Prune-and-Search

The prune-and-search method consists of several iterations, and in each iteration, a fraction of the solution space is pruned.

The pruning will continue until the remaining solution space is small enough, in which the desired solution can be found quickly.

In general, the time complexity can be expressed as

$$T(n) \leq T((1-\alpha)n) + p(n),$$

where $0 < \alpha < 1$ is the fraction to prune the solution space in each iteration, and $p(n)$ is the time needed for each iteration.

Ex. The binary search

$$\alpha = 1/2, \qquad p(n) = O(1)$$

Ex. The selection problem

Given n distinct numbers $a_1, a_2, \ldots, a_n$, determine the kth smallest one.

Approach 1. Choose a number p arbitrarily from these n numbers and partition them into three subsets:

$$S_1 = \{x \mid x < p\}, \quad S_2 = \{p\}, \quad S_3 = \{x \mid x > p\}.$$

The kth smallest number is located in one of these three subsets, and the partitioning is repeated for the designated subset until the kth smallest number is found.

This approach cannot guarantee that a fraction of the solution space is always discarded after each iteration.

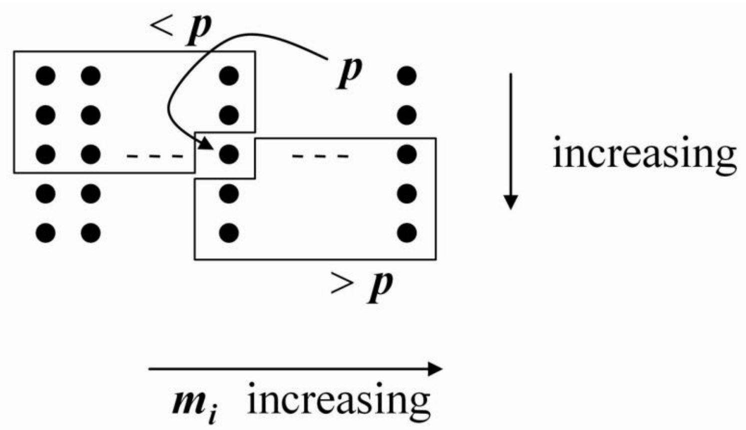
In fact, $O(n^2)$ time is needed in the worst case.

Approach 2. **First, divide the *n* numbers into n/r groups each of *r* numbers, where $r > 3$ is odd.**

If *n* is not a multiple of *r*, sufficient numbers are added artificially.

Then, sort every group and let m_i be the median of group *i*, where $1 \leq i \leq \lceil n/r \rceil$. Choose the median of m_i's as *p* and perform the partitioning repeatedly until the solution space is small enough.

Ex. Assume *r* = 5.



At least one fourth of the solution space is discarded after each iteration.

Thus,

$$T(n) \le T(n/5) + T(3n/4) + O(n),$$

from which $T(n) = O(n)$ can be derived (refer to *Computer Algorithms*, by Horowitz, Sahni, and Rajasekaran)

Ex. The 2-dimensional linear programming problem.

$$\begin{aligned} &\textbf{minimize} \quad a'x' + b'y' \\ &\textbf{subject to} \quad a_i'x' + b_i'y' + c_i' \le 0 \qquad i = 1, 2, \ldots, n. \end{aligned}$$

To simplify the problem, we transform the original form into the following equivalent form by letting $y = a'x' + b'y'$ and $x = x'$ (座標旋轉).

$$\begin{aligned} &\textbf{minimize} \quad y \\ &\textbf{subject to} \quad a_i x + b_i y + c_i \le 0 \qquad i = 1, 2, \ldots, n. \end{aligned}$$

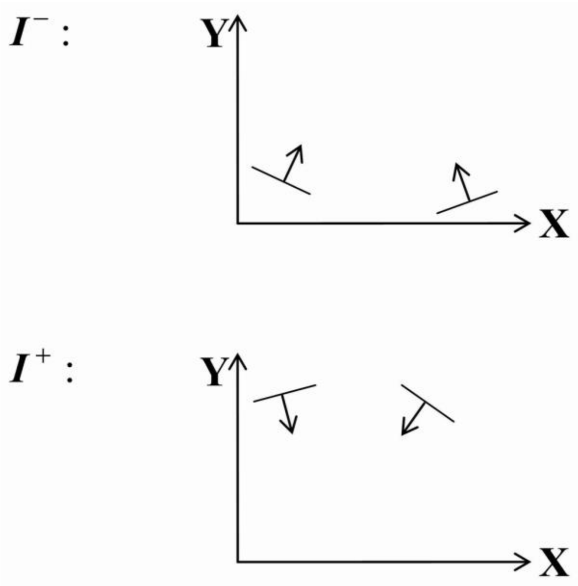
The transformation takes $O(n)$ time.

Thus, we only have to compute the smallest y-coordinate in the feasible region.

The n constraints may be partitioned into three classes I^0, I^-, I^+, depending on whether b_i is zero, negative, or positive.

