Set Covering Problem & Set Partitioning Problem

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Course No: 546 U6110

Agenda

- Introduction
- Types of set covering (SC) and set partition (SP) problems
- Properties of SC and SP

Set Covering Problem (SC)

$$\begin{aligned} \min\ & cx \\ \text{s.t.}\ & Ex \geq e \\ & x_j = 0 \text{ or } 1\ (j = 1, \ldots, n) \end{aligned}$$

Where E=(e_{ij}) is an $m*n$ matrix whose entries are 0 or 1.

- If the inequality constraints is replaced by equalities, the problem is referred to as a set partitioning problem (SP).

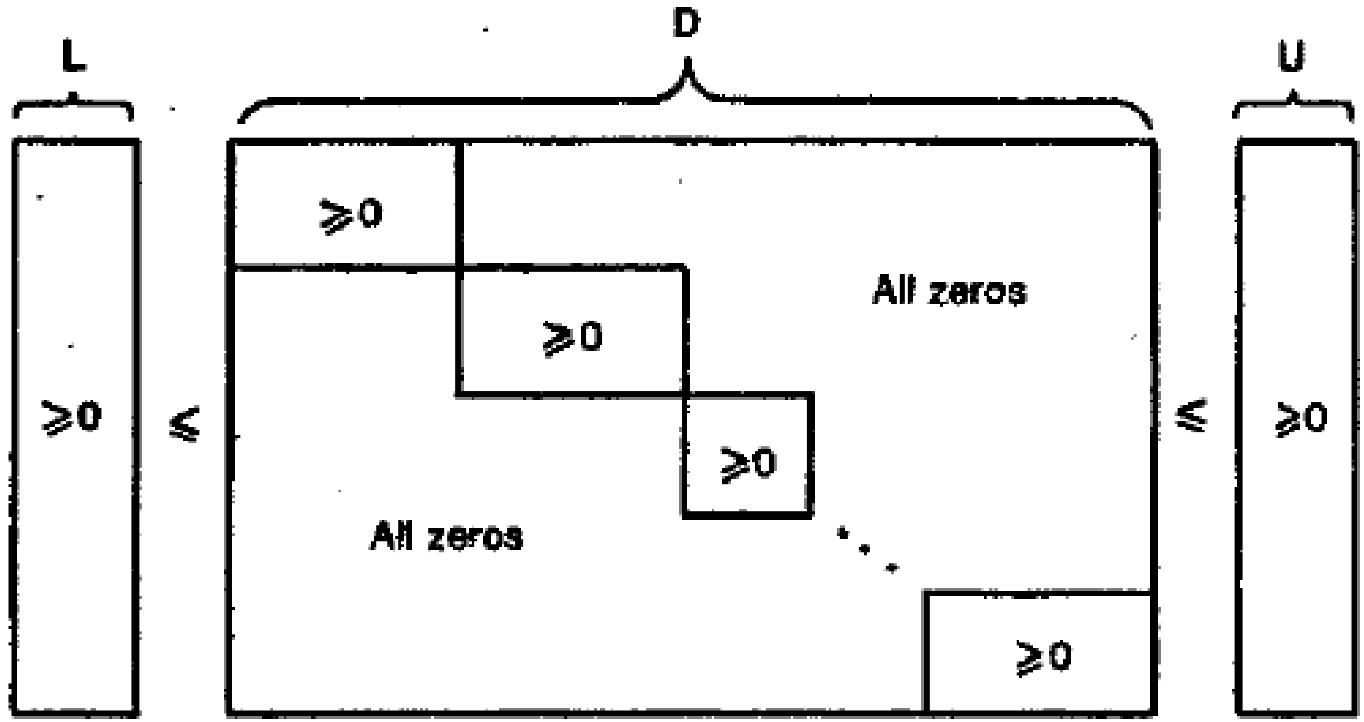
$$Ex = e$$

Introduction

- The set covering problem is to find a cheapest union of sets from **E** that covers every component of **e**, where component *i* of **e** is covered if at least one columns of **E** has a 1 in row *i*.
- The set partitioning problem is to find a cheapest disjoint of sets from **E** which equals to **e**.
- Extensions of SC/SP problem:
  - Positive integers of RHS instead of **e**.
  - Decision variables are integers, not necessary binary.
  - The constraints are in the form of

