What is GAME THEORY?

The study of multi-person decision problem.

Classification of GAMES

- ✓ Static / Dynamic
  是否同時決策？
- ✓ Complete Information / Incomplete Information
  是否所有人皆知他人之收益函數？

Elements of a GAME

Players / Strategies / Payoffs / Information / Rationality

Normal Model of GAME

- ✓ A set of N players $I = \{1, 2, \cdots, N\}$
- ✓ Each player $i \in I$ has an action set
$$A_i = \{a_1^i, a_2^i, \cdots, a_N^i\}$$
- ✓ Each player i has a payoff function $\pi(a^1, a^2, \cdots, a^n)$
- ✓ $a^{-i} = \{a^1, a^2, \cdots, a^{i-1}, a^{i+1}, \cdots, a_N\}$ denotes other players action

Definition: Dominant Actions

An action a^{i*} is a dominant action for i , if, no matter what all other players are playing, playing a^{i*} maximizes the payoff.

$$\Longrightarrow \pi^i(a^{i*}, a^{-i}) \geq \pi^i(a^i, a^{-i})$$

- ✓ Dominant actions may not always exist

Definition: Best Response

In a N players game, i's best response is
$$R^i(a^{-i}) = \arg\max \pi^i(a^i, a^{-i})$$

Nash Equilibrium (NE)

Each player's predicted action must be that player's best response to the predicted actions of the other players. Such a prediction can be called strategically stable or self-enforcing, because no single player wants to deviate from his or her predicted action.

- ✓ $a^* = (a^{1*}, \cdots, a^{N*})$ is a **NE**. If, for each i, a^{i*} is i's best response to other player's action
$a^{-i*} = (a^{1*}, \cdots, a^{i-1*}, a^{i+1*}, \cdots, a^{N*})$
$$\pi^i(a^{i*}, a^{-i*}) \geq \pi^i(a^i, a^{-i*})$$

Applications of Static Games: Duopoly Models

- ✓ 2 firms sell a homogeneous good.
- ✓ Marginal cost of producing each unit of the good: c_1, c_2
- ✓ Market price is determined by (inverse) market demand

$$P = \begin{cases} a - Q & \text{, if } 0 \leq a \leq Q \\ 0 & \text{, if } a > Q \end{cases}$$

- ✓ Cournot Model: Firms set quantities simultaneously.
- ✓ Bertrand Model: Firms set prices simultaneously.

Case of Duopoly

(1) Pepsi vs. Coca-Cola in Beverages Markets.
(2) Airbus vs. Boeing in Commercial Aircraft Markets.
(3) CPC and Formosa in petroleum market in Taiwan.

Cournot Competition: Firm's Best Response

Suppose Firms 2 purchase q_2, Firm1's payoff:
$$\pi_1 = (P - c_1)q_1 = [a - (q_1 + q_2) - c_1] \cdot q_1$$

(1) F.O.C. (First Order Condition)

$$\frac{\partial \pi_1}{\partial q_1} = a - 2q_1 - q_2 - c_1 = 0$$

$$\Longrightarrow q_1 = \frac{a - c_1 - q_2}{2} = R^1(q_2)$$

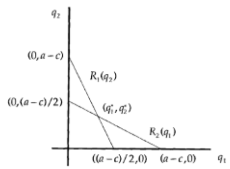
Firm1's best response to Firm2's action

(2) S.O.C. (Second Order Condition)

$$\frac{\partial^2 \pi_1}{\partial {q_1}^2} = -2 < 0$$

Firm1's payoff function is a concave function

Graph Solution of Cournot Competition



- ✓ Delete the strategies that are never a best response, and then delete strategies that are never best response to anything that is a best response and so on and so forth.

