V. Wigner Distribution Function

V-A Wigner Distribution Function (WDF)

Definition 1: $$W_x(t, f) = \int_{-\infty}^{\infty} x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j2\pi\tau f} d\tau$$

Definition 2: $$W_x(t, \omega) = \int_{-\infty}^{\infty} x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j\omega\tau} d\tau$$

Another way for computation from the frequency domain

Definition 1: $$W_x(t,f) = \int_{-\infty}^{\infty} X\left(f+\eta/2\right) \cdot X^*\left(f-\eta/2\right) e^{j2\pi\eta t} d\eta$$

where $X(f)$ is the Fourier transform of $x(t)$

Definition 2: $$W_x(t,\omega) = \int_{-\infty}^{\infty} X\left(\omega+\eta/2\right) \cdot X^*\left(\omega-\eta/2\right) e^{j\eta t} d\eta$$

Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 5, Prentice Hall, N.J., 1996.

Other References

[Ref] E. P. Wigner, "On the quantum correlation for thermodynamic equilibrium," *Phys. Rev.*, vol. 40, pp. 749-759, 1932.

[Ref] T. A. C. M. Classen and W. F. G. Mecklenbrauker, "The Wigner distribution–A tool for time-frequency signal analysis; Part I," *Philips J. Res.*, vol. 35, pp. 217-250, 1980.

[Ref] F. Hlawatsch and G. F. Boudreaux–Bartels, "Linear and quadratic time-frequency signal representation," *IEEE Signal Processing Magazine*, pp. 21-67, Apr. 1992.

[Ref] R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley-Interscience, NJ, 2004.

The operators that are related to the WDF:

(a) Signal auto-correlation function:

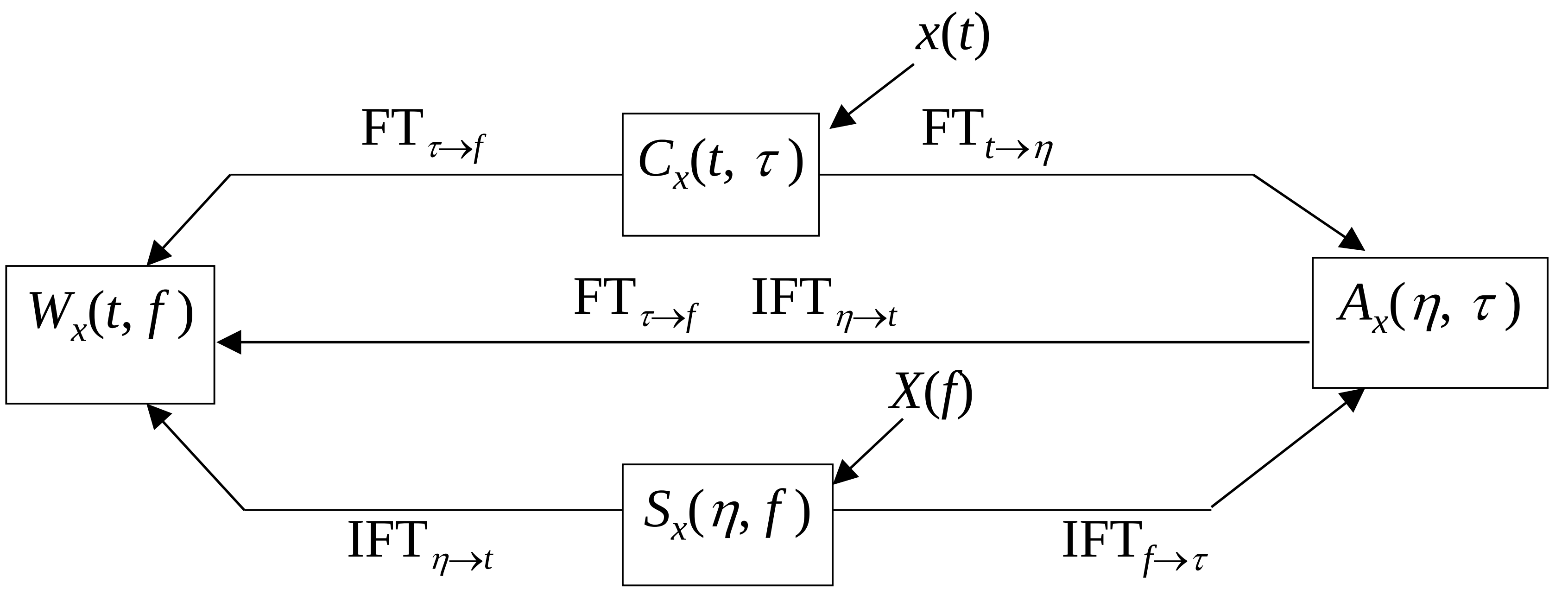
$$C_x(t,\tau) = x(t+\tau/2)\cdot x^*(t-\tau/2)$$

(b) Spectrum auto-correlation function:

$$S_x(\eta,f) = X(f+\eta/2)\cdot X^*(f-\eta/2)$$

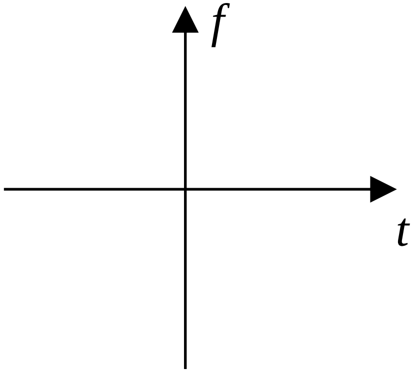
(c) Ambiguity function (AF):

$$A_x(\eta,\tau) = \int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt$$



V-B Why the WDF Has Higher Clarity?

Due to signal auto-correlation function



If $x(t) = 1$

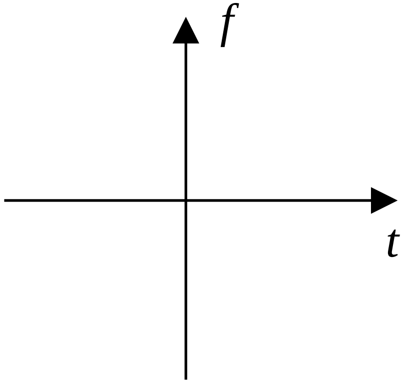
If $x(t) = \exp(j2\pi h\, t)$

$$W_x\left(t, f\right) = \int_{-\infty}^{\infty} e^{j2\pi h(t+\tau/2)} e^{-j2\pi h(t-\tau/2)} \cdot e^{-j2\pi\tau f} d\tau$$

