Convex Sets (II)

Lecture 2, Nonlinear Programming

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Intersection Preserves Convexity

Intersection Preserves Convexity

If S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex.

Intersection of an infinite number of sets

If S_α is convex for every $\alpha \in \mathcal{A}$, then

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha$$

is convex.

- Example: A polyhedron is the intersection of halfspaces and hyperplanes (which are convex), and therefore is convex.

Positive Semidefinite Cone

Positive Semidefinite

The positive semidefinite cone $\mathbf{S}_+^n$ can be expressed as

$$\bigcap_{z \neq 0} \left\{ X \in \mathbf{S}^n \mid z^T X z \geq 0 \right\}$$

and is convex.

- For each $z \neq 0$, $z^T X z$ is a linear function of X, so the set

$$\left\{ X \in \mathbf{S}^n \mid z^T X z \geq 0 \right\}$$

is a halfspace in $\mathbf{S}^n$.

An Example

- Consider the set

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = \sum_{k=1}^{m} x_k \cos kt$.

- The set S can be expressed as the intersection of an infinite number of **slabs**:

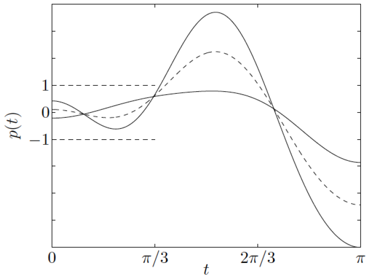
$$S = \bigcap_{|t| \leq \pi/3} S_t$$

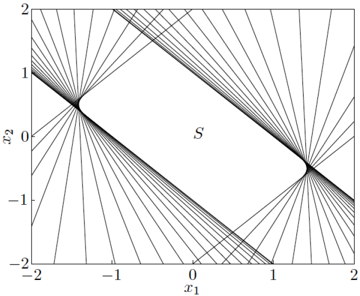
where

$$S_t = \left\{x \mid -1 \leq [\cos t, \cdots, \cos mt]^T x \leq 1\right\}.$$

- So, S is convex.

An Example





$$p(t) = \sum_{k=1}^{m} x_k \cos kt$$

$S = \bigcap_{|t| \leq \pi/3} S_t$
where $S_t =$
$\{x \mid -1 \leq [\cos t, \cdots, \cos mt]^T x \leq 1\}$

Convex Sets as Intersection of Halfspaces

- We have seen that the intersection of (possibly infinite) halfspaces is convex.
- It will be shown that a converse is true: every closed convex set S is the intersection of (usually infinite) halfspaces.
- A closed convex set S is the intersection of all halfspaces that contain it:

$$S = \bigcap \{\mathcal{H} \mid \mathcal{H} \text{ halfspace}, S \subseteq \mathcal{H}\}.$$

Affine functions

Affine function

A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is **affine** if it is a sum of a linear function and a constant. That is, it has the form

$$f(x) = Ax + b$$

where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$.

Affine functions preserve convexity

Suppose $S \subseteq \mathbf{R}^n$ is convex and $f : \mathbf{R}^n \to \mathbf{R}^m$ is an affine function. Then the **image** of S under f,

$$f(S) = \{f(x) \mid x \in S\},$$

is convex.

Affine functions

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Inverse Image under Affine functions

Suppose $S \subseteq \mathbf{R}^n$ is convex and $f : \mathbf{R}^k \to \mathbf{R}^n$ is an affine function. Then the **inverse image** of S under f,

$$f^{-1}(S) = \{x \mid f(x) \in S\},$$

is convex.

Examples – Scaling, Translation, and Projection

- **Scaling**: If $S \subseteq \mathbf{R}^n$ is convex, then for any $\alpha \in \mathbf{R}$, the set

$$\alpha S = \{\alpha x \mid x \in S\}$$

  is convex.

