Chapter 3
Continuous Random Variables

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Outline

Continuous Sample Space

- A continuous set of numbers, sometimes referred to as an **interval**, contains all the real numbers between two limits.
- For the limits x_1 and x_2 with $x_1 < x_2$, there are 4 different intervals distinguished by which of the limits are contained in the interval.
  - $(x_1, x_2) = \{x \mid x_1 < x < x_2\}$
  - $[x_1, x_2] = \{x \mid x_1 \leq x \leq x_2\}$
  - $[x_1, x_2) = \{x \mid x_1 \leq x < x_2\}$
  - $(x_1, x_2] = \{x \mid x_1 < x \leq x_2\}$

Continuous Sample Space (2)

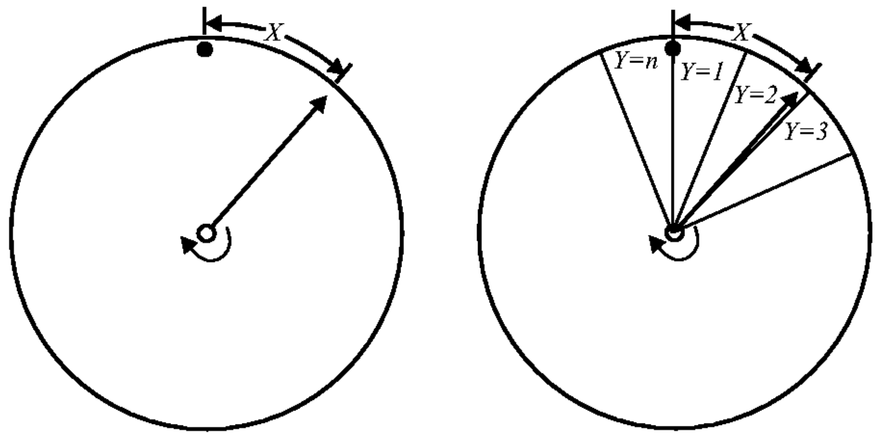
- Many experiments lead to random variables with a range that is a continuous interval. (**continuous random variables**)
  - The arrival time T of a particle ($S_T = \{t \mid 0 \leq t < \infty\}$)
  - The voltage V across a resistor ($S_V = \{v \mid -\infty < v < \infty\}$)
  - The phase angle A of a sinusoidal radio wave ($S_A = \{a \mid 0 \leq a < 2\pi\}$)
- A distinguishing feature is the probability of each individual outcome is zero!
- We refer to **probability density functions** to describe probabilities related to continuous random variables.
  - Mass in continuous volume, no mass at a single point.
  - Probability mass functions for discrete random variables

Continuous Sample Space (3)

- The sample space of **uniform random variable** is an interval with finite limits.
- The probability model of a uniform random variable states that any two intervals of equal size within the sample have equal probability.
- To introduce many concepts of continuous random variables, we will refer frequently to a uniform random variable with limits 0 and 1.
- (Pseudo) random number generators
  - rand() in Matlab: pseudo-random number generator
  - runif() in R

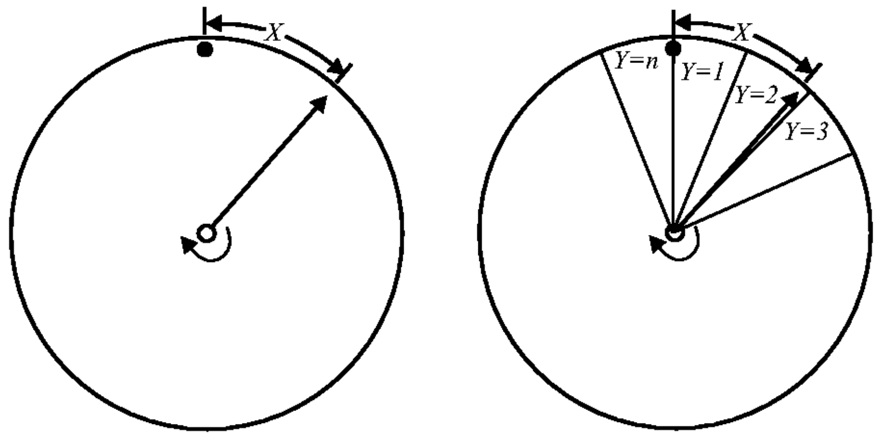
Example 3.1 Problem

Suppose we have a wheel of circumference one meter and we mark a point on the perimeter at the top of the wheel. In the center of the wheel is a radial pointer that we spin. After spinning the pointer, we measure the distance, X meters, around the circumference of the wheel going clockwise from the marked point to the pointer position as shown in Figure 3.1. Clearly, $0 \leq X < 1$. Also, it is reasonable to believe that if the spin is hard enough, the pointer is just as likely to arrive at any part of the circle as at any other. For a given x, what is the probability $P[X = x]$?



Example 3.1

- We have a wheel of circumference 1 meter and mark a point on the perimeter at the top of the wheel.
- Spin a radial pointer
- Measure the distance, X meters around the circumference of the wheel going clockwise from the mark point to the pointer position in Figure 3.1. ($0 \le X < 1$)
- For a given x, what is the probability $P[X = x]$?



Example 3.1 Solution

- The problem is surprisingly difficult.
- Reasonable approach: find a discrete approximation to X.
- Mark the perimeter with n equal-length arcs numbered 1 to n.
- Let Y denote the number of the arc in which the pointer stops.
- Y: discrete RV with range $S_Y = \{1, 2, \ldots, n\}$.

$$P_Y(y) = \begin{cases} 1/n & y = 1,2,\cdots,n \\ 0 & \text{otherwise} \end{cases}$$

- If $X = x$, $y = \lceil nx \rceil$

$$P[X = x] \le P[Y = \lceil nx \rceil] = \frac{1}{n}$$

$$P[X = x] \le \lim_{n\to\infty} P[Y = \lceil nx \rceil] = \lim_{n\to\infty} \frac{1}{n} = 0$$

- First axiom of probability states $P[X = x] \ge 0$.
- Therefore $P[X = x] = 0$.

3.1 The Cumulative Distribution Function

Definition 3.1 (CDF)

The cumulative distribution function *(CDF) of random variable X is*

$$F_X(x) = P[X \leq x].$$

- Same as the Definition 2.11
- It is impossible to define a probability mass function $P_X(x)$. On the other hand, we will see that the cumulative distribution function, $F_X(x)$ in Definition 2.11, is a very useful probability model for a continuous random variable.

Theorem 3.1

For any random variable X,

(a) $F_X(-\infty) = 0$

(b) $F_X(\infty) = 1$

(c) $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

- Similar to Theorem 2.2
- For a discrete RV X, $F_X(x)$ has zero slope everywhere except at value of x with nonzero probability. At these points, the function has a discontinuity in the form of a jump of magnitude $P_X(x)$.
- For a continuous RV X, $F_X(x)$ is a continuous function of X.

Definition 3.2 Continuous Random Variable

X *is a* continuous random variable *if the CDF* $F_X(x)$ *is a continuous function.*

Example 3.2 Problem

In the wheel-spinning experiment of Example 3.1, find the CDF of X.

- Solution
  - Outcome: $x \in S_X = [0, 1)$
  - $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x \geq 1$
  - $F_X(x) = P[X \leq x]$ is the probability that the pointer stops somewhere in the arc. It grows from 0 to 1. It grows in **proportion** to the fraction of the circle occupied by the arc $X \leq x$. The fraction is simply x.
  - The circle divided into n arcs.
  - $\{Y \leq \lceil nx \rceil - 1\} \subset \{X \leq x\} \subset \{Y \leq \lceil nx \rceil\}$
  - $F_Y(\lceil nx \rceil - 1) \leq F_X(x) \leq F_Y(\lceil nx \rceil)$

[Continued]

Example 3.2 Solution

We begin by observing that any outcome $x \in S_X = [0, 1)$. This implies that $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x \geq 1$. To find the CDF for x between 0 and 1 we consider the event $\{X \leq x\}$ with x growing from 0 to 1. Each event corresponds to an arc on the circle in Figure 3.1. The arc is small when $x \approx 0$ and it includes nearly the whole circle when $x \approx 1$. $F_X(x) = P[X \leq x]$ is the probability that the pointer stops somewhere in the arc. This probability grows from 0 to 1 as the arc increases to include the whole circle. Given our assumption that the pointer has no preferred stopping places, it is reasonable to expect the probability to grow in proportion to the fraction of the circle occupied by the arc $X \leq x$. This fraction is simply x. To be more formal, we can refer to Figure 3.1 and note that with the circle divided into n arcs,

$$\{Y \leq \lceil nx \rceil - 1\} \subset \{X \leq x\} \subset \{Y \leq \lceil nx \rceil\}.$$

Therefore, the probabilities of the three events satisfy

$$F_Y(\lceil nx \rceil - 1) \leq F_X(x) \leq F_Y(\lceil nx \rceil).$$

[Continued]

Example 3.2 Solution (continued)

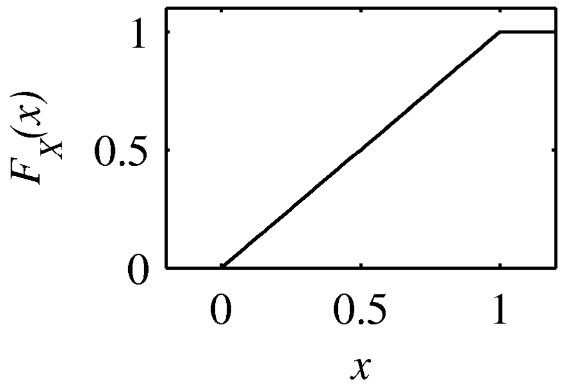
Note that Y is a discrete random variable with CDF

$$P_Y(y) = \begin{cases} 1/n & y = 1,2,\cdots,n, \\ 0 & \text{otherwise.} \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 1, \\ k/n & k \le y < k+1, \;\; k = 1,2,3,\ldots,n, \\ 1 & 1 \le y. \end{cases}$$

Thus for $x \in [0, 1)$ and for all n, we have

$$\frac{\lceil nx \rceil - 1}{n} \le F_X(x) \le \frac{\lceil nx \rceil}{n}.$$

In Problem 3.1.4, we ask the reader to verify that $\lim_{n\to\infty} \lceil nx \rceil / n = x$. This implies that as $n \to \infty$, both fractions approach x. The CDF of X is



$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \le x < 1, \\ 1 & x \ge 1. \end{cases}$$

Quiz 3.1

The cumulative distribution function of the random variable Y is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \le y \le 4, \\ 1 & y > 4. \end{cases}$$

Sketch the CDF of Y and calculate the following probabilities:

(1) $P[Y \le -1]$

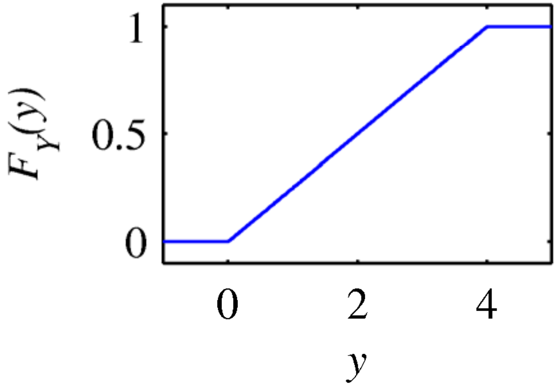
(2) $P[Y \le 1]$

(3) $P[2 < Y \le 3]$

(4) $P[Y > 1.5]$

Quiz 3.1 Solution

The CDF of Y is



$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/4 & 0 \le y \le 4 \\ 1 & y > 4 \end{cases}$$

From the CDF $F_Y(y)$, we can calculate the probabilities:

(1) $P[Y \le -1] = F_Y(-1) = 0$

(2) $P[Y \le 1] = F_Y(1) = 1/4$

(3) $P[2 < Y \le 3] = F_Y(3) - F_Y(2) = 3/4 - 2/4 = 1/4$

(4) $P[Y > 1.5] = 1 - P[Y \le 1.5] = 1 - F_Y(1.5) = 1 - (1.5)/4 = 5/8$

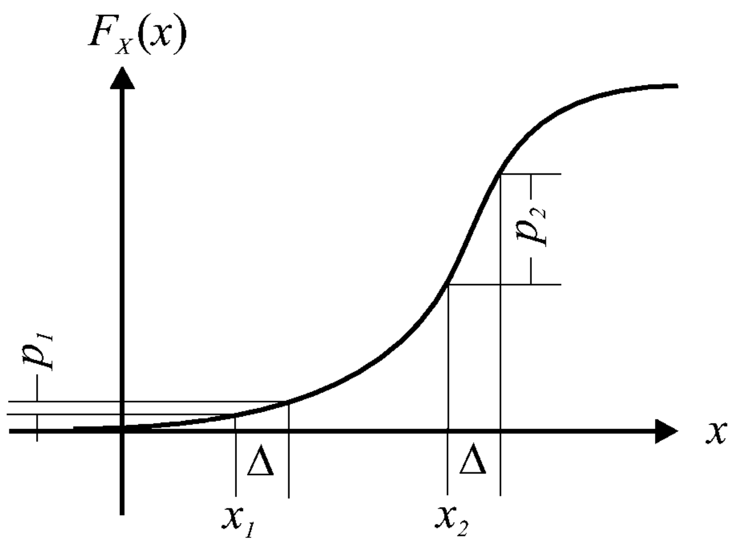
3.2 Probability Density Function

Slope of CDF

- The slop at any point x indicates the probability that X is near x.
- The limit of the average slope as $\Delta \to 0$ is the derivative of $F_X(x)$ evaluated at x_i.
- **Probability density**: the slope of CDF.

$$p_i = P\left[x_i < X \le x_i + \Delta\right] = F\left(x_i + \Delta\right) - F\left(x_i\right)$$

$$P\left[x_i < X \le x_i + \Delta\right] = \frac{F\left(x_i + \Delta\right) - F\left(x_i\right)}{\Delta}\Delta$$



Definition 3.3 (PDF)

The probability density function *(PDF) of a continuous random variable* X *is*

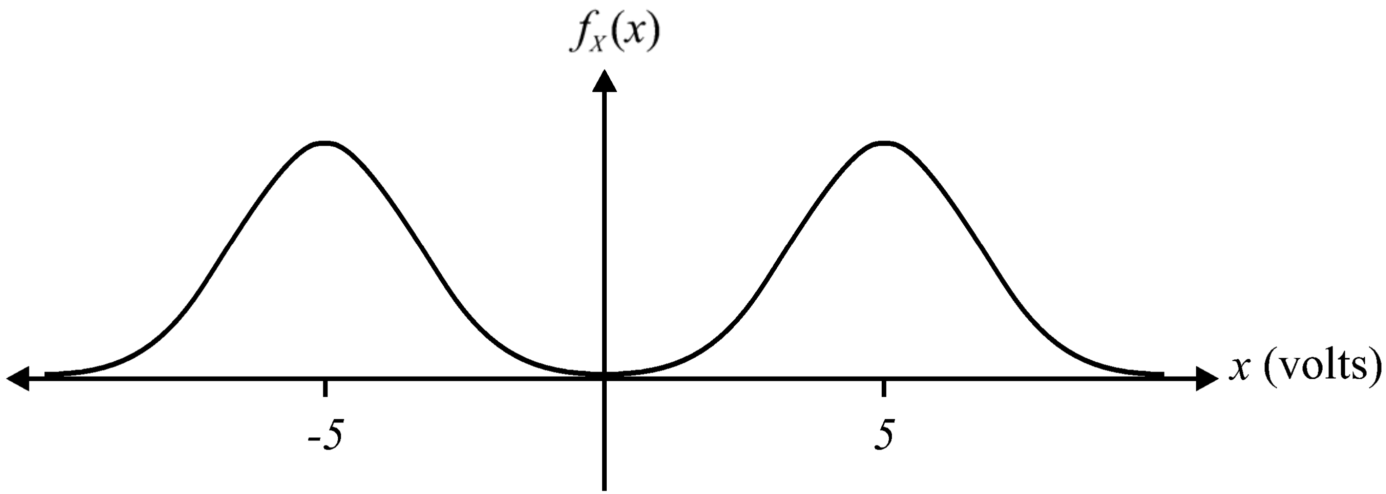
$$f_X(x) = \frac{dF_X(x)}{dx}.$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Example 3.3 Problem

Figure 3.3 depicts the PDF of a random variable X that describes the voltage at the receiver in a modem. What are probable values of X?

