Duality (III)

Lecture 10, Nonlinear Programming

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December 6, 2016

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Equivalent Reformulations of Optimization Problems

- We show by example that simple equivalent reformulations of a problem can lead to very different dual problems.
- We consider the following types of reformulations:
  1. Introducing new variables and associated equality constraints.
  2. Replacing the objective with an increasing function of the original objective.
  3. Making explicit constraints implicit, i.e., incorporating them into the domain of the objective.

Introducing new variables and equality constraints (1/2)

- Consider an unconstrained problem of the form

$$\text{minimize} \quad f_0(Ax+b).$$

  - The Lagrange dual function is the constant p^*.
  - The strong duality holds, i.e., $p^* = d^*$, but the Lagrangian dual is neither useful nor interesting.
- Now let us reformulate the problem as

$$\begin{aligned} \text{minimize} \quad & f_0(y) \\ \text{subject to} \quad & Ax+b=y \end{aligned}$$

  where we introduced new variables y and new equality constraints. Note that the two problems are equivalent.
- The Lagrangian of the reformulated problem is

$$L(x,y,\nu) = f_0(y) + \nu^T(Ax+b-y).$$

Introducing new variables and equality constraints (2/2)

- So, the dual function is

$$g(\nu) = b^T\nu + \inf_y (f_0(y) - \nu^T y) = b^T\nu - f_0^*(\nu),$$

  where f_0^* is the conjugate of f_0.
- Therefore, the dual problem of

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & Ax + b = y \end{array}$$

  can be expressed as

$$\begin{array}{ll} \text{maximize} & b^T\nu - f_0^*(\nu) \\ \text{subject to} & A^T\nu = 0. \end{array}$$

- The dual of the reformulated problem is considerably more useful than the dual of the original problem.

Example – Unconstrained Geometric Program (1/2)

- Consider the unconstrained geometric program

$$\text{minimize} \quad \log\left(\sum_{i=1}^{m} \exp(a_i^T x + b_i)\right).$$

- We first reformulate it by introducing new variables and equality constraints:

$$\begin{array}{ll} \text{minimize} & f_0(y) = \log\left(\sum_{i=1}^{m} \exp y_i\right) \\ \text{subject to} & Ax + b = y, \end{array}$$

where A is the matrix whose ith row is a_i^T.

- The conjugate of the log-sum-exp function is

$$f_0^*(\nu) = \begin{cases} \sum_{i=1}^{m} \nu_i \log \nu_i & \nu \succeq 0, \mathbf{1}^T\nu = 1 \\ \infty, & \text{otherwise} \end{cases}.$$

Example – Unconstrained Geometric Program (2/2)

- So the dual of the reformulated problem can be expressed as

$$\begin{array}{ll} \text{maximize} & b^T\nu - \sum_{i=1}^{m} \nu_i \log \nu_i \\ \text{subject to} & \mathbf{1}^T\nu = 1 \\ & A^T\nu = 0 \\ & \nu \succeq 0, \end{array}$$

which is an entropy maximization problem.

Example – Norm approximation problem

- We consider the unconstrained norm approximation problem

$$\text{minimize} \quad ||Ax - b||,$$

where $||\cdot||$ is any norm.

- Again, the Lagrange dual function is constant (equal to the optimal value of the problem) and not useful.
- We can reformulate the problem as

$$\begin{array}{ll} \text{minimize} & ||y|| \\ \text{subject to} & Ax - b = y. \end{array}$$

- We will introduce definitions and basic properties of dual norms and come back to this example later.

Introduction to Dual Norms (1/3)

- Let $||\cdot||$ be a norm on $\mathbf{R}^n$. The associated **dual norm**, denoted $||\cdot||_*$, is defined as

$$||z||_* = \sup\left\{z^T x \mid ||x|| \leq 1\right\}.$$

- It can be shown that

$$||z||_* = \sup\left\{|z^T x| \mid ||x|| \leq 1\right\}$$

  and

$$||z||_* = \sup_{x \neq 0} \frac{z^T x}{||x||}.$$

- A dual norm is also a norm.
  - *Hint*: $||u+v||_* = \sup\left\{(u+v)^T x \mid ||x|| \leq 1\right\}$

Introduction to Dual Norms (2/3)

- From the definition of dual norm we have the inequality

$$z^T x \leq ||x||\ ||z||_*,$$

  for all x and z.
- The dual of the dual norm is the original norm: we have $||x||_{**} = ||x||$ for all x.
  - *Hint*: $||x||_{**} = \sup_{z \neq 0} \frac{x^T z}{||z||_*}$
- The dual of the Euclidean norm is the Euclidean norm, since $\sup \left\{ z^T x \mid ||x||_2 \leq 1 \right\} = ||z||_2$.
  - This follows from the Cauchy-Schwarz inequality.
  - For nonzero z, the value of x that maximizes $z^T x$ over $||x||_2 \leq 1$ is $z/||z||_2$.

