Integer and Combinatorial Optimization
整數與組合最佳化

Instructor: Kwei-Long Huang

Course No: 546 U6110

About the course

- Instructor: Kwei-Long Huang (黃奎隆)
  - Email: craighuang@ntu.edu.tw; Phone: 33669508
  - Office: 102 國青大樓
  - Office Hours: by appointment
- Teaching Assistant
  - 翁浩宸 r04546022@ntu.edu.tw,
  - 顏甄霓 r04546023@ntu.edu.tw
  - TA hours: 11:30~12:30 Monday at 國青大樓 102 lobby

About the course

- Graduate level course
- Three credits
- Course Meetings
  - Lecture: 9:10- 12:10 Thursday
  - 國青233
- Please use CEIBA to download course material, and check the course announcements. (https://ceiba.ntu.edu.tw/)

Course Description

- The course covers fundamental integer programming techniques, including cutting plane methods, branch-and-bound enumeration, Bender's decomposition, Lagrangain relaxation/decomposition, and heuristic/meta-heuristic programming.
- It also covers special techniques for solving well-known combinatorial problems, such as knapsack problem and the set covering/partition problem.

Course Objectives

- The course primarily focuses on study of Integer Programming and gives an overview of classical methods about problem formulations and solving.
- The goal of this course is to provide students some understanding such as why some problems are difficult to solve, how they can be reformulated to yield better results, and how effective different algorithms can be.

Textbook

- Textbook
  - H.M. Salkin and K. Mathur, *Foundations of Integer Programming*, North-Holland, New York, 1989
- Reference
  - D.-S. Chen, R. G. Batson, Y. Dang, *Applied Integer Programming: Modeling and Solution*, Wiley 2010
  - L.A. Wolsey, *Integer Programming*, Wiley 1998.
  - G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization*, Wiley 1988. (台北圖書有限公司, 02-23620811)

Grading

- Your grade in the course will be determined by homework (20%), midterm exam (35%), final exam (30%) and project (15%). The requirements in details are described as follows:

Homework (20%)

- Homework will be assigned on a biweekly basis and the assignments are to be done independently.
- Late submissions are not accepted except prior approval is received from the instructor.

Exams

- Midterm exam (35%)
  - On **April 20**
- Final exam (30%)
  - On **June 22**
- Project presentation (15%)
  - On **June 15**
- Study day (No class)
  - On **April 6**

Final Project (15%)

- There is a final project which helps students comprehend the class material and apply them to practical problems or real cases.
- Students can choose any problem which is related to the theories or can be applied the techniques taught in class.
- 2~3 students (may vary upon the class size) form a group.
- Prepare a 25-min presentation and submit a report in the last class. (on **June 15**)

Course Outline

- Introduction to Integer Programming (IP)
- IP modeling and applications
- The beale Tableau
- Using Linear programming to solve IP problems
- Cutting plane techniques
- Branch-and-Bound enumeration
- Search enumeration
- Bender's Decomposition
- Lagrangian Relaxation/Decomposition
- Heuristic Algorithms
- Combinatorial problems: knapsack problem and the set covering/partition problem

Introduction

- Airline crew scheduling
- Production planning
- Telecommunications
- Cutting problems

What is an integer program?

- An integer linear program is a mathematical (optimization) model where the objective function and constraints are linear, and all the variables or a subset of the variables are restricted to take integer values.

Integer Program

If *all* variables are integer: we have a *(Linear) Integer Program:*

$$\text{(IP)} \quad \max_{x \in \mathbb{R}^n} \{c^T x : Ax \leq b, \ x \geq 0 \text{ and integer}\}.$$

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, \dots, m,$$

$$x_j \geq 0 \text{ , integer,} \qquad j = 1, \dots, n,$$

Mixed Integer Program

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j + \sum_{k=1}^{p} d_k y_k ,$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{p} g_{ik} y_k \le b_i , \qquad i = 1, \ldots, m,$$

$$x_j \ge 0 \text{, integer} , \qquad j = 1, \ldots, n,$$

$$y_k \ge 0 , \qquad k = 1, \ldots, p,$$

Binary Integer Program

If each variable is only allowed to take on values 0 or 1, i.e., all variables are binary, we have a 0-1, or *Binary, Integer Program*:

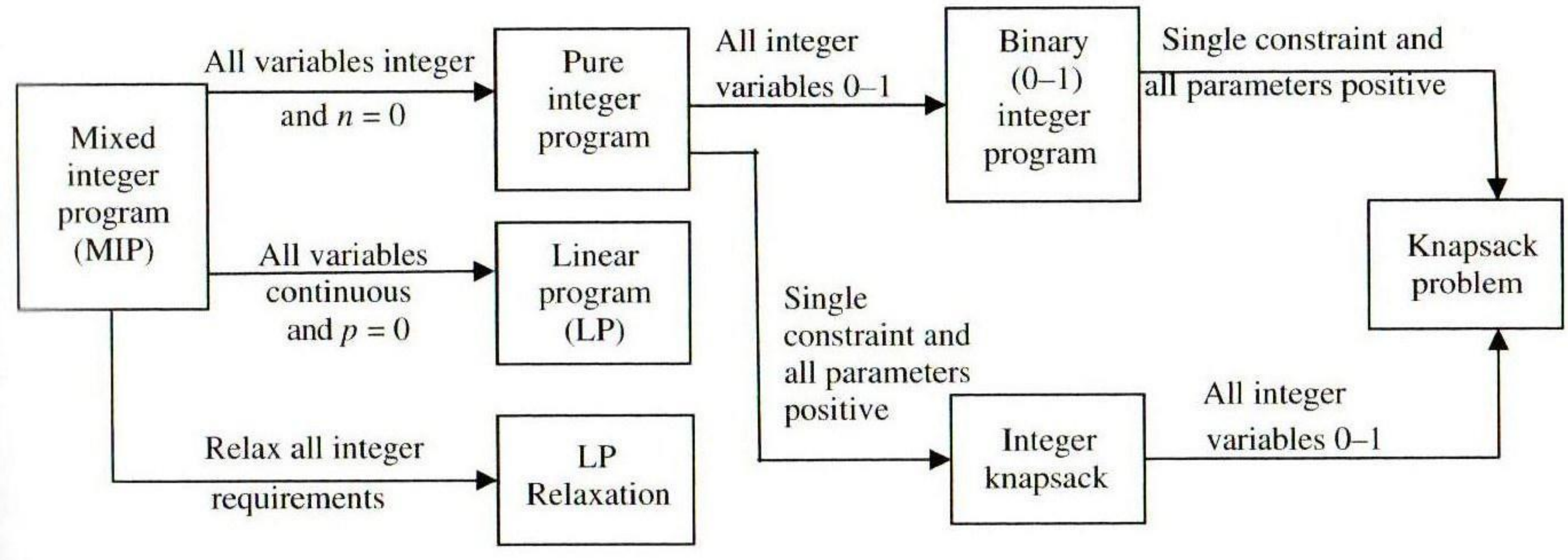
$$\text{(BIP)} \quad \max_{x \in \mathbb{R}^n} \{ c^T x : Ax \leq b, \; x \in \{0,1\}^n \}$$

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, \ldots, m,$$

$$x_j \in \{0,1\} \qquad j = 1, \ldots, n,$$

A Simple Classification of Integer Programs



n: number of continuous variables
p: number of integer variables

Source: Applied Integer Programming: Modeling and Solution, WILEY

ILP and MILP

- Some ILP and MILP models have all integer extreme points e.g., assignment problems, transportation problems with integer supplies and demands, min-cost network flow problems with integer supplies and demands, etc. In these cases, there always exist extreme points that are optimal and linear programming (LP) techniques can be used to find optimal solutions.

Assignment Problem

- There are *n* workers available to carry out *n* jobs.
- One-to-one assignment
- Each worker has his own expertise, so that each assignment has different costs (denoted as c_{ij}).

0-1 Knapsack Problem

- Suppose that a plane has cargo weight capacity b and is to be loaded with items each with weight a_j and relative value c_j.
- The problem is to load the plane so as to maximize its total relative value.

Set Covering Problem

- The problem is to decide where to install a set of service centers, so that a number of given regions can be covered.
- Let c_j be the cost to install a service center in region j, and a_{ij} be a indicator which shows whether region i is within the service range of the center j.