Figure 3.3



The PDF of the modem receiver voltage X.

Example 3.3 Solution

Note that there are two places where the PDF has high values and that it is low elsewhere. The PDF indicates that the random variable is likely to be near -5 V (corresponding to the symbol 0 transmitted) and near $+5$ V (corresponding to a 1 transmitted). Values far from ± 5 V (due to strong distortion) are possible but much less likely.

Theorem 3.2

For a continuous random variable X with PDF $f_X(x)$,

(a) $f_X(x) \geq 0$ for all x,

(b) $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$,

(c) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.

Proof: Theorem 3.2

The first statement is true because $F_X(x)$ is a nondecreasing function of x and therefore its derivative, $f_X(x)$, is nonnegative. The second fact follows directly from the definition of $f_X(x)$ and the fact that $F_X(-\infty) = 0$. The third statement follows from the second one and Theorem 3.1(b).

Theorem 3.3

$$P\left[x_1 < X \leq x_2\right] = \int_{x_1}^{x_2} f_X(x)\, dx.$$

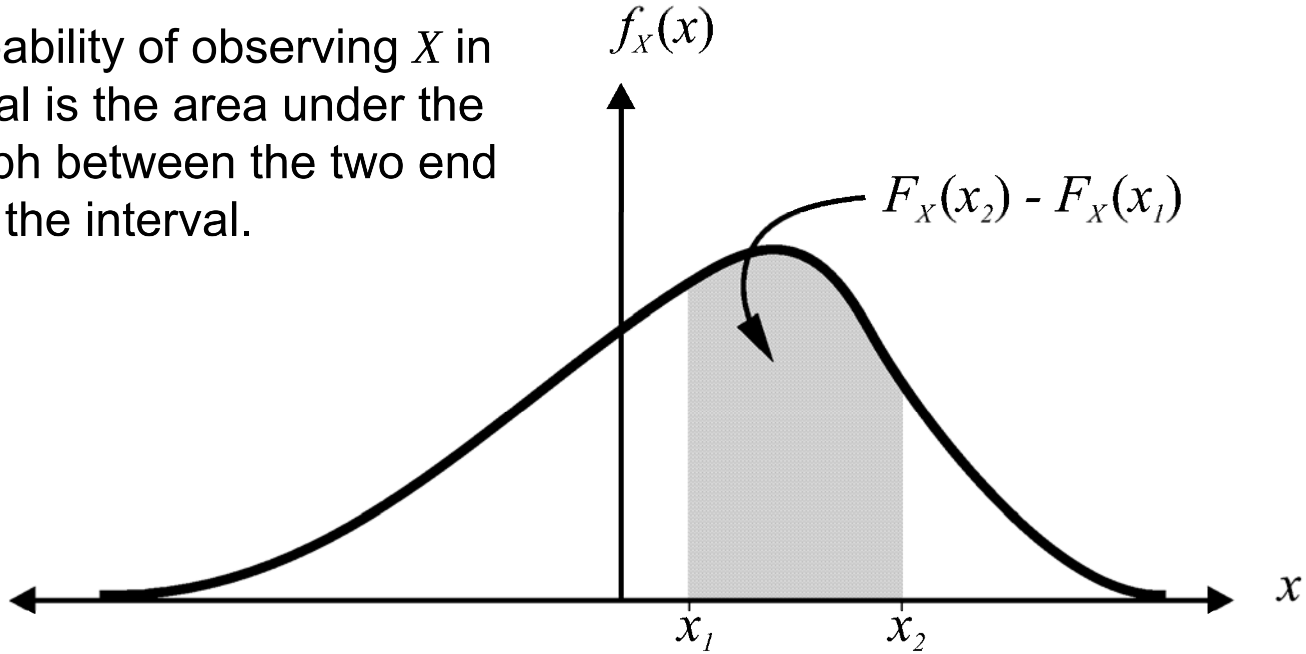
Proof: Theorem 3.3

From Theorem 3.2(b) and Theorem 3.1,

$$\begin{aligned} P\left[x_1 < X \leq x_2\right] &= P\left[X \leq x_2\right] - P\left[X \leq x_1\right] \\ &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} f_X(x)\, dx. \end{aligned}$$

Figure 3.4

The probability of observing X in an interval is the area under the PDF graph between the two end points of the interval.



The PDF and CDF of X.

$$P[x < X \leq x + dx] = f_X(x)dx$$

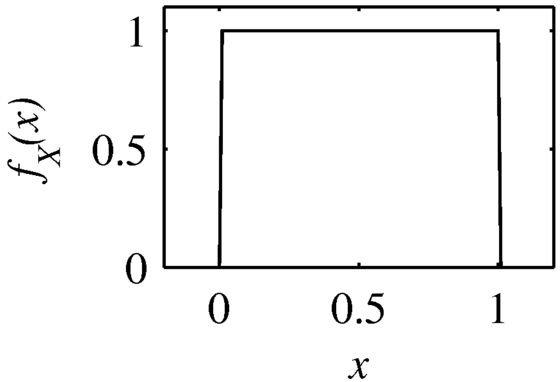
$P[]$ interpret the integral of Theorem 3.3 as the limiting case of a sum of probabilities of events $\{x < X \leq x+dx\}$

Example 3.4 Problem

For the experiment in Examples 3.1 and 3.2, find the PDF of X and the probability of the event $\{1/4 < X \leq 3/4\}$.

Example 3.4 Solution

Taking the derivative of the CDF in Equation (3.8), $f_X(x) = 0$, when $x < 0$ or $x \geq 1$. For x between 0 and 1 we have $f_X(x) = dF_X(x)/dx = 1$. Thus the PDF of X is



$$f_X(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that the PDF is constant over the range of possible values of X reflects the fact that the pointer has no favorite stopping places on the circumference of the circle. To find the probability that X is between $1/4$ and $3/4$, we can use either Theorem 3.1 or Theorem 3.3. Thus

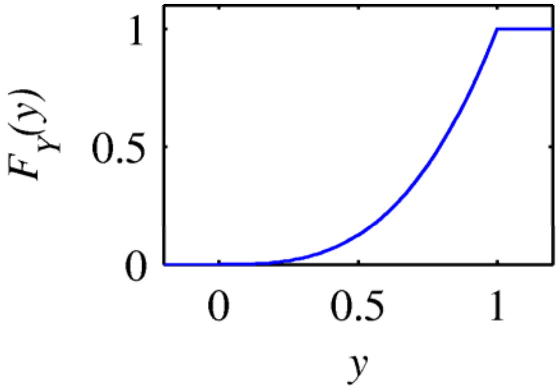
$$P[1/4 < X \leq 3/4] = F_X(3/4) - F_X(1/4) = 1/2,$$

and equivalently,

$$P[1/4 < X \leq 3/4] = \int_{1/4}^{3/4} f_X(x)\, dx = \int_{1/4}^{3/4} dx = 1/2.$$

Example 3.5 Problem

Consider an experiment that consists of spinning the pointer in Example 3.1 three times and observing Y meters, the maximum value of X in the three spins. In Example 5.8, we show that the CDF of Y is

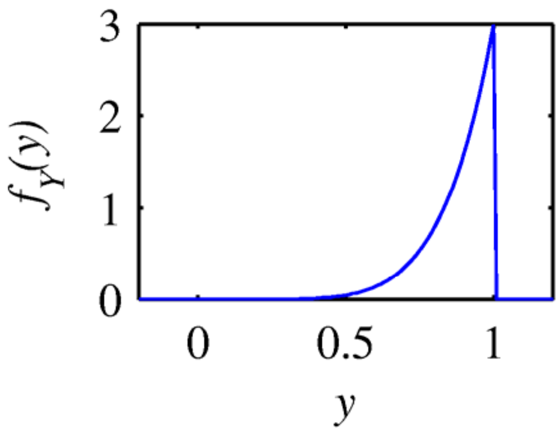


$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y^3 & 0 \le y \le 1, \\ 1 & y > 1. \end{cases}$$

Find the PDF of Y and the probability that Y is between $1/4$ and $3/4$.

Example 3.5 Solution

Applying Definition 3.3,



$$f_Y(y) = \begin{cases} dF_Y(y)/dy = 3y^2 & 0 < y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the PDF has values between 0 and 3. Its integral between any pair of numbers is less than or equal to 1. The graph of $f_Y(y)$ shows that there is a higher probability of finding Y at the right side of the range of possible values than at the left side. This reflects the fact that the maximum of three spins produces higher numbers than individual spins. Either Theorem 3.1 or Theorem 3.3 can be used to calculate the probability of observing Y between $1/4$ and $3/4$:

$$P[1/4 < Y \leq 3/4] = F_Y(3/4) - F_Y(1/4) = (3/4)^3 - (1/4)^3 = 13/32,$$

and equivalently,

$$P[1/4 < Y \leq 3/4] = \int_{1/4}^{3/4} f_Y(y)\,dy = \int_{1/4}^{3/4} 3y^2\,dy = 13/32.$$

Comments on PDF and PMF

- In Example 3.2, there are 4 different sets of numbers defined by the word "X is between ¼ and ¾":
  - A = (¼ , ¾), B = (¼ , ¾], C = [¼ , ¾), D = [¼ , ¾]
- While the are all different events, they all have the same probability because they differ only in whether they include $\{X = 1/4\}$, $\{X = 3/4\}$, or both. Since the two sets have zero probability, their inclusion or exclusion does not affect the probability of the range of numbers.
- The is quite different from the situation we encounter with discrete random variables.

Compare Discrete RV and Continuous RV

- When X is discrete,
  (Theorem 2.1(c))

$$P[B] = \sum_{x \in B} P_X(x)$$

- When X is continuous and $B = [x_1, x_2]$,
  (Theorem 3.3)

$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(u)\,du$$

- "sums to discrete RV" v.s. "integrals to continuous RV"

Quiz 3.2

Random variable X has probability density function

$$f_X(x) = \begin{cases} cxe^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the PDF and find the following:

(1) the constant c

(2) the CDF $F_X(x)$

(3) $P[0 \leq X \leq 4]$

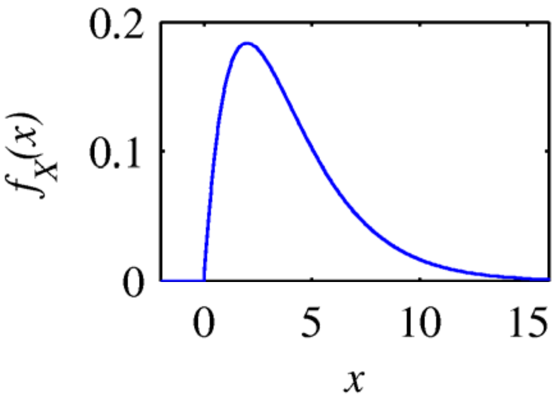
(4) $P[-2 \leq X \leq 2]$

Quiz 3.2 Solution

(1) First we will find the constant c and then we will sketch the PDF. To find c, we use the fact that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$. We will evaluate this integral using integration by parts:

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)\,dx &= \int_0^{\infty} cxe^{-x/2}\,dx \\ &= \underbrace{-2cxe^{-x/2}\Big|_0^{\infty}}_{=0} + \int_0^{\infty} 2ce^{-x/2}\,dx \\ &= -4ce^{-x/2}\Big|_0^{\infty} = 4c\end{aligned}$$

Thus $c = 1/4$ and X has the Erlang $(n = 2, \lambda = 1/2)$ PDF



$$f_X(x) = \begin{cases} (x/4)e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

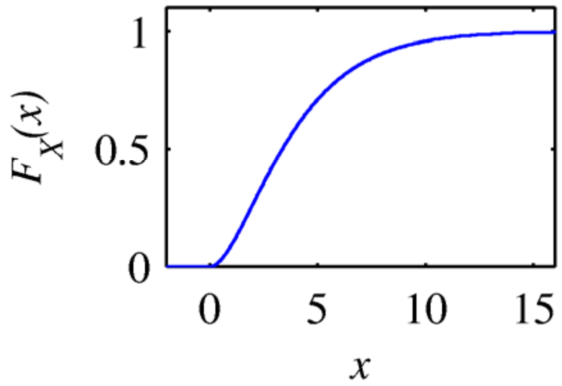
(2) To find the CDF $F_X(x)$, we first note X is a nonnegative random variable so that $F_X(x) = 0$ for all $x < 0$. For $x \geq 0$,

[Continued]

Quiz 3.2 Solution (continued)

$$F_X(x) = \int_0^x f_X(y)\,dy = \int_0^x \frac{y}{4}e^{-y/2}\,dy$$
$$= -\frac{y}{2}e^{-y/2}\Big|_0^x - \int_0^x -\frac{1}{2}e^{-y/2}\,dy$$
$$= 1 - \frac{x}{2}e^{-x/2} - e^{-x/2}$$

The complete expression for the CDF is



$$F_X(x) = \begin{cases} 1 - (\frac{x}{2}+1)e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(3) From the CDF $F_X(x)$,

$$P[0 \leq X \leq 4] = F_X(4) - F_X(0) = 1 - 3e^{-2}.$$

(4) Similarly,

$$P[-2 \leq X \leq 2] = F_X(2) - F_X(-2) = 1 - 3e^{-1}.$$

3.3 Expected Values

From Discrete RV to Continuous RV

- For a discrete RV Y, $$E[Y] = \sum_{y_i \in S_Y} y_i P_Y(y_i)$$

- For a continuous RV X, examining a discrete approximation of X.
  - For a small Δ, let $Y = \Delta \left\lfloor \frac{X}{\Delta} \right\rfloor$
  - Y is an approximation to X in that $Y = k\Delta$ iff $k\Delta \le X < k\Delta + \Delta$
  - Range of Y is $S_Y = \{\ldots, -\Delta, 0, \Delta, 2\Delta, \ldots\}$

$$E[Y] = \sum_{k=-\infty}^{\infty} k\Delta P[Y = k\Delta] = \sum_{k=-\infty}^{\infty} k\Delta P[k\Delta \le X < k\Delta + \Delta]$$

$$E[X] \approx \sum_{k=-\infty}^{\infty} k\Delta f_X(k\Delta)\Delta$$

Definition 3.4 Expected Value

The expected value *of a continuous random variable* X *is*

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

- **Law of Large Numbers** (Theorem 7.8)
  As $n \to \infty$, $E[Y] \to E[X]$

$$E[Y] = \sum_{k=-\infty}^{\infty} k\Delta P[Y = k\Delta] = \sum_{k=-\infty}^{\infty} k\Delta P[k\Delta \le X < k\Delta + \Delta]$$

Example 3.6 Problem

In Example 3.4, we found that the stopping point X of the spinning wheel experiment was a uniform random variable with PDF

$$f_X(x) = \begin{cases} 1 & 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected stopping point $E[X]$ of the pointer.