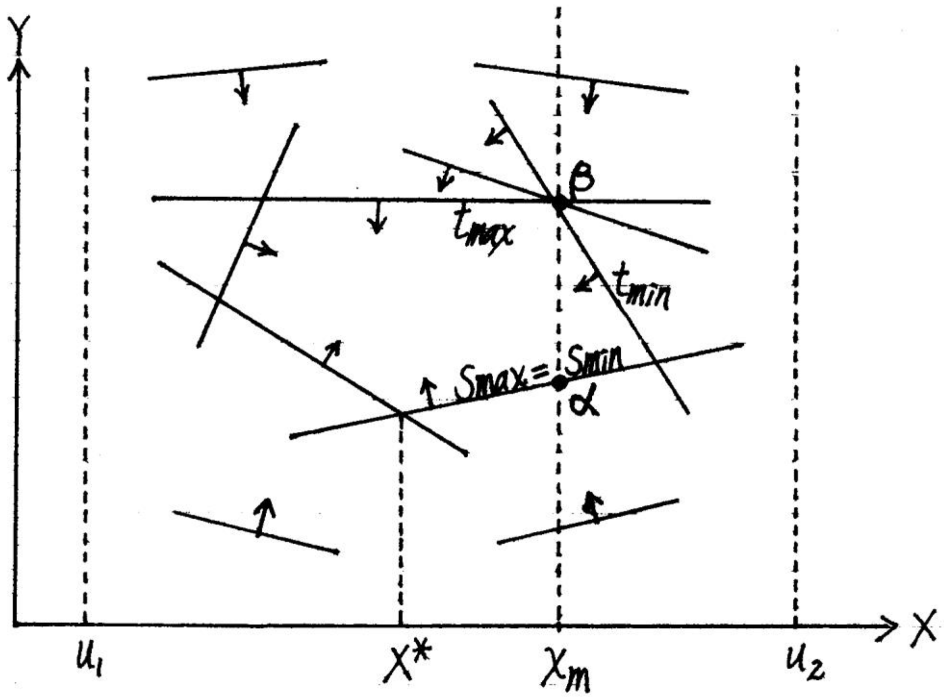
I^0: may be combined into a single constraint

$$u_1 \le x \le u_2$$



Observation (1) : The feasible region, if not empty, is a convex polygon with possibly one open side.

Observation (2) :

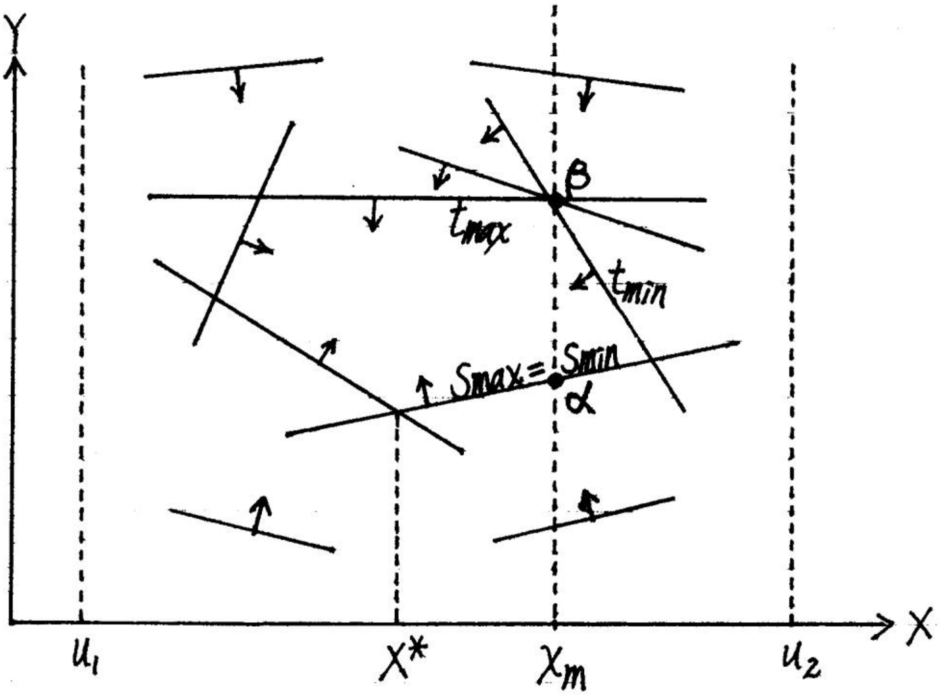


x_m : any x-value in $[u_1, u_2]$;

x^* : the optimal x-value;