Cournot Equilibrium

$$\begin{cases} q_1 = \frac{a - c_1 - q_2}{2} \\ q_2 = \frac{a - c_2 - q_1}{2} \end{cases} \Longrightarrow \begin{cases} {q_1}^c = \frac{a - 2c_1 + c_2}{3} \\ {q_2}^c = \frac{a - 2c_2 + c_1}{3} \end{cases}$$

Then we can get $Q^c = {q_1}^c + {q_2}^c = \frac{2a - c_1 - c_2}{3}$

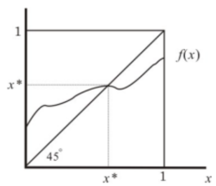
If $c_1 = c_2 = c$, $Q^c = {q_1}^c + {q_2}^c = \frac{2(a-c)}{3}$

- ✓ **Total Market Quantity in symmetric case.**

Perfect Comp.	Cournot Comp.	Monopoly
$(a-c) > \frac{2(a-c)}{3} > \frac{(a-c)}{2}$		

Cournot Equilibrium: The Fixed Point Theorem

- ✓ Nash Equilibrium solutions are the fixed points of the composite function $R^1(R^2(\cdots))$ and $R^2(R^1(\cdots))$
- ✓ The Fixed Point Theorem: Suppose $f(x)$ is a continuous function with domain $[0, 1]$ and range $[0, 1]$. Then, The fixed Point Theorem guarantees that there exists at least on fixed point – $x^* \in [0, 1]$ such that $f(x^*) = x^*$



Cournot Equilibrium with N Firms

Prices $P = a - Q = a - \sum_{i=1}^N q_i$, Firm i's payoff:
$$\pi_i(q_i, q_{j \neq i}) = [a - (q_i + \textstyle\sum_{i \neq j} q_j) - c_i] \cdot q_i$$

(1) F.O.C. (First Order Condition)

$$\frac{\partial \pi_i}{\partial q_i} = a - 2q_i - \textstyle\sum_{i \neq j} - c_i = a - q_i - Q - c_i = 0$$

Note that $Q = \sum_{i=1}^N q_i = (q_i + \sum_{j \neq i} q_j)$

(2) Sum over N

$$Na - \textstyle\sum_{i=1}^N q_i - N \cdot Q - \sum_{i=1}^N c_i$$
$$= Na - Q - NQ - \textstyle\sum_{i=1}^N c_i = 0$$
$$\Longrightarrow Q^c = \frac{N \cdot a - \sum c_i}{N+1}, \ P^c = a - Q = \frac{a + \sum c_i}{N+1}$$

(3) Find q_i^c

Consider F.O.C for symmetric case: $c_i = c$

$$\Longrightarrow {q_i}^c = \frac{Q^c}{N} = \frac{a-c}{N+1}, \ P^c = a - Q = \frac{a + NC}{N+1}$$

Bertrand Equilibrium Model

- ✓ Firms set prices rather than quantities. $P = a - Q$
- ✓ Customs buy from the firm with the cheapest price.
- ✓ The market is split evenly if firms offer the same price

$$q_i \begin{cases} 0 & , P_i > P_j \\ a - P_i & , P_i < P_j \\ \frac{a - P_i}{2} & , P_i = P_j \end{cases}$$

- ➢ Firm1 should choose $c_1 \leq P_1 \leq P_2$
- ➢ Firm2 should choose $c_2 \leq P_2 \leq P_1$

Bertrand Equilibrium

(1) Consider a symmetric case: $c_1 = c_2 = c$
$$P = c, \ q_1 = q_2 = \frac{a-c}{2}$$

(2) If $c_1 < c_2, p_2 \geq c_2$
$$P_1 = c_2 - \epsilon \Longrightarrow q_1 = a - (c_2 - \epsilon), q_2 = 0$$

Homogeneous Product

Assume a symmetric case:
$q_1(P_1, P_2) = a - P_1 + bP_2$
$\pi_1(P_1, P_2) = q_1(P_1, P_2) \cdot [P_1 - c]$
F.O.C

$$\frac{\partial \pi_1}{\partial P_1} = 0 \Longrightarrow {P_1}^* = \frac{a + bP_2 + c}{2}$$
$$\frac{\partial \pi_2}{\partial P_2} = 0 \Longrightarrow {P_2}^* = \frac{a + bP_1 + c}{2}$$

Then ${P_1}^* = {P_2}^* = \frac{a+c}{2-b}$

Cournot Behavior In General Function

By Chain Rule

$$q_1 = \frac{(c' - P)}{P'}, \ \frac{dq_1}{dq_2} = \frac{(p - c')P'' - (P')^2}{2(P')^2 - c''P' - (P - c')P''}$$

Mixed-Strategy

A randomization over your pure strategies

NE in Mixed-Strategy

In the two-player normal-form game, the mixed strategy $({P_1}^*, {P_2}^*)$ is a Nash Equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy.

