Dual Fractional Integer Programming

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Course No: 546 U6110

Agenda – Part 1

- Beale Tableau
- The Basic Approach
- The Form of the Gomory cut
- Illustrations

Beale Tableau

- This tableau is often used in ILP algorithms.
- The tableau represents the following system of equations

$$x_i = \bar{b}_i + \sum_{j=m+1}^{n} \bar{a}_{ij} \left(-x_j\right), \qquad i = 1, \ldots, m,$$

$$\left(-z\right) = -\bar{z} + \sum_{j=m+1}^{n} \bar{c}_j \left(-x_j\right).$$

- In addition to the above equations, the Beale tableau displays the following equations:

$$x_i = -(-x_i), \; i = m+1, \ldots, n.$$

Beale Tableau

All variable	Constant values	$(-x_{m+1})$	$\cdots$	$(-x_{m+j})$	$\cdots$	$(-x_n)$
$(-z)$	$(-\bar{z})$	$\bar{c}_{m+1}$	$\cdots$	$\bar{c}_{m+j}$	$\cdots$	$\bar{c}_n$
x_{m+1}	0	-1	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
x_{m+j}	0	0	$\cdots$	-1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
x_n	0	0	$\cdots$	0	$\cdots$	-1
x_1	$\bar{b}_1$	$\bar{a}_{1,m+1}$	$\cdots$	$\bar{a}_{1,m+j}$	$\cdots$	$\bar{a}_{1,n}$
x_2	$\bar{b}_2$	$\bar{a}_{2,m+1}$	$\cdots$	$\bar{a}_{2,m+j}$	$\cdots$	$\bar{a}_{2,n}$
$\cdots$	$\cdots$	$\vdots$		$\vdots$		$\vdots$
x_r	$\bar{b}_r$	$\bar{a}_{r,m+1}$	$\cdots$	$\bar{a}_{r,m+j}$	$\cdots$	$\bar{a}_{r,n}$
$\cdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
x_m	$\bar{b}_m$	$\bar{a}_{m,m+1}$	$\cdots$	$\bar{a}_{m,m+j}$	$\cdots$	$\bar{a}_{m,n}$

Beale Tableau

$$\alpha_j = \begin{pmatrix} \bar{c}_{m+j} \\ 0 \\ \cdots \\ -1 \\ \cdots \\ 0 \\ \bar{a}_{1,m+j} \\ \cdots \\ \bar{a}_{m,m+j} \end{pmatrix} \quad \begin{array}{l} (n-m) \text{ elements -1 in } j^{th} \text{ row} \\ , j = 1, \ldots, n\text{-}m \\ \\ m \text{ elements} \end{array} \qquad \alpha_0 = \begin{pmatrix} -\bar{z} \\ 0 \\ \cdots \\ 0 \\ \cdots \\ 0 \\ \bar{b}_1 \\ \cdots \\ \bar{b}_m \end{pmatrix}$$

The Beale tableau represents the following equations:

$$\begin{pmatrix} -z \\ x \end{pmatrix} = \alpha_0 + \sum_{j=1}^{n-m} \alpha_j (-x_{m+j})$$

$$where \quad x = (x_{m+1} \;\; x_{m+2} \cdots x_n \;\; x_1 \cdots x_m)^T.$$

Lexicographic Dual Simplex Method (LDS)

- The dual lexicographic simplex method requires lexicographic primal optimality, i.e., $\alpha_j \succ 0,\ j = 1, \ldots, n-m,$ where $x \prec y$ if the first nonzero term of $x-y$ is negative. (vector x is lexicographically smaller than vector y)
- The pivot row satisfies $\bar{b}_r = \min_{\bar{b}_i < 0} \bar{b}_i$.
- Using lexicographic ordering, the pivot column k satisfies

$$\frac{\alpha_k}{|\bar{a}_{r,m+k}|} \prec \frac{\alpha_j}{|\bar{a}_{r,m+j}|}, \text{ for all j such that } \bar{a}_{r,m+j} < 0.$$

LDS Method

- This selection of the pivot column prevents cycling, because column zero (α_0) is guaranteed to lexicographically decrease at each iteration ,even though the objective function value ($\overline{z}$) remains unchanged when degeneracy occurs.
- This selection rule also maintains the lexicographic primal optimality of the tableaus $\alpha_j \succ 0, j=1,\ldots, n-m$.

LDS Method

- Pivot rules:
  - List basic variables: basic variable x_r is replaced by x_{m+k}.
  - Update all columns α_j by $\hat{\alpha}_j$:

$$\hat{\alpha}_0 = \alpha_0 - \frac{\bar{b}_r}{\bar{a}_{r,m+k}} \alpha_k,$$

$$\hat{\alpha}_k = \frac{-1}{\bar{a}_{r,m+k}} \alpha_k,$$

$$\hat{\alpha}_j = \alpha_j - \frac{\bar{a}_{r,m+j}}{\bar{a}_{r,m+k}} \alpha_k, \qquad j = 1, \ldots, m, \; j \neq k.$$

Example (1/3)

$$\min \; z = x_1 + 4x_2 + 3x_4$$
$$s.t. \quad -x_1 - 2x_2 + x_3 - x_4 + x_5 = -3,$$
$$2x_1 + x_2 - 4x_3 - x_4 + x_6 = -2,$$
$$x_j \geq 0, \forall j$$

		α_0	α_1	α_2	α_3	α_4
			$-x_1$	$-x_2$	$-x_3$	$-x_4$
	$(-z)$	0	1	4	0	3
	x_1	0	-1	0	0	0
x_N	x_2	0	0	-1	0	0
	x_3	0	0	0	-1	0
	x_4	0	0	0	0	-1
	x_5	-3	(-1)	-2	1	-1
	x_6	-2	2	1	-4	-1

$$min\left\{\frac{1}{1}, \frac{4}{2}, \frac{3}{1}\right\} = 1$$
$$\Longrightarrow \; x_1 \text{ becomes basic}$$