$$= \int_{-\infty}^{\infty} e^{j2\pi h\tau} \cdot e^{-j2\pi\tau f} d\tau$$

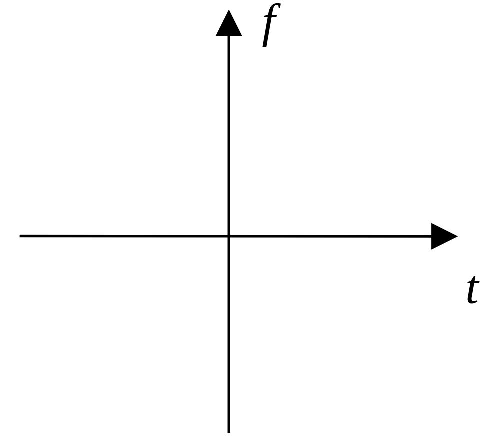
$$= \int_{-\infty}^{\infty} e^{-j2\pi\tau(f-h)} d\tau$$

$$= \delta(f-h)$$



Comparing: for the case of the STFT

If $x(t) = \exp(j2\pi k\, t^2)$



If $x(t) = \delta(t)$

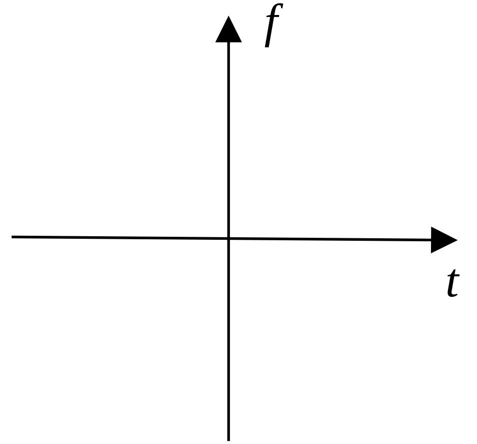
$$W_x(t, f) = \int_{-\infty}^{\infty} \delta\left(t + \tau/2\right) \cdot \delta\left(t - \tau/2\right) e^{-j2\pi\tau f} d\tau$$

$$= 4\int_{-\infty}^{\infty} \delta\left(2t + \tau\right) \cdot \delta\left(2t - \tau\right) e^{-j2\pi\tau f} d\tau$$

$$= 4\delta\left(4t\right) e^{j4\pi t f} = \delta\left(t\right) e^{j4\pi t f} = \delta\left(t\right)$$

Page 118 公式(2)

Page 118 公式(5), $t_0 = 0$



V-C The WDF is not a Linear Distribution

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

If $h(t) = \alpha\, g(t) + \beta\, s(t)$

$$W_h(t,f)=\int_{-\infty}^{\infty} h(t+\tau/2)\cdot h^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$$=\int_{-\infty}^{\infty}\left[\alpha\, g(t+\tau/2)+\beta\, s(t+\tau/2)\right]\left[\alpha^*\, g^*(t-\tau/2)+\beta^*\, s^*(t-\tau/2)\right]e^{-j2\pi\tau f}\, d\tau$$

$$=\int_{-\infty}^{\infty}\Big[|\,\alpha\,|^2\, g(t+\tau/2)g^*(t-\tau/2)+|\,\beta\,|^2\, s(t+\tau/2)s^*(t-\tau/2)$$

$$+\alpha\beta^* g(t+\tau/2)s^*(t-\tau/2)+\alpha^*\beta\, g^*(t-\tau/2)s(t+\tau/2)\Big]e^{-j2\pi\tau f}\, d\tau$$

$$=\ |\,\alpha\,|^2\, W_g(t,f)+|\,\beta\,|^2\, W_s(t,f)$$

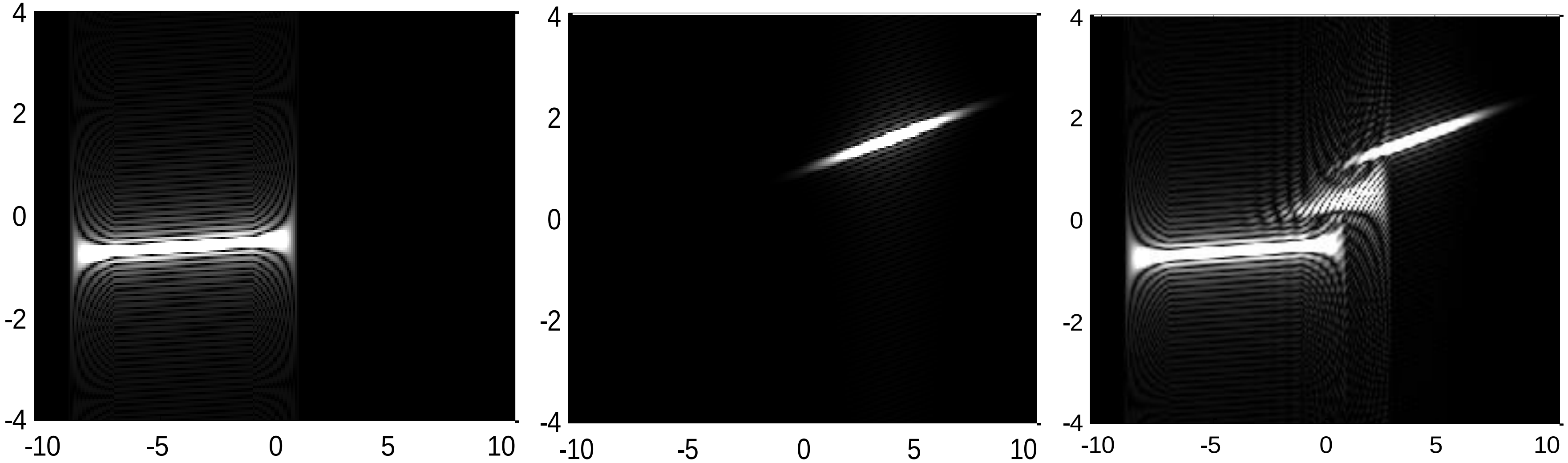
$$+\int_{-\infty}^{\infty}\underbrace{\left[\alpha\beta^* g(t+\tau/2)s^*(t-\tau/2)+\alpha^*\beta\, g^*(t-\tau/2)s(t+\tau/2)\right]}_{\text{cross terms}} e^{-j2\pi\tau f}d\tau$$

