■ Standard Form and Conversion

標準形式：①目標函數極大 ② 所有限制皆為 $\leq$ ③ 整數決策函數有下界 0 和上界 ∞ ④ 連續決策變數非負且無界

非標準形式與轉換方法：

(1) Minimization Problem: 左右同乘 (-1) 並改為極大
(2) Inequality of $\geq$ form: 左右同乘 (-1) 並變號
(3) Equality Constraint: 等號改為不等號，轉為 $\leq$ 形式
(4) Unrestricted Variable: 將未限制變數轉為兩新變數的差值，如 $x_i = x_i^+ - x_i^-$，其中 $x_i^+ = x_i$ if $x_i > 0$, else $x_i^+ = 0$ 且 $x_i^- = -x_i$ if $x_i < 0$, else $x_i^- = 0$
(5) Variable with finite upper bound: $x'_j = x_j - l_j$

■ Linear and Convex Combinations

DEF：Given p vectors, $\mathbf{a}^1, \mathbf{a}^2, \cdots, \mathbf{a}^p$ in $\mathbf{E}^n$, and p scalars $\lambda_1, \lambda_2, \cdots, \lambda_p$, then $\lambda_1\mathbf{a}^1, \lambda_2\mathbf{a}^2, \cdots, \lambda_p\mathbf{a}^p$ is called a linear combination. The scalars are real numbers that can be positive, negative or zero. The linear combination becomes a convex combination when $\lambda_1 + \lambda_2 + \cdots + \lambda_p = 1$ and $0 \leq \lambda_1, \lambda_2, \cdots, \lambda_p \leq 1$.

■ Primal and Dual Formulation

Maximization Problem	Minimization Problem
Constraint i	Variable I
$\leq$	≥ 0
$\geq$	≤ 0
$=$	Unrestricted in sign
Variable j	Constraint J
≥ 0	$\geq$
≤ 0	$\leq$
Unrestricted in sign	$=$
Objective Row	RHS Column
RHS Column	Objective Row

■ Simplex Method and Tableau

(1) Obtain the canonical system and fill it in a tableau.
(2) Choose a NBV (select the most negative value) to be the entering variable and BV (select that it has a minimum positive ratio) to be the leaving variable.
(3) Iterate until there is no negative value in the First Z Row.

■ Dual Fractional Integer Programming

(1) 加入寬鬆變數並建立表格，先選取 Pivot Row (選寬鬆變數中負較多者)，再決定 Pivot Column (選對應比值絕對值較小者)，若為 Infeasible 或存在整數解則停止。
(2) 將選定的 Pivot Column 進行單位化並置換變數，使 Pivot Element 為 -1 後進行基本行運算使同一 Row 上其他元素為 0，若仍存在寬鬆變數仍為負值則反覆迭代。
(3) 若所得決策變數不為整數，選擇 Source Row 並創建 Cut 方程式，其中建立方式如下，建立後繼續上述操作至求得整數最佳解。.

✓ Form of a Gomory Cut

The k^th Gomory cut: the Gomory slack variable will be

$$x_{n+m+k} = -f_{v,0} + \sum_{j=1}^{n}(-f_{v,0})(-x_{J(j)}) \geq 0$$

$$f_{v,j} = a_{v,j} - \lfloor a_{v,j} \rfloor, j = 0, 1, \cdots, n$$

■ Dual Fractional Mixed Integer Programming

步驟與上述雷同，但要注意 Cut 方程的構造方式有所不同，必須根據變數性質有以下不同計算。

✓ Form of a Gomory Cut for MIP

The k^th Gomory cut: the Gomory slack variable will be

$$x_{n+m+k} = -f_{v,0} + \sum_{j=1}^{n}(-g_{v,0})(-x_{J(j)}) \geq 0$$

$$g_{v,j} = \begin{cases} a_{v,j}, & \text{if } a_{v,j} \geq 0, x_{J(j)} \text{ continuous} \\ \frac{f_{v,0}}{f_{v,0}-1} a_{v,j}, & \text{if } a_{v,j} < 0, x_{J(j)} \text{ continuous} \\ f_{v,j}, & \text{if } f_{v,j} \leq f_{v,0}, x_{J(j)} \text{ integer} \\ \frac{f_{v,0}}{1-f_{v,0}}(1 - f_{v,j}), & \text{if } f_{v,j} > f_{v,0}, x_{J(j)} \text{ integer} \end{cases}$$

$$f_{v,j} = a_{v,j} - \lfloor a_{v,j} \rfloor, j = 0, 1, \cdots, n$$

■ Dual Fractional Mixed Integer Programming

(1) 加入寬鬆變數並建立表格，先選定 Source Row (選寬鬆變數中負較大者)，再根據 α_p 決定 Pivot Column (選 α_p 較小者)，並令 $u_p = 1$ 並計算決定 u_j (選其中最大的整數)。
(2) 根據下述公式決定 λ。
(3) 將選定的 Pivot Column 進行單位化並置換變數，使 Pivot Element 為 -1 後進行基本行運算使同一 Row 上其他元素為 0，若仍存在寬鬆變數仍為負值則反覆迭代。