- **Translation**: If $S \subseteq \mathbf{R}^n$ is convex, then for any $a \in \mathbf{R}^n$, the set

$$S + a = \{x + a \mid x \in S\}$$

  is convex.

- **Projection** onto some coordinates: If $S \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is convex, then

$$T = \{x_1 \in \mathbf{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbf{R}^n\}$$

  is convex.

Examples – Sums of Sets

Sum of two sets

The **sum** of two sets is defined as

$$S_1 + S_2 = \{x + y \mid x \in S_1, y \in S_2\}.$$

- If S_1 and S_2 are convex, then $S_1 + S_2$ is convex.

Partial sum of two sets

The **partial sum** of $S_1, S_2 \subseteq \mathbf{R}^n \times \mathbf{R}^m$ is defined as

$$S = \{(x, y_1 + y_2) \mid (x, y_1) \in S_1, (x, y_2) \in S_2, x \in \mathbf{R}^n, y_i \in \mathbf{R}^m\},$$

- Partial sums of convex sets are convex.
- Partial sums are general cases for set intersection ($m = 0$) and set addition ($n = 0$).

Examples – Polyhedra

- The polyhedron $\{x \mid Ax \preceq b\}$ can be expressed as the inverse image of the nonnegative orthant under the affine function $f(x) = b - Ax$:

$$\{x \mid Ax \preceq b\} = \left\{x \mid f(x) \in \mathbf{R}^m_+\right\}.$$

- More generally, the polyhedron $\{x \mid Ax \preceq b,\ Cx = d\}$ can be expressed as the inverse image of the Cartesian product of the nonnegative orthant and the origin under the affine function $f(x) = (b - Ax, d - Cx)$:

$$\{x \mid Ax \preceq b,\ Cx = d\} = \left\{x \mid f(x) \in \mathbf{R}^m_+ \times \{0\}\right\}.$$

Examples – Hyperbolic Cone

- The set

$$\left\{x \;\middle|\; x^T P x \leq (c^T x)^2,\ c^T x \geq 0\right\}$$

where $P \in \mathbf{S}_+^n$ and $c \in \mathbf{R}^n$, is convex.

- It is the inverse image of the second-order cone

$$\left\{(z,t) \mid z^T z \leq t^2, t \geq 0\right\}$$

under the affine function

$$f(x) = (P^{1/2}x, c^T x).$$

Examples – Ellipsoid

- The ellipsoid

$$\mathcal{E} = \left\{ x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1 \right\},$$

where $P \in \mathbf{S}^n_{++}$ is the image of the unit Euclidean ball $\{u \mid ||u||_2 \leq 1\}$ under the affine mapping $f(u) = P^{1/2} u + x_c$.

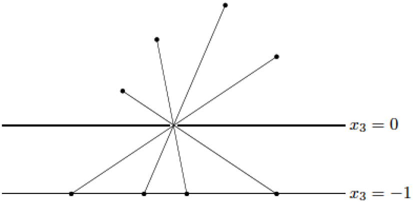
- It is also the inverse image of the unit Euclidean ball under the affine mapping $g(x) = P^{-1/2}(x - x_c)$.

Perspective Functions

Perspective function

The **perspective function** $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$, with domain $\mathbf{dom}\ P = \mathbf{R}^n \times \mathbf{R}_{++}$, is defined as $P(z, t) = z/t$.

The perspective function can be interpreted as the action of a pin-hole camera.



Perspective Functions Preserve Convexity

- Let $C \subseteq \textbf{dom } P = \mathbf{R}^n \times \mathbf{R}_{++}$ be convex, then its image under the perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$, defined as $P(z, t) = z/t$, i.e.,

$$P(C) = \{P(x) \mid x \in C\}$$

  is also convex.

Proof idea: a line segment in C is mapped to a line segment $P(C)$ under $P(\cdot)$.

