XIII. Continuous WT with Discrete Coefficients

13-A Definition

The parameters *a* and *b* are not chosen arbitrarily.

For example,

$a = n2^{-m}$ and $b = 2^{-m}$.

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt \qquad \begin{array}{ll} n \in Z, & n \in (-\infty, \infty) \\ m \in Z, & m \in (-\infty, \infty) \end{array}$$

註：某些文獻把這個式子稱作是 discrete wavelet transform，實際上仍然是 continuous wavelet transform 的特例

- Main reason for constrain *a* and *b* to be $n2^{-m}$ and 2^{-m} :

Easy to implementation

$X_w(n, m)$ can be computed from $X_w(n, m-1)$ by digital convolution.

13-B Inverse Wavelet Transform

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi_1\left(2^m t - n\right) X_w(n,m)$$

$\psi_1(t)$ is the dual function of $\psi(t)$.

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^m \psi_1\left(2^m t - n\right) \psi\left(2^m t_1 - n\right) = \delta\left(t - t_1\right)$$

i.e., $$\int_{-\infty}^{\infty} 2^m \psi_1\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

should be satisfied.

We often desire that $\psi_1(t) = \psi(t)$.

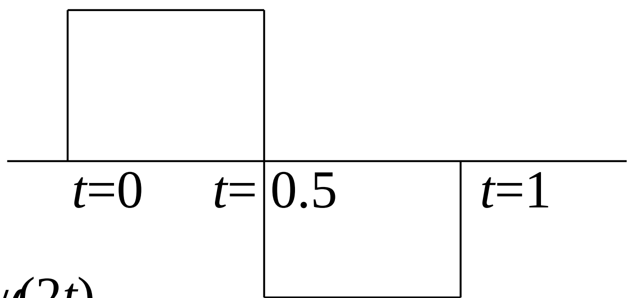
$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w(m,n)$$

The mother wavelet should satisfies

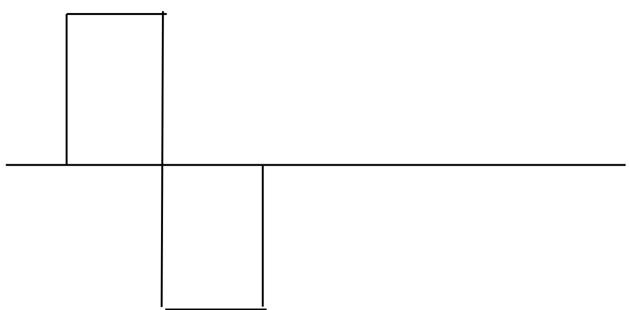
$$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

13-C Haar Wavelet

$\psi(t)$ mother wavelet (wavelet function)