Example (2/3)

	α_0	α_1	α_2	α_3	α_4
		$-x_5$	$-x_2$	$-x_3$	$-x_4$
$(-z)$	-3	1	2	1	2
x_1	3	-1	2	-1	1
x_2	0	0	-1	0	0
x_3	0	0	0	-1	0
x_4	0	0	0	0	-1
x_5	0	-1	0	0	0
x_6	-8	2	-3	(-2)	-3

$$min\left\{\frac{2}{3},\frac{1}{2},\frac{2}{3}\right\} = \frac{1}{2}$$

$\Longrightarrow$ x_3 becomes basic

Example (3/3)

		α_0	α_1	α_2	α_3	α_4
			$-x_5$	$-x_2$	$-x_6$	$-x_4$
$(-z)$		-7	2	$1/2$	$1/2$	$1/2$
x_1		7	-2	$7/2$	$-1/2$	$5/2$
x_2		0	0	-1	0	0
x_3		4	-1	$3/2$	$-1/2$	$3/2$
x_4		0	0	0	0	-1
x_5		0	-1	0	0	0
x_6		0	0	0	-1	0

Optimal solution:

$$x_1{}^* = 7$$
$$x_3{}^* = 4$$
$$z^* = 7$$

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Procedures

The there main steps are:

- Solve the LP relaxation with the simplex method. If the problem is an ILP ,start with an all-integer tableau (or with tableau of rational numbers). If the problem is infeasible or has an integer solution, stop. Otherwise, go to step 2.
- Whenever the solution is noninteger, the integrality constraints imply new additional constraints (or "cuts"), which cut off the current optimal point. Add a new constraint to the tableau, which will produce primal infeasibility.
- Re-optimize with the lexicographic dual simplex (LDS) method. If the new problem is infeasible or has an integer solution, stop. Otherwise, go to 2.

Concept of cuts

- ABCDE is the feasible region

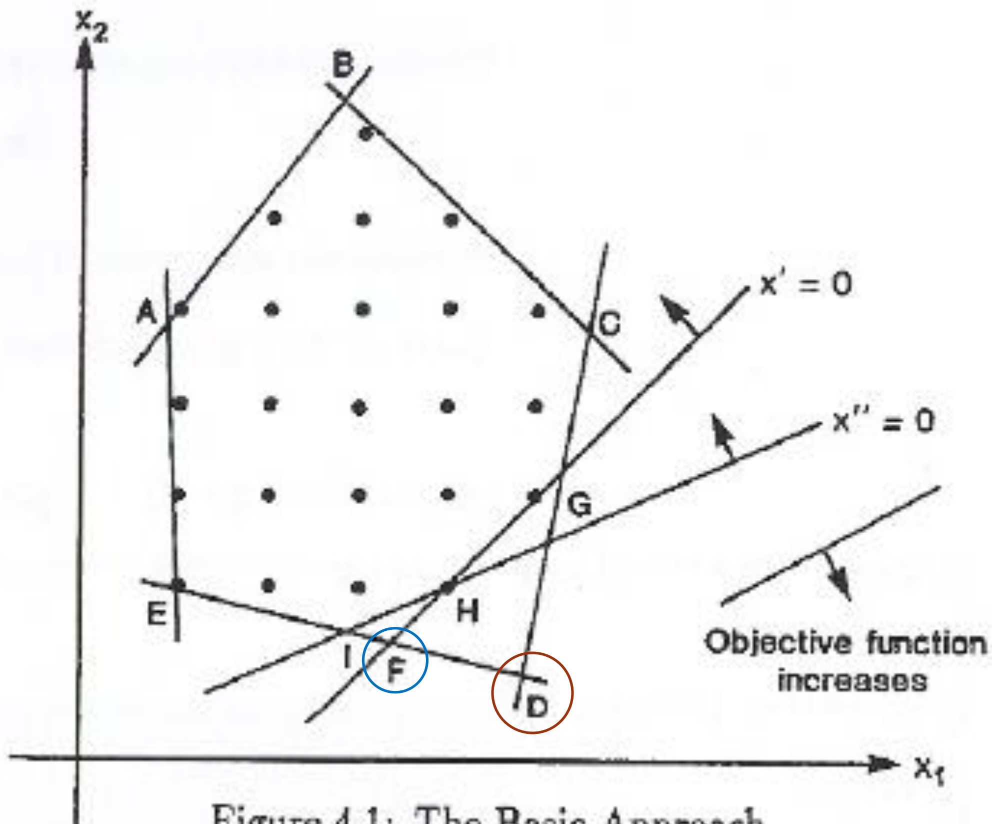

Figure 4.1: The Basic Approach

Beale Tableau

$$
\begin{array}{rc|c|ccccc|l}
\text{Variables} & & \text{values} & & & \text{Non-basic variables} & & & \\
 & & 1 & (-x_1) & \cdots & (-x_j) & \cdots & (-x_n) & \\
\hline
x_0 & = & a_{00} & a_{01} & \cdots & a_{0j} & \cdots & a_{0n} & \\
\hline
x_1 & = & 0 & -1 & & & & & \geq 0 \\
\vdots & & \vdots & & \ddots & & & & \vdots \\
x_j & = & 0 & & & -1 & & & \geq 0 \\
\vdots & & \vdots & & & & \ddots & & \vdots \\
x_n & = & 0 & & & & & -1 & \geq 0 \\
\hline
x_{n+1} & = & a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,j} & \cdots & a_{n+1,n} & \geq 0 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
x_{n+m} & = & a_{n+m,0} & a_{n+m,1} & \cdots & a_{n+m,j} & \cdots & a_{n+m,n} & \geq 0 \\
\hline
\end{array}
$$

