VII. Other Time Frequency Distributions (II)

The trend of time-frequency analysis in recent years:

(1) S transform and its generalization

(2) Time-variant signal expansion

(3) Improvement for the Hilbert-Huang transform

VII-A S Transform

(Gabor transform 的修正版)

$$S_x(t,f)=|f|\int_{-\infty}^{\infty}x(\tau)\exp\left[-\pi(t-\tau)^2 f^2\right]\exp(-j2\pi f\tau)\,d\tau$$

closely related to the wavelet transform

advantages and disadvantages

[Ref] R. G. Stockwell, L. Mansinha, and R. P. Lowe, "Localization of the complex spectrum: the S transform," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 998–1001, Apr. 1996.

S transform 和 Gabor transform 相似。

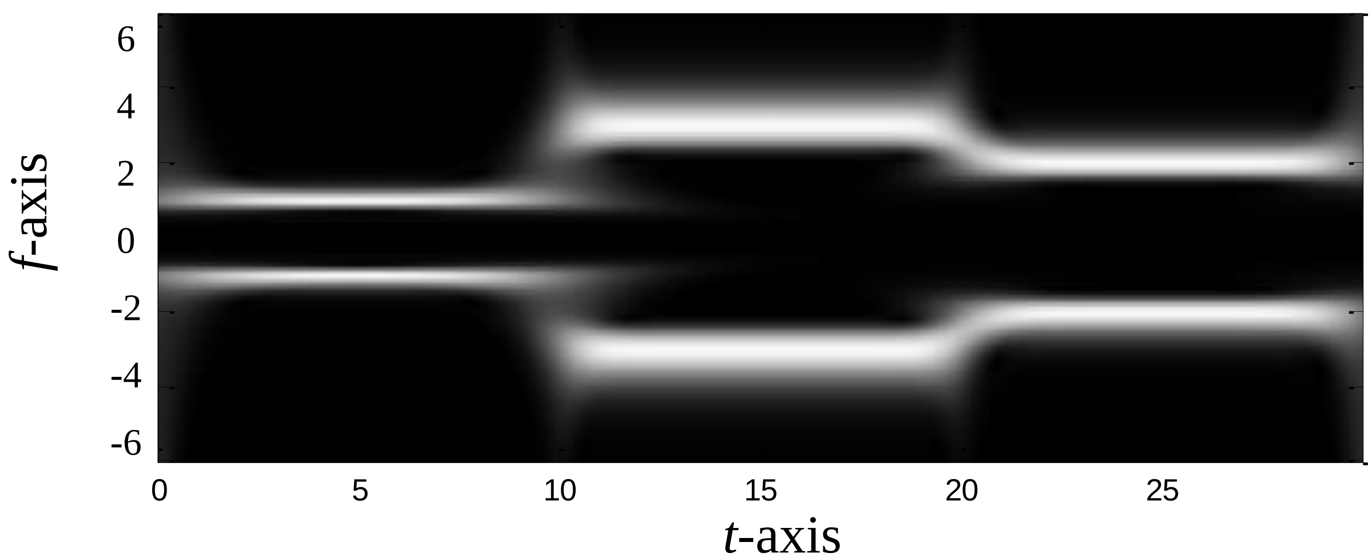
但是 Gaussian window 的寬度會隨著 f 而改變

$$w(t) = \exp\left[-\pi t^2\right] \qquad\qquad w(t) = |f|\exp\left[-\pi t^2 f^2\right]$$

低頻： worse time resolution, better frequency resolution

高頻： better time resolution, worse frequency resolution

The result of the S transform (compared with page 83)



- General form

$$S_x(t,f) = \left|s(f)\right| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 s^2(f)\right] \exp(-j2\pi f\tau)\, d\tau$$

$s(f)$ increases with f

C. R. Pinnegar and L. Mansinha, “The S-transform with windows of arbitrary and varying shape,” *Geophysics*, vol. 68, pp. 381-385, 2003.