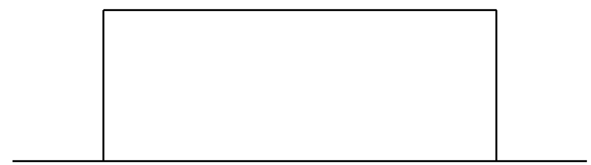
$\psi(2t)$



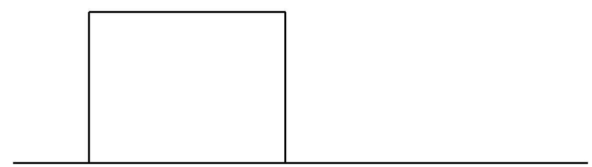
$\phi(t) = \phi(2t) + \phi(2t–1)$

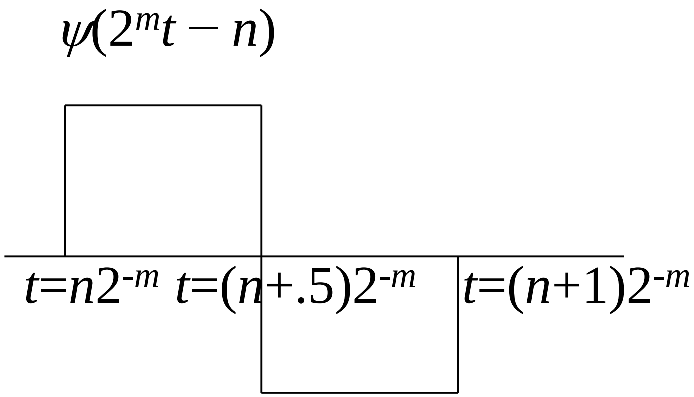
$\psi(t) = \phi(2t) – \phi(2t–1)$

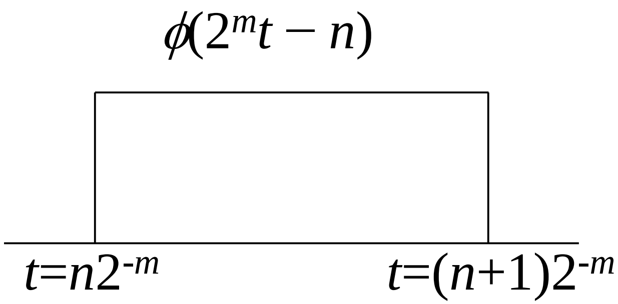
$\phi(t)$ scaling function



$\phi(2t)$ $\quad 2\phi(2t) = \phi(t) + \psi(t)$







- Advantages of Haar wavelet

(1) Simple

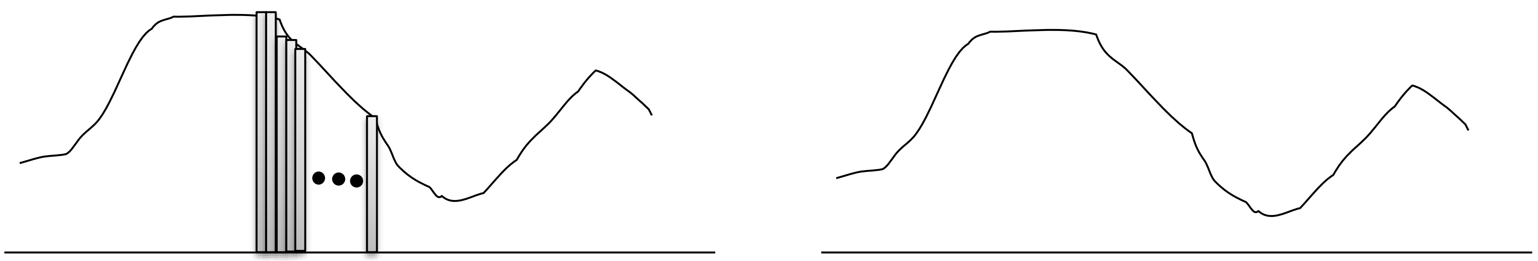
(2) Fast algorithm

(3) Orthogonal →reversible

(4) Compact, real, odd

(5) Vanish moment

(1) 任何 function 都可以由 $\phi(t)$, $\phi(2t)$, $\phi(4t)$, $\phi(8t)$, $\phi(16t)$, ……….. 以及它們的位移所組成



(2) 任何平均為 0 的function 都可以由 $\psi(t)$, $\psi(2t)$, $\psi(4t)$, $\psi(8t)$, $\psi(16t)$, ……….. 所組成

換句話說……. 任何 function 都可以由 constant, $\psi(t)$, $\psi(2t)$, $\psi(4t)$, $\psi(8t)$, $\psi(16t)$, ……….. 所組成

(3) Orthogonal

$$\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m_1}t-n_1\right)\psi\left(2^{m}t-n\right)dt=\delta\left(m-m_1\right)\delta\left(n-n_1\right)$$

The dual function of $\psi(t)$ is $\psi(t)$ itself.

(4) 不同寬度 (也就是不同 m) 的 wavelet / scaling functions 之間會有一個關係

$$\boxed{\phi(t) = \phi(2t) + \phi(2t-1)}$$

$$\phi(t-n) = \phi(2t-2n) + \phi(2t-2n-1)$$

$$\phi(2^m t-n) = \phi(2^{m+1}t-2n) + \phi(2^{m+1}t-2n-1)$$

$$\boxed{\psi(t) = \phi(2t) - \phi(2t-1)}$$

$$\psi(t-n) = \phi(2t-2n) - \phi(2t-2n-1)$$

$$\psi(2^m t-n) = \phi(2^{m+1}t-2n) - \phi(2^{m+1}t-2n-1)$$

(5) 可以用 m+1 的 coefficients 來算 m 的 coefficients

若 $\chi_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^m t - n)dt$

$$\chi_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1}t - 2n)dt + 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1}t - 2n - 1)dt$$

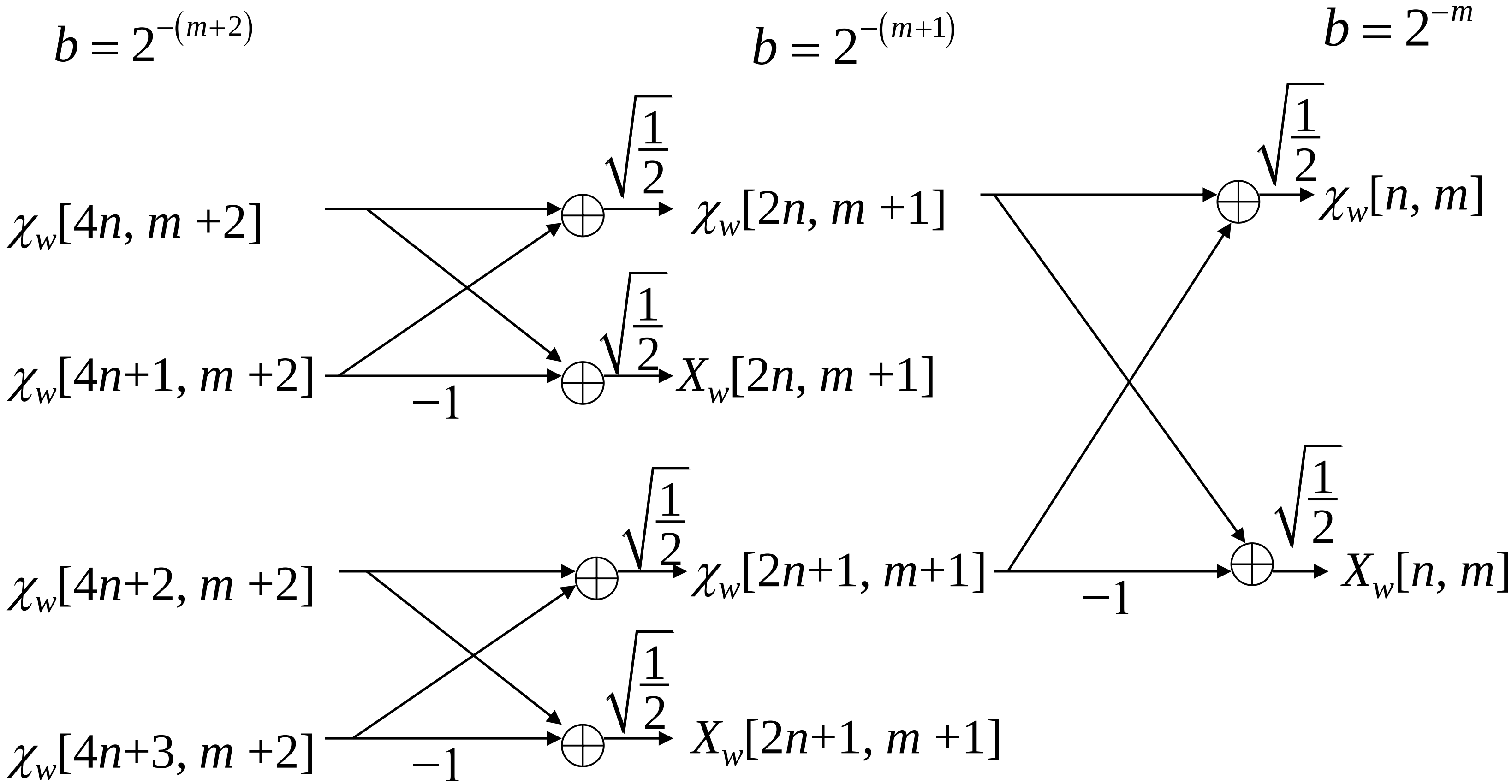
$$= \sqrt{\frac{1}{2}}\left(\chi_w(2n, m+1) + \chi_w(2n+1, m+1)\right)$$

