Integer and Combinatorial Optimization – Solution Methods

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Agenda

- Cutting Plane
- Brand and Bound
- Bender Decomposition
- Lagrangian Relaxation

Branch and Bound Method

- Traditional approach to solving integer programming problems.
- Based on principle that total set of feasible solutions can be partitioned into smaller subsets of solutions.
- Smaller subsets evaluated until best solution is found.
- Method is a tedious and complex mathematical process.

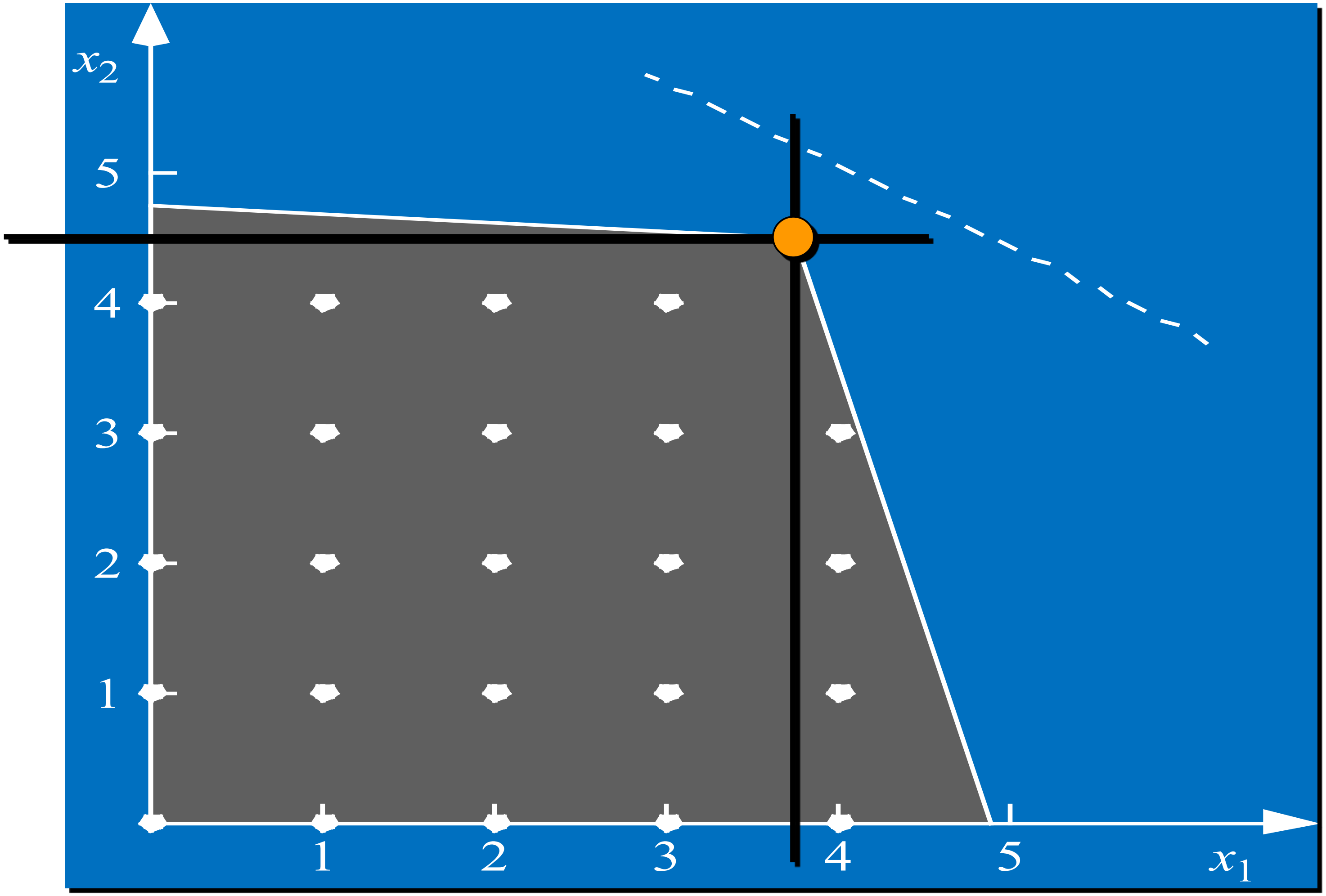
Branch and Bound

Problems are solved as LP. If not integers, one of them is chosen and 2 new constraints are added.