Example 3.6 Solution

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x\, dx = 1/2 \text{ meter.}$$

With no preferred stopping points on the circle, the average stopping point of the pointer is exactly half way around the circle.

Example 3.7

In Example 3.5, find the expected value of the maximum stopping point Y of the three spins:

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 y(3y^2)\, dy = 3/4 \text{ meter.}$$

Example 3.8

Let X be a uniform random variable with PDF

$$f_X(x) = \begin{cases} 1 & 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = g(X) = 0$ if $X \le 1/2$, and $W = g(X) = 1$ if $X > 1/2$. W is a discrete random variable with range $S_W = \{0, 1\}$.

- A function of a continuous random variable is also a random variable; however, this random variable is **not** necessarily continuous!
- Regardless of the nature of the random variable $W = g(x)$, its expected value can be calculated by an integral that is analogous to the sum in **Theorem 2.10** for discrete RV.

Theorem 3.4

The expected value of a function, $g(X)$, of random variable X is

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)\, f_X\left(x\right)\, dx.$$

Theorem 3.5

For any random variable X,

(a) $E[X - \mu_X] = 0$,

(b) $E[aX + b] = aE[X] + b$,

(c) $\text{Var}[X] = E[X^2] - \mu_X^2$,

(d) $\text{Var}[aX + b] = a^2 \text{Var}[X]$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$$

$$\text{Var}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)dx$$

Example 3.9 Problem

Find the variance and standard deviation of the pointer position in Example 3.1.

Example 3.9 Solution

To compute $\text{Var}[X]$, we use Theorem 3.5(c): $\text{Var}[X] = E[X^2] - \mu_X^2$. We calculate $E[X^2]$ directly from Theorem 3.4 with $g(X) = X^2$:

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2\, dx = 1/3.$$

In Example 3.6, we have $E[X] = 1/2$. Thus $\text{Var}[X] = 1/3 - (1/2)^2 = 1/12$, and the standard deviation is $\sigma_X = \sqrt{\text{Var}[X]} = 1/\sqrt{12} = 0.289$ meters.

Example 3.10 Problem

Find the variance and standard deviation of Y, the maximum pointer position after three spins, in Example 3.5.

Example 3.10 Solution

We proceed as in Example 3.9. We have $f_Y(y)$ from Example 3.5 and $E[Y] = 3/4$ from Example 3.7:

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^1 y^2 \left(3y^2\right) dy = 3/5.$$

Thus the variance is

$$\operatorname{Var}[Y] = 3/5 - (3/4)^2 = 3/80 \text{ m}^2,$$

and the standard deviation is $\sigma_Y = 0.194$ meters.

Quiz 3.3

The probability density function of the random variable Y is

$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the PDF and find the following:

(1) the expected value $E[Y]$

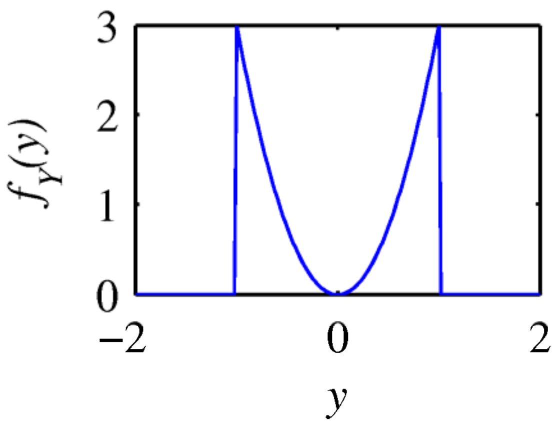
(2) the second moment $E[Y^2]$

(3) the variance $\mathrm{Var}[Y]$

(4) the standard deviation σ_Y

Quiz 3.3 Solution

The PDF of Y is



$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(1) The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-1}^{1} (3/2) y^3\, dy = (3/8) y^4 \Big|_{-1}^{1} = 0.$$

Note that the above calculation wasn't really necessary because $E[Y] = 0$ whenever the PDF $f_Y(y)$ is an even function (i.e., $f_Y(y) = f_Y(-y)$).

(2) The second moment of Y is

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_{-1}^{1} (3/2) y^4\, dy = (3/10) y^5 \Big|_{-1}^{1} = 3/5.$$

(3) The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 3/5$.

(4) The standard deviation of Y is $\sigma_Y = \sqrt{\text{Var}[Y]} = \sqrt{3/5}$.

3.4 Families of Continuous Random Variables

3 families of continuous random variables

Uniform, Exponential, Erlang

Gaussian (Section 3.5)

Definition 3.5 Uniform Random Variable

X *is a uniform* (a, b) *random variable if the PDF of* X *is*

$$f_X(x) = \begin{cases} 1/(b-a) & a \leq x < b, \\ 0 & \textit{otherwise}, \end{cases}$$

where the two parameters are $b > a$.

- X is a uniform random variable
  = X is uniformly distributed
  = X has a uniform distribution

Theorem 3.6

If X is a uniform (a, b) random variable,

(a) The CDF of X is

$$F_X(x) = \begin{cases} 0 & x \leq a, \\ (x-a)/(b-a) & a < x \leq b, \\ 1 & x > b. \end{cases}$$

Theorem 3.2(b)

(b) The expected value of X is $E[X] = (b + a)/2$. Theorem 3.4

(c) The variance of X is $\text{Var}[X] = (b - a)^2/12$. Theorem 3.5

Example 3.11 Problem

The phase angle, Θ, of the signal at the input to a modem is uniformly distributed between 0 and 2π radians. Find the CDF, the expected value, and the variance of Θ.

Example 3.11 Solution

From the problem statement, we identify the parameters of the uniform (a, b) random variable as $a = 0$ and $b = 2\pi$. Therefore the PDF of Θ is

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \le \theta < 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

The CDF is

$$F_\Theta(\theta) = \begin{cases} 0 & \theta \le 0, \\ \theta/(2\pi) & 0 < x \le 2\pi, \\ 1 & x > 2\pi. \end{cases}$$

The expected value is $E[\Theta] = b/2 = \pi$ radians, and the variance is $\text{Var}[\Theta] = (2\pi)^2/12 = \pi^2/3$ rad^2.

Theorem 3.7

Let X be a uniform (a, b) random variable, where a and b are both integers. Let $K = \lceil X \rceil$. Then K is a discrete uniform $(a+1, b)$ random variable.

Proof: Theorem 3.7

Recall that for any x, $\lceil x \rceil$ is the smallest integer greater than or equal to x. It follows that the event $\{K = k\} = \{k - 1 < x \leq k\}$. Therefore,

$$P[K = k] = P_K(k) = \int_{k-1}^{k} P_X(x)\, dx = \begin{cases} 1/(b-a) & k = a+1, a+2, \ldots, b, \\ 0 & \text{otherwise.} \end{cases}$$

This expression for $P_K(k)$ conforms to Definition 2.9 of a discrete uniform $(a+1, b)$ PMF.

Definition 3.6 Exponential Random Variable

X *is an* exponential (λ) random variable *if the PDF of* X *is*

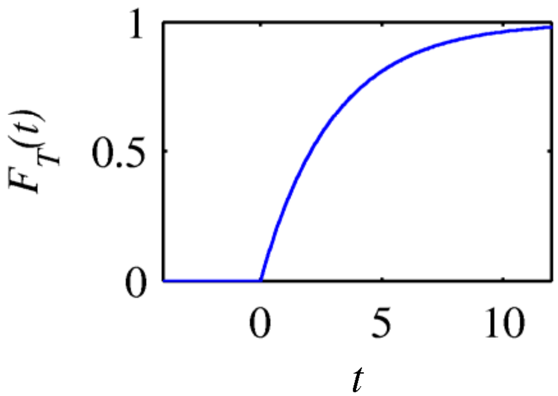
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \textit{otherwise}, \end{cases}$$

where the parameter $\lambda > 0$*.*

- The continuous relatives of the family of geometric random variables, Definition 2.6, are the members of the family of exponential random variables.

Example 3.12 Problem

The probability that a telephone call lasts no more than t minutes is often modeled as an exponential CDF.

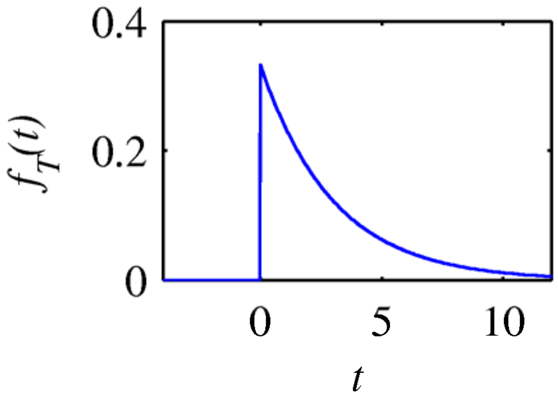


$$F_T(t) = \begin{cases} 1 - e^{-t/3} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is the PDF of the duration in minutes of a telephone conversation? What is the probability that a conversation will last between 2 and 4 minutes?

Example 3.12 Solution

We find the PDF of T by taking the derivative of the CDF:



$$f_T(t) = \frac{dF_T(t)}{dt} = \begin{cases} (1/3)e^{-t/3} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, observing Definition 3.6, we recognize that T is an exponential $(\lambda = 1/3)$ random variable. The probability that a call lasts between 2 and 4 minutes is

$$P[2 \leq T \leq 4] = F_4(4) - F_2(2) = e^{-2/3} - e^{-4/3} = 0.250.$$

Example 3.13 Problem

In Example 3.12, what is $E[T]$, the expected duration of a telephone call? What are the variance and standard deviation of T? What is the probability that a call duration is within ± 1 standard deviation of the expected call duration?

Example 3.13 Solution

Using the PDF $f_T(t)$ in Example 3.12, we calculate the expected duration of a call:

$$E[T] = \int_{-\infty}^{\infty} t f_T(t)\, dt = \int_0^{\infty} t \frac{1}{3} e^{-t/3}\, dt.$$

Integration by parts (Appendix B, Math Fact B.10) yields

$$E[T] = -te^{-t/3}\Big|_0^{\infty} + \int_0^{\infty} e^{-t/3}\, dt = 3 \text{ minutes.}$$

To calculate the variance, we begin with the second moment of T:

$$E\left[T^2\right] = \int_{-\infty}^{\infty} t^2 f_T(t)\, dt = \int_0^{\infty} t^2 \frac{1}{3} e^{-t/3}\, dt.$$

[Continued]

Example 3.13 Solution (continued)

Again integrating by parts, we have

$$E\left[T^2\right] = -t^2 e^{-t/3}\Big|_0^\infty + \int_0^\infty (2t)e^{-t/3}\,dt = 2\int_0^\infty te^{-t/3}\,dt.$$

With the knowledge that $E[T] = 3$, we observe that $\int_0^\infty te^{-t/3}\,dt = 3E[T] = 9$. Thus $E[T^2] = 6E[T] = 18$ and

$$\text{Var}[T] = E\left[T^2\right] - (E[T])^2 = 18 - 3^2 = 9.$$

The standard deviation is $\sigma_T = \sqrt{\text{Var}[T]} = 3$ minutes. The probability that the call duration is within 1 standard deviation of the expected value is

$$P[0 \le T \le 6] = F_T(6) - F_T(0) = 1 - e^{-2} = 0.865$$

Theorem 3.8

If X is an exponential (λ) random variable,

(a) $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$ Theorem 3.2(b)

(b) $E[X] = 1/\lambda$. Theorem 3.4

(c) $\text{Var}[X] = 1/\lambda^2$. Theorem 3.5

Theorem 3.9

If X is an exponential (λ) random variable, then $K = \lceil X \rceil$ is a geometric (p) random variable with $p = 1 - e^{-\lambda}$.

Proof: Theorem 3.9

As in the proof of Theorem 3.7, the definition of K implies

$$P_K(k) = P[k - 1 < X \leq k].$$

Referring to the CDF of X in Theorem 3.8, we observe

$$P_K(k) = F_X(k) - F_X(k-1) = e^{-\lambda(k-1)} - e^{-\lambda k} = (e^{-\lambda})^{k-1}(1 - e^{-\lambda}).$$

If we let $p = 1 - e^{-\lambda}$, we have $P_K(k) = p(1-p)^{k-1}$, which conforms to Definition 2.6 of a geometric (p) random variable with $p = 1 - e^{-\lambda}$.

Example 3.14 Problem

Phone company A charges $\$0.15$ per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone Company B also charges $\$0.15$ per minute. However, Phone Company B calculates its charge based on the exact duration of a call. If T, the duration of a call in minutes, is an exponential $(\lambda = 1/3)$ random variable, what are the expected revenues per call $E[R_A]$ and $E[R_B]$ for companies A and B?

Example 3.14 Solution

Because T is an exponential random variable, we have in Theorem 3.8 (and in Example 3.13), $E[T] = 1/\lambda = 3$ minutes per call. Therefore, for phone company B, which charges for the exact duration of a call,

$$E[R_B] = 0.15E[T] = \$0.45 \text{ per call.}$$

Company A, by contrast, collects \$0.15$\lceil T \rceil$ for a call of duration T minutes. Theorem 3.9 states that $K = \lceil T \rceil$ is a geometric random variable with parameter $p = 1 - e^{-1/3}$. Therefore, the expected revenue for Company A is

$$E[R_A] = 0.15E[K] = 0.15/p = (0.15)(3.53) = \$0.529 \text{ per call.}$$

Definition 3.7 Erlang Random Variable

X *is an* Erlang (n, λ) *random variable if the PDF of* X *is*

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

where the parameter $\lambda > 0$*, and the parameter* $n \geq 1$ *is an integer.*

- n is the **order** of an Erlang random variable.
- The Erlang $(n = 1, \lambda)$ RV is identical to the exponential (λ) RV.
- Exponential RV ⇔ Geometric RV
Erlang RV ⇔ Pascal RV
- In **Theorem 6.11**, we show that the sum of a set of independent identically distributed **exponential** RV is an **Erlang** RV.
- Problem 3.4.10. Verify that the integral of the Erlang PDF over all x is 1.

Problem 3.4.10

In this problem we verify that an Erlang (n, λ) PDF integrates to 1. Let the integral of the nth order Erlang PDF be denoted by

$$I_n = \int_0^\infty \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \, dx.$$

First, show directly that the Erlang PDF with $n = 1$ integrates to 1 by verifying that $I_1 = 1$. Second, use integration by parts (Appendix B, Math Fact B.10) to show that $I_n = I_{n-1}$.

Problem 3.4.10 Solution

The integral I_1 is

$$I_1 = \int_0^\infty \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^\infty = 1$$

For $n > 1$, we have

$$I_n = \int_0^\infty \underbrace{\frac{\lambda^{n-1}x^{n-1}}{(n-1)!}}_{u} \underbrace{\lambda e^{-\lambda x}\,dt}_{dv}$$

We define u and dv as shown above in order to use the integration by parts formula $\int u\,dv = uv - \int v\,du$. Since

$$du = \frac{\lambda^{n-1}x^{n-2}}{(n-2)!}\,dx \qquad v = -e^{-\lambda x}$$

we can write

$$\begin{aligned} I_n &= uv|_0^\infty - \int_0^\infty v\,du \\ &= -\frac{\lambda^{n-1}x^{n-1}}{(n-1)!}e^{-\lambda x}\Big|_0^\infty + \int_0^\infty \frac{\lambda^{n-1}x^{n-2}}{(n-2)!}e^{-\lambda x}\,dx = 0 + I_{n-1} \end{aligned}$$

Hence, $I_n = 1$ for all $n \geq 1$.

