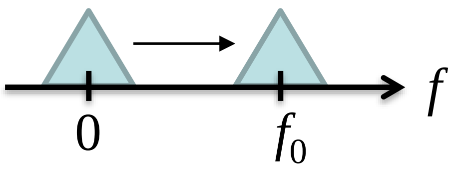
VIII. Motions on the Time-Frequency Distribution

Fourier spectrum 為 1-D form，只有二種可能的運動或變形：

Modulation $e^{j2\pi f_0 t}x(t) \xrightarrow{FT} X(f-f_0)$

Scaling $x(t/a) \xrightarrow{FT} |a|X(af)$



Time-frequency analysis 為 2-D，在 2-D 平面上有多種可能的運動或變形

(1) Horizontal shifting

(2) Vertical shifting

(3) Dilation

(4) Shearing

(5) Generalized Shearing

(6) Rotation

(7) Twisting

8-1 Basic Motions

(1) Horizontal Shifting

$$x(t-t_0) \to S_x(t-t_0, f)e^{-j2\pi f t_0} \quad \text{,STFT, Gabor}$$

$$\to W_x(t-t_0, f) \quad \text{,Wigner}$$

(2) Vertical Shifting

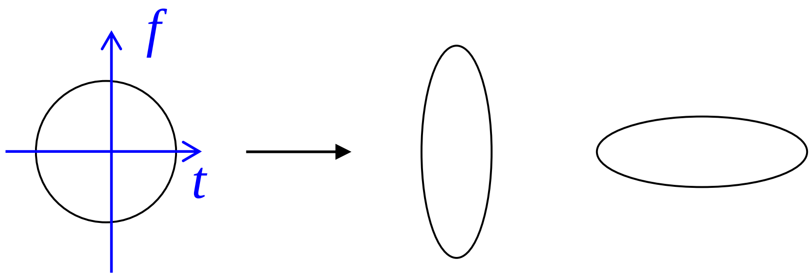
$$e^{j2\pi f_0 t}x(t) \to S_x(t, f-f_0) \quad \text{,STFT,Gabor}$$

$$\to W_x(t, f-f_0) \quad \text{,Wigner}$$

(3) Dilation (scaling)

$$\frac{1}{\sqrt{|a|}}x\left(\frac{t}{a}\right) \to \approx S_x\left(\frac{t}{a}, af\right) \text{ ,STFT,Gabor}$$

$$\to W_x\left(\frac{t}{a}, af\right) \text{ ,WDF}$$

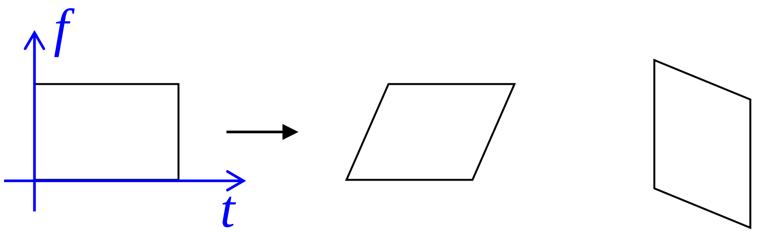


(4) Shearing

$$x(t) = e^{j\pi a t^2} y(t)$$

$$S_x(t, f) \approx S_y(t, f - at) \text{ ,STFT,Gabor}$$

$$W_x(t, f) = W_y(t, f - at) \text{ ,WDF}$$



$$x(t) = e^{j\pi \frac{t^2}{a}} * y(t)$$

$$S_x(t, f) \approx S_y(t - af, f) \text{ ,STFT,Gabor}$$

$$W_x(t, f) = W_y(t - af, f) \text{ ,WDF}$$

(Proof): When $x(t)=e^{j\pi at^2}y(t)$,

$$
\begin{aligned}
W_x(t,f) &= \int_{-\infty}^{\infty} x(t+\tau/2)\,x^*(t-\tau/2)\,e^{-j2\pi\tau f}\cdot d\tau \\
&= \int_{-\infty}^{\infty} e^{j\pi a(t+\tau/2)^2} e^{-j\pi a(t-\tau/2)^2} y(t+\tau/2)\,y^*(t-\tau/2)\,e^{-j2\pi\tau f}d\tau \\
&= \int_{-\infty}^{\infty} e^{j2\pi at\tau} y(t+\tau/2)\,y^*(t-\tau/2)\,e^{-j2\pi\tau f}d\tau \\
&= \int_{-\infty}^{\infty} y(t+\tau/2)\,y^*(t-\tau/2)\,e^{-j2\pi\tau(f-at)}d\tau \\
&= W_y(t,f-at)
\end{aligned}
$$

(5) Generalized Shearing

$$x(t) = e^{j\phi(t)}y(t)$$ 的影響?

$$\phi(t) = \sum_{k=0}^{n} a_k t^k$$

$$S_x(t, f) \cong S_y(t, f - \qquad\qquad\qquad)$$,STFT,Gabor

$$W_x(t, f) \cong W_y(t, f - \qquad\qquad\qquad)$$,WDF

J. J. Ding, S. C. Pei, and T. Y. Ko, "Higher order modulation and the efficient sampling algorithm for time variant signal," *European Signal Processing Conference*, pp. 2143-2147, Bucharest, Romania, Aug. 2012.