Homework 01-02

Consider the following game:

	Z	W
X	5, 5	-8, 8
Y	-7, -8	0, 0

(1) What is the Nash equilibrium?
A Nash Equilibrium in a game is a list of strategies, one for each player, such that no player can get a better payoff by switching to some other strategy that is available to her while all other players adhere to the strategies specified for them in the list

(2) Justify your answer in (1).
考慮 Player 1 選策略 X 時，Player 2 將選擇策略 W；
考慮 Player 1 選策略 Y 時，Player 2 亦選擇策略 W；
因此考慮當 Player 2 選擇優勢策略 W 時，Player 1 將選擇 Y，此時 (Y,W) 為 Nash Equilibrium。

(3) Change only one of the eight entries in the table such that there is no equilibrium.
只要使其策略組合中 Player 2 的報酬改為非零，即沒有均衡。如：改變 Player 2 中 $(Y,W)<-8$。

Homework 01-03

(1) Find the fixed point of function $f(x)=x^2-3x+4$
By **Brouwer Fixed Point Theorem**, Let

$$f(x)=x^2-3x+4=x$$

$$\Longrightarrow x^2-4x+4=(x-2)^2=0$$

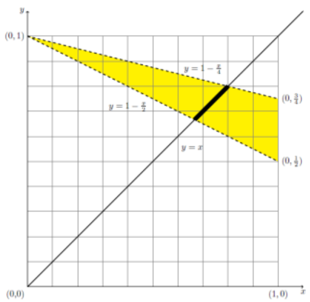
$$\Longrightarrow x=2$$

Then we can get the fixed point $(x,y)=(2,2)$

(2) Does the function, $f(x)=x+1$, have a fixed point? Why or why not?
It has no fixed point because that the lines $y=x$ is parallel to the line $y=x+1$.

(3) For each $x\in\mathbb{R}$ define $F(x)=(x,\infty)=\{y\in\mathbb{R}:y>x\}$. Then $F:\mathbb{R}\to\mathbb{R}$ is a correspondence. What is the correspondence $F(2)$?
$\forall x\in\mathbb{R},\ F(x)=(x,\infty)=\{y\in\mathbb{R}:y>x\}$
Then $F(2)=(2,\infty)$

(4) Let C be a correspondence defined on the closed interval $[0,1]$ that maps a point x to the closed interval $[1-x/2,1-x/4]$. Draw all fixed points on the graph.
As the figure, we can find all the fixed points on the thick segment (intersection by the line and the shadow area).



Homework 02-01

Consider the following bargaining game. Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have – s_1 and s_2, where $0\le s_1,s_2\le 1$, if $s_1+s_2\le 1$, then the players receive the shares they named; if $s_1+s_2>1$, then both players receive zero. What are the pure strategy Nash equilibria of this game?
當 $s_1+s_2\le 1$ 時，雙方各得 s_1 和 s_2
當 $s_1+s_2>1$ 時，雙方皆沒有得到任何收益

在不失一般性的狀況下，作以下討論：
給定任意的 $s_2\in[0,1)$ 則對於另一位決策者所得之最佳回應為 $R_1(s_2)=1-s_2$；對於 $s_2=1$ 時，另一位決策者所得之最佳回應為 $[0,1]$（無論如何選擇，報酬皆為零）

$$s_1=R_1(s_2)=\begin{cases}1-s_2 & \text{, if } 0\le s_2<1\\ [0,1] & \text{, if } s_2=1\end{cases}$$

$$s_2=R_2(s_1)=\begin{cases}1-s_1 & \text{, if } 0\le s_1<1\\ [0,1] & \text{, if } s_1=1\end{cases}$$