$$L \leq Dx \leq U$$

Structure of Base Constraints



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Set Covering and Networks

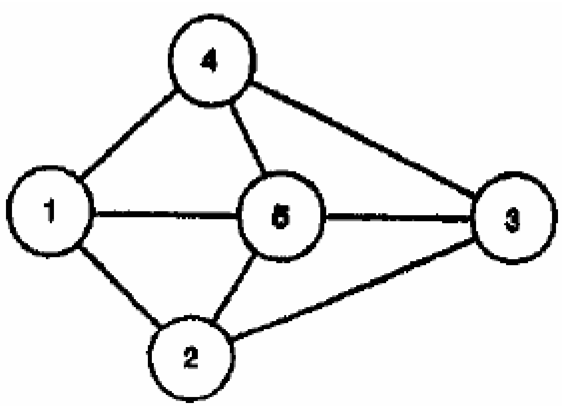
- The node covering problem
- The matching problem
- Disconnecting paths
- The Maximum flow problem

Minimal Cost Covering Problem

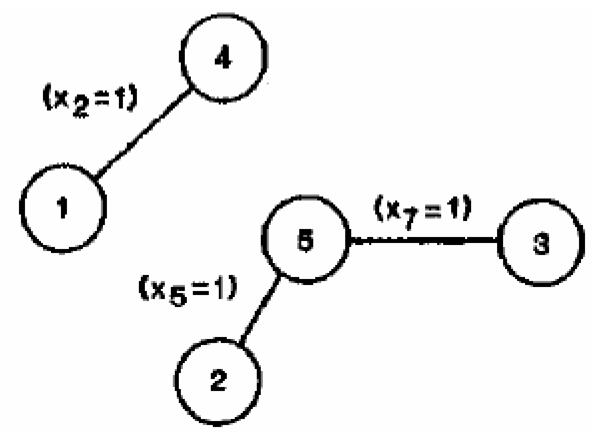
- A subset of arcs in the network such that each node is an end point of at least one the arcs in the subset.

$$
\begin{array}{c|cccccccc|l}
\text{Binary variable } x_j & (x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8) & = \mathbf{x} \\
\text{Costs} & (1 & 1 & 1 & 1 & 1 & 1 & 1 & 1) & = \mathbf{c} \\
\text{Node/arc} & (1,2) & (1,4) & (1,5) & (2,3) & (2,5) & (3,4) & (3,5) & (4,5) & \\
\hline
1 & 1 & 1 & 1 & & & & & & \geq 1 \\
2 & 1 & & & 1 & 1 & & & & \geq 1 \\
3 & & & & 1 & & 1 & 1 & & \geq 1 \\
4 & & 1 & & & & 1 & & 1 & \geq 1 \\
5 & & & 1 & & 1 & & 1 & 1 & \geq 1
\end{array}
$$

The matrix of node rows is labelled $\mathbf{E}$; the right-hand side column of 1s is $= \mathbf{e}$.



Example

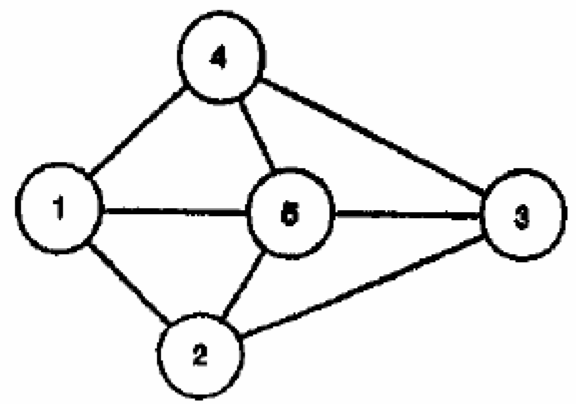


- By inspection, any three variables are at value 1.
- One such set is $x_2=x_5=x_7=1$, and others are at 0.