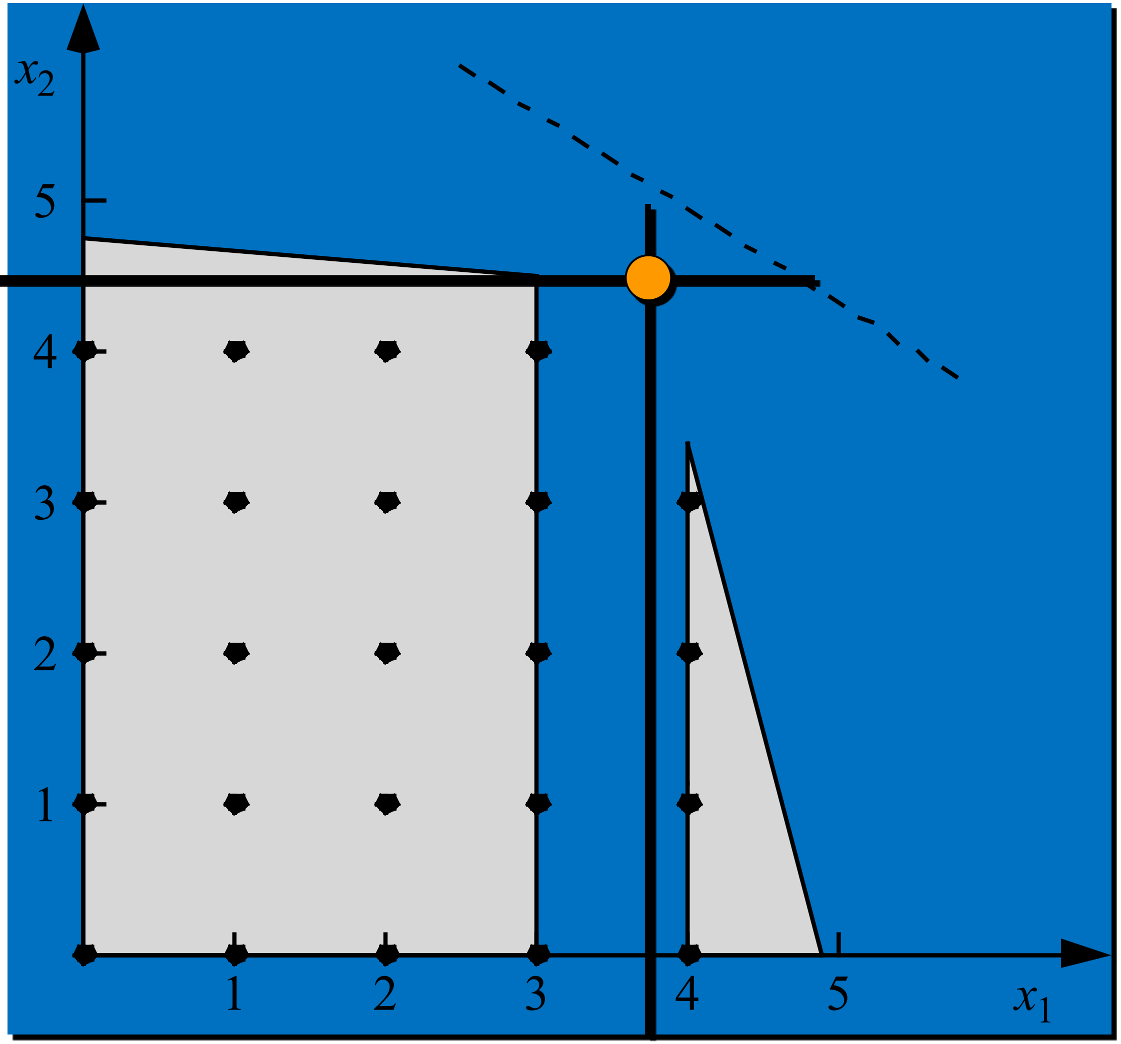
$$x_i \le \lfloor v \rfloor \text{ and } x_i \ge \lfloor v \rfloor + 1$$

Proceeds until optimal.

How Integer Programs are Solved: Original Graph



After Branch and Bound on x_1



Branch & Bound 1/3

Step 1: Solve problem using LP. If solution is integer—finished. Otherwise, next.

Step 2: Branch on non-integer variable from step 1. Split problem into two pieces: integer above, and integer below.

Branch & Bound 2/3

Step 3: Create nodes of these branches and solve the new LP problems.

Step 4:

a) Infeasible, terminate branch;
b) Feasible, not integer, back to Step 2;
c) Feasible and integer, go to Step 5.

Branch & Bound 3/3

Step 5: Check branches.

1) The feasible solution is a _____ bound of the optimum (Max problem).
2) If the feasible solution is better than the LP solution of a node, the branch of that node is _______ .
3) If there are no remaining branches, the feasible solution is the solution to the problem.

First Branch

Original Problem

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \end{aligned}$$

$x_1 = 3.75$

$x_2 = 1.50$

$Z = 35.25$

Sub-problem A

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \ge 4 \end{aligned}$$

Sub-problem B

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \le 3 \end{aligned}$$

Second Branch

Sub-problem A

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\ & 6x_1 + 5x_2 \leq 30 \\ & x_1 \geq 4 \end{aligned}$$

$\Rightarrow$

$x_1 = 4$

$x_2 = 1.20$

$Z = 35.2$

Sub-problem C

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\ & 6x_1 + 5x_2 \leq 30 \\ & x_1 \geq 4 \\ & x_2 \geq 2 \end{aligned}$$

Sub-problem D

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\ & 6x_1 + 5x_2 \leq 30 \\ & x_1 \geq 4 \\ & x_2 \leq 1 \end{aligned}$$

Third Branch

Sub-problem B

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\ & 6x_1 + 5x_2 \leq 30 \\ & x_1 \leq 3 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 3 \\ x_2 &= 2 \\ Z &= 33 \end{aligned}$$

Integer solution $\Rightarrow$
No more branch is needed along this sub-problem.