V-D Examples of the WDF

$$s(t) = \exp\left(jt^2/10 - j3t\right) \quad \text{for } -9 \le t \le 1,\ s(t) = 0 \text{ otherwise,}$$

$$r(t) = \exp\left(jt^2/2 + j6t\right)\exp\left[-(t-4)^2/10\right]$$

$$f(t) = s(t) + r(t)$$



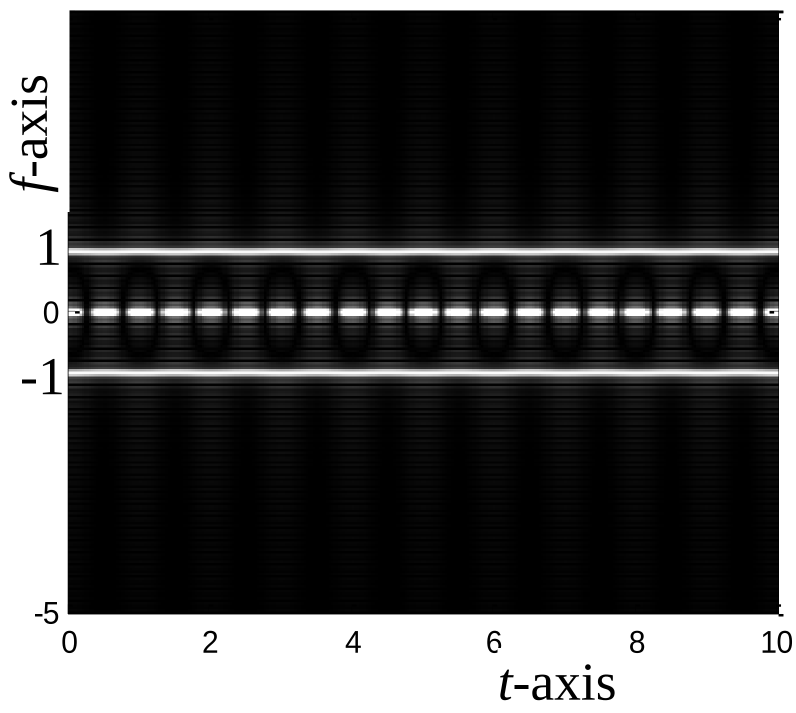
WDF of $s(t)$, WDF of $r(t)$, WDF of $s(t) + r(t)$