Introduction to Dual Norms (3/3)

- The dual of the ℓ_∞-norm is the ℓ_1-norm:

$$\sup\left\{z^T x \mid ||x||_\infty \leq 1\right\} = \sum_{i=1}^{n} |z_i| = ||z||_1.$$

- The dual of the ℓ_1-norm is the ℓ_∞-norm.
- More generally, the dual of the ℓ_p-norm is the ℓ_q-norm, where q satisfies

$$\frac{1}{p} + \frac{1}{q} = 1,$$

  i.e., $q = p/(p-1)$.
  - *Hint*: Hölder's inequality: $u^T v \leq ||u||_p ||v||_q$.

Example – Norm approximation problem (1/2)

- We consider the unconstrained norm approximation problem

$$\text{minimize} \quad ||Ax - b||,$$

  where $||\cdot||$ is any norm.
- The Lagrange dual function is constant and is not useful, so we reformulated the problem as

$$\begin{array}{ll} \text{minimize} & ||y|| \\ \text{subject to} & Ax - b = y. \end{array}$$

Example – Norm approximation problem (2/2)

- The Lagrange dual function is

$$\begin{aligned} g(\nu) &= \inf_{x,y} \left\{ ||y|| + \nu^T (Ax - b - y) \right\}, \\ &= \begin{cases} -b^T \nu + \inf_y \left\{ ||y|| - \nu^T y \right\}, & A^T \nu = 0 \\ -\infty, & A^T \nu \neq 0 \end{cases} . \end{aligned}$$

- So, the Lagrange dual problem is,

$$\begin{aligned} \text{maximize} \quad & -b^T \nu \\ \text{subject to} \quad & ||\nu||_* \leq 1 \\ & A^T \nu = 0, \end{aligned}$$

where we used the fact that $\nu^T y \leq ||y||\ ||\nu||_*$, and hence,

$$||y|| \left(1 - ||\nu||_*\right) \leq ||y|| - \nu^T y \leq ||y|| \left(1 + ||\nu||_*\right).$$

Transforming the objective

- In an optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, ..., m \\ & h_i(x) = 0, i = 1, ..., p, \end{array}$$

  if we replace the objective f_0 by an increasing function of f_0, the resulting problem is clearly equivalent.
- The dual of this equivalent problem, however, can be very different from the dual of the original problem.

Example – minimum norm problem

- We consider the minimum norm problem

$$\text{minimize} \quad ||Ax - b||,$$

where $||\cdot||$ is some norm. We reformulate this problem as

$$\begin{array}{ll} \text{minimize} & (1/2)||y||^2 \\ \text{subject to} & Ax - b = y. \end{array}$$

Here we not only have introduced new variables, but also have replaced the objective by half its square. Evidently it is equivalent to the original problem.

Dual Norms and Conjugate functions of Norms (1/3)

- Let $||\cdot||$ be a norm on $\mathbf{R}^n$, with dual norm $||\cdot||_*$. We show that the conjugate of $f(x) = ||x||$ is

$$f^*(y) = \begin{cases} 0, & ||y||_* \leq 1 \\ \infty, & \text{otherwise} \end{cases}.$$

- If $||y||_* > 1$, then there is a $z \in \mathbf{R}^n$ with $||z|| \leq 1$ and $y^T z > 1$. Taking $x = tz$ and letting $t \to \infty$, we have

$$y^T x - ||x|| = t(y^T z - ||z||) \to \infty,$$

which shows that $f^*(y) = \infty$.

- If $||y||_* \leq 1$, then we have $y^T x \leq ||x||\ ||y||_*$ for all x, which implies for all x, $y^T x - ||x|| \leq 0$.

