Lecture 2: Dynamic Games of Complete Information

Concept – dynamic games of complete information

- When players interact by playing a similar stage game numerous times, the game is called a dynamic or repeated game. Unlike simultaneous games, players have at least some information about the strategies chosen by others and thus their play may be contingent on past moves.
- Dynamic games that have not only complete but also perfect information: at each move in the game, the player knows the full history of the play of the game and only one player makes a decision.
- Dynamic games of complete but imperfect information: at some move, the player does not know the history of the game or multiple players simultaneously make decisions.

First example-Cash in a hat

- Player 1 can put \$0, \$10, or \$30 in a hat.
- The hat is passed to player 2.
- Player 2 can either add the same amount ("match") or take the cash away.
- Payoffs:
  - Player 1 has three choices

    \$0 payoff: \$0

    \$10 payoff: double if player 2 matches, -10 if not

    \$30 payoff: double if player 2 matches, -30 if not
  - Player 2

    Net payoff: \$15 if match \$10

    Net payoff: \$20 if match \$30

    The cash in the hat if takes
- It is not only a "Cash in a hat" game. Player 1 (Investment Bank), Player 2 (New Firm). Playing the game…

- What is the major difference between the game of cash in a hat and the games we talked about so far?

Game tree-Cash in a hat

Discussion

- Moral hazard: If player 2 takes the investment away, or simply shirks, it called “Moral hazard.”
  Moral hazard: An agent has incentives to do things that are bad for the principal.
  e.g., car insurance
- How to induce player 1 and player 2 to choose (“30”, “match”)?
  - Redistribution of net payoff

- Collateral

Stackelberg model of duopoly (Leader-follower)

- Two players: Players 1 and 2; Two-stage game.
- Two competing firms, selling a homogeneous good
- The marginal cost of producing each unit of the good: c_1 and c_2.
- Firm 1 moves first and decides on the quantity to sell: q_1.
- Firm 2 moves next after seeing q_1 and decides on the quantity to sell: q_2.
- $Q = q_1 + q_2$ the total market demand.
- The market price, P is determined by (inverse) market demand: $P = a - Q$ if $a > Q$, $P = 0$ otherwise.
- Both firms seek to maximize profits
- Q_j: the space of feasible q_j's, $j = 1, 2$

Analysis:

- First mover advantage: the advantage gained by the initial occupant of a market segment.

  $q_1 > q_2 \quad \Rightarrow \pi_1 > \pi_2 \quad (\because c_1 = c_2$ in this particular case)
- Example: Stackelberg competition
- P= 130-(q_1+q_2), c_1=c_2=c =10

- In the Stackelberg game, it was easy to see how player 2 "should" play. Because once q_1 was fixed, player 2 faced a simple decision problem. This allows us to solve for player 2's optimal second-stage choice for any arbitrary q_1 and then work backward to find the optimal choice for player 1.
- Though the Stackelberg outcome may seem the natural prediction in this game, there are many other Nash equilibria.
  - Cournot equilibrium (40, 40) is also an equilibrium for the Stackelberg game!
  - Condition on q_1 = 40, what is firm 2's best response to q_1 = 40? $s_2(q_1)$=40.
  - Condition on q_2 = 40, firm 1's best response to q_2 = 40 is q_1=40.

    If player 1 "anticipates" that player 2 will choose Cournot quantity q_2 = 40, then player 1's best response is q_1=40.
  - Note: sequential-move games sometimes have multiple Nash equilibria, only one of which is associated with the backward induction outcome of the game.

Multi-stage games with observed actions

- Definition:
  - All players knew the actions chosen at all previous stages 0, 1, 2,…, k-1 when choosing their actions at stage k.
  - All players move “simultaneously” in each stage k.
- We adopt the convention that the first stage is “stage 0” in order to simplify the notation.
- Note: players move simultaneously in stage k if each player chooses his or her action at stage k without knowing the stage-k action of any other player.
- “Simultaneous moves” does not exclude games where players move in alternation, as we allow for the possibility that some of the players have the one-element choice set “do nothing.”
  - For example, the Stackelberg game has two stages: in the first stage, the leader chooses an output level (and the follower “does nothing”). In the second stage, the follower knows the leader’s output and chooses an output level of his own (and the leader “does nothing”).
- Notation:
  - Let $h^0 = \varnothing$ be the “history” at the start of play and $a^k \equiv (a_1^k,...,a_N^k)$ be the stage-k action profile.
  - h^{k+1}: History at the end of stage k, $h^{k+1} = (a^0, a^1,..., a^k)$
  - Let K+1 denote the total number of stages in the game.
  - $A_i(h^{k+1})$: player i’s feasible actions in stage k+1 when the history is h^{k+1}.
  - H^k: the set of all stage-k histories
  - s_i: Pure strategy for player i that specifies an action $a \in A_i(h^k)$ for each k and each history h^k (a contingent plan of how to play in each stage k for possible history h^k)
  - s: Pure strategy for all players is a contingent plan of how to play in each stage k for possible history h^k.
- Since all players know the history h^k of moves before stage k, we can view the game from stage k on with history h^k as a game in its own right. (subgame)
- G(h^k): game from stage k on with history h^k
- Any strategy profile s of the whole game induces a strategy profile $s|h^k$ on any $G(h^k)$. (For each player i, $s_i|h^k$ is simply the restriction of s_i to the histories consistent with h^k.
- Example: the Stackelberg game
- $h^0 = \varnothing$, $h^1 = (a^0)$: (firm 1 (the leader) chose q_1 and firm 2 (the follower) “did nothing” in stage 0)

- What is $G(h^1)$? What is $s_2|h^1$? (firm 2's strategy given history h^1?)

Hungry lions game[1]

Dynamic Games of Complete and Perfect Information

- A multistage game has perfect information if
  - For every stage k and history h^k, exactly one player has a nontrivial action set, and all other players have one-element action set "do nothing." A simple example of such a game has player 1 moving in stages 0, 2, 4, etc. and player 2 moving in stages 1, 3, 5, and so on.
  - Each player knows all previous moves when making a decision.
- In a finite game of perfect information, the number of stages and the number of actions at any stage are finite.

Two-Stage Games of Complete and Perfect Information

Application: Wages and Employment in a Unionized Firm **(Leontief 1946)** (The firm and the union bargain over wages.)