Fourth Branch

Sub-problem C

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \ge 4 \\ & x_2 \ge 2 \end{aligned}$$

No feasible solution

No feasible solution $\Rightarrow$

No more branch is needed along this sub-problem.

Fifth Branch

Sub-problem D

$$\begin{aligned}
\text{Max} \quad & 7x_1 + 6x_2 \\
\text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\
& 6x_1 + 5x_2 \leq 30 \\
& x_1 \geq 4 \\
& x_2 \leq 1
\end{aligned}$$

$$\Rightarrow \quad x_1 = 4.16 \qquad x_2 = 1 \qquad Z = 35.12$$

Sub-problem E

$$\begin{aligned}
\text{Max} \quad & 7x_1 + 6x_2 \\
\text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\
& 6x_1 + 5x_2 \leq 30 \\
& x_1 \geq 4,\ x_1 \leq 4 \\
& x_2 \leq 1
\end{aligned}$$

Sub-problem F

$$\begin{aligned}
\text{Max} \quad & 7x_1 + 6x_2 \\
\text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\
& 6x_1 + 5x_2 \leq 30 \\
& x_1 \geq 4,\ x_1 \geq 5 \\
& x_2 \leq 1
\end{aligned}$$

Sixth Branch

Sub-problem E

$$\begin{aligned}
\text{Max} \quad & 7x_1 + 6x_2 \\
\text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\
& 6x_1 + 5x_2 \leq 30 \\
& x_1 \geq 4,\ x_1 \leq 4 \\
& x_2 \leq 1
\end{aligned}$$

$\Rightarrow$ $x_1 = 4$, $x_2 = 1$, $Z=34$

Sub-problem F

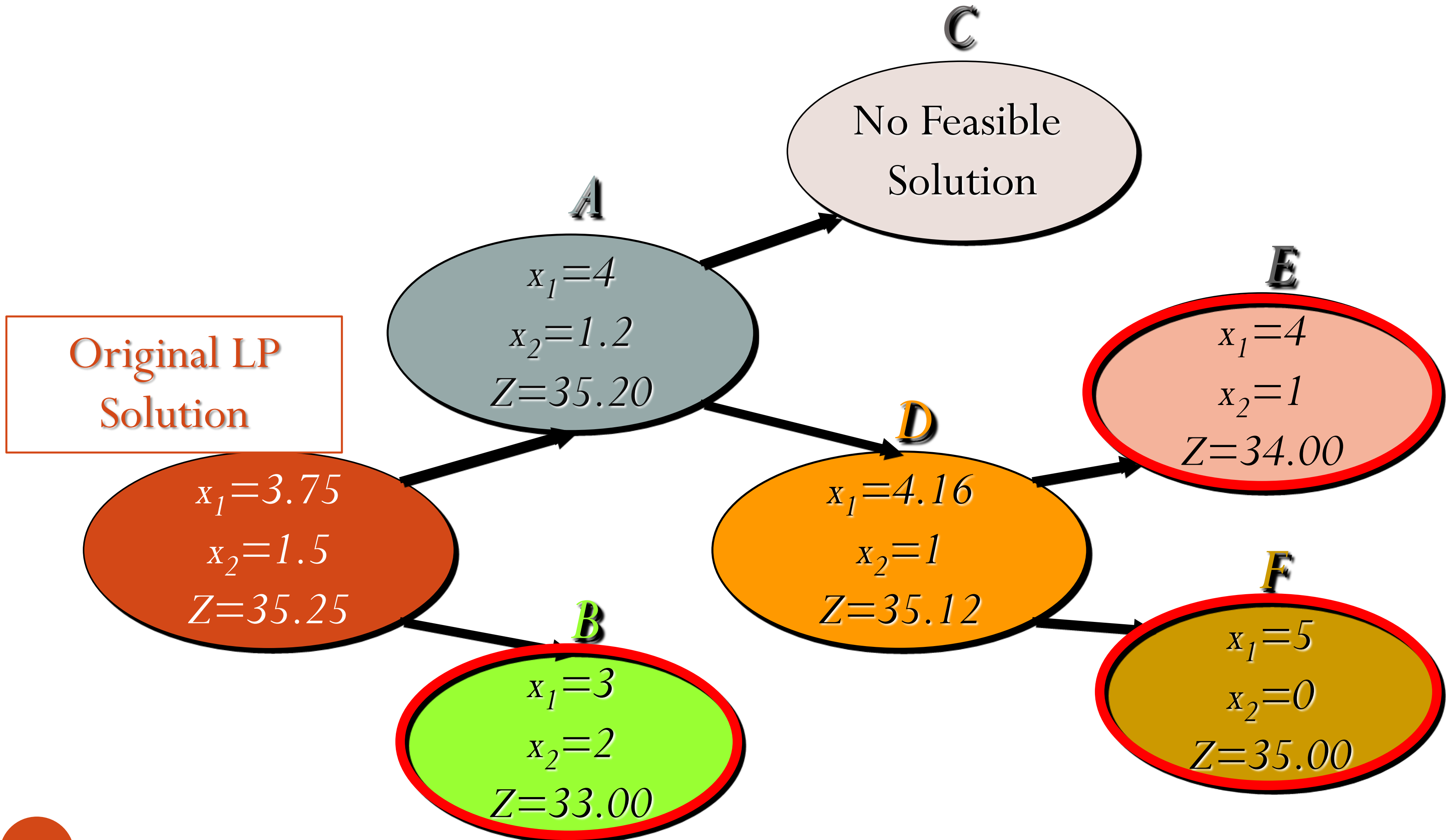
$$\begin{aligned}
\text{Max} \quad & 7x_1 + 6x_2 \\
\text{st.} \quad & 2x_1 + 3x_2 \leq 12 \\
& 6x_1 + 5x_2 \leq 30 \\
& x_1 \geq 4,\ x_1 \geq 5 \\
& x_2 \leq 1
\end{aligned}$$