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$$

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1}t - 2n)dt - 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1}t - 2n - 1)dt$$

$$= \sqrt{\frac{1}{2}}\left(\chi_w(2n, m+1) - \chi_w(2n+1, m+1)\right)$$

layer:



structure of multiresolution analysis (MRA)

13-D General Methods to Define the Mother Wavelet and the Scaling Function

Constraints:
- (a) nearly compact support
- (b) fast algorithm
- (c) real
- (d) vanish moment
- (e) orthogonal

和 continuous wavelet transform 比較：

(1) compact support 放寬為 "near compact support"

(2) 沒有 even, odd symmetric 的限制

(3) 由於只要是 complete and orthogonal, 必定可以 reconstruction

所以不需要 admissibility criterion 的限制

(4) 多了對 fast algorithm 的要求

13-E Fast Algorithm Constraints

Higher and lower resolutions 的 recursive relation 的一般化

$$\phi(t) = 2\sum_k g_k \phi(2t-k)$$ 稱作 dilation equation

$$\psi(t) = 2\sum_k h_k \phi(2t-k)$$

$\psi(t)$: mother wavelet, $\phi(t)$: scaling function

這些關係式成立，才有fast algorithms

$$\boxed{\phi(t) = 2\sum_k g_k \phi(2t-k)}$$

$$\boxed{\psi(t) = 2\sum_k h_k \phi(2t-k)}$$

If $\quad \chi_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^m t - n)dt$

then $\quad \chi_w(n,m) = \sum_k 2^{\frac{m}{2}+1}\int_{-\infty}^{\infty} x(t) g_k \phi(2^{m+1}t - 2n - k)dt$

$$= 2^{\frac{1}{2}}\sum_k g_k \chi_w(2n+k, m+1)$$

If $\quad X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$

then $\quad X_w(n,m) = \sum_k 2^{\frac{m}{2}+1}\int_{-\infty}^{\infty} x(t) h_k \phi(2^{m+1}t - 2n - k)dt$

$$= 2^{\frac{1}{2}}\sum_k h_k \chi_w(2n+k, m+1)$$

(Step 1) convolution

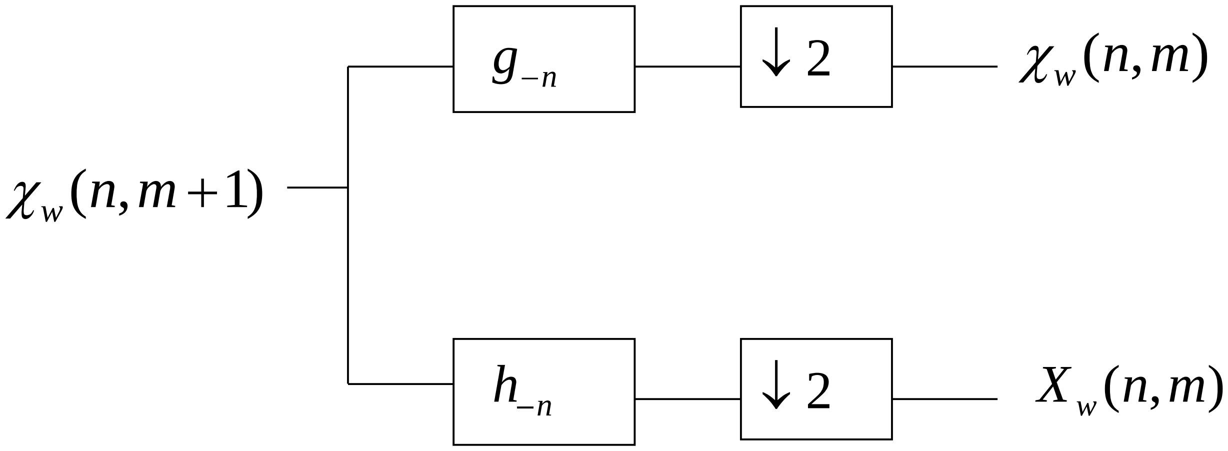
$$\tilde{\chi}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{g}_k \chi_w(n-k, m+1) \qquad \tilde{g}_k = g_{-k}$$

$$\tilde{X}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{h}_k \chi_w(n-k, m+1) \qquad \tilde{h}_k = h_{-k}$$

