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Recall the 2-player game mentioned in the class.

The normal-form is shown as follows:

Player 2

Player 1

	(H) Head (q)	(H) Head ($1-q$)
(H) Head (p)	-1,1	1,-1
(T) Tail ($1-p$)	1,-1	-1,1

Let us follow the notations introduced in class to verify several important conditions mentioned in the Kakutani fixed-point theorem.

Notation:

- The action (or strategy) profile: $\sigma = (p, q)$
- The action (or strategy) profile except player i 's action: σ_{-i}
- The space of action (or strategy) profile: Σ
- Player i 's payoff function: $u_i(\sigma)$
- Player i 's *best response correspondence*, r_i , maps each strategy profile σ to the set of mixed strategies that maximize player i 's payoff when his opponents play σ_{-i} .
- The correspondence $r : \Sigma \rightarrow \Sigma$ to be the Cartesian product of the r_i .
- The graph G_R of the correspondence r : $G_R = \{(p, q, \hat{p}, \hat{q}) : (\hat{p}, \hat{q}) \in r(p, q)\}$

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1. Please explicitly write the following terms in this example.
   (a) The space of action (or strategy) profile Σ
   (b) Player 1's expected payoff function $u_1(\sigma)$

(a)

$$\Sigma = [0,1] \times [0,1]$$

(b) $u_1(p,1) = (-1) \times p + 1 \times (1-p) = 1 - 2p$

$u_1(p,0) = 1 \times p + (-1) \times (1-p) = -1 + 2p$

$\implies E(u_1) = (1-2p) \times q + (-1+2p) \times (1-q) = 2p + 2q - 4pq - 1$

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2. Suppose $\sigma' = (p', q') \in r(p,q)$ and $\sigma'' = (p'', q'') \in r(p,q)$ where $\sigma = (p,q)$. Show that $\lambda p' + (1-\lambda)p''$ is player 1's best response to q for $\lambda \in (0,1)$.

Consider the different state below:

I. $\boxed{\text{As } q \le \frac{1}{2}}$

$p' = p'' = \lambda p' + (1-\lambda)p'' \implies p = 1$. (Player 1 choose H)

II. $\boxed{\text{As } q = \frac{1}{2}}$

$p', p'' \in [0,1]$ and $\lambda p' + (1-\lambda)p''$ is between p' and p''.

III. $\boxed{\text{As } q > \frac{1}{2}}$

$p' = p'' = \lambda p' + (1-\lambda)p'' \implies p = 0$.

In any case, $\lambda p' + (1-\lambda)p''$ is Player 1's best response to q for $\lambda \in (0,1)$. Additionally, $\forall i = 1,2$ $u_i(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_i) = \lambda u_i(\sigma_i', \sigma_{-i}') + (1-\lambda)u_i(\sigma_i'', \sigma_{-i})$ holds.

So $\lambda\sigma' + (1-\lambda)\sigma'' \in r(\sigma) = r(p,q) \implies \lambda p' + (1-\lambda)p'' \in r_1(q)$ for $\lambda \in (0,1)$

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3. Show that correspondence $r(p,q)$ in this example is convex for all $(p,q) \in \Sigma$.

Assume $r(p,q)$ is not convex $\forall (p,q) \in \Sigma$ for contradiction.
Then we have $\sigma', \sigma'' \in r(\sigma)$ and $\lambda\sigma' + (1-\lambda)\sigma'' \notin r(\sigma) = r(p,q)$ $\forall \lambda \in (0,1)$.

However $\forall i = 1,2$, $u_i(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}) = \lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda)u_i(\sigma_i'', \sigma_{-i})$. $\lambda\sigma' + (1-\lambda)\sigma''$ is exactly i's best response to σ_{-i}.

Therefore we can proof that $r(p,q)$ is convex $\forall (p,q) \in \Sigma$ by contradiction.

4. Let sequence $(p_n) = \left(\frac{1}{4^n}\right)$ and $(q_n) = \left(\frac{1}{4^n}\right)$. What are the best response sequence $(\hat{p}_n)$ and $(\hat{q}_n)$?

We can know that the best response sequence $(\hat{q_n}) = (0, 0, \cdots, 0)$ since $(p_n) = \left(\frac{1}{4}, \frac{1}{4^2}, \cdots, \frac{1}{4^n}\right)$.

We can know that the best response sequence $(\hat{p_n}) = (1, 1, \cdots, 1)$ since $(q_n) = \left(\frac{1}{4}, \frac{1}{4^2}, \cdots, \frac{1}{4^n}\right)$.

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5. What is the limit point of the sequence $(p_n, q_n, \hat{p}_n, \hat{q}_n)$, where $p_n = \dfrac{1}{4^n}$ and $q_n = \dfrac{1}{4^n}$ for all n?

