Linear Programming

1. Simplex Method

Let's start with an example.

Minimize $x_1 - 3x_2$

subject to $-x_1 + 2x_2 \le 6$

$$x_1 + x_2 \le 5$$
$$x_1 \quad x_2 \ge 0$$

To make it be a standard format

Minimize $x_1 - 3x_2$

subject to
$$\begin{aligned} -x_1 + 2x_2 + x_3 \quad &= 6 \\ x_1 + x_2 \quad + x_4 &= 5 \\ x_1,\ x_2,\ x_3,\ x_4 \quad &\ge 0 \end{aligned}$$

where x_3 and x_4 are slack variables

define $x_3 = 6 + x_1 - 2x_2$ (The form illustrated below is called "a dictionary.")

$$x_4 = 5 - x_1 - x_2$$
$$z = x_1 - 3x_2$$

Initial solution

$x_1 = 0,\ \ x_2 = 0,\ \ x_3 = 6,\ \ x_4 = 5,\ \ z = 0$, x_1 x_2 are nonbasic variables, x_3 x_4 are basic variables

Choosing the entering variable x_2

$$x_3 = 6 + x_1 - 2x_2 \ge 0 \quad \Rightarrow \quad x_2 \le 3 \qquad (1)$$
$$x_4 = 5 - x_1 - x_2 \ge 0 \quad \Rightarrow \quad x_2 \le 5 \qquad (2)$$

(1) is the most stringent. Increasing x_2 up to (1).

$\therefore\ x_1 = 0,\ \ x_2 = 3,\ \ x_3 = 0,\ \ x_4 = 3$ (x_3 leaves the basis)

To construct the new system, we shall begin with the new comer to the left hand side, Namely, the variable x_2. The desired formula for x_2 in terms of x_1, x_3, x_4 is

$$x_2 = 3 + \frac{x_1}{2} - \frac{x_3}{2}$$

Next, in order to express x_4 and z in terms of $x_1, x_{3,}$ we simply substitute

$$x_4 = 5 - x_1 - \left(3 + \frac{x_1}{2} - \frac{x_3}{2}\right) = 2 - \frac{3}{2}x_1 + \frac{x_3}{2}$$

$$z = x_1 - 3\left(3 + \frac{x_1}{2} - \frac{x_3}{2}\right) = -9 - \frac{x_1}{2} + \frac{3}{2}x_3$$

Hence our new system

$$x_2 = 3 + \frac{x_1}{2} \qquad - \frac{x_3}{2}$$
$$x_4 = 2 - \frac{3}{2}x_1 \qquad + \frac{x_3}{2}$$
$$z = -9 - \frac{x_1}{2} \qquad + \frac{3}{2}x_3$$

Increase x_1 (x_1 enters the basis)

$$x_2 = 3 + \frac{x_1}{2} - \frac{x_3}{2} \geq 0 \quad \Rightarrow \qquad x_1 \text{ can be infinity}$$

$$x_4 = 2 - \frac{3}{2}x_1 + \frac{x_3}{2} \geq 0 \quad \Rightarrow \qquad x_1 \leq \frac{4}{3} \qquad (x_4 \text{ leaves the basis})$$

$$\therefore \quad x_1 = \frac{4}{3}, \quad x_2 = \frac{11}{3}, \quad x_3 = 0, \quad x_4 = 0$$

$$\Rightarrow \quad x_1 = 2 \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3}x_3 - \frac{2}{3}x_4 = \frac{4}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4$$

and $x_2 = 3 + \frac{1}{2}\left(\frac{4}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4\right) - \frac{x_3}{2} = \frac{11}{3} - \frac{1}{3}x_3 - \frac{1}{3}x_4$

$$z = -9 - \frac{1}{2}\left(\frac{4}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4\right) + \frac{3}{2}x_3 = -\frac{29}{3} + \frac{4}{3}x_3 + \frac{1}{3}x_4$$

Hence

$$\begin{aligned} x_1 &= \frac{4}{3} && + \frac{1}{3}x_3 - \frac{2}{3}x_4 \\ x_2 &= \frac{11}{3} && - \frac{1}{3}x_3 - \frac{1}{3}x_4 \\ z &= -\frac{29}{3} && + \frac{4}{3}x_3 + \frac{1}{3}x_4 \end{aligned}$$

$\therefore$ x_3 and x_4 in the objective function are with positive coefficients.