J. J. Ding and C. H. Lee, "Noise removing for time-variant vocal signal by generalized modulation," *APSIPA ASC*, Kaohsiung, Taiwan, Oct. 2013

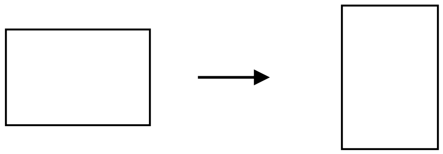
8-2 Rotation by $\pi/2$: Fourier Transform

$$X(f) = FT(x(t))$$

$$|S_X(t,f)| \approx |S_x(-f,t)| \qquad \text{,STFT}$$

$$G_X(t,f) = G_x(-f,t)e^{-j2\pi ft} \qquad \text{,Gabor}$$

$$W_X(t,f) = W_x(-f,t) \qquad \text{,WDF}$$



(clockwise rotation by 90°)

Strictly speaking, the rec-STFT have no rotation property.

For Gabor transforms, if

$$\boxed{G_x(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau} \;,$$

$$G_X(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} X(\tau)\,d\tau \qquad X(f)=FT\left[x(t)\right]=\int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$

then $\boxed{G_X(t,f)=G_x(-f,t)e^{-j2\pi t f}}$

(clockwise rotation by 90° for amplitude)

If we define the Gabor transform as

$$\boxed{G_x(t,f)=e^{j\pi f t}\int_{-\infty}^{\infty}e^{-\pi(\tau-t)^2}e^{-j2\pi f\tau}x(\tau)\,d\tau},$$

and $G_X(t,f)=e^{j\pi f t}\int_{-\infty}^{\infty}e^{-\pi(\tau-t)^2}e^{-j2\pi f\tau}X(\tau)\,d\tau$

then $\boxed{G_X(t,f)=G_x(-f,t)}$

If $W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j2\pi\tau f} \cdot d\tau$ is the WDF of $x(t)$,

$W_X(t,f) = \int_{-\infty}^{\infty} X(t+\tau/2) \cdot X^*(t-\tau/2) e^{-j2\pi\tau f} \cdot d\tau$ is the WDF of $X(f)$,

then $W_X(t,f) = W_x(-f,t)$

(clockwise rotation by 90°)

還有哪些 time-frequency distribution 也有類似性質？

- If $X(f) = IFT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{j2\pi f t} dt$, then

$$W_X(t, f) = W_x(f, -t), \qquad G_X(t, f) = G_x(f, -t) e^{j2\pi t f}$$

(counterclockwise rotation by 90°).

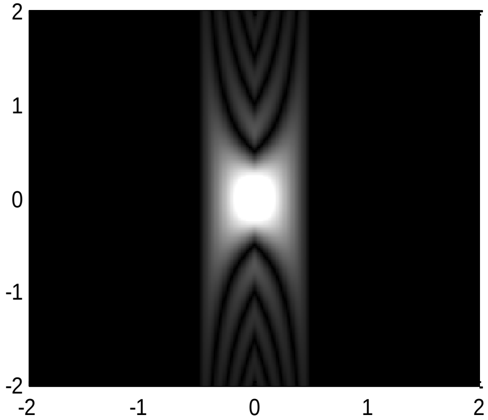
- If $X(f) = x(-t)$, then

$$W_X(t, f) = W_x(-t, -f) \ , \qquad G_X(t, f) = G_x(-t, -f) \, .$$

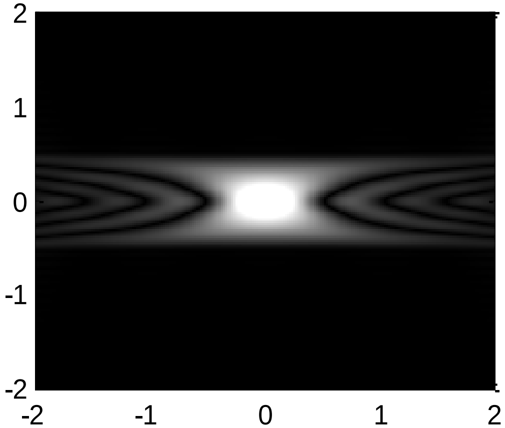
(rotation by 180°).

Examples: $x(t) = \Pi(t)$, $X(f) = FT[x(t)] = \text{sinc}(f)$.

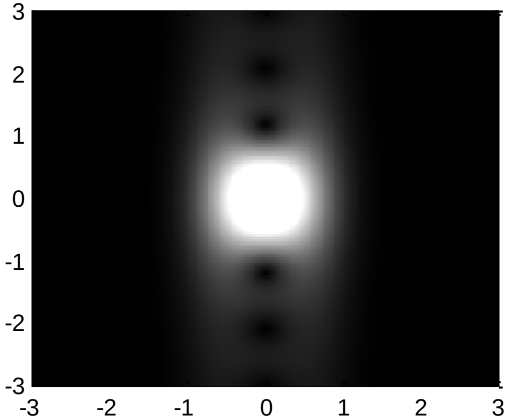
WDF of $\Pi(t)$



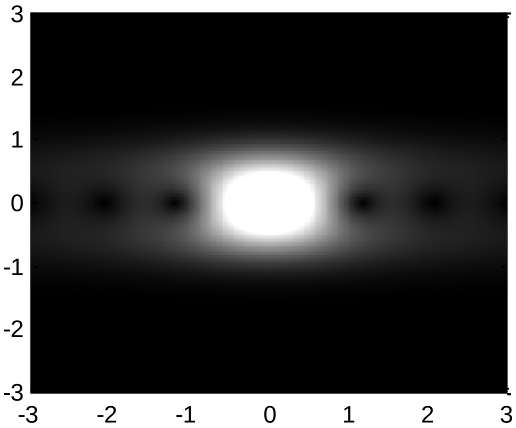
WDF of sinc(f)



Gabor transform of $\Pi(t)$



Gabor transform of sinc(f)