(Step 2) down sampling

$$\chi_w(n,m) = \tilde{\chi}_w(2n)$$

$$X_w(n,m) = \tilde{X}_w(2n)$$



m 越大，越屬於細節

- To satisfy $\phi(t) = 2\sum_k g_k \phi(2t-k)$,

$$\phi(t/2) = 2\sum_k g_k \phi(t-k)$$

$$\Big\downarrow \text{FT} \qquad\qquad \Big\downarrow \text{FT}$$

$$2\Phi(2f) = 2G(f)\Phi(f)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

where $\Phi(f) = FT\left[\phi(t)\right] = \int_{-\infty}^{\infty} \phi(t) e^{-j2\pi f t} dt$

$$G(f) = FT\left[\sum_k g_k \delta(t-k)\right]$$

$$= \sum_k g_k \int_{-\infty}^{\infty} \delta(t-k) e^{-j2\pi f t} dt$$

$$= \sum_k g_k e^{-j2\pi f k}$$

$\Phi(f)$ 是 $\phi(t)$ 的 continuous Fourier transform

$G(f)$ 是 $\{g_k\}$ 的 discrete time Fourier transform

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)\Phi\left(\frac{f}{4}\right) = G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)G\left(\frac{f}{8}\right)\Phi\left(\frac{f}{8}\right) = \cdots\cdots$$

$$\Phi(f) = \Phi\left(\frac{f}{2^\infty}\right)\underbrace{\prod_{q=1}^{\infty}}_{\text{連乘}}G\left(\frac{f}{2^q}\right) = \Phi(0)\prod_{q=1}^{\infty}G\left(\frac{f}{2^q}\right)$$

$$\Phi(0) = \int_{-\infty}^{\infty}\phi(t)\,dt$$

(可以藉由 normalization, 讓 $\Phi(0) = 1$)

$$\boxed{\Phi(f) = \prod_{q=1}^{\infty}G\left(\frac{f}{2^q}\right)}$$

若 $G(f)$ 決定了，則 $\Phi(f)$ 可以被算出來

constraint 1

$G(f)$: 被稱作 generating function

- 同理

$$\psi(t)=2\sum_k h_k \phi(2t-k) \qquad \Psi(f)=\int_{-\infty}^{\infty}\psi(t)e^{-j2\pi f t}dt$$

$$\Psi(f)=H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right) \qquad H(f)=\sum_k h_k e^{-j2\pi f k}$$

$$\boxed{\Psi(f)=H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty}G\left(\frac{f}{2^q}\right)}$$

constraint 2

- 另外，由於

$$\Phi(f)=G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right) \qquad \Phi(0)=G(0)\Phi(0) \qquad (f=0 \text{ 代入})$$

$$\boxed{G(0)=1}$$ 必需滿足

constraint 3

13-F Real Coefficient Constraints

Since $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ $\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$

If $\boxed{G(f) = G^*(-f)}$ $\boxed{H(f) = H^*(-f)}$ are satisfied,

constraint 4 constraint 5

then $\Phi(f) = \Phi^*(-f)$, $\Psi(f) = \Psi^*(-f)$, and $\phi(t)$, $\psi(t)$ are real.

Note: If these constraints are satisfied, g_k, h_k on page 383 are also real.

13-G Vanish Moment Constraint

If $\psi(t)$ has p vanishing moments,

$$\int t^k \psi(t)\,dt = 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1$$

Since $$FT\left[t^k\right] = \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\delta(f)$$

from the Parseval's Theorem,

$$\int t^k \psi(t)\,dt = \int \Psi^*(f)\left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\delta(f)\,df = 0$$

$$\left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\Psi^*(f)\bigg|_{f=0} = 0$$

(Here, we use the fact that

$$\int x(u)\frac{d^k}{du^k}\delta(u-u_0)\,du = \frac{d^k}{du^k}x(u)\bigg|_{u=u_0}$$

general form of the sifting property)

Therefore, $\left.\dfrac{d^k}{df^k}\Psi^*(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$

Taking the conjugation on both sides, $\left.\dfrac{d^k}{df^k}\Psi(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$

Since $\Psi(f) = H\left(\dfrac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\dfrac{f}{2^q}\right)$

if $\boxed{\left.\dfrac{d^k}{df^k}H(f)\right|_{f=0} = 0}$ for $k = 0, 1, 2, \ldots, p-1$ is satisfied,

constraint 6

then $\left.\dfrac{d^k}{df^k}\Psi(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$ are satisfied

and the wavelet function has p vanishing moments.

13-H Orthogonality Constraints

- orthogonality constraint:

$$\boxed{\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)}$$

$\psi(t)$: wavelet function

If the above equality is satisfied,

forward wavelet transform:

$$X_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

inverse wavelet transform:

$$x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w\left(m,n\right)$$

(much easier for inverse) $\qquad$ C = mean of $x(t)$

(證明於後頁)

If $$x(t) = C + \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} 2^{m/2}\psi\left(2^m t - n\right)X_w(m,n)$$

and $$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right)\psi\left(2^m t - n\right)dt = \delta(m - m_1)\delta(n - n_1),$$

then $$2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$$

$$= 2^{m/2}\int_{-\infty}^{\infty}\left[C + \sum_{m_1=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} 2^{m_1/2}\psi\left(2^{m_1} t - n_1\right)X_w(m_1,n_1)\right]\psi(2^m t - n)dt$$

$$= 2^{m/2}\int_{-\infty}^{\infty} C\psi(2^m t - n)dt + 2^{m/2}\sum_{m_1=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} 2^{m_1/2}\int_{-\infty}^{\infty}\psi\left(2^{m_1} t - n_1\right)\psi(2^m t - n)dt\, X_w(m_1,n_1)$$

$$= 0 + \sum_{m_1=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty}\delta(m_1 - m)\delta(n_1 - n)X_w(m_1,n_1)$$

$$= X_w(m,n)$$

due to $$\int_{-\infty}^{\infty}\psi(t)dt = 0$$

Therefore, $2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t-n)dt$ is the inverse operation of

$$C+\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} 2^{m/2}\psi\left(2^m t-n\right)X_w\left(m,n\right) \qquad \#$$