取其交集可得 $\{(s_1,s_2):s_1+s_2=1,S_1\ge 0,S_2\ge 0\}$ 和 $(1,1)$，上述策略即為 Nash Equilibrium 之優勢策略。

Homework 02-02

Consider the Cournot Model we discussed in class:
- Two competing firms, selling a homogeneous good.
- Marginal cost of producing each unit of the good is c.
- The market price, P is determined by (inverse) market demand: $P=a-Q$ if $a>Q$; $P=0$ otherwise.
- Each firm decides on the quantity to sell (market share): q_1 and q_2.
- $Q=q_1+q_2$ is the total market demand.
- Both firms seek to maximize profits.

(1) Solve for the equilibrium quantity $q_1{}^*$ and $q_2{}^*$
設 Firm1 和 Firm2 之利潤和銷售量分別為 π_1,π_2 和 q_1,q_2 且 $c=c_1=c_2$ 為 Firm1 和 Firm2 之邊際單位成本，可知：

$$\pi_1=\mathrm{TR}_1-\mathrm{TC}_1=[a-(q_1+q_2)-c_1]\cdot q_1$$
$$\pi_2=\mathrm{TR}_2-\mathrm{TC}_2=[a-(q_1+q_2)-c_2]\cdot q_2$$

F.O.C

$$\begin{cases}\frac{\partial\pi_1}{\partial q_1} & =(a-2q_1-q_2-c)=0\\ \frac{\partial\pi_2}{\partial q_2} & =(a-q_1-2q_2-c)=0\end{cases}$$

求解上式聯立方程式可得均衡銷售量

$$q^*=q_1=q_2=\frac{a-c}{3}$$

S.O.C

取其二階偏微分，可得：

$$\begin{cases}\frac{\partial^2\pi_1}{\partial q_1{}^2} & =-2<0\\ \frac{\partial^2\pi_2}{\partial q_2{}^2} & =-2<0\end{cases}$$

(2) Please verify your solution in (a) by showing that the statement "In the equilibrium, no one can be better-off by a unilateral change in its solution" is satisfied.
均衡情況下，沒有任何一個決策者會因單一改變選擇而得到更高的收益。由 S.O.C 可知兩者之收益函數皆凹口向下，因此若有其中一家廠商增加產量時而另一家產量固定，會使兩者收益皆為零（由題目可知當 $a>Q$ 時 $P=0$），反而會造成整體收益減少。

Homework 02-03

Following Question 2, suppose that each firm produces the half of monopoly quantity q_m, i.e., $q_1=q_2=\frac{q_m}{2}$

(1) Solve for the monopoly quantity q_m.
若為獨佔(monopoly)廠商，則有：

$$\mathrm{TR}=P\cdot Q=(a-q_m)\cdot q_m$$

$$\mathrm{MR}=\mathrm{MC}=c=\frac{\partial(\mathrm{TR})}{\partial q_m}=a-2q_m$$

由上式可得 $q_m=\frac{a-c}{2}$，故 $q_1=q_2=\frac{q_m}{2}=\frac{a-c}{4}$

(2) Please compare each firm's profit in Question 3 with the solution you obtained in Question 2.

$\boxed{\text{[Case 2(a)]}\ q_1=q_2=\frac{a-c}{3}}$

$$\pi_1=\pi_2=\mathrm{TR}-\mathrm{TC}$$
$$=\left[a-\left(\tfrac{a-c}{3}+\tfrac{a-c}{3}\right)-c\right]\cdot\tfrac{a-c}{3}=\tfrac{(a-c)^2}{9}$$

$\boxed{\text{[Case 3(a)]}\ q_1=q_2=\frac{a-c}{4}}$

$$\pi_1=\pi_2=\mathrm{TR}-\mathrm{TC}$$
$$=\left[a-\left(\tfrac{a-c}{4}+\tfrac{a-c}{4}\right)-c\right]\cdot\tfrac{a-c}{4}=\tfrac{(a-c)^2}{8}$$

(3) Show that $q_1=q_2={}^{q_m}/_2$ is not an equilibrium solution.