$\Rightarrow$ $x_1 = 5$, $x_2 = 0$, $Z=35$

Integer solutions $\Rightarrow$

No more branch is needed along this sub-problem.

Branch & Bound - Overall



Special Case: 1-0 Problem (BIP)

- Now consider an IP problem where all integer variable are either 0 or 1.
- For any integer variable x_i,

$$x_i \leq 0 \Rightarrow x_i = 0,$$

$$x_i \geq 1 \Rightarrow x_i = 1.$$

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Introduction

- For large MILP problems, if there are complicating constraints, we can use **Lagrangian relaxation** or **Lagrangian decomposition** to find near optimal solutions. If there are complicating variables, we can use **Bender's decomposition** to find an optimal solution
- Consider the following MILP problem:

$$(\text{MILP}) \quad \begin{aligned} &\textit{maximizze} && z = c^T x + h^T y, \\ &\textit{subject to} && Ax + Gy \leq b, \\ & && x \geq 0, \text{integer}, \\ & && y \geq 0. \end{aligned}$$

Bender's Decomposition (1/4)

- In Bender's decomposition, the MILP problem is reformulated in terms of the integer(x) variables and only one continuous variable.
- However, the reformulation contains a huge number of constraints. Usually, only a small number of constraints is active in an optimal solution and a natural relaxation is obtained by dropping most of the constraints.
- Constraint Generation Algorithm can be used to solve the problem.

Bender's Decomposition (2/4)

- If we fix the x variables (integer), we obtain the following LP:

$$
\begin{aligned}
(\mathrm{LP(x)}) \quad z_{LP}(x) = \mathit{maximize} \quad & h^T y, \\
\mathit{subject\ to} \quad & Gy \le b - Ax, \quad : \ u \\
& y \ge 0,
\end{aligned}
$$

and its dual is

$$
\begin{aligned}
\mathit{minimize} \quad & u^T (b - Ax), \\
\mathit{subject\ to} \quad & u^T G \ge h^T, \\
& u \ge 0.
\end{aligned}
$$

Bender's Decomposition (3/4)

- The dual polyhedron, $Q = \{u : u^T G \geq h^T, u \geq 0\}$, can be represented in terms of its extreme points and extreme directions

- Set of extreme points of $Q = \{u^T : k \in K\}$
- Set of extreme directions of $Q = \{v^j : j \in J\}$

- Characterization of $z_{LP}(x)$:

  1. If Q=$\varnothing$, then $z_{LP}(x) = \infty$, if $\langle v^j \rangle^T (b - Ax) \geq 0$, for all $j \in J$,

     $z_{LP}(x) = -\infty$, otherwise.

  2. If $Q \neq \varnothing$, then $z_{LP}(x) = \min\left\{ \left(u^k\right)^T (b - Ax), k \in K \right\}$, if $\left(v^j\right)^T (b - Ax) \geq 0$, for all $j \in J$,

     $z_{LP}(x) = -\infty$, otherwise.