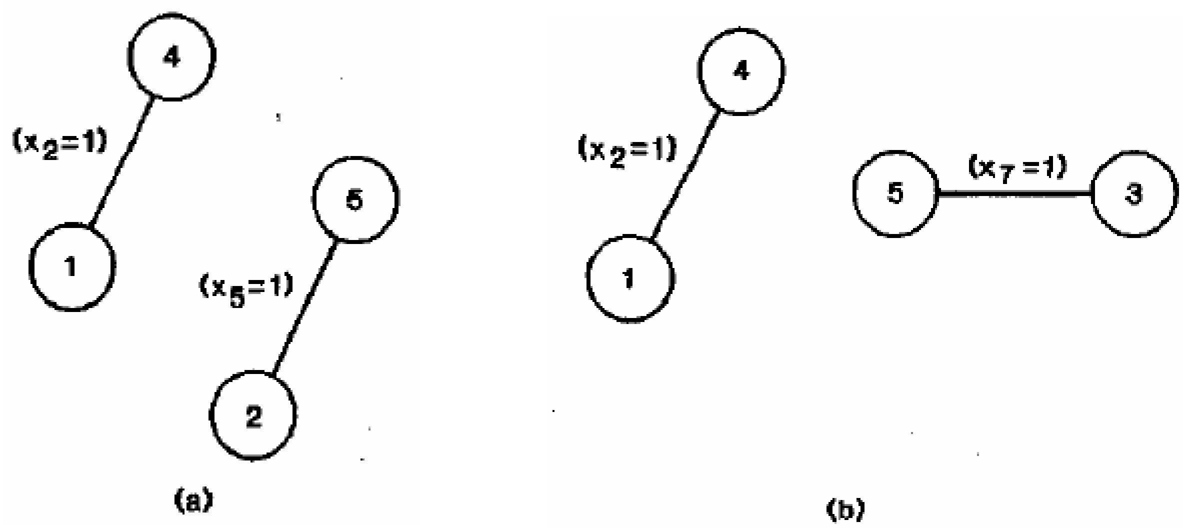
The Matching Problem

- A matching for a network is a subset of the arcs such that no two arcs in the subset have a common end point.
- That is, each node has at most one arc in the subset incident to it.
- Find the maximum number of arcs.

$$\begin{aligned} \max \quad & \sum_j c_j x_j \\ \text{s.t.} \quad & Ex \le e \\ & x_j = 0 \text{ or } 1. \end{aligned}$$



Example



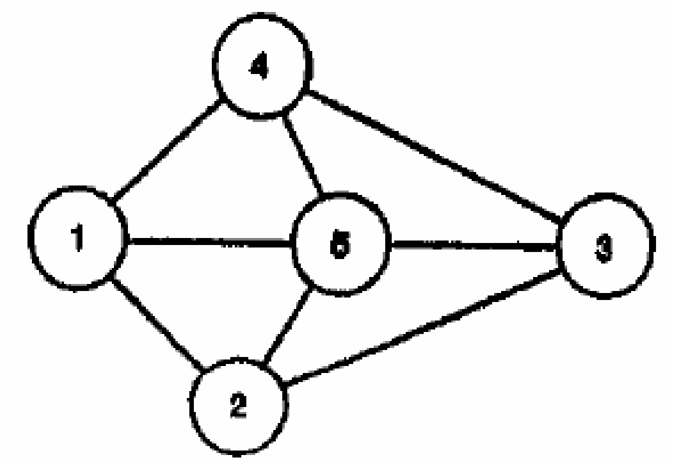
- By inspection, at most two arcs can be selected so that the constraints $Ex \leq e$ are satisfied.
- For example, $x_2 = x_5 = 1$ or $x_2 = x_7 = 1$.