If a function is an eigenfunction of the Fourier transform,

$$\int_{-\infty}^{\infty} e^{-j2\pi f t} x(t)\, dt = \lambda x(f) \qquad \lambda = 1, -j, -1, j$$

then its WDF and Gabor transform have the property of

$$W_x(t, f) = W_x(f, -t) \qquad \left|G_x(t, f)\right| = \left|G_x(f, -t)\right|$$

(轉了 90^{o}之後，和原來還是一樣）

Example: Gaussian function

$$\exp\left(-\pi t^2\right)$$

Hermite-Gaussian function

$$\phi_m(t) = \exp\left(-\pi t^2\right) H_m(t)$$

Hermite polynomials: $H_m(t) = C_m e^{\pi t^2} \dfrac{d^m}{dt^m} e^{-2\pi t^2}$, C_m is some constant,

$$H_0(t) = 1 \qquad H_1(t) = t \qquad H_2(t) = 4\pi t^2 - 1$$

$$H_3(t) = 4\pi t^3 - 3t \qquad H_4(t) = 16\pi^2 t^4 - 24\pi t^2 + 3$$

$\int_{-\infty}^{\infty} e^{-2\pi t^2} H_m(t) H_n(t) = D_m \delta_{m,n}$, D_m is some constant,

$\delta_{m,n} = 1$ when $m = n$, $\delta_{m,n} = 0$ otherwise.

[Ref] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 1990.

Hermite-Gaussian functions are eigenfunctions of the Fourier transform

$$\int_{-\infty}^{\infty} \phi_m(t) e^{-j2\pi f t} dt = (-j)^m \phi_m(f)$$

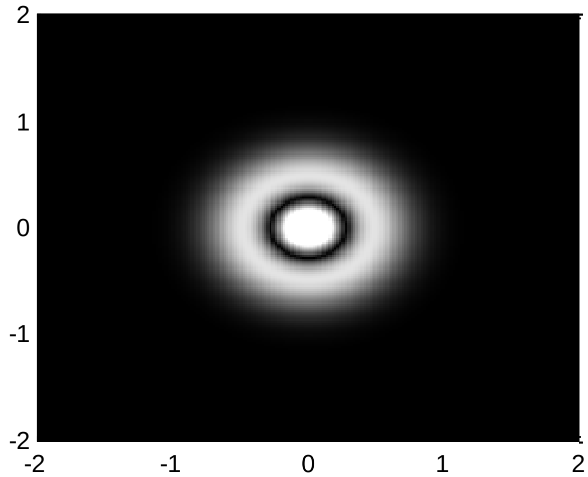
Any eigenfunction of the Fourier transform can be expressed as the form of

$$k(t) = \sum_{q=0}^{\infty} a_{4q+r} \phi_{4q+r}(t)$$

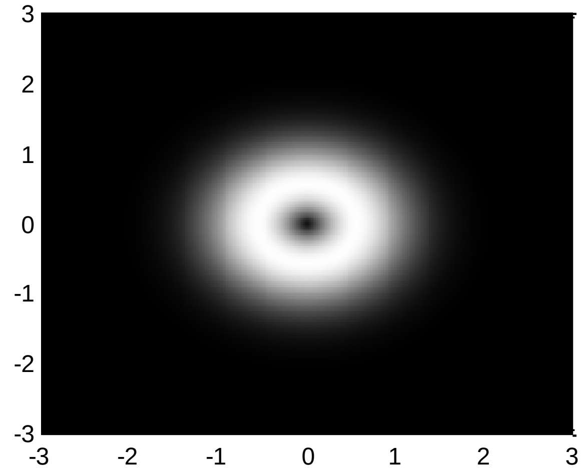
where r = 0, 1, 2, or 3,
a_{4q+r} are some constants

$$\int_{-\infty}^{\infty} k(t) e^{-j2\pi f t} dt = (-j)^r k(f)$$

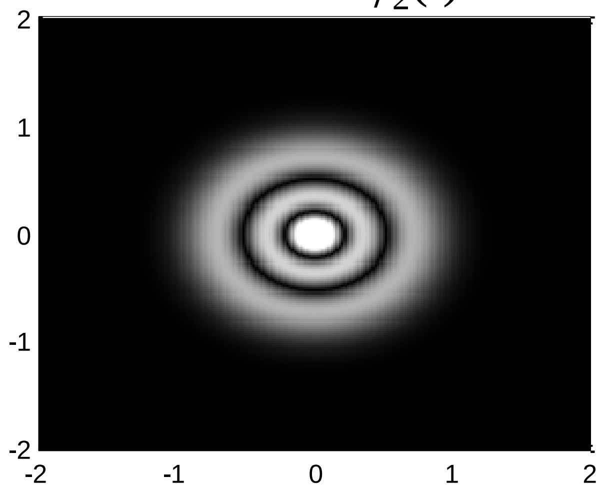
WDF for $\phi_1(t)$



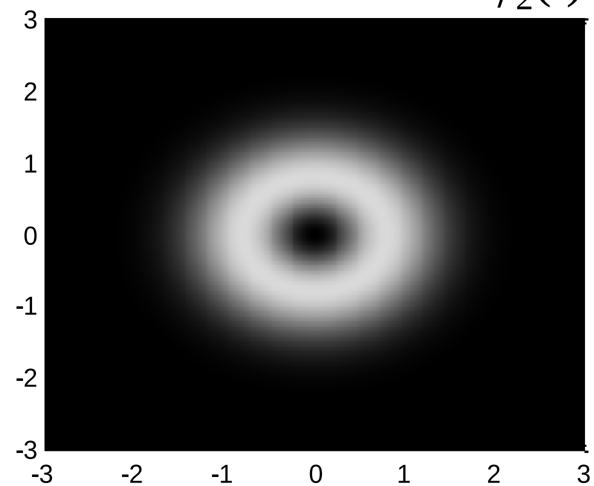
Gabor transform for $\phi_1(t)$



WDF for $\phi_2(t)$



Gabor transform for $\phi_2(t)$



Problem: How to rotate the time-frequency distribution by the angle other than $\pi/2$, π, and $3\pi/2$?