I. 獨佔產量 $q_m={}^{(a-c)}/_2$ 嚴格優於其他更高產量（$\forall x>0$ 且 $\forall q_j\ge 0$）
若 $Q=q_m+x+q_j<a$ 有

$$\pi_i(q_m,q_j)=\tfrac{a-c}{2}\left[\tfrac{a-c}{2}-q_i\right]$$

$$\pi_i(q_m+x,q_j)=\left[\tfrac{a-c}{2}+x\right]\left[\tfrac{a-c}{2}-x-q_i\right]$$

若 $Q=q_m+x+q_j\ge a$ 則有 $P(Q)=0$，生產較低的產出就會提高利潤。

II. 剔除大於獨佔產量 q_m 後，產量 $q_m={}^{(a-c)}/_4$ 嚴格優於其他更低產量

$$\pi_i\left(\tfrac{a-c}{4},q_j\right)=\tfrac{a-c}{4}\left[\tfrac{3(a-c)}{4}-q_i\right]$$

$$\pi_i\left(\tfrac{a-c}{4}-x,q_j\right)=\left[\tfrac{a-c}{4}-x\right]\left[\tfrac{3(a-c)}{4}+x-q_i\right]$$

III. 反覆進行剔除嚴格優勢策略
如上所述，經剔除後，各個企業選擇銷售量的策略空間會逐漸縮小。反覆進行上述操作，可以使得其空間限制越來越小，最終得到均衡銷售量：

$$q_i^*=\tfrac{a-c}{3}$$

Homework 02-04

In an industry there are N firms producing a homogeneous product. Let q_i denote the output level of N firm i, $i=1,2,\cdots,N$, and let Q denote the aggregate industry production level. That is, $Q=\sum_{i=1}^{N}q_i$. Assume that the demand curve facing the industry is $P=100-Q$. Suppose that the cost function of each firm i is given by

$$\mathrm{TC}_i(q_i)=\begin{cases}F+q_i{}^2 & \text{, if } q_i>0\\ 0 & \text{, if } q_i=0\end{cases}$$

Suppose that the number of firms in the industry N is sufficiently small so that all the N firms make above-normal profits. Calculate the output and profit levels of each firm in a Cournot equilibrium. (Hint: you can assume that all firms have identical cost functions.)

令 $\pi_i(q_i,q_{-i})=\left[a-(q_i+\sum_{j\neq i}q_j)\right]\cdot q_i-(F+q_i{}^2)$

F.O.C

$$\frac{\partial\pi_i}{\partial q_i}=a-4q_i-\textstyle\sum_{j\neq i}q_j=0,\forall i=1,2\cdots N$$

又已知 $Q=\sum_{i=1}^{N}q_i=q_i+\sum_{j\neq i}q_j$ 代入整理

可得：

$$a-3q_i-Q=0,\forall i=1,2\cdots N$$

$$\Longrightarrow Na-3\textstyle\sum q_i-NQ=0$$

由上式可得 $Q=\frac{N}{N+3}a$ 和 $P=a-Q=\frac{3}{N+3}a$

故 $q_i=\frac{Q}{N}=\frac{1}{N+3}a,\forall i=1,2\cdots N$

再代回可得利潤

$$\begin{aligned}\pi_i&=\tfrac{3}{N+3}a\cdot\tfrac{1}{N+3}a-\left[F+\left(\tfrac{1}{N+3}a\right)^2\right]\\&=\tfrac{3a^2}{(N+3)^2}-\left[F+\left(\tfrac{1}{N+3}a\right)^2\right]\\&=\tfrac{3\cdot 100^2}{(N+3)^2}-\left[F+\left(\tfrac{100}{N+3}\right)^2\right]\\&=\tfrac{20000}{(N+3)^2}-F\end{aligned}$$