Fast algorithm of the S transform

When f is fixed, the S transform can be expressed as a convolution form:

$$S_x(t, f) = |s(f)| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi(t-\tau)^2 s^2(f)\right] \exp(-j2\pi f \tau) d\tau$$

$$\Downarrow$$

$$S_x(t, f) = |s(f)| \left(x(t)\exp(-j2\pi ft) \underset{\substack{convolution \\ along\ t-axis}}{*} \exp\left[-\pi t^2 s^2(f)\right] \right)$$

(for every fixed f)

Remember: $g(t) * h(t) = \int g(\tau) h(t-\tau) d\tau$

Q: Can we use the FFT-based method on page 100 to implement the S transform?

VI-B Generalized Spectrogram

[Ref] P. Boggiatto, G. De Donno, and A. Oliaro, "Two window spectrogram and their integrals," *Advances and Applications*, vol. 205, pp. 251-268, 2009.

Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)G^*_{x,w_2}(t,f)$

$$G_{x,w_1}(t,f) = \int_{-\infty}^{\infty} w_1(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

$$G_{x,w_2}(t,f) = \int_{-\infty}^{\infty} w_2(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

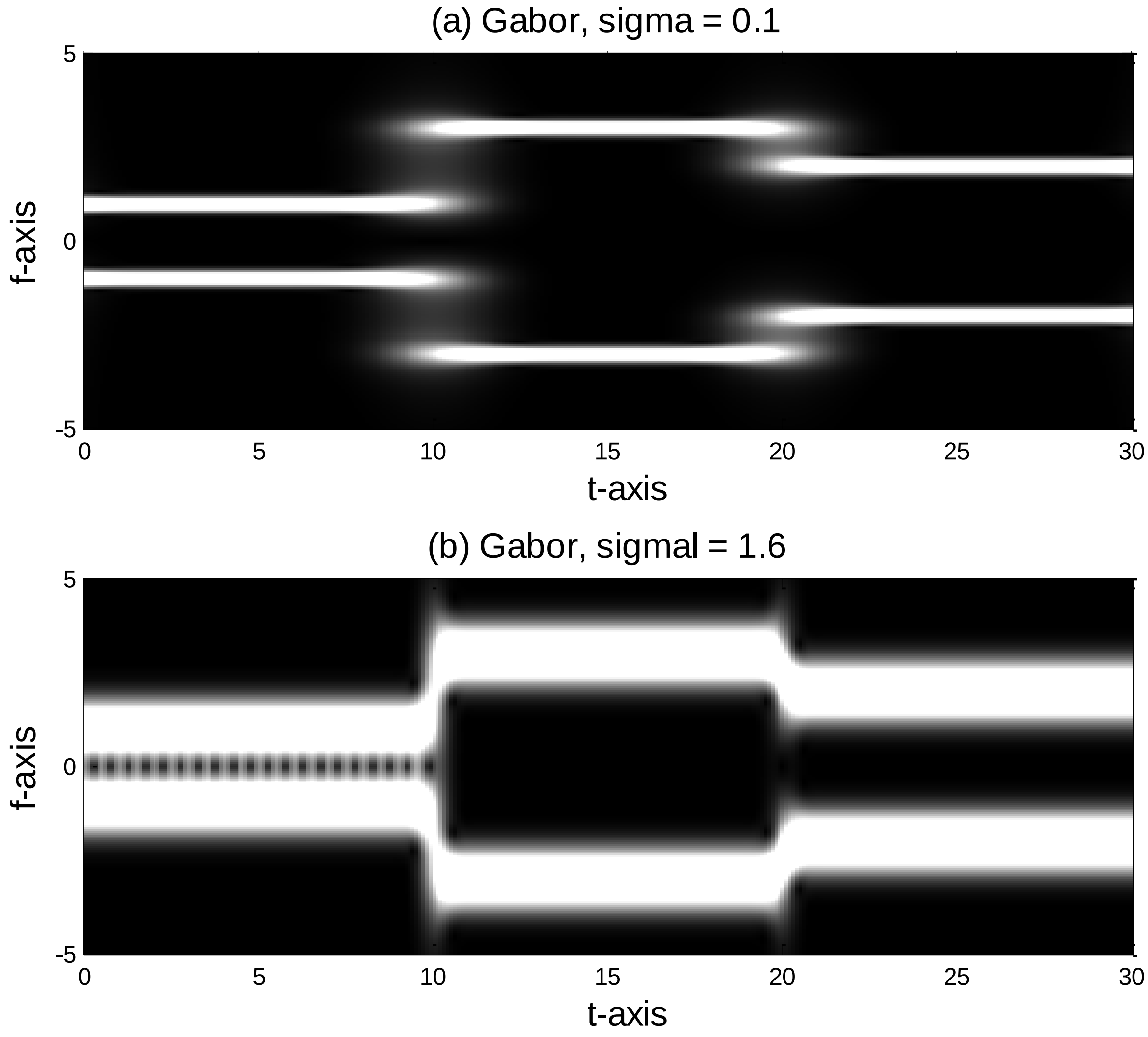
Original spectrogram: $w_1(t) = w_2(t)$

To achieve better clarity, $w_1(t)$ can be chosen as a wider window,
$w_2(t)$ can be chosen as a narrower window.