8-3 Rotation: Fractional Fourier Transforms (FRFTs)

$$X_\phi(u)=\sqrt{1-j\cot\phi}\, e^{j\pi\cot\phi\cdot u^2}\int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t}\, e^{j\pi\cdot\cot\phi\cdot t^2}\, x(t)dt\ , \qquad \phi=0.5a\pi$$

When $\phi = 0.5\pi$, the FRFT becomes the FT.

Additivity property:

If we denote the FRFT as O_F^ϕ (i.e., $X_\phi(u)=O_F^\phi[x(t)]$)

then $O_F^\sigma\left\{O_F^\phi[x(t)]\right\}=O_F^{\phi+\sigma}[x(t)]$

Physical meaning: Performing the FT a times.

Another definition $X_\phi(u)=\sqrt{\frac{1-j\cot\phi}{2\pi}}\,e^{j\frac{\cot\phi}{2}\cdot u^2}\int_{-\infty}^{\infty}e^{-j\csc\phi\cdot u\,t}e^{j\frac{\cot\phi}{2}\cdot t^2}x(t)dt$

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] N. Wiener, "Hermitian polynomials and Fourier analysis," *Journal of Mathematics Physics MIT*, vol. 18, pp. 70-73, 1929.

[Ref] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.

[Ref] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.

[Ref] S. C. Pei and J. J. Ding, "Closed form discrete fractional and affine Fourier transforms," *IEEE Trans. Signal Processing*, vol. 48, no. 5, pp. 1338-1353, May 2000.

$$FT\left[x(t)\right] = X(f)$$

$$FT\left\{FT\left[x(t)\right]\right\} = x(-t)$$

$$FT\left(FT\left\{FT\left[x(t)\right]\right\}\right) = X(-f) = IFT\left[f(t)\right]$$

$$FT\left[FT\left(FT\left\{FT\left[x(t)\right]\right\}\right)\right] = x(t)$$

What happen if we do the FT non-integer times?

Physical Meaning:

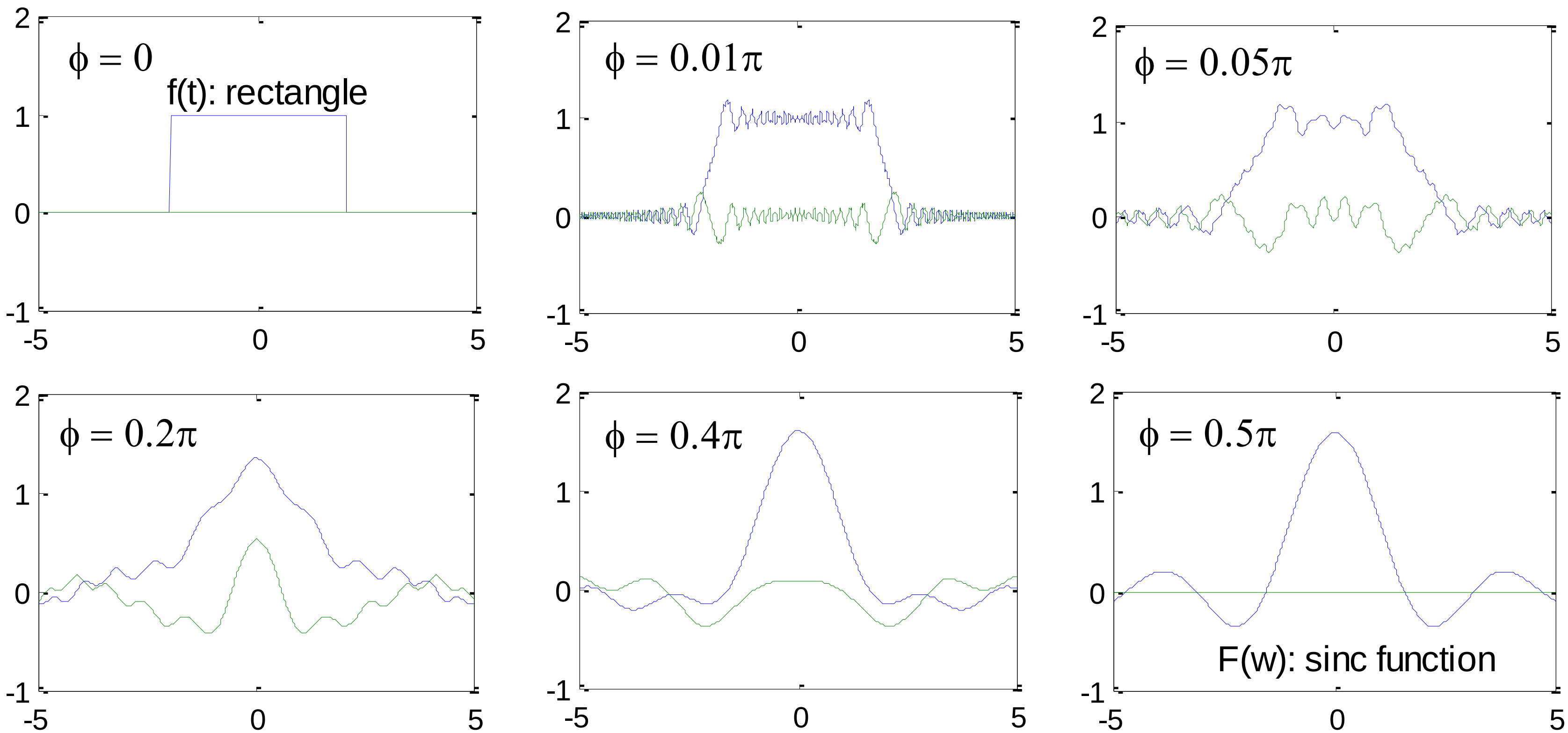
Fourier Transform: time domain $\rightarrow$ frequency domain

