Duality (I)

Lecture 8, Nonlinear Programming

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The Lagrangian of an optimization problem (1/2)

- Consider an optimization problem, not necessarily convex, in the standard form:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, ..., m \\ & h_i(x) = 0, i = 1, ..., p, \end{aligned}$$

with variable $x \in \mathbf{R}^n$, with a nonempty domain

$$\mathcal{D} = \left(\bigcap_{i=0}^{m} \mathbf{dom}\ f_i \right) \cap \left(\bigcap_{i=1}^{p} \mathbf{dom}\ h_i \right),$$

and the optimal value being p^*.

The Lagrangian of an optimization problem (2/2)

- The **Lagrangian** associated with the problem is defined as $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x),$$

  with $\mathbf{dom}\ L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$.
- We refer to λ_i and ν_i as the Lagrange multiplier associated with the ith inequality constraint $f_i(x) \leq 0$ and that with the ith equality constraint $h_i(x) = 0$, respectively.
- The vectors λ and ν are called the **dual variables** or Lagrange multiplier vectors associated with the original problem.

The Lagrange dual function

- The **Lagrange dual function** (or just dual function) is defined as the minimum value of the Lagrangian over x: $g : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$ and for $\lambda \in \mathbf{R}^m$, $\nu \in \mathbf{R}^p$,

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right).$$

- When the Lagrangian is unbounded below in x, the dual function takes on the value $-\infty$.
- The domain of the dual function is set to be

$$\mathbf{dom}\ g = \{(\lambda, \nu) \mid g(\lambda, \nu) > -\infty\}.$$

- The dual function is always concave.

The Dual Function Gives Lower Bounds on Optimal Value

- For any $\lambda \succeq 0$ and any ν we have

$$g(\lambda, \nu) \leq p^*.$$

- Proof: Suppose $\tilde{x}$ is a feasible point for the original problem, i.e., $f_i(\tilde{x}) \leq 0$ and $h_i(\tilde{x}) = 0$, and $\lambda \succeq 0$. Then we have

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq 0,$$

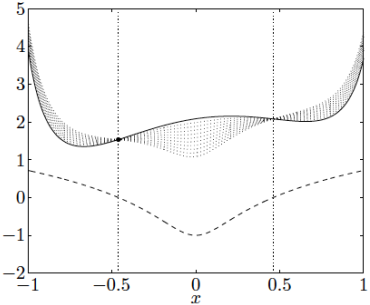
and therefore

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x}).$$

So, $g(\lambda, \nu) \leq f_0(\tilde{x})$ holds for every feasible point $\tilde{x}$.

Example

- A simple problem with $x \in \mathbf{R}$ and one inequality constraint.



- The objective function f_0: in solid curve.
- The constraint function f_1: in dashed curve.
- The feasible set $= [-0.46, 0.46]$.
- The optimal point and value: $x^* = -0.46$, $p^* = 1.54$.
- The dotted curves show $L(x, \lambda)$ for $\lambda = 0.1, 0.2, ..., 1.0$.

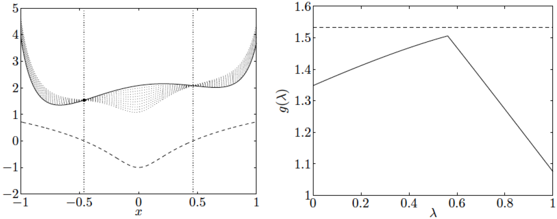
The Lagrange dual function

- When $g(\lambda, \nu) = -\infty$, the inequality

$$g(\lambda, \nu) \leq p^*$$

  holds, but is vacuous.
- The dual function gives a nontrivial lower bound on p^* only when $\lambda \succeq 0$ and $(\lambda, \nu) \in \textbf{dom } g$, i.e., $g(\lambda, \nu) > -\infty$.
- We refer to a pair (λ, ν) with $\lambda \succeq 0$ and $(\lambda, \nu) \in \textbf{dom } g$ as **dual feasible**.

Example



- Left figure: The objective function f_0: in solid curve. The constraint function f_1: in dashed curve. Neither f_0 nor f_1 is convex.
- Right figure: The dual function g for the problem in the left figure. It is concave. The horizontal dashed line shows p^*, the optimal value of the problem.

Linear approximation interpretation (1/3)

- The Lagrangian and lower bound property can be given a simple interpretation, based on a linear approximation of the **indicator functions** of the sets $\{0\}$ and $-\mathbf{R}_+$ (defined below).
- We rewrite the original problem as an unconstrained problem,

$$\text{minimize} \quad f_0(x) + \sum_{i=1}^m I_-(f_i(x)) + \sum_{i=1}^p I_0(h_i(x)),$$

where $I_- : \mathbf{R} \to \mathbf{R}$ is the indicator function for $-\mathbf{R}_+$,

$$I_-(u) = \begin{cases} 0 & u \le 0 \\ \infty & u > 0 \end{cases},$$

and similarly, I_0 is the indicator function of $\{0\}$:

$$I_0(u) = \begin{cases} 0 & u = 0 \\ \infty & u \ne 0 \end{cases}.$$

Linear approximation interpretation (2/3)

