Using LP to solve IP

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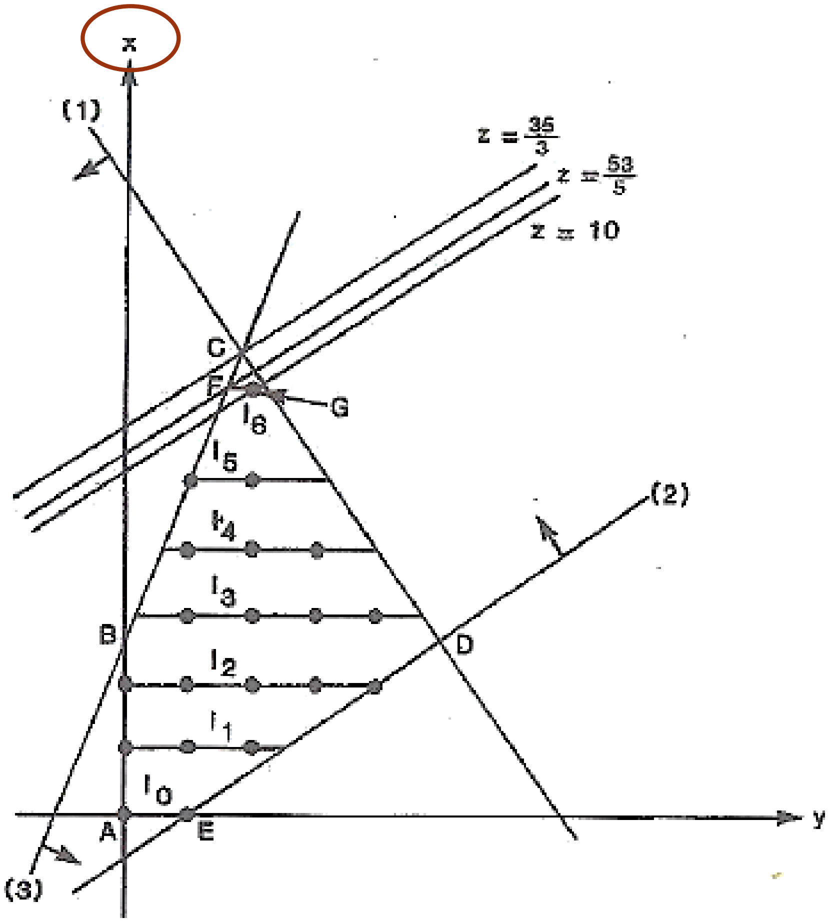
Course No: 546 U6110

Agenda

- Graphical Solutions
- Unimodularity
- Rounding LP Solutions

Graphical Solutions

$$\begin{aligned}
\text{Maximize}\quad & 2x - y = z \\
\text{subject to}\quad & 5x + 7y \le 45, \\
& -2x + y \le 1, \\
& 2x - 5y \le 5, \\
& x,\quad y \ge 0, \\
\text{and}\quad & x \text{ integer}.
\end{aligned}$$