$x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

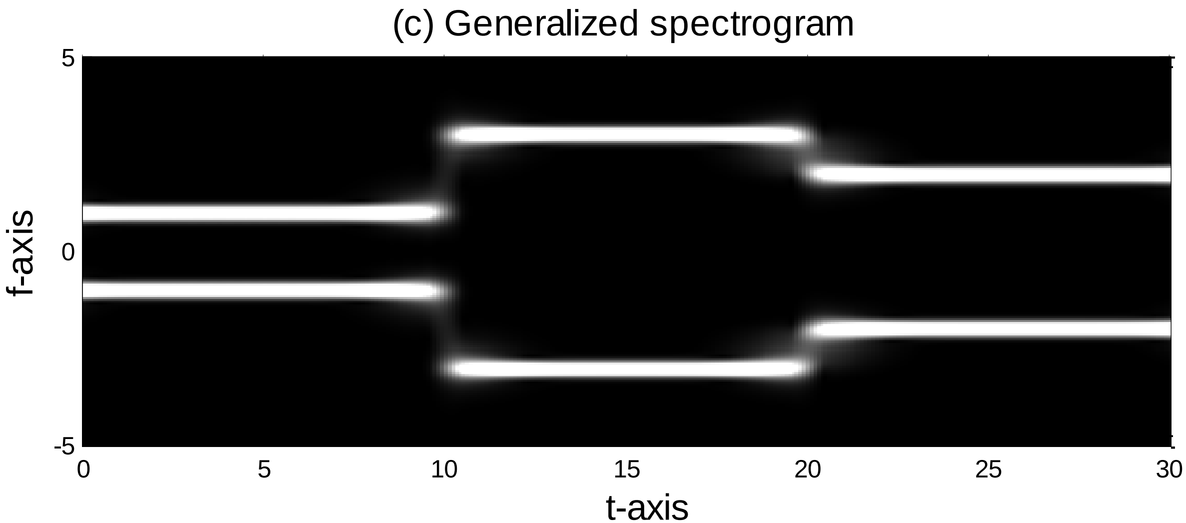
$x(t) = \cos(4\pi t)$ when $t \ge 20$

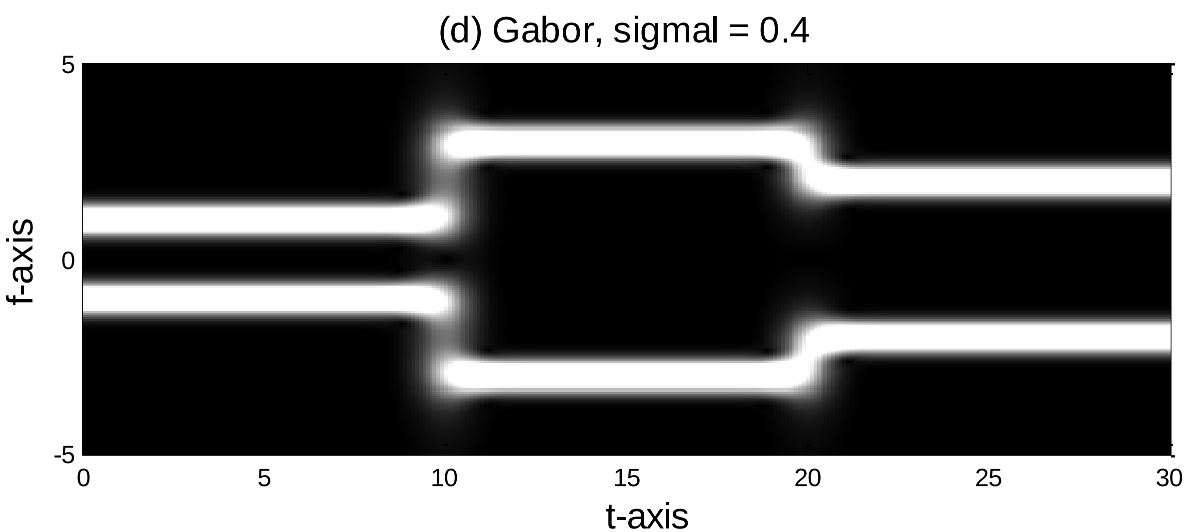


$x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$





Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)G^*_{x,w_2}(t,f)$

Further Generalization for the spectrogram:

$$\boxed{SP_{x,w_1,w_2}(t,f) = G^{\alpha}_{x,w_1}(t,f)\overline{G^{\beta}_{x,w_2}(t,f)}}$$

or

$$\boxed{SP_{x,w_1,w_2}(t,f) = \left|G_{x,w_1}(t,f)\right|^{\alpha}\left|G_{x,w_2}(t,f)\right|^{\beta}}$$

VII-C Basis Expansion Time-Frequency Analysis

就如同

- Fourier series: $\varphi_m(t) = \exp(j2\pi f_m t)$, $\quad x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$

$$a_m = \frac{\left\langle x(t), \varphi_m^*(t) \right\rangle}{\left\langle \varphi_m(t), \varphi_m^*(t) \right\rangle} = \frac{1}{T}\int_0^T x(t)\exp(-j2\pi f_m t)dt \qquad f_m = m/T$$

部分的 Time-Frequency Analysis 也是意圖要將 signal 表示成如下的型態

$$x(t) \approx \sum_{m=1}^{M} a_m \varphi_m(t)$$

並且要求在 M 固定的情形下，

$$\text{approximation error} = \int_{-\infty}^{\infty} \left| x(t) - \sum_{m=1}^{M} a_m \varphi_m(t) \right|^2 dt \quad \text{為最小}$$

將 $\varphi_m(t)$ 一般化，不同的 basis 之間不只是有 frequency 的差異

(1) Three Parameter Atoms

$$x(t) \approx \sum a_{t_0,f_0,\sigma}\varphi_{t_0,f_0,\sigma}(t)$$

$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}}\exp(j2\pi f_0 t - \frac{\pi(t-t_0)^2}{\sigma^2})$$

3 parameters: t_0 controls the central time

f_0 controls the central frequency

σ controls the scaling factor

[Ref] S. G. Mallat and Z. Zhang, "Matching pursuits with time-frequency dictionaries," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3397-3415, Dec. 1993.