Bender's Decomposition (4/4)

- Then, if $Q \neq \emptyset$, MILP is equivalent to

$$z^{*} = \max \left\{ c^{T}x + \min\left\{ \left(u^{k}\right)^{T}(b - Ax), k \in K \right\} \right\},$$

$$\text{subject to} \quad \left(v^{j}\right)^{T}(b - Ax) \geq 0, \text{ for all } j \in J,$$

$$x \geq 0, \text{ integer},$$

and for any Q, MILP is equivalent to (Bender's Reformulation)

$$\text{(MILP')} \quad z^{*} = \text{maximize} \quad \eta,$$

$$\text{subject to } \eta \leq c^{T}x + \left(u^{k}\right)^{T}(b - Ax), \text{ for all } k \in K,$$

$$\left(v^{j}\right)^{T}(b - Ax) \geq 0, \qquad \text{for all } j \in J,$$

$$x \geq 0, \text{ integer}.$$

Example (1/4)

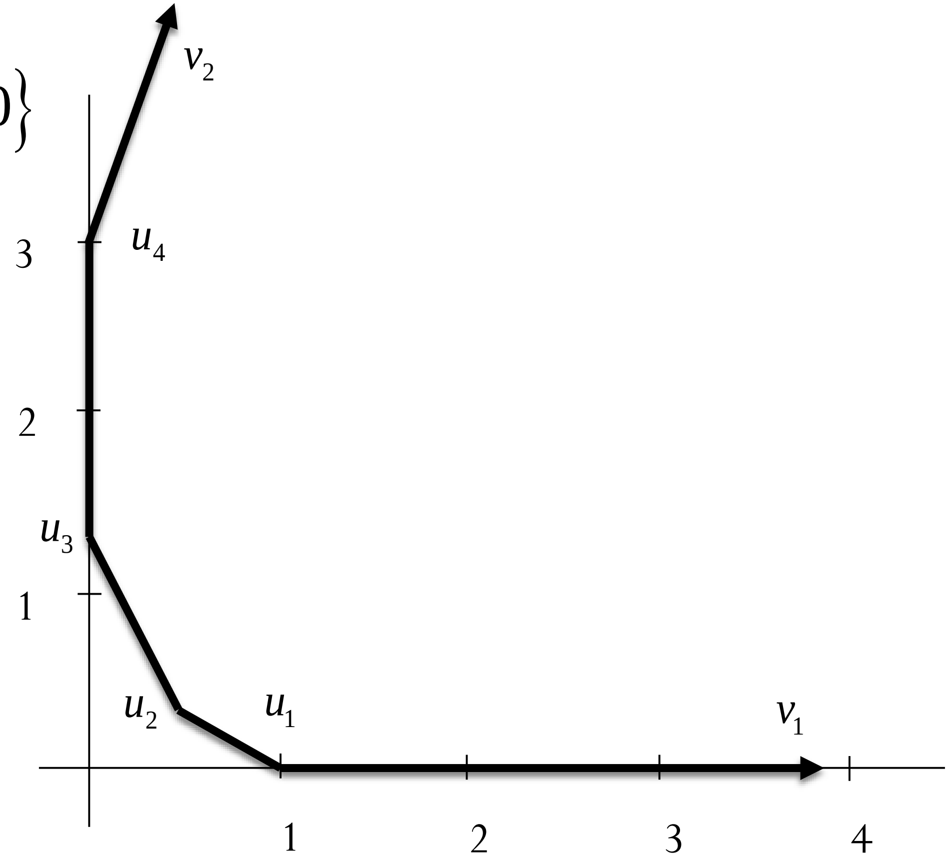
$$
\begin{aligned}
\text{maximize} \quad & z = 5x_1 - 2x_2 + 9x_3 + 2y_1 - 3y_2 + 4y_3, \\
\text{subject to} \quad & 5x_1 - 3x_2 + 7x_3 + 2y_1 + 3y_2 + 6y_3 \le -2, \\
& 4x_1 + 2x_2 + 4x_3 + 3y_1 - y_2 + 3y_3 \le 10, \\
& 0 \le x_j \le 5, \text{ integer}, \quad j = 1, 2, 3, \\
& y_j \ge 0 \qquad\qquad\qquad j = 1, 2, 3.
\end{aligned}
$$

Example (2/4)

The polygon $Q=\{u: u^T G \geq h, u \geq 0\}$

is defined by

$$\begin{aligned} 2u_1 + 3u_2 &\geq 2, \\ 3u_1 - \ u_2 \ &\geq -3, \\ 6u_1 + 3u_2 &\geq 4, \\ u_1, u_2 &\geq 0. \end{aligned}$$



Set of extreme points of Q= $\left\{ u^1 = (1,0)^T, u^2 = (\frac{1}{2}, \frac{1}{3})^T, u^3 = (0, \frac{4}{3})^T, u^4 = (0,3)^T \right\}$,

Set of extreme directions of Q= $\left\{ v^1 = (1,0)^T, v^2 = (1,3)^T \right\}$.