- Two players: a firm and a union (the union has exclusive control over wages, but the firm has exclusive control over employment.)
  - The union determines the level of the wage
  - The firm determines the employment, i.e., the number of employees the firm hires.
- The union's utility function is $U(w, L)$ where w is the wage and L is employment. Assume that $U(w, L)$ increases in both w and L.
- The firm's profit function is $\pi(w,L) = R(L) - wL$, where $R(L)$ is the revenue the firm can earn if it hires L workers. Assume that $R(L)$ is increasing and concave (diminishing returns in marginal revenue).
- The timing of the game: (1) the union makes a wage demand, w; (2) the firm observes (and accepts) w and then chooses employment, L; (3) payoffs are $U(w, L)$ and $\pi(w,L)$.
- First, we can characterize the firm's best response in stage (2), $L^*(w)$, to an arbitrary wage demanded by the union in stage (1), w. Given w, the firm chooses $L^*(w)$ (employment's best response) to solve

[1] The example is available at Yale open course, game theory.

Analysis:

- Firm's isoprofit curves

- Union's indifference curves

- The union's problem at the first stage:

- The equilibrium point is inefficient.

Two-stage games of complete but imperfect information

Application 1: The model of strategic investment in a duopoly

- Firm 1 and Firm 2 currently both have a constant average cost of 2 per unit.
- Firm 1 can install a new technology with an average cost of 0 per unit; installing the technology costs f.
- Firm 2 will observe whether or not firm 1 invests in the new technology.
- Once firm 1's investment decision is observed, the two firms will simultaneously choose output levels q_1 and q_2 as in Cournot competition. Thus, this is a two-stage game. (why is it a two-stage games with *imperfect information*?)
- We suppose that the demand is $p(q) = 14 - q$ and that each firm's goal is to maximize its net revenue minus costs.
- Firm 1's payoff is then $[12 - (q_1 + q_2)]q_1$ if it does not invest, and $[14 - (q_1 + q_2)]q_1 - f$ if it does;
- Firm 2's payoff is $[12 - (q_1 + q_2)]q_2$.
- Backward induction:

Application 2: Tariffs and Imperfect International Competition

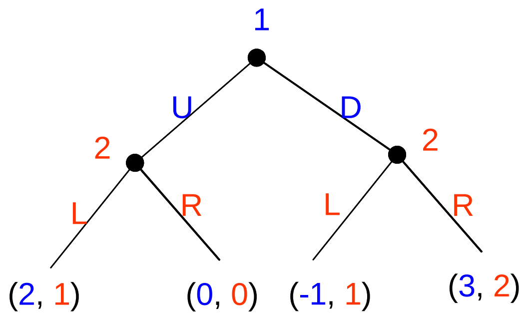
- Consider two identical countries, denoted by $i = 1, 2$. Each country has a government that chooses a tariff rate, a firm that produces output for both home consumption and export, and consumers who buy on the home market from either the home firm or the foreign firm.
- Total quantity on the market in country i is Q_i. The market-clearing price is $P_i(Q_i) = a - Q_i$.
- The firm in country i (hereafter called firm i) produces h_i for home consumption and e_i for export. Thus, $Q_\text{i} = h_i + e_j$.
- The firms have a constant marginal cost, c, and no fixed costs. Thus, the total cost of production for firm i is $C_i(h_i, e_i) = c(h_i + e_i)$. (a function of the total production quantity)
- The firms also incur tariff costs on exports: if firm i exports e_i to country j when government j has set the tariff rate t_j, then firm i must pay t_je_i to government j. t_i : tariff rate charged by country i for goods imported from country j.
- First, the governments simultaneously choose tariff rates, t_1 and t_2. Second, the firms observe the tariff rates and simultaneously choose quantities for home consumption and for export, (h_1, e_1) and (h_2, e_2). Third, payoffs are profit to firm i and total welfare to government i, where total welfare to country i = consumers' surplus enjoyed by the consumers in country i + the profit earned by firm i + the tariff revenue collected by government i from firm j.

Extensive-form representation of games

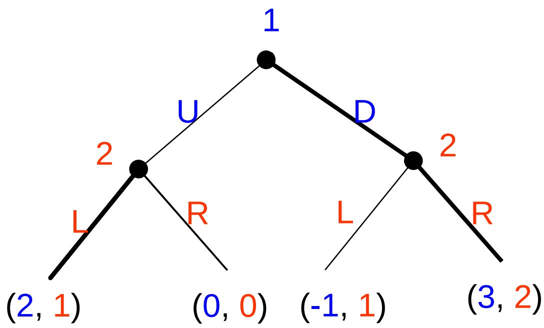
- The extensive-form representation of a game specifies:
  - The players in the game
  - When each player has the move,
  - The players' payoffs as a function of the moves that were made,
  - The set of actions available to the players when they move,
  - Each player's information when he makes his move

Example of extensive-form

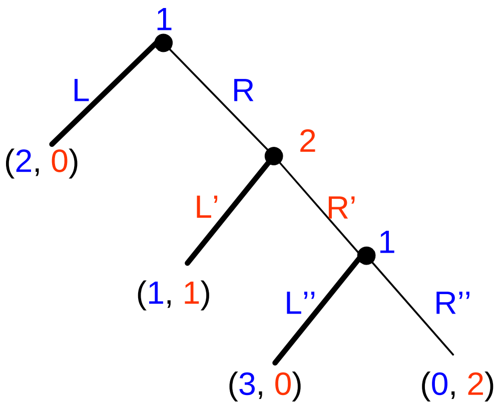
- Player 1 chooses an action a_1 from the feasible set A_1={U, D}.
- Player 2 observes a_1 and then chooses an action a_2 from the set A_2={L, R}.
- Payoffs are $\pi_1(a_1, a_2)$ and $\pi_2(a_1, a_2)$, as shown in the game tree.
- Strategy: A complete plan of action-it specifies a feasible action for the player in every contingency in which the player might be called upon to act.



- Player 1 strategies: {U, D}
- Player 2 strategies: {L, L}, {L, R}, {R, L}, {R, R}
  - e.g. {L, R}: if player 1 chooses U, player 2 chooses L; if player 1 chooses D, player 2 chooses R.
- Backward induction:



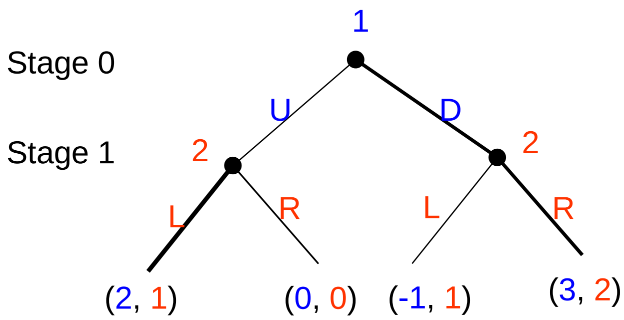
- **Example**:
  - Player 1 chooses *L* or *R*, where *L* ends the game with payoffs of 2 to player 1 and 0 to player 2.
  - Player 2 observes 1's choice. If 1 chose *R* then 2 chooses *L*' or *R*', where *L*' ends the game with payoffs of 1 to both players.
  - Player 1 observes 2's choice (and recalls his or her own choice in the first stage). If the earlier choices were *R* and *R*' then 1 chooses *L*'' or *R*'', both of which end the game, *L*'' with payoffs of 3 to player 1 and 0 to player 2 and *R*'' with analogous payoffs of 0 and 2.
- Extensive form and backward induction:



Subgame-perfect equilibrium

- Any strategy profile s of the whole game induces a strategy profile $s|h^k$ on any $G(h^k)$.
- Difference between an action and a strategy:
  - A player's strategy is a complete plan of action-it specifies a feasible action for the player in every contingency in which the player might be called upon to act.
  - In a static game of complete information, for example, a strategy is simply an action.
- A strategy profile s of a multistage game with observed actions is a ***subgame-perfect Nash equilibrium*** if, for every h^k, the restriction $s|h^k$ is a Nash equilibrium of subgame $G(h^k)$.
- A strategy profile is a subgame perfect equilibrium of a game, G, if this strategy profile is also a Nash equilibrium for every subgame of G.
- For each player j, $s_j|h^k$ is the restriction of s_j to the histories consistent with h^k.

Example:



- Backward induction outcome?

- Subgame perfect Nash equilibrium?

Example: Stackelberg competition

- P= 130-(q_1+q_2), c_1=c_2=c =10
- Backward induction
  - Firm 2 strategy: $s_2(q_1) = q_2 = 60 - q_1 / 2$
  - Firm 1 strategy: substitute q_2 into firm 1's payoff function and solve $\max_{q_1} \pi_1(q_1, s_2(q_1))$

    to obtain $q_1 = 60$
  - The outcome (60, 30) is a Nash equilibrium in the Stackelberg game.
- Is (60, 30) the unique equilibrium in this game?
  - Cournot equilibrium (40, 40) is also an equilibrium for the Stackelberg game!
  - Condition on q_1 = 40, $s_2(q_1)$=40.
  - Condition on q_2 = 40, the best response of player 1 is q_1=40.

    If player 1 "anticipates" that player 2 will choose Cournot quantity q_2 = 40, then player 1's best response is q_1=40.
  - Note: sequential-move games sometimes have multiple Nash equilibria, only one of which is associated with the backward induction outcome of the game.
  - However, the outcome (40, 40) is NOT ***subgame perfect***, because the strategy $s_2(q_1) = 40$ that does not induce a Nash equilibrium in stage 2 for player 2 for histories other than q_1= 40.
  - How about (60,30)? (60, 30) is not subgame perfect because the strategy $s_2(q_1) = 30$ that does not induce a Nash equilibrium in stage 2 for player 2 for histories other than q_1= 30.
  - What is the Subgame perfect Nash equilibrium of the Stackelberg game?

Repeated Games

- In this section we analyze whether future behavior can influence current behavior in repeated relationships.
- Two-stage repeated games
- Consider the Prisoners' Dilemma game again. Suppose two players play this simultaneous-move game ***twice***, observing the outcome of the first play before the second play begins, and suppose the payoff for the entire game is simply the sum of the payoffs from the two stages (i.e., there is no discounting).

	Player 2		
Player 1		L2	R2
	L1	1, 1	5, 0
	R1	0, 5	4, 4

- We analyze the first stage game by taking into account that the outcome of the game remaining in the second stage will be the Nash equilibrium of that remaining game.
- The payoff pair (1, 1) for the second stage has been added to each first-stage payoff pair.

	Player 2		
Player 1		L2	R2
	L1		
	R1		

- The unique subgame-perfect outcome of the two-stage Prisoners' Dilemma is (L1, L2) in the first stage, followed by (L1, L2) in the second stage.
- Here we depart from the two-period case to allow for any finite number of repetitions.
- Let $G = \{A_1,\ldots,A_n; u_1,\ldots,u_n\}$ denote a static game of complete information in which players 1 through n simultaneously choose actions a_1 through a_n from the action spaces A_1 through A_n, respectively, and payoffs are $u_1(a_1,\ldots,a_n)$ through $u_n(a_1,\ldots,a_n)$. The game G will be called the **stage game** of the repeated game.
- **Definition** Given a stage game G, let $G(T)$ denote the **finitely repeated game** in which G is played T times, with the outcomes of all preceding plays observed before the next play begins. The payoffs for $G(T)$ are simply the sum of the payoffs from the T stage games.
- **Proposition** If the stage game G has a ***unique*** Nash equilibrium then, for any finite T, the repeated game $G(T)$ has a unique subgame-perfect outcome: the Nash equilibrium of G is played in every stage.
- What if the stage game has multiple Nash equilibria?

- **Example**

		Player 2		
		L	M	R
	L	<u>1</u>, <u>1</u>	<u>5</u>, 0	0, 0
Player 1	M	0, <u>5</u>	4, 4	0, 0
	R	0, 0	0, 0	<u>3</u>, <u>3</u>

- The stage game is played twice.
- The first-stage outcome is observed before the second stage begins.
- In the first stage, the players anticipate that the second-stage outcome will be a Nash equilibrium of the stage game. Since this stage game has more than one Nash equilibrium, the different first-stage outcomes will be followed by different stage-game equilibria in the second stage.
- In the finitely repeated game $G(T)$, a player's strategy specifies the player's actions in each stage, for each possible history of play through the previous stages.
- One example of a partial strategy for stage 2:
  - Play R in stage 2 if stage 1 outcome is (M, M); otherwise, play L in stage 2.
- Suppose both players chose this ***strategy***. (Recall the definition of a strategy: a complete plan of action-it specifies a feasible action for the player in every contingency in which the player might be called upon to act.)
- Modified stage 1 game based on the strategy:
  - (3, 3) has been added to the (M, M) cell and (1,1) has been added to the eight other cells.

			Player 2	
		L	M	R
	L			
Player 1	M			
	R			

- Three pure-strategy Nash equilibria:

- **Observation** Let G be a static game of complete information with multiple Nash equilibria. There may be subgame-perfect outcomes of the repeated game $G(T)$ in which for any $t < T$, the outcome in stage t is <u>***not***</u> a Nash equilibrium of G.
- Implication is that credible threats or promises about the future can induce cooperation in the present.