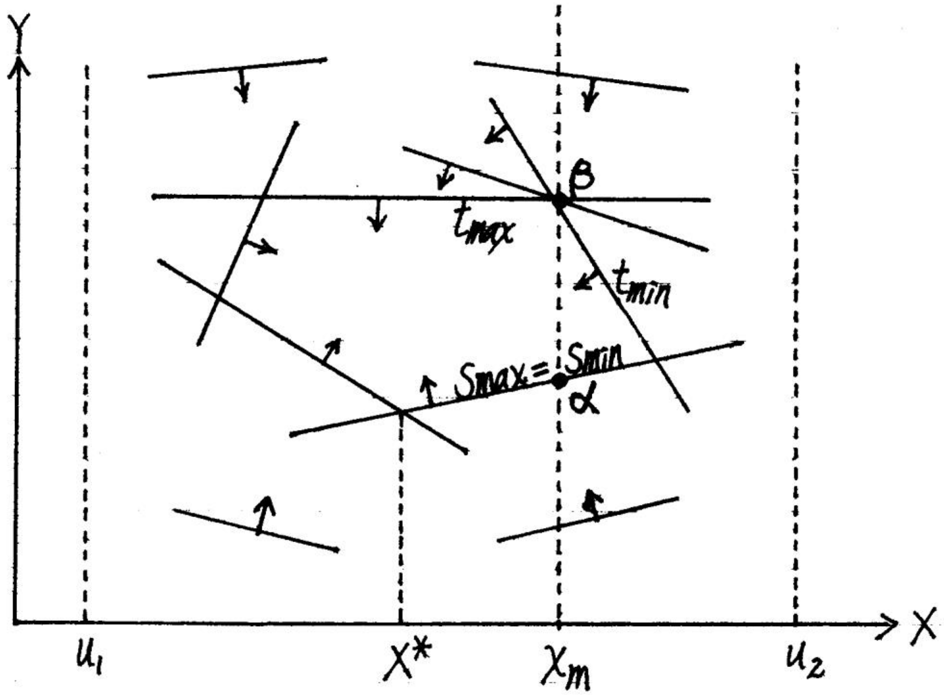
α : the boundary point of the feasible region that is represented by I^-, where $x = x_m$ passes;

β : the boundary point of the feasible region that is represented by I^+, where $x = x_m$ passes;



s_{max} (s_{min}) : the *maximum* (*minimum*) slope of the bounding lines (in I^-) that pass through α;

t_{max} (t_{min}) : the maximum (minimum) slope of the bounding lines (in I^+) that pass through β.



$\alpha = (\alpha_x, \alpha_y)$ and $\beta = (\beta_x, \beta_y)$ can be determined in $O(n)$ time as follows:

$\alpha_x \leftarrow x_m$; $\beta_x \leftarrow x_m$;

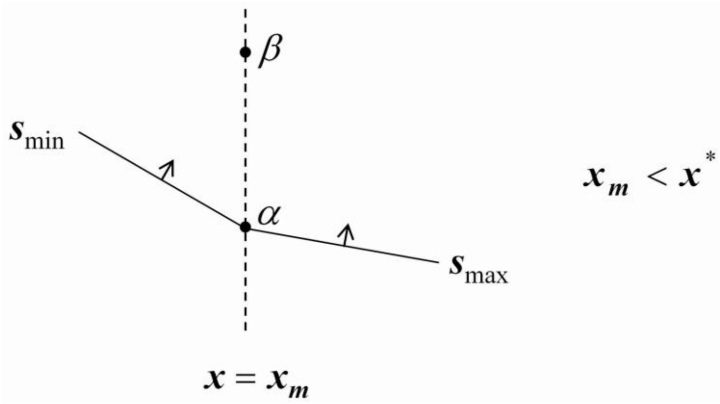
$\alpha_y \leftarrow \max\{(-a_i x_m - c_i)/b_i \mid a_i x + b_i y + c_i \le 0 \in I^-\}$;

$\beta_y \leftarrow \min\{(-a_i x_m - c_i)/b_i \mid a_i x + b_i y + c_i \le 0 \in I^+\}$.

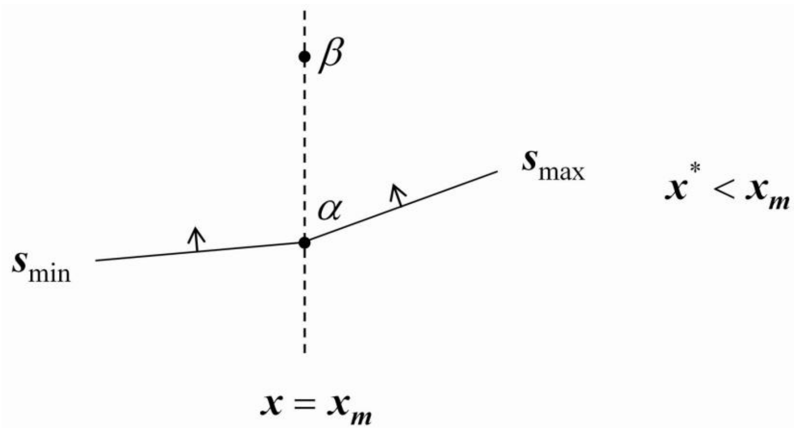
Also, s_{max}, s_{min}, t_{max} and t_{min} can be determined as well.

Depending on the values of (α_x, α_y), (β_x, β_y), s_{max}, s_{min}, t_{max} and t_{min}, the following six cases are discussed.

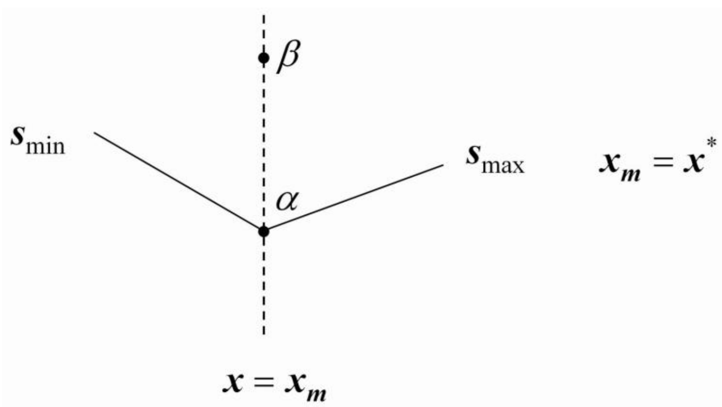
Case 1. $\alpha_y \le \beta_y$ **and** $(s_{\min} \le)\ s_{\max} < 0$.



