最佳化概論(Introduction to Optimization)
Homework 03 – 2017/10/05

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1. Use the simplex method to describe all the optimal solutions of the following problem.

$$
\begin{array}{rccccccccc}
\max & 2x_1 & + & 3x_2 & + & 5x_3 & + & 4x_4 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 5 \\
 & x_1 & + & 2x_2 & + & 2x_3 & + & 3x_4 & \leq & 3 \\
 & & & & x_1, x_2, x_3, x_4 & & & & \geq & 0
\end{array}
$$

Take x_3 and s_2 leaves:

$$
\begin{array}{rcccccccccc}
\max & 2x_1 & + & 3x_2 & + & 5x_3 & + & 4x_4 & & & \\
\text{s.t.} & S_1 & = & 5 & - & x_1 & - & 2x_2 & - & 3x_3 & - & x_4 \\
 & S_2 & = & 3 & - & x_1 & - & x_2 & - & 3x_3 & - & 3x_4 \\
 & & & & s_1, s_2, x_1, x_2, x_3, x_4 & & & & & & \geq & 0 \\
\hline
 & 2x_3 & = & 3 & - & x_1 & - & x_2 & - & 3x_4 & + & S_2
\end{array}
$$

Take x_2 and s_1 leaves:

$$
\begin{array}{rcccccccccc}
\max & \frac{15}{2} & - & \frac{1}{2}x_1 & + & \frac{1}{2}x_2 & - & \frac{7}{2}x_4 & - & \frac{5}{2}S_2 & & \\
\text{s.t.} & S_1 & = & \frac{1}{2} & + & \frac{1}{2}x_1 & - & \frac{1}{2}x_2 & + & \frac{7}{2}x_4 & + & \frac{3}{2}S_2 \\
 & x_3 & = & \frac{3}{2} & - & \frac{1}{2}x_1 & - & \frac{1}{2}x_2 & - & \frac{3}{2}x_4 & - & \frac{1}{2}S_2 \\
 & & & & s_1, s_2, x_1, x_2, x_3, x_4 & & & & & & \geq & 0 \\
\hline
 & \frac{1}{2}x_2 & = & \frac{1}{2} & + & \frac{1}{2}x_1 & + & \frac{7}{2}x_4 & + & \frac{3}{2}S_2 & - & S_1
\end{array}
$$

Finally, we can obtain that the max $z = 8,\ s_1, s_2 = 0$

$$
\begin{array}{rcccccccccc}
\max & 8 & - & S_1 & - & S_2 & + & 0x_1 & + & 0x_4 & & \\
\text{s.t.} & x_2 & = & 1 & + & x_1 & + & 7x_4 & + & 3S_2 & - & 2S_1 \\
 & x_3 & = & 1 & - & x_1 & - & 5x_4 & - & 2S_4 & + & S_1
\end{array}
$$

Thus

$$
\left\{
\begin{array}{ccccccc}
 & & \forall x_i & \geq & 0 & & \\
x_1 & + & 7x_4 & - & x_2 & = & -1 \\
x_1 & + & x_3 & + & 5x_4 & = & -1
\end{array}
\right.
$$

It's a polygon. We find its vertices assigning 2 coordinates to 0 and get only 3 vertices:

$$(1, 2, 0, 0),\ (0, 1, 1, 0),\ (0, 2.4, 0, 0.2)$$

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2. Solve the following LP problem.

$$\begin{array}{rrcrcrcl} \max & x_1 & + & 3x_2 & - & x_3 & & \\ \text{s.t.} & 2x_1 & + & 2x_2 & - & x_3 & \leq & 10 \\ & 3x_1 & - & 2x_2 & + & x_3 & \leq & 10 \\ & x_1 & - & 3x_2 & + & x_3 & \leq & 10 \\ & & & x_1, x_2, x_3 & & & \geq & 0 \end{array}$$

Take x_2 enters and s_1 leaves:

$$\begin{array}{rrcrcrcrcl} \max & & & x_1 & + & 3x_2 & - & x_3 & & \\ \text{s.t.} & S_1 & = & 10 & - & 2x_1 & - & 2x_2 & + & x_3 \\ & S_2 & = & 10 & - & 3x_1 & + & x_2 & - & x_3 \\ & S_3 & = & 10 & - & x_1 & + & 3x_2 & - & x_3 \\ & & & s_1, s_2, s_3, x_1, x_2, x_3 & & & & & \geq & 0 \end{array}$$