※ 要滿足

$$\boxed{\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m_1}t-n_1\right)\psi\left(2^{m}t-n\right)dt=\delta\left(m-m_1\right)\delta\left(n-n_1\right)}$$

之前，需要滿足以下三個條件

(1) $\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\psi\left(t-n\right)dt=\delta\left(n_1-n\right)$ for mother wavelet

這個條件若滿足， $\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m}t-n_1\right)\psi\left(2^{m}t-n\right)dt=\delta\left(n-n_1\right)$

對所有的 m 皆成立

(2) $\int_{-\infty}^{\infty}\phi\left(t-n_1\right)\phi\left(t-n\right)dt=\delta\left(n_1-n\right)$ for scaling function

嚴格來說，這並不是必要條件，但是可以簡化 第 (3) 個條件的計算

(3) $\int_{-\infty}^{\infty} \psi(t-n_1)\psi\left(2^{-k}t-n\right)dt = 0$ for any n, n_1 if $k > 0$

若 (1) 和 (3) 的條件滿足，則

$$\boxed{\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m_1}t-n_1\right)\psi\left(2^{m}t-n\right)dt = \delta(m-m_1)\delta(n-n_1)}$$

也將滿足

(Proof): Set $t_1 = 2^{m}t$, $dt_1 = 2^{m}dt$

$$\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m_1}t-n_1\right)\psi\left(2^{m}t-n\right)dt = \int_{-\infty}^{\infty} \psi\left(2^{m_1-m}t_1-n_1\right)\psi(t_1-n)dt_1$$

If (3) is satisfied,

$$\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m_1}t-n_1\right)\psi\left(2^{m}t-n\right)dt = 0 \quad \text{when } m \neq m_1$$

In the case where $m = m_1$, if (1) is satisfied, then

$$\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m}t-n_1\right)\psi\left(2^{m}t-n\right)dt = \int_{-\infty}^{\infty} \psi(t_1-n_1)\psi(t_1-n)dt_1 = \delta(n_1-n)$$

#

- 由 Page 396 的條件 (1)

$$\int_{-\infty}^{\infty}\psi(t-n_1)\psi(t-n)dt$$

$$=\int_{-\infty}^{\infty}e^{-j2\pi n_1 f}\Psi(f)e^{j2\pi n f}\Psi^*(f)df$$

Parseval's theorem

$$\int_{-\infty}^{\infty}x(t)y^*(t)dt=\int_{-\infty}^{\infty}X(f)Y^*(f)df$$

$$=\int_{-\infty}^{\infty}e^{j2\pi(n-n_1)f}\Psi(f)\Psi^*(f)df$$

$$=\sum_{p=-\infty}^{\infty}\int_0^1 e^{j2\pi(n-n_1)(f+p)}\Psi(f+p)\Psi^*(f+p)df$$

$$e^{j2\pi(n-n_1)(f+p)}=e^{j2\pi(n-n_1)f}$$

if p is an integer

$$=\int_0^1 e^{j2\pi(n-n_1)f}\sum_{p=-\infty}^{\infty}|\Psi(f+p)|^2 df=\delta(n-n_1)$$

Therefore,

$$\int_0^1 e^{-j2\pi n_2 f}\sum_{p=-\infty}^{\infty}|\Psi(f+p)|^2 df=\delta(-n_2)=\delta(n_2)$$

$$\boxed{\sum_{p=-\infty}^{\infty}|\Psi(f+p)|^2=1}$$

for all f should be satisfied

- 同理，由 Page 396 的條件 (2)

$$\int_{-\infty}^{\infty}\phi(t-n_1)\phi(t-n)dt=\delta(n_1-n)$$ for scaling function

$\big\downarrow$ 推導過程類似 page 398

$$\boxed{\sum_{p=-\infty}^{\infty}|\Phi(f+p)|^2=1}$$ for all f should be satisfied

衍生的條件：將 $\Psi(f)=H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ 代入 $\sum_{p=-\infty}^{\infty}|\Psi(f+p)|^2=1$

(page 398)

$$\sum_{p=-\infty}^{\infty}\left|H\left(\frac{f}{2}+\frac{p}{2}\right)\Phi\left(\frac{f}{2}+\frac{p}{2}\right)\right|^2=1$$

$$\sum_{q=-\infty}^{\infty}\left|H\left(\frac{f}{2}+q\right)\Phi\left(\frac{f}{2}+q\right)\right|^2+\sum_{q=-\infty}^{\infty}\left|H\left(\frac{f}{2}+q+\frac{1}{2}\right)\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2=1$$

因為 h_k 是 discrete sequence, $H(f)$ 是 h_k 的 discrete-time Fourier transform

$$H(f)=H(f+1)=H(f+2)=\cdots\cdots$$

$$\left|H\left(\frac{f}{2}\right)\right|^2\sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q\right)\right|^2+\left|H\left(\frac{f}{2}+\frac{1}{2}\right)\right|^2\sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2=1$$

$$|H\left(\frac{f}{2}\right)|^2 \sum_{q=-\infty}^{\infty} |\Phi\left(\frac{f}{2}+q\right)|^2 + |H\left(\frac{f}{2}+\frac{1}{2}\right)|^2 \sum_{q=-\infty}^{\infty} |\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)|^2 = 1$$