橫軸: t-axis, 縱軸: f-axis

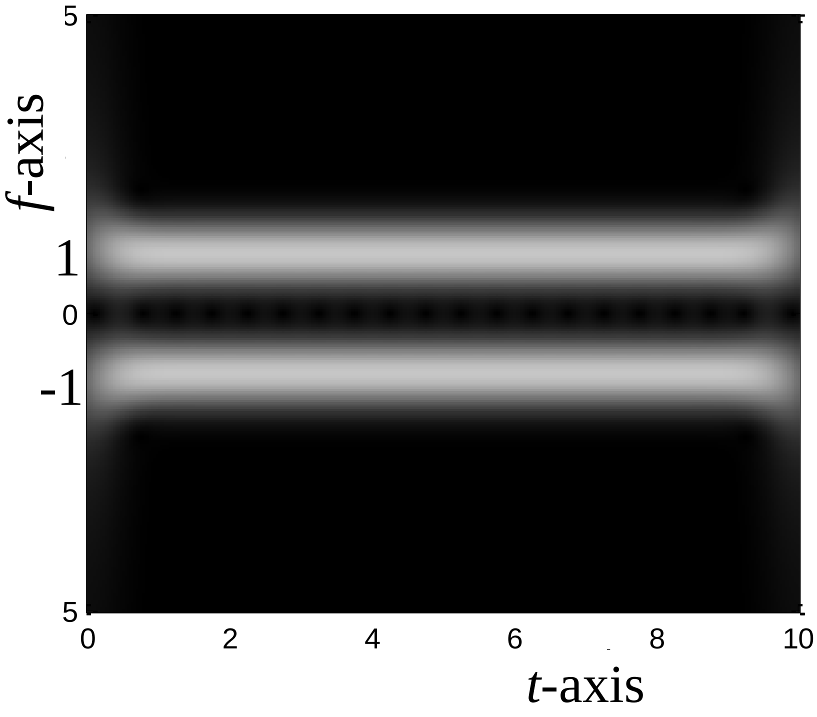
Simulations

$$x(t) = \cos(2\pi t) = 0.5[\exp(j2\pi t) + \exp(-j2\pi t)]$$

by the WDF

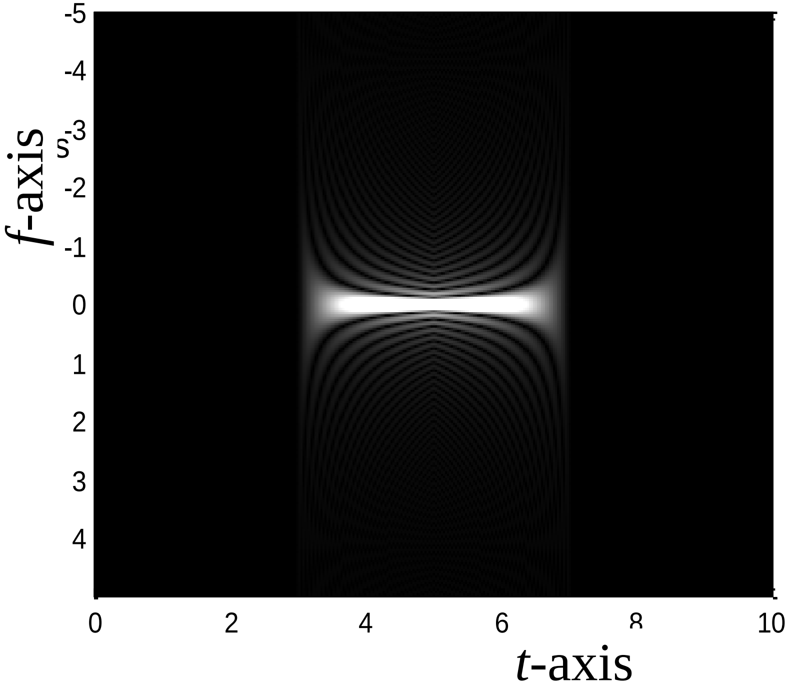


by the Gabor transform

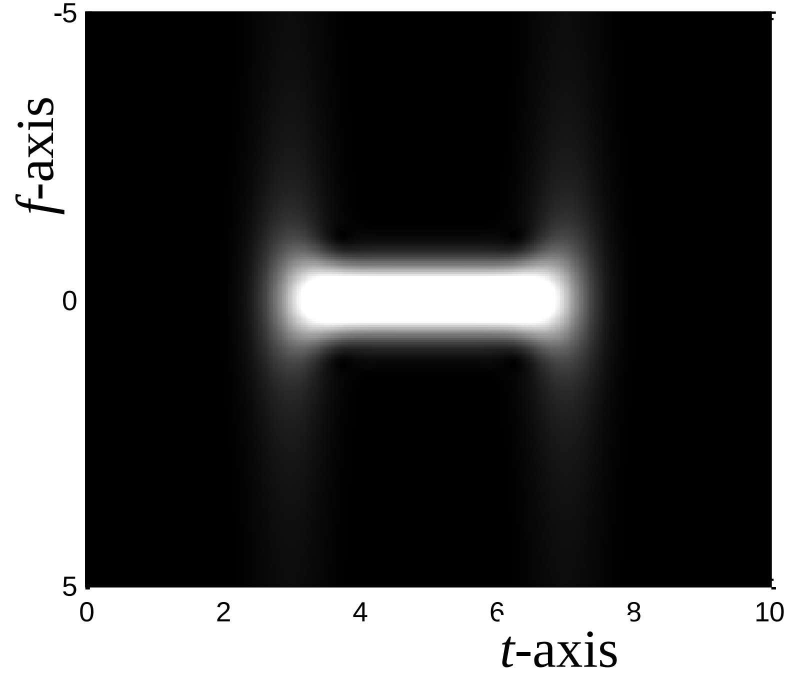


$x(t) = \Pi((t-5)/4)$ $\quad\quad$ Π: rectangular function

by the WDF

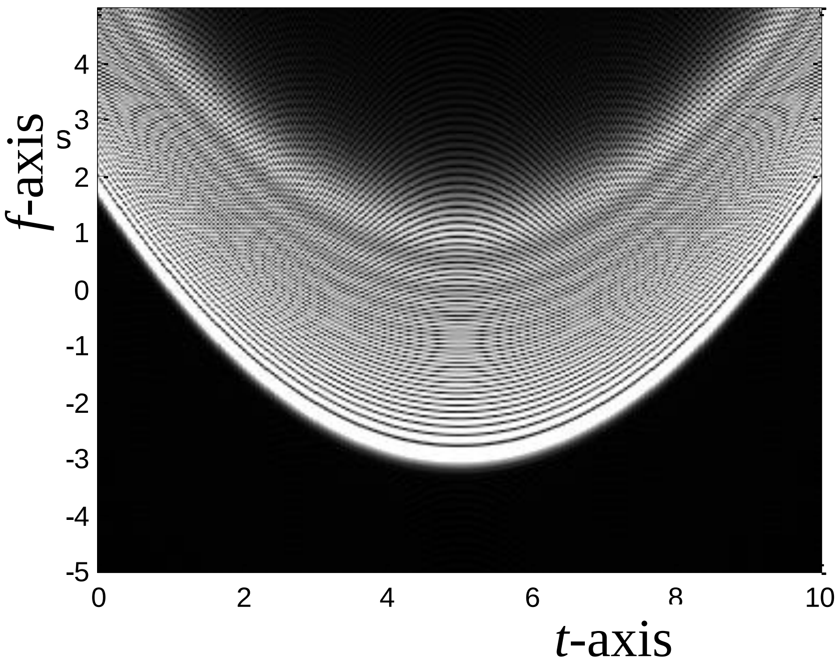


by the Gabor transform

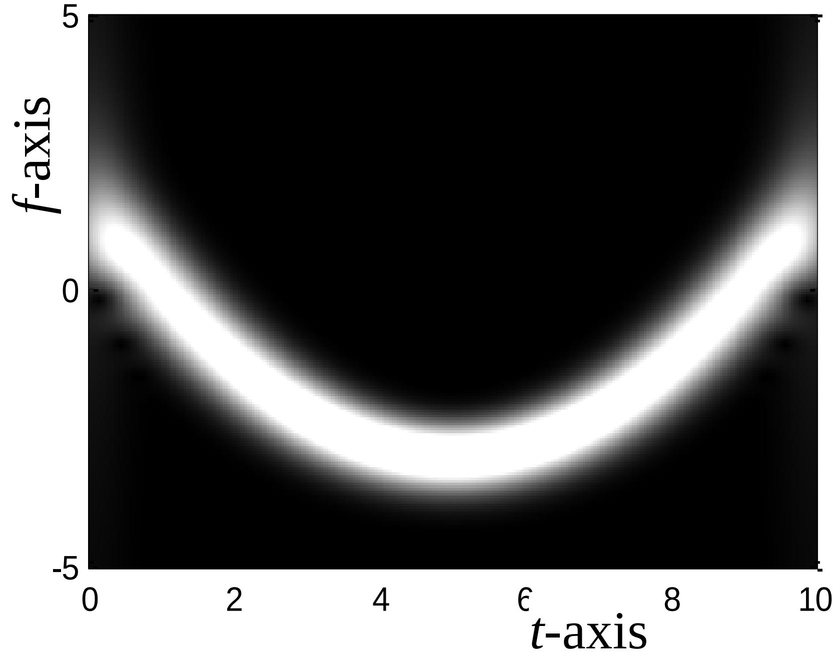


$$x(t) = \exp(j(t-5)^3 - j6\pi t)$$

by the WDF

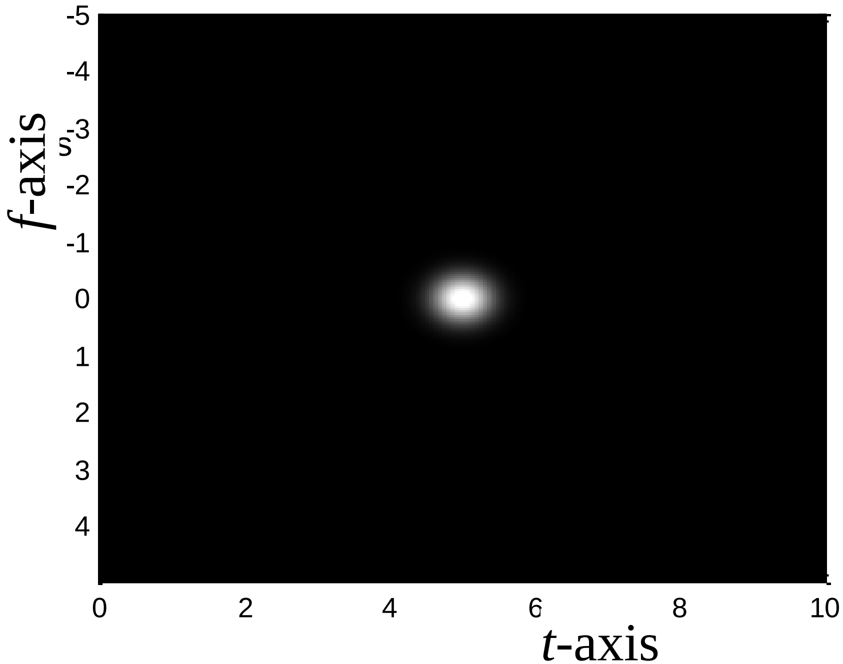


by the Gabor transform

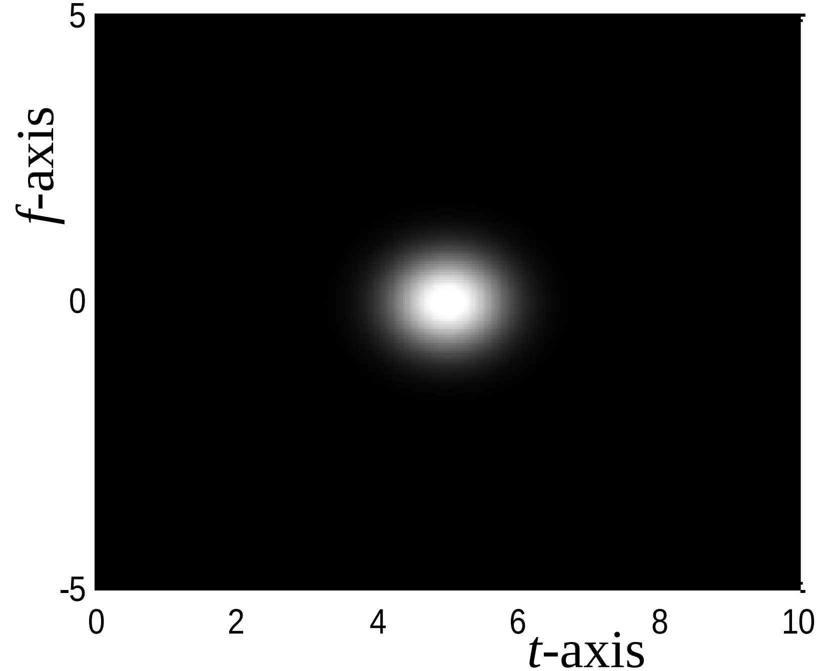


$$x(t) = \exp\left[-\pi(t-5)^2\right]$$

by the WDF



by the Gabor transform



Gaussian function: $e^{-\pi t^2} \xrightarrow{FT} e^{-\pi f^2}$

Gaussian function's T-F area is minimal.

V-E Digital Implementation of the WDF

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j2\pi\tau f} \cdot d\tau \ ,$$

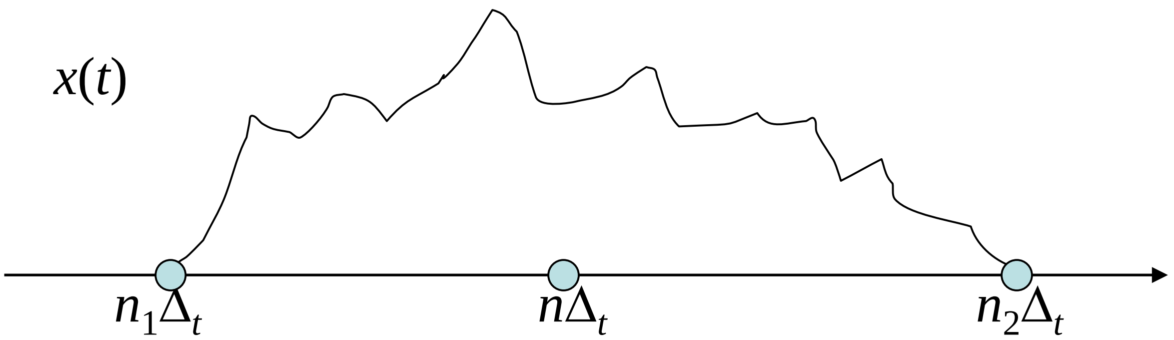
$$W_x(t, f) = 2\int_{-\infty}^{\infty} x(t+\tau') \cdot x^*(t-\tau') e^{-j4\pi\tau' f} \cdot d\tau' \quad \text{(using } \tau' = \tau/2 \text{)}$$

Sampling: $t = n\Delta_t$, $f = m\Delta_f$, $\tau' = p\Delta_t$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\sum_{p=-\infty}^{\infty} x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t\Delta_f)\Delta_t}$$