$\therefore$ Stop.

In general from

Minimize

$$Z = \sum_{j=1}^{n} c_j x_j$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n$$

Some notes:

- x_s is chosen to increase the objective function, and x_r is chosen to maintain feasibility.
- A change from one dictionary to the next dictionary is called a pivot, and the equation corresponding to x_r is called the pivoting row. The row with the smallest ratio $\frac{s}{r}$ is called the pivot row and corresponds to the leaving variable.

2. Pitfall of the simplex method

The examples illustrating the simplex method in the preceding lecture were purposely smooth. They did not point out the dangers that can occur. The purpose of this section, therefore is to rigorously analyze the method by scrutinizing its every step.

INITIALIZATION

Given a standard form LP

Maximize

$$Z = \sum_{j=1}^{n} c_j x_j$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \ldots, m$$

$$x_j \ge 0, \qquad j = 1, 2, \ldots, n$$

If all b_i are nonnegative the initial dictionary is feasible, the basic simplex algorithm solves the problem.

If some $b_i < 0$, how to find an initial feasible solution?

An example

Maximize $x_1 - x_2 + x_3$

s.t. $2x_1 - x_2 + 2x_3 \le 4$

$2x_1 - 3x_2 + x_3 \le -5$

$-x_1 + x_2 - 2x_3 \le -1$

$x_1, x_2, x_3 \ge 0.$

Following the previous discussion, here is the trouble to find an initial feasible solution.

$$x_4 = 4 - 2x_1 + x_2 - 2x_3$$
$$x_5 = -5 - 2x_1 + 3x_2 - x_3$$
$$x_6 = -1 + x_1 - x_2 + 2x_3$$

One way of getting rid of this trouble is solving the auxiliary problem for finding the initial feasible solution. To avoid unnecessary confusion, we write the auxiliary problem in its maximization form:

Maximize $-x_0$

s.t. $2x_1 - x_2 + 2x_3 - x_0 \leq 4$

$2x_1 - 3x_2 + x_3 - x_0 \leq -5$

$-x_1 + x_2 - 2x_3 - x_0 \leq -1$

$x_0, x_1, x_2, x_3 \geq 0.$

where x_0 is called an *artificial variable*.

Writing down the formulas defining the slack variables x_4, x_5, x_6 and the objective function w, we obtain the dictionary

$$
\begin{aligned}
x_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\
x_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\
x_6 &= -1 + x_1 - x_2 + 2x_3 + x_0 \\
w &= -x_0
\end{aligned}
$$

which is infeasible. Nevertheless, this infeasible dictionary can be transformed into a feasible one by a single pivot, with x_0 entering and x_5 leaving the basis: (x_5 is the most infeasible basic variable among x_4, x_5, x_6)

$$
\begin{aligned}
x_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
x_4 &= 9 - 2x_2 - x_3 + x_5 \\
x_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\
w &= -5 - 2x_1 + 3x_2 - x_3 - x_5.
\end{aligned}
$$

After the first iteration, with x_2 entering and x_6 leaving:

$$
\begin{aligned}
x_2 &= 1 + 0.75x_1 + 0.75x_3 + 0.25x_5 - 0.25x_6 \\
x_0 &= 2 - 0.25x_1 - 1.25x_3 + 0.25x_5 + 0.75x_6 \\
x_4 &= 7 - 1.5x_1 - 2.5x_3 + 0.5x_5 + 0.5x_6 \\
w &= -2 + 0.25x_1 + 1.25x_3 - 0.25x_5 - 0.75x_6.
\end{aligned}
$$

