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1. Consider the following bargaining game. Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, s_1 and s_2, where $0 \le s_1, s_2 \le 1$, if $s_1 + s_2 \le 1$, then the players receive the shares they named; if $s_1 + s_2 > 1$, then both players receive zero. What are the pure strategy Nash equilibria of this game?.

當 $S_1 + S_2 \le 1$ 時，雙方各得 S_1 和 S_2
當 $S_1 + S_2 > 1$ 時，雙方皆沒有得到任何收益

在不失一般性的狀況下，作以下討論：
給定任意的 $S_2 \in [0,1)$ 則對於另一位決策者所得之最佳回應為 $R_1(S_2) = 1 - S_2$
對於 $S_2 = 1$ 時，另一位決策者所得之最佳回應為 $[0,1]$ （無論如何選擇，報酬皆為零）

$$S_1 = R_1(S_2) = \begin{cases} 1 - S_2 & \text{, if } 0 \le s_2 < 1 \\ [0,1] & \text{, if } S_2 = 1 \end{cases} \qquad S_2 = R_2(S_1) = \begin{cases} 1 - S_1 & \text{, if } 0 \le s_1 < 1 \\ [0,1] & \text{, if } S_1 = 1 \end{cases}$$

取其交集可得 $\{(S_1, S_2) : S_1 + S_2 = 1, S_1 \ge 0, S_2 \ge 0\}$ 和 $(1,1)$
上述策略即為 Nash Equilibrium 之優勢策略。

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2. Consider the Cournot model we discussed in class:
    - Two competing firms, selling a homogeneous good.
    - The marginal cost of producing each unit of the good is c.
    - The market price, P is determined by (inverse) market demand: $P = a - Q$ if $a > Q$, $P = 0$ otherwise.
    - Each firm decides on the quantity to sell (market share): q_1 and q_2.
    - $Q = q_1 + q_2$ is the total market demand.
    - Both firms seek to maximize profits.

    (a) Solve for the equilibrium quantity q_1^* and q_2^*.

    (b) Please verify your solution in (a) by showing that the statement "In the equilibrium, no one can be better-off by a unilateral change in its solution" is satisfied.

(a) 設 Firm1 和 Firm2 之利潤和銷售量分別為 π_1, π_2 和 q_1, q_2
且 $c = c_1 = c_2$ 為 Firm1 和 Firm2 之邊際單位成本，可知：

$$\pi_1 = \mathrm{TR}_1 - \mathrm{TC}_1 = [a - (q_1 + q_2) - c_1] \cdot q_1$$
$$\pi_2 = \mathrm{TR}_2 - \mathrm{TC}_2 = [a - (q_1 + q_2) - c_2] \cdot q_2$$

F.O.C

取其一階偏微分並令 $\dfrac{\partial \pi_1}{\partial q_1} = 0$ 和 $\dfrac{\partial \pi_2}{\partial q_2} = 0$ 可得：

$$\begin{cases} \dfrac{\partial \pi_1}{\partial q_1} = (a - 2q_1 - q_2 - c) = 0 \\ \dfrac{\partial \pi_2}{\partial q_2} = (a - q_1 - 2q_2 - c) = 0 \end{cases}$$

求解上式聯立方程式可得均衡銷售量 $q^* = q_1 = q_2 = \dfrac{a-c}{3}$

S.O.C

取其二階偏微分，可得：

$$\begin{cases} \dfrac{\partial^2 \pi_1}{\partial {q_1}^2} & = -2 < 0 \\ \dfrac{\partial^2 \pi_2}{\partial {q_2}^2} & = -2 < 0 \end{cases}$$

(b) 在均衡情況下，沒有任何一個決策者會因單一改變選擇而得到更高的收益。由上述 S.O.C 可知兩者之收益函數皆為凹口向下，因此若有其中一家廠商增加產量時而另一家產量固定，會使兩者收益皆為零（由題目可知當 $a > Q$ 時 $P = 0$），反而會造成整體收益減少。

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3. Following Question 2, suppose that each firm produces the half of monopoly quantity q_m, I.e., $q_1 = q_2 = \frac{1}{2} q_m$.
   (a) Solve for the monopoly quantity q_m.
   (b) Please compare each firm's profit in Question 3 with the solution you obtained in Question 2.
   (c) Show that $q_1 = q_2 = \frac{1}{2} q_m$ is not an equilibrium solution.

