Modeling and Models

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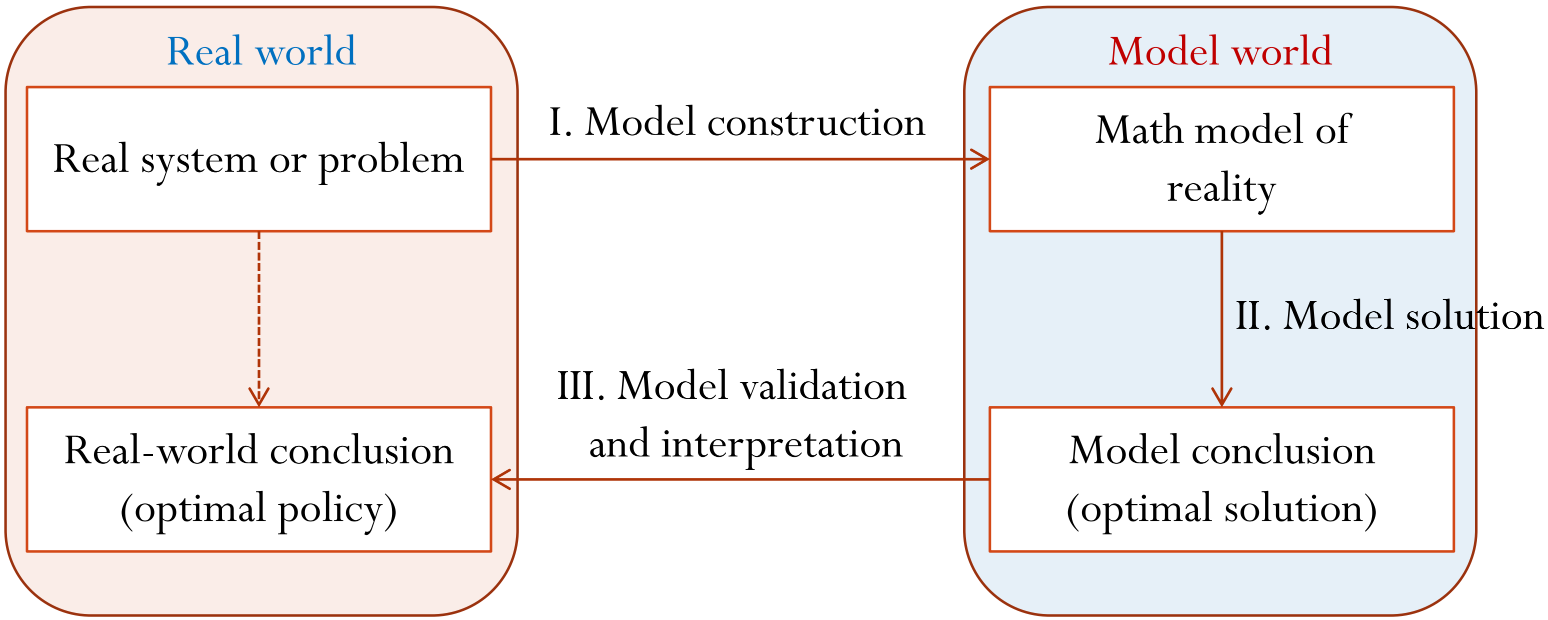
Course No: 546 U6110

Agenda

- Modeling Process
- Models
  - Project Selection
  - Production Planning Problems
  - Staff Scheduling
  - Transportation and Distribution Problems
  - Transshipment Models

Modeling Process

- Construct of the model
- Solution of the model
- Validation and interpretation



Source: Applied Integer Programming: Modeling and Solution, WILEY

Model Construction Process

- Step 1: verbally identify and define decision variables, parameters, constraints and the objective from the problem description. Assign appropriate symbols.
- Step 2: translate the verbal description into functions, equations, and inequalities.
- Step 3: check whether the non-MIP factors can be transformed into equivalent mathematical expressions. If yes, an MIP is obtained.