Disconnecting Paths

- A path is from a node s to a node t as a sequence of distinct nodes: $s, i_1, i_2, \ldots, i_r, t$.
- Suppose all paths in a network are known, and there is a cost associated with removing an arc form the network.
- The problem is to discard a set of arcs which will disconnect all paths from s to t with the minimum removing costs.

Example (1/2)

- Consider the following network, suppose we want to disconnect all path from node 1 to node 3.
- List all possible paths:



i	paths
1	1, 4, 3
2	1, 5, 3
3	1, 2, 3
4	1, 4, 5, 3
5	1, 5, 4, 3
6	1, 2, 5, 3
7	1, 5, 2, 3
8	
9	

Example (2/2)

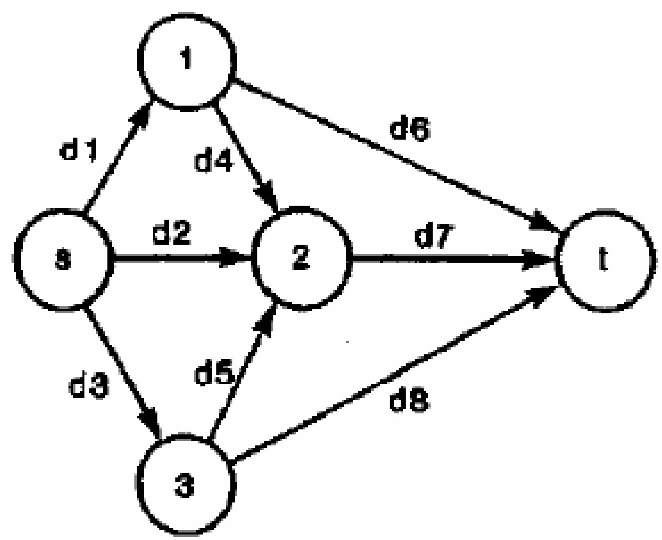
Path/ Arc	(1, 2) (2, 1)	(1, 4) (4, 1)	(1, 5) (5, 1)	(4, 5) (5, 4)	(2, 5) (5, 2)	(4, 3) (3, 4)	(3, 2) (2, 3)	(3, 5) (5, 3)
1		1				1		
2			1					1
3	1						1	
4		1		1				1
5			1	1		1		
6	1				1			
7			1		1		1	1
8		1		1	1		1	
9	1			1	1	1		

The Maximum Flow Problem

- Consider a directed network depicting (e.g.) a pipeline network between a refinery s and a terminal t.
- Let d_j be the maximum flow rate.
- The problem is to find the maximum flow from s to t through the network without exceeding the arc capacities.
- Define a directed path is from a node s to a node t as a sequence of distinct nodes: $s, i_1, i_2, \ldots, i_r, t$ with a directed arc between each successive pair of nodes.

Example

- One possible formulation: flow in = flow out for each node.
- The other: consider all path from s to t.



dj is the capacity of arc j

Arc (j)	path(i) 1	2	3	4	5
$1:(s,1)$	1	1	0		
$2:(s,2)$	0	0	1		
$3:(s,3)$	0	0	0		
$4:(1,2)$	0	1	0		
$5:(3,2)$	0	0	0		
$6:(1,t)$	1	0	0		
$7:(2,t)$	0	1	1		
$8:(3,t)$	0	0	0		

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Facts 1&2

- If any row r of ***E*** has all zero's, there is ______________.
- If in a row of ***E*** there is only one 1 and it occurs in the *k*th column, the $x_k=1$. The other constraints satisfied by $x_k=1$ are also dropped.
- Example: consider a SP problem (***E***x = e),