Fractional Fourier transform: time domain $\rightarrow$ fractional domain

Fractional domain: the domain between time and frequency

(partially like time and partially like frequency)

Experiment:



ϕ = 0
f(t): rectangle
ϕ = 0.01π
ϕ = 0.05π
ϕ = 0.2π
ϕ = 0.4π
ϕ = 0.5π
F(w): sinc function
2
1
0
-1
-5
0
5

Time domain	Frequency domain	fractional domain
Modulation	Shifting	Modulation + Shifting
Shifting	Modulation	Modulation + Shifting
Differentiation	$\times j2\pi f$	Differentiation and $\times j2\pi f$
$\times -j2\pi f$	Differentiation	Differentiation and $\times -j2\pi f$

$$\frac{d}{dt}x(t) \xrightarrow{FT} j2\pi f\, X(f)$$

$$\frac{d}{dt}x(t) \xrightarrow{fractional\ FT} j2\pi u\, X(u)\sin\phi + \frac{d}{du}X(u)\cos\phi$$

[Theorem] The fractional Fourier transform (FRFT) with angle ϕ is equivalent to the clockwise rotation operation with angle ϕ for the Wigner distribution function (or for the Gabor transform)

$$\boxed{\textbf{FRFT}_{\phi} = \curvearrowright \textbf{ with angle } \phi}$$

For the WDF

If $W_x(t, f)$ is the WDF of $x(t)$, and $W_{X\phi}(u, v)$ is the WDF of $X_\phi(u)$, ($X_\phi(u)$ is the FRFT of $x(t)$), then

$$\boxed{W_{X_\phi}(u,v) = W_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

For the Gabor transform (with standard definition)

If $G_x(t, f)$ is the Gabor transform of $x(t)$,

and $G_{X\phi}(u, v)$ is the Gabor transform of $X_\phi(u)$, then

$$\boxed{G_{X_\phi}(u,v) = e^{j[-2\pi uv\sin^2\phi + \pi(u^2-v^2)\sin(2\phi)/2]} G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

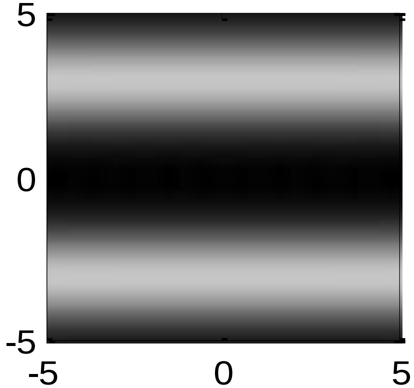
$$\boxed{\left|G_{X_\phi}(u,v)\right| = \left|G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)\right|}$$

For the Gabor transform (with another definition on page 216)

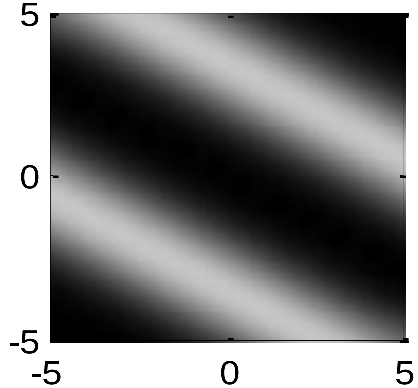
$$\boxed{G_{X_\phi}(u,v) = G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

The Cohen's class distribution and the Gabor-Wigner transform also have the rotation property

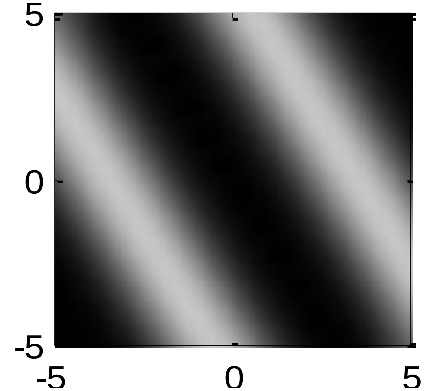
The Gabor Transform for the FRFT of a cosine function