Example (3/4)

- For extreme point $(1, 0)$:

  According to $\eta \leq c^T x + \left(u^k\right)^T (b - Ax)$, for all $k \in K$,

  we have

- For extreme direction $(1, 0)$:

  Based on $\left(v^j\right)^T (b - Ax) \geq 0$, for all $j \in J$,

  we have

Example (4/4)

The resulting Bender's reformulation is

$$
\begin{aligned}
z = \text{maximize}\quad & \eta, \\
\text{subject to}\quad & \eta \le 5x_1 - 2x_2 + 9x_3 + (-2 - 5x_1 + 3x_2 - 7x_3), \\
& \eta \le 5x_1 - 2x_2 + 9x_3 + \frac{1}{2}(-2 - 5x_1 + 3x_2 - 7x_3) + \frac{1}{3}(10 - 4x_1 - 2x_2 - 4x_3), \\
& \eta \le 5x_1 - 2x_2 + 9x_3 + \frac{4}{3}(10 - 4x_1 - 2x_2 - 4x_3), \\
& \eta \le 5x_1 - 2x_2 + 9x_3 + 3\,(10 - 4x_1 - 2x_2 - 4x_3), \\
& (-2 - 5x_1 + 3x_2 - 7x_3) \ge 0, \\
& (-2 - 5x_1 + 3x_2 - 7x_3) + 3\,(10 - 4x_1 - 2x_2 - 4x_3) \ge 0, \\
& 0 \le x_j \le 5, \text{ integer}, \quad j = 1,2,3.
\end{aligned}
$$

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Integer Programming Problem

- Consider the following ILP problem:

$$z_{\mathrm{IP}} = \text{ maximize } \mathrm{c}^T x$$

$$\left.\begin{array}{ll} & A^1 x \le b^1 \quad \text{(complicating constraints) : } \lambda \ge 0 \\ \text{subject to} & A^2 x \le b^2 \quad \text{(nice(easy) constraints)} \\ & x \ge 0, \ \text{ integer} \end{array}\right\} = S$$

$$\text{where } A = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}_{m\times n}, \qquad A^1_{m_1\times n}, \qquad A^2_{m_2\times n}, \qquad b = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}.$$

Relax Hard Constraint

- The problem is easy to solve when the complicating constraints are dropped.
- For any m_1-vector $\lambda \geq 0$, the **Lagrangian relaxation** of ILP with respect to any $A^1x \leq b^1$ is

$$LR(\lambda){:}\ z_{LR}(\lambda) =$$
$$\text{maximize } z(\lambda, x) = c^T x + \lambda^T \left(b^1 - A^1 x\right)$$

Subject to

$$A^2 x \leq b^2,$$
$$x \geq 0, \text{integer}$$

- Define $Q = \left\{x{:}\ A^2 x \leq b^2, x \geq 0, \text{integer}\right\}$.
- $LR(\lambda)$ does not contain the complicating constraints.

$LR(\lambda)$

- **Theorem**: $LR(\lambda)$ is a relaxation of ILP for all $\lambda \geq 0$.
- **Proof**: For any x feasible in ILP, x is in $LR(\lambda)$ and

$$z(\lambda,\ x) = c^T x + \underset{\geq 0}{\lambda^T} \left(\underbrace{b^1 - A^1 x}_{} \right) \qquad c^T x.$$

Lagrangian Dual

- The **Lagrangian dual** of ILP with respect to $A^1 x \le b^1$ is

(LD) $$z_{LD} = minimize \quad z_{LR}(\lambda),$$

$$subject\ to \quad \lambda \ge 0$$

Example(1/3)

Maximize $\quad x_1 + 2x_2,$

Subject to

$$x_1 + 2x_2 \le 4 \quad : \lambda$$

$$\left.\begin{aligned} 5x_1 + x_2 &\le 20 \\ -2x_1 - 2x_2 &\le -7 \\ -x_1 \qquad &\le -2 \\ x_2 &\le 4 \\ x_1, x_2 \ge 0, &\text{ integer} \end{aligned}\right\} = Q$$

Example(2/3)

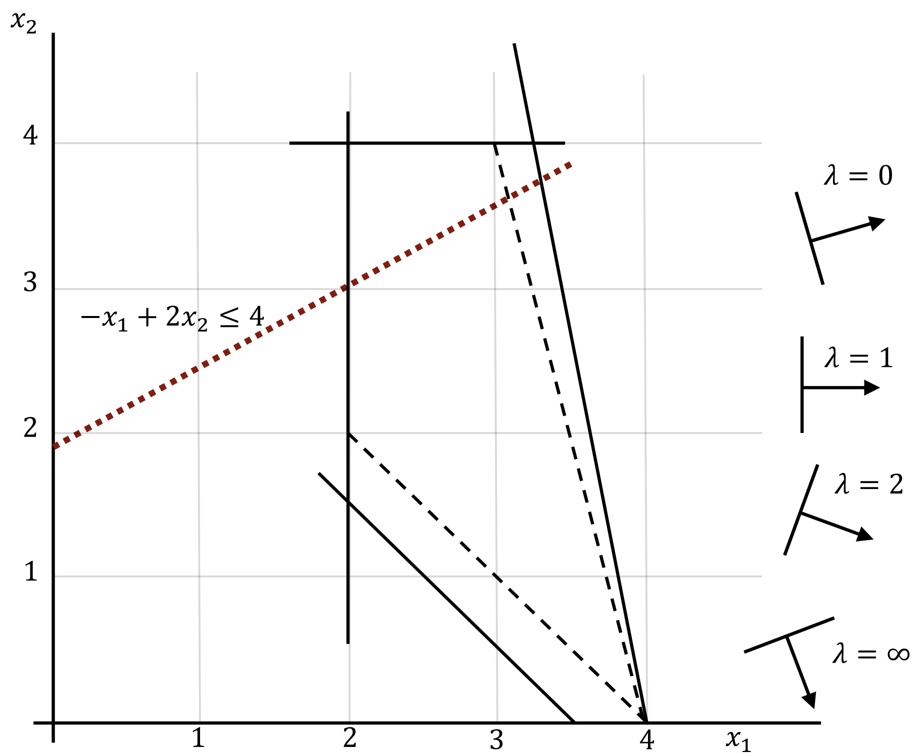
- The Lagrangian relaxation with respect to $-x_1 + 2x_2 \leq 4$ is