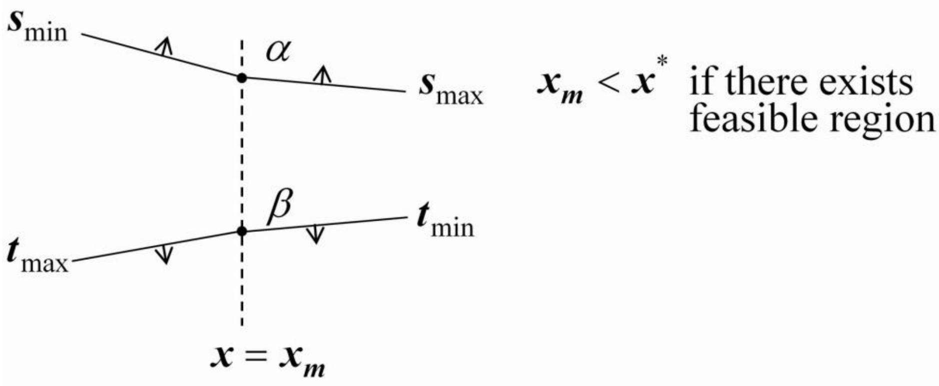
Case 2. $\alpha_y \le \beta_y$ **and** $(s_{\max} \ge)\ s_{\min} > 0$.



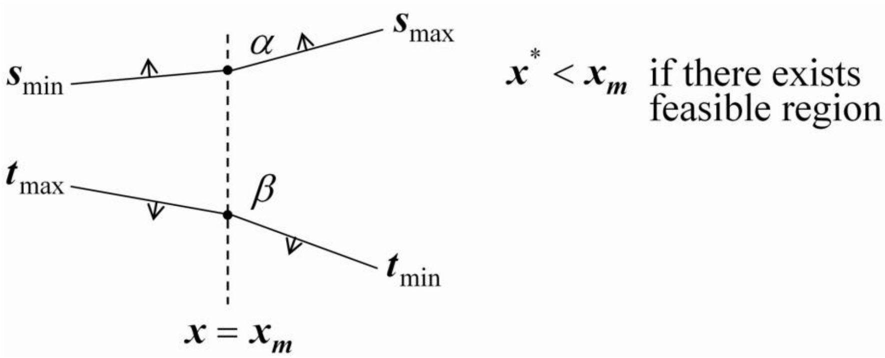
Case 3. $\alpha_y \le \beta_y$ **and** $s_{\min} \le 0 \le s_{\max}$.



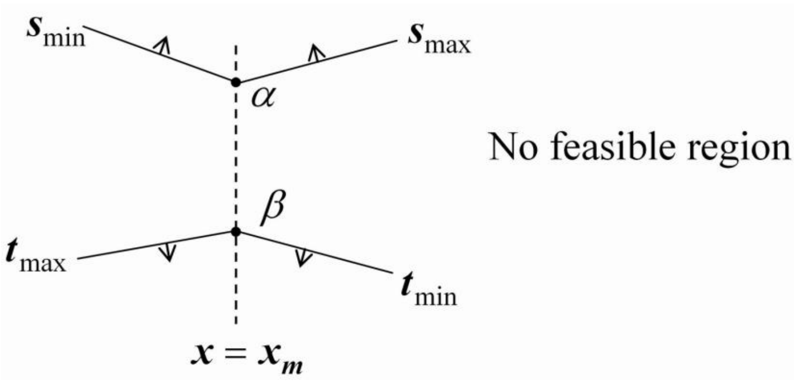
Case 4. $\alpha_y > \beta_y$ **and** $s_{\max} < t_{\min}$**.**



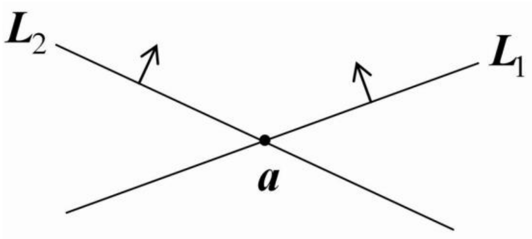
Case 5. $\alpha_y > \beta_y$ **and** $s_{\min} > t_{\max}$**.**



Case 6. $\alpha_y > \beta_y$ **and** $(s_{\max} \geq t_{\min}$ **and** $s_{\min} \leq t_{\max})$**.**

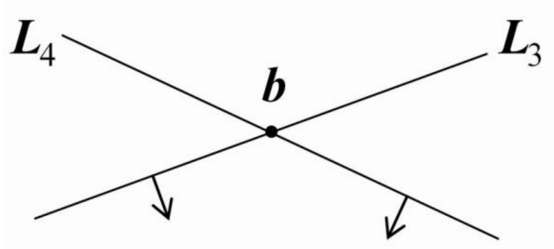


Observation (3) :



If $x^* < a_x$, **then** L_1 **can be removed.**

If $a_x < x^*$, **then** L_2 **can be removed.**



If $x^* < b_x$, **then** L_4 **can be removed.**

If $b_x < x^*$, **then** L_3 **can be removed.**

A prune-and-search algorithm.

