II. Short-time Fourier Transform

II-A Definition

Short-time Fourier transform (STFT)

$$X(t,f) = \int_{-\infty}^{\infty} w(t-\tau)\,x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

Alternative definition

$$X(t,\omega) = \int_{-\infty}^{\infty} w(t-\tau)\,x(\tau)\,e^{-j\omega\tau}\,d\tau$$

參考資料

[1] S. Qian and D. Chen, Section 3-1 in *Joint Time-Frequency Analysis: Methods and Applications*, Prentice-Hall, 1996.

[2] S. H. Nawab and T. F. Quatieri, “Short time Fourier transform,” in *Advanced Topics in Signal Processing*, pp. 289-337, Prentice Hall, 1987.

STFT $$X(t, f) = \int_{-\infty}^{\infty} w(t-\tau)\, x(\tau)\, e^{-j2\pi f \tau} d\tau$$

$$X(t, \omega) = \int_{-\infty}^{\infty} w(t-\tau)\, x(\tau)\, e^{-j\omega\tau} d\tau$$

Inverse of the STFT: To recover $x(t)$,

$$x(t) = w^{-1}(t_1 - t) \int_{-\infty}^{\infty} X(t_1, f)\, e^{j2\pi f t} df$$

where $w(t_1 - t) \neq 0$.

For the alternative definition,

$$x(t) = \frac{1}{2\pi} w^{-1}(t_1 - t) \int_{-\infty}^{\infty} X(t_1, \omega)\, e^{j\omega t} d\omega$$

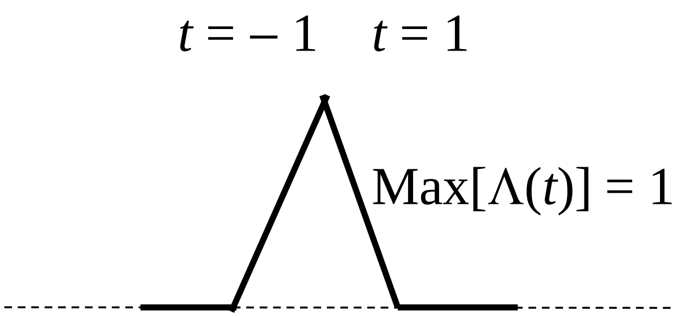
The mask function $w(t)$ always has the property of

(a) even: $w(t) = w(-t)$,   (通常要求這個條件要滿足)

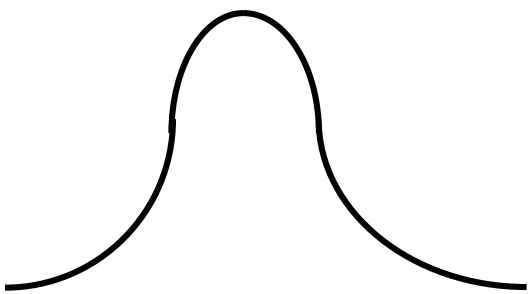
(b) $\max(w(t)) = w(0)$, $w(t_1) \geq w(t_2)$ if $|t_2| > |t_1|$

(c) $w(t) \approx 0$ when $|t|$ is large

$w(t) = \Lambda(t)$ (triangular function)



$w(t) = \exp(-a|t|^b)$
(hyper-Laplacian function)



II-B Rec-STFT

Rectangular mask STFT (rec-STFT)

$$X(t, f) = \int_{t-B}^{t+B} x(\tau) e^{-j2\pi f \tau} d\tau$$

Inverse of the rec-STFT

$$x(t) = \int_{-\infty}^{\infty} X(t_1, f) e^{j2\pi f t} df$$

where $t - B < t_1 < t + B$

The simplest form of the STFT

Other types of the STFT may require more computation time than the rec-STFT.

