# 整數與組合最佳化(Integer and Combinatorial Optimization)
Homework 01 – 2017/03/16

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1. Consider the following LP:

$$
\begin{array}{rrcrcr}
\max & 4x_1 & + & 3x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \leq & 12 \\
 & -x_1 & + & 2x_2 & \leq & 4 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

(a) Solve the problem graphically and indicate each basic feasible solution.

(b) Solve the problem using the primal simplex method.

(c) Associate each tableau with basic feasible solution in (a)

(d) The dual model of the problem and draw the dual problem graphically.

(a) Show as the Figure 1. We can get the solution $z^* = 28$ at $(x_1, x_2) = (4, 4)$, and the basic feasible solution are $(0,0)$, $(0,2)$, $(6,0)$ and $(4,4)$.

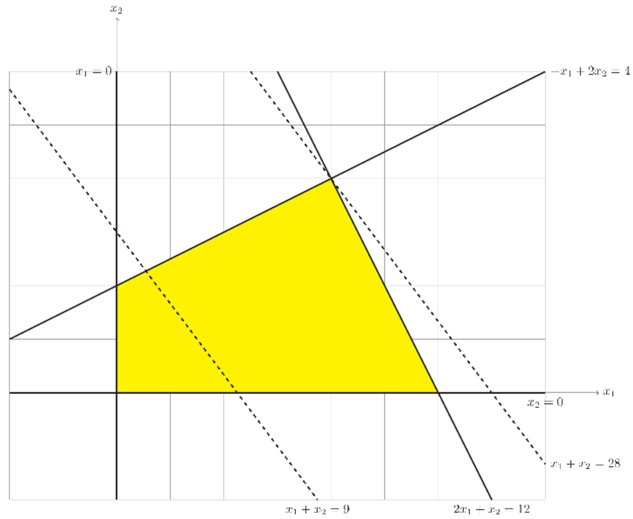


Figure 1

(b) Rewrite the problem:

$$
\begin{array}{rrcrcrcrcrcr}
\max & z & = & 4x_1 & + & 3x_2 & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & & & & & = & 12 \\
 & -x_1 & + & 2x_2 & & & + & x_4 & & & = & 4
\end{array}
$$

$(x_1, x_2) = (0, 0)$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| $z$ | $-4$ | $-3$ | 0 | 0 | 0 |
| $x_3$ | 2 | 1 | 1 | 0 | 12 |
| $x_4$ | $-1$ | 0 | 0 | 1 | 4 |

$(x_1, x_2) = (6, 0)$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| $z$ | 0 | $-1$ | 2 | 0 | 24 |
| $x_1$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 6 |
| $x_4$ | 0 | $\frac{5}{2}$ | $\frac{1}{2}$ | 1 | 10 |

$(x_1, x_2) = (4, 4)$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| $z$ | 0 | 0 | $\frac{11}{5}$ | $\frac{2}{5}$ | 28 |
| $x_1$ | 1 | 0 | $\frac{2}{5}$ | $-\frac{1}{5}$ | 4 |
| $x_2$ | 0 | 1 | $\frac{1}{5}$ | $\frac{2}{5}$ | 4 |

(c) Show above.

(d) Consider the problem below and the Figure 2.

$$
\begin{array}{rrcrcr}
\max & w & = & 12y_1 & + & 4y_2 \\
\text{s.t.} & 2y_1 & - & y_2 & \geq & 4 \\
 & y_1 & + & 2y_2 & \geq & 3 \\
 & & & y_1 & \geq & 0 \\
 & & & y_2 & \geq & 0
\end{array}
$$

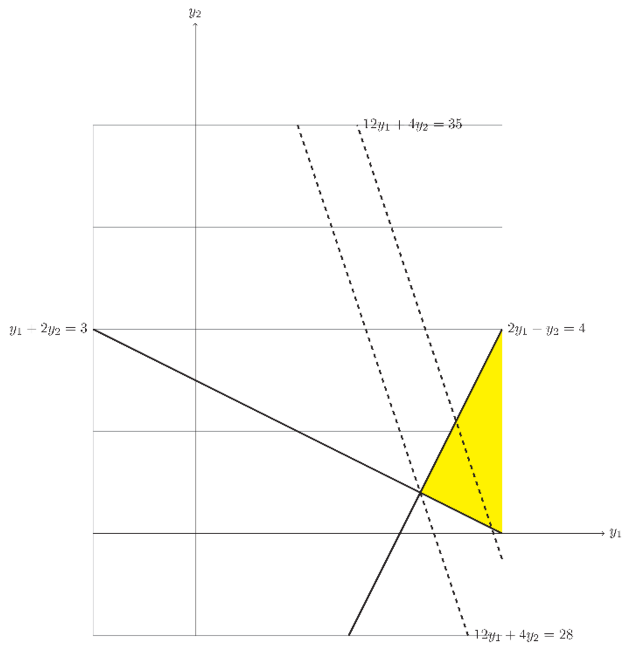


Figure 2

We can get the solution $w^* = 28$ at $(y_1, y_2) = (\frac{11}{5}, \frac{2}{5})$

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2. Given the following LP:

$$
\begin{array}{lrcrcr}
\max & 4x_1 & + & 2x_2 & & \\
\text{s.t.} & 2x_1 & & & \leq & 16 \\
 & x_1 & + & 3x_2 & \leq & 17 \\
 & & & x_2 & \leq & 5 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

(a) Solve the problem graphically

(b) Determine how many additional units of resource 1 (constraint 1) would be needed to increase the optimal value by 15. Justify your answer.

(1) Show as the Figure 3.

