Search Enumeration

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Course No: 546 U6110

Agenda

- Enumeration Tree
- Implicit Enumeration
- Branching Rules

Binary Integer Program

- Consider the zero-one integer program

$$
\begin{aligned}
\text{minimize} \quad & z = \mathbf{cx} \\
\text{subject to} \quad & \mathbf{Ax} \le \mathbf{b}\ , \\
& x_j \in \{0,1\} \quad j = 1,\ \dots,\ n,
\end{aligned}
$$

Introduction

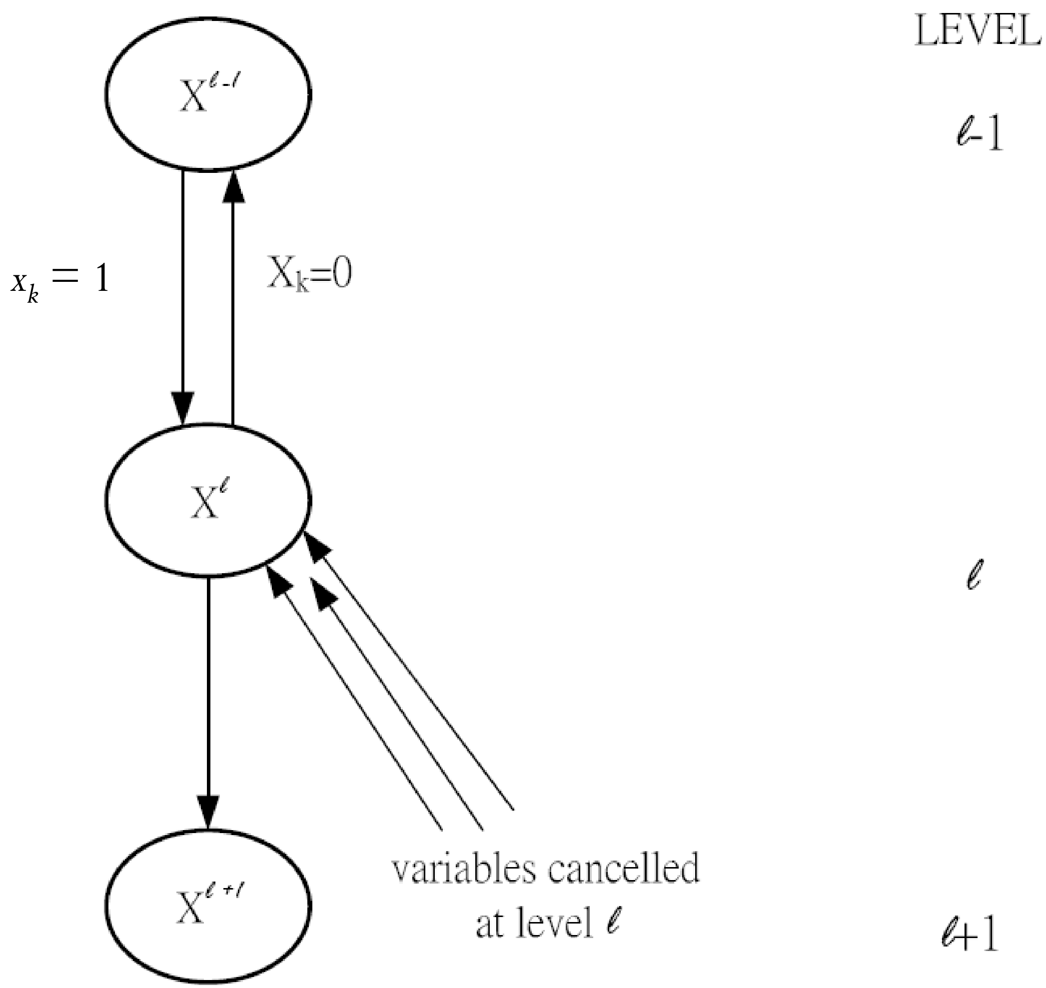
- There are ___ possible 0-1 vectors x.
- Like in branch and bound, a search enumeration procedure is also related to a tree.
- A **node** corresponds to a particular combination of 0-1 values for x.
- A **branch** joints two nodes reachable from one another. The two nodes differ in the state of one variable.
- A variable can be fixed at 1, fixed at 0, or free. A new node is defined by fixing a variable to 1 (**forward step**), and a node is revised after fixing a variable to 0 (**backward step**).
- A point x^{ℓ} is a node x with ℓ variables fixed at 1; ℓ is the level of the node.