Notation

- Notation:

$$x_0 = z\ ,$$
$$a_{0,0} = 0,$$
$$c = (c_j) = (-a_{0j}),$$
$$b = (b_j) = (a_{n+i\ ,0}),$$
$$\boldsymbol{A} = (a_{i,\,j})\ , \quad a_{ij}\ (\text{in } \boldsymbol{A}) \quad a_{n+i,\,j}\ (\text{in tableau}).$$

New notation: $\boldsymbol{J}$ = set containing the indices of the current nonbasic variables,

$\boldsymbol{J}(j) = j^{\text{th}}$ element is set $\boldsymbol{J}$ (the index of the j^{th} nonbasic variable).

LDS in Beale Tableau

- The LDS method will product an optimal tableau such that

$$\alpha_j \succ 0, j = 1, \dots, n$$
$$\Rightarrow a_{0,j} \geq 0, j = 1, \dots, n,$$
$$a_{i,0} \geq 0, i = 1, \dots, n + m.$$

- In addition, if a_{i0} is integer, i=1,…, n+m , the ILP is solved

$$z^* = a_{0,0}, x_i^* = a_{i,0}, i = 1, \dots, n + m.$$

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Form of a Gomory Cut

There is a row v,

$x_v = a_{v,0} + \sum_{j=1}^{n} a_{v,j} \left(-x_{J(j)}\right)$, with $a_{v,0}$ fractional

k^{th} Gomory cut,

$x_{n+m+k} = -f_{v,0} + \sum_{j=1}^{n}(-f_{v,j}) \left(-x_{J(j)}\right) \geq 0,$ (Gomory,1958)

where x_{n+m+k} is the Gomory slack variables,

$$f_{v,j} = a_{v,j} - \lfloor a_{v,j} \rfloor, j = 0, \ldots n.$$

Note that $0 \leq f_{v,j} < 1, j = 0, \ldots n, 0 < f_{v,0} < 1.$

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Example 1 (1/6)

- Primal

$$\begin{aligned} \text{Maximize} \quad & -4x_1 - 5x_2 = x_0 \\ \text{subject to} \quad & -x_1 - 4x_2 \le -5, \\ & -3x_1 - 2x_2 \le -7, \\ \text{and} \quad & x_1, \quad x_2 \ge 0, \quad \text{integer.} \end{aligned}$$

- Dual

$$\begin{aligned} \text{Minimize} \quad & -5w_1 - 7w_2 = w_0 \\ \text{subject to} \quad & -w_1 - 3w_2 \ge -4 \;\; (\mathrm{w}_{,1}), \\ & -4w_1 - 2w_2 \ge -5 \;\; (\mathrm{w}_{,2}), \\ \text{and} \quad & \mathrm{w}_1, \quad \mathrm{w}_2 \ge 0\,; \end{aligned}$$

Example 1 (2/6)

#1	1	$(-x_1)$	$(-x_2)$
x_0	0	4	5
x_1	0	-1	0
x_2	0	0	-1
x_3	-5	-1	-4
x_4	-7	-3	-2

- $x_1=x_2=0$
- Pivot row selection:

  Min{-5, -7} = -7,

  or x_4 becomes nonbasic,
- Pivot column selection:

  Min{4/|-3|, 5/|-2|} = 4/3,

  or x_1 becomes basic.

Example 1 (3/6)

#2	1	$(-x_4)$	$(-x_2)$
x_0	$-28/3$	$4/3$	$7/3$
x_1	$7/3$	$-1/3$	$2/3$
x_2	0	0	-1
x_3	$-8/3$	$-1/3$	$-10/3$
x_4	0	-1	0

$(x_1 = 7/3, x_2 = 0)$

Pivot column selection:

Min$\{(4/3)/|-1/3|, (7/3)/|-10/3|\} = 0.7$,

or x_2 becomes basic.

Example 1 (4/6)

#3	1	$(-x_4)$	$(-x_3)$
x_0	$-112/10$	$11/10$	$7/10$
$\rightarrow x_1$	$18/10$	$-4/10$	$2/10$
x_2	$8/10$	$1/10$	$-3/10$
x_3	0	0	-1
x_4	0	-1	0

Example 1 (5/6)

#4	1	$(-x_5)$	$(-x_3)$
x_0	$-76/6$	$11/6$	$2/6$
x_1	$14/6$	$-4/6$	$2/6$
$\rightarrow x_2$	$4/6$	$1/6$	$-2/6$
x_3	0	0	-1
x_4	$8/6$	$-10/6$	$2/6$
x_5	0	-1	0