When $x(t)$ is not a time-limited signal, it is hard to implement.

Suppose that $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$



$$x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right) = 0$$

if $n + p \notin [n_1, n_2]$

or $n - p \notin [n_1, n_2]$

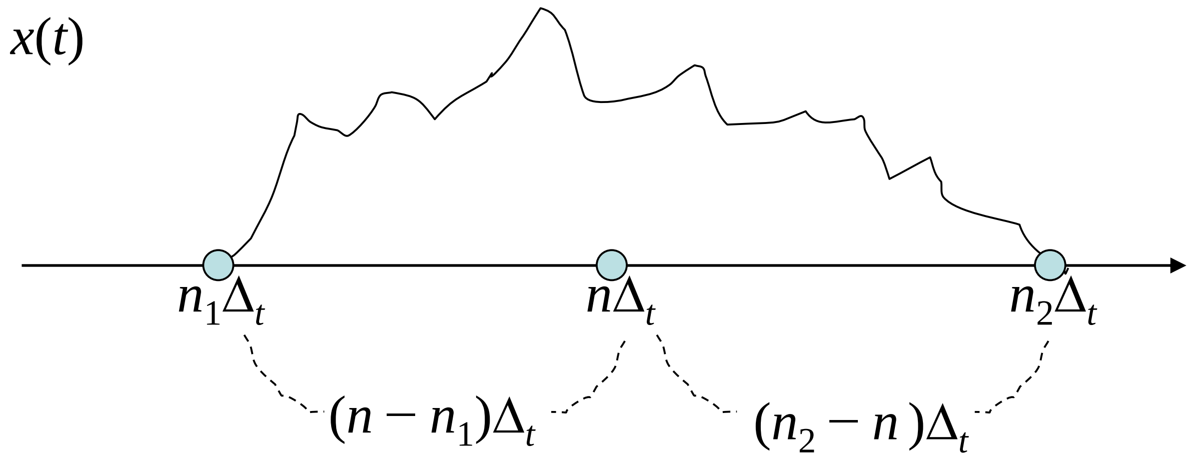
- **p 的範圍的問題 (當 n 固定時)**

$n_1 \le n + p \le n_2 \longrightarrow n_1 - n \le p \le n_2 - n$

$n_1 \le n - p \le n_2 \longrightarrow n_1 - n \le -p \le n_2 - n, \quad n - n_2 \le p \le n - n_1$

$$\max(n_1 - n, n - n_2) \le p \le \min(n_2 - n, n - n_1)$$

$$-\min(n_2 - n, n - n_1) \le p \le \min(n_2 - n, n - n_1)$$



$$\underbrace{-\min(n_2-n\,,\,n-n_1)}_{-Q} \leq p \leq \underbrace{\min(n_2-n\,,\,n-n_1)}_{Q}$$

$(n_2-n)\Delta_t$， $(n-n_1)\Delta_t$： 離兩個邊界的距離

注意：當 $n > n_2$ 或 $n < n_1$ 時，

將沒有 p 能滿足上面的不等式

If $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$

$$\boxed{W_x\big(\underset{T點}{n\Delta_t}, \underset{F點}{m\Delta_f}\big) = 2\sum_{p=-Q}^{Q} x\big((n+p)\Delta_t\big)x^*\big((n-p)\Delta_t\big)\exp\big(-j4\pi\, mp\Delta_t\Delta_f\big)\Delta_t}$$

$Q = \min(n_2-n, n-n_1)$.