Basic Approach

1. Fix a free variable x_k from x^ℓ (initially, $x^\ell = x^0$) at value 1.
2. Resolve the subproblem for the remaining free variable then
3. Fix x_k at value 0 (or cancel x_k at level ℓ), then
4. Resolve the subproblem for the remaining free variables, then
5. Repeat this process for the problem with x_k at value 0.

- In a new node, ℓ variables are fixed at 1, c variables are fixed at 0, and we are concerned in solving the 0-1 ILP subproblem for the remaining $f = n - (\ell + c)$ free variables. This requires enumerating, implicitly or explicitly, __ points x^ℓ with (ℓ + c) variables fixed.

Illustration of the Tree

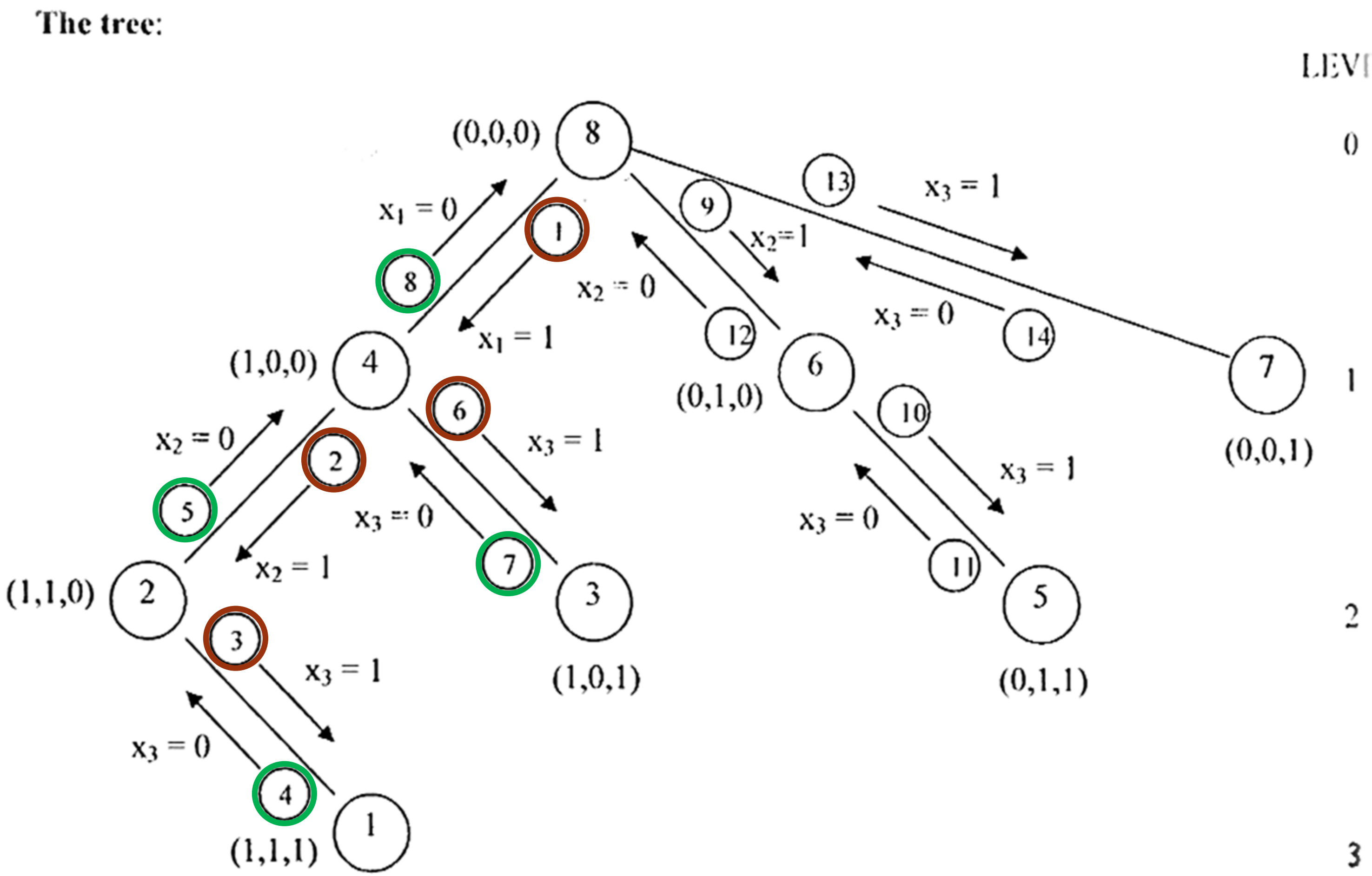


Example: 3-variable BIP

- There are 2^3 nodes corresponding to vectors.