✓ Form of a Gomory Cut for Dual All-Integer

The k^th Gomory cut: the Gomory slack variable will be

$$x' = \lfloor \frac{a_{v,0}}{\lambda} \rfloor + \sum_{j=1}^{n} \lfloor \frac{a_{v,j}}{\lambda} \rfloor (-x_{J(j)})$$

$$\alpha_p < \frac{\alpha_j}{u_j} \text{ to find } u_j$$

$$\lambda_j = \frac{a_{v,j}}{u_j}, \text{let } \lambda = \max\{\lambda\}$$

■ 0-1 Binary Transformation

✓ $2y_{jk} \leq y_j + y_k \leq y_{jk} + 1$
$y_j, y_k, y_{jk} = 0$ or 1

✓ $3y_{ijk} \leq y_i + y_j + y_k \leq y_{ijk} + 2$
$y_i, y_j, y_k, y_{ijk} = 0$ or 1

■ Homework 01-02

$$\begin{array}{lllll} \max & 4x_1 & + & 2x_2 & & \\ \text{s.t.} & 2x_1 & & & \leq & 16 \\ & x_1 & + & 3x_2 & \leq & 17 \\ & \multicolumn{4}{l}{x_2 \leq 5, x_2 \geq 0, x_2 \geq 0} \end{array}$$

Determine how many additional units of resource 1 (constraint 1) would be needed to increase the optimal value by 15. Justify your answer.

《解》

While adding 1 unit of resource 1

$$\begin{cases} 2x_1 & = 17 \\ x_1 + 3x_2 & = 17 \end{cases} \implies x_1 = \frac{17}{2}, x_2 = \frac{17}{6}$$

$$\implies z' = \frac{119}{3} \implies \Delta z = z' - z = \frac{5}{3}$$

■ Homework 01-03

Show the feasible region of a LP model, $\{A\mathbf{X} \leq \mathbf{0}, \mathbf{X} \geq \mathbf{0}\}$ is convex.

《解》

Let $x_1, x_2 \in A\mathbf{X}$ and $0 \leq \lambda \leq 1$, $\exists\, t_1, t_2 \in x$ such that $x_1 = A \cdot t_1$ and $x_2 = A \cdot t_2$
Then $\lambda x_1 + (1-\lambda)x_2 = \lambda(A \cdot t_1) + (1-\lambda)(A \cdot t_2)$
$= A[\lambda t_1 + (1-\lambda)t_2]$. Since $t_1, t_2 \in x$ and x is **Convex**, $A[\lambda t_1 + (1-\lambda)t_2] \in A\mathbf{X}$.

■ Homework 01-04

A caterer to "The Ritz" motel collects the dirty napkins and sends them to laundry every day. Due to different room occupation levels during a week, the number of napkins needed on day i is d_i. The caterer can wash and dry at most u napkins every day and the cleaned napkins will be ready for use next day. If the dirty napkin is not cleaned, a new one is purchased at the price of p. If the laundry room is used on day i, a fixed cost of f_i is incurred. Assume that at the beginning of a week, there are n clean napkins and no dirty napkins left. Find the best laundry plan for the caterer so that the entire week's cost is minimized.

《解》

Let w_i be the number of napkins cleaned, c_i be the number of clean napkins, uw_i be the number of uncleaned napkins, and b_i be the number of napkins purchased on day i. Let y_i be the binary variable when $y_i = 1$, napkins are cleaned on day i.