$$z_{LR}(\lambda) = maximize\ \ 7x_1 + 2x_2 + \lambda(4 + x_1 - 2x_2)$$
$$= (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda,$$

$subject\ to$

$$\begin{aligned} 5x_1 + \ \ x_2 &\leq 20, \\ -2x_1 - \ 2x_2 &\leq -7, \\ -x_1 \qquad\quad &\leq -2\,, \\ x_2 &\leq 4, \\ x_1, x_2 \geq 0,&\ \ integer. \end{aligned}$$

Example(3/3)



$$Q = \left\{ x^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, x^2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, x^3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, x^4 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, x^5 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, x^6 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, x^7 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, x^8 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right\}.$$

Convex Hull

- Note that $z(\lambda, x)$ is an affine function for a fixed λ, and $z_{LR}(\lambda)$ is determined by solving LP.

$$
\begin{aligned}
z_{LR}(\lambda) = maximize \quad & z(\lambda, x) \\
\text{subject to} \quad & x \in \mathrm{conv}(\mathrm{Q}),
\end{aligned}
$$

where $\mathrm{conv}(\mathrm{Q})$ is the convex hull of Q (which is the set of all points that are convex combinations of points in S.)

Example for Points 7 & 8

- For example,

$$\mathrm{conv}(Q) = \{x \in R_+^2 : -x_1 \le -2, x_2 \le 4, -x_1 - x_2 \le -4, 4x_1 + x_2 \le 16\}.$$

Thus,

$$z_{LR}(0) = \max\{7x_1 + 2x_2 : x \in conv(Q)\} = z(0, x^7) = 29 \quad ,$$

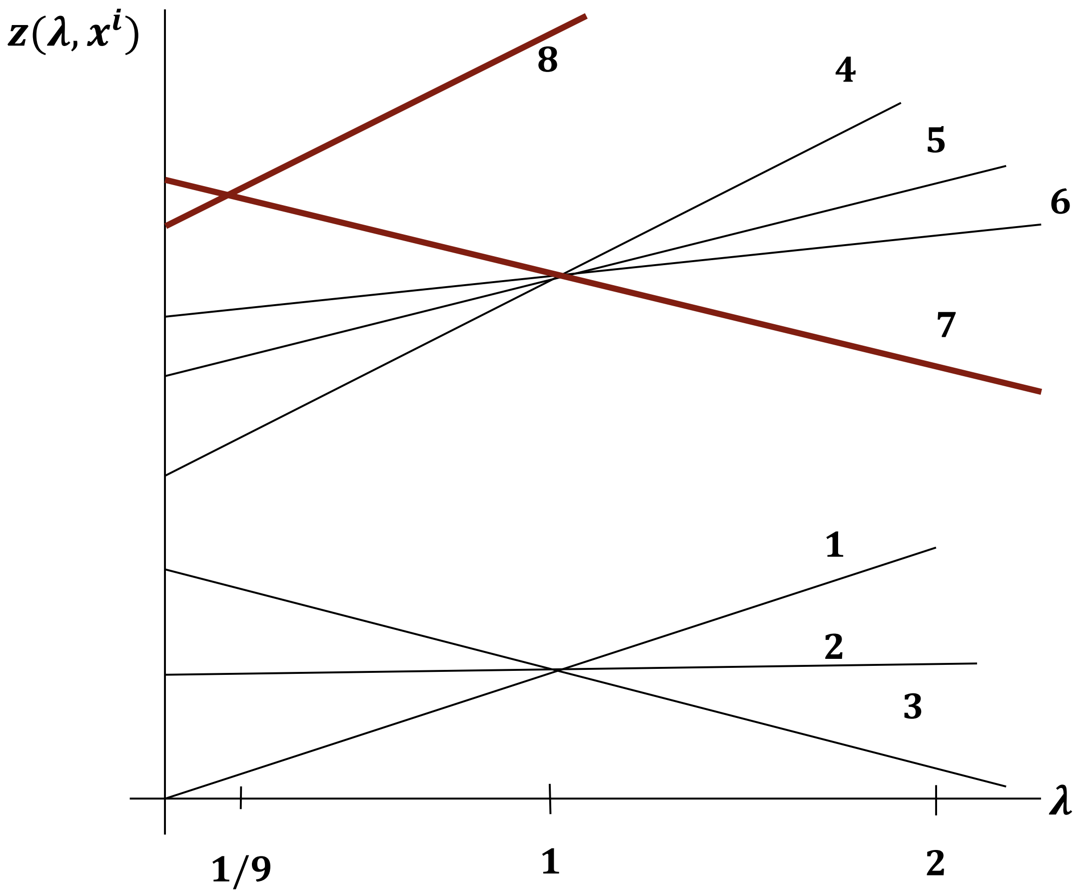
$$z_{LR}(1) = \max\{8x_1 + 0x_2 + 4 : x \in conv(Q)\} = z(1, x^8) = 36 \quad .$$