Perspective Functions Preserve Convexity

- The inverse image of a convex set under the perspective function is also convex:
- If $C \subseteq \mathbf{R}^n$ is convex, then

$$P^{-1}(C) = \left\{(x, t) \in \mathbf{R}^{n+1} \mid x/t \in C, \ t > 0\right\}$$

is convex.

Linear-fractional functions

- A **linear-fractional function** is formed by composing the perspective function with an affine function.

Linear-fractional functions

Let $g : \mathbf{R}^n \to \mathbf{R}^{m+1}$ be affine:

$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix},$$

where $A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m, c \in \mathbf{R}^n$, and $d \in \mathbf{R}$. The function $f : \mathbf{R}^n \to \mathbf{R}^m$ given by $f = P \circ g$, i.e.,

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \mathbf{dom}\ f = \left\{ x \mid c^T x + d > 0 \right\},$$

is called a **linear-fractional** (or **projective**) function.

- Affine functions and linear functions are special cases of linear-fractional functions.

Projective Interpretation

- A linear-fractional function can be represented as a matrix

$$Q = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \in \mathbf{R}^{(m+1)\times(n+1)}.$$

- The matrix Q maps the point $\begin{bmatrix} x \\ 1 \end{bmatrix}$ to $\begin{bmatrix} Ax + b \\ c^T x + d \end{bmatrix}$, a scalar multiple of $\begin{bmatrix} f(x) \\ 1 \end{bmatrix}$.

Projective Interpretation

- Let us associate $\mathbf{R}^n$ with a set of rays in $\mathbf{R}^{n+1}$ as follows.
- For any $z \in \mathbf{R}^n$ we associate the (open) ray

$$\mathcal{P}(z) = \left\{ t \begin{bmatrix} z \\ 1 \end{bmatrix} \mid t > 0 \right\}$$

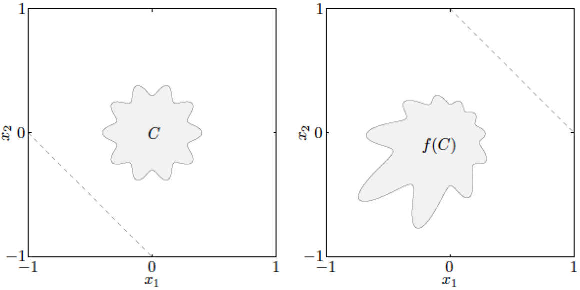
  in $\mathbf{R}^{n+1}$.
- Conversely, any ray in $\mathbf{R}^{n+1}$, with base at the origin and last component which takes on positive value, can be written as $\mathcal{P}(v) = \left\{ t \begin{bmatrix} v \\ 1 \end{bmatrix} \mid t \geq 0 \right\}$ for some $v \in \mathcal{R}^n$.
- The correspondence $\mathcal{P}$ is therefore one-to-one and onto.
- The linear-fractional function f can be expressed as

$$f(x) = \mathcal{P}^{-1}(Q\mathcal{P}(x)).$$

Linear-fractional Functions Preserve Convexity

- Linear-fractional functions preserve convexity.
- If C is convex and $C \subseteq \textbf{dom } f = \{x \mid c^T x + d > 0\}$, then its image $f(C)$ is convex.
    - Proof idea: $f = P \circ g$ where P is the perspective function and g is an affine function.
- Similarly, if $C \subseteq \mathbf{R}^n$ is convex, then the inverse image $f^{-1}(C)$ is convex.

Linear-fractional functions – An Example



- A set $C \subseteq \mathbf{R}^2$ and its image under the linear-fractional function

$$f(x) = \frac{x}{x_1 + x_2 + 1}, \quad \mathbf{dom}\ f = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \; x_1 + x_2 + 1 > 0 \right\}.$$

Linear-fractional Functions – An Example

- Suppose u and v are random variables that take on values in $\{1, ..., n\}$ and $\{1, ..., m\}$, respectively.
- Let $p_{ij} = \mathbf{prob}(u = i, v = j)$. Then the conditional probability $f_{ij} = \mathbf{prob}(u = i | v = j)$ is

$$f_{ij} = \frac{p_{ij}}{\sum_{k=1}^{n} p_{kj}}.$$

- Then f is obtained by a linear-fractional mapping from p. (what is the mapping?)