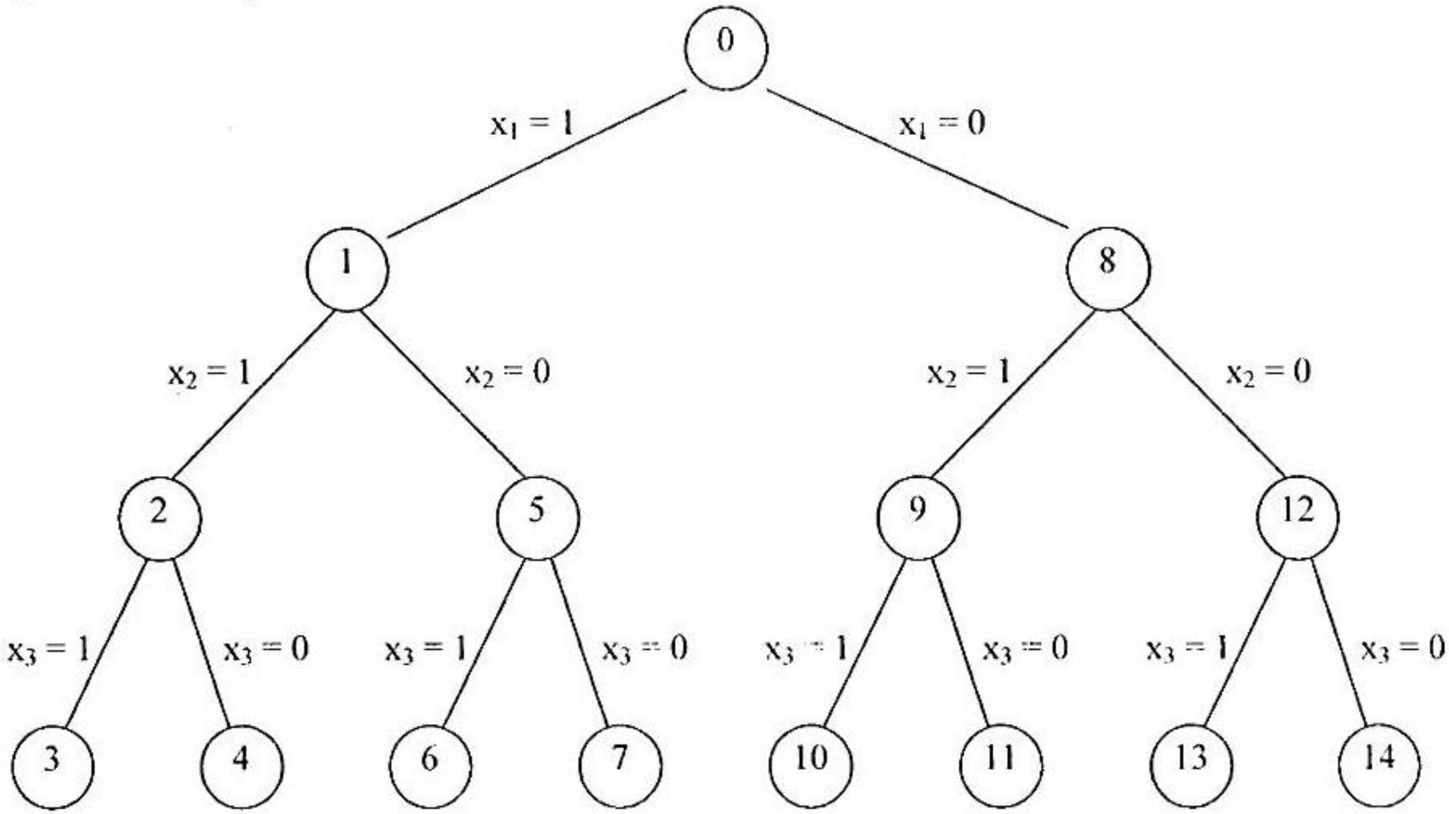
Iteration	Occurrence	Step	Level	State of Node	Node Number
0	-	-	0	(x_1, x_2, x_3)	
1	$x_1 = 1$	Forward(F)	1	$(1, x_2, x_3)$	
2	$x_2 = 1$	F	2	$(1, 1, x_3)$	
3	$x_3 = 1$	F	3	(1, 1, 1)	(1)
4	$x_3 = 0$	Backward(B)	2	(1, 1, 0)	(2)
5	$x_2 = 0$	B	1	$(1, 0, x_3)$	
6	$x_3 = 1$	F	2	(1, 0, 1)	(3)
7	$x_3 = 0$	B	1	(1, 0, 0)	(4)
8	$x_1 = 0$	B	0	$(0, x_2, x_3)$	
9	$x_2 = 1$	F	1	$(0, 1, x_3)$	
10	$x_3 = 1$	F	2	(0, 1, 1)	(5)
11	$x_1 = 0$	B	1	(0, 1, 0)	(6)
12	$x_2 = 0$	B	0	$(0, 0, x_3)$	
13	$x_3 = 1$	F	1	(0, 0, 1)	(7)
14	$x_3 = 0$	B	0	(0, 0, 0)	(8)
15	Stop				