In general, we obtain

$$z_{LR}(\lambda) = \begin{cases} z(\lambda, x^7) = (7 + \lambda) \times 3 + (2 - 2\lambda) \times 4 + 4\lambda, & 0 \le \lambda \le \frac{1}{9} \\ z(\lambda, x^8) = (7 + \lambda) \times 3 + (2 - 2\lambda) \times 0 + 4\lambda, & \lambda \ge \frac{1}{9} \end{cases}$$

Hence, $z_{LD} = z_{LR}\left(\frac{1}{9}\right) = z\left(\frac{1}{9}, x^7\right) = z\left(\frac{1}{9}, x^8\right) = 28\frac{8}{9}$ *and* $\lambda^* = \frac{1}{9}$.

Affine functions for Set Q



Linear and Convex

- Equivalently,

$z_{LR}(\lambda)$= maximize $z(\lambda, x^i)$, $\equiv$ minimize w,

subject to $x^i \in Q$, subject to w $z(\lambda, x^i)$, i=1,…,8,

- Which shows that $z_{LR}(\lambda)$ is the maximum of a finite number of affine (linear) functions and is therefore piecewise linear and convex.
- We want to know the relationship between the solution of the Lagrangian dual (LD) and the original ILP.

Example

- In the example,

$$Z_{LD} = 28\frac{8}{9} = z\left(\frac{1}{9}, x^7\right) = z\left(\frac{1}{9}, x^8\right),$$

$$= z\left(\frac{1}{9}, \frac{8}{9}x^7 + \frac{1}{9}x^8\right),$$

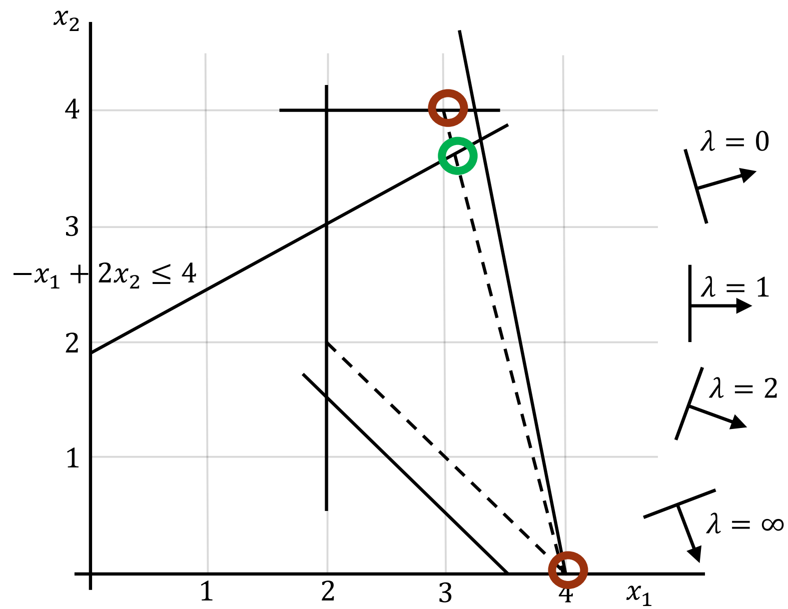
since $z(\lambda,\ x\ )$is affine in x.

$$= z\left(\frac{1}{9}, \frac{8}{9}\binom{3}{4} + \frac{1}{9}\binom{4}{0}\right),$$

$$= z\left(\frac{1}{9}, \begin{pmatrix} 28/9 \\ 32/9 \end{pmatrix}\right) = z\left(\frac{1}{9}, x^*\right),$$

$$= c^T x^* + \frac{1}{9}(4 + x_1^* - 2x_2^*),$$

$$= c^T x^*, \quad \text{since } \ -x_1^* + 2x_2^* = 4.$$



Theorems

- It can be shown that, in general, we can find a convex combination of points in Q that can generate a point x^* satisfying the complicating constraint $A^1 x \leq b^1$ such that $c^T x^* = z_{LD}$

- **Theorem**: $z_{LD} = \text{maximize} \quad c^T x,$

$$\text{subject to} \quad A^1 x \leq b^1, \quad x \in conv(Q)$$

- **Theorem:** $\mathbf{z_{IP} \leq z_{LD}}$.

Questions?

- Next week (1/3), the classroom is still room 101.
- 20% of the final exam will cover the lecture of IP.