Lecture 4 The Revised Simplex Method

In each iteration of the simplex method, one basic feasible solution is replaced by another. When the old solution is represented by a dictionary, the new solution is easy to find, and only a small part of the dictionary is actually used for that purpose. This part may be reconstructed directly from the original data and the new solution found without any reference to dictionaries. The resulting implementations of the simplex method are known under the generic name of the *revised simplex method*; the implementation of the simplex method that updates a dictionary in each iteration is known as the *standard simplex method*.

1. Matrix description of dictionaries

$$\begin{array}{lrl} \text{maxmize} & 19x_1 + 13x_2 + 12x_3 + 17x_4 & \\ \text{subject to} & 3x_1 + 2x_2 + x_3 + 2x_4 & \le 225 \\ & x_1 + x_2 + x_3 + x_4 & \le 117 \\ & 4x_1 + 3x_2 + 3x_3 + 4x_4 & \le 420 \\ & x_1, x_2, x_3, x_4 & \ge 0 \end{array}$$

After two iterations of the simplex method

$$\begin{aligned} x_1 &= 54 - 0.5x_2 - 0.5x_4 - 0.5x_5 + 0.5x_6 \\ x_3 &= 63 - 0.5x_2 - 0.5x_4 + 0.5x_5 - 1.5x_6 \\ x_7 &= 15 + 0.5x_2 - 0.5x_4 + 0.5x_5 + 2.5x_6 \end{aligned}$$

$$z_1 = 1782 - 2.5x_2 + 1.5x_3 - 3.5x_4 - 8.5x_6$$

More generally, consider an arbitrary LP problem in the standard form

maxmize $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \dots, m)$

$x_j \ge 0 \qquad (j = 1, 2, \dots, n).$

After the introduction of the slack variables x_{n+1}, x_{n+2},…, x_{n+m},this problem may be recorded as

maxmize $\mathbf{cx}$

subject to $\mathbf{Ax} = \mathbf{b}$

$\mathbf{x} \ge \mathbf{0}.$

2. The revised simplex method

In each iteration of the simplex method, we first choose the entering variable, then find the leaving variable, and finally update the current basic feasible solution. An examination of the way these tasks are carried out in the standard simplex method will lead us to the alternative, the revised simplex method. For illustration, we shall consider the update of the feasible dictionary on page 1 in the standard simplex method. The corresponding iteration of the revised simplex method begins with

$$\mathbf{x}_{\mathbf{B}}^{*} = \begin{bmatrix} \mathrm{x}_1^* \\ \mathrm{x}_3^* \\ \mathrm{x}_7^* \end{bmatrix} = \begin{bmatrix} 54 \\ 63 \\ 15 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}.$$

The entering variable may be any nonbasic variable with a positive coefficient in the last row of the dictionary. As previously observed, the coefficients in this row form the vector $\mathbf{c_N} - \mathbf{c_B}\mathbf{B}^{-1}\mathbf{A_N}$. If the standard simplex method is used, then this vector is readily available as part of the dictionary; in our example, we have

$$z = \cdots - 2.5x_2 + 1.5x_3 - 3.5x_4 - 8.5x_6.$$

Iteration of the revised simplex method

Step 1. Solve the system $\mathbf{yB} = \mathbf{c_B}$

Step 2. Choose an entering column. This may be any column $\mathbf{a}$ of $\mathbf{A_N}$ such that $\mathbf{ya}$ is less than the corresponding component of $\mathbf{c_N}$. If there is no such column, then the current solution is optimal.

Step 3. Solve the system $\mathbf{Bd} = \mathbf{a}$.

Step 4. Find the largest t such that $\mathbf{x_B^*} - t\mathbf{d} \geq \mathbf{0}$. If there is no such t, then the problem is unbounded; otherwise, at least one component of $\mathbf{x_B^*} - t\mathbf{d}$ equals zero and the corresponding variable is leaving the basis.

Step 5. Set the value of the entering variable at t and replace the values $\mathbf{x_B^*}$ of the basic variables by $\mathbf{x_B^*} - t\mathbf{d}$. Replace the leaving column of $\mathbf{B}$ by the entering column and, in the basis heading, replace the leaving variable by the entering variable.