Since $\varphi_{t_0,f_0,\sigma}(t)$ are not orthogonal, $a_{t_0,f_0,\sigma}$ should be determined by a matching pursuit process.

(2) Four Parameter Atoms (Chirplet)

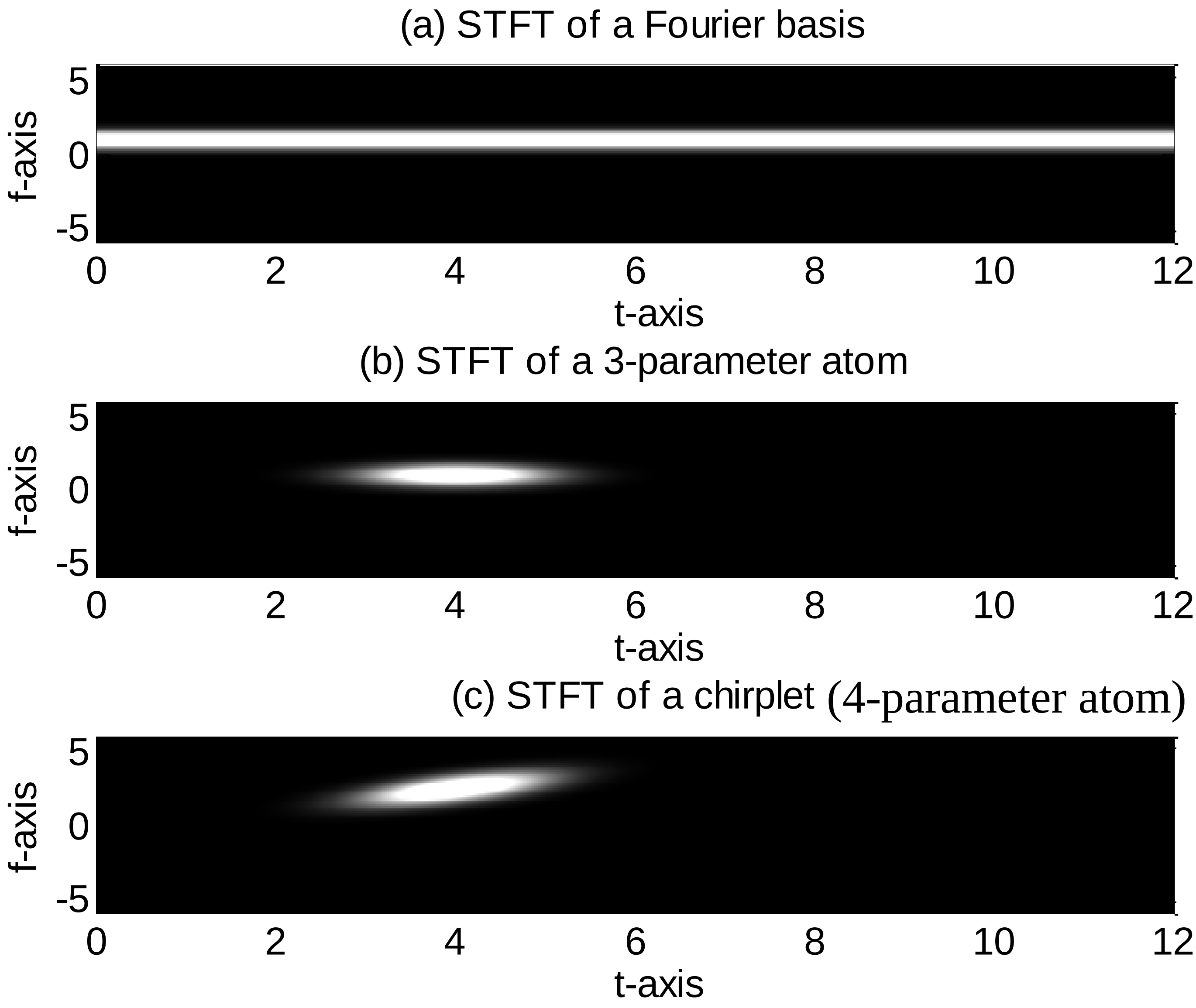
$$x(t) \approx \sum a_{t_0, f_0, \sigma, \eta} \varphi_{t_0, f_0, \sigma, \eta}(t)$$