Theorem 3.10

If X is an Erlang (n, λ) random variable, then

$$E[X] = \frac{n}{\lambda}, \qquad \mathrm{Var}[X] = \frac{n}{\lambda^2}.$$

- X: Erlang (n, λ) random variable
  Y: exponential (λ) random variable
- $E[X] = nE[Y]$, $\mathrm{Var}[X] = n\mathrm{Var}[Y]$

Theorem 3.11

Let K_α denote a Poisson (α) random variable. For any $x > 0$, the CDF of an Erlang (n, λ) random variable X satisfies

$$F_X(x) = 1 - F_{K_{\lambda x}}(n-1) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}.$$

- Connect Erlang and Poisson random variables.

Problem 3.4.12

In this problem, we outline the proof of Theorem 3.11.

(a) Let X_n denote an Erlang (n, λ) random variable. Use the definition of the Erlang PDF to show that for any $x \geq 0$,

$$F_{X_n}(x) = \int_0^x \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\,dt.$$

(b) Apply integration by parts (Appendix B, Math Fact B.10) to this integral to show that for $x \geq 0$,

$$F_{X_n}(x) = F_{X_{n-1}}(x) - \frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}.$$

(c) Use the fact that $F_{X_1}(x) = 1 - e^{-\lambda x}$ for $x \geq 0$ to verify the claim of Theorem 3.11.

Problem 3.4.12 Solution

In this problem, we prove Theorem 3.11 which says that for $x \geq 0$, the CDF of an Erlang (n, λ) random variable X_n satisfies

$$F_{X_n}(x) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}.$$

We do this in two steps. First, we derive a relationship between $F_{X_n}(x)$ and $F_{X_{n-1}}(x)$. Second, we use that relationship to prove the theorem by induction.

(a) By Definition 3.7, the CDF of Erlang (n, λ) random variable X_n is

$$F_{X_n}(x) = \int_{-\infty}^{x} f_{X_n}(t)\, dt = \int_0^x \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\, dt.$$

(b) To use integration by parts, we define

$$u = \frac{t^{n-1}}{(n-1)!} \qquad dv = \lambda^n e^{-\lambda t}\, dt$$

$$du = \frac{t^{n-2}}{(n-2)!} \qquad v = -\lambda^{n-1} e^{-\lambda t}$$

[Continued]

Problem 3.4.12 Solution (continued)

Thus, using the integration by parts formula $\int u\,dv = uv - \int v\,du$, we have

$$F_{X_n}(x) = \int_0^x \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\,dt = -\frac{\lambda^{n-1} t^{n-1} e^{-\lambda t}}{(n-1)!}\bigg|_0^x + \int_0^x \frac{\lambda^{n-1} t^{n-2} e^{-\lambda t}}{(n-2)!}\,dt$$

$$= -\frac{\lambda^{n-1} x^{n-1} e^{-\lambda x}}{(n-1)!} + F_{X_{n-1}}(x)$$

(c) Now we do proof by induction. For $n = 1$, the Erlang (n, λ) random variable X_1 is simply an exponential random variable. Hence for $x \geq 0$, $F_{X_1}(x) = 1 - e^{-\lambda x}$. Now we suppose the claim is true for $F_{X_{n-1}}(x)$ so that

$$F_{X_{n-1}}(x) = 1 - \sum_{k=0}^{n-2} \frac{(\lambda x)^k e^{-\lambda x}}{k!}.$$

Using the result of part (a), we can write

$$F_{X_n}(x) = F_{X_{n-1}}(x) - \frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}$$

$$= 1 - \sum_{k=0}^{n-2} \frac{(\lambda x)^k e^{-\lambda x}}{k!} - \frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}$$

which proves the claim.

Quiz 3.4

Continuous random variable X has $E[X] = 3$ and $\text{Var}[X] = 9$. Find the PDF, $f_X(x)$, if

(1) X has an exponential PDF,

(2) X has a uniform PDF.

Quiz 3.4 Solution

(1) When X is an exponential (λ) random variable, $E[X] = 1/\lambda$ and $\text{Var}[X] = 1/\lambda^2$. Since $E[X] = 3$ and $\text{Var}[X] = 9$, we must have $\lambda = 1/3$. The PDF of X is

$$f_X(x) = \begin{cases} (1/3)e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) We know X is a uniform (a, b) random variable. To find a and b, we apply Theorem 3.6 to write

$$E[X] = \frac{a+b}{2} = 3 \qquad \text{Var}[X] = \frac{(b-a)^2}{12} = 9.$$

This implies

$$a + b = 6, \qquad b - a = \pm 6\sqrt{3}.$$

The only valid solution with $a < b$ is

$$a = 3 - 3\sqrt{3}, \qquad b = 3 + 3\sqrt{3}.$$

The complete expression for the PDF of X is

$$f_X(x) = \begin{cases} 1/(6\sqrt{3}) & 3 - 3\sqrt{3} \leq x < 3 + 3\sqrt{3}, \\ 0 & \text{otherwise.} \end{cases}$$

3.5 Gaussian Random Variables

Gaussian Random Variables

- Bell-shaped curves.
- Chapter 6 contains a mathematical explanation for the prevalence of Gaussian RV in models of practical phenomena.
- Gaussian RV ➔ Normal RV

Definition 3.8 Gaussian Random Variable

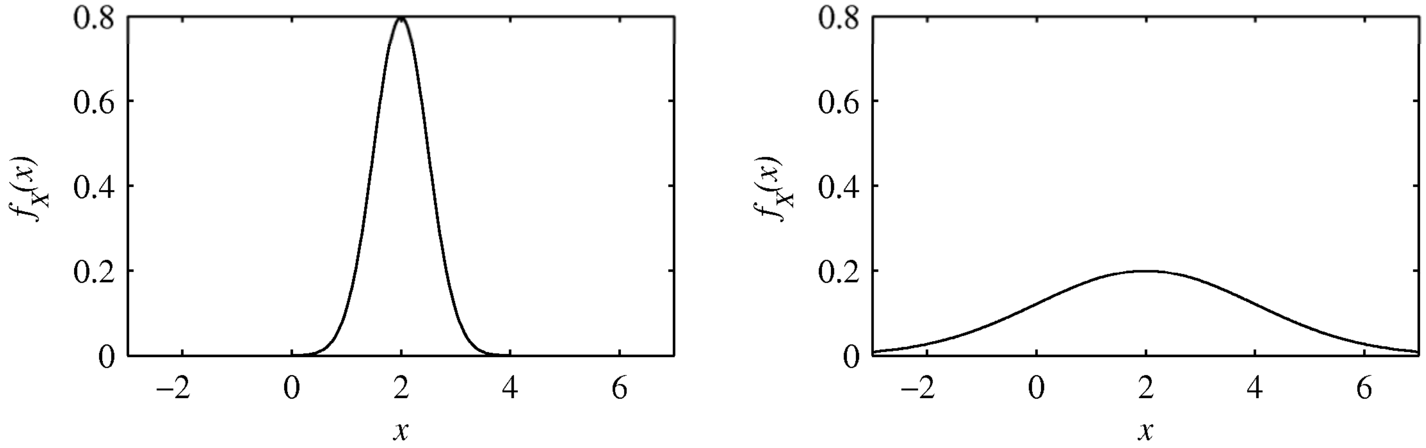
X is a Gaussian (μ, σ) random variable if the PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter μ can be any real number and the parameter $\sigma > 0$.

- $N[\mu, \sigma^2]$ is a shorthand for Gaussian(μ, σ)
  N: normal
- $x = \mu$: the center of the bell shape
  σ: the width of the bell shape
- The height of the peak is $\dfrac{1}{\sigma\sqrt{2\pi}}$

Figure 3.5



(a) $\mu = 2, \sigma = 1/2$ **(b)** $\mu = 2, \sigma = 2$

Two examples of a Gaussian random variable X with expected value μ and standard deviation σ.

Theorem 3.12

If X is a Gaussian (μ, σ) random variable,

$$E[X] = \mu \qquad \mathrm{Var}[X] = \sigma^2.$$

– Problem 3.5.9.Verify that the integral of the Gaussian PDF over all x is 1.

Theorem 3.13

If X is Gaussian (μ, σ), $Y = aX + b$ is Gaussian $(a\mu + b, a\sigma)$.

Problem 3.5.9

This problem outlines the steps needed to show that the Gaussian PDF integrates to unity. For a Gaussian (μ, σ) random variable W, we will show that

$$I = \int_{-\infty}^{\infty} f_W(w)\, dw = 1.$$

(a) Use the substitution $x = (w - \mu)/\sigma$ to show that

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx.$$

(b) Show that

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy.$$

(c) Change the integral for I^2 to polar coordinates to show that it integrates to 1.

Problem 3.5.9 Solution

First we note that since W has an $N[\mu, \sigma^2]$ distribution, the integral we wish to evaluate is

$$I = \int_{-\infty}^{\infty} f_W(w)\, dw = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(w-\mu)^2/2\sigma^2}\, dw$$

(a) Using the substitution $x = (w-\mu)/\sigma$, we have $dx = dw/\sigma$ and

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$$

(b) When we write I^2 as the product of integrals, we use y to denote the other variable of integration so that

$$\begin{aligned} I^2 &= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy\right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy \end{aligned}$$

(c) By changing to polar coordinates, $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$ so that

$$\begin{aligned} I^2 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1 \end{aligned}$$

Integral of Gaussian PDF

- It is impossible to express the integral of a Gaussian PDF between non-infinite limits as a function that appears on most scientific calculators.
- Instead, we usually find integrals of the Gaussian PDF by referring to tables, such as Table 3.1, that have been obtained by numerical integration.

Standard Normal Random

Definition 3.9 Variable

The standard normal random variable Z *is the Gaussian* $(0, 1)$ *random variable.*

- $E[Z] = 0$, $\mathrm{Var}[Z] = 1$
- $F_Z(z)$: the CDF of Z. Introduce the special notation $\Phi(z)$ for $F_Z(z)$.

Definition 3.10 Standard Normal CDF

The CDF of the standard normal random variable Z *is*

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

Theorem 3.14

If X is a Gaussian (μ, σ) random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that X is in the interval $(a, b]$ is

$$P[a < X \leq b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

– In using this theorem, we transform values of a **Gaussian** RV, X, to equivalent values of the **standard normal** RV, Z. For a sample value x of the RV X, the corresponding sample value of Z is

$$z = \frac{x-\mu}{\sigma}$$

– Note that z is dimensionless.

Example 3.15 Problem

Suppose your score on a test is $x = 46$, a sample value of the Gaussian $(61, 10)$ random variable. Express your test score as a sample value of the standard normal random variable, Z.

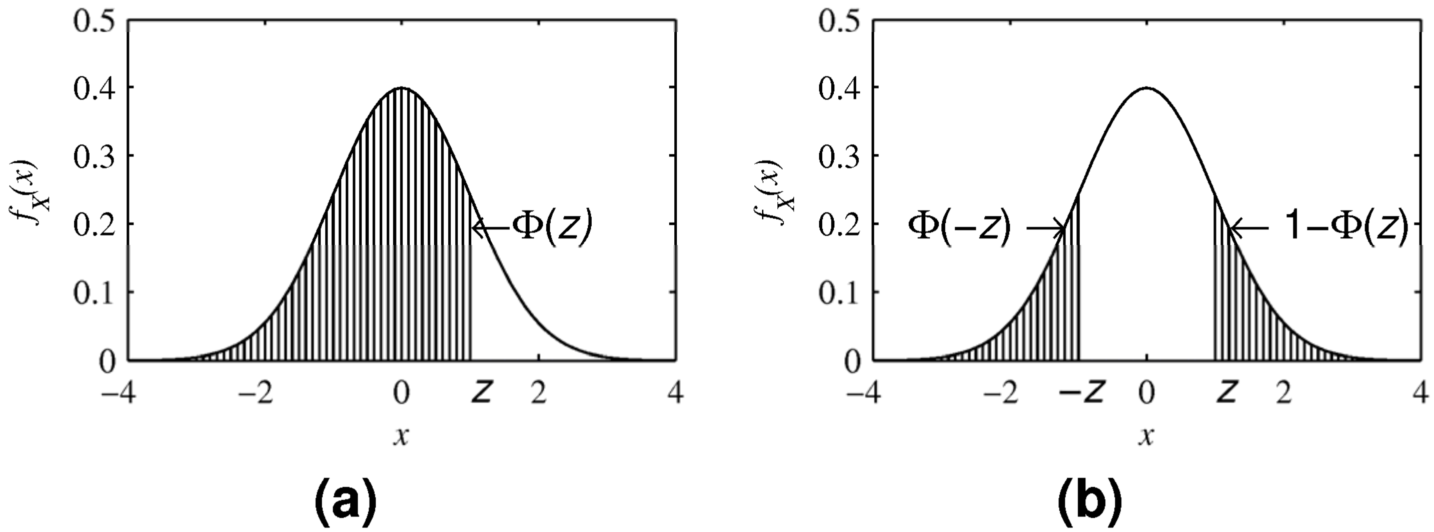
Example 3.15 Solution

Equation (3.54) indicates that $z = (46 - 61)/10 = -1.5$. Therefore your score is 1.5 *standard deviations less than the expected value*.

Theorem 3.15

$$\Phi(-z) = 1 - \Phi(z).$$

Figure 3.6 Symmetry properties of $\Phi(z)$.



Symmetry properties of th Gaussian $(0, 1)$ PDF.

Example 3.16 Problem

If X is the Gaussian $(61, 10)$ random variable, what is $P[X \leq 46]$?

Example 3.16 Solution

Applying Theorem 3.14, Theorem 3.15 and the result of Example 3.15, we have

$$P[X \leq 46] = F_X(46) = \Phi(-1.5) = 1 - \Phi(1.5) = 1 - 0.933 = 0.067.$$

This suggests that if your test score is 1.5 standard deviations below the expected value, you are in the lowest 6.7% of the population of test takers.

Example 3.17 Problem

If X is a Gaussian random variable with $\mu = 61$ and $\sigma = 10$, what is $P[51 < X \leq 71]$?

Example 3.17 Solution

Applying Equation (3.54), $Z = (X - 61)/10$ and the event $\{51 < X \leq 71\}$ corresponds to the event $\{-1 < Z \leq 1\}$. The probability of this event is

$$P[-1 < Z \leq 1] = \Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1 = 0.683.$$

Comments on Integral of Gaussian PDF

- In an experiment with a Gaussian probability model,
  - **68.3%** (about 2/3) of the outcomes are within ± 1 standard deviation of the expected value.
  - About **95%** ($2\Phi(2) - 1$) of the outcomes are within two standard deviations of the expected value.
  - Regions further than 3 standard deviations from the expected value (corresponding to $|z| \geq 3$) are in the tails of the PDF.
  - When $|z| > 3$, $\Phi(z)$ is very close to 1
    - $\Phi(3) = 0.9987$, $\Phi(4) = 0.9999768$
- The properties of $\Phi(z)$ for extreme values of z are apparent in the standard normal complementary CDF.
  - Table 3.1 and Table 3.2

Standard Normal

Definition 3.11 Complementary CDF

The standard normal complementary CDF *is*

$$Q(z) = P[Z > z] = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-u^2/2}\, du = 1 - \Phi(z).$$

- $\Phi(3) = 0.9987 \quad \Rightarrow Q(3) = 1.35 \times 10^{-3}$
- $\Phi(4) = 0.9999768 \quad \Rightarrow Q(4) = 3.17 \times 10^{-5}$
- Q(3) is almost two orders of magnitude larger than Q(4)

Example 3.18 Problem

In an optical fiber transmission system, the probability of a binary error is $Q(\sqrt{\gamma/2})$, where γ is the signal-to-noise ratio. What is the minimum value of γ that produces a binary error rate not exceeding 10^{-6}?