Note that $x \leq 5$ and obtain that $x_2 = 5 - x_1 + \frac{1}{2}x_3 - \frac{1}{2}S_1$

$$\begin{array}{rrcrcrcrcrcl} \max & & & 15 & - & 2x_1 & - & \frac{3}{2}S_1 & + & \frac{1}{2}x_3 & & \\ \text{s.t.} & x_2 & = & 5 & - & x_1 & - & S_1 & + & \frac{1}{2}x_3 & - & \frac{1}{2}S_1 \\ & S_2 & = & 20 & - & 5x_1 & - & S_2 & & & & \\ & S_3 & = & 25 & - & 4x_1 & - & \frac{3}{2}S_1 & + & \frac{1}{2}x_3 & & \\ & & & s_1, s_2, s_3, x_1, x_2, x_3 & & & & & & & \geq & 0 \end{array}$$

It is unbound. Then we can make x_3 as large as we wish.

$$x_1 = 0,\ x_2 = 5 + \tfrac{1}{2}t,\ x_3 = t,\ s_1 = 0,\ s_2 = 20,\ s_3 = 25 + \tfrac{1}{2}t,\ z = 15 + \tfrac{1}{2}t$$

Where t can be any positive number.

3. Solve the following problems by the two-phase simplex method:

(a)

$$
\begin{array}{rrcrcl}
\max & 3x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & - & x_2 & \leq & -1 \\
 & -x_1 & - & x_2 & \leq & -3 \\
 & 2x_1 & + & x_2 & \leq & 4 \\
 & & x_1, x_2 & & \geq & 0
\end{array}
$$

(b)

$$
\begin{array}{rrcrcl}
\max & 3x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & - & x_2 & \leq & -1 \\
 & -x_1 & - & x_2 & \leq & -3 \\
 & 2x_1 & + & x_2 & \leq & 2 \\
 & & x_1, x_2 & & \geq & 0
\end{array}
$$

(c)

$$
\begin{array}{rrcrcl}
\max & 3x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & - & x_2 & \leq & -1 \\
 & -x_1 & - & x_2 & \leq & -3 \\
 & 2x_1 & - & x_2 & \leq & 2 \\
 & & x_1, x_2 & & \geq & 0
\end{array}
$$

(a)

$$
\begin{array}{lllllllll}
\max & 3x_1 & + & x_2 & & & & & \\
\text{s.t.} & S_1 & = & -1 & - & x_1 & + & x_2 & \\
 & S_2 & = & -3 & + & x_1 & + & x_2 & \\
 & S_3 & = & 4 & - & 2x_1 & - & x_2 & \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0 &
\end{array}
\Longrightarrow
\begin{array}{llllllllll}
\max & \multicolumn{9}{c}{\{-x_0\}\ \text{artificial obj}} \\
\text{s.t.} & S_1 & = & x_0 & - & 1 & - & x_1 & + & x_2 \\
 & S_2 & = & x_0 & - & 3 & + & x_2 & + & x_2 \\
 & S_3 & = & x_0 & + & 4 & - & 2x_2 & - & x_2 \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0
\end{array}
$$

$$
\Longrightarrow
\begin{array}{llllllllll}
\max & -S_2 & - & x_1 & + & x_2 & & & & \\
\text{s.t.} & S_1 & = & 2 & - & 2x_1 & + & S_2 & & \\
 & x_0 & = & 3 & + & S_2 & - & x_2 & & \\
 & S_3 & = & 7 & - & 3x_1 & - & 2x_2 & + & S_2 \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0 & & \\
\hline
 & x_1 & = & 1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_1 & &
\end{array}
\Longrightarrow
\begin{array}{llllllllll}
\max & -2 & + & x_2 & - & \frac{1}{2}S_1 & - & \frac{1}{2}S_2 & & \\
\text{s.t.} & x_1 & = & 1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_1 & & \\
 & x_0 & = & 2 & - & x_2 & + & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 \\
 & S_3 & = & 4 & - & 2x_2 & + & \frac{3}{2}S_1 & - & \frac{1}{2}S_2 \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0 & & \\
\hline
 & x_2 & = & 2 & + & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 & - & x_0
\end{array}
$$

$$
\Longrightarrow
\begin{array}{llllllllll}
\max & 5 & - & S_1 & + & 2S_2 & & & & \\
\text{s.t.} & x_1 & = & 1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_1 & & \\
 & x_2 & = & 2 & + & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 & - & x_0 \\
 & S_3 & = & \frac{1}{2}S_1 & - & \frac{3}{2}S_2 & + & 2x_0 & & \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0
\end{array}
$$