- Now, in the unconstrained problem

$$\text{minimize} \quad f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) + \sum_{i=1}^{p} I_0(h_i(x)),$$

  suppose we replace
  - the function $I_-(u)$ with the linear function $\lambda_i u$, where $\lambda_i \geq 0$,
  - and the function $I_0(u)$ with $\nu_i u$.
- Then, the objective becomes the Lagrangian function $L(x, \lambda, \nu)$, and the dual function value $g(\lambda, \nu)$ is the optimal value of the problem

$$\text{minimize} \quad L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x).$$

Linear approximation interpretation (3/3)

- In the above formulation, we use a linear or "soft" displeasure function in place of I_- and I_0.
- For an inequality constraint, our displeasure is zero when $f_i(x) = 0$, and is positive when $f_i(x) > 0$ (assuming $\lambda_i > 0$); our displeasure grows as the constraint becomes "more violated". Further, we derive pleasure from constraints that have margin, i.e., from $f_i(x) < 0$.
- Although such a linear approximation of the indicator function $I_-(u)$ is rather poor, it is at least an underestimator of the indicator function.
- Since $\lambda_i u \leq I_-(u)$ and $\nu_i u \leq I_0(u)$ for all u, we see immediately that the dual function yields a lower bound on the optimal value of the original problem.

Least-squares solution of linear equations (1/2)

- Consider the problem

$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax = b, \end{array}$$

  where $A \in \mathbf{R}^{p\times n}$. This problem has no inequality constraints and p (linear) equality constraints.
- The Lagrangian is $L(x,\nu) = x^T x + \nu^T(Ax - b)$, with domain $\mathbf{R}^n \times \mathbf{R}^p$.
- The dual function is given by $g(\nu) = \inf_x L(x,\nu)$. Since $L(x,\nu)$ is a convex quadratic function of x, we can find the minimizing x from the optimality condition

$$\nabla_x L(x,\nu) = 2x + A^T\nu = 0,$$

  which yields $x = -(1/2)A^T\nu$.

Least-squares solution of linear equations (2/2)

- Therefore, the dual function is

$$g(\nu) = L(-(1/2)A^T\nu, \nu) = -(1/4)\nu^T AA^T\nu - b^T\nu,$$

which is a concave quadratic function, with domain $\mathbf{R}^p$.

- The lower bound property

$$g(\lambda, \nu) \leq p^*$$

states that for any $\nu \in \mathbf{R}^p$, we have

$$-(1/4)\nu^T AA^T\nu - b^T\nu \leq \inf\left\{x^T x \mid Ax = b\right\}.$$

Standard form LP (1/2)

- Consider an LP in standard form,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0, \end{array}$$

  which has inequality constraint functions $f_i(x) = -x_i, i = 1, ..., n$.

- The Lagrangian is

$$L(x, \lambda, \nu) = c^T x - \sum_{i=1}^{n} \lambda_i x_i + \nu^T (Ax - b) = -b^T \nu + (c + A^T \nu - \lambda)^T x.$$

Standard form LP (2/2)

- The dual function is

$$
\begin{aligned}
g(\lambda, \nu) &= \inf_x L(x, \lambda, \nu) \\
&= -b^T \nu + \inf_x (c + A^T \nu - \lambda)^T x. \\
&= \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} .
\end{aligned}
$$

- Note that the dual function g is finite only on a proper affine subset of $\mathbf{R}^m \times \mathbf{R}^p$.
- The lower bound property $g(\lambda, \nu) \leq p^*$ is nontrivial only when λ and ν satisfy $\lambda \succeq 0$ and $A^T \nu - \lambda + c = 0$.
- When this occurs, $-b^T \nu$ is a lower bound on the optimal value of the LP in standard form.

Two-way partitioning problem (1/3)

- We consider the (nonconvex) problem

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, i = 1, \ldots, n, \end{array}$$

  where $W \in \mathbf{S}^n$.
- The constraints restrict the values of x_i to 1 or -1, so the problem is equivalent to finding the vector x with components ± 1 that minimizes $x^T W x$.
- The feasible set here is finite (it contains 2^n points) so this problem can in principle be solved by simply checking the objective value of each feasible point.
  - However, for a large n, it is very difficult to solve.
- We can interpret the problem as a two-way partitioning problem on a set of n elements, say, $\{1, \ldots, n\}$: A feasible x corresponds to the partition

$$\{1, \ldots, n\} = \{i \mid x_i = -1\} \bigcup \{i \mid x_i = 1\}.$$

Two-way partitioning problem (2/3)

- The Lagrangian is

$$L(x,\nu) = x^T W x + \sum_{i=1}^{n} \nu_i (x_i^2 - 1) = x^T (W + \textbf{diag}\ (\nu)) x - \mathbf{1}^T \nu.$$

- We obtain the Lagrange dual function by minimizing over x:

$$\begin{aligned} g(\nu) &= \inf_x x^T (W + \textbf{diag}\ (\nu)) x - \mathbf{1}^T \nu \\ &= \begin{cases} -\mathbf{1}^T \nu & W + \textbf{diag}\ (\nu) \succeq 0 \\ -\infty & \text{otherwise,} \end{cases} \end{aligned}$$

where we use the fact that the infimum of a quadratic form $u^T A u$ is either zero (if $A \succeq 0$) or $-\infty$ (if $A \not\succeq 0$).