1. $x_l \leftarrow u_1,\ \ x_r \leftarrow u_2.$

2. **Partition I^- and I^+ each into pairs of constraints (bounding lines) and determine the x-coordinates r_x's of their intersections.**

   If the pair of bounding lines is in parallel or $r_x \notin [x_l, x_r]$, then one can be removed.

3. **For the set of r_x's $\in [x_l, x_r]$, find their median.**

4. **Let x_m be the median. If $x_m = x^*$ or no feasible region, then terminate the algorithm. If $x^* < x_m$, then $x_r \leftarrow x_m$. If $x_m < x^*$, then $x_l \leftarrow x_m$.**

5. Prune redundant constraints.

Since half of r_x's are known to be smaller ($x_m < x^*$) or greater ($x^* < x_m$) than x^*, about one fourth of the constraints can be removed.

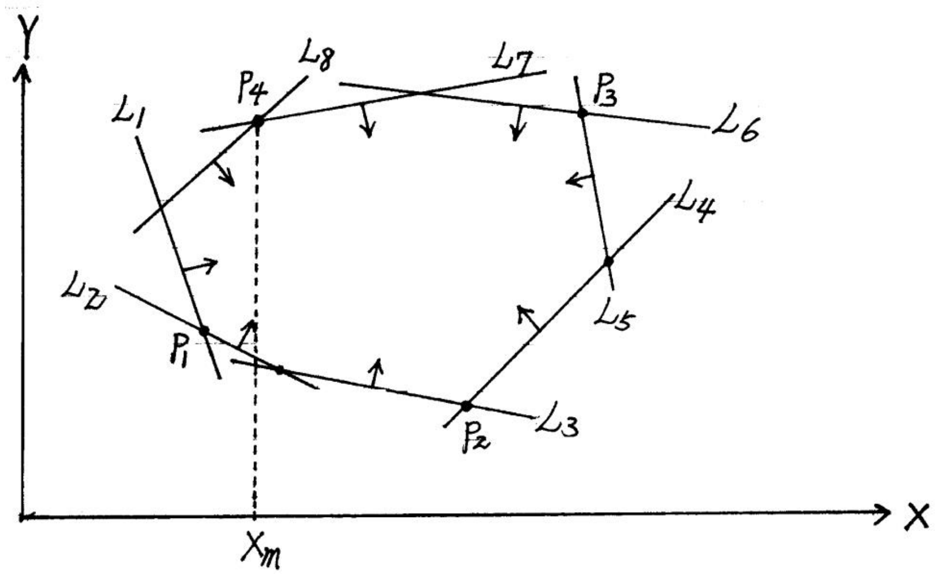
6. Repeat 2~5 until the number of remaining constraints is small enough.

7. Solve the reduced problem directly.

Time complexity :

$$\begin{aligned} T(n) &\le T(3n/4) + O(n) \\ &\le O(n). \end{aligned}$$

Ex.



1. $x_l \leftarrow -\infty$, $x_r \leftarrow \infty$.

2. **Form pairs :** (L_1, L_2), (L_3, L_4), (L_5, L_6) **and** (L_7, L_8).

3. $x_m \leftarrow x$**-coordinate of the intersection of** L_7 **and** L_8.

   $x_m < x^*$, $x_l \leftarrow x_m$

4. L_8 **and** L_1 **are pruned.**

Ex. The smallest circle enclosing *n* points.

(The Euclidean 1–center problem in the plane)

Given *n* points $p_1, p_2, \ldots, p_n$ in the Euclidean plane, find the smallest circle enclosing these *n* points.

Let (a_i, b_i) be the coordinate of p_i, $1 \le i \le n$.

The problem is to determine a point so as to minimize

$$f(x, y) = \max\{((x-a_i)^2 + (y-b_i)^2)^{1/2} \mid 1 \le i \le n\}.$$

To begin with, we consider a restricted case (1-d 1-center problem) where the center of the circle lies on the line $y = y_m$.

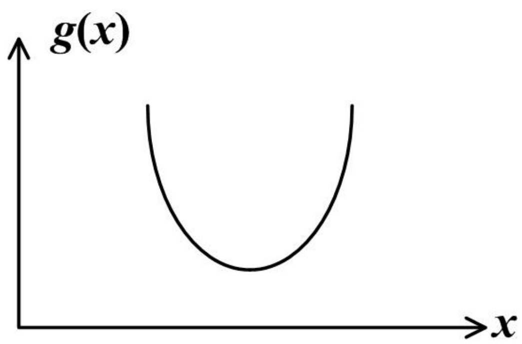
The restricted problem is to minimize

$$g(x) = \max \{((x - a_i)^2 + (y_m - b_i)^2)^{1/2} \mid 1 \leq i \leq n\}.$$

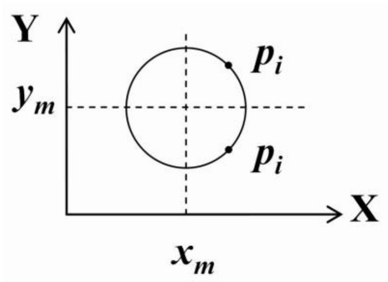
Observation (1) :

Let x^* be the x-value that minimizes $g(x)$, x_m be any x-value, and define I to be the set of points that are farthest from (x_m, y_m).