$p \in [-Q, Q]$, $n \in [n_1, n_2]$,

possible for implementation

Method 1: Direct Implementation (brute force method)

唯一的限制條件？

Method 2: Using the DFT

3 大限制條件

When $\Delta_t\Delta_f = \dfrac{1}{2N}$ and $N \ge 2Q+1$

$$\boxed{W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t \sum_{p=-Q}^{Q} x\big((n+p)\Delta_t\big)x^*\big((n-p)\Delta_t\big)e^{-j\frac{2\pi mp}{N}}}$$

$q = p+Q \rightarrow p = q - Q$

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{2Q} x\big((n+q-Q)\Delta_t\big)x^*\big((n-q+Q)\Delta_t\big)e^{-j\frac{2\pi mq}{N}}$$

$$\boxed{W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{N-1} c_1(q)e^{-j\frac{2\pi mq}{N}}}$$

$Q = \min(n_2 - n,\ n - n_1)$.

$n \in [n_1, n_2]$,

$c_1(q) = x\big((n+q-Q)\Delta_t\big)x^*\big((n-q+Q)\Delta_t\big)$ for $0 \le q \le 2Q$

$c_1(q) = 0$ for $2Q+1 \le q \le N-1$

假設 $t = n_0\Delta_t,\ (n_0+1)\,\Delta_t,\ (n_0+2)\,\Delta_t,\ \cdots\cdots,\ n_1\Delta_t$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ m_1\,\Delta_f$

Step 1: Calculate n_0, n_1, m_0, m_1, N

Step 2: $n = n_0$

Step 3: Determine Q

Step 4: Determine $c_1(q)$

Step 5: $C_1(m) = \mathrm{FFT}[c_1(q)]$

Step 6: Convert $C_1(m)$ into $C(n\Delta_t, m\Delta_f)$

Step 7: Set $n = n+1$ and return to Step 3 until $n = n_1$.

Method 3: Using the Chirp Z Transform

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\sum_{p=-Q}^{Q} x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)\exp\left(-j4\pi mp\Delta_t\Delta_f\right)\Delta_t$$

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t\, e^{-j2\pi m^2\Delta_t\Delta_f}\sum_{p=-Q}^{Q} x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)e^{-j2\pi p^2\Delta_t\Delta_f}e^{j2\pi(p-m)^2\Delta_t\Delta_f}$$

Step 1 $x_1\left(n, p\right) = x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)e^{-j2\pi p^2\Delta_t\Delta_f}$

Step 2 $X_2\left[n,m\right] = \sum_{p=-Q}^{Q} x_1\left[n,p\right]c\left[m-p\right]$ $\qquad c\left[m\right] = e^{j2\pi m^2\Delta_t\Delta_f}$

Step 3 $X\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t\, e^{-j2\pi m^2\Delta_t\Delta_f} X_2\left[n,m\right]$

思考：Method 1 的複雜度為多少

思考：Method 2 的複雜度為多少

思考：Method 3 的複雜度為多少

The computation time of the WDF is more than those of the rec-STFT and the Gabor transform.

V-F Properties of the WDF

(1) Projection property	$\lvert x(t)\rvert^2 = \int_{-\infty}^{\infty} W_x(t,f)\,df \qquad \lvert X(f)\rvert^2 = \int_{-\infty}^{\infty} W_x(t,f)\,dt$
(2) Energy preservation property	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dtdf = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$
(3) Recovery property	$\int_{-\infty}^{\infty} W_x(t/2,f)e^{j2\pi ft}\,df = x(t)\cdot x^*(0)$ $x^*(0)$ 已知 $\int_{-\infty}^{\infty} W_x(t,f/2)e^{-j2\pi ft}\,dt = X(f)\cdot X^*(0)$
(4) Mean condition frequency and mean condition time	If $x(t) = \lvert x(t)\rvert \cdot e^{j2\pi\phi(t)}$, $X(f) = \lvert X(f)\rvert \cdot e^{j2\pi\Psi(f)}$ then $\phi'(t) = \lvert x(t)\rvert^{-2} \cdot \int_{-\infty}^{\infty} f\cdot W_x(t,f)\cdot df$ $-\Psi'(f) = \lvert X(f)\rvert^{-2} \int_{-\infty}^{\infty} t\cdot W_x(t,f)\cdot dt$
(5) Moment properties	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n W_x(t,f)\,dtdf = \int_{-\infty}^{\infty} t^n \lvert x(t)\rvert^2 dt$, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^n W_x(t,f)\,dtdf = \int_{-\infty}^{\infty} f^n \lvert X(f)\rvert^2 df$

(6) $W_x(t, f)$ is real	$\overline{W_x(t, f)} = W_x(t, f)$
(7) Region properties	If $x(t) = 0$ for $t > t_2$ then $W_x(t, f) = 0$ for $t > t_2$ If $x(t) = 0$ for $t < t_1$ then $W_x(t, f) = 0$ for $t < t_1$
(8) Multiplication theory	If $y(t) = x(t)h(t)$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(t, \rho) W_h(t, f-\rho) \cdot d\rho$
(9) Convolution theory	If $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(t-\rho, f) \cdot d\rho$
(10) Correlation theory	If $y(t) = \int_{-\infty}^{\infty} x(t+\tau)h^*(\tau)d\tau$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(-t+\rho, f) \cdot d\rho$

(11) Time-shifting property	If $y(t) = x(t - t_0)$, then $W_y(t, f) = W_x(t - t_0, f)$
(12) Modulation property	If $y(t) = \exp(j2\pi f_0 t)x(t)$, then $W_y(t, f) = W_x(t, f - f_0)$

The STFT (including the rec-STFT, the Gabor transform) does not have real region, multiplication, convolution, and correlation properties.

- Why the WDF is always real?

  What are the advantages and disadvantages it causes?

- Try to prove of the projection and recovery properties

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

- Proof of the region properties

If $x(t) = 0$ for $t < t_0$,

$x(t + \tau/2) = 0$ for $\tau < (t_0 - t)/2 = -(t - t_0)/2$,

$x(t - \tau/2) = 0$ for $\tau > (t - t_0)/2$,

Therefore, if $t - t_0 < 0$, the nonzero regions of $x(t + \tau/2)$ and $x(t - \tau/2)$ does not overlap and $x(t + \tau/2)\, x^*(t - \tau/2) = 0$ for all τ.