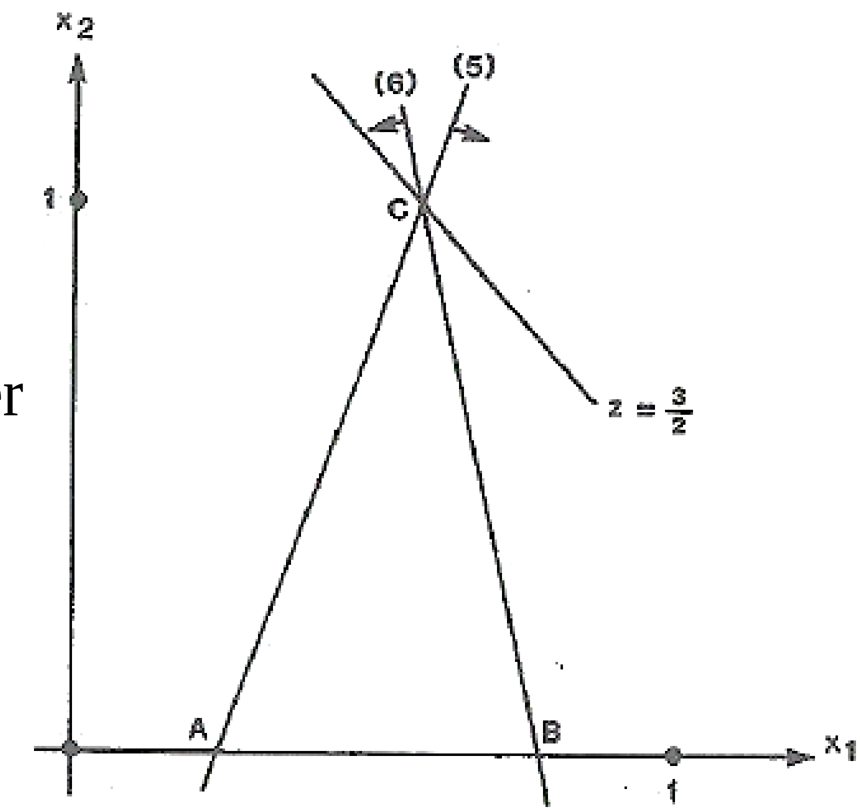
No Solution

Maximize $\quad x_1 + x_2 = z$

subject to $\quad -4x_1 + x_2 \le -1,$

$\quad 4x_1 + x_2 \le 3,$

and $\quad x_1,\ \ x_2 \ge 0 \quad$ and integer



Some Insights

- The maximal value of the objective function to the MIP (IP) solved as a LP is ______________ on the value of any feasible solution to MIP (IP).
- If the optimal solution to the MIP solved as a LP is integer in its integer constrained variables, it solves the MIP
- If the MIP solved as a linear one is infeasible, so is the MIP.

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Unimodularity

- Matrix A is totally unimodular *if and only if* the determinant of every square of A is either 0,+1 or -1.

Examples

- The following matrices are TU.

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- The following matrices are not TU

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Simplest Determinant

For a 2*2 matrix A, the determinant, denoted by $|A|$ or $det(A)$, is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example

$$A = \begin{bmatrix} -5 & 4 \\ 2 & 7 \end{bmatrix}$$

$$|A| = ?$$

$$|A| = -5*7 - 2*4 = -43$$

Cofactor Expansion

Formal definition of determinants of $n*n$ matrices will **adopt** the definition of the determinant of a **2*2** matrix.

Minor of a_{ij}: M_{ij}

For a square $n*n$ matrix A, M_{ij} is the **determinants** of the matrix after **remove** row $\boldsymbol{i}$ and column $\boldsymbol{j}$.

Example

$$A = \begin{bmatrix} & & & \\ a_{21} & & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix}$$

$$M_{12} = ?$$

Example

$$A = \begin{bmatrix} 1 & 0 & \\ & & \\ 0 & -2 & \end{bmatrix} \qquad M_{23} = ?$$

$$M_{23} = \left\| \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \right\| = -2$$

Cofactor of a_{ij}: C_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

C_{ij} and M_{ij} might be different only by **a sign**.

Example

$$A = \begin{bmatrix} 1 & 0 & \\ & & \\ 0 & -2 & \end{bmatrix} \qquad C_{23} = ?$$

$$C_{23} = (-1)^{2+3} \left| \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \right| = 2$$

Cofactor Expansions on first row (column)

For an $n*n$ matrix, A, the determinant is the sum of the products of the elements of the **first** row (column) and their cofactors.

Cofactor Expansions on First Row/Column

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n}C_{1n}$$

Or

$$|A| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + \ldots + a_{n1}C_{n1}$$

Example

$$A = \begin{bmatrix} \boxed{1 \quad 0 \quad 3} \\ 4 \quad -1 \quad 2 \\ 0 \quad -2 \quad 1 \end{bmatrix} \qquad |A| = ?$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 1(-1)^{1+1}(-1+4) + 0 +$$

$$3(-1)^{1+3}(-8+0) = -21$$

Cofactor Expansions on *i*th Row or *j*th Column

*i*th row expansion:

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

*j*th column expansion:

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

Example

$$A = \begin{bmatrix} 1 & \boxed{\begin{matrix} 0 \\ -1 \\ -2 \end{matrix}} & 3 \\ 4 & & 2 \\ 0 & & 1 \end{bmatrix} \qquad |A| = ?$$

$$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= 0 + (-1)(-1)^{2+2}(1-0) +$$

$$(-2)(-1)^{3+2}(2-12) = -21$$

Example

$$A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix} \qquad |A| = ?$$

$$|A| = 3(-1)^{3+3}[2(-1)^{1+1}(3-2)]$$

$$= 6$$

Matrix of Cofactors

For a square $n*n$ matrix A,

$$\text{Matrix of Cofactor} = \begin{bmatrix} C_{11} & C_{12} \cdots & C_{1n} \\ C_{21} & C_{22} \cdots & C_{2n} \\ \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} \cdots & C_{nn} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$ Matrix of Cofactor = ?

