**整數與組合最佳化**(Integer and Combinatorial Optimization)
Homework 05 – 2017/06/01

R05546030 **彭新翔**

1. Consider the problem:

$$\begin{array}{lrcrcl} \max & z & = & x_1 & + & 2x_2 \\ \text{s.t.} & 3x_1 & + & x_2 & \geq & 6 \\ & -x_1 & + & x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 8 \\ & & & x_1, x_2 \geq 0 & & \end{array}$$

Assume that the first constraint $(3x_1 + x_2 \geq 6)$ is relaxed.

(a) Formulate the Lagrangian dual problem.

(b) Show that $z_R(\lambda) = 6\lambda + \min\{0, 4-2\lambda, 13-14\lambda, 8-24\lambda\}$

(c) Plot $z_R(\lambda)$ for each value of $\lambda$.

(d) For part (c) locate the optimal solution of the Lagrangian dual problem

(e) For part (d) find the optimal solution to the primal problem. 2.

(a)

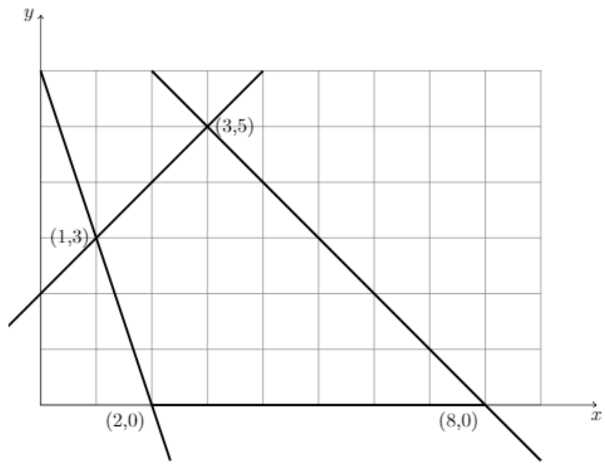
**LR**

$$\begin{aligned} \max \quad & z_{LR}(\lambda) = x_1 + 2x_2 + \lambda(6 - 3x_1 - x_2) = 6\lambda + (-3\lambda+1)x_1 + (-\lambda+2)x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq 2 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

**LD**

$$\begin{aligned} \min \quad & z_{LD}(\lambda) = (3\lambda - 1)x_1 + (\lambda - 2)x_2 - 6\lambda \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

(b)



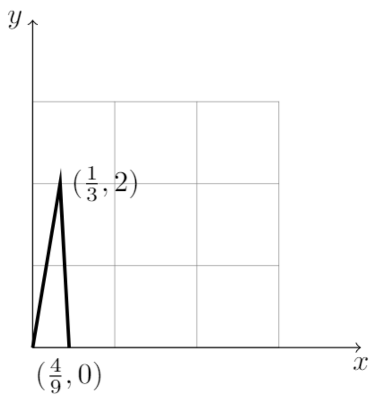

$$\text{conv}(Q) = \{x \in \mathbb{R}^{2+} : -x_1 + x_2 \leq 2, x_1 + x_2 \leq 8\}$$

如圖所示，將 $(x^1, x^2, x^3, x^4)$ 四點代入可得：

$$z_{LR}(\lambda, x^1) = 6\lambda + 0$$
$$z_{LR}(\lambda, x^2) = 6\lambda + 4 - 2\lambda$$
$$z_{LR}(\lambda, x^3) = 6\lambda + 13 - 14\lambda$$
$$z_{LR}(\lambda, x^4) = 6\lambda + 8 - 24\lambda$$

故可知 $z_{LR} = \min z = 6\lambda + \min\{0, 4-2\lambda, 13-14\lambda, 8-24\lambda\}$

(c)