II-C Properties of the Rec-STFT

(1) Integration (recovery):

(a)
$$\int_{-\infty}^{\infty} X(t,f)\,df = \int_{t-B}^{t+B} x(\tau)\int_{-\infty}^{\infty} e^{-j2\pi f\tau}df\,d\tau$$
$$= \int_{t-B}^{t+B} x(\tau)\delta(\tau)\,d\tau$$
$$= \begin{cases} x(0) & \text{when } t - B < 0 < t + B, \quad -B < t < B \\ 0 & \text{otherwise} \end{cases}$$

(b) $\int_{-\infty}^{\infty} X(t,f)e^{j2\pi f v}\,df = x(v)$ when $v - B < t < v + B$,

$= 0$ otherwise

(2) Shifting property (橫的方向移動)

$$\int_{t-B}^{t+B} x(\tau+\tau_0)e^{-j2\pi f\tau}d\tau = X(t+\tau_0, f)e^{j2\pi f\tau_0}$$

(3) Modulation property (縱的方向移動)

$$\int_{t-B}^{t+B} [x(\tau)e^{j2\pi f_0\tau}]e^{-j2\pi f\tau}d\tau = X(t, f-f_0)$$

(4) Special inputs:

(1) When $x(t) = \delta(t)$,

$$X(t, f) = 1 \text{ when } -B < t < B, \quad X(t, f) = 0 \text{ otherwise}$$

(2) When $x(t) = 1$

$$X(t, f) = 2B\,\mathrm{sinc}(2Bf)e^{-j2\pi f t}$$

思考： B 值的大小，對解析度的影響是什麼？

(5) Linearity property

If $h(t) = \alpha\, x(t) + \beta\, y(t)$ and $H(t, f)$, $X(t, f)$ and $Y(t, f)$ are their rec-STFTs, then

$$H(t, f) = \alpha\, X(t, f) + \beta\, Y(t, f).$$

(6) Power integration property

$$\int_{-\infty}^{\infty} \left| X(t, f) \right|^2 df = \int_{t-B}^{t+B} \left| x(\tau) \right|^2 d\tau$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| X(t, f) \right|^2 df\, dt = 2B \int_{-\infty}^{\infty} \left| x(\tau) \right|^2 d\tau$$

(7) Energy sum property (Parseval's theorem)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(t, f) Y^*(t, f) df\, dt = 2B \int_{-\infty}^{\infty} x(\tau) y^*(\tau) d\tau$$

$$\int_{-\infty}^{\infty} X(t, f) Y^*(t, f) df = \int_{t-B}^{t+B} x(\tau) y^*(\tau) d\tau$$

思考:

(1) 哪些性質 Fourier transform 也有？

(2) 其他型態的 STFT 是否有類似的性質？

Shifting

$$\int_{-\infty}^{\infty} w(t-\tau)x(\tau-\tau_0)e^{-j2\pi f\tau}d\tau$$

$$=\int_{-\infty}^{\infty} w(t-\tau-\tau_0)x(\tau)e^{-j2\pi f\tau}e^{-j2\pi f\tau_0}d\tau$$

