Introduction to Optimization

Homework #3 – Due Wednesday, October 25

1. Use the simplex method to describe all the optimal solutions of the following problem.

$$\begin{aligned} \text{Maximize } & 2x_1 + 3x_2 + 5x_3 + 4x_4 \\ \text{s.t. } & x_1 + 2x_2 + 3x_3 + x_4 \leq 5 \\ & x_1 + x_2 + 2x_3 + 3x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

2. Solve the following LP problem.

$$\begin{aligned} \text{Maximize } & x_1 + 3x_2 - x_3 \\ \text{s.t. } & 2\,x_1 + 2x_2 - x_3 \leq 10 \\ & 3\,x_1 - 2x_2 + x_3 \leq 10 \\ & x_1 - 3x_2 + x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

3. Solve the following problems by the two-phase simplex method:

(a)

$$\begin{aligned} \text{Maximize } & 3x_1 + x_2 \\ \text{s.t. } & x_1 - x_2 \leq -1 \\ & -x_1 - x_2 \leq -3 \\ & 2x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} \text{Maximize } & 3x_1 + x_2 \\ \text{s.t. } & x_1 - x_2 \leq -1 \\ & -x_1 - x_2 \leq -3 \\ & 2x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(c)

$$\begin{aligned} \text{Maximize } & 3x_1 + x_2 \\ \text{s.t. } & x_1 - x_2 \leq -1 \\ & -x_1 - x_2 \leq -3 \\ & 2x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

4. Consider the following dictionaries in a cycling example.
The initial dictionary:

$$\begin{aligned}
x_5 &= \quad -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= \quad -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1 \\
z &= \quad 10x_1 - 57x_2 - 9x_3 - 24x_4.
\end{aligned}$$

After the first iteration:

Dictionary D:

$$\begin{aligned}
x_1 &= \quad 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= \quad -4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 \\
z &= \quad 53x_2 + 41x_3 - 204x_4 - 20x_5.
\end{aligned}$$

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After the fifth iteration:

Dictionary D*:

$$\begin{aligned}
x_5 &= 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_4 &= -x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_7 &= 1 - x_1 \\
z &= 24x_6 + 22x_1 - 93x_2 - 21x_3.
\end{aligned}$$

Let B be the index set of the basic variables in D, and let B^* be the index set of the basic variables in D*. At the first iteration, let x_2 be the entering variable, while x_6 is obviously the leaving variable. Therefore, $x_2 = t,\ x_3 = x_4 = x_5 = 0,\ x_1 = 11t,\ x_6 = -4t,\ x_7 = 1 - 11t,$ $z = 53t$. Verify that at Dictionary D*, $z = 53t$ if x_j in D* are substituted by x_j in D and $c_j^* = 0\ \forall j \in B^*$.

5. For a maximization problem, let D denote a dictionary in which x_t leaves the basis and x_s will enter the basis.

$$D: \quad x_i = b_i - \sum_{j \notin B} a_{ij}x_j \qquad i \in B$$

$$z = v + \sum_{j \notin B} c_j x_j$$

where B is the index set of basis variables in D, and $s \notin B$ and $t \in B$.

Now let D^* be a dictionary in which x_t enters the basis.

$$D^*: \quad x_i = b_i^* - \sum_{j \notin B^*} a_{ij}^* x_j \qquad i \in B^*$$

$$z = v^* + \sum_{j \notin B^*} c_j^* x_j$$

Show $c_t^* a_{ts} > 0$. (Hint: you need to show $c_t^* > 0$ and $a_{ts} > 0$)