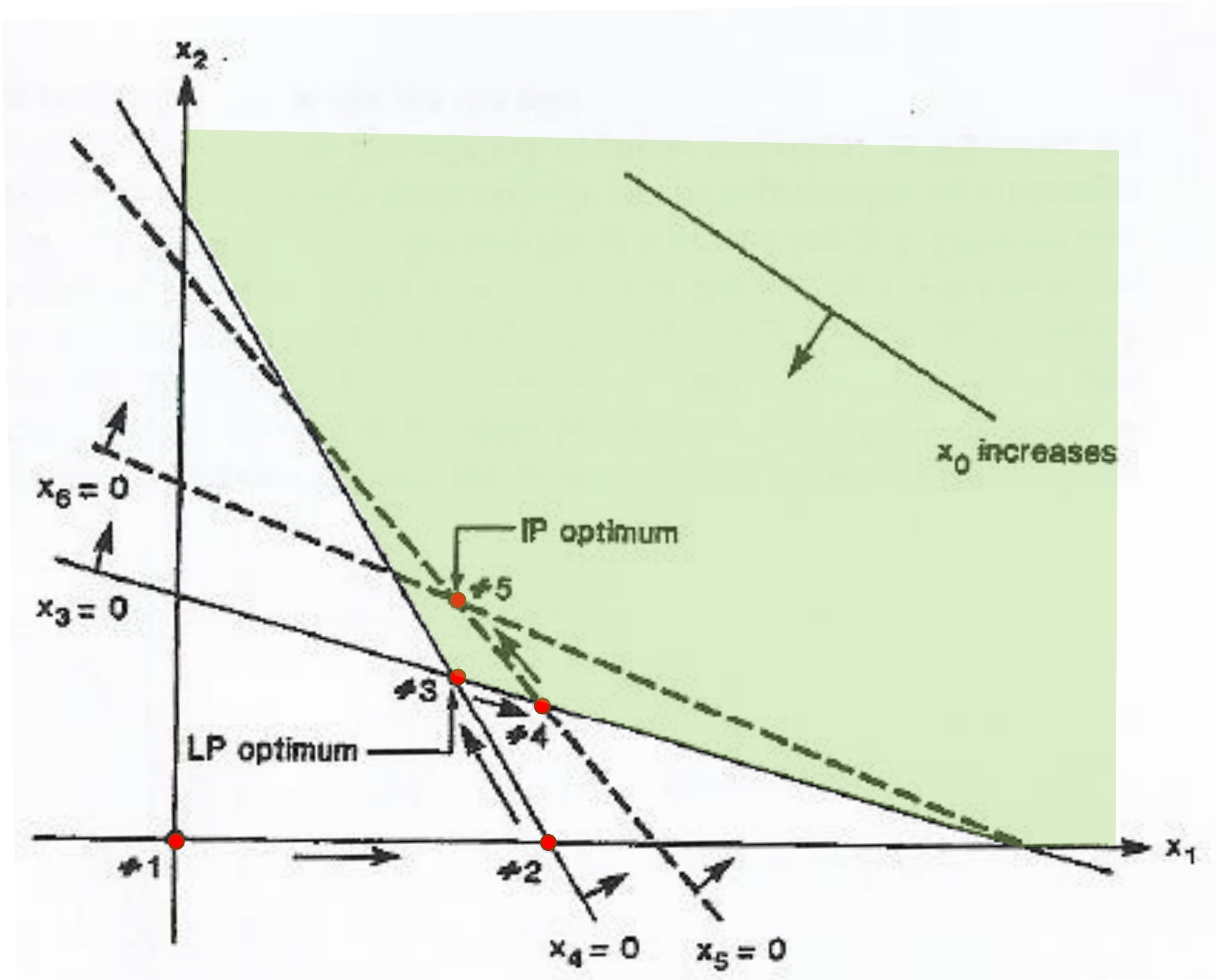
$(x_1 = 14/6, x_2 = 4/6)$

#5	1	$(-x_5)$	$(-x_6)$
$\rightarrow x_0$	-13	$7/4$	$2/4$
x_1	2	$-3/4$	$2/4$
x_2	1	$1/4$	$-2/4$
x_3	1	$1/4$	$-6/4$
x_4	1	$-7/4$	$2/4$
x_5	0	-1	0
x_6	0	0	-1

$(x_1 = 2, x_2 = 1)$

Optimal tableau

Example 1 (6/6)


x_2
x_1
x_0 increases
$x_6 = 0$
$x_3 = 0$
IP optimum
#5
#3
#4
LP optimum
#1
#2
$x_4 = 0$
$x_5 = 0$

Example 2 (1/4)

$$
\begin{array}{llllllll}
\text{maximize } z = 2x_1 + x_2, & & & & & & & \\
\text{subject to} & x_1 + & x_2 + & x_3 & & & = 5, & (*) \\
 & -x_1 + & x_2 & & + x_4 & & = 0, & (**) \\
 & 6x_1 + & 2x_2 & & & + x_5 & = 21, & (***)
\end{array}
$$

$$x_j \geq 0, \text{ integer}, \qquad j = 1, \ldots, 5.$$

Example 2 (2/4)

We obtain the following solution by solving the LP relaxation:

		$(-x_3)$	$(-x_5)$
x_0	$31/4$	$1/2$	$1/4$
x_1	$11/4$	$-1/2$	$1/4$
x_2	$9/4$	$-3/2$	$-1/4$
x_3	0	-1	0
x_4	$1/2$	-2	$1/2$
x_5	0	0	-1

Possible cuts

$$-\frac{3}{4}+\frac{1}{2}x_3+\frac{1}{4}x_5 \geq 0 \quad (0)$$

$$-\frac{3}{4}+\frac{1}{2}x_3+\frac{1}{4}x_5 \geq 0 \quad (1)$$

$$-\frac{1}{4}+\frac{1}{2}x_3+\frac{3}{4}x_5 \geq 0 \quad (2)$$

$$-\frac{1}{2}+\frac{1}{2}x_5 \geq 0 \quad (4)$$

Cuts (0) and (1) are the same.