$$=X(t-\tau_0, f)e^{-j2\pi f\tau_0}$$

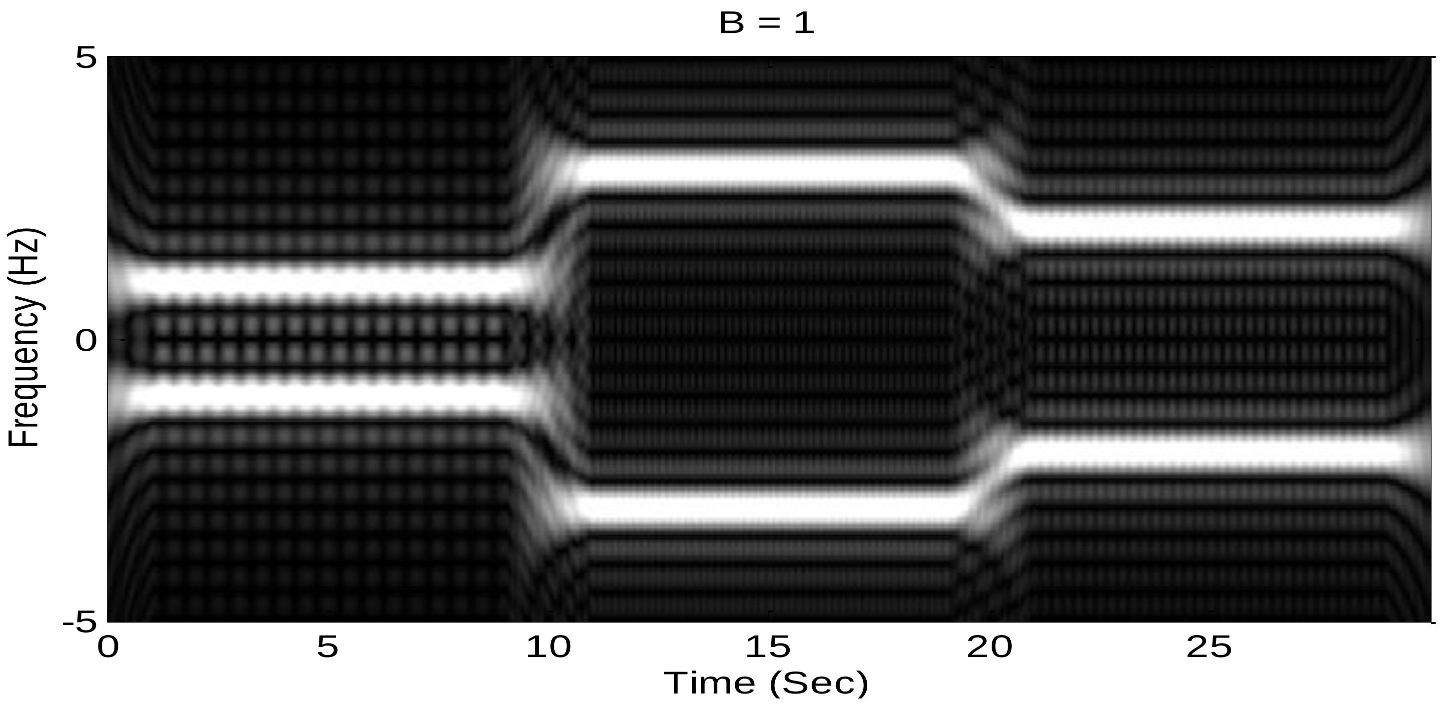
Modulation

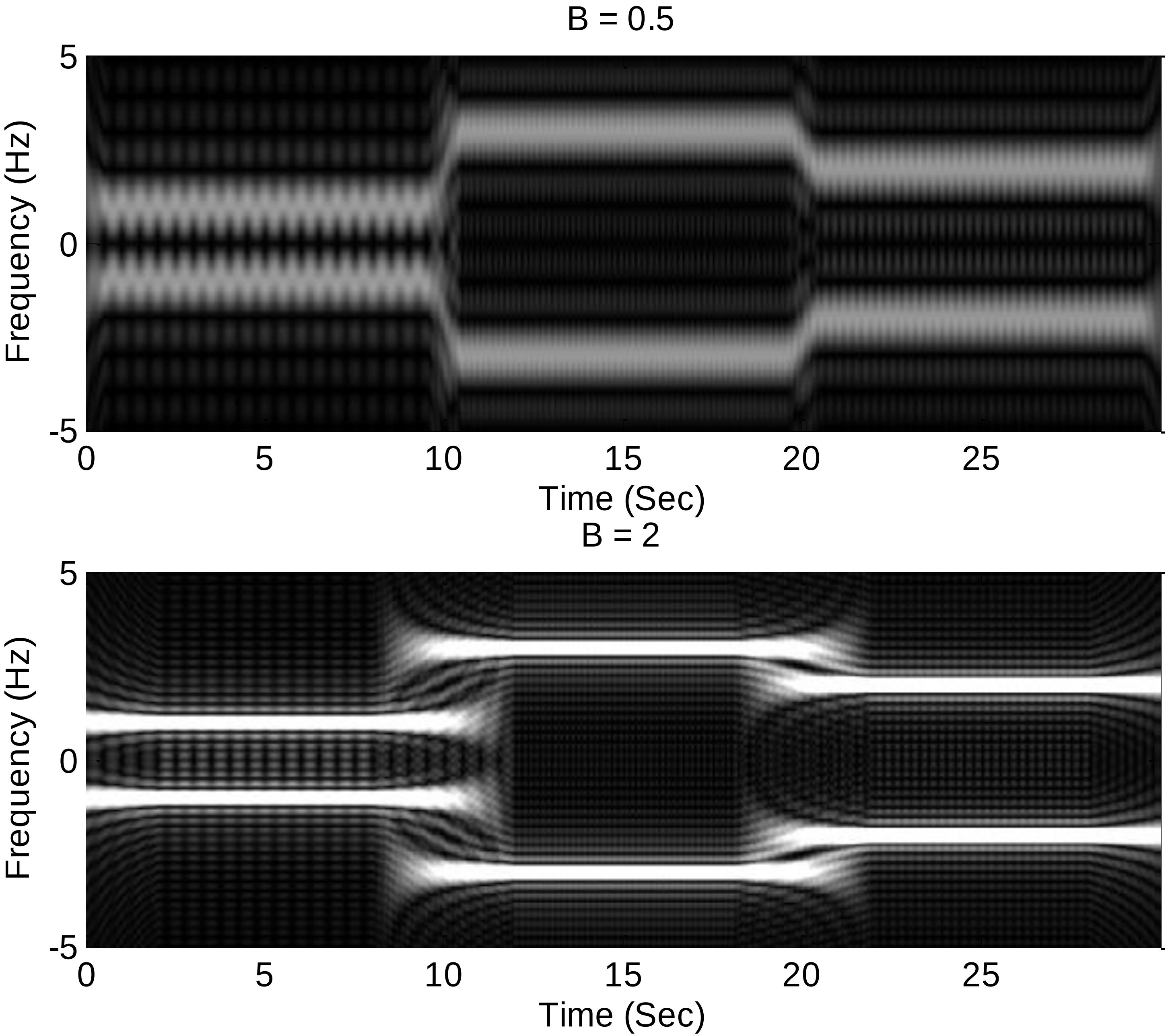
$$\int_{t-B}^{t+B} w(t-\tau)[x(\tau)e^{j2\pi f_0\tau}]e^{-j2\pi f\tau}d\tau = X(t, f-f_0)$$

Example: $x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$




B = 0.5
5
0
-5
Frequency (Hz)
0
5
10
15
20
25
Time (Sec)
B = 2
5
0
-5
Frequency (Hz)
0
5
10
15
20
25
Time (Sec)

II-D Advantage and Disadvantage

- Compared with the Fourier transform:

All the time-frequency analysis methods has the advantage of:

The instantaneous frequency can be observed.

All the time-frequency analysis methods has the disadvantage of:

Higher complexity for computation

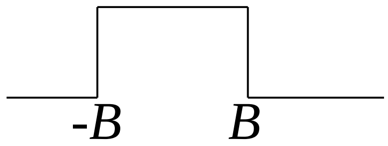
- Compared with other types of time-frequency analysis:

The rec-STFT has an advantage of the least computation time for digital implementation

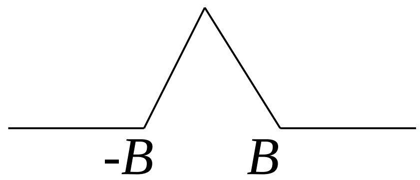
but its performance is worse than other types of time-frequency analysis.