Two-way partitioning problem (3/3)

- This dual function

$$g(\nu) = \begin{cases} -\mathbf{1}^T\nu & W + \textbf{diag}\ (\nu) \succeq 0 \\ -\infty & \text{otherwise,} \end{cases},$$

  provides lower bounds on the optimal value of the two-way partitioning problem.

- For example, we can take the specific value of the dual variable $\nu = -\lambda_{min}(W)\mathbf{1}$, which is dual feasible, since

$$W + \textbf{diag}\ (\nu) = W - \lambda_{min}(W)I \succeq 0.$$

  This yields the bound on the optimal value p^*

$$p^* \geq -\mathbf{1}^T\nu = n\lambda_{min}(W).$$

The Lagrange dual function and conjugate functions (1/2)

- Recall that the conjugate f^* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is given by
$$f^*(y) = \sup_{x \in \mathbf{dom}\ f} \left(y^T x - f(x) \right).$$
- The conjugate function and Lagrange dual function are closely related. As a simple example, consider the problem
$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x = 0. \end{array}$$
- This problem has Lagrangian $L(x, \nu) = f(x) + \nu^T x$, and dual function
$$g(\nu) = \inf_x \left(f(x) + \nu^T x \right) = -\sup_x \left((-\nu)^T x - f(x) \right) = -f^*(-\nu).$$

The Lagrange dual function and conjugate functions (2/2)

- More generally, consider an optimization problem with linear inequality and equality constraints,

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax \preceq b \\ & Cx = d. \end{array}$$

- Using the conjugate of f_0 we can write the dual function for the problem as

$$\begin{aligned} g(\lambda, \nu) &= \inf_x \left(f_0(x) + \lambda^T (Ax - b) + \nu^T (Cx - d) \right) \\ &= -b^T \lambda - d^T \nu + \inf_x \left(f_0(x) + (A^T \lambda + C^T \nu)^T x \right) \\ &= -b^T \lambda - d^T \nu - f_0^*(-A^T \lambda - C^T \nu). \end{aligned}$$

- The domain of g follows from the domain of f_0^* :

$$\textbf{dom } g = \left\{ (\lambda, \nu) \mid -A^T \lambda - C^T \nu \in \textbf{dom } f_0^* \right\}.$$

Example – Equality constrained norm minimization

- Consider the problem

$$\begin{array}{ll} \text{minimize} & ||x||_2 \\ \text{subject to} & Ax = b. \end{array}$$

- The conjugate of $f_0 = ||\cdot||_2$ is given by

$$f_0^*(y) = \begin{cases} 0, & ||y||_2 \leq 1 \\ \infty, & \text{otherwise} \end{cases}.$$

- The dual function for the problem is given by

$$g(\nu) = -b^T\nu - f_0^*(-A^T\nu) = \begin{cases} -b^T\nu, & ||A^T\nu||_2 \leq 1 \\ -\infty, & \text{otherwise} \end{cases}.$$

The Lagrange dual problem (1/2)

- For each pair (λ, ν) with $\lambda \succeq 0$, the Lagrange dual function $g(\lambda, \nu)$ gives us a lower bound on the optimal value p^* of the optimization problem (called the **primal problem**)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, ..., m \\ & h_i(x) = 0, i = 1, ..., p, \end{array}$$

- The **Lagrange dual problem** associated with the original problem, defined as

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0, \end{array}$$

would give the best lower bound that can be obtained from the Lagrange dual function.

The Lagrange dual problem (2/2)

- We use the term **dual feasible** to describe a pair (λ, ν) with $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$.
- We refer to (λ^*, ν^*) as **dual optimal** or optimal Lagrange multipliers if they are optimal for the dual problem.
- The Lagrange dual problem is a convex optimization problem, since the objective to be maximized is concave and the constraint is convex, regardless of the convexity of the primal problem.

Making dual constraints explicit

- It is not uncommon for the domain of the dual function,

$$\mathbf{dom}\ g = \{(\lambda, \nu) \mid g(\lambda, \nu) > -\infty\},$$

to have dimension smaller than $m + p$.

- In many cases we can identify the affine hull of **dom** g, and describe it as a set of linear equality constraints. This means we can identify the equality constraints that are 'hidden' or 'implicit' in the objective g of the dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

- In this case we can form an equivalent problem, in which these equality constraints are given explicitly as constraints.

Example – Lagrange dual of standard form LP (1/3)

- The Lagrange dual function for the standard form LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \succeq 0 \end{aligned}$$

is given by

$$g(\lambda, \nu) = \begin{cases} -b^T \nu, & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

- Strictly speaking, the Lagrange dual problem of the standard form LP is to maximize this dual function g subject to $\lambda \succeq 0$:

$$\begin{aligned} \text{maximize} \quad & g(\lambda, \nu) = \begin{cases} -b^T \nu, & A^T \nu - \lambda + c = 0 \\ -\infty, & \text{otherwise} \end{cases} \\ \text{subject to} \quad & \lambda \succeq 0. \end{aligned}$$

Example – Lagrange dual of standard form LP (2/3)

- Here g is finite only when $A^T\nu - \lambda + c = 0$. So the Lagrange dual problem is equivalent to

$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu - \lambda + c = 0 \\ & \lambda \succeq 0. \end{array}$$

- This problem, in turn, can be expressed as

$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu + c \succeq 0, \end{array}$$

which is an LP in inequality form.