Proper Cones

Proper Cone

A cone $K \subseteq \mathbf{R}^n$ is called proper cone if it satisfies the following:

- K is convex.
- K is closed.
- K is solid: it has nonempty interior.
- K is pointed: it contains no line ($x \in K, -x \in K \Longrightarrow x = 0$).

Generalized Inequality – Definitions

- A proper cone K can be used to define a **generalized inequality**, a partial ordering on $\mathbf{R}^n$.
- Specifically, we associate the proper cone K with the partial ordering on $\mathbf{R}^n$ defined by

$$x \preceq_K y \iff y - x \in K.$$

Also, $x \succeq_K y$ means $y \preceq_K x$.

- Similarly, define an associated strict partial ordering by

$$x \prec_K y \iff y - x \in \mathbf{int}\ K$$

and write $x \succ_K y$ for $y \prec_K x$.

Generalized Inequality – Examples

- When $K = \mathbf{R}_+$, the partial ordering $\preceq_K$ is the usual ordering $\leq$ on $\mathbf{R}$, and the strict partial ordering $\prec_K$ is the same as the usual strict order $<$ on $\mathbf{R}$.
- Let $K = \mathbf{R}^n_+$ (the nonnegative orthant) in $\mathbf{R}^n$. Then K is a proper cone. The associated generalized inequality $\preceq_K$ corresponds to component-wise inequality between vectors:

$$x \preceq_{\mathbf{R}^n_+} y \iff x_i \leq y_i, \ i = 1, \ldots, n$$

$$x \prec_{\mathbf{R}^n_+} y \iff x_i < y_i, \ i = 1, \ldots, n$$

- The subscript is usually dropped when the proper cone is chosen as $K = \mathbf{R}^n_+$: $\preceq$ means $\preceq_{\mathbf{R}^n_+}$, and $\prec$ means $\prec_{\mathbf{R}^n_+}$.

Positive Semidefinite Cones and Matrix Inequalities

- Let $K = \mathbf{S}_+^n$. Then K is a proper cone in $\mathbf{S}^n$.
- The associated generalized inequality $\preceq_K$ is the usual matrix inequality:
  - $X \preceq_K Y$ means $Y - X$ is positive semidefinite.
  - $X \prec_K Y$ means $Y - X$ is positive definite.
- Note that the interior of $\mathbf{S}_+^n$ consists of the positive definite matrices: $\textbf{int } \mathbf{S}_+^n = \mathbf{S}_{++}^n$.
- Similarly, the subscript is usually dropped when the proper cone in $\mathbf{S}^n$ is chosen as $K = \mathbf{S}_+^n$:
  $\preceq$ means $\preceq_{\mathbf{S}_+^n}$, and $\prec$ means $\prec_{\mathbf{S}_+^n}$.

Cone of Polynomials Nonnegative on $[0, 1]$

- Let $K \subseteq \mathbf{R}^n$ be defined as

$$K = \left\{c \in \mathbf{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\right\}.$$

- K is a proper cone. And its interior is:

$$\textbf{int } K = \left\{c \in \mathbf{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} > 0 \text{ for } t \in [0, 1]\right\}.$$

- For any two vectors $c, d \in \mathbf{R}^n$, we have $c \preceq_K d$ if and only if

$$c_1 + c_2 t + \cdots + c_n t^{n-1} \leq d_1 + d_2 t + \cdots + d_n t^{n-1}$$

for all $t \in [0, 1]$.