Constraint	Variable x_1	x_2	x_3	x_4	x_5	x_6	
1	1	0	0	1	0	0	= 1
2	0	1	1	0	0	0	= 1
3	0	1	0	0	0	0	= 1
4	1	1	1	0	1	0	= 1
5	0	0	0	1	0	1	= 1

E

Fact 3

- Row dominance: suppose row s and row r are two rows of ***E*** such that row r > row s, if x_k has a nonzero coefficient in the *s*th constraint, it has a nonzero coeff. in the *r*th constraint.
- For SC problem, row __ of ***E*** may be deleted.

$$\begin{array}{lccccccl} & & & x_k & & & & \\ Row\ r & 1 & 1 & 1 & 0 & 1 & 1 & \geq (=) 1 \\ Row\ s & 1 & 0 & 1 & 0 & 0 & 1 & \geq (=) 1 \end{array}$$

- For SP problem, what do you conclude based on the following example?

$$\begin{array}{lccccccl} & & x_h & & & x_l & & \\ Row\ r & 1 & 1 & 1 & 0 & 1 & 1 & =1 \\ Row\ s & 1 & 0 & 1 & 0 & 0 & 1 & =1 \end{array}$$

Fact 4

- Column dominance: suppose for some column $\boldsymbol{E}_j$ of $\boldsymbol{E}$ in a SC problem there exists a set of S of other columns of $\boldsymbol{E}$ whose sum is $\geq \boldsymbol{E}_j$, and the cost of $x_j \geq$ sum of the costs of the variables corresponding to the columns in S. Then, $\boldsymbol{E}_j$ may be ______.
- The same result for SP problem. Suppose for some column $\boldsymbol{E}_j$ of E in a SC problem there exists a set of S of other columns of E whose sum is equal to $\boldsymbol{E}_j$, and the cost of $x_j \geq$ sum of the costs of the variables corresponding to the columns in S.

Fact 5

- Any SP problem can be converted to a SC problem, and hence an algorithm which solves SC will also solve SP.
- For a SP problem, solving its corresponding SC problem if all slack variables (i.e., s= $\boldsymbol{E}$x−e) are equal to____, then the SP problem solved.
- By assigning positive costs to slack variables, finding a minimal SC solution will indicate that either obtaining an optimal solution to SP or show its infeasibility.

Fact 5 (con't)

$$\begin{array}{lll} \text{minimize} & cx + Me^T s & \\ \text{subject to} & Ex - \quad Is = e, & \\ & \qquad s \geq 0, & \\ \text{and} & x_j = 0 \text{ or } 1 & (\text{j=1,....,n}), \end{array}$$

- Replace $Me^T s$ in objective by $Me^T Ex$ - $Me^T e$

$$\begin{array}{lll} -Me^T e + \text{minimize} & \bar{c}x & \\ \text{subject to} & Ex - \ Is = e, & \\ & \qquad s \geq 0, & \\ \text{and} & x_j = 0 \text{ or } 1 & (\text{j=1,...,n}) \\ & \bar{c} = c + Me^T E & \end{array}$$

Example

$$\begin{array}{ll} \text{minimize} & 5x_1 + 4x_2 + 1x_3 + 2x_4 \\ \text{subject to} & x_1 + x_4 = 1, \\ & x_2 + x_3 = 1, \\ & x_1 + x_3 + x_4 = 1, \\ \text{and} & x_1, x_2, x_3, x_4 = 0 \text{ or } 1 \ . \end{array}$$

$$\begin{array}{ll} \text{minimize} & 5x_1 + 4x_2 + 1x_3 + 2x_4 + 12s_1 + 12s_2 + 12s_3 \\ \text{s.t. (i)} & x_1 + x_4 - s_1 = 1, \\ \text{(ii)} & x_2 + x_3 - s_2 = 1, \\ \text{(iii)} & x_1 + x_3 + x_4 - s_3 = 1, \\ \text{and} & x_1, x_2, x_3, x_4 = 0 \text{ or } 1,\ s_1, s_2, s_3 \geq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & \\ \text{subject to} & x_1 + x_4 \geq 1, \\ & x_2 + x_3 \geq 1, \\ & x_1 + x_3 + x_4 \geq 1, \\ \text{and} & x_1, x_2, x_3, x_4 = 0 \text{ or } 1 \ . \end{array}$$