Example – Lagrange dual of inequality form LP

- Conversely, let's find the Lagrange dual problem of an LP in inequality form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b. \end{array}$$

- The Lagrangian is

$$L(x, \lambda) = c^T x + \lambda^T (Ax - b) = -b^T \lambda + (A^T \lambda + c)^T x,$$

so the dual function is

$$g(\lambda) = \inf_x L(x, \lambda) = -b^T \lambda + \inf_x (A^T \lambda + c)^T x.$$

- The infimum of a linear function is $-\infty$, except in the special case when it is identically zero, so the dual function is

$$g(\lambda) = \begin{cases} -b^T \lambda, & A^T \lambda + c = 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

Example – Lagrange dual of inequality form LP

- The dual variable λ is dual feasible if $\lambda \succeq 0$ and $A^T\lambda + c = 0$.
- So, the dual problem can be written as

$$\begin{array}{ll} \text{maximize} & -b^T\lambda \\ \text{subject to} & A^T\lambda + c = 0 \\ & \lambda \succeq 0, \end{array}$$

  which is an LP in standard form.
- Note that the dual of a standard form LP is an LP with only inequality constraints, and vice versa.

Weak Duality (1/3)

- The optimal value of the Lagrange dual problem, which we denote d^*, is the best lower bound on p^* that can be obtained from the Lagrange dual function.
- Therefore, we have the simple but important inequality

$$d^* \leq p^*,$$

  which holds even if the original problem is not convex. This property is called **weak duality**.
- The weak duality inequality holds when d^* and p^* are infinite.
  - For example, if the primal problem is unbounded below, so that $p^* = -\infty$, we must have $d^* = -\infty$, i.e., the Lagrange dual problem is infeasible.
  - Conversely, if the dual problem is unbounded above, so that $d^* = \infty$, we must have $p^* = \infty$, i.e., the primal problem is infeasible.

Weak Duality (2/3)

- The difference $p^* - d^*$ is called the **optimal duality gap** of the original problem, which is always nonnegative.
- The bound can sometimes be used to find a lower bound on the optimal value of a problem that is difficult to solve, since the dual problem is always convex, and in many cases can be solved efficiently, to find d^*.
  - As an example, consider the two-way partitioning problem

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, i = 1, \ldots, n, \end{array}$$

    where $W \in \mathbf{S}^n$.
  - The dual problem is a semidefinite program (SDP),

$$\begin{array}{ll} \text{maximize} & -\mathbf{1}^T \nu \\ \text{subject to} & W + \mathbf{diag}\,(\nu) \succeq 0, \end{array}$$

    with variable $\nu \in \mathbf{R}^n$.

Weak Duality (3/3)

- This problem can be solved efficiently, even for relatively large values of n, such as $n = 1000$.
- The optimal value of the dual problem

$$\begin{array}{ll} \text{maximize} & -\mathbf{1}^T \nu \\ \text{subject to} & W + \mathbf{diag}\,(\nu) \succeq 0, \end{array}$$

is a lower bound on the optimal value of the two-way partitioning problem, and is always at least as good as the lower bound given as

$$p^* \geq d^* \geq n\lambda_{min}(W).$$

Strong duality

- If the equality $d^* = p^*$ holds, i.e., the optimal duality gap is zero, then we say that **strong duality** holds.
- This means that the best bound that can be obtained from the Lagrange dual function is tight.
- Strong duality does not always hold. But if the primal problem is convex, i.e., of the form

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, ..., m, \\ & Ax = b, \end{aligned}$$

with $f_0, ..., f_m$ convex, we usually (but not always) have strong duality.

Strong duality and Slater's Condition (1/2)

- One simple constraint qualification is **Slater's condition**: There exists an $x \in \textbf{relint } \mathcal{D}$ such that

$$f_i(x) < 0, i = 1, ..., m, \quad Ax = b.$$

  Such a point is sometimes called strictly feasible.
- Slater's theorem: if the problem is convex and Slater's condition holds, then the strong duality holds.
- Slater's condition can be refined when some of the inequality constraint functions f_i are affine:
  - If the first k constraint functions $f_1, ..., f_k$ are affine, then strong duality holds if the following condition holds: (Weaker Slater's condition) There exists an $x \in \textbf{relint } D$ with

$$f_i(x) \le 0, i = 1, ..., k, \quad f_i(x) < 0, i = k+1, ..., m, \quad Ax = b.$$

Strong duality and Slater's Condition (2/2)

- Note that the refined (weaker) Slater's condition reduces to feasibility when
  - the constraints are all linear equalities and inequalities, and,
  - **dom** f_0 is open.
- Slater's condition not only implies strong duality for convex problems. It also implies that the dual optimal value is attained when $d^* > -\infty$, i.e., there exists a dual feasible (λ^*, ν^*) with $g(\lambda^*, \nu^*) = d^* = p^*$.
- We will prove later on that strong duality obtains, when the primal problem is convex and Slater's condition holds.

Examples – Least-squares solution for linear equations

- Recall the least-squares problem

$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax = b, \end{array}$$

  whose associated dual problem is

$$\text{maximize} \quad -(1/4)\nu^T A A^T \nu - b^T \nu,$$

  an unconstrained concave quadratic maximization problem.
- We always have strong duality. Slater's condition is simply that the primal problem is feasible, so $p^* = d^*$ provided $b \in \mathcal{R}(A)$, i.e., $p^* < \infty$.

