賽局理論與應用(Game Theory with Applications)
Homework 03 – 2016/11/10

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1. Let

$$x_n = \frac{1}{n}, \quad n \ge 1$$

By the definition of sequence convergence, prove the sequence x_n converges to 0.

Calculate the limit

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\forall \epsilon > 0$, there is a corresponding positive integer $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \epsilon \text{ (by the Archimedean Property)}$$

Whenever $n \ge N$, we have that

$$|x_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} \le \frac{1}{N} < \epsilon$$

Q.E.D

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2. Let X and Y be sequences in $\mathbb{R}^p$ that converge to x and y respectively. Prove that $X+Y$ converges to $x+y$.

Since the sequences X and Y are convergent. $\forall\epsilon > 0$, there are corresponding positive integers $N_1, N_2 \in \mathbb{N}$ so that

$$\text{If } n > N_1 \text{ then } |X - x| < \frac{\epsilon}{2} \text{ and if } n > N_2 \text{ then } |Y - y| < \frac{\epsilon}{2}$$

Thus whenever we take $n > \max\{N_1, N_2\} = N$, then

$$|(X+Y) - (x+y)| \leq |X - x| + |Y - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Q.E.D

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3. Closed set

(a) True or False? $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$ is closed in $\mathbb{R}$.

(b) True or False? $\{x \in \mathbb{R} : x \geq 0\}$ is closed in $\mathbb{R}$.

(c) True or False? $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is closed in $\mathbb{R}^2$.

(d) True or False? $\{x \in \mathbb{R} : 0 \leq x < 1\}$ is closed in $\mathbb{R}$.

Definition

A subset S of a metric space (X, d) is **closed** if it is the complement of an open set.

Definition

A subset S of a metric space (X, d) is **open** if it contains an open ball about each of its points – i.e., if

$$\forall x \in S : \exists \epsilon > 0 : B(x, \epsilon) \subseteq S$$

By the definition above.

(a) True.

(b) True.

(c) True.

(d) False.

4. Please give an example of the sequence and closed set to demonstrate that a set S is closed if and only if for any convergent sequence of points $\{x_k\}$ contained in S with limit point $\overline{x}$, we also have that $\overline{x} \in S$.

We can choose the sequence $\{x_k\} = \frac{1}{k^2}$ and the closed set $[0, \infty)$. So that the sequence converges to $0 \in [0, \infty)$. In fact, $[0, \infty)$ is closed, since every sequence of positive numbers converging to a limit would have an non-negative limit which is in $[0, \infty)$.

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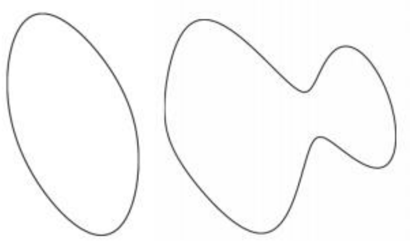
5. Disprove that $\{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = 1\}$ is convex.

Definition

A set C is **convex** if the line segment between any two points in C lies in C, i.e. $\forall x_1, x_2 \in C, \forall\theta \in [0,1]$:

$$\theta x_1 + (1-\theta)x_2 \in C$$



Example of a convex set (left) and a non-convex set (right).

By the definition above. Let arbitrary two point $x_1, x_2 \in \left\{x \in \mathbb{R}^n : \sum_{i=1}^{n} {x_i}^2 = 1\right\}$, $0 \le \theta \le 1$.

$$|\theta x_1 + (1-\theta)x_2| \le \theta|x_1| + (1-\theta)|x_2|$$

Since that $|\theta x_1 + (1-\theta)x_2| \le \theta|x_1| + (1-\theta)|x_2| \le \theta + (1-\theta) = 1$ only hold when $x_1, x_2 \in \left\{x \in \mathbb{R}^n : \sum_{i=1}^{n} {x_i}^2 \le 1\right\}$. The set $\left\{x \in \mathbb{R}^n : \sum_{i=1}^{n} {x_i}^2 = 1\right\}$ is not convex.