II-E STFT with Other Windows

(1) Rectangle



(2) Triangle



(3) Hanning

$$w(t) = \begin{cases} 0.5 + 0.5\cos(\pi t / B) & \text{when } |t| \le B \\ 0 & \text{otherwise} \end{cases}$$

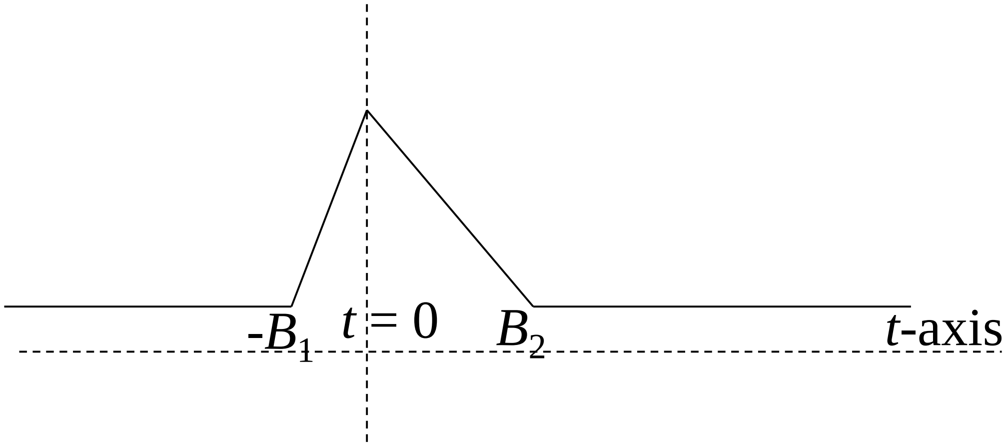
(4) Hamming

$$w(t) = \begin{cases} 0.54 + 0.46\cos(\pi t / B) & \text{when } |t| \le B \\ 0 & \text{otherwise} \end{cases}$$

(5) Gaussian

$$w(t) = \exp\left(-\pi\sigma t^2\right)$$

(6) Asymmetric window



$B_1 \neq B_2$

應用： seismic wave analysis, collision detection

(The applications that require real-time processing)

onset detection

動腦思考：

(1) Are there other ways to choose the mask of the STFT?

(2) Which mask is better?

沒有一定的答案

II-F Spectrogram

STFT 的絕對值平方，被稱作 Spectrogram

$$SP_x(t,f)=\left|G_x(t,f)\right|^2=\left|\int_{-\infty}^{\infty}w(t-\tau)e^{-j2\pi f\tau}x(\tau)d\tau\right|^2$$

文獻上，spectrogram 這個名詞出現的頻率多於 STFT

但實際上， spectrogram 和 STFT 的本質是相同的

附錄二：使用 Matlab 將時頻分析結果 Show 出來

可採行兩種方式：

(1) 使用 mesh 指令畫出立體圖

(但結果不一定清楚，且執行時間較久)

(2) 將 amplitude 變為 gray-level，用顯示灰階圖的方法將結果表現出來

假設 y 是時頻分析計算的結果

```
image(abs(y)/max(max(abs(y)))*C)   % C 是一個常數，我習慣選 C=400
colormap(gray(256))                % 變成 gray-level 的圖
set(gca,'Ydir','normal')           % 若沒這一行, y-axis 的方向是倒過來的
```

```
set(gca,'Fontsize',12)       % 改變橫縱軸數值的 font sizes
xlabel('Time (Sec)','Fontsize',12)           % x-axis
ylabel('Frequency (Hz)','Fontsize',12)       % y-axis
title('STFT of x(t)','Fontsize',12)          % title
```

計算程式執行時間的指令：

tic (這指令如同按下碼錶)

toc (show 出碼錶按下後已經執行了多少時間)

註：通常程式執行第一次時，由於要做程式的編譯，所得出的執行時間會比較長

程式執行第二次以後所得出的執行時間，是較為正確的結果