Example 3.18 Solution

Referring to **Table 3.2,** we find that $Q(z) < 10^{-6}$ when $z \geq 4.75$. Therefore, if $\sqrt{\gamma/2} \geq 4.75$, or $\gamma \geq 45$, the probability of error is less than 10^{-6}.

Comments on Standard Normal Complementary CDF

- $Q(z)$ is the probability that a Gaussian random variable exceeds its expected value by more than z standard deviations.
- In conversation we refer to the event $\{X - \mu_X > 3\sigma_X\}$ as a three-sigma event.
  - A 5σ event is on the order of 10^{-7}
  - What is the "6σ"? (http://en.wikipedia.org/wiki/Six_Sigma)
    - Six Sigma is a business management strategy, originally developed by Motorola (1981), that today enjoys widespread application in many sectors of industry.
    - "Six Sigma" process in fact corresponds to 4.5 sigmas, namely 6 sigmas minus the 1.5 sigma shift introduced to account for long-term variation. $Q(4.5) = 3.4 \times 10^{-6}$

Quiz 3.5

X is the Gaussian $(0, 1)$ random variable and Y is the Gaussian $(0, 2)$ random variable.

(1) Sketch the PDFs $f_X(x)$ and $f_Y(y)$ on the same axes.

(2) What is $P[-1 < X \leq 1]$?

(3) What is $P[-1 < Y \leq 1]$?

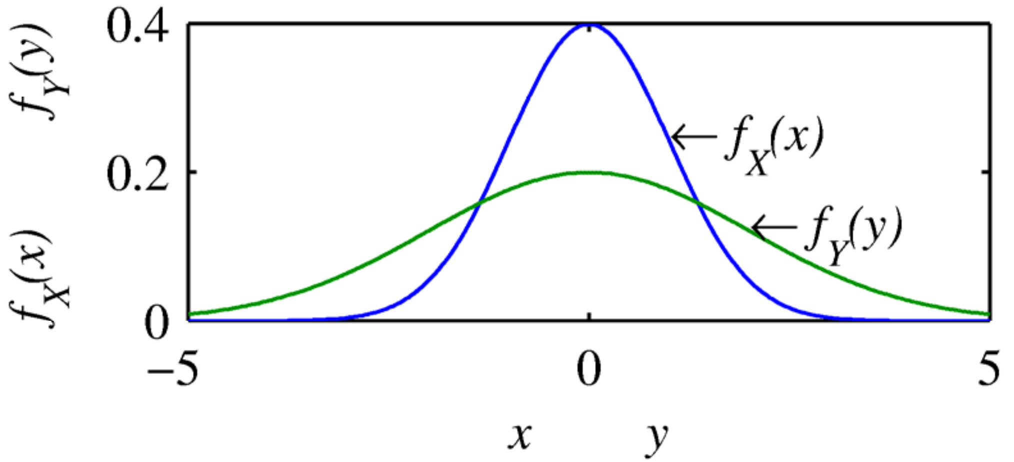
(4) What is $P[X > 3.5]$?

(5) What is $P[Y > 3.5]$?

Quiz 3.5 Solution

Each of the requested probabilities can be calculated using $\Phi(z)$ function and Table 3.1 or $Q(z)$ and Table 3.2. We start with the sketches.

(1) The PDFs of X and Y are shown below. The fact that Y has twice the standard deviation of X is reflected in the greater spread of $f_Y(y)$. However, it is important to remember that as the standard deviation increases, the peak value of the Gaussian PDF goes down.



[Continued]

Quiz 3.5 Solution (continued)

(2) Since X is Gaussian $(0, 1)$,

$$\begin{aligned} P[-1 < X \leq 1] &= F_X(1) - F_X(-1) \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826. \end{aligned}$$

(3) Since Y is Gaussian $(0, 2)$,

$$\begin{aligned} P[-1 < Y \leq 1] &= F_Y(1) - F_Y(-1) \\ &= \Phi\left(\frac{1}{\sigma_Y}\right) - \Phi\left(\frac{-1}{\sigma_Y}\right) = 2\Phi\left(\frac{1}{2}\right) - 1 = 0.383. \end{aligned}$$

(4) Again, since X is Gaussian $(0, 1)$, $P[X > 3.5] = Q(3.5) = 2.33 \times 10^{-4}$.

(5) Since Y is Gaussian $(0, 2)$, $P[Y > 3.5] = Q(\frac{3.5}{2}) = Q(1.75) = 1 - \Phi(1.75) = 0.0401$.

3.6 Delta Functions, Mixed Random Variables

Delta Functions

Random variable	Probability model	Calculation
Discrete	PMF	Sum
Continuous	PDF	Integral

- **Unit impulse function** $\delta(x)$
  - A mathematical tool that unites the analyses of discrete and continuous random variables.
  - The **delta function**: the unit impulse.
  - Use the same formulas to describe calculations with both types of random variables.
  - Provide a new notation for describing them. Especially convenient when we refer to a **mixed** random variable, which has properties of both continuous and discrete random variables.

Definition 3.12 Unit Impulse (Delta) Function

Let

$$d_\epsilon(x) = \begin{cases} 1/\epsilon & -\epsilon/2 \le x \le \epsilon/2, \\ 0 & \textit{otherwise.} \end{cases}$$

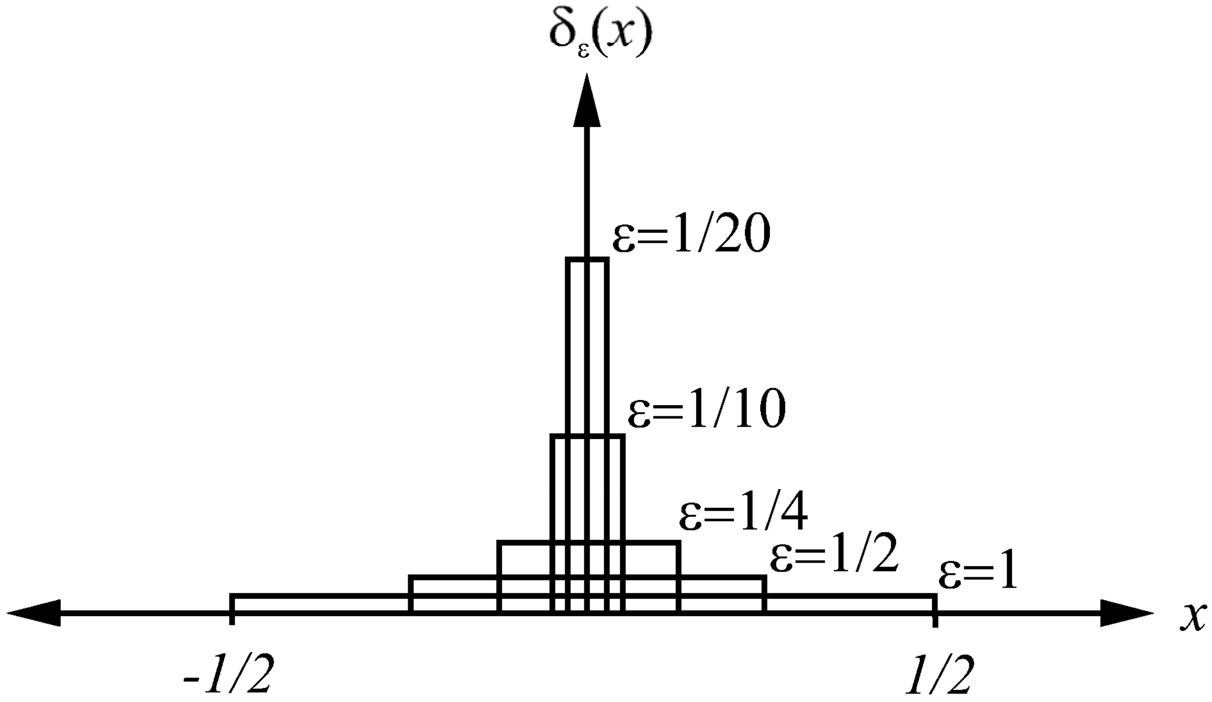
The unit impulse function *is*

$$\delta(x) = \lim_{\epsilon \to 0} d_\epsilon(x).$$

- $d_\varepsilon(x)$ has no limit at $x = 0$.
  $d_\varepsilon(x)$ just gets bigger and bigger as $\varepsilon \to 0$.
- Properties of delta function: $\int_{-\infty}^{\infty} d_\varepsilon(x)dx = \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon} dx = 1$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Figure 3.7



As $\epsilon \to 0$, $d_\epsilon(x)$ approaches the delta function $\delta(x)$. For each ϵ, the area under the curve of $d_\epsilon(x)$ equals 1.

Theorem 3.16

For any continuous function $g(x)$,

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)\, dx = g(x_0).$$

- Theorem 3.16 is often called th **Sifting property** of the delta function.
- Consider the integral $$\int_{-\infty}^{\infty} g(x) d_\varepsilon (x - x_0) dx = \frac{1}{\varepsilon} \int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2} g(x) dx = g(x_0)$$
- On the right side, we have the average value of $g(x)$ over the interval $[x_0 - \varepsilon/2, x_0 + \varepsilon/2]$. As $\varepsilon \to 0$, this average value must converge to $g(x_0)$
- The delta function has a close connection to the **unit sep function**.

Definition 3.13 Unit Step Function

The unit step function *is*

$$u(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Theorem 3.17

$$\int_{-\infty}^{x} \delta(v)\, dv = u(x).$$

Comments on Unit Step Function

- For any $x > 0$, we can choose $\varepsilon \le 2x$ so that

$$\int_{-\infty}^{-x} d_{\varepsilon}(v)dv = 0, \qquad \int_{-\infty}^{x} d_{\varepsilon}(v)dv = 1$$

- The for any $x \neq 0$, in the limit as $\varepsilon \to 0$, $\int_{-\infty}^{x} d_{\varepsilon}(v)dv = u(x)$
  then for $x = 0$?
- Theorem 3.17 allow us to write $\delta(x) = \dfrac{du(x)}{dx}$

  $\delta(x)$ does not really exist at $x = 0$.

- Consider the CDF of a discrete RV, X. Using the definition of unit step function,
  we can write the CDF of X as $F_X(x) = \sum_{x_i \in S_X} P_X(x_i)u(x - x_i)$

  PDF as $f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\delta(x - x_i)$ Ref. pic in p.127

$$E[X] = \int_{-\infty}^{\infty} x \sum_{x_i \in S_X} P_X(x_i)\delta(x - x_i) = \sum_{x_i \in S_X} \int_{-\infty}^{\infty} xP_X(x_i)\delta(x - x_i) = \sum_{x_i \in S_X} xP_X(x_i)$$

Example 3.19

Suppose Y takes on the values $1, 2, 3$ with equal probability. The PMF and the corresponding CDF of Y are

$$P_Y(y) = \begin{cases} 1/3 & y = 1, 2, 3, \\ 0 & \text{otherwise}, \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 1, \\ 1/3 & 1 \le y < 2, \\ 2/3 & 2 \le y < 3, \\ 1 & y \ge 3. \end{cases}$$

Using the unit step function $u(y)$, we can write $F_Y(y)$ more compactly as

$$F_Y(y) = \frac{1}{3}u(y-1) + \frac{1}{3}u(y-2) + \frac{1}{3}u(y-3).$$

The PDF of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{3}\delta(y-1) + \frac{1}{3}\delta(y-2) + \frac{1}{3}\delta(y-3).$$

[Continued]

Example 3.19 (continued)

We see that the discrete random variable Y can be represented graphically either by a PMF $P_Y(y)$ with bars at $y = 1, 2, 3$, by a CDF with jumps at $y = 1, 2, 3$, or by a PDF $f_Y(y)$ with impulses at $y = 1, 2, 3$. These three representations are shown in Figure 3.8. The expected value of Y can be calculated either by summing over the PMF $P_Y(y)$ or integrating over the PDF $f_Y(y)$. Using the PDF, we have

$$
\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\, dy \\
&= \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-1)\, dy + \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-2)\, dy + \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-3)\, dy \\
&= 1/3 + 2/3 + 1 = 2.
\end{aligned}
$$

Comments on Discontinuity

- When $F_X(x)$ has discontinuity at x we will use $F_X(x^+)$ and $F_X(x^-)$ to denote the upper and lower limit as x. That is,

$$F_X\left(x^-\right) = \lim_{h \to 0^+} F_X\left(x - h\right), \qquad F_X\left(x^+\right) = \lim_{h \to 0^+} F_X\left(x + h\right)$$

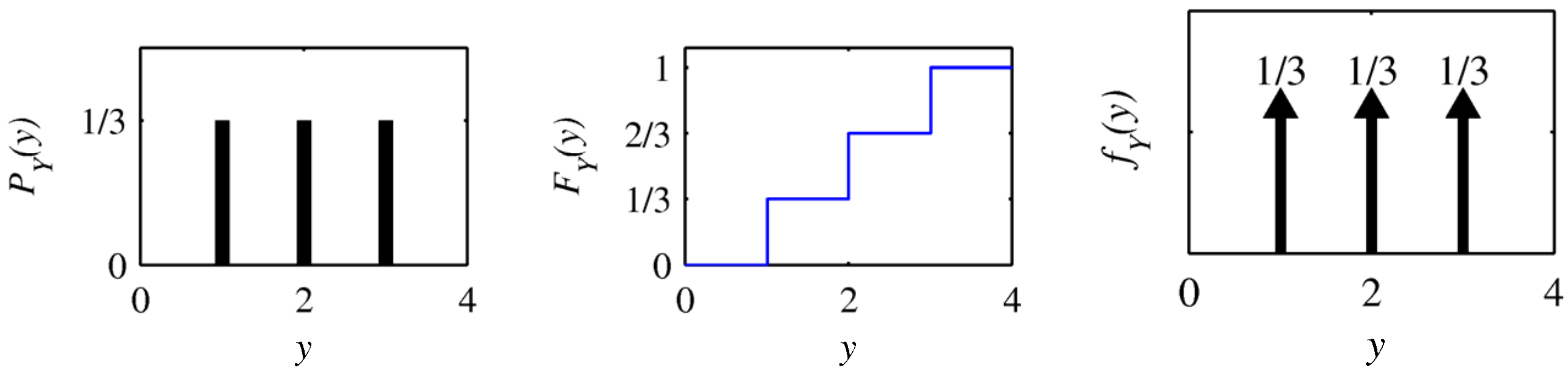
- Using this notation, we can say that if the CDF $F_X(x)$ has a jump at x_0, then $F_X(x)$ has a impulse at x_0 weighted by the height of the discontinuity $F_X(x^+) - F_X(x^-)$

Example 3.20

For the random variable Y of Example 3.19,

$$F_Y\left(2^-\right) = 1/3, \qquad F_Y\left(2^+\right) = 2/3.$$

Figure 3.8



The PMF, CDF, and PDF of the mixed random variable Y.