After the second iteration, with x_3 entering and x_0 leaving:

$$
\begin{aligned}
x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 - 0.8x_0 \\
x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 - 0.6x_0 \\
x_4 &= 3 - x_1 - x_6 + 2x_0 \\
w &= -x_0.
\end{aligned}
$$

The last dictionary is optimal. Since the optimal value of the auxiliary problem is zero, the last dictionary points out a feasible solution of the original problem: $x_1 = 0$, $x_2 = 2.2$, $x_3 = 1.6$. Furthermore, the last dictionary can be easily converted into the desired feasible dictionary of the original problem. To obtain the first three rows of the desired dictionary, we simply copy down the first three rows of the last dictionary, omitting all the terms involving x_0:

$$\begin{aligned}
x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 \\
x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 \\
x_4 &= 3 \quad - \quad x_1 \qquad\quad - \quad x_6.
\end{aligned}$$

To obtain the last row, we have to express the original objective function

$$z = x_1 - x_2 + x_3$$

in terms of the nonbasic variable x_1, x_5, x_6.

$$\begin{aligned}
z &= x_1 - (2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6) + (1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6) \\
&= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6.
\end{aligned}$$

In short, the desired dictionary reads

$$\begin{aligned}
x_3 &= \quad 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 \\
x_2 &= \quad 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 \\
x_4 &= \quad 3 \quad - \quad x_1 \qquad\quad - \quad x_6 \\
z &= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6.
\end{aligned}$$

General case

- **Original problem**

Maximize

$$Z = \sum_{j=1}^{n} c_j x_j$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, ..., m$$

$$x_j \ge 0, \qquad j = 1, 2, ..., n$$

- **Auxiliary problem**

Phase 1:

Theorem: The original LP is feasible if and only if the optimum objective value for the auxiliary problem is 0.

Proof:

ITERATION

Given some feasible dictionary, we have to select an entering variable, to find a leaving variable, and to construct the next feasible dictionary by pivoting.

Unbounded dictionary

$$
\begin{aligned}
x_2 &= 5 + 2x_3 - x_4 - 3x_1 \\
x_5 &= 7 - 3x_4 - 4x_1 \\
z &= 5 + x_3 - x_4 - x_1.
\end{aligned}
$$

The entering variable is x_3, but neither of the two basic variables x_2 and x_5 imposes an upper bound on its increase. Therefore, we can make x_3 as large as we wish (maintaining $x_1 = x_4 = 0$) and still retain feasibility: setting $x_3 = t$ for any positive *t*, we obtain a feasible solution with $x_1 = 0$, $x_2 = 5 + 2t$, $x_4 = 0$, $x_5 = 7$, and $z = 5 + t$. Since *t* can be made arbitrarily large, *z* can be made arbitrarily large. We conclude that the problem is unbounded.

Degeneracy

The presence of more than one candidate for leaving the bases has interesting consequences. For illustration, consider the dictionary.

$$
\begin{aligned}
x_4 &= 1 - 2x_3 \\
x_5 &= 3 - 2x_1 + 4x_2 - 6x_3 \\
x_6 &= 2 + x_1 - 3x_2 - 4x_3 \\
z &= 2x_1 - x_2 + 8x_3.
\end{aligned}
$$

Having chosen x_3 to enter the basis, we find that each of the three basic variables x_4, x_5, x_6 limits the increase of x_3 to 1/2. Hence each of these three variables is a candidate for leaving the basis. We arbitrarily choose x_4. Pivoting as usual, we obtain the dictionary.

$$
\begin{aligned}
x_3 &= 0.5 \qquad\qquad\qquad - 0.5x_4 \\
x_5 &= \quad\; - 2x_1 + 4x_2 + \quad 3x_4 \\
x_6 &= \quad\; + \;\; x_1 - 3x_2 + \quad 2x_4 \\
z \; &= \;\; 4 + 2x_1 - \;\; x_2 - \;\; 4x_4.
\end{aligned}
$$