Lagrange dual of LP

- By the weaker form of Slater's condition, we find that strong duality holds for any LP (in standard or inequality form) provided the primal problem is feasible.
- Applying this result to the dual problem, we conclude that strong duality holds for LPs if the dual problem is feasible.
- This leaves only one possible situation in which strong duality for LPs can fail: both the primal and dual problems are infeasible.

Lagrange dual of QCQP (1/2)

- We consider the QCQP

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \le 0, i = 1, \ldots, m, \end{array}$$

with $P_0 \in \mathbf{S}^n_{++}$, and $P_i \in \mathbf{S}^n_{+}$, $i = 1, \ldots, m$.

- The Lagrangian is

$$L(x, \lambda) = (1/2)x^T P(\lambda) x + q(\lambda)^T x + r(\lambda),$$

where

$$P(\lambda) = P_0 + \sum_{i=1}^m \lambda_i P_i, \;\; q(\lambda) = q_0 + \sum_{i=1}^m \lambda_i q_i, \;\; r(\lambda) = r_0 + \sum_{i=1}^m \lambda_i r_i.$$

Lagrange dual of QCQP (2/2)

- If $\lambda \succeq 0$, we have $P(\lambda) \succ 0$ and the dual function can be written as

$$g(\lambda) = \inf_x L(x, \lambda) = -(1/2)q(\lambda)^T P(\lambda)^{-1} q(\lambda) + r(\lambda).$$

- We can therefore express the dual problem as

$$\begin{array}{ll} \text{maximize} & -(1/2)q(\lambda)^T P(\lambda)^{-1} q(\lambda) + r(\lambda) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

- The Slater's condition says that strong duality holds if the quadratic inequality constraints are strictly feasible, i.e., there exists an x with

$$(1/2)x^T P_i x + q_i^T x + r_i < 0, i = 1, ..., m.$$

Weak and strong duality via set of values (1/5)

- For a given optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p, \end{array}$$

we define the set

$$\mathcal{G} = \{(f_1(x), \dots, f_m(x), h_1(x), \dots, h_p(x), f_0(x)) \in \mathbf{R}^m \times \mathbf{R}^p \times \mathbf{R} \mid x \in \mathcal{D}\}.$$

- Then, the optimal value p^* of the problem is easily expressed in terms of $\mathcal{G}$ as

$$p^* = \inf \{t \mid (u, v, t) \in \mathcal{G}, u \preceq 0, v = 0\}.$$

Weak and strong duality via set of values (2/5)

- The value of the dual function at (λ, ν) can be written as

$$g(\lambda, \nu) = \inf \left\{ (\lambda, \nu, 1)^T (u, v, t) \mid (u, v, t) \in \mathcal{G} \right\}$$

by minimizing the affine function

$$(\lambda, \nu, 1)^T (u, v, t) = \sum_{i=1}^{m} \lambda_i u_i + \sum_{i=1}^{p} \nu_i v_i + t$$

over $(u, v, t) \in \mathcal{G}$.

- In particular, we see that if the infimum is finite, then the inequality

$$(\lambda, \nu, 1)^T (u, v, t) \geq g(\lambda, \nu)$$

defines a supporting hyperplane to $\mathcal{G}$. This is sometimes referred to as a nonvertical supporting hyperplane, because the last component of the normal vector is nonzero.

Weak and strong duality via set of values (3/5)

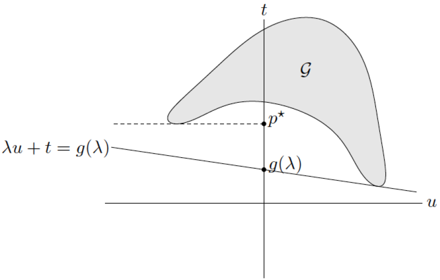
- Now suppose $\lambda \succeq 0$. Then, $t \geq (\lambda, \nu, 1)^T(u, v, t)$ if $u \preceq 0$ and $v = 0$.
- Therefore,

$$\begin{aligned} p^* &= \inf \{t \mid (u, v, t) \in \mathcal{G}, u \preceq 0, v = 0\} \\ &\geq \inf \left\{(\lambda, \nu, 1)^T(u, v, t) \mid (u, v, t) \in \mathcal{G}, u \preceq 0, v = 0\right\} \\ &\geq \inf \left\{(\lambda, \nu, 1)^T(u, v, t) \mid (u, v, t) \in \mathcal{G}\right\} \\ &= g(\lambda, \nu), \end{aligned}$$

i.e., we have weak duality.

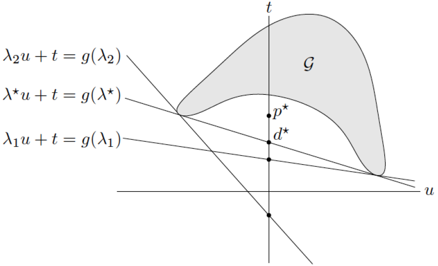
Weak and strong duality via set of values (4/5)

- Consider a problem with one (inequality) constraint. Given λ, we minimize $(\lambda, 1)^T(u, t)$ over $\mathcal{G} = \{(f_1(x), f_0(x)) \mid x \in \mathcal{D}\}$.
- This yields a supporting hyperplane with slope $-\lambda$. The intersection of this hyperplane with the $u = 0$ axis gives $g(\lambda)$.