- Definition:
  - In the finitely repeated game $G(T)$, a player's strategy specifies the player's actions in each stage, for each possible history of play through the previous stages.
  - In the finitely repeated game $G(T)$, a subgame beginning at stage t+1 is the repeated game in which G is played T-t times, denoted by $G(T$-$t)$.
- In the example, all possible outcomes (histories) at the end of the first stage: (L, L) (L, M) (L, R) (M, L) (M, M) (M, R) (R, L) (R, M) (R, R)
- A considered strategy:

	(L, L)	(L, M)	(L, R)	(M, L)	(M, M)	(M, R)	(R, L)	(R, M)	(R, R)
(M;	L,	L,	L,	L,	R,	L,	L,	L,	L)

- Is the considered strategy a Nash equilibrium?

  To show that it is a Nash equilibrium of the two-stage game for both players to adopt the considered strategy, we will assume that player *i* has adopted the considered strategy and then show that it is a best response for player *j* to adopt the considered strategy also.

Infinitely Repeated Games

- As in the finite-horizon case, the main theme is that credible threats or promises about future behavior can influence current behavior. In the finite-horizon case we saw that if there are multiple Nash equilibia of the stage game G then there are may be subgame-perfect outcomes of the repeated game $G(T)$ in which, for $t<T$, the outcome of stage t is not a Nash equilibrium of G.
- A stronger result is true in infinitely repeated games: if the stage game has a unique Nash equilibrium, there may be subgame-perfect outcomes of the infinitely repeated game in which ***no*** stage's outcome is a Nash equilibrium of G.
- Consider the infinitely repeated Prisoners' Dilemma game.

	Prisoner 2		
Prisoner 1		C (cooperate)	D (defect)
	C (cooperate)	4, 4	0, 5
	D (defect)	5, 0	1, 1

- The game is repeated infinitely with the outcomes of all previous stages observed before the current stage begins.
- For each t, the outcomes of the previous t-1 stage games are observed
- Payoffs?

Discounted payoffs

- Let discount factor δ be the value today of a dollar to be received on stage later.
  - E.g., $\delta = 1/(1+r)$ where r is the interest rate per stage
- Given the discount factor δ, the present value of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, \ldots$ is

Average payoffs

- $V = \pi_1 + \delta\pi_2 + \delta^2\pi_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t$
- If we received an "average" payoff of π in every stage, then
  (A geometric series: successive term is produced by multiplying the previous term by a constant number)

Infinitely repeated games

- Given a stage game *G*, let $G(\infty, \delta)$ denote the infinitely repeated game in which *G* is repeated forever and the players share discount factor δ. For each *t*, the outcomes of the *t*-1 preceding plays of the stage game are observed before the t^{th} stage begins. Each player's payoff in $G(\infty, \delta)$ is the present value of the player's payoffs from the infinite sequence of stage games.

Infinitely repeated Prisoner's Dilemma

Definition In an infinitely repeated game $G(\infty, \delta)$, a player's *strategy* specifies the player's actions in each stage, for each possible history of play through the previous stages.

- Difference between an ***action*** and a ***strategy***:
  - A player's strategy is a complete plan of action – it specifies a feasible action for the player in every contingency in which the player might be called upon to act.
  - In a static game of complete information, for example, a strategy is simply an action.

	Player 2		
Player 1		C (cooperate)	D (defect)
	C (cooperate)	4, 4	0, 5
	D (defect)	5, 0	1, 1

- The Cooperation (C, C) can occur in every stage of a subgame-perfect outcome of the infinitely repeated game even though the only Nash equilibrium in the stage game is non-cooperation (D, D).
- Strategy:
  - Play C in the first stage. In the t^{th} stage, if the outcome of all *t*-1 preceding stages has been (C, C), then play C; otherwise, play D.
- Interpretation of the above strategy: if the players ***cooperate*** today then they play a high-payoff equilibrium tomorrow; otherwise they play a low-payoff equilibrium tomorrow.
- This strategy is an example of a trigger strategy because player i cooperates until someone fails to cooperate, which triggers a switch to non-cooperation forever after.
- In the infinitely repeated game $G(\infty, \delta)$, each *subgame* beginning at stage *t*+1 is identical to the original game $G(\infty, \delta)$.
- Statement: if the discount factor, δ, is close enough to one then it is a Nash equilibrium of the infinitely repeated game for both players to adopt this strategy. We then argue that such a Nash equilibrium is subgame-perfect.

Trigger strategies for Prisoner's Dilemma

- Proof sketch: to show that it is a Nash equilibrium of the infinitely repeated game for both players to adopt the trigger strategy, we assume that player *i* has adopted the trigger strategy and then show that, provided δ is close enough to one, it is a best response for player *j* to adopt the strategy as well.
- Assuming player 1 adopts the trigger strategy, what is the best response for player 2?

- We now want to argue that such a Nash equilibrium is subgame-perfect. We must show that the trigger strategies constitute a Nash equilibrium on every subgame of that infinitely repeated game.
- Recall that every subgame of an infinitely repeated game is identical to the game as a whole.
- The subgames can be grouped into two classes:

  (i) Subgames where the outcomes of all previous stages have been (C, C)

  The players' strategies in a subgame in (i) are again the trigger strategy, which we have shown to be a Nash equilibrium of the game as a whole.

  (ii) Subgames where the outcome of, at least one, earlier stage differs from (C, C)

  Player's strategies in a subgame in (ii) are simply to repeat (D, D) forever, which is also a Nash equilibrium of the game as a whole.

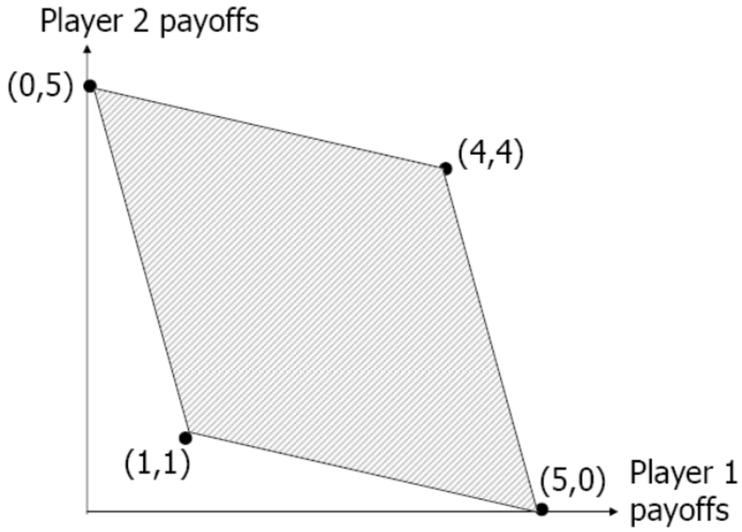
- **Observation**: Even if the stage game G has a unique Nash equilibrium, there may be subgame-perfect outcomes of the infinitely repeated game in which ***no*** stage's outcome is a Nash equilibrium of G.
- We next apply analogous arguments in the infinitely repeated game $G(\infty, \delta)$. These arguments lead to Friedman's (1971) Theorem for infinitely repeated games. To state the theorem, we need the following definition.