Dual Norms and Conjugate functions of Norms (2/3)

- Now consider the function $f(x) = (1/2)||x||^2$, where $||\cdot||$ is a norm, with dual norm $||\cdot||_*$. We will show that its conjugate is $f^*(y) = (1/2)||y||_*^2$.
- From $y^T x \leq ||y||_* ||x||$, we conclude

$$y^T x - (1/2)||x||^2 \leq ||y||_* ||x|| - (1/2)||x||^2$$

  for all x. The righthand side is a quadratic function of $||x||$, which has maximum value $(1/2)||y||_*^2$.
- Therefore for all x, we have

$$y^T x - (1/2)||x||^2 \leq (1/2)||y||_*^2,$$

  which shows that $f^*(y) \leq (1/2)||y||_*^2$.

Dual Norms and Conjugate functions of Norms (3/3)

- On the other hand, let x be any vector with $y^T x = ||y||_* ||x||$, scaled so that $||x|| = ||y||_*$.
- Then we have, for this x, $y^T x - (1/2)||x||^2 = (1/2)||y||_*^2$, which shows that $f^*(y) \geq (1/2)||y||_*^2$.
- Since $f^*(y) \leq (1/2)||y||_*^2$ and $f^*(y) \geq (1/2)||y||_*^2$, we conclude that

$$f^*(y) = (1/2)||y||_*^2,$$

  where $f(x) = (1/2)||x||^2$.
- Compare with the case when $f(x) = ||x||$, we have

$$f^*(y) = \begin{cases} 0, & ||y||_* \leq 1 \\ \infty, & \text{otherwise}, \end{cases}.$$

Example – minimum norm problem (1/2)

- We consider again the minimum norm problem

$$\text{minimize} \quad ||Ax - b||,$$

where $||\cdot||$ is some norm. We reformulate this problem as

$$\begin{array}{ll} \text{minimize} & (1/2)||y||^2 \\ \text{subject to} & Ax - b = y. \end{array}$$

Here we have introduced new variables, and replaced the objective by half its square. Evidently it is equivalent to the original problem.

Example – minimum norm problem (2/2)

- The dual of the reformulated problem is

$$\begin{array}{ll} \text{maximize} & -(1/2)||\nu||_*^2 - b^T\nu \\ \text{subject to} & A^T\nu = 0, \end{array}$$

  where we use the fact that the conjugate of $(1/2)||\cdot||^2$ is $(1/2)||\cdot||_*^2$.
- Note that this dual problem is not the same as the dual problem derived earlier:

$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & ||\nu||_* \leq 1 \\ & A^T\nu = 0. \end{array}$$

Implicit constraints

- We can reformulate a problem by including some of the constraints in the objective function, by modifying the objective function to be infinite when the constraint is not satisfied.

Example – Linear program with box constraints (1/4)

- We consider the linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & l \preceq x \preceq u \end{array}$$

  where $A \in \mathbf{R}^{p \times n}$ and $l \prec u$.

- The constraints $l \preceq x \preceq u$ are sometimes called box constraints or variable bounds.

Example – Linear program with box constraints (2/4)

- By rewriting the primal problem as

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \preceq u \\ & -x \preceq -l \\ & Ax = b, \end{array}$$

we find that the dual problem is

$$\begin{array}{ll} \text{maximize} & -b^T \nu - \lambda_1^T u + \lambda_2^T l \\ \text{subject to} & A^T \nu + \lambda_1 - \lambda_2 + c = 0 \\ & \lambda_1 \succeq 0, \\ & \lambda_2 \succeq 0. \end{array}$$

Example – Linear program with box constraints (3/4)

- Now, let us reformulate the original problem as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax = b, \end{array}$$

  where we define

$$f_0(x) = \begin{cases} c^T x & l \preceq x \preceq u \\ \infty & \text{otherwise.} \end{cases}$$

- Then, the reformulated problem is clearly equivalent to the original problem: we have merely made the explicit box constraints implicit.

Example – Linear program with box constraints (4/4)

- The dual function for the new problem is

$$
\begin{aligned}
g(\nu) &= \inf_{l \preceq x \preceq u} \left(c^T x + \nu^T (Ax - b) \right) \\
&= -b^T \nu - u^T (A^T \nu + c)^- + l^T (A^T \nu + c)^+
\end{aligned}
$$

  where $y_i^+ = \max\{y_i, 0\}, y_i^- = \max\{-y_i, 0\}$.
- So here we are able to derive an analytical formula for g, which is a concave piecewise-linear function.
- The dual problem is the unconstrained problem

$$
\text{maximize} \quad -b^T \nu - u^T (A^T \nu + c)^- + l^T (A^T \nu + c)^+,
$$

  which has a quite different form from the dual of the original problem.