Generalized Inequality – Properties

Properties of generalized inequalities

1. $\preceq_K$ is **preserved under addition**: if $x \preceq_K y$ and $u \preceq_K v$, then $x + u \preceq_K y + v$.
2. $\preceq_K$ is **transitive**: if $x \preceq_K y$ and $y \preceq_K z$, then $x \preceq_K z$.
3. $\preceq_K$ is **preserved under nonnegative scaling**: if $x \preceq_K y$ and $\alpha \geq 0$, then $\alpha x \preceq_K \alpha y$.
4. $\preceq_K$ is **reflexive**: $x \preceq_K x$.
5. $\preceq_K$ is **antisymmetric**: if $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
6. $\preceq_K$ is **preserved under limits**: if $x_i \preceq_K y_i$ for $i = 1, 2, \ldots$, and as $i \to \infty$, $x_i \to x$ and $y_i \to y$, then $x \preceq_K y$.

Generalized Inequality – Properties

Properties of (strict) generalized inequalities

1. if $x \prec_K y$, then $x \preceq_K y$.
2. if $x \prec_K y$ and $u \preceq_K v$, then $x + u \prec_K y + v$.
3. if $x \prec_K y$ and $\alpha > 0$, then $\alpha x \prec_K \alpha y$.
4. $x \not\prec_K x$.
5. if $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.

Minimum and Maximum elements

- For the linear ordering "$\leq$" on $\mathbf{R}$, any two points are comparable: either $x \leq y$ or $y \leq x$. This is not always true for a partial ordering like "$\preceq_K$" on $\mathbf{R}^n$.

Minimum and Maximum Elements

We say that $x \in S$ is the **minimum** element of S (w.r.t. $\preceq_K$) if $x \preceq_K y$ for all $y \in S$.
Similarly, $x \in S$ is said to be the **maximum** element of S if $x \succeq_K y$ for all $y \in S$.

- Note: If S has a minimum (or maximum) element, then it is unique.

Minimal and Maximal Elements

Minimal and Maximal Elements

We say that $x \in S$ is a **minimal** element of S (w.r.t $\preceq_K$) if $y \in S$, $y \preceq_K x$ only if $y = x$.
Similarly, $x \in S$ is a **maximal** element of S if $y \in S$, $y \succeq_K x$ only if $y = x$.

- A set can have many different minimal (maximal) elements.

Minimum and Minimal elements

Minimum Element

A point $x \in S$ is the minimum element of S if and only if

$$S \subseteq x + K.$$

Minimal Element

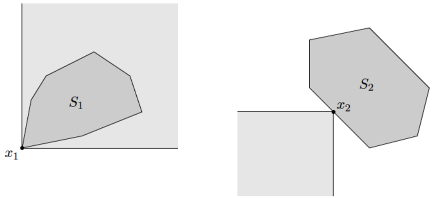
A point $x \in S$ is the minimal element of S if and only if

$$(x - K) \cap S = \{x\}.$$

- For $K = \mathbf{R}_+$, the concepts of minimal and minimum are the same.

Example – Component-wise inequality in $\mathbf{R}^2$

- Consider the cone $\mathbf{R}^2_+$, which induces componentwise inequality in $\mathbf{R}^2$.
- The inequality $x \preceq y$ means y is above and to the right of x.
- $x \in S$ being the minimum element of a set S means that all other points of S lie above and to the right.
- x being a minimal element of a set S means that no other point of S lies to the left and below x.



Example – Minimum and Minimal Elements of $\mathbf{S}^n$

- Consider an ellipsoid centered at the origin and associated with $A \in \mathbf{S}^n_{++}$:

$$\mathcal{E}_A = \left\{ x \mid x^T A^{-1} x \leq 1 \right\}.$$

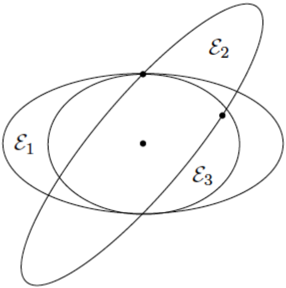
- $A \preceq B$ if and only if $\mathcal{E}_A \subseteq \mathcal{E}_B$.
- Let $v_1, \dots, v_k \in \mathbf{R}^n$ be given and define the set of ellipsoids that contain points $v_1, \dots, v_k$:

$$S = \left\{ P \in \mathbf{S}^n_{++} \;\middle|\; v_i^T P^{-1} v_i \leq 1, i = 1, \dots, k \right\}.$$

- What is the minimum element of S?