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- Modeling Process
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  - Project Selection
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Project Selection

- A single time period
  - Knapsack problem (Cargo loading problem)
- Multiple time periods
  - Capital budgeting problem

Capital Budgeting Problem

- Assume project j has a present value of c_j dollars and requires an investment of a_{tj} dollars in time period t (t=1,…, T). The capital available in time period t is b_t dollars.
- The objective of this problem is to maximize the total present value subject to the budgetary constraint in each time period over a prescribed planning horizon T.

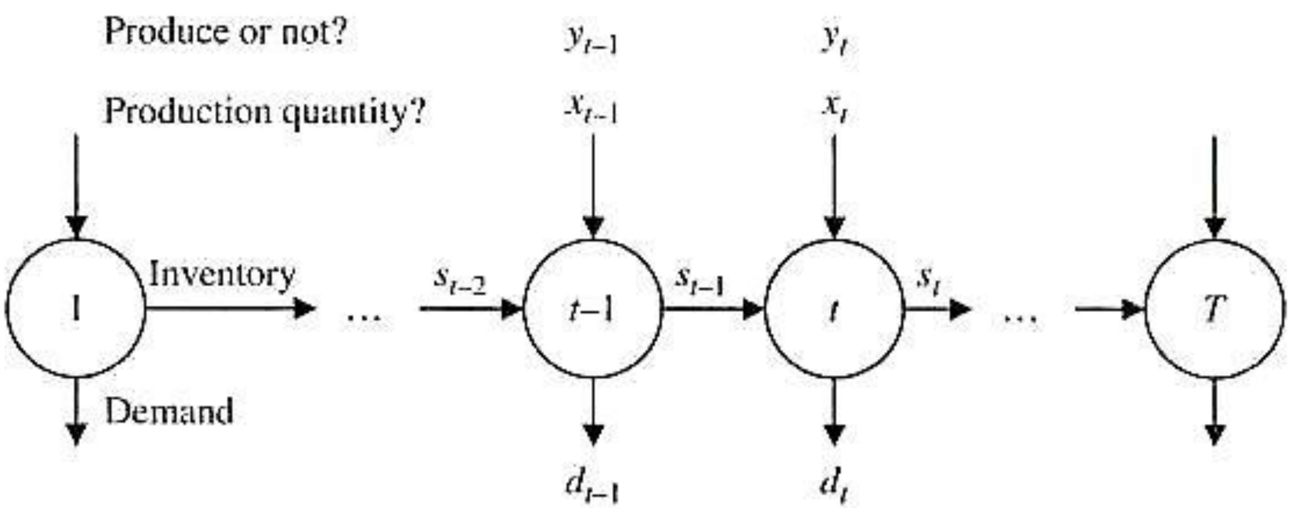
Maximize $z = \sum_j c_j y_j$

subject to $\qquad t = 1, \ldots, T$

$y_j = 0$ or $1 \qquad j = 1, 2, \ldots, n$

Production Planning Problems

- Uncapacitated lot sizing
- Capacitated lot sizing
- Just-in-Time production planning



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Uncapacitated Lot Sizing (1/2)

Input parameters:	number of periods (T), demand in each period (d_t), setup cost for each period (f_t), unit production cost (c_t), unit holding cost (h_t)
Decision variables:	whether or not to produce in each time period $(y_t = 1 \text{ or } 0)$ and how much if the decision is to produce (x_t)
Constraints:	satisfy the demand in each period t
State variables:	inventory level at the end of each period (S_t), assuming the beginning inventory level $s_0 = 0$
Objective:	minimize the total production and inventory costs

Uncapacitated Lot Sizing (2/2)

Let M be a "sufficiently" large number (say, M ______). Note that $y_t = 1$ if and only if ______. The problem can be formulated as follows: Find values of x_t and y_t $(t = 1, \ldots, T)$ so as to

Minimize $\sum_t (c_t x_t + f_t y_t + h_t s_t)$

subject to

		for all t
		for all t
	$x_t \geq 0$	for all t
	$s_t \geq 0$	$t = 0, 1, \ldots, T$
	$y_t = 0$ or 1	for all t