Feasible payoffs in the stage game

- The payoffs $(\pi^1, \pi^2, ..., \pi^n)$ are feasible in the stage game G if they are a convex combination (i.e., a weighted average, where the weights are all nonnegative and sum to one) of the pure-strategy payoffs of G.

	Prisoner 2		
		C (cooperate)	D (defect)
Prisoner 1	C (cooperate)	4, 4	0, 5
	D (defect)	5, 0	1, 1

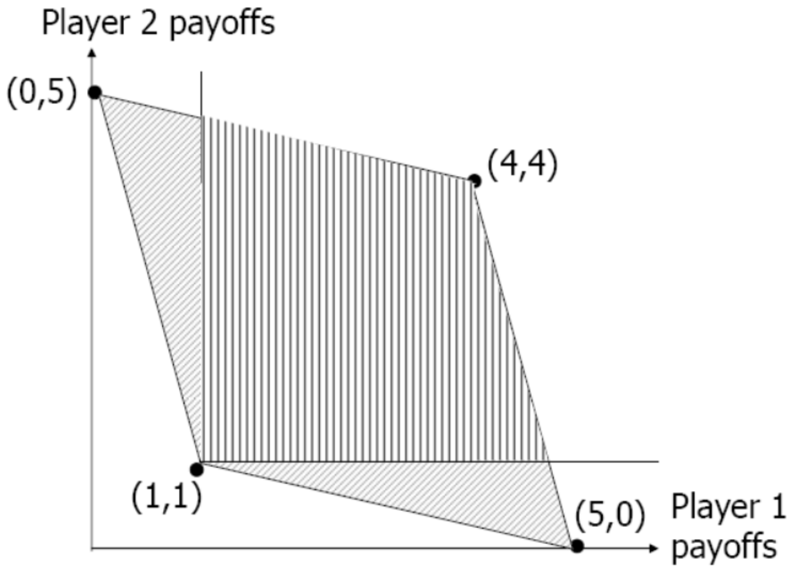
- What are the pure-strategy payoffs?
  - (4, 4) (0, 5) (5, 0) (1, 1)
- Feasible payoffs in the Prisoner's Dilemma



Friedman's Theorem

- Let G be a finite static game of complete information. Let (e^1, e^2,…, e^n) denote the payoffs from a Nash equilibrium of G and let (x^1, x^2,…,x^n) denote any other feasible payoffs from G. If $x^j > e^j$ for every player j and if δ is sufficiently close to 1, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty, \delta)$ that achieves (x^1, x^2,…,x^n) as the average payoff.

- Feasible payoffs in the Prisoner's Dilemma



- Friedman (1971) was the first to show that cooperation could be achieved in an infinitely repeated game by using trigger strategies that switch forever to the stage-game Nash equilibrium following any deviation.
- Proof of Friedman's Theorem

It remains to show that this Nash equilibirm is subgame-perfect. That is, the trigger strategies must constitute a Nash equilibrium in every subgame.) Recall that every subgame of $G(\infty,\delta)$ is identical to $G(\infty,\delta)$ itself. In the trigger strategy Nash equilibrium, these subgames can be grouped into two classes:

(i) subgames in which all the outcomes of earlier stages have been ($\boldsymbol{a}_{\mathrm{x}_1}, \boldsymbol{a}_{x_2}, \ldots, \boldsymbol{a}_{x_n}$)

The players' strategies in a subgame in (i) are again the trigger strategy, which we have just shown to be a Nash equilibrium of the game as a whole.

(ii) subgames in which the outcome of at least one earlier stage differs from ($\boldsymbol{a}_{\mathrm{x}_1}, \boldsymbol{a}_{x_2}, \ldots, \boldsymbol{a}_{x_n}$)

The players' strategies in a subgame in (ii) are simply to repeat the stage-game equilibrium $\boldsymbol{a}_{\mathrm{e}_1}, \boldsymbol{a}_{\mathrm{e}_2}, \ldots, \boldsymbol{a}_{\mathrm{e}_n}$ forever, which is also a Nash equilibrium of the game as a whole.

Repeated Cournot Game (collusion between Cournot duopolists)

- Two competing firms, selling a homogeneous good
- The marginal cost of producing each unit of the good: c
- The market price, P is determined by (inverse) market demand: $P=a-Q$ if $a>Q$, $P=0$ otherwise.
- Each firm decides on the quantity to sell q_1 and q_2.
- $Q = q_1+q_2$ total market demand
- Both firms seek to maximize profits
- Unique NE of the stage game: $q^c= (a-c)/3$ $Q=2(a-c)/3$
- Monopoly quantity:

  $$\Pi = (p-c)Q = (a-Q)Q - cQ\,, \quad \Pi \text{ is concave } = \frac{\partial^2 \Pi}{\partial Q^2} = -2 < 0$$

  $$\frac{\partial \Pi}{\partial Q} = a - 2Q - c = 0 \Rightarrow Q = {(a-c)}\big/{2}\,,\ P = {(a+c)}\big/{2}$$

  $q^M=(a-c)/2$
- Consider the infinitely repeated game based on this Cournot stage game when both firms have the discount factor δ.
- We now compute the values of δ for which it is a subgame-perfect Nash equilibrium of this infinitely repeated game for both firms to play the following trigger strategy:
  - Produce half the monopoly quantity, $q^M/2$, in the first stage. In the t^{th} stage, produce $q^M/2$ if both firms have produced $q^M/2$ in all previous stages; otherwise, produce q^c.
- Show that the trigger strategy induces a subgame perfect Nash equilibrium (SPNE).

What if $\delta \le 9/17$?

- Such trigger strategies cannot support a quantity as low as half the monopoly quantity, but for any value of δ it is a subgame-perfect Nash equilibrium simply to repeat the Cournot quantity forever. Therefore, the most-profitable quantity that trigger strategies can support is between $q^M/2$ and q^c.
- To compute this quantity, consider the following trigger strategy:
  - Produce q^* in the first stage. In the t^{th} stage, produce q^* if both firms have produced q^* in all previous stages; otherwise, produce q^c.

Other strategies?