Weak and strong duality via set of values (5/5)

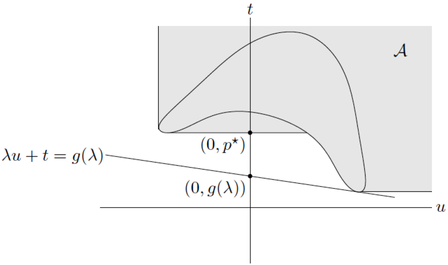
- Supporting hyperplanes corresponding to three dual feasible values of λ, including the optimum λ^*. Strong duality does not hold; the optimal duality gap $p^* - d^*$ is positive.



Epigraph variation (1/3)

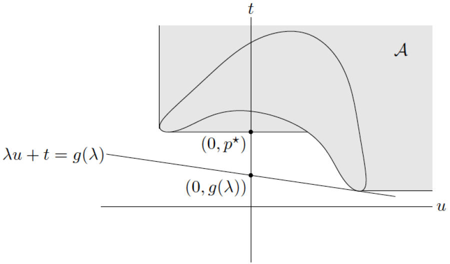
- We define the set $\mathcal{A} \subseteq \mathbf{R}^m \times \mathbf{R}^p \times \mathbf{R}$ as

$$
\begin{aligned}
\mathcal{A} &= \mathcal{G} + \left(\mathbf{R}_+^m \times \{0\} \times \mathbf{R}_+\right) \\
&= \{(u, v, t) \mid \exists x \in \mathcal{D},\ f_i(x) \le u_i, i = 1, \dots, m, \\
&\quad h_i(x) = v_i, i = 1, \dots, p, \quad f_0(x) \le t\},
\end{aligned}
$$



- The set $\mathcal{A}$ is considered as an epigraph-like form of $\mathcal{G}$.

Epigraph variation (2/3)



- We can express the optimal value in terms of $\mathcal{A}$ as

$$p^* = \inf \{t \mid (0, 0, t) \in \mathcal{A}\}.$$

- The dual function at a point (λ, ν) with $\lambda \succeq 0$ is

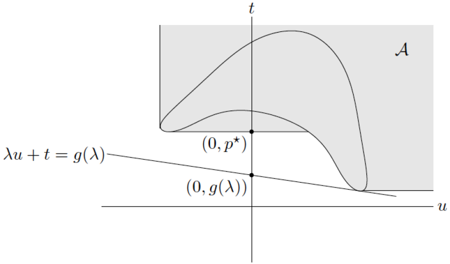
$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf \left\{ (\lambda, \nu, 1)^T (u, v, t) \mid (u, v, t) \in \mathcal{A} \right\}.$$

Epigraph variation (3/3)

- If $g(\lambda, \nu)$ is finite, then

$$(\lambda, \nu, 1)^T(u, v, t) \geq g(\lambda, \nu)$$

defines a nonvertical supporting hyperplane to $\mathcal{A}$.



- In particular, since $(0, 0, p^*) \in \textbf{bd}\ \mathcal{A}$, we have

$$p^* = (\lambda, \nu, 1)^T(0, 0, p^*) \geq g(\lambda, \nu),$$

the weak duality lower bound.
- Strong duality holds if and only if the equality holds for some dual feasible (λ, ν).

Strong Duality Under Slater's Constraint Qualification

- We want to prove that Slater's constraint qualification, i.e., there exists an $\tilde{x} \in \textbf{relint } \mathcal{D}$ such that

$$f_i(\tilde{x}) < 0, i = 1, ..., m, \quad A\tilde{x} = b,$$

  guarantees strong duality (and that the dual optimum is attained) for a convex problem.
- The following assumptions are made to simplify the proof:
  - $\mathcal{D}$ has nonempty interior (hence, $\textbf{relint } \mathcal{D} = \textbf{int } \mathcal{D}$).
  - rank $A = p$.
  - p^* is finite. (Since there is a feasible point, we can only have $p^* = -\infty$ or p^* finite; if $p^* = -\infty$, then $d^* = -\infty$ by weak duality.)

The Proof (1/5)

- The set $\mathcal{A}$ defined as

$$\mathcal{A} = \mathcal{G} + \left(\mathbf{R}_+^m \times \{0\} \times \mathbf{R}_+\right)$$

can be shown to be convex if the underlying problem is convex.

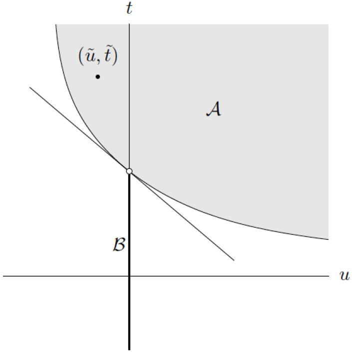
- We define a second convex set $\mathcal{B}$ as

$$\mathcal{B} = \left\{(0, 0, s) \in \mathbf{R}^m \times \mathbf{R}^p \times \mathbf{R} \mid s < p^*\right\}.$$

- We have that $\mathcal{A} \cap \mathcal{B} = \phi$.