Complete Search Tree



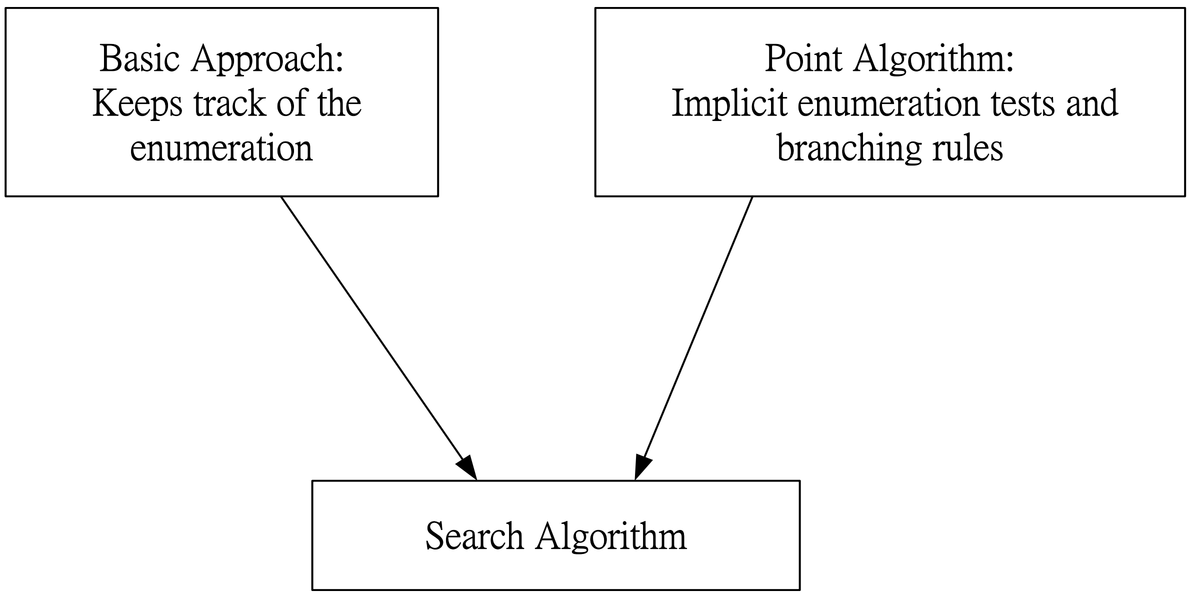
Binary Tree

Equivalent binary tree:



Composition of Search Algorithm

- The efficiency of an enumerative technique relies on the ability to enumerate large portion of the tree implicitly.



Composition of Search Algorithm

- Tests for implicit enumeration are applied to the current subproblem (node).
- These tests are based on the following criteria:
  - Current 0-1 solution cannot be improved.
  - Current constraints are inconsistent.
  - Certain free variables must take 0 or 1 values for feasibility.
- These tests with the branching rules form the **Point or Node Algorithm**. The basic approach along with a point algorithm forms a **Search Algorithm**.