因為 $\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1$ for all f

(page 398 的條件)

$$|H\left(\frac{f}{2}\right)|^2 + |H\left(\frac{f}{2}+\frac{1}{2}\right)|^2 = 1$$

$$\boxed{|H(f)|^2 + |H\left(f+\frac{1}{2}\right)|^2 = 1}$$

constraint 7

同理，將 $\Phi(f)=G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ 代入 $\sum_{p=-\infty}^{\infty}|\Phi(f+p)|^2=1$

(page 398)

經過運算可得

$$\boxed{|G(f)|^2+\left|G\left(f+\frac{1}{2}\right)\right|^2=1}$$

constraint 8

- Page 397 條件 (3) 的處理

由於

$\psi\left(2^{-k}t-n\right)$ 是 $\phi\left(2^{-k+1}t-n_1\right)$ 的 linear combination

$\phi\left(2^{-k+1}t-n_1\right)$ 是 $\phi\left(2^{-k+2}t-n_2\right)$ 的 linear combination

$\phi\left(2^{-k+2}t-n_2\right)$ 是 $\phi\left(2^{-k+3}t-n_3\right)$ 的 linear combination

$\vdots$

$\vdots$

$\phi\left(2^{-1}t-n_{k-1}\right)$ 是 $\phi\left(t-n_k\right)$ 的 linear combination

$$\psi(t)=2\sum_k h_k\phi(2t-k)$$

$$\phi(t)=2\sum_k g_k\phi(2t-k)$$

所以

$\psi\left(2^{-k}t-n\right)$ 必定可以表示成 $\phi\left(t-n_k\right)$ 的 linear combination

$$\psi\left(2^{-k}t-n\right)=\sum_{n_k}b_{n_k}\phi\left(t-n_k\right)$$

$$\psi\left(2^{-k}t-n\right)=\sum_{n_k} b_{n_k}\phi(t-n_k)$$

所以，若 $\int_{-\infty}^{\infty}\psi(t-n_1)\phi(t-n_k)dt=0$ for any n_1, n_k 可以滿足

則 $\int_{-\infty}^{\infty}\psi(t-n_1)\psi\left(2^{-k}t-n\right)dt=0$ for any n_1, n_k 必定能夠成立

Page 397 條件 (3) 可改寫成

$$\boxed{\int_{-\infty}^{\infty}\psi(t-n_1)\phi(t-n_k)dt=0}$$

$$\int_{-\infty}^{\infty}\psi(t)\phi(t-\tau)dt=0$$ (將 $t-n_1$ 變成 t, $\tau = n_k - n_1$)

$$\int_{-\infty}^{\infty}\Psi(f)\Phi^*(f)e^{j2\pi\tau f}\,df=0$$ (from Parseval’s theorem)

$$\int_{-\infty}^{\infty} \Psi(f)\Phi^*(f)e^{j2\pi\tau f}\,df = 0$$

Since $\Psi(f) = H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ $\quad \Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$

$$\int_{-\infty}^{\infty} H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\left|\Phi\left(\frac{f}{2}\right)\right|^2 e^{j2\pi\tau f}\,df = 0$$

$$\sum_{p=-\infty}^{\infty}\int_0^1 H\left(\frac{f+p}{2}\right)G^*\left(\frac{f+p}{2}\right)\left|\Phi\left(\frac{f+p}{2}\right)\right|^2 e^{j2\pi\tau(f+p)}\,df = 0$$

$e^{j2\pi\tau(f+p)} = e^{j2\pi\tau f}$ (since from page 404, τ is an integer)

$$\sum_{q=-\infty}^{\infty}\int_0^1 H\left(\frac{f}{2}+q\right)G^*\left(\frac{f}{2}+q\right)\left|\Phi\left(\frac{f}{2}+q\right)\right|^2 e^{j2\pi\tau f}\,df$$

$$+\sum_{q=-\infty}^{\infty}\int_0^1 H\left(\frac{f}{2}+q+\frac{1}{2}\right)G^*\left(\frac{f}{2}+q+\frac{1}{2}\right)\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 e^{j2\pi\tau f}\,df = 0$$

Since $H(f) = H(f+1) = H(f+2) = \cdots\cdots$

$$G(f) = G(f+1) = G(f+2) = \cdots\cdots$$

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\int_0^1 \sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q\right)\right|^2 e^{j2\pi\tau f}\,df$$

$$+H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right)\int_0^1 \sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 e^{j2\pi\tau f}\,df = 0$$

Since $\sum_{p=-\infty}^{\infty}|\Phi(f+p)|^2 = 1$ for all f (page 398)

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)+H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right)=0$$

$$\boxed{H(f)G^*(f)+H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right)=0}$$

constraint 9

13-I Nine Constraints

整理： 設計 mother wavelet 和 scaling function 的九大條件
(皆由 page 382 的 constraints 衍生而來)

(1) $\Phi(f)=\prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm , page 388

(2) $\Psi(f)=H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm , page 389

(3) $G(0)=1$ for fast algorithm , page 389

(4) $H(f)=H^*(-f)$ for real , page 390

(5) $G(f)=G^*(-f)$ for real , page 389

(6) $\left.\frac{d^k}{df^k}H(f)\right|_{f=0}=0$ for p vanish moments , page 392

for $k = 0, 1, \ldots, p$-1

(7) $|H(f)|^2 + \left|H\left(f+\frac{1}{2}\right)\right|^2 = 1$ for orthogonal , page 401

(8) $|G(f)|^2 + \left|G\left(f+\frac{1}{2}\right)\right|^2 = 1$ for orthogonal , page 402

(9) $H(f)G^*(f) + H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right) = 0$ for orthogonal , page 406