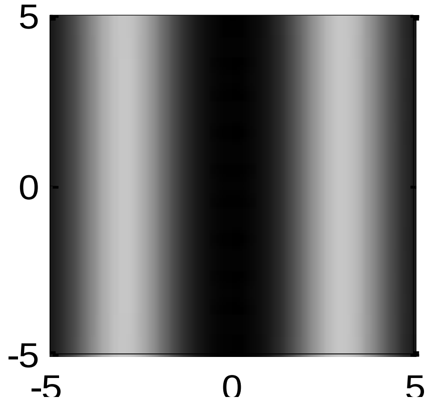
(a) $\phi = 0$



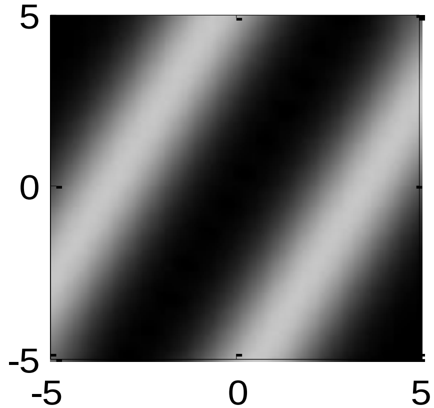
(b) $\phi = \pi/6$



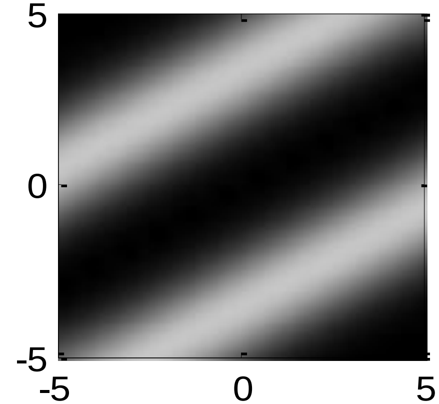
(c) $\phi = 2\pi/6$



(d) $\phi = 3\pi/6$

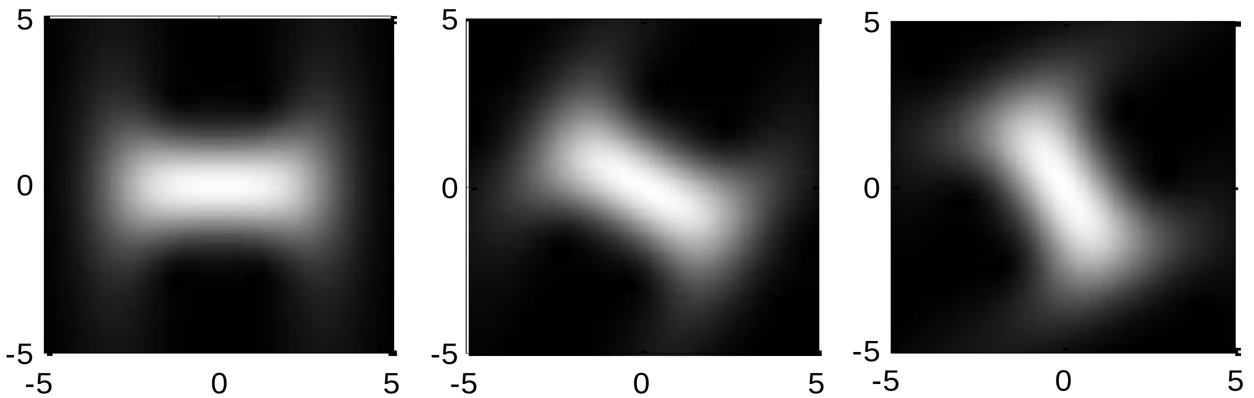


(e) $\phi = 4\pi/6$

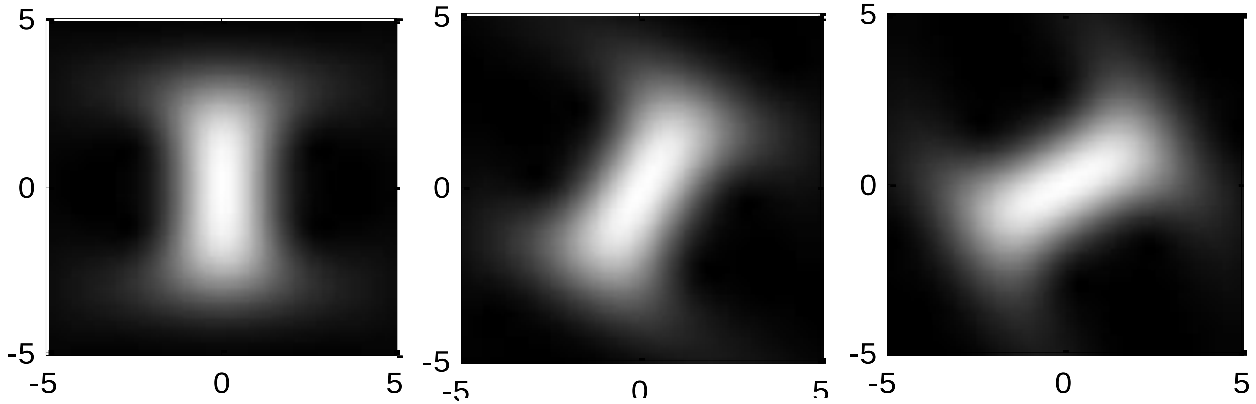


(f) $\phi = 5\pi/6$

The Gabor Transform for the FRFT of a rectangular function.



(a) $\phi = 0$ (b) $\phi = \pi/6$ (c) $\phi = 2\pi/6$



(d) $\phi = 3\pi/6$ (e) $\phi = 4\pi/6$ (f) $\phi = 5\pi/6$

8-4 Twisting: Linear Canonical Transform (LCT)

$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}}\, e^{j\pi \frac{d}{b} u^2} \int_{-\infty}^{\infty} e^{-j2\pi \frac{1}{b} u t} e^{j\pi \frac{a}{b} t^2}\, x(t)\, dt$$ when $b \neq 0$

$$X_{(a,0,c,d)}(u) = \sqrt{d} \cdot e^{j\pi c d u^2}\, x(d u)$$ when $b = 0$

$ad - bc = 1$ should be satisfied

Four parameters a, b, c, d

Additivity property of the WDF

If we denote the LCT by $O_F^{(a,b,c,d)}$, i.e., $X_{(a,b,c,d)}(u) = O_F^{(a,b,c,d)}[x(t)]$

then $$O_F^{(a_2,b_2,c_2,d_2)}\left\{O_F^{(a_1,b_1,c_1,d_1)}[x(t)]\right\} = O_F^{(a_3,b_3,c_3,d_3)}[x(t)]$$

where $$\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

[Ref] K. B. Wolf, "*Integral Transforms in Science and Engineering*," Ch. 9: Canonical transforms, New York, Plenum Press, 1979.