The importance of region property

V-G Advantages and Disadvantages of the WDF

Advantages: clarity

many good properties

suitable for analyzing the random process

Disadvantages: cross-term problem

more time for computation, especial for the signal with long time duration

not one-to-one

not suitable for $\exp\left(jt^n\right),\ n \neq 0,1,2$

V-H Windowed Wigner Distribution Function

When $x(t)$ is not time-limited, its WDF is hard for implementation

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

with mask

$$W_x(t,f) = \int_{-\infty}^{\infty} w(\tau) x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$w(\tau)$ is real and time-limited

Advantages: (1) reduce the computation time

(2) may reduce the cross term problem

Disadvantages:

$$W_x(t,f) = 2\int_{-\infty}^{\infty} w(2\tau')x(t+\tau')\cdot x^*(t-\tau')e^{-j4\pi\tau' f}\cdot d\tau'$$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\sum_{p=-\infty}^{\infty} w(2p\Delta_t)x((n+p)\Delta_t)x^*((n-p)\Delta_t)e^{-j4\pi mp\Delta_t\Delta_f}\Delta_t}$$

Suppose that $w(t) = 0$ for $|t| > B$

$$w(2p\Delta_t) = 0 \quad \text{for } p < -Q \text{ and } p > Q$$

$$Q = \frac{B}{2\Delta_t}$$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\sum_{p=-Q}^{Q} w(2p\Delta_t)x((n+p)\Delta_t)x^*((n-p)\Delta_t)e^{-j4\pi mp\Delta_t\Delta_f}\Delta_t}$$

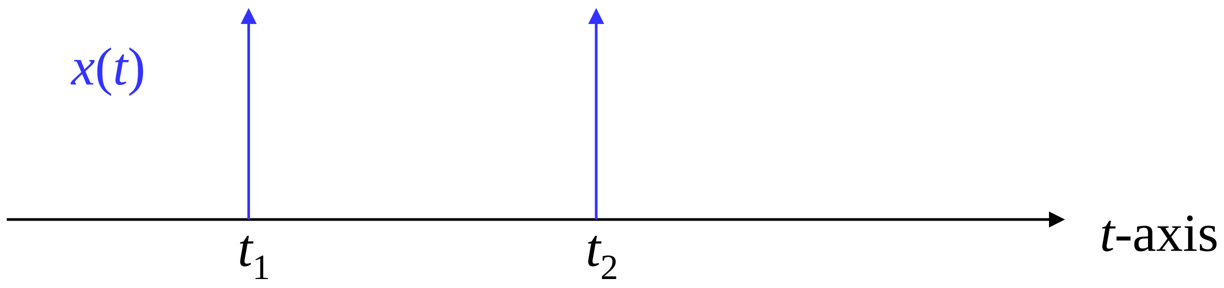
當然，乘上 mask 之後，有一些數學性質將會消失

(B) Why the cross term problem can be avoided ?

$$W_x(t,f)=\int_{-\infty}^{\infty} w(\tau)\,x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$w(\tau)$ is real

Viewing the case where $x(t) = \delta(t - t_1) + \delta(t - t_2)$



理想情形： $W_x(t,f)=0$ for $t \neq t_1, t_2$

然而，當 mask function $w(\tau)=1$ 時 (也就是沒有使用 mask function)

$$y(t,\tau)=x(t+\tau/2) \qquad y^*(t,-\tau)=x^*(t-\tau/2)$$

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\,x^*(t-\tau/2)\,e^{-j2\pi\tau f}\cdot d\tau$$

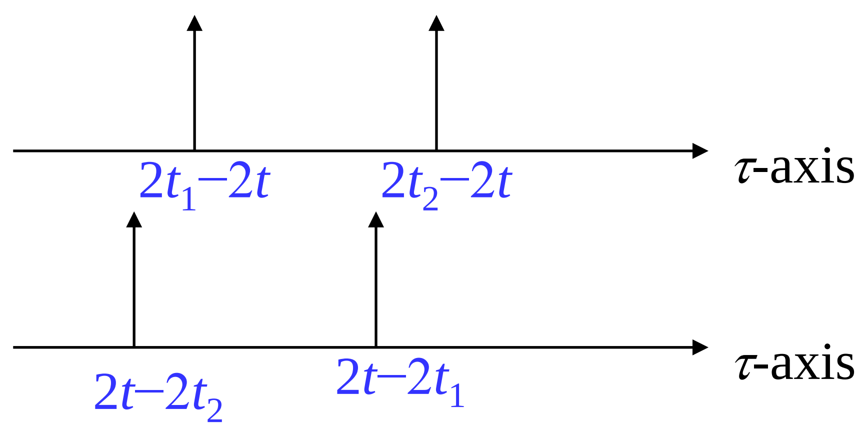
$$=\int_{-\infty}^{\infty}\left[\delta\left(t+\frac{\tau}{2}-t_1\right)+\delta\left(t+\frac{\tau}{2}-t_2\right)\right]\left[\delta\left(t-\frac{\tau}{2}-t_1\right)+\delta\left(t-\frac{\tau}{2}-t_2\right)\right]e^{-j2\pi\tau f}\cdot d\tau$$

from page 118, property 2

$$=4\int_{-\infty}^{\infty}\left[\underbrace{\delta(\tau+2t-2t_1)+\delta(\tau+2t-2t_2)}_{\text{第一項}}\right]\left[\underbrace{\delta(\tau-2t+2t_1)+\delta(\tau-2t+2t_2)}_{\text{第二項}}\right]e^{-j2\pi\tau f}\cdot d\tau$$