(d) 因為 $z_{LD} = \max z_{LR}(\lambda)$ 故知拉格朗日對偶問題的極值為：

$$z_{LD} = z_{LR}(\tfrac{1}{3}) = 2$$

(e) 由於上述四個極值點皆為整數，因此 $z_{LP} = z_{LD} = 2$ 故可知其極值為：

$$z = 2, x_1 = 2, x_2 = 0$$

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2. Consider two different Lagrangian duals for the generalized assignment problem.

$$\begin{array}{lcccccccccc}
\max & Z & = & x_1 & + & x_2 & + & y_1 & & & \\
\text{s.t.} & x_1 & + & x_2 & + & y_1 & - & z & = & 0 \\
 & 2x_1 & + & x_2 & + & y_1 & \le & 6 & & \\
 & x_1 & + & 2x_2 & + & y_1 & \le & 6 & & \\
 & x_1 & + & x_2 & + & 2y_1 & \le & 6 & & \\
 & x_1 & + & x_2 & + & y_1 & \le & 4 & & \\
 & y_1 & \ge & 0 & & & & & & \\
 & & x_1, x_2 & & \ge & & 0 & , \text{integer.} & &
\end{array}$$

Write out these two Lagrangian relaxation problems and discuss their relative merits according to the following three criteria:

(a) Ease of solution of the Lagrangian subproblem.

(b) Ease of solution of the Lagrangian Dual.

(c) Strength of the upper bound obtained by solving the Lagrangian Dual.

$$\begin{array}{ll}
\max & w^1(u) = \max \sum_{j=1}^{n} \sum_{i=1}^{m} (c_{ij} - u_i) x_{ij} + \sum_{i=1}^{m} u_i \\
\text{s.t.} & \sum_{i=1}^{m} a_{ij} x_{ij} \le b_j, \; j = 1, \cdots, n \\
 & x_{ij} \in \{0, 1\}
\end{array}$$

$$\begin{array}{ll}
\max & w^2(v) = \max \sum_{j=1}^{n} \sum_{i=1}^{m} (c_{ij} - a_{ij} v_j) x_{ij} + \sum_{j=1}^{n} v_j b_j \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} \le 1, \; i = 1, \cdots, m \\
 & x_{ij} \in \{0, 1\}
\end{array}$$

(a) $w^1$ can be considered as an $0-1$ Knapspack Problem.
$w^2$ can be calculated by inspection
For example, $x_{ij} = 1$ only when $(c_{ij} - a_{ij} v_j) > 0$

(b) $w_{LD}^1 = \min_{u \ge 0} w^1(u)$, $w_{LD}^2 = \min_{v \ge 0} w^2(v)$

The number of dual variable $(u, v)$ is one of measure of the difficulty of LD.
For example, if $m > n$, $w_{LD}^2$ may be easier.

(c) $w_{LD}^2 = Z_{LP}$ because of integral extreme points of $w^2(v)$.
Thus, $w_{LD}^1 \le w_{LD}^2 = Z_{LP}$

3. Consider the uncapacitated location problem.

- $x_j = 1$, if a facility (factory) is placed at $j (j = 1, \cdots, n)$.
- $y_{ij}$ is the fraction of the demand of client $i$ (store) $(i = 1, \cdots, m)$ satisfied from facility $j$.
- $f_j$ is the operating cost of facility $j$.
- $c_{ij}$ is the revenue of fulfilling demand of client $i$ from facility $j$.

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} \eta_i - \sum_{j=1}^{n} f_j x_j \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} y_{ij} = 1, \text{ for } i = 1, \cdots, m \\ & y_{ij} - x_j \leq 0, \text{ for } i = 1, \cdots, m \text{ ; } j = 1, \cdots, n \\ & x_j = 0 \text{ or } 1 \text{ , } y_{ij} \geq 0 \end{aligned}$$

(a) Please explain how the following reformulation is obtained by Bender decomposition.