If $W_{X_{(a,b,c,d)}}(u,v)$ is the WDF of $X_{(a,b,c,d)}(u)$, where $X_{(a,b,c,d)}(u)$ is the LCT of $x(t)$, then

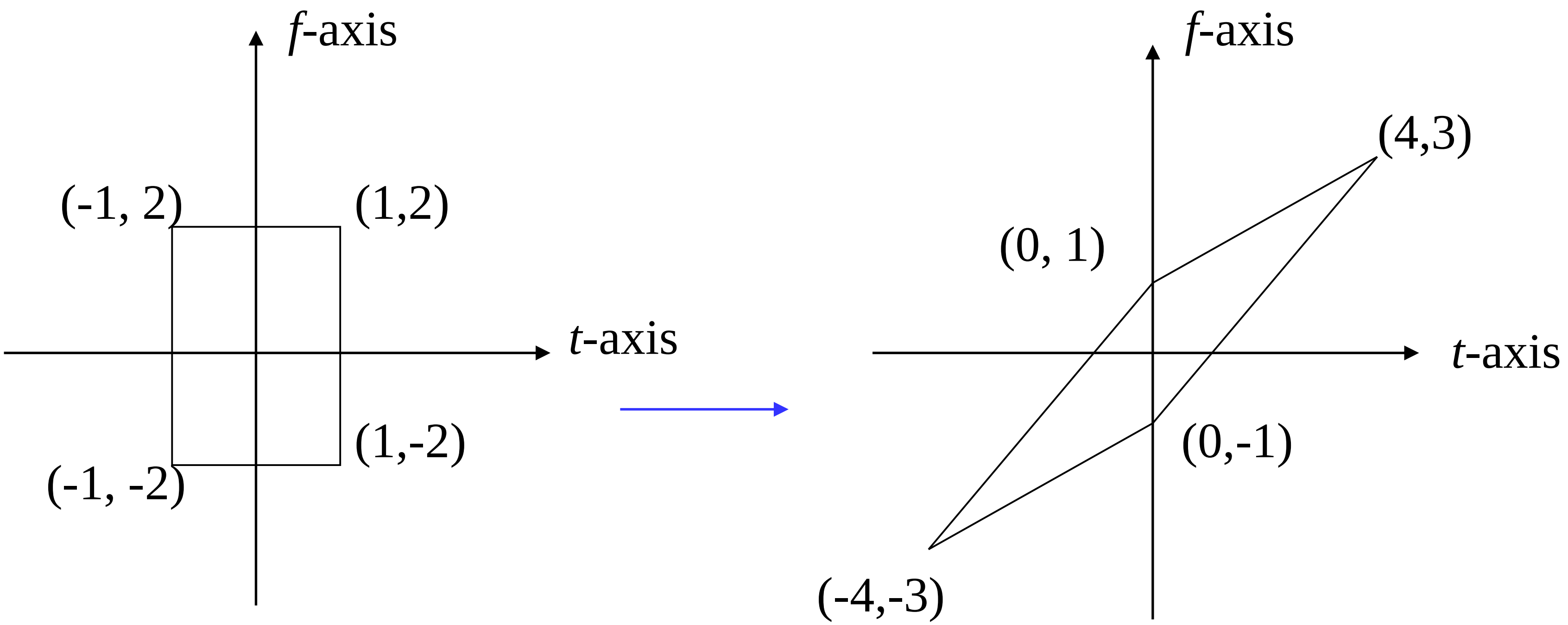
$$W_{X_{(a,b,c,d)}}(u,v) = W_x(du - bv\,, -cu + av)$$

$$W_{X_{(a,b,c,d)}}(au + bv\,, cu + dv) = W_x(u,v)$$

LCT == twisting operation for the WDF

The Cohen's class distribution also has the twisting operation.

我們可以自由的用 LCT 將一個中心在 (0, 0) 的平行四邊形的區域，扭曲成另外一個面積一樣且中心也在 (0, 0) 的平行四邊形區域。



linear canonical transform

- fractional Fourier transform: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$
  - $\phi = \pi/2$: Fourier transform
  - $\phi = 0$: identity operation
  - $\phi = -\pi/2$: inverse Fourier transform
- Fresnel transform: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$
- chirp multiplication: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}$

  $$X_{(a,0,c,d)}(u) = e^{j\pi \tau u^2} x(u)$$
- scaling: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/\sigma & 0 \\ 0 & \sigma \end{bmatrix}$

附錄八　Linear Canonical Transform 和光學系統的關係

(1) Fresnel Transform (電磁波在空氣中的傳播)

$$U_o(x,y) = -\frac{i}{\lambda}\frac{e^{ikz}}{z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}\left[(x-x_i)^2+(y-y_i)^2\right]} U_i(x_i,y_i)\,dx_i dy_i$$

$k = 2\pi/\lambda$: wave number　　λ: wavelength　　z: distance of propagation

$$U_o(x,y) = e^{ikz}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}(y-y_i)^2}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}(x-x_i)^2} U_i(x_i,y_i)\,dx_i dy_i$$

(2 個 1-D 的 LCT)