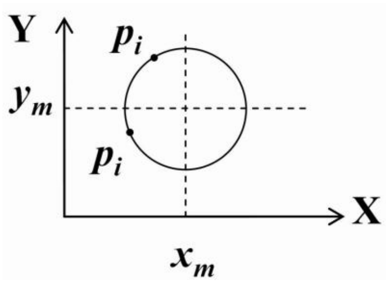
We can recognize whether $x_m < x^*$, $x_m = x^*$, or $x_m > x^*$ as follows ($g(x)$ is convex).



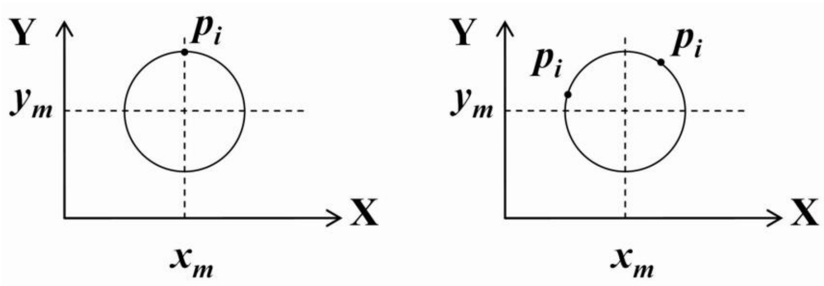
Case 1. $x_m < x^*$, **if** $x_m < a_i$ **for every** $p_i \in I$.



Case 2. $x^* < x_m$, **if** $a_i < x_m$ **for every** $p_i \in I$.



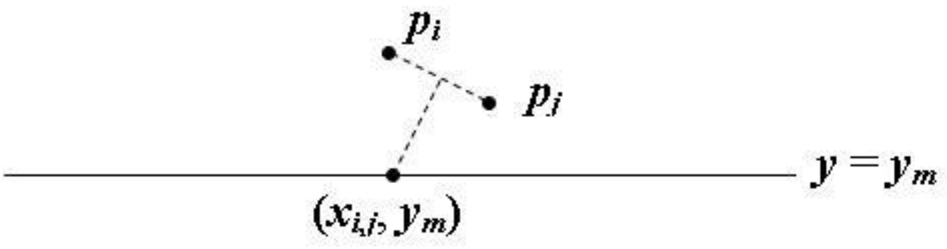
Case 3. $x_m = x^*$, **otherwise.**



Observation (2) :

Let ($x_{i,j}$, y_m) have equal distance from p_i and p_j.

Then, p_i (p_j) can be discarded if $x^* < x_{i,j}$ ($x_{i,j} < x^*$).



A prune-and-search algorithm.

1. **Partition points into pairs (p_1, p_2), (p_3, p_4), …**

2. **For each pair (p_i, p_{i+1}), if $a_i = a_{i+1}$, then discard the point that is closer to the line $y = y_m$.**

   Otherwise, find the point $(x_{i,i+1}, y_m)$ that has equal distance from p_i and p_j.

3. **Let x_m be the median of those $x_{i,i+1}$'s.**

4. **Compute the set I of the points that are farthest from (x_m, y_m), and then determine whether $x_m < x^*$, $x_m = x^*$, or $x_m > x^*$.**

5. **Prune redundant points.**

   Since at least half of $x_{i,i+1}$'s are smaller or greater (depending on $x_m < x^*$ or $x^* < x_m$) than x^*, at least one quarter of the points can be removed.

6. **Repeat 1 ~ 5 until the number of the remaining points is small enough.**

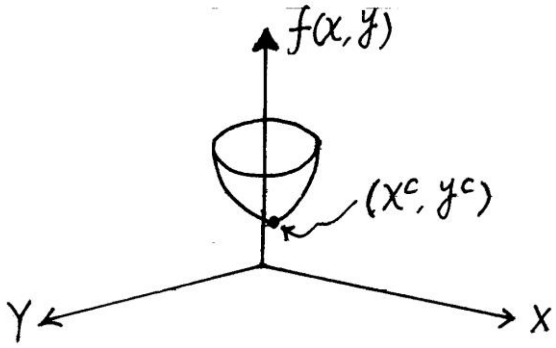
7. **Solve the reduced problem directly.**

Time complexity :

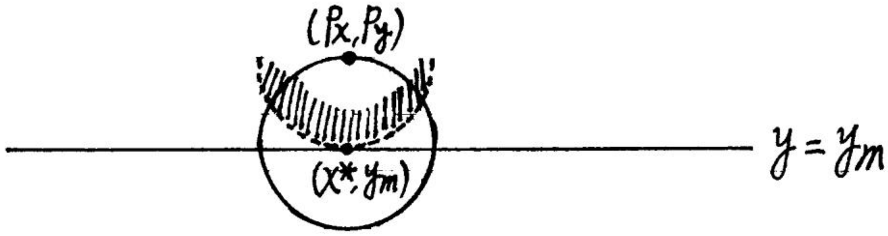
$$\begin{aligned} T(n) &\le T(3n/4) + O(n) \\ &\le O(n). \end{aligned}$$

By slight modifications, the algorithm can find the optimal center that lies on any line $y = ax + b$.

Let (x^c, y^c) be the optimal solution to the original problem. Since $f(x, y)$ is convex, we can determine where to find (x^c, y^c) by checking the set I.



