最佳化概論(Introduction to Optimization)
Homework 02 – 2017/10/11

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1. Answer the following short questions:

(a) Classify each of the following sets as open, closed, neither, or both.

(i) $\{x : |x-5| \leq \frac{1}{2}\}$

(ii) $\{x : x^2 > 0\}$

(b) Find the interior of $[0,3] \cup (3,5)$.

(c) Find the boundary points of $[0,3] \cup (3,5)$.

(d) Find the closure of $\{x : x^2 > 0\}$.

(e) Find all cluster points of $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ or } x = 2\}$.

(f) Find all cluster points of $S = \{(x,y) \in \mathbb{R}^2 : y < x^2 + 1\}$.

(a)

(i) closed.

(ii) open.

(b) $\{x : 0 < x < 5\}$

(c) $\{x : x = 0, x = 5\}$

(d) $\{x : x \in \mathbb{R}\}$

(e) $\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1; y \in \mathbb{R}\}$

(f) $\{(x,y) \in \mathbb{R}^2 : y \leq x^2 + 1\}$

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2. Let $0 < b < 1$ and $x_n = b^n,\ n \geq 1$. Show that the sequence (x_n) converges to 0. (Hint: we may write for some for some $a > 0,\ b = \frac{1}{1+a}$ and use the Bernoulli's inequality if $a > -1,\ a \in \mathbb{R}\ (1+a)^n \geq 1 + na,\ n \geq 1$.)

$$\text{Note that } x_n = b^n = \left(\frac{1}{1+a}\right)^n = \frac{1}{(1+a)^n} \to 0 \text{ and } a = \frac{1-b}{b}$$

$$\text{Since } (1+a)^n \geq 1 + na \implies \frac{1}{(1+a)^n} \leq \frac{1}{1+na}$$

$$\text{Let } \epsilon > 0,\ \exists\, k(\epsilon) \in \mathbb{N} \text{ s.t. } n > k(\epsilon),\ |x_n - x| < \epsilon$$

$$\implies n \geq k(\epsilon),\ \left|\frac{1}{(1+a)^n} - 0\right| = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} \leq \frac{1}{1+k(\epsilon)a} \leq \epsilon$$

$$\implies \frac{1}{1+k(\epsilon)a} \leq \epsilon \implies 1 + k(\epsilon)a > \frac{1}{\epsilon} \implies k(\epsilon) > \frac{1}{a}\left(\frac{1}{\epsilon} - 1\right) = \frac{b}{1-b}\left(\frac{1}{\epsilon} - 1\right)$$

$$\text{Let } k(\epsilon) \text{ be any number that } k(\epsilon) > \frac{1}{a}\left(\frac{1}{\epsilon} - 1\right) = \frac{b}{1-b}\left(\frac{1}{\epsilon} - 1\right).$$

$$\forall\, n > k(\epsilon),\ |b^n - 0| = \left|\frac{1}{(1+a)^n - 0}\right| = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{1+k(\epsilon)a} < \epsilon$$

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3. Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ which is convergent to x, and let $c \in \mathbb{R}$. Show that $\lim(cx_n) = cx$.

$$\lim x_n = x \implies \exists\ k(\epsilon) \text{ s.t. } n > k(\epsilon),\ |x_n - x| < \epsilon$$

$$\lim |cx_n - cx| = \lim c|x_n - x| \leq (c\epsilon) = \epsilon' \text{ where } \epsilon = \frac{\epsilon'}{c}$$

$$\text{Let } \epsilon > 0,\ \forall\ n > k(\frac{\epsilon}{c}),\ |cx_n - cx| = c|x_n - x| \leq c(\frac{\epsilon}{c}) = \epsilon$$

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4. If $X = (x_n)$ and $Y = (y_n)$ are sequences of real numbers which both converge to c and if $Z = (z_n)$ is a sequence such that $x_n \leq z_n \leq y_n$ for $n \in \mathbb{N}$, then Z also converges to c.

Let $\epsilon > 0$ and

$$\text{①}\ \exists\, kx(\epsilon) \text{ s.t. } \forall\, n > kx(\epsilon),\ |x_n - L| < \epsilon \implies L - \epsilon < x_n < L + \epsilon$$
$$\text{②}\ \exists\, ky(\epsilon) \text{ s.t. } \forall\, n > ky(\epsilon),\ |y_n - L| < \epsilon \implies L - \epsilon < y_n < L + \epsilon$$
$$\text{③}\ \forall\, n,\ x_n \leq z_n \leq y_n$$

Let $kz(\epsilon) = \max\{kx(\epsilon), ky(\epsilon)\}$. $\forall\, n > kz(\epsilon),\ L - \epsilon < x_n \leq z_n \leq y_n < L + \epsilon \implies |z_n - n| < \epsilon$