Fact 6

- LPC and LPP are the linear programs associated with SC and SP by relaxing the integer constraint.
- SC has a binary solution if and only if LPC has a feasible solution.
- Assume the feasible solution to LPC is x^0_j, Then, the solution to SC can be constructed:
  - Set $x_j = 0$, for x^0_j
  - Set $x_j = 1$, for x^0_j
- The solution to LPP imply the existence of an integer solution to SP.
- Example: the solution to LPP is (1/2,1/2,1/2) where

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Fact 7

- Dual problem of LPC:

Primal	Dual
$\min \; cx$	$\max \; we$
s.t. $\; Ex \geq e$	s.t. $\; wE \leq c$
$x_j \geq 0$	$w \geq 0$

- If c > 0, an initial feasible solution can be immediately obtained (w, slack variables) = (0, c).
- Dual Simplex method can be applied.
- The same for LPP problems except w are ____________.

Facts 8&9

- At any LPC and LPP extreme point, $0 \le x_j \le 1$ is satisfied.
- For LPP, $\boldsymbol{E}x=e$ guarantee that every extreme point has $x_j \le 1$.
- For LPC, let x denote any extreme point. Form the matrix $\boldsymbol{B}$ (current basis matrix) as follows: take columns of $\boldsymbol{E}$ corresponding to positive x_j's; columns of $-$I corresponding to positive s_j's (where slacks s are given by: $\boldsymbol{E}x - \mathrm{I}s = \mathrm{e}$);

$$Ex - Is = B\begin{pmatrix} \bar{x} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} B_{11} & O \\ B_{12} & -I \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} e \\ e \end{pmatrix}$$

$\boldsymbol{B}_{11}$ and e contain only 0's and 1's, with $\bar{x} > 0$.

Because of $B_{11}\bar{x} = e$, no positive component of $x > 1$.

Fact 10

- If an integer solution to SC is not an extreme point of LPC, the current solution can be reduced to another integer solution which is an extreme point of LPC.
- Example: consider c=(1,2,1,1,2,3,1) and

$$\mathbf{E}=\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

  - An integer solution $\overline{x}=(1,1,0,1,1,0,0)$
  - Slack variables $\overline{s}=(0,0,0,2,1)$
  - The current objective is 6.

$$E\overline{x}-I\overline{s}=\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}\overline{x}-\begin{pmatrix} O \\ I \end{pmatrix}\overline{s}=\begin{pmatrix} e \\ e \end{pmatrix},\ \text{ with } \overline{s}\geq e$$

Example (con't)

- Arrange E based on the basic variables $(x_1, x_2, x_4, x_5, x_3, x_6, x_7)$

$$\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{pmatrix} \bar{\mathbf{x}} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{11} \\ \mathbf{E}_{21} \end{pmatrix} \mathbf{e};$$

$$\begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{pmatrix} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 4 & 5 & 3 & 6 & 7 \end{array} \\ \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