Theorem 3.18

For a random variable X, we have the following equivalent statements:

(a) $P[X = x_0] = q$

(b) $P_X(x_0) = q$

(c) $F_X(x_0^+) - F_X(x_0^-) = q$

(d) $f_X(x_0) = q\delta(0)$

$\delta(x-x_0)$

- $P_X(x_1) = p_1$, $P_X(x_2) = p_2$, $P_X(x_3) = p_3$

Definition 3.14 Mixed Random Variable

X *is a* mixed *random variable if and only if* $f_X(x)$ *contains both impulses and nonzero, finite values.*

Example 3.21 Problem

Observe someone dialing a telephone and record the duration of the call. In a simple model of the experiment, $1/3$ of the calls never begin either because no one answers or the line is busy. The duration of these calls is 0 minutes. Otherwise, with probability $2/3$, a call duration is uniformly distributed between 0 and 3 minutes. Let Y denote the call duration. Find the CDF $F_Y(y)$, the PDF $f_Y(y)$, and the expected value $E[Y]$.

Example 3.21 Solution

Let A denote the event that the phone was answered. Since $Y \geq 0$, we know that for $y < 0$, $F_Y(y) = 0$. Similarly, we know that for $y > 3$, $F_Y(y) = 1$. For $0 \leq y \leq 3$, we apply the law of total probability to write

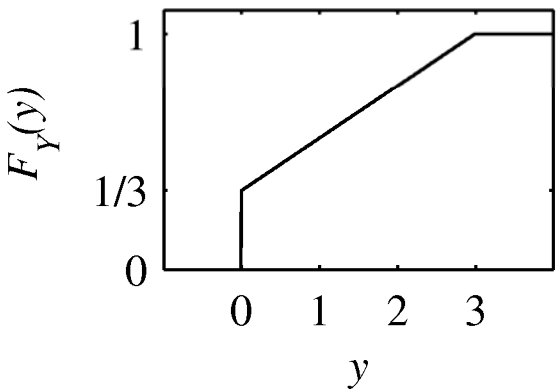
$$F_Y(y) = P[Y \leq y] = P\left[Y \leq y|A^c\right] P\left[A^c\right] + P[Y \leq y|A] P[A].$$

When A^c occurs, $Y = 0$, so that for $0 \leq y \leq 3$, $P[Y \leq y|A^c] = 1$. When A occurs, the call duration is uniformly distributed over $[0, 3]$, so that for $0 \leq y \leq 3$, $P[Y \leq y|A] = y/3$. So, for $0 \leq y \leq 3$,

$$F_Y(y) = (1/3)(1) + (2/3)(y/3) = 1/3 + 2y/9.$$

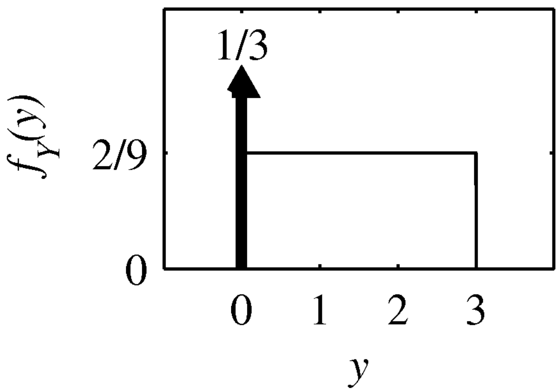
Finally, the complete CDF of Y is [Continued]

Example 3.21 Solution (continued)



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ 1/3 + 2y/9 & 0 \le y < 3, \\ 1 & y \ge 3. \end{cases}$$

Consequently, the corresponding PDF $f_Y(y)$ is



$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

For the mixed random variable Y, it is easiest to calculate $E[Y]$ using the PDF:

$$E[Y] = \int_{-\infty}^{\infty} y \frac{1}{3}\delta(y)\, dy + \int_0^3 \frac{2}{9} y\, dy = 0 + \frac{2}{9}\frac{y^2}{2}\bigg|_0^3 = 1.$$

What we learned about random variables

For any random variable X,

- X always has a CDF $F_X(x) = \mathrm{P}[X \leq x]$
- If $F_X(x)$ is piecewise flat with discontinuous jumps, then X is discrete.
- If $F_X(x)$ is a continuous function, then X is continuous.
- If $F_X(x)$ is a piecewise continuous function with discontinuities, then X is mixed.
- When X is discrete or mixed, the PDF $f_X(x)$ contains one ore more delta functions.

Quiz 3.6

The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/4 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases}$$

Sketch the CDF and find the following:

(1) $P[X \le 1]$

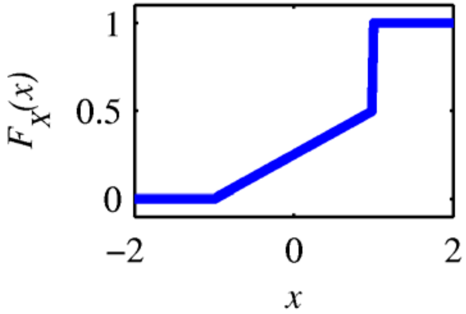
(2) $P[X < 1]$

(3) $P[X = 1]$

(4) the PDF $f_X(x)$

Quiz 3.6 Solution

The CDF of X is



$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/4 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases}$$

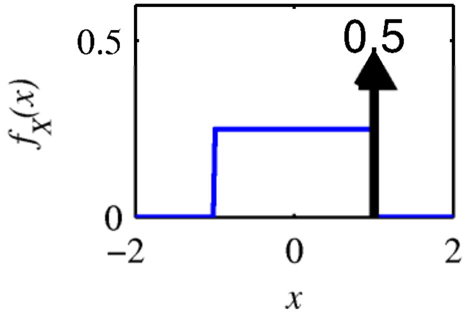
The following probabilities can be read directly from the CDF:

(1) $P[X \le 1] = F_X(1) = 1.$

(2) $P[X < 1] = F_X(1^-) = 1/2.$

(3) $P[X = 1] = F_X(1^+) - F_X(1^-) = 1 - 1/2 = 1/2.$

(4) We find the PDF $f_Y(y)$ by taking the derivative of $F_Y(y)$. The resulting PDF is



$$f_X(x) = \begin{cases} 1/4 & -1 \le x < 1, \\ (1/2)\delta(x-1) & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.7 Probability Models of Derived Random Variables

Probability Models of Derived Random Variables

- If $Y = g(X)$, we discuss methods of determining $f_Y(y)$ from $g(X)$ and $f_X(x)$. The approach is considerably different from the task of determining a derived PMF of a discrete random variable.

- In the discrete case we derive the new PMF directly from the original one.

- For continuous RV, we follow a a two-step procedure.
  - (It always work and is easy to remember)

1. Find the CDF

$$F_Y(y) = P[Y \le y]$$

2. Compute the PDF by calculating the derivative

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Example 3.22 Problem

In Example 3.2, Y centimeters is the location of the pointer on the 1-meter circumference of the circle. Use the solution of Example 3.2 to derive $f_Y(y)$.

Example 3.22 Solution

The function $Y = 100X$, where X in Example 3.2 is the location of the pointer measured in meters. To find the PDF of Y, we first find the CDF $F_Y(y)$. Example 3.2 derives the CDF of X,

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \le x < 1, \\ 1 & x \ge 1. \end{cases}$$

We use this result to find the CDF $F_Y(y) = P[100X \le y]$. Equivalently,

$$F_Y(y) = P[X \le y/100] = F_X(y/100) = \begin{cases} 0 & y/100 < 0, \\ y/100 & 0 \le y/100 < 1, \\ 1 & y/100 \ge 1. \end{cases}$$

We take the derivative of the CDF of Y over each of the three intervals to find the PDF:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 1/100 & 0 \le y < 100, \\ 0 & \text{otherwise.} \end{cases}$$

We see that Y is the uniform $(0, 100)$ random variable.

Theorem 3.19

If $Y = aX$, where $a > 0$, then Y has CDF and PDF

$$F_Y(y) = F_X(y/a), \qquad f_Y(y) = \frac{1}{a} f_X(y/a).$$

– Theorem 3.19 states that multiplying a random variables by a positive constant stretches ($a > 1$) or shrinks ($a < 1$) the original PDF.

Proof: Theorem 3.19

First, we find the CDF of Y,

$$F_Y(y) = P[aX \le y] = P[X \le y/a] = F_X(y/a).$$

We take the derivative of $F_Y(y)$ to find the PDF:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X(y/a).$$

Example 3.23 Problem

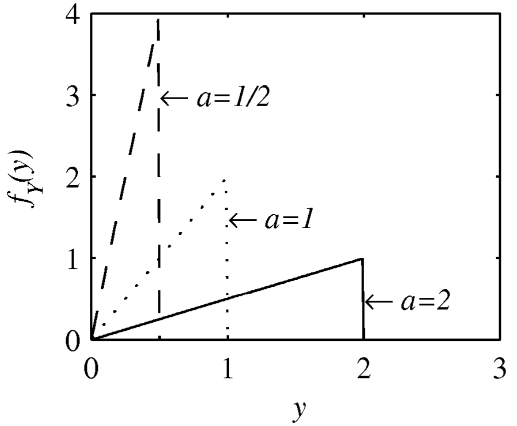
Let X have the triangular PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the PDF of $Y = aX$. Sketch the PDF of Y for $a = 1/2, 1, 2$.

Example 3.23 Solution

For any $a > 0$, we use Theorem 3.19 to find the PDF:



$$f_Y(y) = \frac{1}{a} f_X(y/a)$$
$$= \begin{cases} 2y/a^2 & 0 \le y \le a, \\ 0 & \text{otherwise.} \end{cases}$$

As a increases, the PDF stretches horizontally.

Theorem 3.20

$Y = aX$, where $a > 0$.

(a) If X is uniform (b, c), then Y is uniform (ab, ac).

(b) If X is exponential (λ), then Y is exponential (λ/a).

(c) If X is Erlang (n, λ), then Y is Erlang $(n, \lambda/a)$.

(d) If X is Gaussian (μ, σ), then Y is Gaussian $(a\mu, a\sigma)$.

- Using Theorem 3.19

Theorem 3.21

If $Y = X + b$,

$$F_Y(y) = F_X(y - b), \qquad f_Y(y) = f_X(y - b).$$

Proof: Theorem 3.21

First, we find the CDF of V,

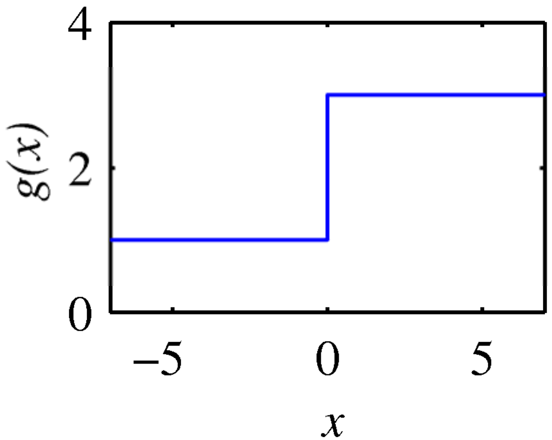
$$F_Y(y) = P[X + b \leq y] = P[X \leq y - b] = F_X(y - b).$$

We take the derivative of $F_Y(y)$ to find the PDF:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(y - b).$$

Example 3.24 Problem

Let X be a random variable with CDF $F_X(x)$. Let Y be the output of a clipping circuit with the characteristic $Y = g(X)$ where



$$g(x) = \begin{cases} 1 & x \le 0, \\ 3 & x > 0. \end{cases}$$

Express $F_Y(y)$ and $f_Y(y)$ in terms of $F_X(x)$ and $f_X(x)$.

Example 3.24 Solution

Before going deeply into the math, it is helpful to think about the nature of the derived random variable Y. The definition of $g(x)$ tells us that Y has only two possible values, $Y = 1$ and $Y = 3$. Thus Y is a discrete random variable. Furthermore, the CDF, $F_Y(y)$, has jumps at $y = 1$ and $y = 3$; it is zero for $y < 1$ and it is one for $y \geq 3$. Our job is to find the heights of the jumps at $y = 1$ and $y = 3$. In particular,

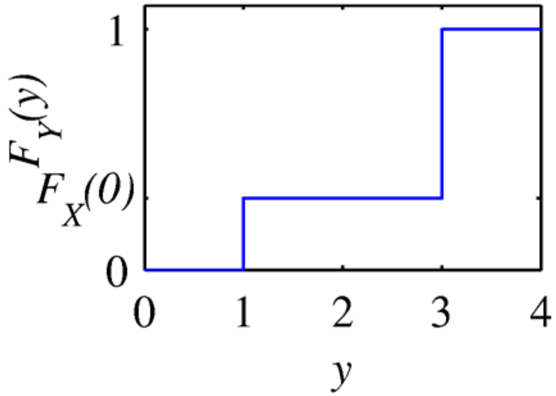
$$F_Y(1) = P[Y \leq 1] = P[X \leq 0] = F_X(0).$$

This tells us that the CDF jumps by $F_X(0)$ at $y = 1$. We also know that the CDF has to jump to one at $y = 3$.

[Continued]

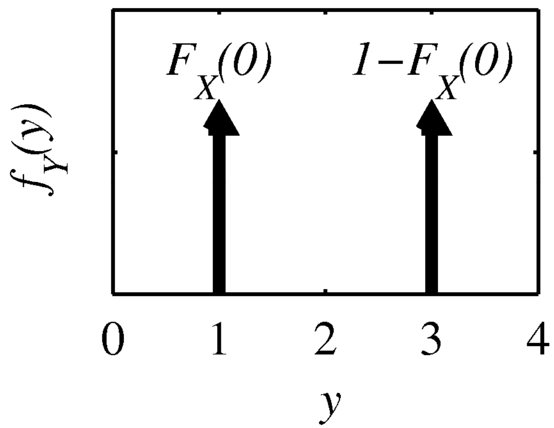
Example 3.24 Solution (continued)

Therefore, the entire story is



$$F_Y(y) = \begin{cases} 0 & y < 1, \\ F_X(0) & 1 \le y < 3, \\ 1 & y \ge 3. \end{cases}$$

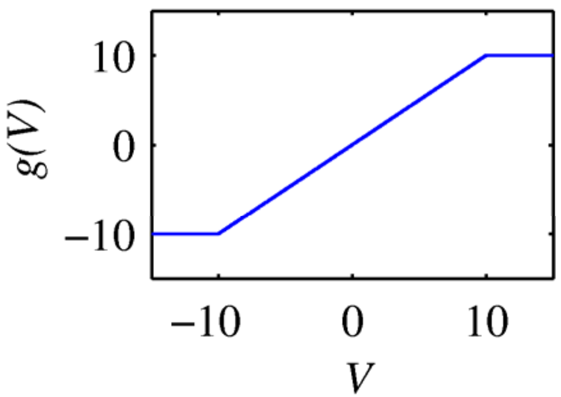
The PDF consists of impulses at $y = 1$ and $y = 3$. The weights of the impulses are the sizes of the two jumps in the CDF: $F_X(0)$ and $1 - F_X(0)$, respectively.



$$f_Y(y) = F_X(0)\delta(y-1) + [1 - F_X(0)]\delta(y-3).$$

Example 3.25 Problem

The output voltage of a microphone is a Gaussian random variable V with expected value $\mu_V = 0$ and standard deviation $\sigma_V = 5$ V. The microphone signal is the input to a limiter circuit with cutoff value ± 10 V. The random variable W is the output of the limiter:



$$W = g(V) = \begin{cases} -10 & V < -10, \\ V & -10 \le V \le 10, \\ 10 & V > 10. \end{cases}$$

What are the CDF and PDF of W?