- 條件的簡化

有時，令

$$H(f)=-e^{-j2\pi f}G^*(f+1/2) \qquad h_k=(-1)^k g_{1-k}$$

此時，若 $|G(f)|^2+\left|G\left(f+\frac{1}{2}\right)\right|^2=1$

$$G(f)=G^*(-f) \qquad \text{(條件 (5), (8) 滿足)}$$

則 $|H(f)|^2+\left|H\left(f+\frac{1}{2}\right)\right|^2=\left|G\left(f+\frac{1}{2}\right)\right|^2+|G(f)|^2=1$

$$H(f)G^*(f)+H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right)$$

$$=-e^{-j2\pi f}G^*\left(f+\frac{1}{2}\right)G^*(f)-e^{-j2\pi(f+\frac{1}{2})}G^*(f)G^*\left(f+\frac{1}{2}\right)$$

$$=-e^{-j2\pi f}G^*\left(f+\frac{1}{2}\right)G^*(f)+e^{-j2\pi f}G^*(f)G^*\left(f+\frac{1}{2}\right)=0$$

$$H^*(-f)=-e^{-j2\pi f}G(-f+1/2)=-e^{-j2\pi f}G^*(f-1/2)=H(f)$$

條件 (4), (7), (9) 也將滿足

整理： 設計 mother wavelet 和 scaling function 的幾個要求 (簡化版)

(1) $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm

(2) $\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm

(3) $G(0) = 1$ for fast algorithm

(4) $G(f) = G^*(-f)$ for real

(5) $\left.\frac{d^k}{df^k}H(f)\right|_{f=0} = 0$ for p vanish moments

for $k = 0, 1, \ldots, p$-1

(6) $|G(f)|^2 + \left|G\left(f+\frac{1}{2}\right)\right|^2 = 1$ for orthogonal

(7) $H(f) = -e^{-j2\pi f}G^*(f+1/2)$

13-J Design Process

設計時，只要 $G(f)$ $(0 \leq f \leq 1/4)$ 決定了，mother wavelet 和scaling function 皆可決定

$G(f)$: 被稱作 generating function

Design Process (設計流程):

(Step 1): 給定 $G(f)$ $(0 \leq f \leq 1/4)$，滿足以下的條件

(a) $G(0) = 1$

(b) $\left.\dfrac{d^k}{df^k} G(f)\right|_{f=\frac{1}{2}} = 0$ for $k = 0, 1, 2, \ldots, p\text{-}1$

(Step 2) 由 $G(f) = G^*(-f)$ 決定$G(f)$ $(3/4 \le f < 1)$

(Step 3) 由 $|G(f)|^2 + |G\left(f + \frac{1}{2}\right)|^2 = 1$ 決定$G(f)$ $(1/4 < f < 3/4)$

再根據 $G(f) = G(f+1)$，決定所有的 $G(f)$ 值

(Step 4) 由 $H(f) = -e^{-j2\pi f} G^*(f + 1/2)$ 決定$H(f)$

(Step 5) 由 $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$

$$\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$ 決定$\Phi(f)$, $\Psi(f)$

註： (1) 當 Step 1 的兩個條件滿足，由於 $|G(f)|^2+|G(f+1/2)|^2=1$

$$\left.\frac{d^k}{df^k}G(f)\right|_{f=1/2}=0 \qquad \text{for } k=0,1,2,\ldots,p\text{-}1$$

又由於 $H(f)=-e^{-j2\pi f}G^*(f+1/2)$

$$\left.\frac{d^k}{df^k}H(f)\right|_{f=0}=0 \qquad \text{for } k=0,1,2,\ldots,p\text{-}1$$

(2) $\left|G(f)\right|^2+\left|G(f+1/2)\right|^2=1 \qquad |G(f)|^2=|G(-f)|^2$

所以當 $G(f)$ $(0\leq f\leq 1/4)$ 給定，$|G(f)|$ 有唯一解

13-K Several Continuous Wavelets with Discrete Coefficients

(1) Haar Wavelet

$g[0] = 1,\ g[1] = 1$ $\qquad G(f) = 1 + \exp(-j2\pi f)$

$h[0] = 1,\ h[1] = -1$ $\qquad H(f) = 1 - \exp(-j2\pi f)$

或

$g[0] = 1/2,\ g[1] = 1/2$ $\qquad G(f) = [1 + \exp(-j2\pi f)]/2$

$h[0] = 1/2,\ h[1] = -1/2$ $\qquad H(f) = [1 - \exp(-j2\pi f)]/2$

vanish moment = ?

(2) Sinc Wavelet

$$G(f) = 1 \qquad \text{for } |f| \leq 1/4$$

$$G(f) = 0 \qquad \text{otherwise}$$

vanish moment = ?