Agenda

- Enumeration Tree
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- Branching Rules

Notations (1/2)

- Subproblem at point x^l:

minimize $z = cTxl,$

subject to

$$Ax^l \leq b,$$

$$x_j \in \{0,1\}, j = 1, \ldots, n.$$

where vector x^l contains l variables fixed to 1, c variables fixed at 0 and $f = n–(\mathrm{l} + c)$ free variables. Let F be the set of indices for the free variables. Then, the subproblem is equivalent to

Notations (2/2)

(Problem P^l) minimize $z = z^l + c^{fT} x^f$,

subject to

$$A^f x^f \leq bl,$$

$$x_j \in \{0,1\}, j \in F.$$

where vector x^f is the vector of free variables, c^f and A^f are the associated costs and coefficients, b^l is the updated b, and z^l is the sum of the costs of the l variables fixed to 1. In addition, let

P^ℓ: subproblem at point x^ℓ,

LP^ℓ: LP associated to P^ℓ,

$\hat{z}$: optimal solution of LP^ℓ including z^ℓ, and

z^*: current minimal integer solution.

Initially, we may set z^* equal to the ______________________________.

Ceiling Tests

- It is possible to improve z* only when

$$z^l + \sum_{j \in \bar{F}} c_j < z^*,$$

where $\bar{F} = \{ j \in F : c_j < 0 \}$.

- We can cancel any x_t with objective c_t , $t \in F$, for which

$$c_t + z^l + \sum_{j \in \bar{F}} c_j \square z^*.$$

Example

- Suppose $z^{*}=12$, $z^{\ell}=25$, and the objective function with respect to the free variable is

$$15x_2 - 30x_3 + 18x_7$$

- Suppose $z^{*}=8$, $z^{\ell}=10$, and the objective function with respect to the free variable is

$$30x_3 + 2x_{17} - 5x_{20}$$

Nonnegative Costs (Zero Completion)

- Transfer the problem with negative costs to the one with all nonnegative costs (Again, for minimization problems).
- In this case, set $\bar{F}$ is an empty set.
- Set all free variables at zero.
- If the node is feasible, it is ______ to the subproblem and the solution (z^ℓ) is a ___________ to the original problem.
  - How to check its feasibility?
- If all the costs are ≥ 0 and the node is feasible, then a signal for the _______ step.

Infeasibility Test

- From the current subproblem's constraints, the slack variable s^ℓ may be written as

$$s^l = \mathbf{b}^l - \mathbf{A}^f \mathbf{x}^f \geq 0.$$

Define $F_i^- = \{j \in F: a_{ij} < 0\}$,

and $P_i = b_i^\ell - \sum_{j \in F_i^-} a_{ij}$, i=1,...,m.

P_i is the largest possible value for slack variable s_i.

Thus ,since $x_j \leq 1$, for a 0-1 solution to be possible, we must have

$P_i \geq 0$, i=1,...,m.

Note that

$$b_i^\ell \geq 0,\ i = 1, \ldots, m. \quad => \quad P_i \quad 0,\ i = 1, \ldots, m.$$

Example

$$-6x_1 + 15x_2 - 30x_8 \leq -20, i = 1$$

$$7x_1 - \ \ 2x_2 + \ \ \ \ x_8 \leq -5, i = 2$$

Then,

$$P_1 = -20 - (-6 - 30) = 16$$
$$P_2 = -5 - (-2) = -3$$

Cancellation Zero Test

It is possible that by setting a free variable x_j with positive coefficients in certain rows to 1 may result in some P_i<0. Therefore, for any $j \in F$ with a_{ij}>0, cancel x_j if

$$P_i - a_{ij} < 0, \text{ for some } i,\ i=1,\ldots,m.$$

That is, x_j can't be one (or $x_j = 0$).

Drop x_j, update P_i and repeat the test .

Example

$$-6x_1 + 15x_2 - 30x_8 \le -20, i = 1$$

$$7x_1 - \ 6x_2 + \quad x_8 \le -5, i = 2$$

Then,

$$P_1 = -20 - (-6 - 30) = 16$$
$$P_2 = -5 - (-6) = 1$$

Cancellation One Test

Some free variables x_j may have to be set to 1 to obtain a feasible solution ($P_i \geq 0$, $i=1,\ldots,m$). Hence, for any $j \in F$ with $a_{ij} < 0$, set x_j to 1 if

$$P_i + a_{ij} < 0, \text{ for some } i, \ i=1,\ldots,m.$$

That is, x_j can't be zero (or $x_j = 1$).