Along with the nonbasic variables, the basic variables x_5 and x_6 have value zero in the associated solution. Basic solutions with one or more basic variables at zero are called ***degenerate***. Degeneracy may have annoying side effects. These are illustrated on the next iteration in the example. There, x_1 enters the basis and x_5 leaves; because of degeneracy, the constraint $x_5 \geq 0$ limits the increment of x_1 to zero. Hence the value of x_1 will remain unchanged, and so will the values of the remaining variables and the value of the objective function z.

$$
\begin{aligned}
x_1 &= \qquad\quad 2x_2 + 1.5x_4 - 0.5x_5 \\
x_3 &= 0.5 \qquad\;\; - 0.5x_4 \\
x_6 &= \quad\; - \;\; x_2 + 3.5x_4 - 0.5x_5 \\
z \; &= \;\; 4 + 3x_2 - \qquad x_4 - \qquad x_5.
\end{aligned}
$$

In this particular iteration, pivoting changes the dictionary as above, but it does not affect the associated solution at all. Simplex iterations that do not change the basic solution are called degenerate.

TERMINATION: CYCLING

Can the simplex method go through an endless sequence of iterations without ever finding an optimal solution? Yes, it can. To justify this claim, let us consider the initial dictionary

$$\begin{aligned}
x_5 &= \quad -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= \quad -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1 \\
z &= \quad 10x_1 - 57x_2 - 9x_3 - 24x_4.
\end{aligned}$$

and let us agree on the following:

(i) The entering variable will always be the nonbasic variable that has the largest coefficient in the z-row of the dictionary.

(ii) If two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript will be made to leave.

Now the sequence of dictionaries constructed in the first six iterations goes as follows.

After the first iteration:

$$\begin{aligned}
x_1 &= \quad 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= \quad -4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 \\
z &= \quad 53x_2 + 41x_3 - 204x_4 - 20x_5.
\end{aligned}$$

After the second iteration:

$$\begin{aligned}
x_2 &= \quad -0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_1 &= \quad -0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6 \\
z &= \quad 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6.
\end{aligned}$$

After the third iteration:

$$\begin{aligned}
x_3 &= \quad 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1 \\
x_2 &= \quad -2x_4 - 0.5x_5 + 2.5x_6 + x_1 \\
x_7 &= 1 \qquad - x_1 \\
z &= \quad 18x_4 + 15x_5 - 93x_6 - 29x_1.
\end{aligned}$$

After the fourth iteration:

$$\begin{aligned}
x_4 &= \quad -0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2 \\
x_3 &= \quad -0.5x_5 + 4.5x_6 + 2x_1 - 4x_2 \\
x_7 &= 1 \qquad - x_1 \\
z &= \quad 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2.
\end{aligned}$$

After the fifth iteration:

$$\begin{aligned}
x_5 &= \quad 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_4 &= \quad -x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_7 &= 1 - x_1 \\
z &= \quad 24x_6 + 22x_1 - 93x_2 - 21x_3.
\end{aligned}$$

After the sixth iteration:

$$\begin{aligned}
x_6 &= \quad -0.5x_1 + 1.5x_2 + 0.5x_3 - \quad x_4 \\
x_5 &= \quad -0.5x_1 + 5.5x_2 + 2.5x_3 - \ 9x_4 \\
x_7 &= 1 - \quad x_1 \\
z &= \qquad 10x_1 - 57x_2 - \ 9x_3 - 24x_4.
\end{aligned}$$

Since the dictionary constructed after the sixth iteration is identical with the initial dictionary, the method will go through the same six iterations again and again without ever finding the optimal solution (which, as we shall see later, has z = 1). This phenomenon is known as cycling. More precisely, we say that the simplex method *cycles* if one dictionary appears in two different iterations (and so the sequence of iterations leading from the dictionary to itself can be repeated over and over without end). Note that cycling can occur only in the presence of degeneracy: since the value of the objective function increases with each nondegenerate iteration and remains unchanged after each degenerate one, all the iterations in the sequence leading from a dictionary to itself must be degenerate. Cycling is one reason why the simplex method may fail to terminate.