$C_{11}=3$, $C_{12}=-4$, $C_{13}=-8$,
$C_{21}=-6$, $C_{22}=1$, $C_{23}=2$,
$C_{31}=3$, $C_{32}=10$, $C_{33}=-1$

$$\begin{bmatrix} 3 & -4 & -8 \\ -6 & 1 & 2 \\ 3 & 10 & -1 \end{bmatrix}$$

Adjoint of A

For a square $n*n$ matrix A,

$$\boldsymbol{Adj(A)} = (\text{Matrix of Cofactors})^t$$

$$\boldsymbol{Adj(A)} = \begin{bmatrix} \boxed{\begin{matrix} C_{11} \\ C_{12} \\ \vdots \\ C_{1n} \end{matrix}} & \boxed{\begin{matrix} C_{21} \\ C_{22} \\ \vdots \\ C_{2n} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \boxed{\begin{matrix} C_{n1} \\ C_{n2} \\ \vdots \\ C_{nn} \end{matrix}} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix} \quad Adj(A) = ?$$

Matrix of Cofactors =

$$\begin{bmatrix} 3 & -4 & -8 \\ -6 & 1 & 2 \\ 3 & 10 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -6 & 3 \\ -4 & 1 & 10 \\ -8 & 2 & -1 \end{bmatrix}$$

The Inverse and the Determinant

Let A be a **square** matrix with **$|A| \neq 0$** (A is invertible)

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

Example 3

- Use the formula for the inverse of a matrix to compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Cofactor of A

$$C_{11} = \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = 14 \qquad C_{12} = -\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix} = 3 \qquad C_{13} = \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = -1$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ -3 & 5 \end{vmatrix} = -9 \qquad C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7 \qquad C_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix} = -12 \qquad C_{32} = -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = -1 \qquad C_{33} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8$$

The adjoint of A is the transpose of this matrix.

$$\operatorname{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

Solution

$|A| = 25$, so the inverse of A exists. Thus,

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{25}\begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix} = \begin{bmatrix} \frac{14}{25} & \frac{-9}{25} & \frac{-12}{25} \\ \frac{3}{25} & \frac{7}{25} & \frac{1}{25} \\ \frac{-1}{25} & \frac{6}{25} & \frac{8}{25} \end{bmatrix}$$

Cramer's Rule

Given $AX=B$ and $|A| \neq 0$,

$$x_1 = \frac{|A_1|}{|A|},\ x_2 = \frac{|A_2|}{|A|},\ \ldots.\ x_n = \frac{|A_n|}{|A|}$$

where A_i is from A by replacing column i with B.

$|A_i|$

$$A = \begin{bmatrix} \boxed{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \boxed{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{matrix}} & \begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$|A_1| = ? \qquad |A_j| = ?$$

Unimodularity Theorem

Consider

maximize $z = c^T x,$

subject to $A\, x = b,$

$x \geq 0$, integer,

Where A is unimodular and b is integer

Unimodularity Theorem:

Let A be an mxn integer matrix with rank(A) = m which is smaller than n.

The following are equivalent:

- For each basis matrix B of A, det B = +1 or -1.
- For each integer vector b, every feasible extreme point of $\mathrm{X} = \{x \mid Ax = b, x \geq 0\}$ is integer.
- For each basis matrix B of A, B^{-1} is integer.

Summary

- If an ILP has a coefficient matrix A that is totally unimodular and the right-hand-side vector *b* is integer, then every basic feasible solution (BFS) is integer. Thus, the integrality constraints can be relaxed and the ILP can be solved as an LP.

- A sufficient condition for TU:
  - Let *A* be an *m*x*n* matrix
  - All entries in *A* are +1, -1 or 0
  - Each column contain at most 2 nonzero entries
  - The row of *A* can be partitioned into 2 sets. For each column, if the entries are of the same sign, there are in different sets; otherwise, there are in the same set.