The Proof (2/5)

- To see that $\mathcal{A} \cap \mathcal{B} = \phi$, suppose $(u, v, t) \in \mathcal{A} \cap \mathcal{B}$. Since $(u, v, t) \in \mathcal{B}$ we have $u = 0, v = 0$, and $t < p^*$. Since $(u, v, t) \in \mathcal{A}$, there exists an x with $f_i(x) \leq 0, i = 1, \ldots, m$, $Ax - b = 0$, and $f_0(x) \leq t < p^*$, which is impossible since p^* is the optimal value of the primal problem.



The Proof (3/5)

- By the separating hyperplane theorem, there exists $(\tilde{\lambda}, \tilde{\nu}, \mu) \neq 0$ and α such that

$$(u, v, t) \in \mathcal{A} \Longrightarrow \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \geq \alpha,$$

and

$$(u, v, t) \in \mathcal{B} \Longrightarrow \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \leq \alpha.$$

- The first condition implies that $\tilde{\lambda} \succeq 0$ and $\mu \geq 0$.
- The second condition means that $\mu t \leq \alpha$ for all $t < p^*$, and hence, $\mu p^* \leq \alpha$.
- Therefore, we conclude that for any $x \in \mathcal{D}$,

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T (Ax - b) + \mu f_0(x) \geq \alpha \geq \mu p^*.$$

The Proof (4/5)

- For any $x \in \mathcal{D}$,

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T(Ax - b) + \mu f_0(x) \geq \alpha \geq \mu p^*.$$

- Assume that $\mu > 0$. Then

$$\mu L(x, \tilde{\lambda}/\mu, \tilde{\nu}/\mu) = \sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T(Ax - b) + \mu f_0(x) \geq \mu p^*$$

  for all $x \in \mathcal{D}$, from which it follows, by minimizing over x, that $g(\lambda, \nu) \geq p^*$, where we define $\lambda = \tilde{\lambda}/\mu, \nu = \tilde{\nu}/\mu$.
- By weak duality we have $g(\lambda, \nu) \leq p^*$, so in fact $g(\lambda, \nu) = p^*$. This shows that strong duality holds, and that the dual optimum is attained, when $\mu > 0$.

The Proof (5/5)

- Now consider the case $\mu = 0$. We have that for all $x \in \mathcal{D}$,

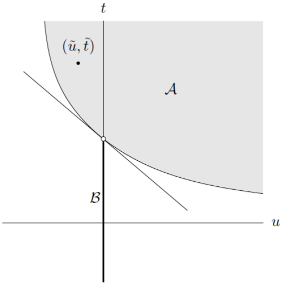
$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T(Ax - b) \geq 0.$$

- For any point $\tilde{x}$ that satisfies the Slater's condition, we have

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(\tilde{x}) \geq 0.$$

  - Since $f_i(\tilde{x}) < 0$ and $\tilde{\lambda}_i \geq 0$, we obtain that $\tilde{\lambda} = 0$. Since $\tilde{\lambda} = 0, \mu = 0$ but $(\tilde{\lambda}, \tilde{\nu}, \mu) \neq 0$, we conclude that $\tilde{\nu} \neq 0$.
- Then we have that for all $x \in \mathcal{D}$, $\tilde{\nu}^T(Ax - b) \geq 0$.
  - But $\tilde{x}$ satisfies $\tilde{\nu}^T(A\tilde{x} - b) = 0$, and since $\tilde{x} \in \textbf{int } \mathcal{D}$, there are points in $\mathcal{D}$ with $\tilde{\nu}^T(Ax - b) < 0$ unless $A^T\tilde{\nu} = 0$. This contradicts the assumption that rank $A = p$.

Geometric idea behind the proof



- The above figure illustrates a simple problem with one inequality constraint.
- The hyperplane separating $\mathcal{A}$ and $\mathcal{B}$ defines a supporting hyperplane to $\mathcal{A}$ at $(0, p^*)$.
- Slater's constraint qualification is used to establish that the hyperplane must be nonvertical (i.e., has a normal vector of the form $(\lambda^*, 1)$).

An example in which Slater's condition does not hold

- Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq 0 \end{array}$$

  with variables x and y, and domain $\mathcal{D} = \{(x, y) \mid y > 0\}$.
- It is a convex optimization problem with the optimal value 1.
- Slater's condition does not hold for this problem since there does not exist $x \in \textbf{int}\ \mathcal{D}$ s.t. $x^2/y < 0$.
- The Lagrange dual function is

$$g(\lambda) = \inf_{x,y} L(x, \lambda) = \inf_{x,y} \left(e^{-x} + \lambda(x^2/y)\right) = \begin{cases} 0, & \lambda \geq 0 \\ -\infty, & \lambda < 0 \end{cases}.$$

- The optimal solution and optimal value of the dual problem is $\lambda^* \in [0, \infty)$ and $d^* = 0$, respectively.
- The optimal duality gap is $p^* - d^* = 1$.

Summary of dual problems and Slater's condition

- Given an optimization problem (convex or nonconvex) with an optimal value p^*, we defined
  - The Lagrangian $L(x, \lambda, \nu)$.
  - The dual function $g(\lambda, \nu)$.
  - The dual problem.
- Weak duality: The optimal value of the dual problem is always a lower bound of the optimal value of the primal problem: $d^* \leq p^*$.
- Strong duality: When certain conditions are met, we will have $d^* = p^*$. A sufficient (but not necessary) condition for strong duality is
  - The primal problem is convex.
  - The primal problem meets the Slater's condition.