$$\varphi_{t_0, f_0, \sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}} \exp\left(j2\pi\left(f_0 t + \frac{\eta}{2} t^2\right) - \frac{\pi(t - t_0)^2}{\sigma^2}\right)$$

4 parameters: t_0 controls the central time
f_0 controls the central frequency
σ controls the scaling factor
η controls the chirp rate

[Ref] A. Bultan, "A four-parameter atomic decomposition of chirplets," *IEEE Trans. Signal Processing*, vol. 47, no. 3, pp. 731–745, Mar. 1999.

[Ref] C. Capus, and K. Brown. "Short-time fractional Fourier methods for the time-frequency representation of chirp signals," *J. Acoust. Soc. Am.* vol. 113, issue 6, pp. 3253-3263, 2003.


(a) STFT of a Fourier basis
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis
(b) STFT of a 3-parameter atom
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis
(c) STFT of a chirplet (4-parameter atom)
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis

(3) Prolate Spheroidal Wave Function (PSWF)

$$x(t) \cong \sum_{n,T,\Omega,t_0,f_0} a_{n,T,\Omega,t_0,f_0} \psi_{n,T,\Omega}(t-t_0) \exp(j2\pi f_0 t)$$

where $\psi_{n,T,\Omega}(t)$ is the prolate spheroidal wave function

[Ref] D. Slepian and H. O. Pollak, “Prolate spheroidal wave functions, Fourier analysis and uncertainty-I,” *Bell Syst. Tech. J.*, vol. 40, pp. 43-63, 1961.

Concept of the prolate spheroidal wave function (PSWF):

- FT: $X(f)=\int_{-\infty}^{\infty}\exp(-j2\pi f t)x(t)dt$, $x, f \in (-\infty, \infty)$.

energy preservation property (Parseval's property)

$$\int_{-\infty}^{\infty}|X(f)|^2 df = \int_{-\infty}^{\infty}|x(t)|^2 dt$$

- finite Fourier transform (fi-FT):

$$X_{fi}(f)=\int_{-T}^{T}\exp(-j2\pi f t)x(t)dt$$

space interval: $t \in [-T, T]$,   frequency interval: $f \in [-\Omega, \Omega]$

$$0 < \text{energy preservation ratio} = \frac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2 df}{\int_{-T}^{T}|x(t)|^2 dt} < 1$$

The PWSF $\psi_{0,T,\Omega}$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2 df}{\int_{-T}^{T}|x(t)|^2 dt}$

The PWSF $\psi_{0,T,\Omega}$ can maximize $\dfrac{\int_{-\Omega}^{\Omega} \left| X_{fi}(f) \right|^2 df}{\int_{-T}^{T} \left| x(t) \right|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$

$\psi_{1,T,\Omega}$ can maximize $\dfrac{\int_{-\Omega}^{\Omega} \left| X_{fi}(f) \right|^2 df}{\int_{-T}^{T} \left| x(t) \right|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$ and $\psi_{1,T,\Omega}$

$\psi_{2,T,\Omega}$ can maximize $\dfrac{\int_{-\Omega}^{\Omega} \left| X_{fi}(f) \right|^2 df}{\int_{-T}^{T} \left| x(t) \right|^2 dt}$

and so on.