Example 2 (3/4)

		$(-x_3)$	$(-x_5)$
x_0	$30/4$	$1/2$	$1/4$
x_1	$11/4$	$-1/2$	$1/4$
x_2	$9/4$	$3/2$	$-1/4$
x_3	0	-1	0
x_4	$1/2$	-2	$1/2$
x_5	0	0	-1
x_6	$-1/2$	0	$(-\frac{1}{2})$

		$(-x_3)$	$(-x_6)$
x_0	$30/4$	$1/2$	$1/2$
x_1	$10/4$	$-1/2$	$1/2$
x_2	$10/4$	$3/2$	$-1/2$
x_3	0	-1	0
x_4	0	-2	1
x_5	1	0	-2
x_6	0	0	-1

Possible cuts

$$-\frac{1}{2}+\frac{1}{2}x_3+\frac{1}{2}x_6 \geq 0 \text{ (0')}$$

$$-\frac{1}{2}+\frac{1}{2}x_3+\frac{1}{2}x_6 \geq 0 \text{ (1')}$$

$$-\frac{1}{2}+\frac{1}{2}x_3+\frac{1}{2}x_6 \geq 0 \text{ (2')}$$

Example 2 (4/4)

		$(-x_3)$	$(-x_6)$
x_0	$30/4$	$1/2$	$1/2$
x_1	$10/4$	$-1/2$	$1/2$
x_2	$10/4$	$3/2$	$-1/2$
x_3	0	-1	0
x_4	0	-2	1
x_5	1	0	-2
x_6	0	0	-1
x_7	$-1/2$	$(-\frac{1}{2})$	$-1/2$

		$(-x_3)$	$(-x_6)$
x_0	7	1	0
x_1	3	-1	1
x_2	1	3	-2
x_3	1	-2	1
x_4	2	-4	3
x_5	1	0	-2
x_6	0	0	-1
x_7	0	-1	0

This table produces primal and Dual(optimal) feasible and integer solutions.

$$x_1^* = 3,$$
$$x_2^* = 1,$$
$$x_3^* = 1,$$
$$x_4^* = 2,$$
$$x_5^* = 1,$$
$$z^* = 7,$$

Agenda – Part 2

- The Derivation of the Cut
- Some Properties of the Cuts
- Algorithm Strategies
- Finiteness
- Dantzig Cut

Congruence

- **Definition:** x is congruent to y module k, i.e., $x \equiv y \pmod k$, if there exist an integer I Such that $x-y \equiv k\,I$
- Examples: $3 \equiv 8 \pmod 5$, and $4 \equiv -2 \pmod 3$.
- $x \equiv y \pmod 1$ if and only if x-y __________ .

Properties of Congruence

$i)\ x \equiv x \pmod{K}$.

$ii)$ If $x \equiv y \pmod{K}$ and $y \equiv z \pmod{K}$, then $x \equiv z \pmod{K}$.

$iii)$ If $x \equiv y \pmod{K}$, then $y \equiv x \pmod{K}$.

$iv)$ If $x \equiv y \pmod{K}$ and z is an integer, then $xz \equiv yz \pmod{K}$.

$v)$ If $x \equiv y \pmod{K}$ and $x' \equiv y' \pmod{K}$, then $x + x' \equiv y + y'$.

(mod K). Further, if x and y' are integers, then

$xx' \equiv yy' \pmod{K}$.

Derivation (1/5)

- Assume we want to generate a cut from equation

$$x_v = a_{v,0} + \sum_{j=1}^{n} a_{v,j}\left(-x_{J(j)}\right),$$

where $\mathrm{a}_{\mathrm{v},0} > 0$.

$$
\begin{aligned}
x_v \text{ integer} \quad &\Leftrightarrow \quad 0 \equiv x_v (\bmod 1), \\
&\Leftrightarrow \quad 0 \equiv a_{v,0} + \sum_{j=1}^{n} a_{v,j}\left(-x_{J(j)}\right)(\bmod 1),
\end{aligned}
$$

Derivation (2/5)

- Adding and subtracting integer amounts does not destroy the congruence relationship.

$$x_v \text{ integer} \quad \Leftrightarrow \quad 0 \equiv f_{v,0} + \sum_{j=1}^{n} a_{v,j}\left(-x_{J(j)}\right) (\bmod 1),$$

$$\text{where } f_{v,0} = a_{v,0} - \lfloor a_{v,0} \rfloor, \text{and } 0 < f_{v,0} < 1$$

Derivation (3/5)

- The same operations on variables $x_{J(j)}$

$$x_{J(j)} \text{ integer} \quad \Leftrightarrow \quad 0 \equiv f_{v,0} + \sum_{j=1}^{n} f_{v,j}\left(-x_{J(j)}\right) (\text{mod } 1),$$

where $f_{v,j} = a_{v,j} - \lfloor a_{v,j} \rfloor$, $j = 1, \ldots, n$.

and $0 \le f_{v,j} < 1, \; j = 1, \ldots, n.$

- Which is equivalent to

$$f_{v,0} \equiv \qquad\qquad ,$$

Derivation (4/5)

$$\left.\begin{array}{l} 0 < f_{v,0} < 1 \\ \sum_{j=1}^{n} f_{v,j} x_{J(j)} > 0 \end{array}\right\} \Rightarrow \sum_{j=1}^{n} f_{v,j} x_{J(j)} = f_{v,0},\ f_{v,0}+1,\ f_{v,0}+2,\ f_{v,0}+3, \ldots$$

$$\Rightarrow f_{v,0} \le \sum_{j=1}^{n} f_{v,j} x_{J(j)}$$

Derivation (5/5)

- The Gomory cut is

$$x_{n+m+1} = -f_{v,0} + \sum_{j=1}^{n} \left(-f_{v,j}\right)\left(-x_{J(j)}\right) \geq 0,$$

where x_{n+m+1} is the Gomory slack variable, which is integer.

