Fixed Points

The fixed point problem is this:

Given a set $S \subset \mathbf{R}^n$ and a function $f : S \to S,$ is there an $x \in S$ such that $f(x) = x$?

FIXED POINT THEOREMS

Banach Fixed Point Theorem

The simplest of all fixed point theorems is ascribed to Stefan Banach (1892－1945).

Definition

A function $f : S \to S$ is called a **contraction mapping** if $\|(f(x), f(y))\| \le C\|(x, y)\|$ for all $x, y \in S,$ where $0 \le C < 1$ is a fixed constant.

In the one-dimensional case, the contraction mapping condition is

$$|f(x) - f(y)| \le C|x - y|.$$

Banach's Contraction Mapping Theorem

Let *f* be a contraction with domain $\mathbf{R}^p$ and range contained in $\mathbf{R}^p$. Then *f* has exactly one fixed point.

The Banach Theorem is quite weak. Consider $f : [0,1] \to [0,1],$ where $f(x) = x.$ This function barely misses being a contraction since $|f(x) - f(y)| = |x - y|$ for all $x, y \in [0,1].$ However, every point in $[0,1]$ is a fixed point of this function.

Brouwer Fixed Point Theorem

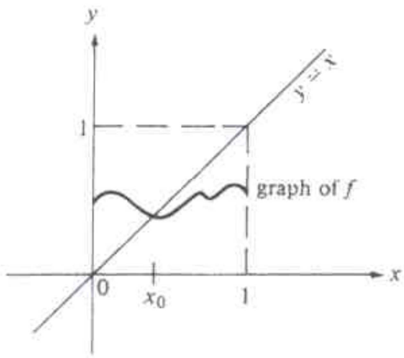
If $S \subset \mathbf{R}^n$ is compact and convex and $f : S \to S$ is continuous there exists $x \in S$ such that $f(x) = x.$

Theorem (a "baby" version of Brouwer fixed-point theorem)

Let *f* be a continuous function mapping [0,1] into [0,1]. In other words, domain of *f* is [0,1] and $f(x) \in [0,1]$ for all $x \in [0,1]$. The function *f* has a fixed pint, i.e., a point $x_0 \in [0,1]$ such that

$f(x_0) = x_0$.

The graph of *f* lies in the unit square. Our assertion is equivalent to the assertion that the graph of *f* crosses the *y*=*x* line, which is almost obvious.



Correspondences

Let S and T be nonempty sets. A *correspondence* F from S into T is a rule that assigns to each element x of S a nonempty subset $F(x)$ of T. We write $F : S \to T$. (Correspondence is an alternative term for a relation between two sets.)

It should be noted that the notation is no difference than that for functions. But of course, functions may be treated as a special kind of correspondence where the subset $F(x)$ of T associated with each element x of S consists of exactly one element. So, we may say that a correspondence is in general multi-valued (each element of $F(x)$ is a value) whereas a function is single-valued (there being one and only one value in $F(x)$.)

Examples.

(1) For each $x \in [0,\infty)$, let $F(x)$ denote the set of real numbers whose squares are x. Thus, $F:[0,\infty) \to \mathbf{R}$ is a correspondence, with for example,

$$F(0) = \{0\},\ F(1) = \{-1,1\},\ F(2) = \{-\sqrt{2},\sqrt{2}\},...$$

(2) For each $x \in \mathbf{R}$ define $F(x) = (x,\infty) = \{y \in \mathbf{R} : y > x\}$. Then $F : \mathbf{R} \to \mathbf{R}$ is a correspondence. Some examples of function values are

$$F(-4) = (-4,\infty),\ F(1) = (1,\infty).$$

(3) Suppose that $\varphi:[a,b]\to\mathbf{R}$ and $\psi:[a,b]\to\mathbf{R}$ are two functions such that for all $x\in[a,b]$, $\varphi(x)\le\psi(x)$. Then we can define a correspondence $F:[a,b]\to\mathbf{R}$ by

$$F(x)=\left[\varphi(x),\ \psi(x)\right]=\left\{y\in\mathbf{R}:\varphi(x)\le y\le\psi(x)\right\}.$$

(4) Now, some constructions in higher dimensions. Fix a nonzero vector $u\in\mathbf{R}^n$, and define $F:\mathbf{R}^n\to\mathbf{R}^n$ by

$$F(x)=\left\{y\in\mathbf{R}^n : y\cdot u\le x\cdot u\right\}.$$

Then F is a correspondence.

(5) Suppose that $p=(p_1,\dots,p_n)$ is a given price system and w the total wealth of a consumer. The set of all feasible consumptions is then

$$\begin{aligned}F(w)&=\left\{(x_1,\dots,x_n)\in\mathbf{R}^n : p_1x_1+\dots+p_nx_n\le w,\ x_i\ge 0\ \ (1\le i\le n)\right\}\\&=\left\{x\in\mathbf{R}^n_+ : p\cdot x\le w\right\}\end{aligned}$$

where we write R_+ for the set $[0,\infty)$ of all non-negative real numbers, then we have a correspondence $F:\mathbf{R}_+\to\mathbf{R}^n_+$.

Graph

Suppose that $F:\mathbf{R}^m\to\mathbf{R}^n$ is a correspondence. Then the graph of F is defined to be

$$\begin{aligned}G_F&=\{(x_1,\dots,x_m,y_1,\dots,y_n)\in\mathbf{R}^{m+n}:(y_1,\dots,y_n)\in F(x_1,\dots,x_m)\}\\&=\{(x,y)\in\mathbf{R}^m\times\mathbf{R}^n : y\in F(x)\}\end{aligned}$$

Examples.