Feasibility of a system of equalities and inequalities (1/2)

- In this section, we apply Lagrange duality theory to the problem of determining feasibility of a system of inequalities and equalities

$$f_i(x) \leq 0, i = 1, \dots, m, \qquad h_i(x) = 0, i = 1, \dots, p.$$

- We assume the domain of the inequality system,

$$\mathcal{D} = \bigcap_{i=1}^{m} \textbf{dom}\ f_i \cap \bigcap_{i=1}^{p} \textbf{dom}\ h_i,$$

is nonempty.

Feasibility of a system of equalities and inequalities (2/2)

- Specifically, consider the feasibility test problem in the form of an optimization problem with objective $f_0 = 0$, i.e.,

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p. \end{array}$$

- This problem has optimal value

$$p^* = \begin{cases} 0 & \text{the problem is feasible} \\ \infty & \text{the problem is infeasible,} \end{cases}$$

so solving the optimization problem is the same as solving the inequality system

$$f_i(x) \leq 0, i = 1, \ldots, m, \qquad h_i(x) = 0, i = 1, \ldots, p.$$

The Dual Function of the Inequality System

- We associate with the inequality system

$$f_i(x) \leq 0, i = 1, \ldots, m, \qquad h_i(x) = 0, i = 1, \ldots, p$$

  the dual function

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \left(\sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right),$$

  the same as the dual function for the feasibility test problem.

- It can be shown that $g(\lambda, \nu)$ is homogeneous:

$$g(\alpha\lambda, \alpha\nu) = \alpha g(\lambda, \nu)$$

  for any $\alpha > 0$.

The Dual Problem of the Feasibility Test Problem

- The dual problem can be written as

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

- Since g is homogeneous, the optimal value of this dual problem is given by

$$d^* = \begin{cases} \infty, & \text{if } \lambda \succeq 0,\ \ g(\lambda, \nu) > 0 \text{ is feasible} \\ 0, & \text{if } \lambda \succeq 0,\ \ g(\lambda, \nu) > 0 \text{ is infeasible.} \end{cases}$$

Primal v.s. Dual Problems

- The primal problem

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{array}$$

has optimal value $p^* = \begin{cases} 0 & \text{the problem is feasible} \\ \infty & \text{the problem is infeasible,} \end{cases}$

- The dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

has optimal value $d^* = \begin{cases} \infty, & \text{if } \lambda \succeq 0, g(\lambda, \nu) > 0 \text{ is feasible} \\ 0, & \text{if } \lambda \succeq 0, g(\lambda, \nu) > 0 \text{ is infeasible.} \end{cases}$

- Note that weak duality implies $d^* \leq p^*$.
  - This implies the primal problem and the inequality system induced by the dual problem cannot be feasible simultaneously.

Weak Alternatives

- Therefore, weak duality implies the following thing:
  If the inequality system
  $$\lambda \succeq 0, \quad g(\lambda, \nu) > 0$$
  is feasible (which means $d^* = \infty$), then the inequality system
  $$f_i(x) \leq 0, i = 1, ..., m, \qquad h_i(x) = 0, i = 1, ..., p$$
  is infeasible (since we then have $p^* = \infty$).
- Any solution (λ, ν) of the inequalities $\lambda \succeq 0, \quad g(\lambda, \nu) > 0$ can be viewed as a proof or certificate of infeasibility of the system
  $$f_i(x) \leq 0, i = 1, ..., m, \qquad h_i(x) = 0, i = 1, ..., p.$$

Weak Alternatives

- Conversely, we can interpret an x which satisfies

$$f_i(x) \leq 0, i = 1, \ldots, m, \qquad h_i(x) = 0, i = 1, \ldots, p$$

as a certificate establishing infeasibility of the inequality system

$$\lambda \succeq 0, \quad g(\lambda, \nu) > 0.$$

- Two systems of inequalities (and equalities) are called **weak alternatives** if at most one of the two is feasible.
- This is true whether or not the inequalities

$$f_i(x) \leq 0, i = 1, \ldots, m, \qquad h_i(x) = 0, i = 1, \ldots, p$$

are convex (i.e., f_i convex, h_i affine); moreover, the alternative inequality system $\lambda \succeq 0, \quad g(\lambda, \nu) > 0$ is always convex (i.e., g is concave and the constraints $\lambda_i \geq 0$ are convex).