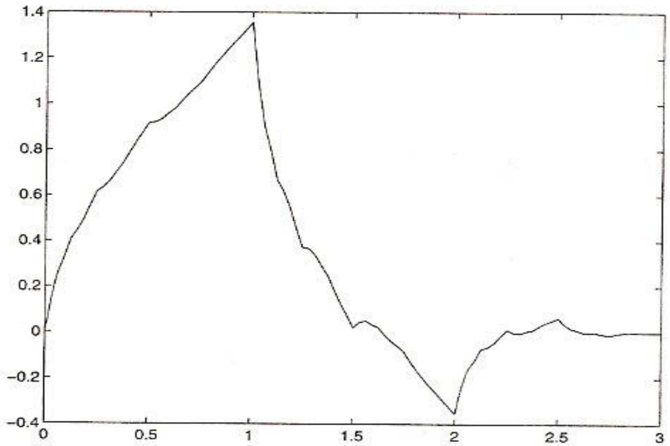
(3) 4-point Daubechies Wavelet

$$g_k : \left[\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8} \right]$$

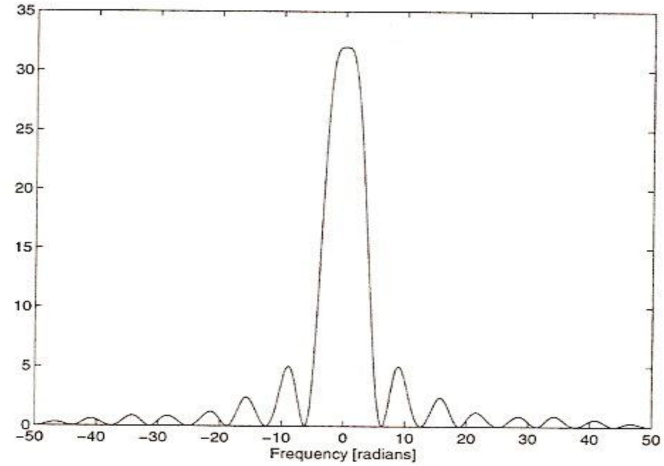
vanish moment = ?

<u>vanish moment</u> VS <u>the number of coefficients</u>

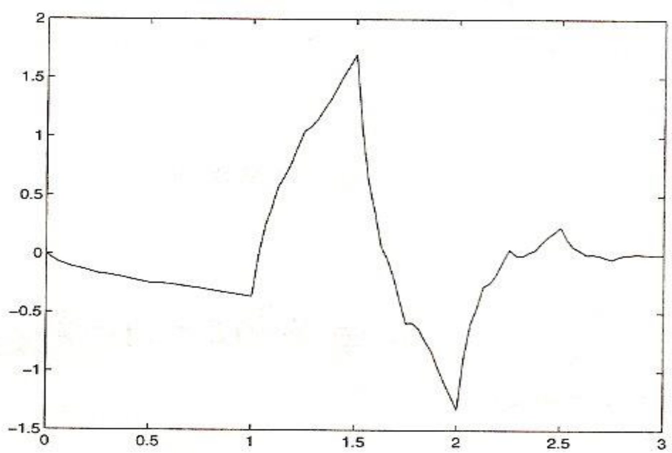
From: S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Prentice Hall, N.J., 1996.



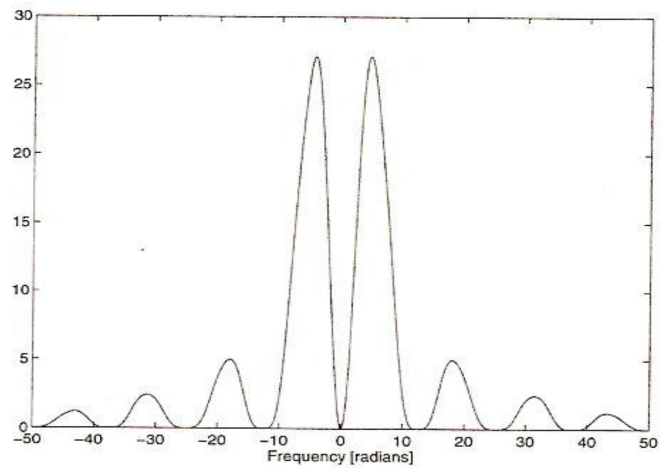
(a) Scaling function $\phi(t)$



(b) $|\Phi(\omega)|$



(c) Daubechies wavelet $\psi(t)$



(d) $|\Psi(\omega)|$

13-L Continuous Wavelet with Discrete Coefficients 優缺點

- Advantages:

(1) Fast algorithm for MRA

(2) Non-uniform frequency analysis

$$\psi\left(2^m t - n\right) \xrightarrow{\text{FT}} 2^{-m} e^{-j2\pi n 2^{-m} f}\,\Psi\left(2^{-m} f\right)$$

(3) Orthogonal

- Disadvantages:

(a) 無限多項連乘

(b) problem of initial

$$\chi_w(n,m),\quad X_w(n,m)\ \text{皆由}\ \chi_w(n,m+1)\ \text{算出}$$

$$\chi_w(n,m)\big|_{m\to\infty}\quad \text{如何算}$$

(c) 難以保證 compact support

(d) 仍然太複雜

附錄十三 幾種常見的影像壓縮格式

(1) JPEG: 使用 discrete cosine transform (DCT) 和 8×8 blocks
是當前最常用的壓縮格式 (副檔名為 *.jpg 的圖檔都是用 JPEG 來壓縮)

可將圖檔資料量壓縮至原來的 1/8 (對灰階影像而言) 或 1/16 (對彩色影像而言)

(2) JPEG2000: 使用 discrete wavelet transform (DWT)
壓縮率是 JPEG 的 5 倍左右

(3) JPEG-LS: 是一種 lossless compression
壓縮率較低，但是可以完全重建原來的影像

(4) JPEG2000-LS: 是 JPEF2000 的 lossless compression 版本

(5) JBIG: 針對 bi-level image (非黑即白的影像) 設計的壓縮格式

(6) GIF: 使用 LZW (Lempel–Ziv–Welch) algorithm (類似字典的建構)

適合卡通圖案和動畫製作，lossless

(7) PNG: 使用 LZ77 algorithm (類似字典的建構，並使用 sliding window)

lossless

(8) JPEG XR (又稱 HD Photo): 使用 Integer DCT，lossless

在 lossy compression 的情形下壓縮率可和 JPEG 2000 差不多

(9) TIFF: 使用標籤，最初是為圖形的印刷和掃描而設計的，lossless