(a) 若為獨佔(monopoly)廠商，則有：

$$\mathrm{TR} = P \cdot Q = (a - q_m) \cdot q_m$$

$$\mathrm{MR} = \mathrm{MC} = c = \frac{\partial(\mathrm{TR})}{\partial q_m} = a - 2q_m$$

由上式可得 $q_m = \dfrac{a-c}{2}$，故 $q_1 = q_2 = \dfrac{q_m}{2} = \dfrac{a-c}{4}$

(b) **[Case 2(a)]** $q_1 = q_2 = \frac{a-c}{3}$

$$\begin{aligned}\pi_1 = \pi_2 &= \mathrm{TR} - \mathrm{TC} \\ &= \left[a - \left(\frac{a-c}{3} + \frac{a-c}{3}\right) - c\right] \cdot \frac{a-c}{3} = \frac{(a-c)^2}{9}\end{aligned}$$

[Case 3(a)] $q_1 = q_2 = \frac{a-c}{4}$

$$\begin{aligned}\pi_1 = \pi_2 &= \mathrm{TR} - \mathrm{TC} \\ &= \left[a - \left(\frac{a-c}{4} + \frac{a-c}{4}\right) - c\right] \cdot \frac{a-c}{4} = \frac{(a-c)^2}{8}\end{aligned}$$

(c) 證明如下：

I 獨佔產量 $q_m = \frac{a-c}{2}$ 嚴格優於其他更高產量（$\forall x > 0$ 且 $\forall q_j \geq 0$）

若 $Q = q_m + x + q_j < a$ 有

$$\pi_i(q_m, q_j) = \frac{a-c}{2}\left[\frac{a-c}{2} - q_i\right]$$

$$\pi_i(q_m + x, q_j) = \left[\frac{a-c}{2} + x\right]\left[\frac{a-c}{2} - x - q_i\right]$$

若 $Q = q_m + x + q_j \geq a$ 則有 $P(Q) = 0$，生產較低的產出就會提高利潤。

II 剔除大於獨佔產量 q_m 後，產量 $q = \frac{a-c}{4}$ 嚴格優於其他更低產量

$$\pi_i(\frac{a-c}{4}, q_j) = \frac{a-c}{4}\left[\frac{3(a-c)}{4} - q_i\right]$$

$$\pi_i(\frac{a-c}{4} - x, q_j) = \left[\frac{a-c}{4} - x\right]\left[\frac{3(a-c)}{4} + x - q_i\right]$$

III 反覆進行剔除嚴格優勢策略

如上所述，經剔除後，各個企業選擇銷售量的策略空間會逐漸縮小。反覆進行上述操作，可以使得其空間限制越來越小，最終得到均衡銷售量：

$$q_i^* = \frac{a-c}{3}$$

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4. In an industry there are N firms producing a homogeneous product. Let q_i denote the output level of firm i, $i=1,2,\cdots,N$, and let Q denote the aggregate industry production level. That is, $Q=\sum_{i=1}^{N} q_i$.
Assume that the demand curve facing the industry is $p=100-Q$. Suppose that the cost function of each firm i is given by

$$TC_i(q_i)=\begin{cases} F+q_i^2 & \text{if} \quad q_i>0 \\ 0 & \text{if} \quad q_i=0 \end{cases}$$

Suppose that the number of firms in the industry N is sufficiently small so that all the N firms make above-normal profits. Calculate the output and profit levels of each firm in a Cournot equilibrium. (Hint: you can assume that all firms have identical cost functions.)

令 $\max \pi_i(q_i, q_{-i}) = \left[a-(q_i+\sum_{j\neq i} q_j)\right]\cdot q_i-(F+{q_i}^2)$

由 F.O.C 取其一階偏微分，並令 $\frac{\partial \pi_i}{\partial q_i}=0$ 可得：

$$\frac{\partial \pi_i}{\partial q_i}=a-4q_i-\sum_{j\neq i} q_j=0, \forall i=1,2\cdots N$$

又已知 $Q=\sum_{i=1}^{N} q_i=q_i+\sum_{j\neq i} q_j$ 代入整理，可得：

$$a-3q_i-Q=0, \forall i=1,2\cdots N$$

$$\Longrightarrow Na-3\sum q_i-NQ=0$$

由上式可得 $Q=\frac{N}{N+3}a$ 和 $P=a-Q=\frac{3}{N+3}a$

故 $q_i=\frac{Q}{N}=\frac{1}{N+3}a, \forall i=1,2\cdots N$

再代回可得利潤

$$\begin{aligned}\pi_i &= \frac{3}{N+3}a\cdot\frac{1}{N+3}a-\left[F+\left(\frac{1}{N+3}a\right)^2\right] \\ &= \frac{3a^2}{(N+3)^2}-\left[F+\left(\frac{1}{N+3}a\right)^2\right] \\ &= \frac{3\cdot 100^2}{(N+3)^2}-\left[F+\left(\frac{100}{N+3}\right)^2\right] \\ &= \frac{20000}{(N+3)^2}-F\end{aligned}$$