Weak Alternatives for Strict Inequalities (1/3)

- We can also study feasibility of the strict inequality system

$$f_i(x) < 0, i = 1, ..., m, \quad h_i(x) = 0, i = 1, ..., p.$$

- With g defined as

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \left(\sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right),$$

we have the alternative inequality system

$$\lambda \succeq 0, \ \lambda \neq 0, \ g(\lambda, \nu) \geq 0.$$

- We show in the next pages that

$$f_i(x) < 0, i = 1, ..., m, \quad h_i(x) = 0, i = 1, ..., p.$$

and $\lambda \succeq 0, \ \lambda \neq 0, \ g(\lambda, \nu) \geq 0$ are weak alternatives.

Weak Alternatives for Strict Inequalities (2/3)

- Suppose there exists an $\tilde{x}$ with

$$f_i(\tilde{x}) < 0, i = 1, \dots, m, \quad h_i(\tilde{x}) = 0, i = 1, \dots, p.$$

- Then for any $\lambda \succeq 0, \lambda \neq 0$, and ν,

$$\lambda_1 f_1(\tilde{x}) + \dots + \lambda_m f_m(\tilde{x}) + \nu_1 h_1(\tilde{x}) + \dots + \nu_p h_p(\tilde{x}) < 0.$$

- It follows that

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} \left(\sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \\ &\leq \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) < 0. \end{aligned}$$

So, $\lambda \succeq 0, \lambda \neq 0, g(\lambda, \nu) \geq 0$ is not feasible.

Weak Alternatives for Strict Inequalities (3/3)

- Therefore, feasibility of

$$f_i(\tilde{x}) < 0, i = 1, ..., m, \quad h_i(\tilde{x}) = 0, i = 1, ..., p$$

implies that there does not exist (λ, ν) satisfying

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad g(\lambda, \nu) \geq 0.$$

- Thus, the two systems cannot be feasible simultaneously. If one of them is shown to be feasible, then the other must be infeasible.

Strong alternatives

- If the inequality system

$$f_i(x) \leq 0, i = 1, ..., m, \qquad h_i(x) = 0, i = 1, ..., p$$

(with its weak alternative $\lambda \succeq 0, \quad g(\lambda, \nu) > 0$) is convex, i.e., f_i are convex and h_i are affine, and some type of constraint qualification holds, then the pairs of weak alternatives described above are **strong alternatives**. That is, each of the inequality systems is feasible if and only if the other is infeasible.

- The inequality system can be expressed as

$$f_i(x) \leq 0, \;\; i = 1, ..., m, \quad Ax = b,$$

where $A \in \mathbf{R}^{p \times n}$.

Strong Alternatives for Strict Inequalities (1/5)

- We first study the strict inequality system

$$f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b,$$

  and its alternative

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad g(\lambda, \nu) \geq 0$$

  with one extra condition: There exists an $x \in$ **relint** $\mathcal{D}$ with $Ax = b$.
- In other words, we assume that the linear equality constraints are consistent, and that they have a solution in **relint** $\mathcal{D}$.
- We will show that these two systems are strong alternatives: exactly one of them is feasible.

Strong Alternatives for Strict Inequalities (2/5)

- Consider the related optimization problem

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & f_i(x) - s \leq 0, i = 1, \ldots, m \\ & Ax = b \end{array}$$

  with variables x, s, and domain $\mathcal{D} \times \mathbf{R}$.
- Then, the optimal value p^* of this problem is negative if and only if there exists a solution to the strict inequality system

$$f_i(x) < 0, \;\; i = 1, \ldots, m, \quad Ax = b.$$

Strong Alternatives for Strict Inequalities (3/5)

- The Lagrange dual function for the problem in the previous page is

$$\inf_{x\in\mathcal{D},s}\left(s+\sum_{i=1}^{m}\lambda_i(f_i(x)-s)+\nu^T(Ax-b)\right)$$
$$=\begin{cases} g(\lambda,\nu), & \mathbf{1}^T\lambda=1 \\ -\infty, & \text{otherwise} \end{cases}.$$

- The dual problem of this can be expressed as

$$\begin{array}{ll} \text{maximize} & g(\lambda,\nu) \\ \text{subject to} & \lambda\succeq 0, \quad \mathbf{1}^T\lambda=1. \end{array}$$