Fresnel transform 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$

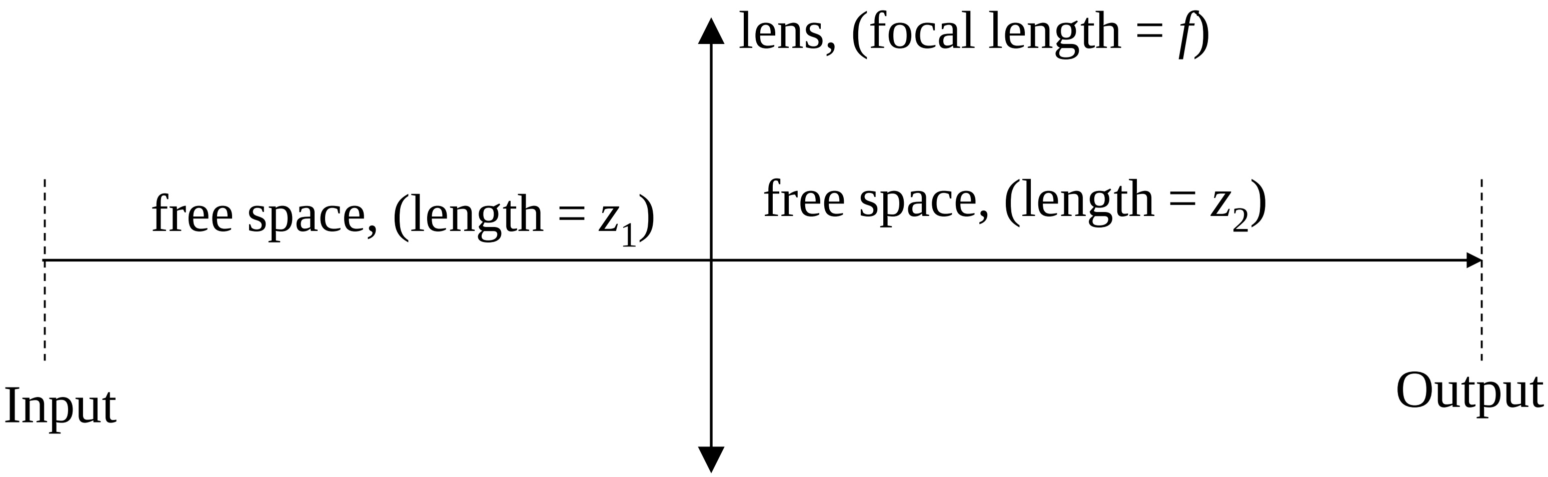
(2) Spherical lens, refractive index = n

$$U_o(x,y) = e^{ikn\Delta} e^{-j\frac{k}{2f}\left[x^2+y^2\right]} U_i(x,y)$$

f : focal length Δ: thickness of length

經過 lens 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix}$ 的情形

(3) Free space 和 Spherical lens 的綜合



Input 和 output 之間的關係，可以用 LCT 表示

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\dfrac{z_2}{f} & \lambda(z_1+z_2)-\dfrac{\lambda z_1 z_2}{f} \\ -\dfrac{1}{\lambda f} & 1-\dfrac{z_1}{f} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1-\dfrac{z_2}{f} & \lambda(z_1+z_2)-\dfrac{\lambda z_1 z_2}{f} \\ -\dfrac{1}{\lambda f} & 1-\dfrac{z_1}{f} \end{bmatrix}$$

$z_1 = z_2 = 2f \rightarrow$ 即高中物理所學的「倒立成像」

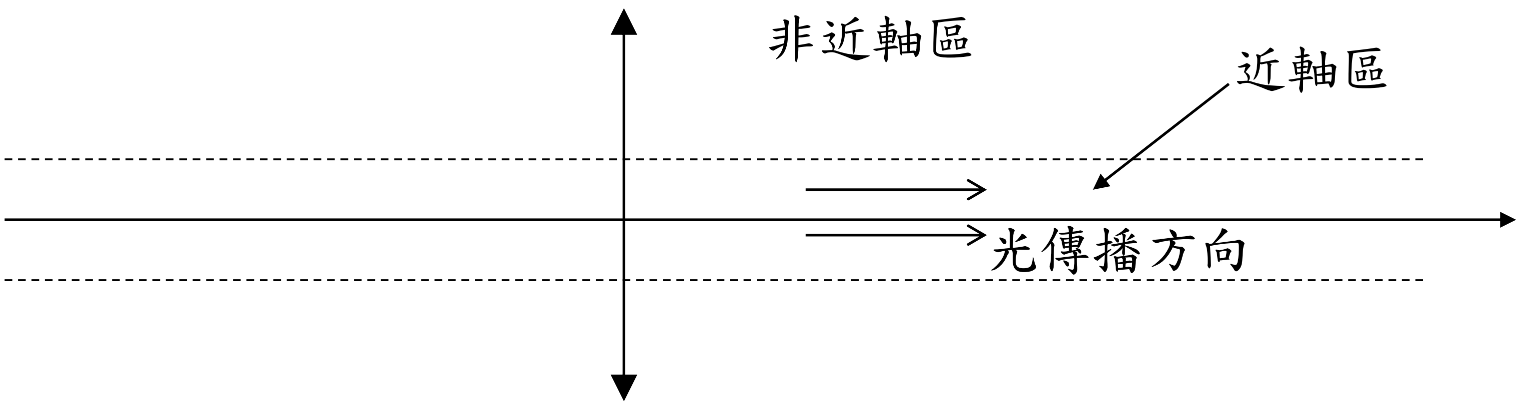
$z_1 = z_2 = f \rightarrow$ Fourier Transform + Scaling

$z_1 = z_2 \rightarrow$ fractional Fourier Transform + Scaling

用 LCT 來分析光學系統的好處：

只需要用到 2×2 的矩陣運算，避免了複雜的物理理論和數學積分

但是 LCT 來分析光學系統的結果，只有在「近軸」的情形下才準確



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