Example – Minimum and Minimal Elements of $\mathbf{S}^n$

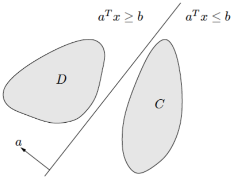
- The set S does not have a minimum element.
- An ellipsoid is minimal if it contains the points, but no smaller ellipsoid does.



Separating Hyperplane Theorem

Separating Hyperplane

The hyperplane $\{x \mid a^T x = b\}$ is called a **separating hyperplane** for the sets C and D, or is said to **separate** the sets C and D if $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$.



Separating Hyperplane Theorem

Suppose C and D are two convex sets that do not intersect, i.e., $C \cap D = \phi$. Then there exist $a \neq 0$ and b such that the hyperplane $\{x \mid a^T x = b\}$ separates C and D.

Separating Hyperplane Theorem – Proof

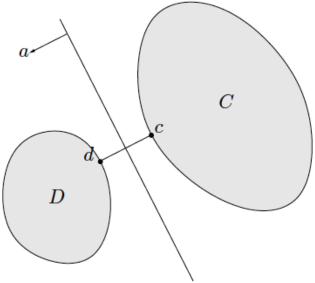
Proof of a special case

- Consider that C and D are both convex, closed, and bounded.
- Assume that the **Euclidean distance** between C and D, defined as

$$\mathbf{dist}(C, D) = \inf \{||u - v||_2 \mid u \in C, v \in D\},$$

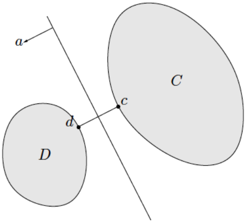
  is positive.
- Since C and D are both closed and bounded, there exist $c \in C$ and $d \in D$ such that

$$||c - d||_2 = \mathbf{dist}(C, D).$$



Separating Hyperplane Theorem – Proof

Proof of a special case



- Let

$$a = d - c, \quad b = \frac{||d||_2^2 - ||c||_2^2}{2}.$$

- Then, it can be shown that the affine function

$$f(x) = a^T x - b = (d - c)^T \left(x - \frac{d + c}{2} \right)$$

is nonpositive on C and nonnegative on D.

Example – A Convex Set and An Affine Set

- Suppose C is convex and D is affine, i.e., $D = \{Fu + g | u \in \mathbf{R}^m\}$, where $F \in \mathbf{R}^{n \times m}$.
- Suppose C and D are disjoint, so by the separating hyperplane theorem there are $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$.
- $\because a^T x \geq b$ for all $x \in D$, $\therefore a^T F u \geq b - a^T g$ for all $u \in \mathbf{R}^m$.
- But a linear function is bounded below on $\mathbf{R}^m$ only when it is zero, so we conclude $a^T F = 0$ (and hence, $b \leq a^T g$).
- Thus we conclude that there exists $a \neq 0$ such that $F^T a = 0$ and $a^T x \leq a^T g$ for all $x \in C$.

Strict Separation of Convex Sets

Strict separation of convex sets

For two sets $C, D \subseteq \mathbf{R}^n$, if there exists $a \in \mathbf{R}^n, b \in \mathbf{R}$ such that

$$a^T x < b \; \forall x \in C \text{ and } a^T x > b \; \forall x \in D,$$

then C and D are said to be **strictly separable**, and the hyperplane $\{x \mid a^T x = b\}$ is called **strict separation** of C and D.

- Remark: The separating hyperplane theorem only dictates that two convex sets that are disjoint to be separated by a hyperplane. A strict separation is not guaranteed (even when the sets are closed).