Example

#3	1	$(-x_4)$	$(-x_3)$
x_0	$-112/10$	$11/10$	$7/10$
$\rightarrow x_1$	$18/10$	$-4/10$	$2/10$
x_2	$8/10$	$1/10$	$-3/10$
x_3	0	0	-1
x_4	0	-1	0
x_5	$-8/10$	$-6/10$	$-2/10$

The source row is $x_1 = 18/10 - (4/10)(-x_4) + (2/10)(-x_3)$

Since $x_1 \equiv 0$, we have $0 \equiv 18/10 - (4/10)(-x_4) + (2/10)(-x_3)$.

subtracting $0 \equiv 1$

yields $0 \equiv 8/10 - (4/10)(-x_4) + (2/10)(-x_3)$;

adding $0 \equiv 1(-x_4)$

gives $0 \equiv 8/10 + (6/10)(-x_4) + (2/10)(-x_3)$.

Or $8/10 \equiv (6/10)x_4 + (2/10)x_3$,

which means that $8/10 \le (6/10)x_4 + (2/10)x_3$.

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Integer Inequality

- Each added inequality becomes an all-integer inequality when expressed in terms of the original nonbasic variables.

- Example: in tableau #3 (in page 24)

$$x_5 = -\frac{8}{10} + \frac{6}{10}x_4 + \frac{2}{10}x_3 \geq 0$$

$$\text{where } x_3 = -5 + x_1 + 4x_2 \text{ and } x_4 = -7 + 3x_1 + 2x_2$$

Integer inequality

- The same reasoning for the second cut

#4	1	$(-x_5)$	$(-x_3)$
x_0	$-76/6$	$11/6$	$2/6$
x_1	$14/6$	$-4/6$	$2/6$
$\rightarrow x_2$	$4/6$	$1/6$	$-2/6$
x_3	0	0	-1
x_4	$8/6$	$-10/6$	$2/6$
x_5	0	-1	0
x_6	$-4/6$	$-1/6$	$-4/6$

Properties

- After integer optimality is achieved ,we may continue to add cuts (derived from rows containing fractions) to obtain all-integer optimal tableau.

- If the hyperplane corresponding to an added inequality goes through an integer point, it goes through an infinite number of integer points.
  - $2x_1+3x_2=5$, given an integer point, (1, 1).
  - We can obtain other integer points, k is integer.

Example (1/6)

$$\max \; 3x_1 - x_2$$

S.t.

$$3x_1 - 2x_2 \le 3$$

$$-5x_1 - 4x_2 \le -10$$

$$2x_1 + x_2 \le 5$$

$$x_1, x_2 \ge 0, \text{integer}$$

Example (2/6)

#1	1	$(-x_1)$	$(-x_2)$
x_0	0	-3	0
x_1	0	-1	0
x_2	0	0	-1
x_3	3	3	-2
x_4	-10	-5	-4
x_5	5	2	1
$\rightarrow\ s$	5	1	1

#2	1	$(-s)$	$(-x_2)$
x_0	15	3	4
x_1	5	1	1
x_2	0	0	-1
$\rightarrow x_3$	-12	-3	-5
x_4	15	5	1
x_5	-5	-2	-1
s	0	-1	0

Example (3/6)

#3	1	$(-s)$	$(-x_3)$
x_0	$27/5$	$3/5$	$4/5$
x_1	$13/5$	$2/5$	$1/5$
x_2	$12/5$	$3/5$	$-1/5$
x_3	0	0	-1
x_4	$63/5$	$22/5$	$1/5$
$\rightarrow x_5$	$-13/5$	$-7/5$	$-1/5$
s	0	-1	0

#4	1	$(-x_5)$	$(-x_3)$
x_0	$30/7$	$3/7$	$5/7$
x_1	$13/7$	$2/7$	$1/7$
x_2	$9/7$	$3/7$	$-2/7$
x_3	0	0	-1
x_4	$31/7$	$22/7$	$-3/7$
x_5	0	-1	0

Example (4/6)

- Suppose use x_1 as a source row,

$$x_1 = \frac{13}{7} + \frac{2}{7}(-x_5) + \frac{1}{7}(-x_3)$$

- The cut

$$x_6' = -\frac{6}{7} - \frac{1}{7}(-x_3) - \frac{2}{7}(-x_5) \geq 0 \text{ , } x_3 = 3 - 3x_1 + 2x_2 \text{ and } x_5 = 5 - 2x_1 - x_2$$

- In terms of non-basic variables

$$2 - 2x_1 \geq 0$$

Example (5/6)