Phase-1 Done

$$
\Longrightarrow
\begin{array}{lllllllll}
\max & 5 & - & S_1 & + & 2S_2 & & & \\
\text{s.t.} & x_1 & = & 1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_1 & \\
 & x_2 & = & 2 & + & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 & \\
 & S_3 & = & \frac{1}{2}S_1 & - & \frac{3}{2}S_2 & & & \\
 & \multicolumn{6}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0 \\
\hline
 & S_2 & = & \frac{1}{3}S_1 & - & \frac{2}{3}S_3 & & &
\end{array}
\Longrightarrow
\begin{array}{llllllll}
\max & 5 & - & \frac{1}{3}S_1 & - & \frac{4}{3}S_3 & & \\
\text{s.t.} & x_1 & = & 1 & - & \frac{1}{3}S_1 & - & \frac{1}{3}S_3 \\
 & x_2 & = & 2 & + & \frac{2}{3}S_1 & - & \frac{1}{3}S_3 \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0
\end{array}
$$

$\Longrightarrow$ Finally $z = 5$

$$x_1 = 1,\ \ x_2 = 2,\ \ S_1 = 0,\ \ S_2 = 0,\ \ S_3 = 0$$

(b)

$$
\begin{array}{lllllllll}
\max & 3x_1 & + & x_2 & & & & & \\
\text{s.t.} & S_1 & = & -1 & - & x_1 & + & x_2 & \\
 & S_2 & = & -3 & + & x_1 & + & x_2 & \\
 & S_3 & = & 2 & - & 2x_1 & - & x_2 & \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0 &
\end{array}
\Longrightarrow
\begin{array}{llllllllll}
\max & \multicolumn{9}{c}{\{-x_0\}} \\
\text{s.t.} & S_1 & = & x_0 & - & 1 & - & x_1 & + & x_2 \\
 & S_2 & = & x_0 & - & 3 & + & x_2 & + & x_2 \\
 & S_3 & = & x_0 & + & 2 & - & 2x_2 & - & x_2 \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0 \\
\hline
 & x_0 & = & 3 & - & x_1 & - & x_2 & + & S_2
\end{array}
$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & -3 & + & x_1 & + & x_2 & - & S_2 & & \\
\text{s.t.} & S_1 & = & 2 & - & 2x_1 & + & S_2 & & \\
 & x_0 & = & 3 & + & x_1 & - & x_2 & + & S_2 \\
 & S_3 & = & 5 & - & 3x_1 & - & 2x_2 & + & S_2 \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0 & & \\ \hline
 & x_1 & = & 1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_1 & &
\end{array}$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & -2 & + & x_2 & - & \frac{1}{2}S_1 & - & \frac{1}{2}S_2 & & \\
\text{s.t.} & x_1 & = & 1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_1 & & \\
 & x_0 & = & 2 & - & x_2 & + & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 \\
 & S_3 & = & 2 & - & 2x_2 & + & \frac{3}{2}S_1 & - & \frac{1}{2}S_2 \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0 & & \\ \hline
 & x_2 & = & 1 & + & \frac{3}{4}S_1 & - & \frac{1}{4}S_2 & - & \frac{1}{2}S_3
\end{array}$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & -1 & + & \frac{1}{4}S_1 & - & \frac{3}{4}S_2 & - & \frac{1}{2}S_3 & & \\
\text{s.t.} & x_1 & = & 1 & - & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 & & \\
 & x_0 & = & 1 & - & \frac{1}{4}S_1 & + & \frac{3}{4}S_2 & + & \frac{1}{2}S_3 \\
 & x_2 & = & 1 & + & \frac{3}{4}S_2 & - & \frac{1}{4}S_2 & - & \frac{1}{2}S_3 \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0 \\ \hline
 & S_1 & = & 2 & + & S_2 & - & 2x_1 & &
\end{array}$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & -0.5 & - & \frac{1}{2}x_1 & - & \frac{1}{2}S_2 & - & \frac{1}{2}S_3 & & \\
\text{s.t.} & S_1 & = & 2 & - & 2x_1 & + & \frac{1}{3}S_3 & & \\
 & x_0 & = & \frac{1}{2} & + & \frac{1}{2}x_1 & + & \frac{1}{2}S_2 & + & \frac{1}{2}S_3 \\
 & x_2 & = & \frac{5}{2} & - & \frac{3}{2}x_1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_3 \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0
\end{array}$$

