' \implies p = 0$

In any case, $\lambda p' + (1-\lambda)p''$ is player 1's best response to q for $\lambda \in (0,1)$. In addition, $\forall i,\ u_i(\lambda p' + (1-\lambda)p'', q)$ $= \lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda)u_i(\sigma_i'', \sigma_{-i})$ holds. So $\lambda\sigma' + (1-\lambda)\sigma'' \in r(\sigma) = r(p,q)$

$\implies \lambda p' + (1-\lambda)p'' \in r_1(q)$ for $\lambda \in (0,1)$

Homework 04-03

Show that correspondence $r(p,q)$ in this example is convex for all $(p,q) \in \Sigma$.

Suppose $r(p,q)$ is not convex for all $(p,q) \in \Sigma$. i.e. $\sigma', \sigma'' \in r(\sigma)$, but $\lambda\sigma' + (1-\lambda)\sigma'' \notin r(\sigma) = r(p,q)$.

However, we have $u_i(\lambda p' + (1-\lambda)p'', q)$ $= \lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda)u_i(\sigma_i'', \sigma_{-i}),\ \forall i = 1,2$

Because $\lambda\sigma_i' + (1-\lambda)\sigma_i''$ is exactly Player 1's best response to σ_{-i}. It gets a contradiction, we can proof the statement by contrapositive.

Q.E.D

Homework 04-04

Let sequence $(p_n) = (1/4^n)$ and $(q_n) = (1/4^n)$. What are the best response sequence $(\hat{p})$ and $(\hat{q})$?

$$(p_n) = (\tfrac{1}{4}, \tfrac{1}{4^2}, \cdots, \tfrac{1}{4^n}) \implies (\hat{q_n}) = (0, 0, \cdots, 0)$$

$$(q_n) = (\tfrac{1}{4}, \tfrac{1}{4^2}, \cdots, \tfrac{1}{4^n}) \implies (\hat{p_n}) = (1, 1, \cdots, 1)$$

Homework 04-05

What is the limit point of the sequence $(p_n, q_n, \hat{p}_n, \hat{q}_n)$, where $p_n = 1/4^n$ and $q_n = 1/4^n$ for all n?

$$(0, 0, 1, 0)$$

Homework 04-06

Suppose that a sequence $(p_n, q_n, \hat{p}_n, \hat{q}_n)$ converges to $(p, q, \hat{p}, \hat{q})$ for all sequence index n, but $\hat{p}$ is not player 1 's best response to q. Show that $\hat{p}_n$ cannot be player 1's best response to $\hat{q}_n$.

Homework 04-07

Show that $r(\cdot)$ has a closed graph.

Problem Set 1.13

Each of two firms has one job opening. Suppose that (for reasons not discussed here but relating to the value of filling each opening) the firms offer different wages: firm i offers the wage w_i, where $(1/2)w_1 < w_2 < 2w_1$. Imagine that there are two workers, each of whom can apply to only one firm. The workers simultaneously decide whether to apply to firm 1 or to firm 2. If only one worker applies to a given firm, that worker gets the job; if both workers apply to one firm, the firm hires one worker at random and the other worker is unemployed (which has a payoff of zero).

(a) Represent this game using the normal form.

		Worker 2	
		Firm 1	**Firm 1**
Worker 1	**Firm 1**	$\frac{1}{2}w_1, \frac{1}{2}w_1$	w_1, w_2
	Firm 2	w_2, w_1	$\frac{1}{2}w_2, \frac{1}{2}w_2$

(b) Solve for the Nash equilibrium of the workers' normal-form game.

The player 1's mixed strategy is $\sigma_1 = (p, 1-p)$

The player 2's mixed strategy is $\sigma_2 = (q, 1-q)$

Compute the expected values.

Player 01

$$E_1(\text{Firm1}|\sigma_2) = q(\tfrac{w_1}{2}) + (1-q)w_1 = w_1(1 - \tfrac{q}{2})$$

$$E_1(\text{Firm2}|\sigma_2) = qw_2 + (1-q)(\tfrac{w_2}{2}) = w_2(\tfrac{1+q}{2})$$

Player 02

$$E_2(\text{Firm1}|\sigma_1) = p(\tfrac{w_1}{2}) + (1-p)w_1 = w_1(1 - \tfrac{p}{2})$$

$$E_2(\text{Firm2}|\sigma_1) = pw_2 + (1-p)(\tfrac{w_2}{2}) = w_2(\tfrac{1+p}{2})$$

Suppose $\sigma_1 = (1,0)$

Player 02

$$E_2(\text{Firm1}|\sigma_1) = \tfrac{w_1}{2} \quad E_2(\text{Firm2}|\sigma_1) = w_2$$

Given that $w_2 > w_1/2$, the best response for player 2 is $\sigma_2 = (0,1)$. Given $\sigma_2 = (0,1)$ the expected payoff of player 1 are:

$$E_1(\text{Firm1}|\sigma_2) = w_1 \quad E_1(\text{Firm2}|\sigma_2) = \tfrac{w_2}{2}$$

Given that $w_1 > w_2/2$, the best response for player 1 is $\sigma_1 = (1,0)$. Then $\sigma_1 = (1,0)$ and $\sigma_2 = (0,1)$ is a NE.

Suppose $\sigma_1 = (0,1)$

Player 02

$$E_2(\text{Firm1}|\sigma_1) = w_1 \quad E_2(\text{Firm2}|\sigma_1) = \tfrac{w_2}{2}$$

Given that $w_1 > w_2/2$, the best response for player 2 is $\sigma_2 = (1,0)$. Given $\sigma_2 = (1,0)$ the expected payoff of player 1 are:

$$E_1(\text{Firm1}|\sigma_2) = \tfrac{w_1}{2} \quad E_1(\text{Firm2}|\sigma_2) = w_2$$

Given that $w_2 > w_1/2$, the best response for player 1 is $\sigma_1 = (0,1)$. Then $\sigma_1 = (0,1)$ and $\sigma_2 = (1,0)$ is a NE.

Suppose $\sigma_1 = (p, 1\text{-}p)$

In this case must be that $E_1(\text{Firm1}|\sigma_2) = E_1(\text{Firm2}|\sigma_2)$:

$$w_1(1 - \tfrac{q}{2}) = w_2(\tfrac{1+q}{2})$$

This is true only if $q = \frac{2w_1 - w_2}{w_1 + w_2}$.

But for player 2, in order to play such a strategy, the condition satisfied $E_2(\text{Firm1}|\sigma_1) = E_2(\text{Firm2}|\sigma_1)$:

$$w_1(1 - \tfrac{p}{2}) = w_2(\tfrac{1+p}{2})$$

This is true only if $p = \frac{2w_1 - w_2}{w_1 + w_2}$.

Result

There are 3 Nash equilibrium:

(1) $\sigma_1 = (1,0)$ and $\sigma_2 = (0,1)$

(2) $\sigma_1 = (0,1)$ and $\sigma_2 = (1,0)$

(3) $\sigma_1 = (\frac{2w_1 - w_2}{w_1 + w_2}, \frac{2w_2 - w_1}{w_1 + w_2}),\ \sigma_2 = (\frac{2w_1 - w_2}{w_1 + w_2}, \frac{2w_2 - w_1}{w_1 + w_2})$