Facility Location Problem

- Assume that Walmark decides to build a set of distribution centers so that new-open stores can be served by these DCs.
- There is a fixed cost f_j associated with the use of DC j, and a transportation costs c_{ij} if the order of store i is delivered from DC j.
- Assume the demand of store i is d_i, and the max capacity of DC j is s_j.

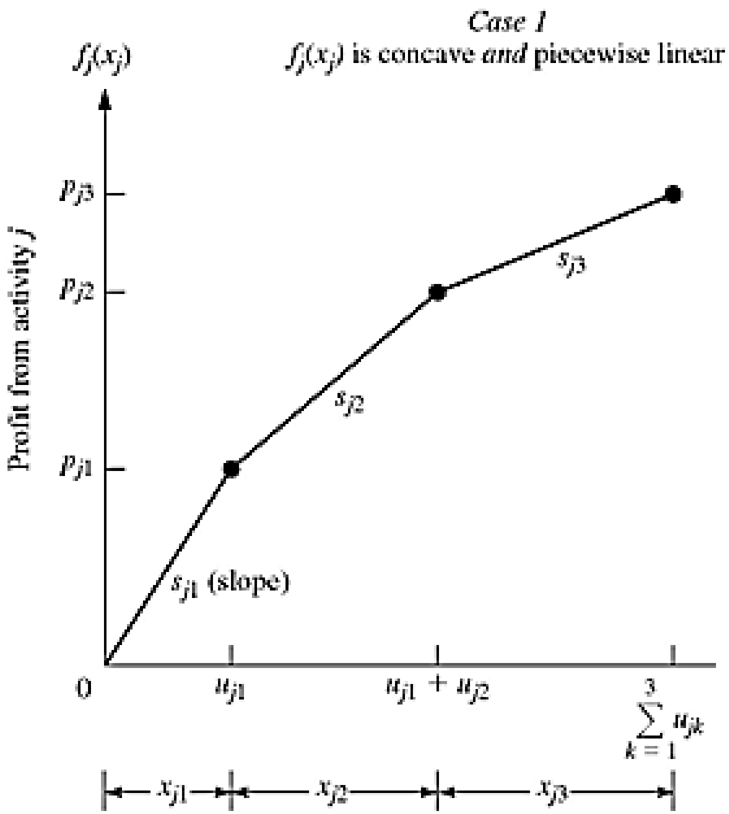
MIP => BIP

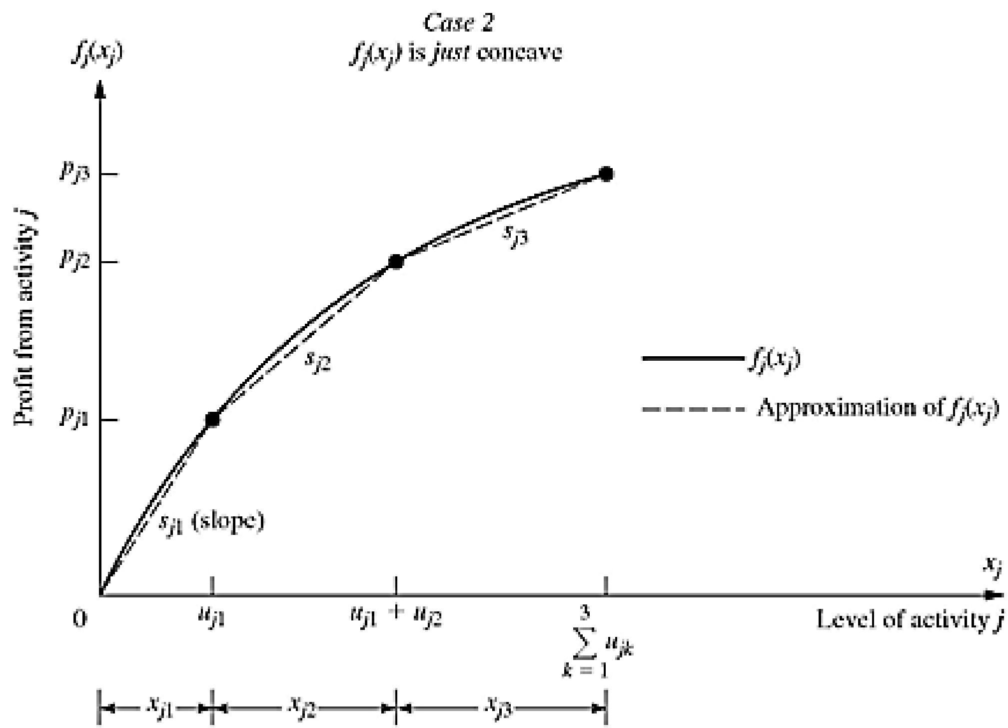
- Suppose in problem MIP each variable has a finite bound (a positive integer); then the MIP is equivalent to a BIP.
- Replace each x_j by $\sum_{k=1}^{u_j} t_{kj}$ where the t_{kj}'s are 0-1 variables.
  - For example: $x_j \leq 107$
- Can we have an improved formulation to reduce the number of variables?