- Suppose use $2x_1$ as a source row,

$$2x_1 = \frac{26}{7} + \frac{4}{7}(-x_5) + \frac{2}{7}(-x_3)$$

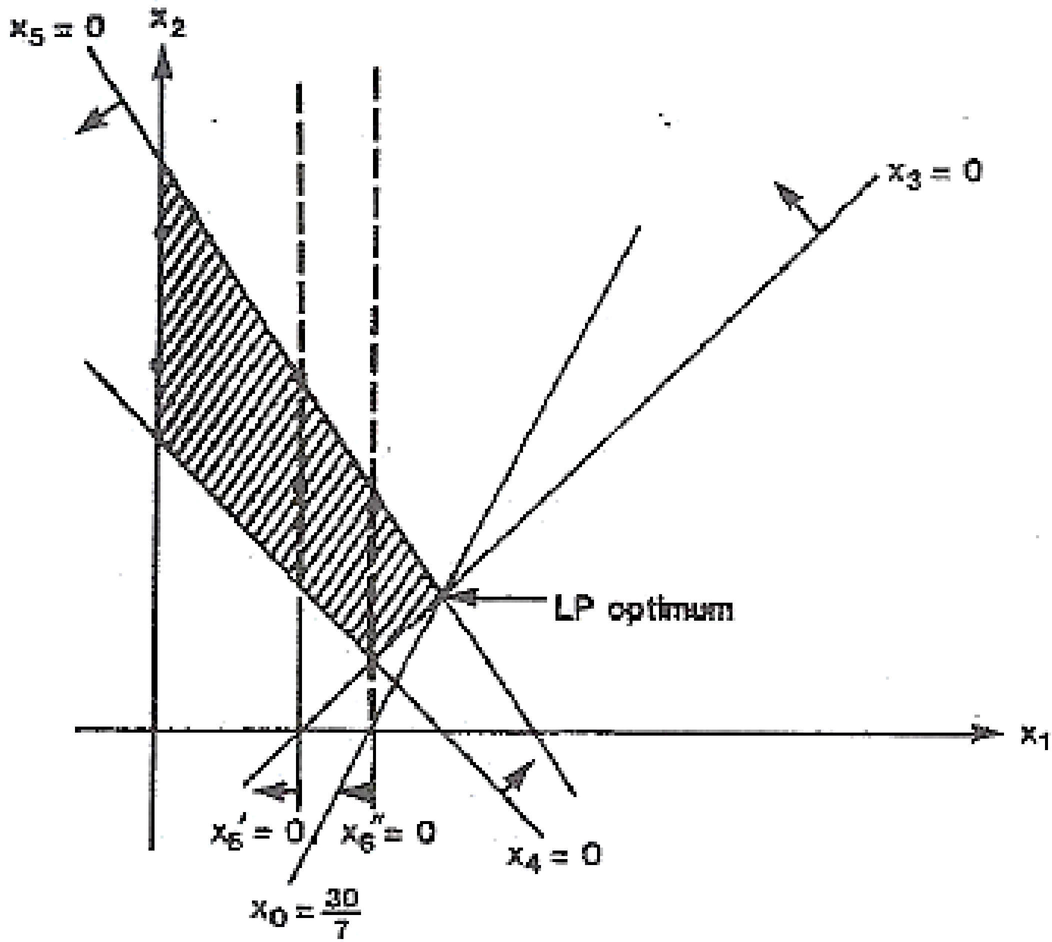
- The cut

$$x_6'' = -\frac{5}{7} - \frac{2}{7}(-x_3) - \frac{4}{7}(-x_5) \geq 0 \quad , x_3 = 3 - 3x_1 + 2x_2 \text{ and } x_5 = 5 - 2x_1 - x_2$$

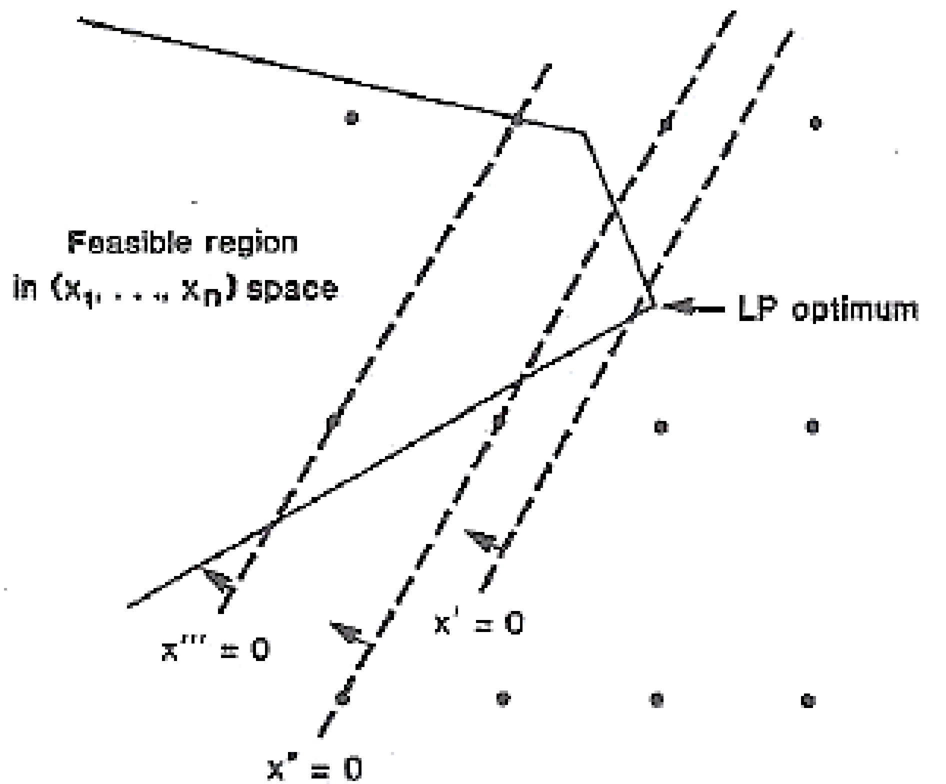
- In terms of non-basic variables

$$3 - 2x_1 \geq 0$$

Example (6/6)



Example 2



Agenda – Part 2

- The Derivation of the Cut
- Some Properties of the Cuts
- Algorithm Strategies
- Finiteness
- Dantzig Cut

Rules to Choose the Source Row

Let $\sum_{j=1}^{n} f_j x_{J(j)} = f_0$

be the hyperplane defined by the cut in the nonnegative region of the

$(X_{J(j)},\ldots.,X_{J(n)})$ space.

The intersection of the hyperplane with the $X_{J(j)}$ axis, j=1,…,n, occurs at

.

- The larger the value of ________, the stronger the cut.

Rules to Choose the Source Row

- **Rule 1:** Generate the cut from the row with the largest f_0 value. If f_0 is large, hopefully f_0/f_j the ratio will also be large.

- **Rule 2:** Choose the first row with a fractional constant f_0.

- **Rule 3:** Add several cuts and let the simplex method choose one. For example, the one that produces the largest decrease in the objective function.