Hence $(p_n) = \left(\dfrac{1}{4}, \dfrac{1}{4^2}, \cdots, \dfrac{1}{4^n}\right)$, $(q_n) = \left(\dfrac{1}{4}, \dfrac{1}{4^2}, \cdots, \dfrac{1}{4^n}\right)$, $(\hat{p_n}) = (1, 1, \cdots, 1)$ and $(\hat{q_n}) = (0, 0, \cdots, 0)$.

We can know that $(p_n, q_n, \hat{p_n}, \hat{q_n}) = (0, 0, 1, 0)$ as $n \to \infty$.

6. Suppose that a sequence $(p_n, q_n, \hat{p}_n, \hat{q}_n)$ converges to $(p, q, \hat{p}, \hat{q})$ for all sequence index n, but $\hat{p}$ is not player 1's best response to q. Show that $\hat{p}_n$ cannot be player 1's best response to q_n.

$\because (p_n, q_n, \hat{p_n}, \hat{q_n})$ converges to $(p, q, \hat{p}, \hat{q})$. $\therefore (p_n, \hat{p_n}) \to (p, \hat{p})$ but $\hat{p} \not\ni r(q) \implies \hat{p_1} \not\ni r_1(q)$. It means that $\forall \epsilon > 0, \exists p_1'$. s.t. $\underline{\underline{u_1(p_1', q) > u_1(\hat{p_1}, q) + 3\epsilon \cdots}}$

$\because u_1$ is continuous and $(p_n, \hat{p_n}) \to (p, \hat{p})$ for sufficiently large n. $\underline{\underline{\therefore u_1(p_1', q) > u_1(p_1', q) - \epsilon > u_1(\hat{p_1}, q) + 2\epsilon}}$

$\because (p_n, \hat{p_n}) \to (p, \hat{p})$, $\therefore \forall \epsilon > 0, \exists k(\epsilon) \in N$ for $n > k(\epsilon)$. $\underline{\underline{\implies u_1(\hat{p_n}, q_n) - u_1(\hat{p_1}, q) < \epsilon}}$

By the equation underlined, we knew that

$$u_1(p_1', q_n) > u_1(\hat{p_1}, q) + 2\epsilon > u_1(\hat{p_n}, q_n) + \epsilon, \forall \epsilon > 0$$

$$\implies \hat{p_n} \text{ is not 1's best response to } q_n.$$

7. Show that $r(\cdot)$ has a closed graph.

$G_r = \{(\sigma, \hat{\sigma}) : \hat{\sigma} \in r(\sigma)\}, ((\sigma_1, \hat{\sigma_1}), (\sigma_2, \hat{\sigma_2}), \cdots, (\sigma_n, \hat{\sigma_n})) \to (\sigma, \hat{\sigma})$ for some Player i, $\hat{\sigma}_i \ni r_i(\sigma)$. $\exists \epsilon > 0, \sigma_i'$. $u_i(\sigma_i', \sigma_i) > u_i(\hat{\sigma}_i, \sigma_i) + 3\epsilon \cdots$.

$\because (\sigma^n, \hat{\sigma^n} \to (\sigma, \hat{\sigma})$ for sufficiently large n.
$\implies u_i(\sigma_i', \sigma_{-i}) - u_i(\sigma_i', \sigma_i^n) < \epsilon$ ($\because u_i$ is continuous)
$\therefore u_i(\sigma_i', \sigma_{-i}^n) > u_i(\sigma_i', \sigma_{-i}) - \epsilon > u_i(\hat{\sigma i}, \sigma_{-i}) + 2\epsilon$

$\because (\sigma^n, \hat{\sigma^n} \to (\sigma, \hat{\sigma})$ for sufficiently large n.
$\implies u_i(\hat{\sigma_i^n}, \sigma_{-i}) + \epsilon < u_i(\hat{\sigma}_i, \sigma_i^n) + 2\epsilon$
$\therefore u_i(\hat{\sigma_i^n}, \sigma_{-i}^n) > u_i(\hat{\sigma_i'}, \sigma_{-i}) - \epsilon < \epsilon$

$$\implies u_i(\sigma_i', \sigma_{-i}^n) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon > u_i(\hat{\sigma_i^n}, \sigma_{-i}^n) + \epsilon$$

$\therefore \sigma_i'$ is obviously better than $\hat{\sigma_i^n}$, it gets a contradiction.
$\therefore \hat{\sigma_i^n} \in r_i(\sigma^n) \implies \hat{\sigma}_i \in r_i(\sigma)$, this is also in the graph.

Therefore we can proof that $r(\cdot)$ has a closed graph by contradiction.