Assignment Problem

$$\text{Minimize} \quad \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{i=1}^{3} x_{ij} = 1 \qquad (j = 1,2,3)$$

$$\sum_{j=1}^{3} x_{ij} = 1 \qquad (i = 1,2,3)$$

$$\text{and} \quad x_{ij} = 0 \text{ or } 1 \qquad (\text{all } i, j).$$

$$\begin{array}{cc} & \begin{array}{ccccccccc} x_{11} & x_{12} & & & \cdots\cdots & & & x_{32} & x_{33} \end{array} \\ \text{Machine} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\ \text{Job} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Minimum Cost Network Flow Problem

$$\text{maximize } z = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{m} x_{ij} - \sum_{k=1}^{m} x_{ki} = b_i, \qquad i = 1, \ldots, m,$$

$$x_{ij} \geq 0 \text{ , integer}$$

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Rounding LP Solutions

- Example: facility location (assume all produced items have to be shipped out)

		Shopping costs Customers (n)					Production Capacity M_i
		1	2	3	4	5	
Sources (m)	1	93	70	48	68	81	**2**
	2	45	89	97	85	96	**3**
	3	92	93	58	37	99	**2**
	4	55	103	55	57	38	**3**
	5	74	60	78	54	52	**2**
	Demands d_j	**1**	**1**	**1**	**1**	**1**	

LP Solution

LP optimal solution, cost = 228

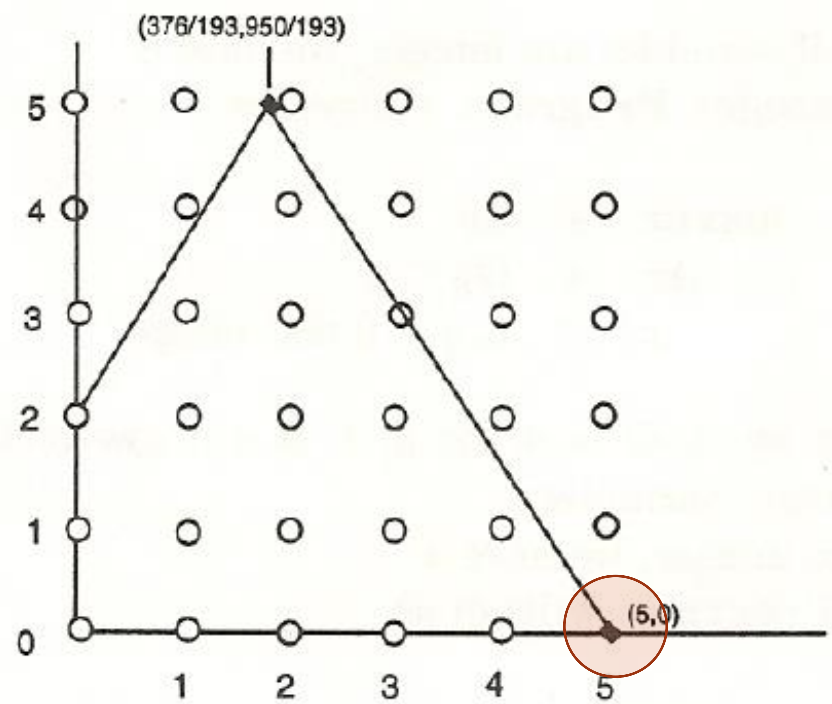
$$x_1 = x_3 = x_5 = 1/2, \quad x_2 = x_4 = 1/3,$$

$$z_{13} = z_{21} = z_{34} = z_{45} = z_{52} = 1$$

Facility variables	Shipping variables	Total cost
$x_1 = x_2 = 1$	$z_{12}=z_{13}=z_{21}=z_{24}=z_{25}=1$	344
$x_1 = x_4 = 1$	$z_{12}=z_{13}=z_{41}=z_{44}=z_{45}=1$	268
$x_3 = x_2 = 1$	$z_{21}=z_{22}=z_{25}=z_{33}=z_{34}=1$	325
$x_3 = x_4 = 1$	$z_{32}=z_{34}=z_{41}=z_{43}=z_{45}=1$	278
$x_5 = x_2 = 1$	$z_{21}=z_{22}=z_{23}=z_{54}=z_{55}=1$	337
$x_5 = x_4 = 1$	$z_{41}=z_{43}=z_{45}=z_{52}=z_{54}=1$	**262**

Example 2

$$\max 1.00x_1 + 0.64x_2$$
$$50x_1 + 31x_2 \leq 250$$
$$3x_1 - 2x_2 \geq -4$$
$$x_1, x_2 \geq 0 \text{ and integer.}$$



Questions

- Homework 2 (due on 3/30). Please download it from Ceiba.