Capacitated Lot Sizing

- Each facility has its own capacity limitation (demoted as u).
- We simply replace the ______ in the uncapacitated lot sizing model with a capacity upper limit u.

$$
\begin{array}{lll}
\text{Minimize} & \sum_t (c_t x_t + f_t y_t + h_t s_t) & \\
\text{subject to} & s_{t-1} + x_t - s_t = d_t & \text{for all } t \\
 & & \text{for all } t \\
 & x_t \geq 0 & \text{for all } t \\
 & s_t \geq 0 & \text{for all } t \\
 & y_t = 0 \text{ or } 1 & \text{for all } t
\end{array}
$$

Just-in-Time Production Planning (1/4)

- Multiple products.
- This type of planning seeks to determine a production level for each product in each time period with the right quantity at the right time.
- Try to maintain zero inventory level.
- Penalty on shortage or surplus.

Just-in-Time Production Planning (2/4)

Input parameters:	number of product types (n), number of periods (T), demand of product j in each period (d_{jt}), prescribed production lot size for each product (l_{jt}), unit penalty of earliness (p_j), unit penalty of lateness (q_j)
Decision variables:	production level of each product in each period $(x_{jt} \geq 0)$, number of production runs in each period t for each product (y_{jt})
Constraints:	satisfy demand of each product j in each period and constraints relating to prescribed lot size, number of production runs per period, and production level
State variables:	surplus and shortage inventory levels for each product in each time period $\underline{(d_{jt}^{+} \text{ and } d_{jt}^{-})}$, ending inventory level of each product (s_{jt})
Objective:	minimize total penalty cost of all products due to earliness and lateness over all periods

Just-in-Time Production Planning (3/4)

- Recall the inventory balancing equation that relates the beginning inventory level, production level, demand level, and the ending level given below:

$$s_{j,t-1} + x_{jt} - d_{jt} = s_{jt} \qquad \text{for all } j,t$$

$$\text{or} \quad s_{j,t-1} + x_{jt} - s_{jt} = d_{jt}$$

- Let d_{jt}^{+} and d_{jt}^{-}, respectively, be a nonnegative amount of surplus and shortage for each period t and each product j.

$$s_{jt} =$$

Just-in-Time Production Planning (4/4)

- Integer lot size number.
- For example, l_{jt} =150 and x_{jt} =700. Then, the number of lots is 700/150=4.67 which is not an integer.
- A pair of constraints required:

and for all j and t

where $y_{jt} \geq 0$ and integer for all j and t.

- The objective is to

Minimize $z =$

Workforce Scheduling Problems (1/3)

- Workforce time windows

	Shift				
Time Window	1	2	3	4	Workers Required
6 a.m.–9 a.m.	X			X	55
9 a.m.–12 noon	X				46
12 noon–3 p.m.	X	X			59
3 p.m.–6 p.m.		X			23
6 p.m.–9 p.m.		X	X		60
9 p.m.–12 a.m.			X		38
12 a.m.–3 a.m.			X	X	20
3 a.m.–6 a.m.				X	30
Wage rate per 9 h shift	$135	$140	$190	$188	

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Workforce Scheduling Problems (2/3)

Input parameters: number of shifts (n), number of time windows (T), number of workers required during each time window $(d_{t,}\ t = 1, 2, ..., T)$, wage rate per shift for a full-time worker (w_j), wage rate per time window per part-time worker (c_t)

Decision variables: number of full-time workers needed for each work shift (y_j), number of part-time workers needed for each time window (x_t)

Constraints: demand within each time window t must be satisfied, restriction on using part-time workers (can be used only if one or more full-time workers are available in the same time window)

Objective: minimize the total wages paid to all workers

Workforce Scheduling Problems (3/3)

- Let parameter a_{jt}=1 if shift j covers time window t, 0 otherwise.