Entropy maximization (1/4)

- Consider the entropy maximization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to} & Ax \preceq b \\ & \mathbf{1}^T x = 1 \end{array}$$

  where $\textbf{dom}\ f_0 = \mathbf{R}^n_{++}$.
- Recall that the conjugate of the negative entropy function $u \log u$, with scalar variable u, is e^{v-1}.
- Then, the conjugate of f_0 can be written as

$$f_0^*(y) = \sum_{i=1}^{n} e^{y_i - 1},$$

  with $\textbf{dom}\ f_0^* = \mathbf{R}^n$.

Entropy maximization (2/4)

- So, the dual function of the entropy maximization problem is given by

$$
\begin{aligned}
g(\lambda, \nu) &= -b^T\lambda - 1 \cdot \nu - f_0^*(-A^T\lambda - \mathbf{1}\nu) \\
&= -b^T\lambda - \nu - \sum_{i=1}^{n} e^{-a_i^T\lambda - \nu - 1} \\
&= -b^T\lambda - \nu - e^{-\nu-1}\sum_{i=1}^{n} e^{-a_i^T\lambda}
\end{aligned}
$$

where a_i is the ith column of A.

Entropy maximization (3/4)

- So, the dual problem for the entropy maximization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to} & Ax \preceq b \\ & \mathbf{1}^T x = 1 \end{array}$$

  is

$$\begin{array}{ll} \text{maximize} & -b^T\lambda - \nu - e^{-\nu-1} \sum_{i=1}^{n} e^{-a_i^T \lambda} \\ \text{subject to} & \lambda \succeq 0, \end{array}$$

  with variables $\lambda \in \mathbf{R}^m, \nu \in \mathbf{R}$.
- The (weaker) Slater's condition tells us that the optimal duality gap is zero if there exists an $x \succ 0$ with $Ax \preceq b$ and $\mathbf{1}^T x = 1$.

Entropy maximization (4/4)

- For fixed λ, the dual objective function

$$-b^T\lambda - \nu - e^{-\nu-1}\sum_{i=1}^{n} e^{-a_i^T\lambda}$$

is maximized when the derivative with respect to ν is zero:

$$\nu = \log\sum_{i=1}^{n} e^{-a_i^T\lambda} - 1.$$

- Substituting this optimal value of ν into the dual problem gives

$$\begin{array}{ll} \text{maximize} & -b^T\lambda - \log\left(\sum_{i=1}^{n} e^{-a_i^T\lambda}\right) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

Minimum volume covering ellipsoid (1/4)

- Consider the problem with variable $X \in \mathbf{S}^n$,

$$\begin{array}{ll} \text{minimize} & f_0(X) = \log \det X^{-1} \\ \text{subject to} & a_i^T X a_i \le 1, \quad i = 1, \ldots, m, \end{array}$$

  where $\mathbf{dom}\ f_0 = \mathbf{S}^n_{++}$.
- With each $X \in \mathbf{S}^n_{++}$ we associate the ellipsoid, centered at the origin,

$$\mathcal{E}_X = \left\{ z \mid z^T X z \le 1 \right\}.$$

- The volume of this ellipsoid is proportional to $\left(\det X^{-1}\right)^{1/2}$, so the objective function is, except for a constant and a factor of two, the logarithm of the volume of $\mathcal{E}_X$.
- The problem is essentially to determine the minimum volume ellipsoid, centered at the origin, that includes the points $a_1, \ldots, a_m$.

Minimum volume covering ellipsoid (2/4)

- Note that the inequality constraints of the problem are affine, and can be expressed as

$$\mathbf{tr}\ \left((a_i a_i^T)X\right) \leq 1.$$

- Also, recall that the conjugate of $f_0(X) = \log\det X^{-1}$ is

$$f_0^*(Y) = \log\det(-Y)^{-1} - n,$$

  with $\mathbf{dom}\ f_0^* = -\mathbf{S}_{++}^n$.
- So, the dual function for the problem is given by

$$g(\lambda) = \begin{cases} \log\det\left(\sum_{i=1}^m \lambda_i a_i a_i^T\right) - \mathbf{1}^T\lambda + n, & \sum_{i=1}^m \lambda_i a_i a_i^T \succ 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

Minimum volume covering ellipsoid (3/4)

- Thus, for any $\lambda \succeq 0$ with $\sum_{i=1}^{m} \lambda_i a_i a_i^T \succ 0$, the number

$$\log\det\left(\sum_{i=1}^{m} \lambda_i a_i a_i^T\right) - \mathbf{1}^T\lambda + n$$

is a lower bound on the optimal value of the primal problem.

Minimum volume covering ellipsoid (4/4)

- So, the dual problem can be expressed as

$$\begin{array}{ll} \text{maximize} & \log\det\left(\sum_{i=1}^m \lambda_i a_i a_i^T\right) - \mathbf{1}^T\lambda + n \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

  where we take $\log\det X = -\infty$ if $X \not\succ 0$.

- The (weaker) Slater's condition for the problem is that there exists an $X \in \mathbf{S}^n_{++}$ with $a_i^T X a_i \leq 1$, for $i = 1, \ldots, m$.
- This is always satisfied, so strong duality always holds.