- Prolate spheroidal wave functions (PSWFs) are the continuous functions that satisfy: $\int_{-T}^{T} K_{F,\Omega}(t_1,t)\psi_{n,T,\Omega}(t)dt = \lambda_{n,T,\Omega}\psi_{n,T,\Omega}(t_1)$,

  where $$K_{F,\Omega}(t_1,t) = \frac{\sin[2\pi\Omega(t_1-t)]}{\pi(t_1-t)}$$

PSWFs are orthogonal and can be sorted according to the values of $\lambda_{n,T,\Omega}$'s:

$$\int_{-T}^{T} \psi_{m,T,\Omega}(t)\psi_{n,T,\Omega}(t)dt = \lambda_{n,T,\Omega}\delta_{m,n}$$

$1 > \lambda_{0,T,\Omega} > \lambda_{1,T,\Omega} > \lambda_{2,T,\Omega} > \ldots\ldots\ldots\ldots > 0.$ (All of $\lambda_{n,T,\Omega}$'s are real)

附錄七： Compressive Sensing and Matching Pursuit 的觀念

Different from orthogonal basis expansion, which applies a complete and orthogonal basis set, compressive sensing is to use an over-complete and non-orthogonal basis set to expand a signal.

Example:

Fourier series expansion is an orthogonal basis expansion method:

$$x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$$

$$\int \exp(j2\pi f_m t)\overline{\exp(j2\pi f_n t)}\, dt = 0 \qquad \text{if } f_m \neq f_n$$

Three-parameter atom expansion, Four-parameter atom (chirplet) expansion, and PSWF expansion are over-complete and non-orthogonal basis expansion methods.

$$x(t) \approx \sum a_{t_0, f_0, \sigma} \varphi_{t_0, f_0, \sigma}(t)$$

$\varphi_{t_0, f_0, \sigma}(t)$ do not form a complete and orthogonal set.

The problems that compressive sensing deals with:

Suppose that $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$ ………….. form an over-complete and non-orthogonal basis set.

(Problem 1) We want to minimize $||c||_0$ （$||\ ||_0$ 是 zero-order norm，$||c||_0$ 意指 c_m 的值不為 0 的個數） such that

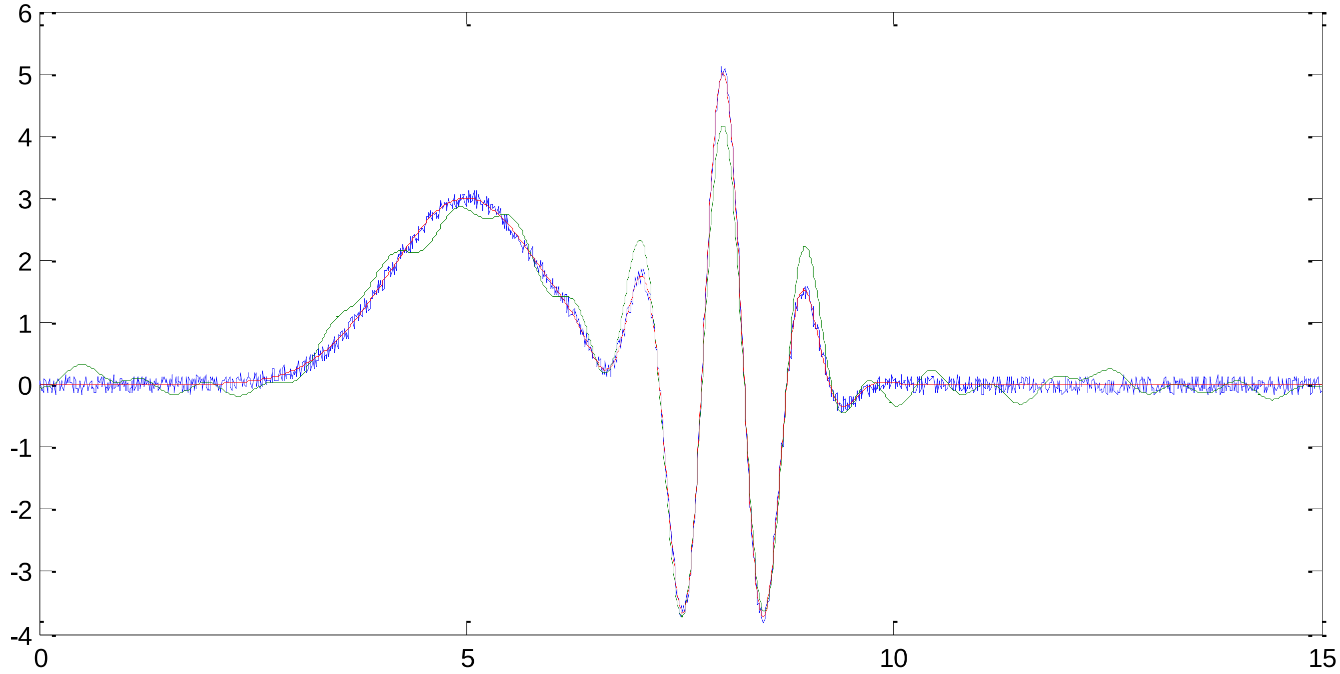
$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_0$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_0$ is limited to M, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$