(1) For $F:[0,\infty)\to\mathbf{R}$ defined by $F(x)$=the set of real numbers whose squares are *x*.

$F(x)=\{y\in\mathbf{R}:y^2=x\}$. Then the graph of *F* is the parabola $y^2=x$.

(2) For $F:\mathbf{R}\to\mathbf{R}$ defined by $F(x)=(x,\infty)$, we have

$G_F=\{(x,y)\in\mathbf{R}^2 : y\in(x,\infty)\}=\{(x,y)\in\mathbf{R}^2 : y>x\}$ which is the open halfplane above the line *y*=*x*.

(3) For $F(x)=\left[\varphi(x),\ \psi(x)\right]=\left\{y\in\mathbf{R}:\varphi(x)\le y\le\psi(x)\right\}$, $x\in[a,b]$ where $\varphi(x)\le\psi(x)$ for all

$x \in [a,b]$, we have $G_F = \{(x,y) \in \mathbf{R}^2 : \varphi(x) \le y \le \psi(x)\}$

That is the region on the plane bounded between the curves $y = \varphi(x)$, $y = \psi(x)$, x=a, and x=b

SEMI-CONTINUITY

Closed and Open Sets

Definition: a set $A \subseteq \mathbf{R}^n$ is open if for each point in *A*, its neighborhood is contained in *A*.

Definition: a set $A \subseteq \mathbf{R}^n$ is closed if its complement, $\mathbf{R}^n \setminus A$ is open.

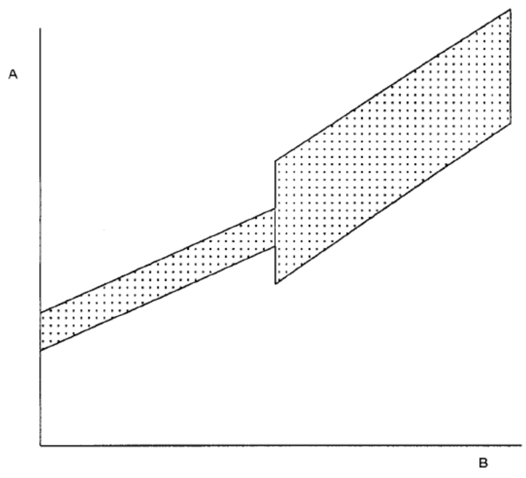
Definition

A correspondence *C* is called **upper semi-continuous** (usc), if the set $\{(x,y): y \in C(x)\}$ is closed. The set $\{(x,y): y \in C(x)\}$ is the **graph** of the correspondence.

Example 1

Here is a correspondence defined on $[-1,1]$. If $x \in [-1,0)$ then $C(x)$=0.5. If $x \in (0,1]$ then $C(x)$= -0.5. If x=0, $C(x)$={ $0.5, -0.5$ }. It is easy to check that this correspondence is usc.

Example 2



The correspondence whose graph is portrayed in the left figure is upper semi-continuous.

Definition

A correspondence *C* on $S \subset \mathbf{R}^n$ is called **convex value**d if $C(x)$ is a convex set for all $x \in S$.

The correspondence in both **Example 1** and **Example 2** above is not convex valued.

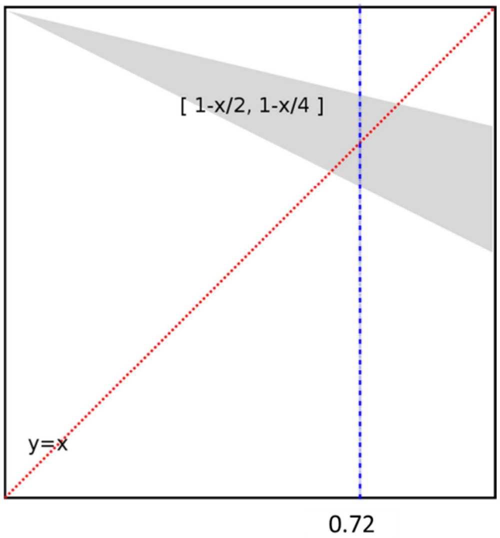
Kakutani's fixed point theorem

Let $S \subset \mathbf{R}^n$ be a compact and convex set. Let C be a correspondence from S into itself that is usc and convex valued. Then, there is an $x^* \in S$ such that $x^* \in C(x^*)$.

Example 1

Let C be a correspondence defined on the closed interval [0, 1] that maps a point x to the closed interval [1-x/2, 1-x/4]. Then C satisfies all the assumptions of the theorem and must have fixed points.

In the diagram, any point on the 45° line (dotted line in red) which intersects the graph of the function (shaded in grey) is a fixed point, so in fact there is an infinity of fixed points in this particular case. For example, x=0.72 (dashed line in blue) is a fixed point since 0.72∈[1−0.72/2, 1−0.72/4].



Example 2

The requirement that φ(x) be convex for all x is essential for the theorem to hold.

Consider the following function defined on [0,1]:

$$C(x) = \begin{cases} 3/4, & 0 \le x < 0.5 \\ \{3/4, 1/4\}, & x = 0.5 \\ 1/4, & 0.5 < x \le 1 \end{cases}$$

The correspondence has no fixed point. Though it satisfies all other requirements of Kakutani's theorem, its value fails to be convex at x = 0.5.