- So that,

$$\mathbf{E}\bar{\mathbf{x}} - \mathbf{I}\mathbf{s} = \begin{pmatrix} \mathbf{E}_{11} \\ \mathbf{E}_{21} \end{pmatrix} \mathbf{e} + \begin{pmatrix} \mathbf{0} \\ -\mathbf{I} \end{pmatrix} \bar{\mathbf{s}} = \begin{pmatrix} \mathbf{e} \\ \mathbf{e} \end{pmatrix}, \quad \bar{\mathbf{s}} \geq \mathbf{e} \qquad (6)$$

$$\mathbf{E}_{11} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E}_{11}^1 & \mathbf{0} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \mathbf{E}_{11} \\ \mathbf{E}_{21} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E}_{11}^1 & \mathbf{0} \\ \mathbf{E}_{21}^1 & \mathbf{E}_{21}^2 \end{pmatrix}$$

$$\mathbf{E}_{11} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E}_{11}^1 & \mathbf{0} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right) \qquad \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E}_{11}^1 & \mathbf{0} \\ \mathbf{E}_{21}^1 & \mathbf{E}_{21}^2 \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 5 \end{array} \\ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \end{array}$$

Example (con't)

- Rewrite (6)

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E}_{11}^{1} & \mathbf{0} \\ \mathbf{E}_{21}^{1} & \mathbf{E}_{21}^{2} \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{e} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{pmatrix} \bar{\mathbf{s}} = \begin{pmatrix} \mathbf{e} \\ \mathbf{e} \\ \mathbf{e} \end{pmatrix}$$

- Set $\overline{x}_4 = 0$, then

$$\begin{pmatrix} \bar{s}'_4 \\ \bar{s}'_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E}_{11}^{1} & \mathbf{0} \\ \mathbf{E}_{21}^{1} & \mathbf{E}_{21}^{2} \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 5 \\ 4 \end{array} \!\! \begin{array}{c} \begin{array}{ccc} 1 & 2 & 5 \end{array} \\ \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \end{array}\right) \end{array}$$

- Set $\overline{x}_5 = 0$, all slack variables are zero and the new integer solution is $(1,1,0,0,0,0,0)$. The new solution is an extreme point of LPC and the new objective is 3.

Facts 11&12

- Roundup any feasible solution to LPC is a solution to SC.
- A roundup solution obtained from a (nonintegral) extreme point x of (LPC) can always be reduced to another feasible solution with a smaller cost.
- Summarize:
  1) Change the LPC extreme point to an SC integer solution point.
  2) Improve the roundup solution to another SC integer point with a lower cost.
  3) Reduce the solution of (2) to an LPC extreme point.

Facts 13&14

- At least one of the constraints satisfied by the positive nonintegral x_j variables holds with strict equality.

$$x^* - \min_i \left[s_i / (E^* e)_i \right] e \geqq 0$$

- If a constraint satisfied by the positive nonintegral minimal LPC variables has a positive slack, then every variable corresponding to a 1 in that row must contribute to the strict equality of another constraint satisfied by these variables.

Example (1/2)

- Consider the columns of $\boldsymbol{E}$ associated with the positive nonintegral x_j variables, and the rows of $\boldsymbol{E}$ corresponding to the constraints that these variables explicitly satisfy. Let the selected matrix denoted as $\boldsymbol{E}^*$

$$E^* = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} = (e^*_{ij}).$$

- Suppose $x^* = (1/3, 1/2, 2/3, 1/2, 1/3, 1/6,)$
- Then, s=?

Example (2/2)

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

- Is fact 14 satisfied?
- s_2 and $s_4 > 0$, and $e_{23}=e_{43}=1$.
- Therefore, set $x_3^* =$
  - $x^* = (\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ & $s = (0, \frac{1}{6}, \frac{1}{6}, 0)$
- Repeat the process
  - $s_3 > 0$
  - Set $x_2^* =$
  - $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ & $s = (0, \frac{1}{6}, 0, 0)$

Reminder

- Final Project (6/15)(15% of your final grade)
  - Please prepare a 20-min presentation.
  - Every student has to attend the class, and for each group, more than half of team members need to do presentation.
  - Also submit a report with at most 10 pages before 6/26, and a draft is required on 6/15.
- Final Exam (6/15) (30%)
  - A cheat sheet with A4 size is allowed.
  - No laptop or smart phones.
  - A calculator is welcomed.