Constraints:

✓ For day 1:
$c_1 = (n + b_1) - d_1 + w_1$
$uw_1 = d_1 - w_1$
$w_1 \leq y_1 \cdot u$

✓ For day i:
$c_i = (c_{i-1} + b_i) - d_i + w_i$
$uw_1 = uw_{i-1} + d_u - w_{i-1}$
$w_i \leq y_i \cdot u$

Objective: $\min p \sum_i (b_i + f_i y_i)$

■ Homework 01-05

Assume that you have spare saving $\$10,000$ each year. There are three investment tools you can choose from:
① deposit ② mutual funds ③ bonds
The annual interest rate is 2% if you deposit your money in a bank. If you buy mutual funds, the investment length is two years and the return is estimated to be 7% after two years. If you invest in bonds, you can get 4% of interest payment every year but the investment length is 4 years. At the end of year, you will re-invest all your available money and renew your portfolio. In addition, you are advised to deposit at least 30% of your available money in a bank and the amount of money invested in mutual funds not greater than twice of the amount invested in bonds throughout the entire investment period. Please formulate a mathematical model to maximize your money at the end of the fifth year. (At the beginning you already have $\$10,000$ and the investment length is five years.)

《解》

Let x_i the amount invested in deposit in year $i = 1, \cdots, 5$.
Let y_i the amount invested in funds in year $i = 1, \cdots, 4$.
Let z_i the amount invested in bonds in year $i = 1, 2$.
Objective: $\max(1.02x_5 + 1.07y_4 + 1.04z_2)$
Subject to: (Note that all variables ≥ 0)

1 Year
$x_1 \geq 3000$
$y_1 \leq 2z_1$
$x_1 + y_1 + z_1 = 10000$

2 Year
$y_2 \geq 0.3(10000 + 1.02x_1 + 0.04z_1)$
$y_1 + y_2 \leq 2(z_1 + z_2)$
$x_2 + y_2 + z_2 = 10000 + 1.02x_1 + 0.04z_1$

3 Year
$x_3 \geq 0.3(10000 + 1.02x_2 + 1.07y_1 + 0.04(z_1 + z_2))$
$y_2 + y_3 \leq 2(z_1 + z_2)$
$x_3 + y_3 = 10000 + 1.02x_2 + 1.07y_1 + 0.04(z_1 + z_2)$

4 Year
$x_4 \geq 0.3(10000 + 1.02x_2 + 1.07y_1 + 0.04(z_1 + z_2))$
$y_3 + y_4 \leq 2(z_1 + z_2)$
$x_4 + y_4 = 10000 + 1.02x_3 + 1.07y_2 + 0.04(z_1 + z_2)$

5 Year
$y_4 \leq 2z_2$
$x_5 = 10000 + 1.02x_3 + 1.07y_2 + 1.04z_1 + 0.04z_2$

■ Homework 02-01

Ford has four automobile plants. Each plant is capable of producing the Taurus, Lincoln, or Escort, but it can only produce one of these cars. The fixed cost of operating each plant for a year and the variable cost of producing a car of each type at each plant are given as follows:

Plant	Fixed Cost	Taurus	Lincoln	Escort
1	7 Billions	12000	16000	9000
2	6 Billions	15000	18000	11000
3	4 Billions	17000	19000	12000
4	2 Billions	19000	22000	14000

Each year, Ford must produce 500,000 of each type of car. Formulate an IP which minimize the annual cost of producing cars based on the following constrains:
(a) Each plant can produce only one type of car.
(b) The total production of each type of car must be at a single plant.
(c) If plant 2 is used, plant 3 cannot be used.
If plant 3 and plant 4 are used, plant 1 must also be used.

《解》

✓ The factory i $(i = 1, 2, 3, 4)$ produce j $(j = 1, 2, 3)$ car type.
✓ Whether i produce j or not, $y_{ij} = 0$ or 1. If $y_{ij} = 1$ how much x_{ij}.
✓ Fixed Cost: $(f_{1j}, f_{2j}, f_{3j}, f_{4j}) = (7, 6, 4, 2)$ billion $\forall j$.
✓ Variable Cost:

$$\mathbf{C} = [c_{ij}] = \begin{bmatrix} 12000 & 16000 & 9000 \\ 15000 & 18000 & 11000 \\ 17000 & 19000 & 12000 \\ 19000 & 22000 & 14000 \end{bmatrix}$$

Then

$$\begin{array}{ll} \min & \sum_{i=1}^{4}\sum_{j=1}^{3}(c_{ij}x_{ij} + f_{ij}y_{ij}) \\ \text{s.t.} & \sum_{j=1}^{3} y_{ij} \leq 1,\ \forall i = 1, 2, 3, 4 \\ & \sum_{i=1}^{4} y_{ij} = 1,\ \forall j = 1, 2, 3 \\ & \sum_{j=1}^{3}(y_{2j} + y_{3j}) \leq 1,\ \forall j = 1, 2, 3 \\ & \sum_{j=1}^{3} y_{1j} \geq \sum_{j=1}^{3} y_{3j} + \sum_{j=1}^{4} y_{4j} - 1 \\ & \sum_{i=1}^{4} x_{ij} \geq 500000 \\ & My_{ij} \geq x_{ij},\ \forall i = 1, 2, 3, 4;\ \forall j = 1, 2, 3 \\ & x_{ij}, y_{ij} \in \{0, 1\},\ \forall i = 1, 2, 3, 4;\ \forall j = 1, 2, 3 \end{array}$$