Example – A Point and A Closed Convex Set

- Let C be a closed convex set and $x_0 \notin C$. Then there exists a hyperplane that strictly separates $\{x_0\}$ from C.
- Proof idea:
  - The two sets C and $B(x_0, \epsilon)$ do not intersect for some $\epsilon > 0$.
  - Apply the separating hyperplane theorem on C and $B(x_0, \epsilon)$ (getting a^T and b), and let $f(x)$.
  - The affine function

$$f(x) = a^T x - b - \epsilon ||a||_2/2$$

    strictly separates C and $\{x_0\}$.
- Corollary: a closed convex set is the intersection of all halfspaces that contain it. (*Hint: proof by contradiction*)

Converse of Separating Hyperplane Theorems

- Question: If there exists a hyperplane that separates convex sets C and D, does this imply C and D are disjoint?
  - (No. Consider $C = D = \{0\} \subseteq \mathbf{R}$.)
- Suppose C and D are convex sets, with C open, and there exists an affine function f that is nonpositive on C and nonnegative on D. Then C and D are disjoint.
  - *Hint: f is negative on C.*

Theorem

Any two convex sets, at least one of which is open, are disjoint if and only if there exists a separating hyperplane.

Theorem of alternatives for strict linear inequalities

Theorem of alternatives for strict linear inequalities

Let $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$. The inequalities

$$Ax \prec b$$

are infeasible if and only if there exists $\lambda \in \mathbf{R}^m$ such that

$$\lambda \neq 0, \quad \lambda \succeq 0, \quad A^T\lambda = 0, \quad \lambda^T b \leq 0.$$

- Proof idea: consider the open convex set

$$D = \mathbf{R}^m_{++} = \{y \in \mathbf{R}^m \mid y \succ 0\}$$

and the affine set (hence convex)

$$C = \{b - Ax \mid x \in \mathbf{R}^n\}.$$

Supporting Hyperplanes

Supporting hyperplanes

Suppose $C \subseteq \mathbf{R}^n$, and x_0 is a point in its boundary $\textbf{bd}\ C$, i.e.,

$$x_0 \in \textbf{bd}\ C = \textbf{cl}\ C \backslash \textbf{int}\ C.$$

If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x \in C$, then the hyperplane $\{x | a^T x = a^T x_0\}$ is called a **supporting hyperplane** to C at the point x_0.

- This is equivalent to the statement that $\{x_0\}$ and C are separated by the hyperplane $\{x | a^T x = a^T x_0\}$.
- The hyperplane is tangent to C at x_0, and the halfspace $\{x | a^T x \leq a^T x_0\}$ contains C.

Supporting Hyperplanes

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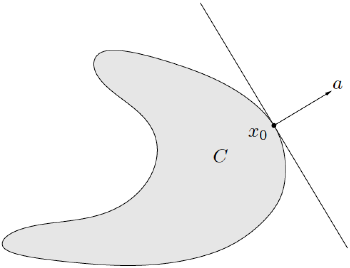

Supporting Hyperplane Theorem

Supporting Hyperplane Theorem

For any nonempty convex set C, and any $x_0 \in \mathbf{bd}\ C$, there exists a supporting hyperplane to C at x_0.

Proof: Use the separating hyperplane theorem.

- If $\mathbf{int}\ C \neq \phi$: then by applying the separating hyperplane theorem on $\{x_0\}$, the statement is proved.
- If $\mathbf{int}\ C = \phi$, then C lies in an affine set of dimension less than n. Then any hyperplane that contains this affine set contains both C and x_0 and therefore is a supporting hyperplane.

(Partial) Converse of the Supporting Hyperplane Theorem

Converse of the Supporting Hyperplane Theorem

If a set C is closed, has nonempty interior, and has a supporting hyperplane at any $x_0 \in \mathbf{bd}\ C$, then C is convex.