Examples

Example 1:

$$3x_1 - 2x_3 - 5x_5 - x_7 \leq -6,\ i = 2,$$

Then,

$$P_2 = -6 - (-2 - 5 - 1) = 2,$$

Example 2:

$$5x_2 - 2x_3 - 4x_4 - x_6 \leq -5,\ i = 5,$$

Linear Programming

- After solving LP^{ℓ}, the corresponding LP for subproblem P^{ℓ}, we can consider the following result:
  - LP^{ℓ} solution is integer => It is an optimal solution for P^{ℓ}.
  - LP^{ℓ} is infeasible => P^{ℓ} is infeasible.
  - $\hat{z}$, the optimal objective function value for LP^{ℓ}, is a lower bound for any 0-1 feasible solution for P^{ℓ}. (minimization)
  - Suppose LP^{ℓ} is solved by the dual simplex algorithm. The value of the objective function is increasing because the dual of LP^{ℓ} is a maximization problem. Therefore, if at any iteration the objective function value reaches or exceeds z^*, a ________ step is allowed.

Post-Optimization, Penalties

- After solving LP^{ℓ}, we may consider the effect on the optimal objective function value $\hat{z}$ of forcing each integer variable to 0 or 1.
- After adding a new constraint(i.e., forcing an integer variable to 0 or 1) and re-optimizing, the objective function value will take a value $\hat{\hat{z}} \geq \hat{z}$.
- Then, $\hat{\hat{z}} - \hat{z}$ is the penalty for forcing a variable to 0 or 1. If the added constraint makes the linear program infeasible, we may set the penalty to ∞.
- The new value $\hat{\hat{z}}$ may be compared with z^*. If $\hat{\hat{z}} \geq z^*$, we must change the value of the variable under consideration, from 0 to 1, or from 1 to 0.

Summary of the LP Post-optimization Analysis

Current value of x_j	New constraint: $x_j = 0$	New constraint: $x_j = 1$	Action when $\hat{z} \geq z^*$
0	X	√	x_j is cancalled
1	√	X	$x_j = 1$
fractional	X	√	x_j is cancelled
fractional	√	X	$x_j = 1$
fractional	√	√	Backward step

Compute Penalties

- To compute the penalties, we need to consider whether x_j is basic or nonbasic. If x_j is basic and fractional, a new constraint is added. In this case, dual simplex iterations are necessary to regain optimality.
- If x_j is nonbasic with value 0, the penalty for forcing it to 1 is a_{0j} (coefficient in row 0), and to 0 is zero.
- Thus, if $\overline{x}^f$ is a particular 0-1 vector value with respect to the free variables, a total penalty for this point is given by

Compute Penalties

$$P\left(\bar{x}^f\right)=\max\left\{\ \max_{j\in B} P_j,\ \sum_{j\in NB_0} a_{0j} + \sum_{j\in NB_1}(-a_{0j})\ \right\},$$

where $B = \{j \in F:\ x_j \text{ is a basic variable}\}$,

$NB_0 = \{j \in F: x_j \text{ is a nonbasic variable with value 0 set to 1 in } \bar{x}^f\}$,

$NB_1 = \{j \in F: x_j \text{ is a nonbasic variable with value 1 set to 0 in } \bar{x}^f\}$,

and $P_j = \hat{\hat{z}} - \hat{z}$, when x_j takes the corresponding value in $\bar{x}^f$.

Then,

- $\hat{z}+P\left(\bar{x}^f\right) \geq z^*$ => by setting x^f to $\bar{x}^f$ we cannot get an improved 0-1 solution.
- If very few completion of x^f are possible, we can compute the total penalty for each one.

Agenda

- Enumeration Tree
- Implicit Enumeration
- Branching Rules

Point Algorithm (Branching Strategies)

- Select which free variable to be one?
- It is possible to eliminate some of the branching candidates.
  - Find a subset which contains the free variables whose values have to be one so that the solution can be improved.
  - That is, the free variables in this subset can be considered as a branching candidates.

Preferred Sets (1/2)

- When a forward step from node x^ℓ is to be made, one of the free variables will be set to 1. Sometimes it is possible to find a subset of the free variables in which at least one must be set to 1 for an improved 0-1 solution at node x^ℓ.