Case 1. *I* contains one point.



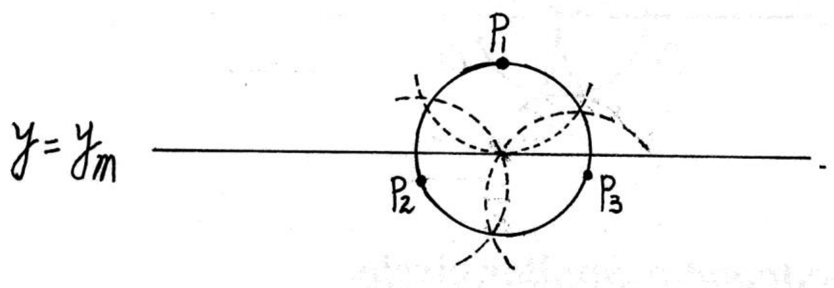
$p_x = x^*$ (otherwise, moving the circle right or left can result in a smaller circle)

Moving the center to the shaded area can result in a smaller circle, i.e., $y^c > y_m$.

Case 2. ***I*** **contains more than one point.**

$O(n)$ time is sufficient to determine if there exists an arc < 180° covering all of the points in I.

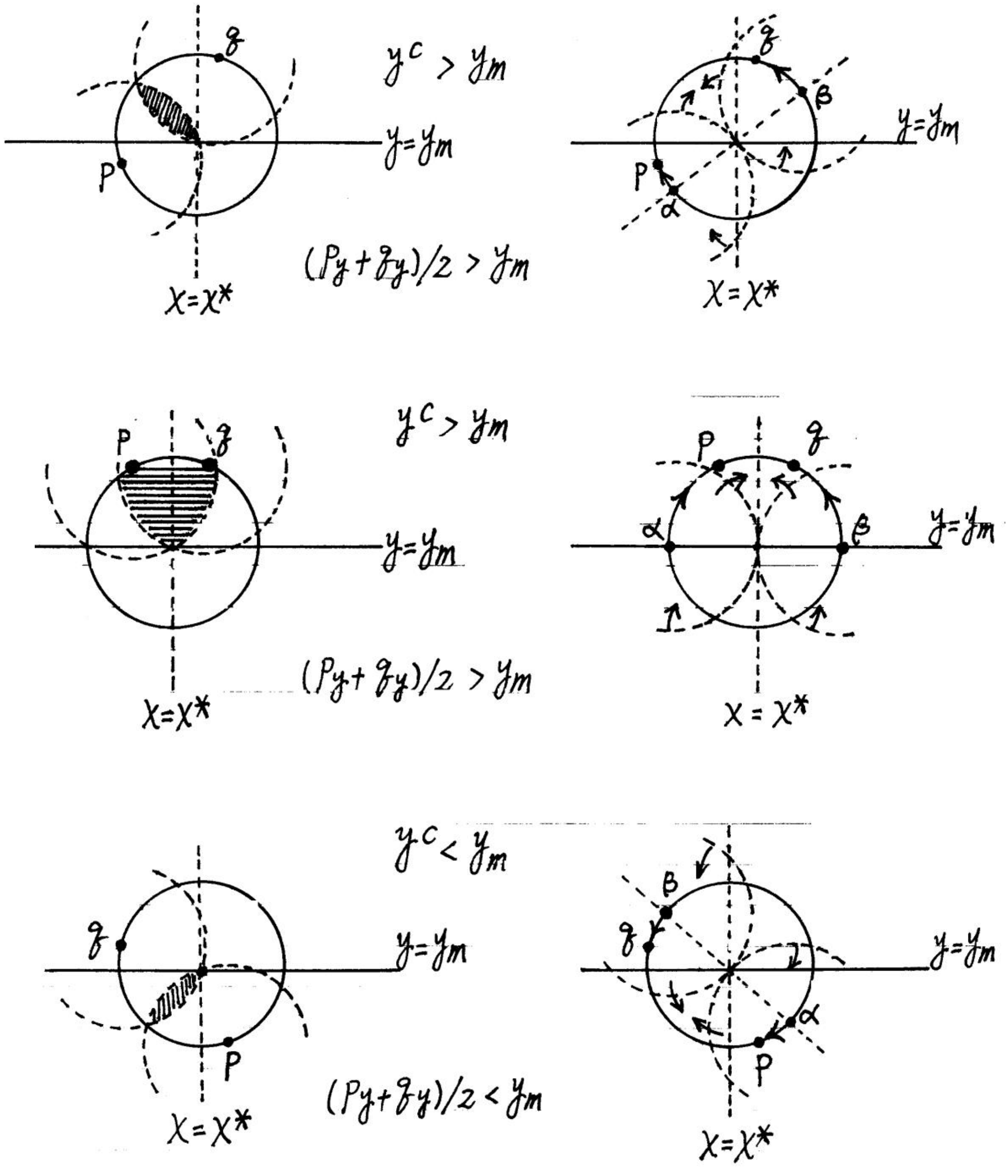
(1) No such arc can be found.



It is impossible to get a smaller circle, i.e., $(x^c, y^c) = (x^*, y_m)$.

(2) Such an arc can be found.