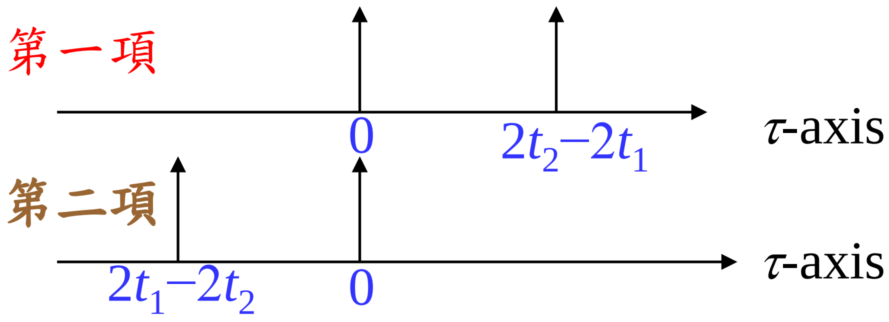
第一項

第二項

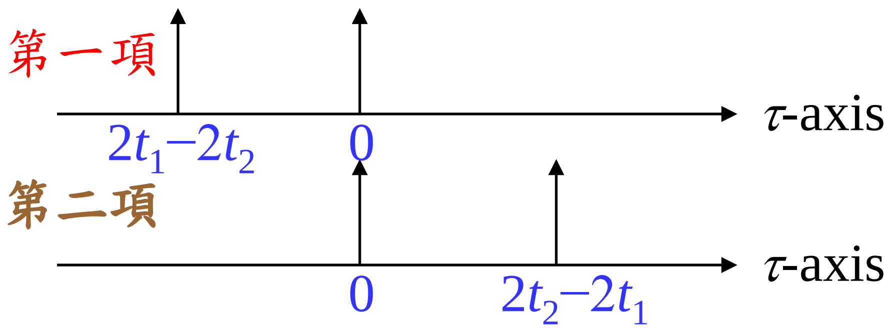


3種情形 $W_x(t, f) \neq 0$

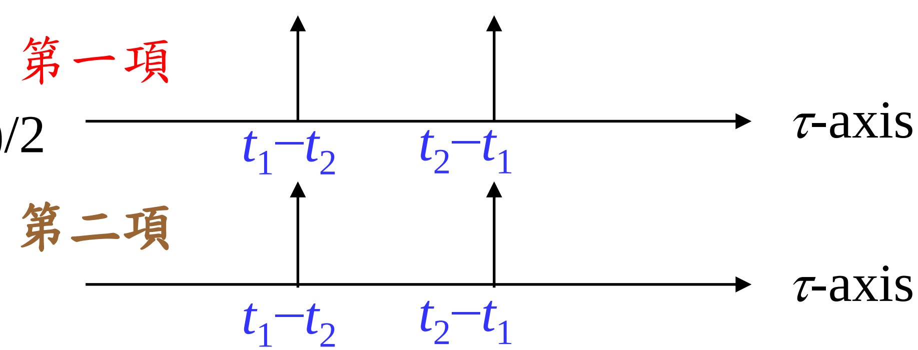
(1) If $t = t_1$



(2) If $t = t_2$



(3) If $t = (t_1 + t_2)/2$



With mask function

$$W_x(t,f)=\int_{-\infty}^{\infty} w(\tau)\,x(t+\tau/2)\,x^*(t-\tau/2)\,e^{-j2\pi\tau f}\cdot d\tau$$

$$=\int_{-\infty}^{\infty} w(\tau)\left[\delta(\tau+2t-2t_1)+\delta(\tau+2t-2t_2)\right]$$

$$\times\left[\delta(\tau-2t+2t_1)+\delta(\tau-2t+2t_2)\right]e^{-j2\pi\tau f}\cdot d\tau$$

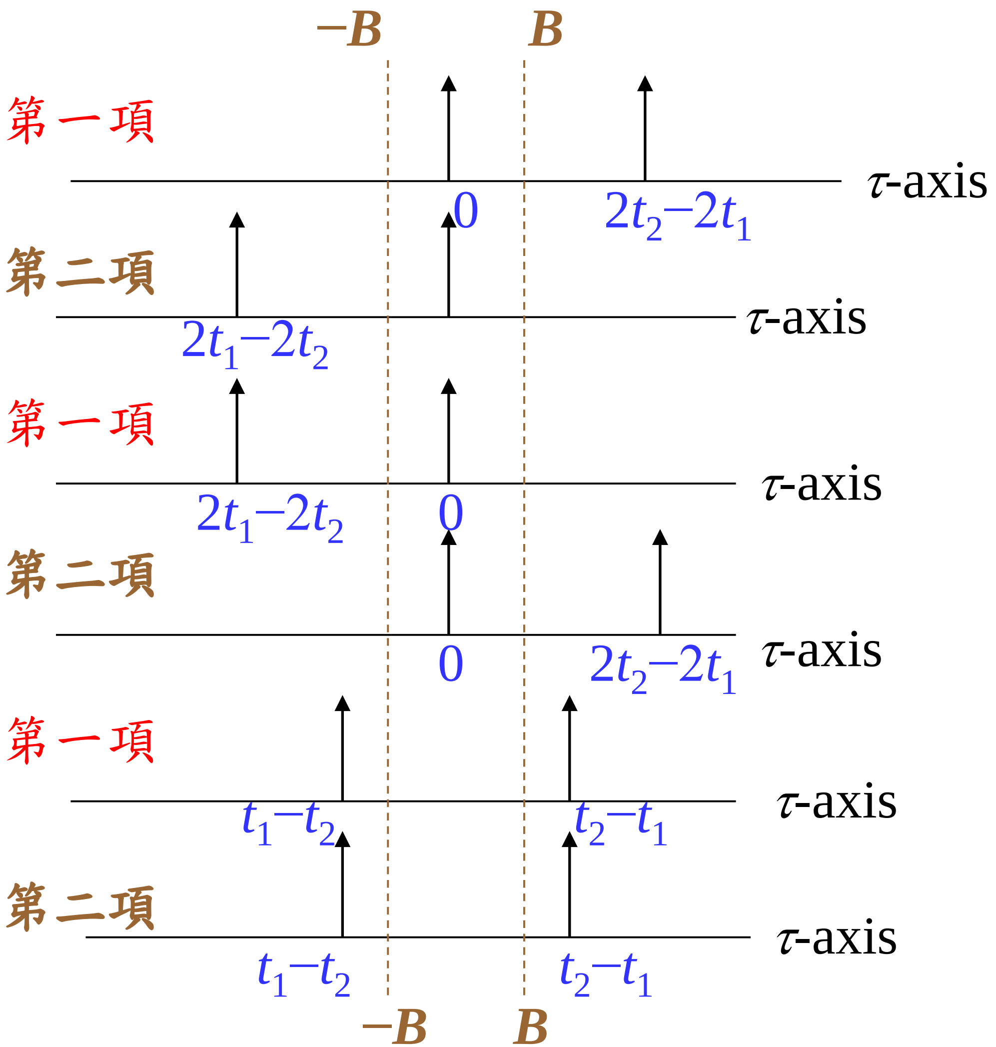
Suppose that $w(\tau) = 0$ for $|\tau| > B$, B is positive.

If $B < t_2 - t_1$

(1) $t = t_1$

(2) $t = t_2$

(3) $t = (t_1 + t_2)/2$



附錄五： 研究所學習新知識把握的要點

(1) Concepts: 這個方法的核心概念、基本精神是什麼

(2) Comparison: 這方法和其他方法之間，有什麼相同的地方？
有什麼相異的地方

(3) Advantages: 這方法的優點是什麼

(3-1) Why? 造成這些優點的原因是什麼

(4) Disadvantages: 這方法的缺點是什麼

(4-1) Why? 造成這些缺點的原因是什麼

(5) Applications: 這個方法要用來處理什麼問題，有什麼應用

(6) Innovations: 這方法有什麼可以改進的地方
或是可以推廣到什麼地方

看過一篇論文或一個章節之後，若能夠回答 (1)-(5) 的問題，就代表你已經學通了這個方法

如果你的目標是發明創造出新的方法，可試著回答 (3-1), (4-1), 和 (6) 的問題