Rule for Dropping Inequalities

- When an inequality is introduced, it is immediately used as pivot row and the slack variable becomes nonbasic. If it becomes basic again, the inequality can be dropped.

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Converge of the Algorithm

- **Theorem:** The algorithm converges to an optimal integer solution after a finite number of iterations (cuts) if

i) There exists a lower bound M for the value of $x_{0.}$

ii) The first row with a fractional constant component is selected as the source row (every finite number of iterations).

Proof (1/6)

- **Proof :** (By contradiction) assume the algorithm is not finite.

$\Rightarrow$ There exists a sequence of LDS tableaus such that

$$\alpha_0^k \overset{\ell}{\succ} \alpha_0^{k+1} \overset{\ell}{\succ} \alpha_0^{k+2} \quad \ldots \text{ etc.}$$

$$\Rightarrow \quad \alpha_{0,0}^k \succ \alpha_{0,0}^{k+1} \succ \alpha_{0,0}^{k+2} \quad \ldots \text{ etc.}$$

Proof (2/6)

- We will show now the a_{00} will eventually remain fixed at some integer value. Assume that row 0 generates the cut

$$x' = -f_{0,0} + \sum_{j=1}^{n} \left(-f_{0,j}\right)\left(-x_{J(j)}\right) \geq 0,\text{where x is the new slack variable.}$$

The cut is used as a pivot row. If p is the pivot column, then

$$a'_{0,0} = a_{0,0} - \frac{a_{0,p} f_{0,0}}{f_{0,p}},$$

where $f_{0,p} > 0$ is the negative value of the pivot element, and $a'_{0,0}$ is the new $a_{0,0}$ .Note that $a_{0,p} \geq 0,$ but since its fractional part $f_{0,p} \neq 0,$ $a_{0,p}$ must be positive ($a_{0,p} > 0$).

Proof (3/6)

$$a_{0,p} = \lfloor a_{0,p} \rfloor + f_{0,p} \geq f_{0,p} \quad or \quad \frac{a_{0,p}}{f_{0,p}} \geq 1$$

$$a'_{0,0} \leq a_{0,0} - f_{0,0} = \lfloor a_{0,0} \rfloor$$

$a_{0,0}$ is decreased at least to the next integer.

$a_{0,0}$ can have a fractional part for only a finite number of tableaus because it has a lower bound M.

Let k, be the iteration at which $a_{0,0}$ is fixed.

Then, for $k \geq k' + 1$, $a_{1,0}$ remains non-increasing due to the lexicographic decrease of α_0.

Proof (4/6)

In addition, $a_{1,0}$ is bounded from below by 0. Using a contradictory argument, note that

If $a_{1,0} < 0$ at certain iteration, row 1 becomes the pivot row and, if p is the pivot column, then

$$a'_{0,0} = a_{0,0} - \frac{a_{0,p} a_{1,0}}{a_{1,p}}.$$

Proof (5/6)

Since $a_{1,0} < 0$ and the pivot element $a_{1,p} < 0$, $\frac{a_{1,0}}{a_{1,p}} > 0.$ But then $a_{0,p}$ must be 0 since ,by assumption, $a'_{0,0} = a_{0,0}.$
This contradicts the lexicographic positivity of a_p

$(a_{0,p} = 0 \text{ and } a_{1,p} < 0).$

Therefore, $a_{1,0}$ is bounded from below by 0 for all $k \geq k' + 1.$

Proof (6/6)

Now, using similar arguments, we can show that there exists $k'' \geq k' + 1$ such that $a_{1,0}$ remains fixed at a nonnegative integer value.

Then, we can repeat the same argument for each of the other m+n+1 original variables.

Thus, at the end, we will have all-integer a_0.

Agenda – Part 2

- The Derivation of the Cut
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Dantzig Cut

- **Argument:** If the primal optimal solution is not integer, then at least one of the nonbasic Variables $x_{J(j)}$ must be positive.

Thus, $$\sum_{j=1}^{n} x_{J(j)} \geq 1, \qquad \text{(Dantzig, 1959)}$$

$$\Leftrightarrow \qquad x' = -1 + \sum_{j=1}^{n} (-1)\left(-x_{J(j)}\right) \geq 0.$$

Dantzig Cut

- In general, these cuts do not yield a finite algorithm.
- It can be shown that a necessary, but not sufficient condition for the algorithm to be finite is that the optimal solution be on an edge (a line joining two adjacent extreme points) of the constraint set.
- There are improved versions of the Dantzig cut for which a finite algorithm exists.

Dantzig Cut

Improved version of the cut with respect to a row v:

Strengthened Danzig cut: $\sum_{\substack{j=1 \\ a_{v,j} \neq 0}}^{n} x_{J(j)} \geq 1.$ (Charnes and Cooper, 1969)

Double strengthened Danzig cut: $\sum_{\substack{j=1 \\ f_{v,j} > 0}}^{n} x_{J(j)} \geq 1.$

(Bowman and Nemhauser, 1969)

- The details of the finite versions of these cutting plane algorithms are unimportant, because it is unlikely they will generate better cuts than the Gomory cut.

Example

$$x_v = \frac{1}{2} + 2\left(-x_1\right) - \frac{1}{2}\left(-x_2\right) + 0\left(-x_3\right) - \frac{1}{4}\left(-x_4\right).$$

Danzig cut: $x_1 + x_2 + x_3 + x_4 \geq 1,$

Charnes and Cooper cut: $x_1 + x_2 + x_4 \geq 1,$

Bowman and Nemhauser cut: $x_2 + x_4 \geq 1.$

Questions

- Please download HW2 on Ceiba (due on 3/30).