Bland's Rule

- The entering variable chosen by the smallest index variables with $c_j > 0$.
- If there is a tie for leaving variables, break the tie by the smallest index.

Recall the previous example if we follow Bland's rule.

$$\begin{aligned}
x_5 &= \quad -0.5x_1 + 5.5x_2 + 2.5x_3 - \ 9x_4 \\
x_6 &= \quad -0.5x_1 + 1.5x_2 + 0.5x_3 - \quad x_4 \\
x_7 &= 1 - \quad x_1 \\
z &= \qquad 10x_1 - 57x_2 - \ 9x_3 - 24x_4.
\end{aligned}$$

After the first iteration:

$$
\begin{aligned}
x_1 &= \quad 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= \quad - 4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 \\
z &= \quad 53x_2 + 41x_3 - 204x_4 - 20x_5.
\end{aligned}
$$

After the second iteration:

$$
\begin{aligned}
x_2 &= \quad - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_1 &= \quad - 0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6 \\
z &= \quad 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6.
\end{aligned}
$$

After the third iteration:

$$
\begin{aligned}
x_3 &= \quad 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1 \\
x_2 &= \quad - 2x_4 - 0.5x_5 + 2.5x_6 + x_1 \\
x_7 &= 1 \quad - x_1 \\
z &= \quad 18x_4 + 15x_5 - 93x_6 - 29x_1.
\end{aligned}
$$

After the fourth iteration:

$$
\begin{aligned}
x_4 &= \quad - 0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2 \\
x_3 &= \quad - 0.5x_5 + 4.5x_6 + 2x_1 - 4x_2 \\
x_7 &= 1 \quad - x_1 \\
z &= \quad 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2.
\end{aligned}
$$

After the fifth iteration:

$$
\begin{aligned}
x_5 &= 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_4 &= - x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_7 &= 1 - x_1 \\
z &= 24x_6 + 22x_1 - 93x_2 - 21x_3.
\end{aligned}
$$

After the sixth iteration: (different from the previous example)

$$
\begin{aligned}
x_1 &= -2x_6 + 3x_2 + x_3 - 2x_4 \\
x_5 &= x_6 + 4x_2 + x_3 - 8x_4 \\
x_7 &= 1 + 2x_6 - 3x_2 - x_3 + 2x_4 \\
z &= -20x_6 - 27x_2 + x_3 - 44x_4.
\end{aligned}
$$

After the seventh iteration: (x_3 enters and x_7 leaves)

$$
\begin{aligned}
x_3 &= 1 + 2x_6 - 3x_2 - x_7 + 2x_4 \\
x_5 &= \cdots \\
x_1 &= \cdots \\
z &= 1 - 18x_6 - 30x_2 - x_7 - 42x_4.
\end{aligned}
$$

Theorem: The simplex method always terminates provided that both the entering and the leaving variable are chosen according to Bland’s rule.

Proof

3. Geometric interpretation of the simplex method

$$
\begin{array}{lrlcl}
\text{Maximize } z = & 4x_1 + 5x_2 & & & \\
\text{s.t.} & 6x_1 + 4x_2 \le 24 & (1) & 6x_1 + 4x_2 + x_3 & = 24 \\
& x_1 + 2x_2 \le 6 & (2) & x_1 + 2x_2 \quad + x_4 & = 6 \\
& -x_1 + x_2 \le 1 & (3) & -x_1 + x_2 \quad + x_5 & = 1 \\
& x_2 \le 2 & (4) & x_2 \quad + x_6 & = 2 \\
& x_1, x_2 \ge 0 & & &
\end{array}
$$

- Choose the entering variable with the most positive objective function coefficient

$$
\begin{aligned}
x_3 &= 24 - 6x_1 - 4x_2 \\
x_4 &= 6 - x_1 - 2x_2 \\
x_5 &= 1 + x_1 - x_2 \\
x_6 &= 2 - x_2 \\
z &= 4x_1 + 5x_2
\end{aligned}
$$