■ Homework 02-02

Consider the following problem:

$$\begin{array}{lllllllll} \max & 4x_1 & - & {x_1}^2 & + & 10{x_2}^2 & - & {x_2}^3 \\ \text{s.t.} & x_1 & + & x_2 & \leq & 3 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0 \text{ and } x_1, x_2 \in \mathbb{Z}} \end{array}$$

This problem can be reformulated as an equivalent pure Binary Integer Programming (BIP) problem, depending on the definitions of the binary variables. Assume that the binary variables are interpreted as: $y_{ii} = 1$ if $x_i \geq j$ $(i = 1, 2$ and $j = 1, 2, 3)$, and $y_{ij} = 0$ otherwise.

《解》

$$\begin{aligned} \max[&4(y_{11} + y_{12} + y_{13}) - (y_1 1 + 3y_{12} + 5y_{13}) \\ &+10(y_{21} + 3y_{22} + 5y_{23}) - (y_2 1 + 7y_{22} + 19y_{23})] \end{aligned}$$

s.t. $y_{12} \leq y_{11},\ y_{13} \leq y_{12},\ y_{22} \leq y_{21},\ y_{23} \leq y_{22},$
$y_{11} + y_{23} \leq 1,\ y_{12} + y_{22} \leq 1,\ y_{13} + y_{21} \leq 1$

■ Homework 02-03

John is buying stocks. His broker suggests six different stocks, namely, $1, 2, \cdots, 6$. Let c_i denote the return of purchasing stock j. Formulate the stock selection problem subject to the following constraints, using $0-1$ variables as needed:

(a) To lower the risk of losing money, John should buy at least two stocks.

(b) Due to John's budget limit, he cannot buy more than four stocks.

(c) Since stocks 3 and 5 belong to the same company, the broker recommends purchase of at most of one of these.

(d) The broker suggests the following two combinations: **either** choose two from stocks $1,2,3$ and 4, **or** at least two from stocks $3,4,5$ and 6.

Stock 4 can only be purchase if stock 1 is bought.

《解》

- ✓ Take y_j represent buy the stock j $(j=1,2,\cdots,6)$ or not. ($y_j=1$ for 'YES' and $y_j=0$ 'NO')
- ✓ Let $k_i \in \{0,1\},\ i=1$

$$
\begin{array}{ll}
\min & \sum_{j=1}^{6}(c_j \times y_j) \\
\text{s.t.} & \sum_{j=1}^{6} y_j \ge 2 \\
 & \sum_{j=1}^{6} y_j \le 4 \\
 & y_3 + y_5 \le 1 \\
 & \sum_{j=1}^{4} y_j \le (2 + k_1 \times M), \text{if } k_1 = 0 \\
 & \sum_{j=1}^{4} y_j \ge (2 - k_1 \times M), \text{if } k_1 = 0 \\
 & \sum_{j=3}^{6} y_j \ge [2 - M(1-k_1)], \text{if } k_1 = 1 \\
 & y_1 - y_4 \ge 0
\end{array}
$$

■ Homework 03-02

Consider a tableau which is neither primal nor dual feasible. Suppose we add the following redundant constraint

$$s = M + \sum_{j=1}^{n} 1(1 - x_{J(j)}) \ge 0$$

to the bottom of the tableau (s is a nonnegative slack variable and M is a large positive number). Show that after a pivot on the 1 element in the new row which is in the lexicographically smallest column, the tableau exhibits dual feasibility.

《解》

Neither dual nor primal are feasible, and thus it means at least one $a_{0j} < 0$ in the tableau. Assume α_u has the smallest lexicographically order for all $a_{0j} < 0$, then we use α_k to pivot, the new $a'_{0k} = a'_{0k} > 0$; and all other $a'_{0j} = a_{0j} - a_{0k} \ge 0$, then the tableau become dual feasible.