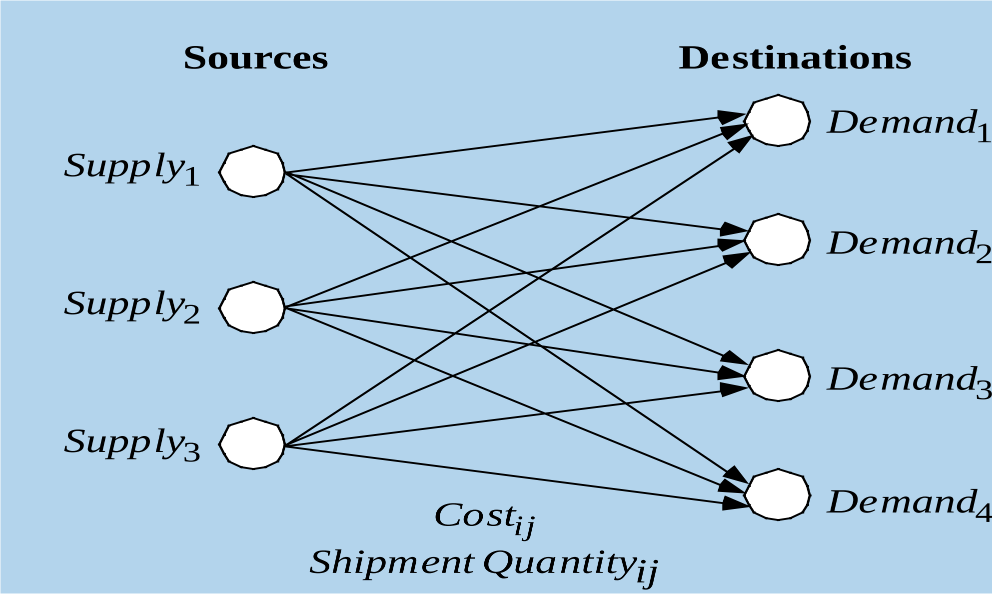
$$\text{Minimize} \quad \sum_j w_j y_j + \sum_t c_t x_t$$

subject to $\qquad t = 1, \ldots, T$

$\qquad t = 1, \ldots, T$

$$x_t, y_j \geq 0 \text{ and integer} \quad j = 1, \ldots, n; \; t = 1, \ldots, T$$

Transportation and Distribution Problems



- Fixed-charge transportation
- Uncapacitated facility location
- Capacitated facility location

Fixed-Charge Transportation (1/2)

- In addition to shipping costs, a fixed cost associated with a route is charged when the route is used.

Decision variables:	whether or not source i will supply destination $j (y_{ij} = 1 \text{ or } 0)$. If yes, how much (x_{ij})
Input parameters:	unit shipping cost (c_{ij}), fixed cost (f_{ij}) from source i to destination j, demand at destination $j (d_j)$
Constraints:	demand at each destination must be satisfied (assuming unlimited product availability at each source node)
Objective:	minimize sum of fixed and variable costs

Fixed-Charge Transportation (2/2)

Let M be a "sufficiently" large number (we can let $M = \sum_j d_j$). Note that $y_{ij} = 1$ if and only if $x_{ij} > 0$. The transportation model can be formulated as

$$\text{Minimize} \quad \sum_i \sum_j (c_{ij} x_{ij} + f_{ij} y_{ij})$$

$$\text{subject to} \quad \sum x_{ij} = d_j \qquad j = 1, \ldots, n$$

$$i = 1, \ldots, m; \; j = 1, \ldots, n$$

$$x_{ij} \geq 0 \qquad i = 1, \ldots, m; \; j = 1, \ldots, n$$

$$\mathrm{y}_{ij} = 0 \text{ or } 1 \qquad i = 1, \ldots, m; \; j = 1, \ldots, n$$

Uncapacitated Facility Location (1/2)

- Determine a set of sources to be open for supply.