Example 3.25 Solution

To find the CDF, we first observe that the minimum value of W is -10 and the maximum value is 10. Therefore,

$$F_W(w) = P[W \leq w] = \begin{cases} 0 & w < -10, \\ 1 & w > 10. \end{cases}$$

For $-10 \leq v \leq 10$, $W = V$ and

$$F_W(w) = P[W \leq w] = P[V \leq w] = F_V(w).$$

Because V is Gaussian $(0, 5)$, Theorem 3.14 states that $F_V(v) = \Phi(v/5)$. Therefore,

$$F_W(w) = \begin{cases} 0 & w < -10, \\ \Phi(w/5) & -10 \leq w \leq 10, \\ 1 & w > 10. \end{cases}$$

Note that the CDF jumps from 0 to $\Phi(-10/5) = 0.023$ at $w = -10$ and that it jumps from $\Phi(10/5) = 0.977$ to 1 at $w = 10$. Therefore,

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 0.023\delta(w+10) & w = -10, \\ \frac{1}{5\sqrt{2\pi}} e^{-w^2/50} & -10 < w < 10, \\ 0.023\delta(w-10) & w = 10, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.26 Problem

Suppose X is uniformly distributed over $[-1, 3]$ and $Y = X^2$. Find the CDF $F_Y(y)$ and the PDF $f_Y(y)$.

Example 3.26 Solution

From the problem statement and Definition 3.5, the PDF of X is

$$f_X(x) = \begin{cases} 1/4 & -1 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Following the two-step procedure, we first observe that $0 \leq Y \leq 9$, so $F_Y(y) = 0$ for $y < 0$, and $F_Y(y) = 1$ for $y > 9$. To find the entire CDF,

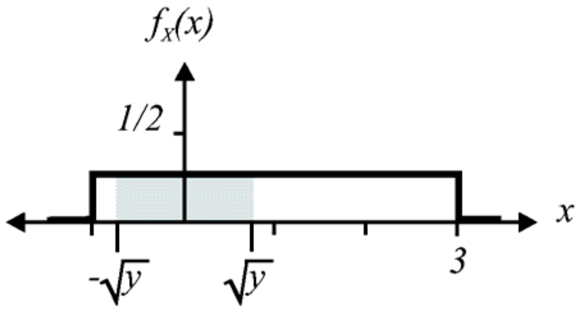
$$F_Y(y) = P\left[X^2 \leq y\right] = P\left[-\sqrt{y} \leq X \leq \sqrt{y}\right] = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\, dx.$$

This is somewhat tricky because the calculation of the integral depends on the exact value of y.

[Continued]

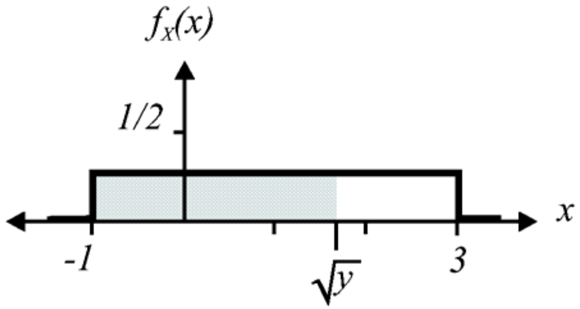
Example 3.26 Solution (continued)

For $0 \leq y \leq 1$, $-\sqrt{y} \leq x \leq \sqrt{y}$ and



$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4}\,dx = \frac{\sqrt{y}}{2}.$$

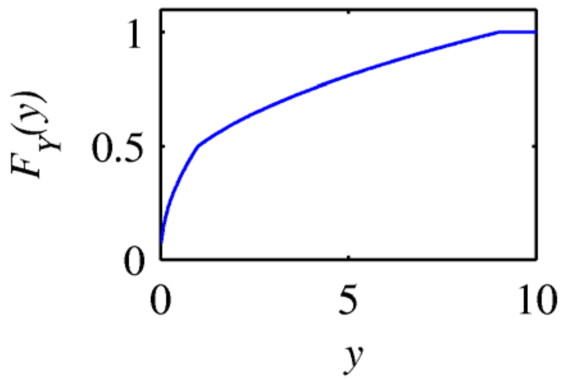
For $1 \leq y \leq 9$, $-1 \leq x \leq \sqrt{y}$ and



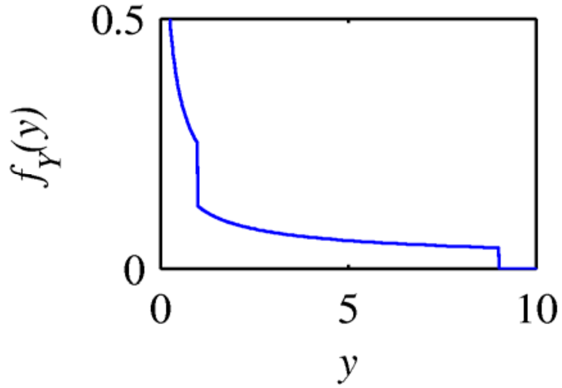
$$F_Y(y) = \int_{-1}^{\sqrt{y}} \frac{1}{4}\,dx = \frac{\sqrt{y}+1}{4}.$$

By combining the separate pieces, we can write a complete expression for $F_Y(y)$. To find $f_Y(y)$, we take the derivative of $F_Y(y)$ over each interval. [Continued]

Example 3.26 Solution (continued)



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ \sqrt{y}/2 & 0 \le y \le 1, \\ (\sqrt{y}+1)/4 & 1 \le y \le 9, \\ 1 & y \ge 9. \end{cases} \tag{1}$$



$$f_Y(y) = \begin{cases} 1/4\sqrt{y} & 0 \le y \le 1, \\ 1/8\sqrt{y} & 1 \le y \le 9, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Theorem 3.22

Let U be a uniform $(0, 1)$ random variable and let $F(x)$ denote a cumulative distribution function with an inverse $F^{-1}(u)$ defined for $0 < u < 1$. The random variable $X = F^{-1}(U)$ has CDF $F_X(x) = F(x)$.

Proof: Theorem 3.22

First, we verify that $F^{-1}(u)$ is a nondecreasing function. To show this, suppose that for $u \geq u'$, $x = F^{-1}(u)$ and $x' = F^{-1}(u')$. In this case, $u = F(x)$ and $u' = F(x')$. Since $F(x)$ is nondecreasing, $F(x) \geq F(x')$ implies that $x \geq x'$. Hence, for the random variable $X = F^{-1}(U)$, we can write

$$F_X(x) = P\left[F^{-1}(U) \leq x\right] = P\left[U \leq F(x)\right] = F(x).$$

Comments on Theorem 3.22

- It's how to derive various types of random variables from the transformation $X = g(U)$.
- The requirement that FX(u) have an inverse for 0 < u < 1 is quite strict.
  - For example, this requirement is not met by the mixed random variables of Section 3.6.
- A generalization of the theorem that does hold for mixed random variables is given in Problem 3.7.18.
- The technique of Theorem 3.22 is particularly useful when the CDF is an easily invertible function.
  - Unfortunately, many CDF is difficult to compute much less to invert. E.q. Gaussian and Erlang.
  - We will need to develop other methods.

Problem 3.7.18

In this problem we prove a generalization of Theorem 3.22. Given a random variable X with CDF $F_X(x)$, define

$$\tilde{F}(u) = \min\{x | F_X(x) \geq u\}.$$

This problem proves that for a continuous uniform $(0, 1)$ random variable U, $\hat{X} = \tilde{F}(U)$ has CDF $F_{\hat{X}}(x) = F_X(x)$.

(a) Show that when $F_X(x)$ is a continuous, strictly increasing function (i.e., X is not mixed, $F_X(x)$ has no jump discontinuities, and $F_X(x)$ has no "flat" intervals (a, b) where $F_X(x) = c$ for $a \leq x \leq b$), then $\tilde{F}(u) = F_X^{-1}(u)$ for $0 < u < 1$.

(b) Show that if $F_X(x)$ has a jump at $x = x_0$, then $\tilde{F}(u) = x_0$ for all u in the interval

$$F_X\left(x_0^-\right) \leq u \leq F_X\left(x_0^+\right).$$

(c) Prove that $\hat{X} = \tilde{F}(U)$ has CDF $F_{\hat{X}}(x) = F_X(x)$.

Problem 3.7.18 Solution

(a) Given $F_X(x)$ is a continuous function, there exists x_0 such that $F_X(x_0) = u$. For each value of u, the corresponding x_0 is unique. To see this, suppose there were also x_1 such that $F_X(x_1) = u$. Without loss of generality, we can assume $x_1 > x_0$ since otherwise we could exchange the points x_0 and x_1. Since $F_X(x_0) = F_X(x_1) = u$, the fact that $F_X(x)$ is nondecreasing implies $F_X(x) = u$ for all $x \in [x_0, x_1]$, i.e., $F_X(x)$ is flat over the interval $[x_0, x_1]$, which contradicts the assumption that $F_X(x)$ has no flat intervals. Thus, for any $u \in (0, 1)$, there is a unique x_0 such that $F_X(x) = u$. Moreiver, the same x_0 is the minimum of all x' such that $F_X(x') \geq u$. The uniqueness of x_0 such that $F_X(x)x_0 = u$ permits us to define $\tilde{F}(u) = x_0 = F_X^{-1}(u)$.

(b) In this part, we are given that $F_X(x)$ has a jump discontinuity at x_0. That is, there exists $u_0^- = F_X(x_0^-)$ and $u_0^+ = F_X(x_0^+)$ with $u_0^- < u_0^+$. Consider any u in the interval $[u_0^-, u_0^+]$. Since $F_X(x_0) = F_X(x_0^+)$ and $F_X(x)$ is nondecreasing,

$$F_X(x) \geq F_X(x_0) = u_0^+, \qquad x \geq x_0.$$

Moreover,

$$F_X(x) < F_X\left(x_0^-\right) = u_0^-, \qquad x < x_0.$$

Thus for any u satisfying $u_o^- \leq u \leq u_0^+$, $F_X(x) < u$ for $x < x_0$ and $F_X(x) \geq u$ for $x \geq x_0$. Thus, $\tilde{F}(u) = \min\{x | F_X(x) \geq u\} = x_0$. [Continued]

Problem 3.7.18 Solution (continued)

(c) We note that the first two parts of this problem were just designed to show the properties of $\tilde{F}(u)$. First, we observe that

$$P\left[\hat{X} \le x\right] = P\left[\tilde{F}(U) \le x\right] = P\left[\min\left\{x' | F_X\left(x'\right) \ge U\right\} \le x\right].$$

To prove the claim, we define, for any x, the events

$$\begin{aligned} A: &\quad \min\left\{x' | F_X\left(x'\right) \ge U\right\} \le x, \\ B: &\quad U \le F_X(x). \end{aligned}$$

Note that $P[A] = P[\hat{X} \le x]$. In addition, $P[B] = P[U \le F_X(x)] = F_X(x)$ since $P[U \le u] = u$ for any $u \in [0, 1]$.
We will show that the events A and B are the same. This fact implies

$$P\left[\hat{X} \le x\right] = P[A] = P[B] = P[U \le F_X(x)] = F_X(x).$$

[Continued]

Problem 3.7.18 Solution (continued)

All that remains is to show A and B are the same. As always, we need to show that $A \subset B$ and that $B \subset A$.

- To show $A \subset B$, suppose A is true and $\min\{x'|F_X(x') \geq U\} \leq x$. This implies there exists $x_0 \leq x$ such that $F_X(x_0) \geq U$. Since $x_0 \leq x$, it follows from $F_X(x)$ being nondecreasing that $F_X(x_0) \leq F_X(x)$. We can thus conclude that

$$U \leq F_X(x_0) \leq F_X(x).$$

  That is, event B is true.
- To show $B \subset A$, we suppose event B is true so that $U \leq F_X(x)$. We define the set

$$L = \left\{x'|F_X\left(x'\right) \geq U\right\}.$$

  We note $x \in L$. It follows that the minimum element $\min\{x'|x' \in L\} \leq x$. That is,

$$\min\left\{x'|F_X\left(x'\right) \geq U\right\} \leq x,$$

  which is simply event A.

Example 3.27 Problem

U is the uniform $(0, 1)$ random variable and $X = g(U)$. Derive $g(U)$ such that X is the exponential (1) random variable.

Example 3.27 Solution

The CDF of X is simply

$$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-x} & x \geq 0. \end{cases}$$

Note that if $u = F_X(x) = 1 - e^{-x}$, then $x = -\ln(1-u)$. That is, for any $u \geq 0$, $F_X^{-1}(u) = -\ln(1-u)$. Thus, by Theorem 3.22,

$$X = g(U) = -\ln(1-U)$$

is an exponential random variable with parameter $\lambda = 1$. Problem 3.7.5 asks the reader to derive the PDF of $X = -\ln(1-U)$ directly from first principles.

Example 3.28 Problem

For a uniform $(0, 1)$ random variable U, find a function $g(\cdot)$ such that $X = g(U)$ has a uniform (a, b) distribution.

Example 3.28 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < a, \\ (x-a)/(b-a) & a \le x \le b, \\ 1 & x > b. \end{cases}$$

For any u satisfying $0 \le u \le 1$, $u = F_X(x) = (x-a)/(b-a)$ if and only if

$$x = F_X^{-1}(u) = a + (b-a)u.$$

Thus by Theorem 3.22, $X = a+(b-a)U$ is a uniform (a, b) random variable. Note that we could have reached the same conclusion by observing that Theorem 3.20 implies $(b-a)U$ has a uniform $(0, b-a)$ distribution and that Theorem 3.21 implies $a+(b-a)U$ has a uniform $(a, (b-a)+a)$ distribution. Another approach, as taken in Problem 3.7.13, is to derive the CDF and PDF of $a+(b-a)U$.

Quiz 3.7

Random variable X has probability density function

$$f_X(x) = \begin{cases} 1 - x/2 & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

A hard limiter produces

$$Y = \begin{cases} X & X \le 1, \\ 1 & X > 1. \end{cases}$$

(1) What is the CDF $F_X(x)$?

(2) What is $P[Y = 1]$?

(3) What is $F_Y(y)$?

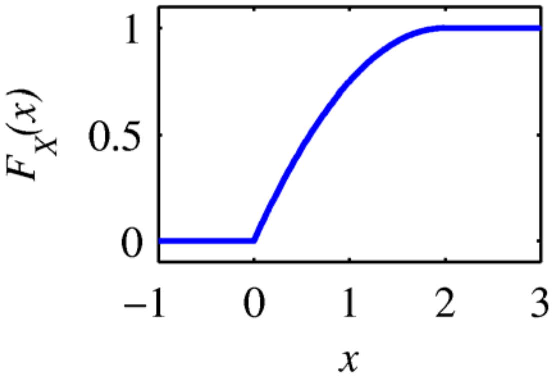
(4) What is $f_Y(y)$?