For Example:

$$
\begin{array}{lccccc}
\max & -2x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \le & 3 \\
 & 2x_1 & + & x_2 & \le & -4 \\
 & x_1, x_2 & & & \ge & 0
\end{array}
$$

■ Homework 03-05

Solving the following problem by the all-integer algorithm.

$$
\begin{array}{lccccc}
\max & -2x_1 & - & 5x_2 & = & x_0 \\
\text{s.t.} & -2x_1 & - & -2x_2 & \le & -9 \\
 & -2x_1 & - & 6x_2 & \le & -22 \\
 & x_1, x_2 \in \mathbb{Z} & & & &
\end{array}
$$

《解》

# 01	1	$(-x_1)$	$(-x_2)$	# 01
x_0	0	2	5	
x_1	0	-1	0	$(x_1 = x_2 = 0)$
x_2	0	0	-1	$p=1$
x_3	-9	-2	-2	$u_1=1,\ u_2=1$
x_4	-22	-2	-6	$\lambda_1=2,\ \lambda_2=3$
x_5	-8	-1	-2	$\lambda=\max(\lambda_1,\lambda_2)=3$

# 02	1	$(-x_5)$	$(-x_2)$	# 02
x_0	-16	2	1	
x_1	8	-1	2	$p=2$
x_2	0	0	-1	$u_1=1,\ u_2=1$
x_3	7	-2	2	$\lambda_1=2,\ \lambda_2=2$
x_4	-6	-2	-2	$\lambda=\max(\lambda_1,\lambda_2)=2$
x_5	0	-1	0	
x_6	-3	-1	-1	

# 03	1	$(-x_5)$	$(-x_6)$	# 03
x_0	-19	1	1	
x_1	2	1	2	$x_0=-19$
x_2	3	1	-1	$x_1=2$
x_3	1	0	2	$x_2=3$
x_4	0	0	-2	
x_5	0	-1	0	
x_6	0	0	-1	

■ Homework 03-03 (2016)

Formulate the following scheduling problem as an IP problem. A set of n jobs are to be processed on m machine. Each job j has a release time (earliest time when the job is ready to be processed) r_j and a processing time p_{ij} on machine i, and no two jobs can be processed on the same machine simultaneously. But during the same time interval, different jobs can be processed on different machines. Schedule the processing order of the jobs so that the makespan z (the completion time of job j on machine i) is minimized. Consider the following two cases:

(a) The m machines are identical, that is, $p_{ij} = p_j$ for each j.

(b) The m machines are different

《解》

- ✓ x_{ij} = the job $j's$ available time on machine i.
- ✓ y_{ij} = if the job j is assigned to machine i. ("Yes" for 1, "NO" for 0).
- ✓ $Z_{i,jk}$ = if the job j is assigned before job k on machine i.

Then

$$
\begin{array}{ll}
\min & C_{max} \\
\text{s.t.} & x_{ij} + y_{ij}p_{ij} \le x_{ik} + MZ_{i,jk} \\
 & x_{ik} + y_{ik}p_{ik} \le x_{ij} + M(1 - Z_{i,jk}) \\
 & x_{ij} \le My_{ij} \\
 & \sum_{i=1}^{m} y_{ij} = 1 \\
 & y_{ij}r_j \le x_{ij} \\
 & C_{max} \ge x_{ij} + y_{ij}p_{ij} \\
 & x_{ij} \ge 0, Z_{i,jk} \in (0,1)
\end{array}
$$

■ Homework 03-04 (2016)

Solve the given MIP Problem using a cutting plane approach

$$
\begin{array}{ll}
\max & z = 5x_1 + 3x_2 + 7y_1 + 2y_2 \\
\text{s.t.} & 7x_1 + 8x_2 + 9y_1 + 3y_2 \le 43 \\
 & 11x_1 + 4x_2 + 4y_1 + 5y_2 \le 51 \\
 & x \ge 0 \\
 & y \ge 0 \text{ and integer}
\end{array}
$$

Solving the LP relaxation, we obtain an LP optimum $y_1 = 43/9$. Please use y_1 as a source row to generate a cut and perform one more iteration.

《解》

The following source row is:

$$(7/9)x_1 + (8/9)x_2 + y_1 + (1/3)y_2 + (1/9)s_1 = 43/9$$

Here, we have all positive coefficients and no negative coefficients for the continuous variables. Compute:

$$f_{r,0} = 7/9,\ f_{r,1} = 0,\ f_{r,2} = 1/3$$

And we obtain a mixed integer cut

$$(7/9)x_1 + (8/9)x_2 + (1/3)y_2 + (1/9)s_1 = 7/9$$