For example, in the above figure, the blue line is the original signal

- When using three-parameter atoms, the expansion result is the red line

$$x(t)=3e^{-0.2\pi(t-5)^2}+2.5e^{-0.4\pi(t-8)^2+j2\pi t}+2.5e^{-0.4\pi(t-8)^2-j2\pi t}$$

Only 3 terms are used and the normalized root square error is 0.39%

- When using Fourier basis, if 31 terms are used, the expansion result is the green line and the normalized root square error is 3.22%

Question: How do we solve the optimization problems on page 203?

Method 1: Matching Pursuit (Greedy Algorithm)

Input: $x(t)$

Initial: $n = 0$, $x_r(t) = x(t)$

Find m such that $\int x_r(t) b_m^*(t)\,dt$ is maximal

Set $\phi_n(t) = b_m(t)$, $\mu_n = \int x(t) b_m^*(t)\,dt$

$x_r(t) = x_r(t) - \mu_n \phi_n(t)$

$n = n+1$

For Problem 1: Whether $x_r(t) = 0$

For Problem 2: Whether $\int x_r^2(t)\,dt < threshold$

For Problem 3: Whether $n = M$

No

Yes

$$x(t) \cong \sum_m \mu_m \phi_m(t)$$

Method 2: Basis Pursuit

Change the zero-order norm into the first order norm

$$||c||_1 = |c_0| + |c_1| + |c_2| + \ldots\ldots$$

(Problem 1) We want to minimize $||c||_1$ such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_1$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_1 \le M$, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

- D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, issue 4, pp. 1289–1306, 2006. (被視為最早提出 compressive sensing 概念的論文)
- E. J. Candès and M. B. Wakin, "An introduction to compressive sampling," *IEEE Signal Processing Magazine*, vol. 25, issue 2, pp. 21-30, 2008. (對 compressive sensing 做 tutorial 式的介紹)
- S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*, Birhauser, Basel, 2013. (以數學的方式介紹 compressive sensing)
- S. G. Mallat and Z. Zhang. "Matching pursuits with time-frequency dictionaries," *IEEE Trans. Signal Processing*, vol. 41, issue 12, pp. 3397-3415, 1993. (最早提出 matching pursuit)
- S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic decomposition by basis pursuit," *SIAM Journal on Scientific Computing*, vol. 20, issue 1, pp. 33-61, 1998. (最早提出 basis pursuit)
- S. Kunis and H. Rauhut, "Random sampling of sparse trigonometric polynomials, II. Orthogonal matching pursuit versus basis pursuit," *Foundations of Computational Mathematics*, vol. 8, issue 6, pp. 737-763, 2008. (將 orthogonal expansion 以及 matching pursuit, basis pursuit 的概念做綜合)