Examples

- If $x_j \leq 107$, then this variable can be represented as
- Therefore, replace x_j by
- And retain the constraint

Separable Programming





Quality Furniture Corporation

The Quality Furniture Corporation manufactures two products: **benches** and **tables**. They employ **three** carpenters. During the next week, 150 hours of labor are available at $8 per hour.

Material and Resource

- 300 pounds of wood is available at a cost of $5 per pound.
- Each bench requires 3 labor hours and 12 pounds of wood. Each table requires 6 labor hours and 38 pounds of wood.
- Completed benches sell for $80 each, and tables sell for $200 each.

Procurement Discount

- Up to 150 pounds of wood can be purchased at \$4 per pound when 300 lb of wood has been purchased.
- Up to 100 pounds of wood can be purchased at \$3.5 per pound when 450 lb of wood has been purchased.

Question: How many benches and how many tables should be produced?

Linear Model

Max $85x1 + 200x2 - 8(3x1 + 6x2) - 5w1 - 4w2 - 3.5w3$

Where

x1 is the number of benches produced,

x2 is the number of tables produced,

w1 is the number of wood purchased at $5,

w2 is the number of wood purchased at $4,

w3 is the number of wood purchased at $3.5.

Linear Model (con't)

- Labor:
  - $3x1 + 6x2 \leq 150, x1, x2 \geq 0$
- Wood:
  - $12x1 + 38x2 \leq w1 + w2 + w3$
  - $0 \leq w1 \leq 300$
  - $0 \leq w2 \leq 150$
  - $0 \leq w3 \leq 100$

Linear Model (con't)

- Discount constraints:
  - Only when w1=300, w2 can be ≥ 0.
  - Only when w2=150, w3 can be ≥ 0.
  -
  -
  -
  -
  -
  - y1, y2, and y3 are binary

Linear Model (con't)

Max $85x1 + 200x2 - 8(3x1 + 6x2) - 5w1 - 4w2 - 3.5w3$

S.t.

$3x1 + 6x2 \leq 150$, $x1, x2 \geq 0$

$12x1 + 38x2 \leq w1 + w2 + w3$

$w1 \leq 300y1$, $w1 \geq 300y2$

$w2 \leq 150y2$, $w2 \geq 150y3$

$w3 \leq 100y3$, $y1$, $y2$, and $y3$ are binary

Piecewise Linear Objective

$$\text{Max } f_1(y_1) + y_2$$

s.t.

$$y_1 + y_2 \le 3$$

$$y_1,\ y_2 \ge 0$$

where

$$f_1(y_1) = \begin{cases} y_1 & \text{for } 0 \le y_1 \le 2 \\ 4 - y_1 & \text{for } 2 \le y_1 \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

Piecewise Linear Objective (con't)

- For a given interval [$\bar{y}_1, \bar{y}_2$],
- y_1 can be represented as
- $f_1(y_1)$ can be represented as
- For the entire interval [$\bar{y}_1, \bar{y}_3$]:
- We need some auxiliary variables to turn on and off intervals.

Piecewise Linear Objective (con't)

$$\text{Max } f_1(y_1) + y_2$$

s.t.

$$y_1 + y_2 \le 3$$

$$y_1,\ y_2 \ge 0$$

$\Rightarrow$

$$\text{Max } 2a_2 + a_3 + y_2$$

s.t.

$$2a_2 + 3a_3 + y_2 \le 3,$$

$$a_1 + a_2 + a_3 = 1,$$

$$a_1,\ a_2, a_3, y_2 \ge 0,$$

Questions?