- We now explore the second approach, which can achieve the monopoly outcome (both firms produce $q^M/2$) in the model when $\delta = 1/2$ (which is less than 9/17).
- Consider the following strategy:
  Produce half the monopoly quantity, $q^M/2$, in the first period. In the t-th period, produce $q^M/2$ if both firms produced $q^M/2$ in period t-1, produce $q^M/2$ if both firms produced x in period t-1, otherwise produce x.
- Note that this strategy involves a (one-period) punishment phase in which the firm produces x and a (potentially infinite) collusive phase in which the firm produces $q^M/2$. If either firm deviates from the collusive phase, then the punishment phase begins. If either firm deviates from the punishment phase, then the punishment phase begins again. If neither firm deviates from the punishment phase, then the collusive phase begins again.
- To show this strategy achieves the monopoly outcome as a subgame-perfect Nash equilibrium is left for your exercise.

Wage setting

- A dynamic model in which firms induce workers to work hard by paying high wages and threatening to fire workers caught shirking
- Stage game: one firm and one worker
- The firm offers the worker a wage, w
- The worker accepts or rejects the firm's offer
  - Reject: the worker becomes self-employed at wage w_0
  - Accept: work (disutility e), or shirk (disutility 0)
    - If the worker works (supplies effort): output is high = y
    - If the worker shirks: output is high with probability p, and low = 0 with probability 1-p.
- The firm does not observe the worker's effort decision
- The output of the worker is observed by both parties.
- Payoffs (Firm, Worker)
  - Work (supply effort)
    - High output (y-w, w-e)
  - Shirk
    - High output: (y-w, w)
    - Low output (-w, w)
  - We assume $y - e > w_0 > py$ so that it is efficient for the worker to be employed by the firm and to supply effort, and also better that the worker be self-employed than employed by the firm and shirking.
- What is the equilibrium in this stage game?
  - For any $w \geq w_0$, worker accepts employment and shirks (because the firm pays w in advance, the worker has no incentive to supply effort)
  - Firm offers w = 0 (or any other $w<w_0$), and the worker chooses self-employment. (if firm offers the wage w, its expected payoff is py-w<0.)
- In the infinitely repeated game, however, the firm can induce effort by paying a wage w in excess of w_0 and threatening to fire the worker if output is ever low. We show that for some parameter values, the firm finds it worthwhile to induce effort by paying such a wage premium.

- Strategies:
  - Firm: offer $w = w^*$ in the first stage.

    In stage t,
    - Offer $w = w^*$ if the history of play is ***high-wage***, ***high-output*** (all previous offers have been w^*, all previous offers have been accepted, and all previous outputs have been high)
    - Otherwise, offer $w = 0$
  - Worker:
    - If $w > w_0$, accept the firm's offer and supply effort if the history of play, including the current offer, is ***high-wage***, ***high-output*** (shirk otherwise)
    - If $w < w_0$, choose self-employment.

Is this a SPNE ?

What are the subgame ?

(i) Subgame beginning after a ***high-wage***, ***high-output*** history.

(ii) Subgames beginning after all other histories.

In subgame (i), we have shown the players' strategies are NE.

In subgame (ii), since the worker will never supply effort (based on the strategy), it is optimal for the firm to induce the worker to choose self-employment; since the firm will offer w=0 in the next stage and forever after, the worker should not supply effort in this stage and should accept the current offer only if $w \geq w_0$.

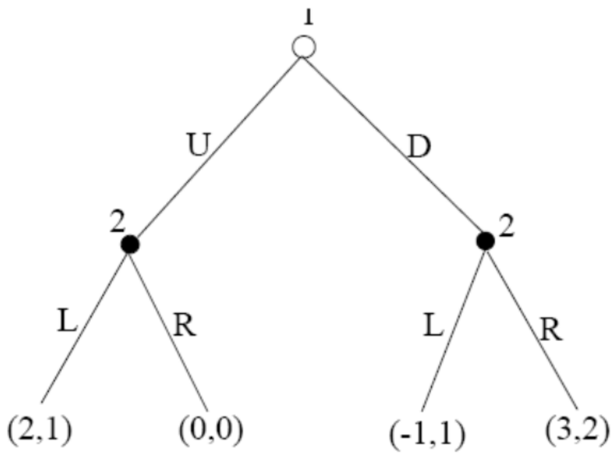
Dynamic Games of Complete but Imperfect Information

Extensive-form representation of games

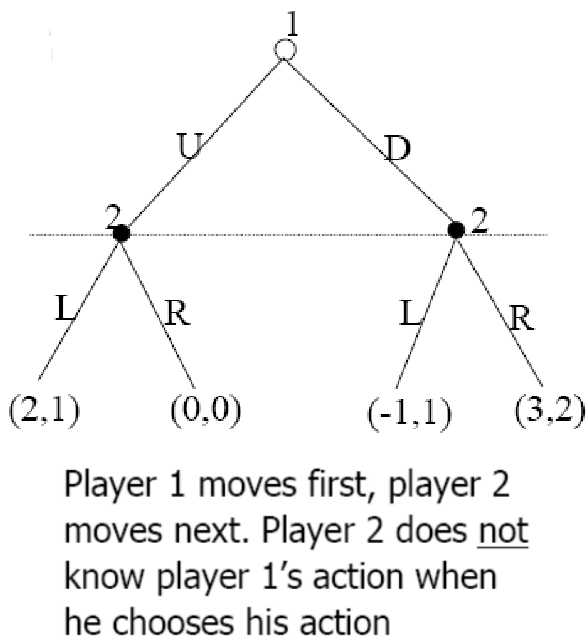
- The set of players
- The order of moves
- The players' payoffs as a function of the moves that were made
- The set of actions available to the players when they move
- Each player's information when he makes his move

Example: Normal form

- Player 1 chooses an action a_1 from the feasible set A_1={U, D}.
- Player 2 observes a_1 and then chooses an action a_2 from the set A_2={L, R}.

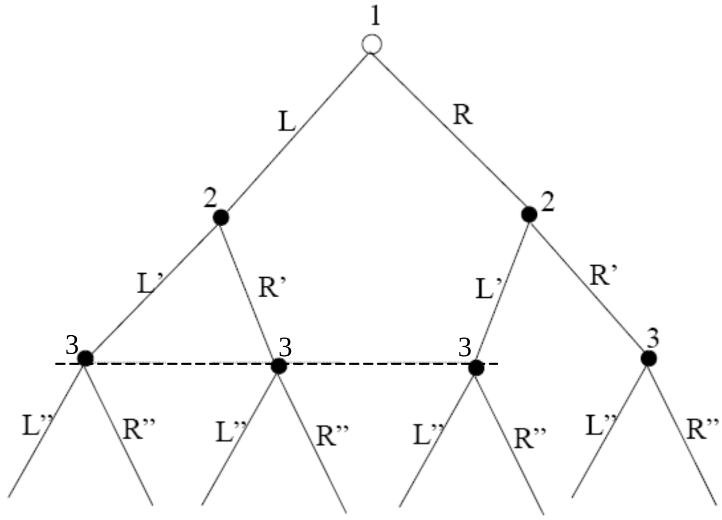


- Player 1 chooses an action a_1 from the feasible set A_1={U, D}.
- Player 2 does not observe player 1's move but chooses an action a_2 from the set A_2={L, R}.