Suppose that $p = (p_x, p_y)$ and $q = (q_x, q_y)$ are the two end points of the arc, where p and q are in the opposite sides of $x = x^*$ (otherwise, moving the circle left or right can result in a smaller circle).



q
p
y^c > y_m
y = y_m
(p_y + q_y)/2 > y_m
x = x*
β
α
y^c < y_m
(p_y + q_y)/2 < y_m

Moving the center to the shaded area will result in a smaller circle.

The shaded area is the intersection of the two circles centered at p and q with the same radius as the enclosing circle.

(1) $(p_y + q_y) / 2 > y_m$.

The shaded area is above $y = y_m$, i.e., $y^c > y_m$.

(2) $(p_y + q_y) / 2 < y_m$.

The shaded area is below $y = y_m$, i.e., $y^c < y_m$.

In brief, we can determine on which side of a given line (x^c, y^c) is located in $O(n)$ time.

A prune-and-search algorithm for the smallest circle problem.

1. **Partition the points into pairs (p_1, p_2), (p_3, p_4), …**

2. **For each pair (p_i, p_{i+1}), find the perpendicular bisector L_i of the line segment $p_i p_{i+1}$.**

3. **Compute the angles α_i ($-\pi/2 \leq \alpha_i < \pi/2$) which L_i forms with the positive direction of the x-axis.**

4. **Compute the median α_m of α_i's.**

5. **Partition the set of L_i's into two equal-size (about $n/4$) subsets L^+ and L^- so that each L_i in L^+ has $\alpha_i \geq \alpha_m$ and each L_j in L^- has $\alpha_j < \alpha_m$.**

6. **Construct disjoint pairs of (l_i^+, l_i^-), $i = 1, 2, \ldots, n/4$, where $l_i^+ \in L^+$ and $l_i^- \in L^-$.**

7. **For each pair (l_i^+, l_i^-), compute their intersection (l_{x_i}, l_{y_i}).**

8. **Find a line (assume $y = ax + b$) parallel to L_m (with angle α_m), which partitions the set of (l_{x_i}, l_{y_i})'s into two equal-size (about $n/8$) subsets.**

9. **Determine on which side of $y = ax + b$ (x^c, y^c) is located.**

Without losing generality, assume that (x^c, y^c) is below $y = ax + b$.

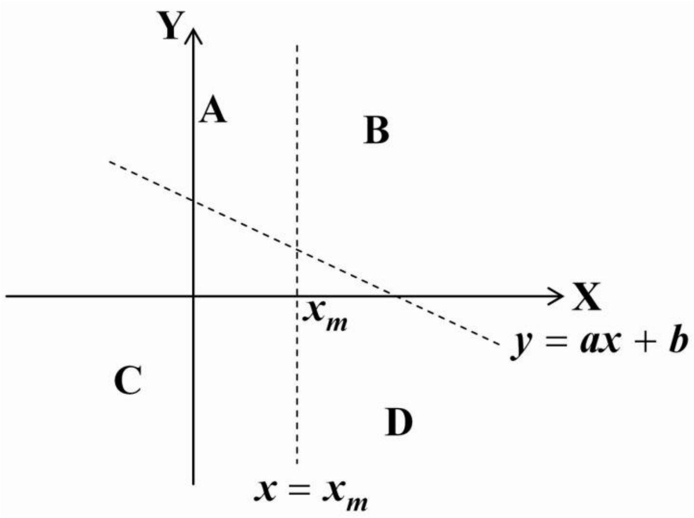
10. **Find a vertical line (assume $x = x_m$), which partitions the set of those (l_{x_i}, l_{y_i})'s that are above $y = ax + b$ into two equal-size (about $n/16$) subsets.**

11. **Determine on which side of $x = x_m$ (x^c, y^c) is located.**

Without losing generality, assume $x^c < x_m$.

12. Prune redundant points.

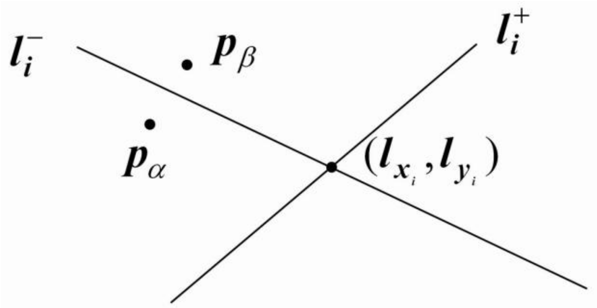
About $n/16$ points can be discarded.



(1) $(x^c, y^c) \in C$.

(2) **There are about $n/16$ (l_{x_i}, l_{y_i})'s in B.**

(3) **For each (l_{x_i}, l_{y_i}) in B, a point (p_α) can be discarded because it is closer to (x^c, y^c) than the other point (p_β).**



l_i^- does not intersect with C

13. Repeat 1 ~ 12 until the number of remaining point is small enough, for which the optimal center (x^c, y^c) can be determined in constant time.

Time complexity :

$$
\begin{aligned}
T(n) &\le T(15n/16) + O(n) \\
&\le O(n)
\end{aligned}
$$

Program Assignment 2 :

Write an executable program to solve the 2-dimensional linear programming problem.

Exercise 2:

Megiddo, "Linear-time algorithms for linear programming in R^3 and related problems," *SIAM Journal on Computing*, vol. 12, no. 4, 1983, pp. 759-776 (only Section 3 is required).