Decision variables:	whether or not distribution center i should be opened ($y_i = 1$ or 0). If opened, how much should be shipped from distribution center to retail store (x_{ij})
Input parameters:	unit shipping cost from center i to retail j (c_{ij}), fixed cost for opening distribution center (f_i)
Constraints:	all demands are to be met at all retail stores
Objective:	minimize total cost of opening and transportation cost

Uncapacitated Facility Location (2/2)

Let M be a "sufficiently" large number (we can let $M = \sum_j d_j$). Note that $y_{ij} = 1$ if and only if $\sum_j x_{ij} > 0$. The uncapacitated facility location problem can be formulated as

$$
\begin{array}{lll}
\text{Minimize} & \sum_i \sum_j c_{ij} x_{ij} + \sum_i f_i y_i & \\
\text{subject to} & \sum_i x_{ij} = d_j & j = 1, \ldots, n \\
& \sum_j x_{ij} \le M y_i & i = 1, \ldots, m \\
& x_{ij} \ge 0 \text{ and integer} & i = 1, \ldots, m;\ j = 1, \ldots, n
\end{array}
$$

Capacitated Facility Location

- Replace "M" with the supply bound u.

$$
\begin{array}{lll}
\text{Minimize} & \sum_i \sum_j c_{ij} x_{ij} + \sum_i f_i y_i & \\
\text{subject to} & \sum_i x_{ij} = d_j & j = 1, \ldots, n \\
 & \sum_j x_{ij} \le u_i y_i & i = 1, \ldots, m \\
 & x_{ij} \ge 0 \text{ and integer} & i = 1, \ldots, m;\ j = 1, \ldots, n \\
 & y_i = 0 \text{ or } 1 & i = 1, \ldots, m
\end{array}
$$

Transshipment Model (1/3)

- Extension of the transportation model.
- Intermediate transshipment points are added between the sources and destinations.
- Items may be transported from:
  - Sources through transshipment points to destinations
  - One source to another
  - One transshipment point to another
  - One destination to another
  - Directly from sources to destinations
  - Some combination of these

Transshipment Model (2/3)

Decision variables:	units of commodity k to be shipped from source i to sink $j(x_{ij}^k)$, from source i to transshipment $t(x_{it}^k)$, and from transshipment t to sink $j(x_{tj}^k)$
Input parameters:	supply s_i^k of each commodity k at each source i, demand d_j^k for each commodity k at each sink j; maximum combined shipping capacity for all commodities from source i to sink $j(u_{ij})$, from source i to transshipment node $t(u_{it})$, from transshipment t to sink $j(u_{tj})$; unit transportation cost for commodity k that can be transported from source i to sink $j(c_{ij}^k)$, from source i to transshipment $t(c_{it}^k)$, from transshipment t to sink $j(c_{tj}^k)$
Constraints:	supply constraints for all sources, demand constraints for all sinks, flow conservation constraints for each transshipment node (total outflow equals total inflow for each commodity), maximum combined flow capacity for all commodities between any two nodes
Objective:	minimize total transportation cost

Transshipment Model (3/3)

Minimize $$z = \sum_{k} \sum_{(i,j)} c_{ij}^{k} x_{ij}^{k} + \sum_{k} \sum_{(i,t)} c_{it}^{k} x_{it}^{k} + \sum_{k} \sum_{(t,j)} c_{tj}^{k} x_{tj}^{k}$$

subject to $$\sum_{t,j} (x_{ij}^{k} + x_{it}^{k}) = s_{i}^{k}$$ for each i, k (node i supplies commodity k)

for each t, k (node t is a transshipment node)

$$\sum_{i,t} (x_{ij}^{k} + x_{tj}^{k}) = d_{j}^{k}$$ for each j, k (sink j demands commodity k)

$$\sum_{k} x_{ij}^{k} \le u_{ij}$$

$$\sum_{k} x_{it}^{k} \le u_{it}$$

$$\sum_{k} x_{tj}^{k} \le u_{tj}$$

$x_{ij}^{k}, x_{it}^{k}, x_{tj}^{k} \ge 0$ and integer

Questions

- Please download HW1 from Ceiba. The due date of HW1 is on 9:00AM 3/16.