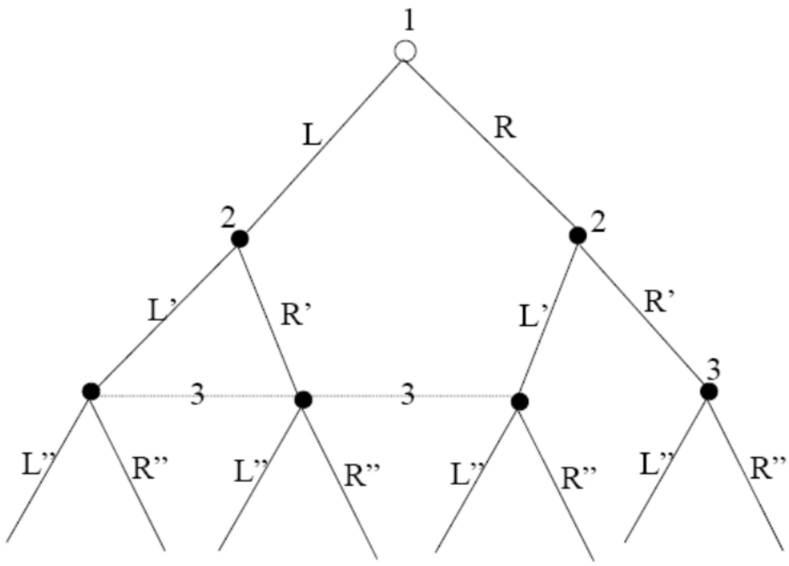
Example to Draw

- Player 1 chooses an action from the feasible set {L, R}
- Player 2 observes player 1's action and then chooses an action from the feasible set {L', R'}
- Player 3 observes whether or not the history of actions is {R, R'} and then chooses an action from the feasible set {L", R"}



Information Set

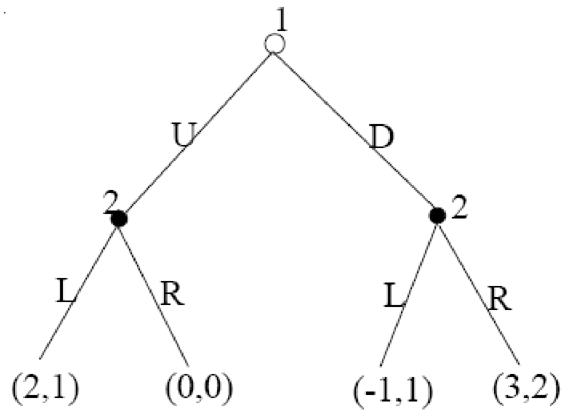
- An information set for a player is a collection of decision nodes satisfying:
  - The player has the move at every node in the information set.
  - When the play of the game reaches a node in the information set, the player with the move does not know which node in the information set has (or has not) been reached.
- In an extensive-form game, we will indicate that a collection of decision nodes constitutes an information set by connecting the nodes by a dotted line.



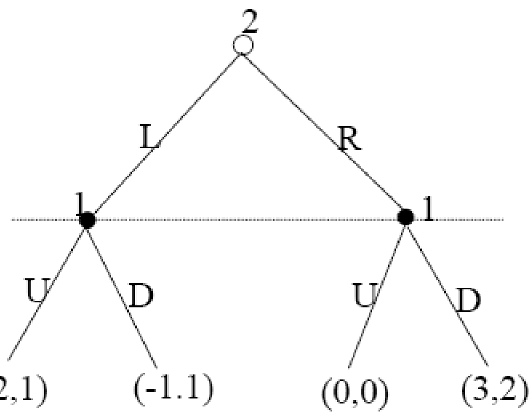
- Player 2 has two information sets, both singletons.
- Player 3 has two information sets, one of them is singleton.

Subgame in an extensive form game

- A subgame in an extensive form game
  - Begins at a decision node *n* that is a singleton information set
  - Includes all the decision and terminal nodes following *n* in the game tree (but no nodes that do not follow *n*), and
  - Does not cut any information sets (i.e., if a decision node *n*' follows *n* in the game tree, then all other nodes in the information set containing *n*' must also follow *n*, and so must be included in the subgame.

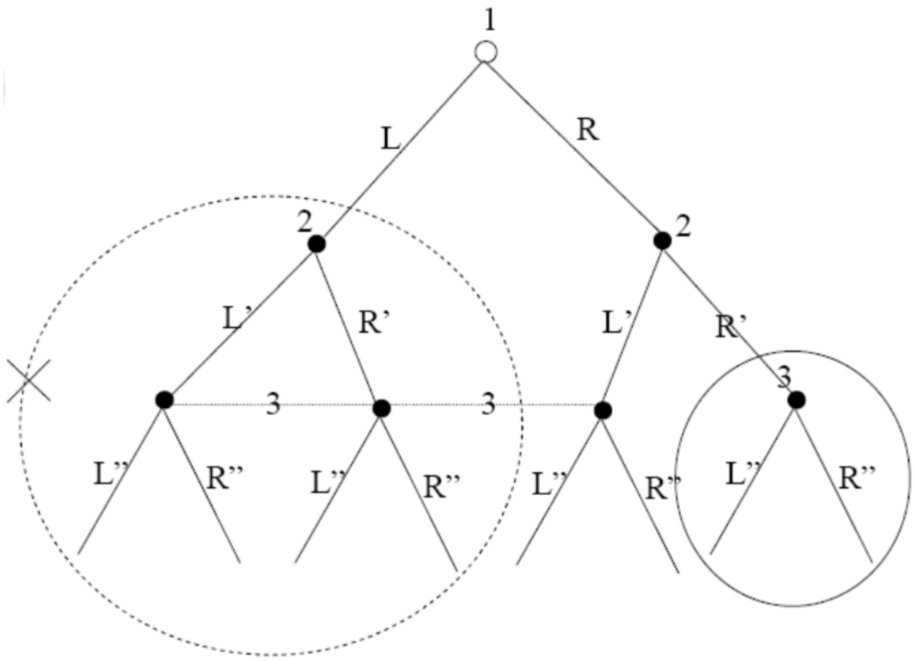


Two subgames, one beginning at each of player 2's decision nodes (+ whole game)



No subgames other than whole game

Example



- A subgame in an extensive form game does not cut any information set.