$\Longrightarrow$ Optimal $\mathrm{obj} < 0$. The Linear Problem is infeasible

(c)

$$\begin{array}{rrcrcrcr}
\text{max} & 3x_1 & + & x_2 & & & & \\
\text{s.t.} & S_1 & = & -1 & - & x_1 & + & x_2 \\
 & S_2 & = & -3 & + & x_1 & + & x_2 \\
 & S_3 & = & 2 & - & 2x_1 & + & x_2 \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0
\end{array}$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & & & \{-x_0\} & & & & & & \\
\text{s.t.} & S_1 & = & x_0 & - & 1 & - & x_1 & + & x_2 \\
 & S_2 & = & x_0 & - & 3 & + & x_2 & + & x_2 \\
 & S_3 & = & x_0 & + & 2 & - & 2x_2 & + & x_2 \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0 \\ \hline
 & x_0 & = & 3 & - & x_1 & - & x_2 & + & S_2
\end{array}$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & -3 & + & x_1 & + & x_2 & - & S_2 & & \\
\text{s.t.} & S_1 & = & 2 & - & 2x_1 & + & S_2 & & \\
 & x_0 & = & 3 & - & x_1 & - & x_2 & + & S_2 \\
 & S_3 & = & 5 & - & 3x_1 & + & S_2 & & \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0 \\ \hline
 & x_1 & = & 1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_1 & &
\end{array}$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & -2 & + & x_2 & - & \frac{1}{2}S_1 & - & \frac{1}{2}S_2 & & \\
\text{s.t.} & x_1 & = & 1 & + & \frac{1}{2}S_2 & - & \frac{1}{2}S_1 & & \\
 & x_0 & = & 2 & - & x_2 & + & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 \\
 & S_3 & = & 2 & + & \frac{3}{2}S_1 & - & \frac{3}{2}S_2 & - & \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0 \\ \hline
 & x_2 & = & 2 & + & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 & - & x_0
\end{array}$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & & & \{-x_0\} & & & & & & \\
\text{s.t.} & x_1 & = & 1 & - & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 & & \\
 & x_2 & = & 2 & + & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 & - & x_0 \\
 & S_3 & = & 2 & + & \frac{3}{2}S_1 & - & \frac{3}{2}S_2 & & \\
 & \multicolumn{7}{c}{S_1, S_2, S_3, x_0, x_1, x_2} & \geq & 0
\end{array}$$

$\Longrightarrow$ Phase-1 Done

$$\Longrightarrow \quad \begin{array}{rrcrcrcr}
\text{max} & 5 & - & S_1 & + & 2S_2 & & \\
\text{s.t.} & x_1 & = & 1 & - & \frac{1}{2}S_1 & + & \frac{1}{2}S_2 \\
 & x_2 & = & 2 & + & \frac{1}{2}S_1 & - & \frac{1}{2}S_2 \\
 & S_3 & = & 2 & + & \frac{3}{2}S_1 & - & \frac{3}{2}S_2 \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0 \\ \hline
 & S_2 & = & \frac{4}{3} & + & S_1 & - & \frac{2}{3}S_3
\end{array}$$

$$\Longrightarrow \quad \begin{array}{rrcrcrcrcr}
\text{max} & \frac{33}{5} & + & S_1 & - & \frac{4}{3}S_3 & & & & \\
\text{s.t.} & x_1 & = & \frac{5}{3} & + & \frac{1}{3}S_3 & & & & \\
 & x_2 & = & \frac{8}{3} & + & S_1 & - & \frac{1}{3}S_3 & & \\
 & S_2 & = & \frac{4}{3} & + & S_1 & - & \frac{2}{3}S_3 & & \\
 & \multicolumn{5}{c}{S_1, S_2, S_3, x_1, x_2} & \geq & 0 & &
\end{array}$$

$$\Longrightarrow \quad \begin{gathered} x_1 = \tfrac{5}{3},\ x_2 = \tfrac{8}{3} + t,\ S_1 = t,\ S_2 = \tfrac{4}{3} + t,\ S_3 = 0 \\ z = \tfrac{33}{5} + t \end{gathered}$$