Quiz 3.7 Solution

(1) Since X is always nonnegative, $F_X(x) = 0$ for $x < 0$. Also, $F_X(x) = 1$ for $x \geq 2$ since its always true that $x \leq 2$. Lastly, for $0 \leq x \leq 2$,

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy = \int_0^x (1 - y/2)\, dy = x - x^2/4.$$

The complete CDF of X is



$$F_X(x) = \begin{cases} 0 & x < 0, \\ x - x^2/4 & 0 \leq x \leq 2, \\ 1 & x > 2. \end{cases}$$

(2) The probability that $Y = 1$ is

$$P[Y = 1] = P[X \geq 1] = 1 - F_X(1) = 1 - 3/4 = 1/4.$$

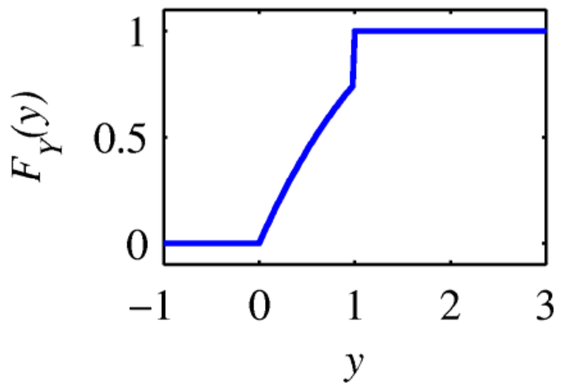
[Continued]

Quiz 3.7 Solution (continued)

(3) Since X is nonnegative, Y is also nonnegative. Thus $F_Y(y) = 0$ for $y < 0$. Also, because $Y \leq 1$, $F_Y(y) = 1$ for all $y \geq 1$. Finally, for $0 < y < 1$,

$$F_Y(y) = P[Y \leq y] = P[X \leq y] = F_X(y).$$

Using the CDF $F_X(x)$, the complete expression for the CDF of Y is



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y - y^2/4 & 0 \leq y < 1, \\ 1 & y \geq 1. \end{cases}$$

As expected, we see that the jump in $F_Y(y)$ at $y = 1$ is exactly equal to $P[Y = 1]$.

(4) By taking the derivative of $F_Y(y)$, we obtain the PDF $f_Y(y)$. Note that when $y < 0$ or $y > 1$, the PDF is zero.

3.8 Conditioning a Continuous Random Variable

Example 3.29

Recall the experiment in which you wait for the professor to arrive for the probability lecture. Let X denote the arrival time in minutes either before ($X < 0$) or after ($X > 0$) the scheduled lecture time. When you observe that the professor is already two minutes late but has not yet arrived, you have learned that $X > 2$ but you have not learned the precise value of X.

Definition 3.15 Conditional PDF given an Event

For a random variable X *with PDF* $f_X(x)$ *and an event* $B \subset S_X$ *with* $P[B] > 0$*, the* conditional PDF of X given B *is*

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{P[B]} & x \in B, \\ 0 & \textit{otherwise.} \end{cases}$$

- It has the same properties as any PDF $f_X(x)$, Thm 3.2(c), 3.3.
- The definition of the conditional PDF follow naturally from the formula for conditional probability $P[A|B] = P[AB]/P[B]$ for the infinitesimal event $A = \{x < X \leq x+dx\}$

$$f_{X|B}(x)dx = P[x < X \leq x + dx|B] = \frac{P[x < X \leq x + dx, B]}{P[B]}$$

Example 3.30 Problem

For the wheel-spinning experiment of Example 3.1, find the conditional PDF of the pointer position for spins in which the pointer stops on the left side of the circle. What are the conditional expected value and the conditional standard deviation?

Example 3.30 Solution

Let L denote the left side of the circle. In terms of the stopping position, $L = [1/2, 1)$. Recalling from Example 3.4 that the pointer position X has a uniform PDF over $[0, 1)$,

$$P[L] = \int_{1/2}^{1} f_X(x)\, dx = \int_{1/2}^{1} dx = 1/2.$$

Therefore,

$$f_{X|L}(x) = \begin{cases} 2 & 1/2 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

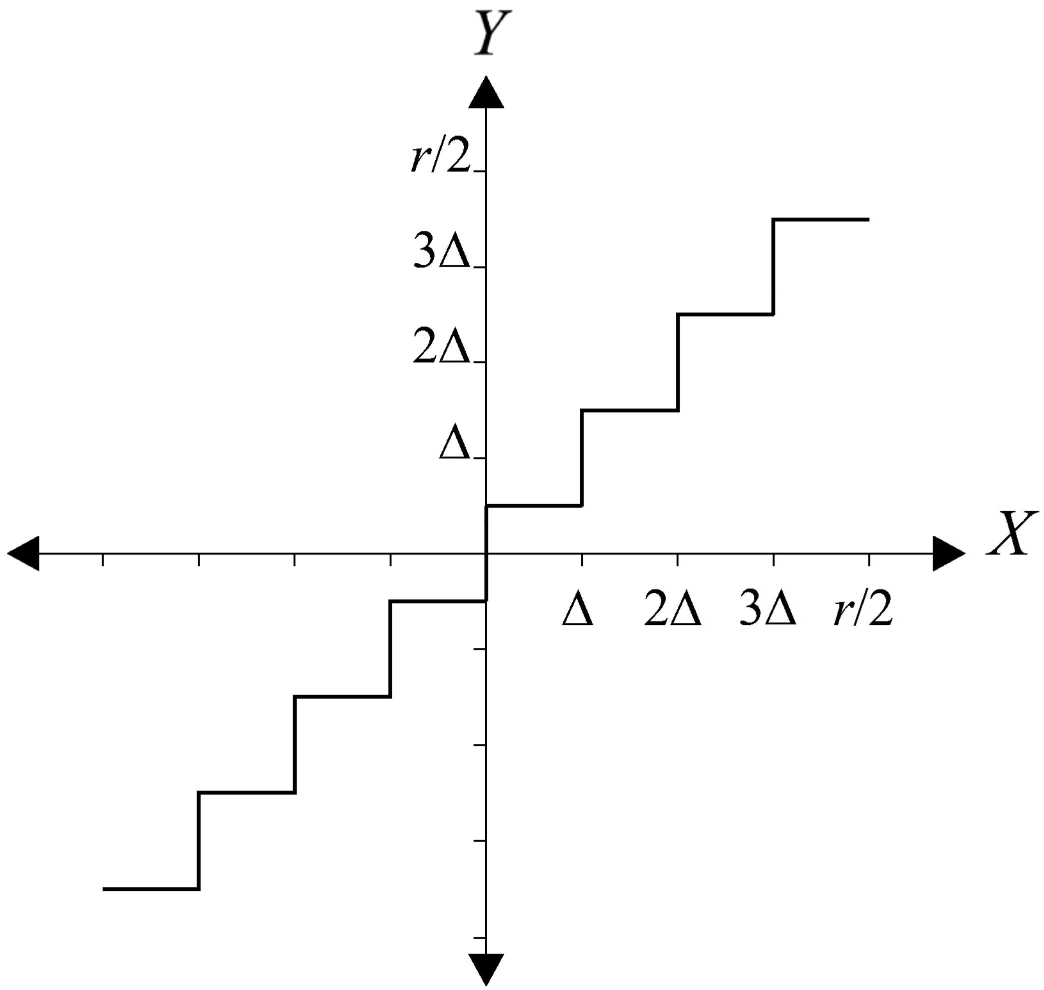
Example 3.31 Problem

The uniform $(-r/2, r/2)$ random variable X is processed by a b-bit uniform quantizer to produce the quantized output Y. Random variable X is rounded to the nearest quantizer level. With a b-bit quantizer, there are $n = 2^b$ quantization levels. The quantization step size is $\Delta = r/n$, and Y takes on values in the set

$$Q_Y = \{y_i = \Delta/2 + i\Delta | i = -n/2, -n/2+1, \ldots, n/2-1\}.$$

This relationship is shown for $b = 3$ in Figure 3.9. Given the event B_i that $Y = y_i$, find the conditional PDF of X given B_i.

Figure 3.9



Example 3.31 Solution

In terms of X, we observe that $B_i = \{i\Delta \leq X < (i+1)\Delta\}$. Thus,

$$P[B_i] = \int_{i\Delta}^{(i+1)\Delta} f_X(x)\, dx = \frac{\Delta}{r} = \frac{1}{n}.$$

By Definition 3.15,

$$f_{X|B_i}(x) = \begin{cases} \dfrac{f_X(x)}{P[B_i]} & x \in B_i, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1/\Delta & i\Delta \leq x < (i+1)\Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Given B_i, the conditional PDF of X is uniform over the ith quantization interval.

Theorem 3.23

Given an event space $\{B_i\}$ and the conditional PDFs $f_{X|B_i}(x)$,

$$f_X(x) = \sum_i f_{X|B_i}(x) \, P[B_i] \, .$$

Example 3.32 Problem

Continuing Example 3.3, when symbol “0” is transmitted (event B_0), X is the Gaussian $(-5, 2)$ random variable. When symbol “1” is transmitted (event B_1), X is the Gaussian $(5, 2)$ random variable. Given that symbols “0” and “1” are equally likely to be sent, what is the PDF of X?

Example 3.32 Solution

The problem statement implies that $P[B_0] = P[B_1] = 1/2$ and

$$f_{X|B_0}(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x+5)^2/8}, \qquad f_{X|B_1}(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x-5)^2/8}.$$

By Theorem 3.23,

$$\begin{aligned} f_X(x) &= f_{X|B_0}(x)\, P\left[B_0\right] + f_{X|B_1}(x)\, P\left[B_1\right] \\ &= \frac{1}{4\sqrt{2\pi}} \left(e^{-(x+5)^2/8} + e^{-(x-5)^2/8} \right). \end{aligned}$$

Problem 3.9.2 asks the reader to graph $f_X(x)$ to show its similarity to Figure 3.3.

Conditional Expected Value

Definition 3.16 Given an Event

If $\{x \in B\}$, *the conditional expected value of* X *is*

$$E[X|B] = \int_{-\infty}^{\infty} x f_{X|B}(x)\, dx.$$

- The conditional expected value of $g(X)$ is

$$E[g(X)|B] = \int_{-\infty}^{\infty} g(x) f_{X|B}(x) dx$$

- The conditional variance is

$$\mathrm{Var}[X|B] = E[(X - \mu_{X|B})^2|B] = E[X^2|B] - \mu_{X|B}^2$$

Example 3.33 Problem

Continuing the wheel spinning of Example 3.30, find the conditional expected value and the conditional standard deviation of the pointer position X given the event L that the pointer stops on the left side of the circle.

Example 3.33 Solution

The conditional expected value and the conditional variance are

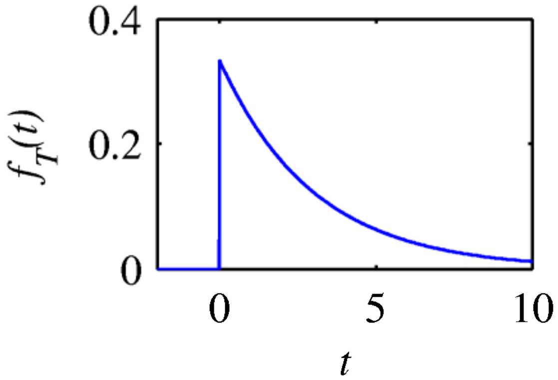
$$E[X|L] = \int_{-\infty}^{\infty} x f_{X|L}(x)\, dx = \int_{1/2}^{1} 2x\, dx = 3/4 \text{ meters.}$$

$$\mathrm{Var}[X|L] = E\left[X^2|L\right] - (E[X|L])^2 = \frac{7}{12} - \left(\frac{3}{4}\right)^2 = 1/48 \text{ m}^2.$$

The conditional standard deviation is $\sigma_{X|L} = \sqrt{\mathrm{Var}[X|L]} = 0.144$ meters. Example 3.9 derives $\sigma_X = 0.289$ meters. That is, $\sigma_X = 2\sigma_{X|L}$. It follows that learning that the pointer is on the left side of the circle leads to a set of typical values that are within 0.144 meters of 0.75 meters. Prior to learning which half of the circle the pointer is in, we had a set of typical values within 0.289 of 0.5 meters.

Example 3.34 Problem

Suppose the duration T (in minutes) of a telephone call is an exponential $(1/3)$ random variable:



$$f_T(t) = \begin{cases} (1/3)e^{-t/3} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

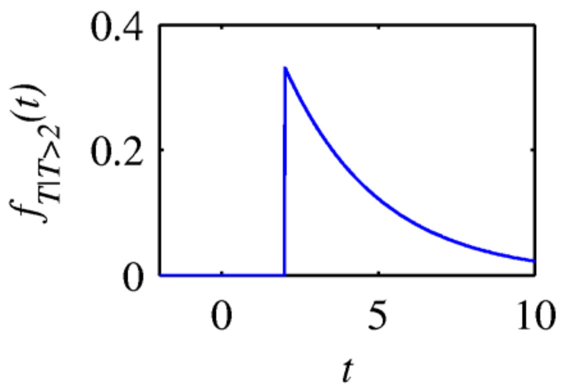
For calls that last at least 2 minutes, what is the conditional PDF of the call duration?

Example 3.34 Solution

In this case, the conditioning event is $T > 2$. The probability of the event is

$$P[T > 2] = \int_2^\infty f_T(t)\, dt = e^{-2/3}.$$

The conditional PDF of T given $T > 2$ is



$$f_{T|T>2}(t) = \begin{cases} \frac{f_T(t)}{P[T>2]} & t > 2, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \begin{cases} \frac{1}{3}e^{-(t-2)/3} & t > 2, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $f_{T|T>2}(t)$ is a time-shifted version of $f_T(t)$. In particular, $f_{T|T>2}(t) = f_T(t-2)$. An interpretation of this result is that if the call is in progress after 2 minutes, the duration of the call is 2 minutes plus an exponential time equal to the duration of a new call.

Quiz 3.8

The probability density function of random variable Y is

$$f_Y(y) = \begin{cases} 1/10 & 0 \le y < 10, \\ 0 & \text{otherwise.} \end{cases}$$

Find the following:

(1) $P[Y \le 6]$

(2) the conditional PDF $f_{Y|Y\le 6}(y)$

(3) $P[Y > 8]$

(4) the conditional PDF $f_{Y|Y>8}(y)$

(5) $E[Y|Y \le 6]$

(6) $E[Y|Y > 8]$

Quiz 3.8 Solution

(1) $P[Y \leq 6] = \int_{-\infty}^{6} f_Y(y)\,dy = \int_0^6 (1/10)\,dy = 0.6$.

(2) From Definition 3.15, the conditional PDF of Y given $Y \leq 6$ is

$$f_{Y|Y\leq 6}(y) = \begin{cases} \frac{f_Y(y)}{P[Y\leq 6]} & y \leq 6, \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} 1/6 & 0 \leq y \leq 6, \\ 0 & \text{otherwise}. \end{cases}$$

(3) The probability $Y > 8$ is

$$P[Y > 8] = \int_8^{10} \frac{1}{10}\,dy = 0.2 .$$

(4) From Definition 3.15, the conditional PDF of Y given $Y > 8$ is

$$f_{Y|Y>8}(y) = \begin{cases} \frac{f_Y(y)}{P[Y>8]} & y > 8, \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} 1/2 & 8 < y \leq 10, \\ 0 & \text{otherwise}. \end{cases}$$

(5) From the conditional PDF $f_{Y|Y\leq 6}(y)$, we can calculate the conditional expectation

$$E[Y|Y \leq 6] = \int_{-\infty}^{\infty} y f_{Y|Y\leq 6}(y)\,dy = \int_0^6 \frac{y}{6}\,dy = 3.$$

(6) From the conditional PDF $f_{Y|Y>8}(y)$, we can calculate the conditional expectation

$$E[Y|Y > 8] = \int_{-\infty}^{\infty} y f_{Y|Y>8}(y)\,dy = \int_8^{10} \frac{y}{2}\,dy = 9.$$