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^{m} \eta_i - \sum_{j=1}^{n} f_j x_j \\ \text{s.t.} \quad & \eta_i \leq c_{ik} + \textstyle\sum_{j=1}^{n} (c_{ij} - c_{ik})^{+} x_j \text{ ,for } k = 1 \cdots, n \text{ ; } i = 1, \cdots, m \\ & \textstyle\sum_{j=1}^{n} x_j \geq 1 \\ & x_j = 0 \text{ or } 1 \text{ , } \eta_i \text{ unrestricted} \end{aligned}$$

(b) Given the following parameters, solve the reformulation by using the constraint generation algorithm. There are $6$ clients and $5$ possible locations for facility. The operation costs for each $j(f_j)$ are $4$, $3$, $4$, $4$, and $7$. The matrix of revenue of each $(c_{ij})$ is

$$\begin{bmatrix} 12 & 13 & 6 & 0 & 1 \\ 8 & 4 & 9 & 1 & 2 \\ 2 & 6 & 6 & 0 & 1 \\ 3 & 5 & 2 & 10 & 8 \\ 8 & 0 & 5 & 10 & 8 \\ 2 & 0 & 3 & 4 & 1 \end{bmatrix}$$

(a)

$$\begin{aligned} \max \quad & z^* = \max\{c^T x + \min(u^k)^T (b - Ax), k \in k\} \\ \text{s.t.} \quad & (v^j)^T (b - Ax) \geq 0 \text{ , } \forall j \in J \end{aligned}$$

($m$ 個需求點，n 個服務設施)

(MILP)

$$\begin{aligned} \max \quad & z^* = \eta \\ \text{s.t.} \quad & \eta \leq c^T x + (u^k)^T (b - Ax) \text{ , } \forall k \in k \\ & (v^j)^T (b - Ax) \geq 0 \text{ , } \forall j \in J \\ & x \geq 0 \text{ , integer , } \eta \text{ unstricted} \end{aligned}$$

經由 Bender's Decomposition 可知，以一連續變數取代原題之決策變數，但會增加限制式數量。求解過程中，只考慮與最佳解有關之限制式。

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(b)

<u>Iteration 01</u>

(1) $\eta_1^1 = 13, \eta_2^1 = 9, \eta_3^1 = 6, \eta_4^1 = 10, \eta_5^1 = 10, \eta_6^1 = 4,\ x^1 = (0,1,0,0,0),\ z^1 = 49$

(2) Separation for each client $i$. $cx^1 + z_{LP}(x^1) = 25$:

$i = 2 : \eta_2 \leq 1 + 7x_1 + 3x_2 + 8x_3 + x_5$ is violated

$i = 4 : \eta_4 \leq 2 + x_1 + 3x_2 + 8x_4 + 6x_5$ is violated

$i = 5 : \eta_5 \leq 0 + 8x_1 + 5x_3 + 10x_4 + 8x_5$ is violated

$i = 6 : \eta_6 \leq 0 + 2x_1 + 3x_3 + 4x_4 + x_5$ is violated

<u>Iteration 02</u>

(1) $\eta_1^2 = 13, \eta_2^2 = 9, \eta_3^2 = 6, \eta_4^2 = 10, \eta_5^2 = 10, \eta_6^2 = 4,\ x^2 = (0,0,1,1,0),\ z^2 = 44$

(2) Separation for each client $i$. $cx^2 + z_{LP}(x^2) = 37$.

$i = 1 : \eta_1 \leq 0 + 12x_1 + 13x_2 + 6x_3 + x_5$ is violated

<u>Iteration 03</u>

(1) $\eta_1^3 = 13, \eta_2^3 = 9, \eta_3^3 = 6, \eta_4^3 = 10, \eta_5^3 = 10, \eta_6^3 = 4,\ x^3 = (0,1,1,1,0),\ z^3 = 41$

(2) Separation for each client $i$. $cx^3 + z_{LP}(x^3) = 37$.

Since the upper bound and lower bounds are equal, the solution $x^3$ is optimal.