Preferred Sets (2/2)

- Consider the following i^{th} constraint:

$$\sum_{j \in F} a_{ij} x_j \le b_i^{\ell}, \quad \text{with } b_i^{\ell} < 0 \text{ and integral,}$$

$$\Leftrightarrow \quad \sum_{j \in F_i^+} a_{ij} x_j + \sum_{j \in F_i^-} a_{ij} x_j \le b_i^{\ell},$$

where $F_i^+ = \left\{ j \in F : \; a_{ij} > 0 \right\}$ and $F_i^- = \left\{ j \in F : \; a_{ij} < 0 \right\}$.

$$\Leftrightarrow \quad \sum_{j \in F_i^-} (-a_{ij}) x_j \ge -b_i^{\ell} + \sum_{j \in F_i^+} a_{ij} x_j \ge -b_i^{\ell} \ge 1, \qquad (*)$$

$$\Rightarrow \quad \sum_{j \in F_i^-} (-a_{ij}) x_j \ge 1 \quad \Rightarrow \quad \sum_{j \in F_i^-} x_j \ge 1.$$

Example

Row i: $7x_3 - 6x_5 + x_6 - 9x_8 - 8x_{10} \leq -15,$

$\Leftrightarrow \quad 6x_5 + 9x_8 + 8x_{10} \geq 15 + 7x_3 + x_6 \geq 15,$

$\Rightarrow \quad x_5 + x_8 + x_{10} \geq 1.$

In this case, note also that

$$(*) \quad \Leftrightarrow \quad 9x_8 \geq 15 + 7x_3 + x_6 - 6x_5 - 8x_{10}$$

$$\geq 15 - 6x_5 - 8x_{10} \geq 1$$

$$\Rightarrow \quad 9x_8 \geq 1 \quad \Rightarrow \quad x_8 \geq \frac{1}{9} \quad \Rightarrow \quad x_8 = 1.$$

Then,

$$7x_3 - 6x_5 + x_6 - 8x_{10} \leq -6$$

$$\Leftrightarrow \quad 6x_5 + 8x_{10} \geq 6 + 7x_3 + x_6 \geq 6,$$

$$\Rightarrow \quad x_5 + x_{10} \geq 1.$$

Complete Reduction

- Subtract from $b_i^l < 0$ the terms a_{ij}, with $a_{ij} < 0$, in order of decreasing a_{ij} until b_i^l becomes nonnegative.
- The remaining variables (including the last one) are in the preferred set.
- The generated inequality is

$$\sum_{j \in P} x_j \geq 1, \text{ where } P \subset F^- \subset F.$$

- Example

$$5y_2 - y_3 - 2y_4 - 4y_1 \leq -5$$
$$-y_3 - 2y_4 - 4y_1 \leq -5 \Rightarrow -2y_4 - 4y_1 \leq -4 \Rightarrow -4y_1 \leq -2$$
$$y_1 \geq 1, \text{ i.e., } P = \{1\}.$$

Summary - Preferred Set

- Only the preferred variables need to be considered as branch candidates. Once all of them have been cancelled at level ℓ, a backward step may be taken.
- It is advantageous to perform complete reduction on each constraint i which has $b_i^\ell < 0$, and choose a preferred set with a minimal number of entries.
- If the (smallest) preferred set contains more than one element, a simple rule is to select the preferred variable with the least cost coefficient.

Balas Test

If a free variable x_j is set to 1, the right-hand-side of each constraint i becomes $b_i^\ell - a_{ij}$. A measure of the total "constraint infeasibility" is given by

$$v_j = \sum_{i \in M_j} \left(b_i^\ell - a_{ij} \right) < 0$$

where $M_j = \left\{ i: b_i^\ell - a_{ij} < 0,\ i=1,\ldots,m \right\},$

and $v_j = 0 \text{ if } M_j = \phi.$

Example (1/2)

- In order to rapidly reach or return to a 0-1solution, we can choose the preferred variable which maximizes v_j for branching.
- Note that $v_j \leq 0$. If $v_j = 0$, a solution exists by fixing $x_j = 1$ and the other free variables to zero.

$$-4x_6 - 2x_9 - 5x_{11} \leq -2$$

$$6x_6 - 2x_9 - x_{11} \leq 4$$

$$-4x_6 - 5x_9 - x_{11} \leq -5$$

Example (2/2)

By doing complete reduction, the following preferred sets are generated:

From row 1, $\{6,9,11\}$,

From row 3, .

Computing v_j for the preferred set with the smallest cardinality yields

$v_6 =$

$v_9 =$

Then, by setting $x_9 = 1$, the constraints become satisfied with $x_6 = x_{11} = 0$.

Questions?

- Homework 4 is due on 5/18.