Integer and Combinatorial Optimization

Spring 2017

Homework 5

(Due at the beginning of the class on June 1)

1. (30%) Consider the problem:

$$\begin{aligned} \text{minimize} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & 3x_1 + x_2 \geq 6, \\ & -x_1 + x_2 \leq 2, \\ & x_1 + x_2 \leq 8, \\ & x_1,\ x_2 \geq 0. \end{aligned}$$

Assume that the first constraint ($3x_1 + x_2 \geq 6$) is relaxed.

(a) Formulate the Lagrangian dual problem.

(b) Show that $z_R(\lambda) = 6\lambda + \min\{0,\ 4 - 2\lambda,\ 13 - 14\lambda,\ 8 - 24\lambda\}$

(c) Plot $z_R(\lambda)$ for each value of λ.

(d) For part (c) locate the optimal solution of the Lagrangian dual problem

(e) For part (d) find the optimal solution to the primal problem.

2. (35%) Consider two different Lagrangian duals for the generalized assignment problem.

$$\begin{aligned} \max \quad & \sum_i^m \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^n x_{ij} \leq 1,\ i = 1,\dots,m \\ & \sum_{i=1}^m a_j x_{ij} \leq b_j,\ j = 1,\dots,n \\ & x_{ij} = 0 \text{ or } 1 \end{aligned}$$

Write out these two Lagrangian relaxation problems and discuss their relative merits according to the following three criteria:

(a) (10%) Ease of solution of the Lagrangian subproblem,

(b) (10%) Ease of solution of the Lagrangian dual,

(c) (15%) Strength of the upper bound obtained by solving the Lagrangian dual.

3. (35%) Consider the uncapacitated location problem.

   x_j=1, if a facility (factory) is placed at j (j=1,...,n).

   y_{ij} is the fraction of the demand of client i (store) (i=1,...,m) satisfied from facility j.

   f_j is the operating cost of facility j.

   c_{ij} is the revenue of fulfilling demand of client i from facility j.

$$
\begin{aligned}
\max\ & \sum_{i}^{m}\sum_{j=1}^{n} c_{ij}y_{ij} - \sum_{j=1}^{n} f_j x_j \\
\text{s.t.}\ & \sum_{j=1}^{n} y_{ij} = 1,\ \text{for } i=1,\ldots,m \\
& y_{ij} - x_j \le 0,\ \text{for } i=1,\ldots,m;\ j=1,\ldots,n \\
& x_j = 0 \text{ or } 1, y_{ij} \ge 0
\end{aligned}
$$

   (a) (15%) Please explain how the following reformulation is obtained by Bender decomposition.

$$
\begin{aligned}
\max\ & \sum_{i=1}^{m} \eta_i - \sum_{j=1}^{n} f_j x_j \\
\text{s.t.}\ & \eta_i \le c_{ik} + \sum_{j=1}^{n} (c_{ij} - c_{ik})^+ x_j\ ,\ \text{for } k=1,\ldots,n; i=1,\ldots,m \\
& \sum_{j=1}^{n} x_j \ge 1 \\
& x_j = 0 \text{ or } 1, \eta_i \text{ unrestricted}
\end{aligned}
$$

   (b) (20%) Given the following parameters, solve the reformulation by using the constraint generation algorithm.

   There are 6 clients and 5 possible locations for facility. The operation costs for each j (f_j) are 4, 3, 4, 4, and 7. The matrix of revenue of each (c_{ij}) is

$$
\begin{bmatrix}
12 & 13 & 6 & 0 & 1 \\
8 & 4 & 9 & 1 & 2 \\
2 & 6 & 6 & 0 & 1 \\
3 & 5 & 2 & 10 & 8 \\
8 & 0 & 5 & 10 & 8 \\
2 & 0 & 3 & 4 & 1
\end{bmatrix}.
$$