The first iteration:

$$
\begin{aligned}
x_2 &= 1 + x_1 - x_5 \\
x_3 &= 20 - 10x_1 + 4x_5 \\
x_4 &= 4 - 3x_1 + 2x_2 \\
x_6 &= 1 - x_1 + x_5 \\
z &= 5 + 9x_1 - 5x_5
\end{aligned}
$$

The second iteration:

$$
\begin{aligned}
x_1 &= 1 + x_5 - x_6 \\
x_2 &= 2 - x_6 \\
x_3 &= 10 - 6x_5 + 10x_6 \\
x_4 &= 1 - x_5 + 3x_6 \\
z &= 14 + 4x_5 - 9x_6
\end{aligned}
$$

The third iteration:

$$
\begin{aligned}
x_1 &= 2 - x_4 + 2x_6 \\
x_2 &= 2 - x_6 \\
x_3 &= 4 + 6x_4 - 8x_6 \\
x_5 &= 1 - x_4 + 3x_6 \\
z &= 18 - 4x_4 + 3x_6
\end{aligned}
$$

The fourth iteration:

$$
\begin{aligned}
x_1 &= 3 - \frac{1}{4}x_3 + \frac{1}{2}x_4 \\
x_2 &= \frac{3}{2} + \frac{1}{8}x_3 - \frac{3}{4}x_4 \\
x_5 &= \frac{5}{2} - \frac{3}{8}x_3 + \frac{5}{8}x_4 \\
x_6 &= \frac{1}{2} - \frac{1}{8}x_3 + \frac{3}{4}x_4 \\
z &= \frac{39}{2} - \frac{3}{8}x_3 - \frac{7}{4}x_4
\end{aligned}
$$

$z = 4x_1 + 5x_2$: A line describes all point satisfying $z = 4x_1 + 5x_2$

The line is perpendicular to the vector $\binom{4}{5}$ (normal vector).

Different values of z lead to different lines, and all of them are parallel to each other. Increasing z corresponds to moving the line along the direction of $\binom{4}{5}$. We would like to move the line as much as possible in the direction of $\binom{4}{5}$.

- Choose the entering variable with the least positive objective function coefficient.

The first iteration:

$$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{6}x_3$$

$$x_4 = 2 - \frac{4}{3}x_2 + \frac{1}{6}x_3$$

$$x_5 = 5 - \frac{5}{3}x_2 - \frac{1}{6}x_3$$

$$x_6 = 2 - x_2$$

$$z = 16 + \frac{7}{3}x_2 - \frac{2}{3}x_3$$

The second iteration:

$$x_1 = 3 - \frac{1}{4}x_3 + \frac{1}{2}x_4$$

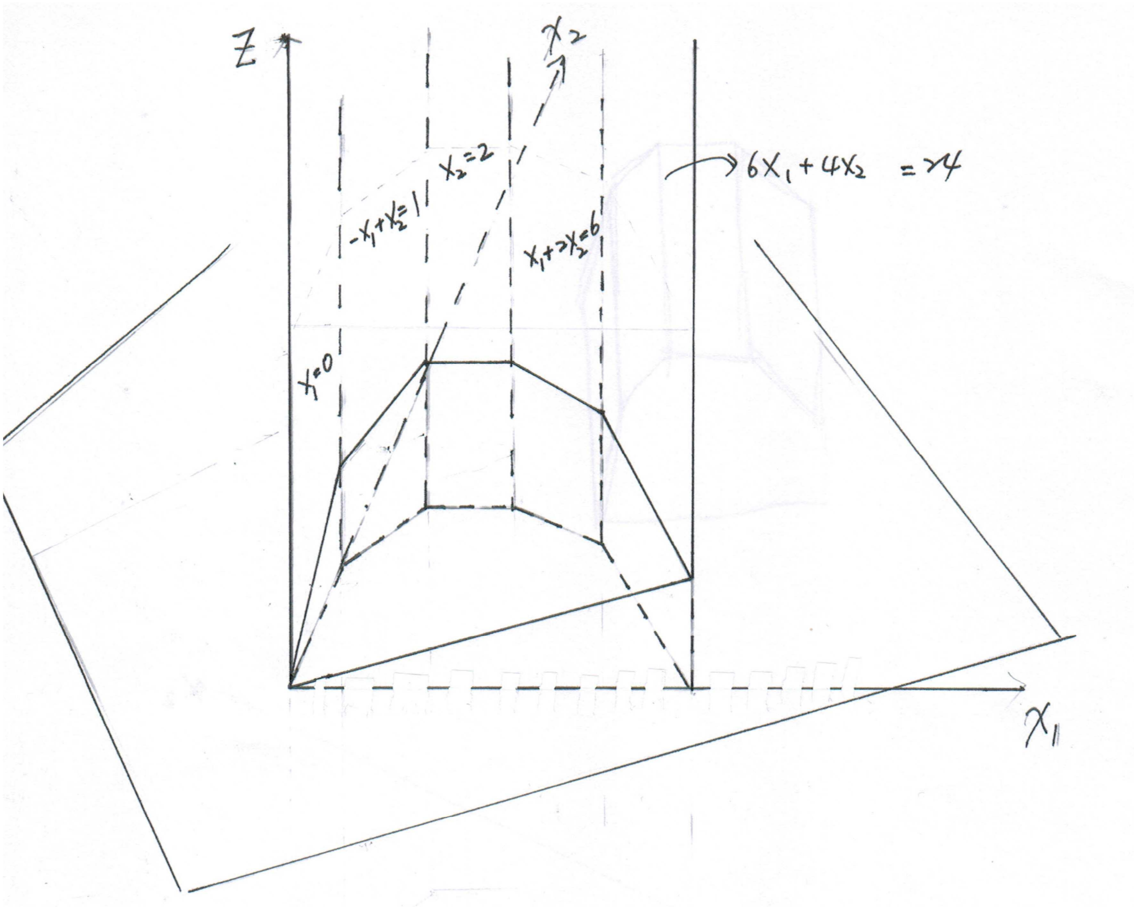
$$x_2 = \frac{3}{2} + \frac{1}{8}x_3 - \frac{3}{4}x_4$$

$$x_5 = \frac{5}{2} - \frac{3}{8}x_3 + \frac{5}{8}x_4$$

$$x_6 = \frac{1}{2} - \frac{1}{8}x_3 + \frac{3}{4}x_4$$

$$z = \frac{39}{2} - \frac{3}{8}x_3 - \frac{7}{4}x_4$$

3-Dimension graph



Degenerate

Maximize $z = 3x_1 + 9x_2$

$$\begin{aligned} \text{s.t.} \quad x_1 + 4x_2 &\le 8 \qquad (1) \qquad x_1 + 4x_2 + x_3 = 8 \\ x_1 + 2x_2 &\le 4 \qquad (2) \qquad x_1 + 2x_2 + x_4 = 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned} x_3 &= 8 - x_1 - 4x_2 \\ x_4 &= 4 - x_1 - 2x_2 \\ z &= 3x_1 + 9x_2 \end{aligned}$$

The first iteration (x_2 enters x_3 leaves)

$$\begin{aligned} x_2 &= 2 - \frac{1}{4}x_1 - \frac{1}{4}x_3 \\ x_4 &= -\frac{1}{2}x_1 + \frac{1}{2}x_3 \\ z &= 18 + \frac{3}{4}x_1 - \frac{9}{4}x_3 \end{aligned}$$

The second iteration (x_1 enters x_4 leaves)

$$\begin{aligned} x_1 &= x_3 - 2x_4 \\ x_2 &= 2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ z &= 18 - \frac{3}{2}x_3 - \frac{3}{2}x_4 \end{aligned}$$