Strong Alternatives for Strict Inequalities (4/5)

- Note that Slater's condition for the problem

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & f_i(x) - s \leq 0, i = 1, \ldots, m \\ & Ax = b, \end{array}$$

  holds since by the hypothesis there exists an $\tilde{x} \in \textbf{relint } \mathcal{D}$ with $A\tilde{x} = b$ and we can choose any $\tilde{s} > \max_i f_i(\tilde{x})$.
- Therefore, we have $d^* = p^*$, and the dual optimum d^* is attained.
- In other words, there exist (λ^*, ν^*) such that

$$g(\lambda^*, \nu^*) = p^*, \quad \lambda^* \succeq 0, \quad \mathbf{1}^T \lambda^* = 1.$$

Strong Alternatives for Strict Inequalities (5/5)

- Now suppose that the strict inequality system

$$f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b$$

is infeasible, which means that $p^* \geq 0$. Then (λ^*, ν^*) from

$$g(\lambda^*, \nu^*) = p^*, \quad \lambda^* \succeq 0, \quad \mathbf{1}^T \lambda^* = 1$$

satisfy the alternate inequality system

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad g(\lambda, \nu) \geq 0$$

- Similarly, if the alternate inequality system

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad g(\lambda, \nu) \geq 0$$

is feasible, then $d^* = p^* \geq 0$, which shows that the strict inequality system

$$f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b$$

is infeasible. Thus, these two inequality systems are strong alternatives; each is feasible if and only if the other is not.

Strong Alternatives for Nonstrict Inequalities (1/3)

- We now consider the nonstrict inequality system

$$f_i(x) \le 0, i = 1, \dots, m, \quad Ax = b,$$

and its alternative

$$\lambda \succeq 0, \quad g(\lambda, \nu) > 0.$$

- We will show these are strong alternatives, provided the following conditions hold:
  1. There exists an $x \in \textbf{relint } \mathcal{D}$ with $Ax = b$, and
  2. the optimal value p^* of the problem

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & f_i(x) - s \le 0, i = 1, \dots, m \\ & Ax = b, \end{array}$$

is attained.

Strong Alternatives for Nonstrict Inequalities (2/3)

- With these assumptions we have that
  1. $p^* = d^*$;
  2. both the primal and dual optimal values are attained.
- Now suppose that the nonstrict inequality system

$$f_i(x) \leq 0, i = 1, \ldots, m, \quad Ax = b$$

  is infeasible, which means that $p^* > 0$. (Here we use the assumption that the primal optimal value is attained.)
- Then (λ^*, ν^*) from

$$g(\lambda^*, \nu^*) = p^*, \quad \lambda^* \succeq 0, \quad \mathbf{1}^T \lambda^* = 1$$

  satisfy the alternate inequality system

$$\lambda \succeq 0, \quad g(\lambda, \nu) > 0.$$

Strong Alternatives for Nonstrict Inequalities (3/3)

- Thus, the inequality systems

$$f_i(x) \leq 0, i = 1, \ldots, m, \quad Ax = b$$

and

$$\lambda \succeq 0, \quad g(\lambda, \nu) > 0$$

are strong alternatives; each is feasible if and only if the other is not.

Example – Linear Inequalities (1/2)

- Consider the system of linear inequalities

$$Ax \preceq b.$$

- The dual function is

$$g(\lambda) = \inf_x \lambda^T (Ax - b) = \begin{cases} -b^T \lambda, & A^T \lambda = 0 \\ -\infty, & \text{otherwise} \end{cases}.$$

- The alternative inequality system is therefore

$$\lambda \succeq 0, \quad A^T \lambda = 0, \quad b^T \lambda < 0.$$

Example – Linear Inequalities (2/2)

- These two systems

$$Ax \preceq b$$

and

$$\lambda \succeq 0, \quad A^T\lambda = 0, \quad b^T\lambda < 0$$

are strong alternatives since the optimum in the related problem

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & a_i^T x - b_i - s \leq 0, i = 1, \ldots, m \end{array}$$

is achieved, unless it is unbounded below.

Example – Strict Linear Inequalities

- We now consider the system of strict linear inequalities

$$Ax \prec b.$$

  Similarly, it can be shown to have the strong alternative system

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad A^T \lambda = 0, \quad b^T \lambda \leq 0.$$

- We have actually studied this before (see p.44 in Lecture2).