Where t can be any positive number, unbound

4. Consider the following dictionaries in a cycling example.

The initial dictionary:

$$\begin{array}{rcrrrrrrrr} x_5 & = & & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\ x_6 & = & & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\ x_7 & = & 1 & - & x_1 & & & & & & \\ z & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \end{array}$$

After the first iteration:

Dictionary D:

$$\begin{array}{rcrrrrrrrr} x_1 & = & & & 11x_2 & + & 5x_3 & - & 18x_4 & - & 2x_5 \\ x_6 & = & & - & 4x_2 & - & 2x_3 & + & 8x_4 & + & x_5 \\ x_7 & = & 1 & - & 11x_2 & - & 5x_3 & + & 18x_4 & + & 2x_5 \\ z & = & & & 53x_2 & + & 41x_3 & - & 204x_4 & - & 20x_5 \end{array}$$

…

…

After the fifth iteration:

Dictionary D^*:

$$\begin{array}{rcrrrrrrrr} x_5 & = & & & 9x_6 & + & 4x_1 & - & 8x_2 & - & 2x_3 \\ x_4 & = & & - & x_6 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 \\ x_7 & = & 1 & & & - & 1x_1 & & & & \\ z & = & & & 24x_6 & + & 22x_1 & - & 93x_2 & - & 21x_3 \end{array}$$

Let B be the index set of the basic variables in D, and let B^* be the index set of the basic variables in D^*. At the first iteration, let x_2 be the entering variable, while x_6 is obviously the leaving variable. Therefore, $x_2 = t$, $x_3 = x_4 = x_5 = 0,\ x_1 = 11t,\ x_6 = -4t,\ x_7 = 1 - 11t,\ z = 53t$. Verify that at Dictionary D^*, $z = 53t$ if x_j in D^* are substituted by x_j in D and $c_j^* = 0\ \forall j \in \mathrm{B}^*$.

Let

$$x_2 = t,\ x_3 = x_4 = x_5 = 0,\ x_1 = 11t,\ x_6 = -4t \text{ and } x_7 = 1 - 11t$$

Then

$$z = 24 \times (-4t) + 22 \times (11t) - 93 \times (t) - 21 \times (0) = 53t$$

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5. For a maximization problem, let D denote a dictionary in which x_t leaves the basis and x_s will enter the basis.

D:

$$x_i = b_i - \sum_{j \notin \mathrm{B}} a_{ij} x_j \qquad i \in \mathrm{B}$$

$$z = v + \sum_{j \notin \mathrm{B}} c_j x_j$$

where B is the index set of basis variables in D, and $s \notin \mathrm{B}$ and $t \in \mathrm{B}$.

Now let D^* be a dictionary in which x_t enters the basis.

D^*:

$$x_i = b_i^* - \sum_{j \notin \mathrm{B}^*} a_{ij}^* x_j \qquad i \in \mathrm{B}^*$$

$$z = v^* + \sum_{j \notin \mathrm{B}^*} c_j^* x_j$$

Show $c_t^* a_{ts} > 0$. (Hint: you need to show $c_t^* > 0$ and $a_{ts} > 0$)

Since C_s is always positive, we can choose S to enter the basis

$$x_t = -a_{ts} x_s + \cdots$$

Therefore $a_{ts} > 0$ and we can obtain the minimal ratio $x_s = \frac{x_t}{a_{ts}}$ if choosing x_t to leave.

Then $c_t^* = C_s \times \frac{x_t}{a_{ts}} a_{ts} > 0$ since $C_s > 0$ and $a_{ts} > 0$.

Finally $c_t^* a_{ts} > 0$ is proven.