Simple Timing Games

- **The War of Attrition:** The game arises when two (or more) firms compete with each other, each one losing money but hoping that the competitor will eventually give up and exit the industry. When playing the game, each firm must decide whether to cut its losses and exit, or alternatively, tough it out in the hope that the competitor will soon exit.
- **A Simple Example**

You and a competitor will battle in rounds for a prize worth \$5. Each round you may choose to either fight or fold, so may your competitor. The first one to fold wins \$0. If the other player doesn't also fold he wins the \$5. In the case you both fold in the same round you each win \$0. If you both choose to fight you both go on to the next round to face a fight or fold choice again. Moving on to each round after round 1 costs \$0.75 per round per player (that is, both players pay \$0.75 per round in which they both choose to fight onward). (Note: the player to fight does not need to pay \$0.75 at the round when the opponent chose to fold.) How many rounds of fighting would you be willing to go? How would your answer change with the size of the prize? With the size of the per-round fee?

- If you know with certainty that your opponent will fold in any round 1-7 then it is rational for you to fight because you will win \$5 and pay less than \$5 in fight fees (\$0.75 per round after round 1, (\$0.75*6<\$5<\$0.75*7)). However, if your opponent intends to fold in any round then it is only sensible for him to do so in round 1. Why pay fight fees only to fold later? One can make the same argument with the roles of the players reversed.
- Hence, if either player is not willing to fight forever he should fold in round 1. If the other player knows this to be the case, the other player should fight. It turns out these are the two pure strategy Nash equilibria in this game: (1) you fight, your opponent folds in round 1 and (2) you fold, your opponent fights in round 1.
- There is an equilibrium strategy; that is to fold in each round with probability ***0.13***; that is, at the equilibrium, both players choose to fight with probability 0.87, but to fold with probability 0.13. How to get it?
- A mixed strategy Nash equilibrium is one in which the pure strategies are mixed probabilistically to make the pure strategy payoffs to each player equivalent. In other words, at the equilibrium, one mixes precisely in a way to make a player indifferent to his options.
- Suppose that the opponent **fold** with probability p and **fight** with probability 1-p such that, at the equilibrium, you are indifferent between "fold" and "fight."
- To make things more general, let's call the prize V (=\$5 in the game) and the per-round fight fee C (=\$0.75 in the game). We can find p in terms of V and C by using the property of a mixed strategy Nash equilibrium given above: p makes a player indifferent to his options. So, the solution approach is to figure out your payoffs under each option and set them equal to

each other.

- If you **fight**,...

- If you **fold**,…

- Setting these two expected payoffs equal gives an equation with one unknown, *p*:

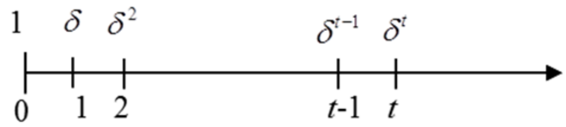
- The solution is $p=C/(V+C)$. Plugging in the values from the game we get that $p=0.75/(5+0.75)=0.13$ (rounded). Notice how *p* changes with *V* and *C*. It decreases with increasing *V* and increases with increasing *C*. This should be consistent with one's intuition. ***The greater the prize the less likely one is not to fight for it. The higher the fight fee the more likely one is to fold..*** Notice also that there is a chance that the fight could go on for a long time, even to the point that the cumulative sum of the fight fees are higher than the prize. In this fight, players are exhausting resources they can never recoup. In real life the player who runs out of resources (money to pay the fight fee) will have to fold, hence the name "war of attrition."
- Examples:
  - The fight between Sky Television and British Satellite Broadcasting for the British satellite TV market
  - Amazon versus Barnes and Noble for the Internet book market
  - Windows CE versus Palm in the market for handheld computers
- Possible solutions to "War of Attrition"
  - Merger
  - Split market
  - Don't fight in the first place
  - Convince other guy not to fight in the first place

- Stationary War of Attrition

In the *discrete-time* version of the stationary war of attrition, two players are fighting for a prize whose current value at any time $t = 0, 1, \ldots$ is $v > 1$; fighting costs 1unit per period. If one player stops fighting in period t, his opponent wins the prize without incurring a fighting cost that period. If both players stop simultaneously, we specify that neither wins the prize. Let's introduce a per-period discount factor δ.

- This stationary game has several Nash equilibria. Here is one: Player 1's strategy is "never stop" and player 2's is "always stop." There is a unique symmetric equilibrium, which is stationary and involves mixed strategies: (i.e. equilibria with mixed strategies that are independent of time)

- For any p, let "always p" be the mixed strategy "if the other player has not stopped before t, then stop at t with probability p." For the stationary symmetric profile **(always *p*, always *p*)** is an equilibrium.



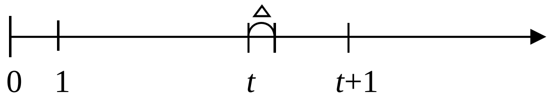
	Stop fighting (p)	Fight (1-p)
Stop fighting	(0, 0)	(0, v)
Fight	(v, 0)	(-1, -1)

- For any t, conditional on the opponent's not having stopped previously

- Another way to arrive at this conclusion is to note that by staying in for one more period, a player gains v with probability p and loses the fighting cost 1 during that period with probability 1-p. For him to be indifferent between staying in for one more period and stopping now, it must be the case that

- The *continuous-time* formulation :
- Let $G_i(t)$ denote the probability that player i stops at or before t (that is, $G_i(\cdot)$ is a cumulative distribution function). As in the discrete-time version of the game, there is a stationary symmetric equilibrium G with the property that at each date the players are indifferent between stopping at time t and waiting a bit longer, until $t+\Delta$, to see if the opponent stops first. Suppose the stationary symmetric equilibrium is ($\boldsymbol{G}$, $\boldsymbol{G}$) and player j adopts this equilibrium.
- Conditional on not stopping before t,

- Thus, the war of attrition does have a symmetric equilibrium in the kind of continuous-time strategies we introduce above. Moreover, this equilibrium is the limit of the symmetric equilibria of the discrete-time game as the interval Δ goes to 0.
- To make the discrete-time and continuous-time formulations comparable, we assume that fighting costs 1 per unit of *real* time. Hence, if in discrete time the real length of each period is Δ (so that there are $1/\Delta$ periods per unit of time), the fighting cost is Δ per period.



- The value of the prize v does not need to be adjusted when the period length changes, as v was taken to be a stock rather than a flow in both formulations.