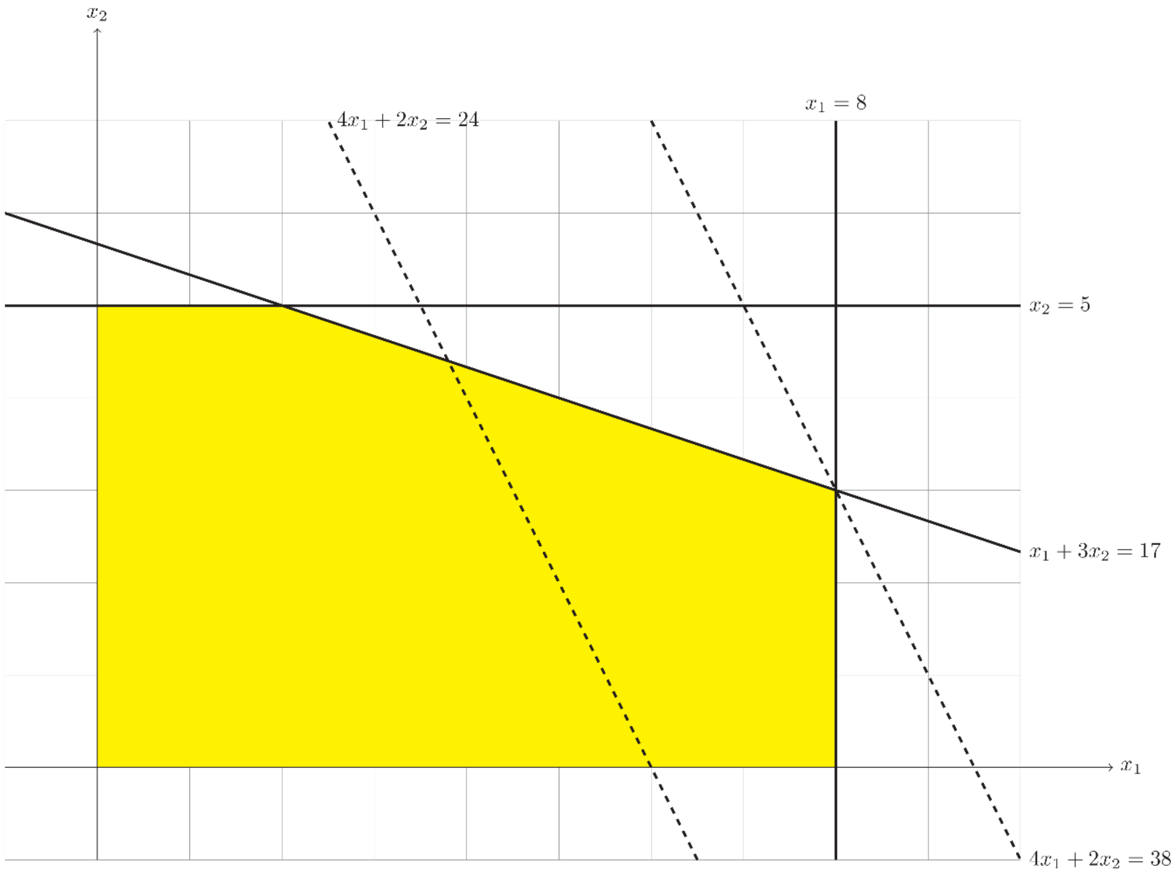


Figure 3

We can get the solution $z^* = 38$ at $(x_1, x_2) = (8, 3)$

(2) While adding 1 unit of resource 1

$$
\begin{cases} 2x_1 & = 17 \\ x_1 + 3x_2 & = 17 \end{cases} \implies x_1 = \frac{17}{2}, x_2 = \frac{17}{6} \implies z' = \frac{119}{3} \implies \Delta z = z' - z = \frac{5}{3}
$$

Then $\implies 15 \div \frac{5}{3} = 9$ units. Trying to add 9 units of resource 1:

$$
\begin{cases} 2x_1 & = 25 \\ x_1 + 3x_2 & = 17 \end{cases} \implies x_1 = \frac{25}{2}, x_2 = \frac{3}{2} \implies z' = 53 \implies \Delta z = z' - z = 15
$$

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3. Show the feasible region of a LP model, $\{A\mathbf{X} \le \mathbf{0}, \mathbf{X} \ge \mathbf{0}\}$ is convex

**Definition of Convex Set**

Let $u, v \in V$. Then the set of all convex combinations of $u$ and $v$ is the set of points

$$\{w_\lambda \in V : w_\lambda = (1-\lambda)u + \lambda v, 0 \le \lambda \le 1\}$$

Let $x_1, x_2 \in A\mathbf{X}$ and $0 \le \lambda \le 1$, $\exists\, t_1, t_2 \in x$ such that $x_1 = A \cdot t_1$ and $x_2 = A \cdot t_2$

Then

$$\lambda x_1 + (1-\lambda)x_2 = \lambda(A \cdot t_1) + (1-\lambda)(A \cdot t_2) = A[\lambda t_1 + (1-\lambda)t_2]$$

Since $t_1, t_2 \in x$ and $x$ is **Convex**, $A[\lambda t_1 + (1-\lambda)t_2] \in A\mathbf{X}$.

Q.E.D

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4. A caterer to "The Ritz" motel collects the dirty napkins and sends them to laundry every day. Due to different room occupation levels during a week, the number of napkins needed on day $i$ is $d_i$. The caterer can wash and dry at most $u$ napkins every day and the cleaned napkins will be ready for use next day. If the dirty napkin is not cleaned, a new one is purchased at the price of $p$. If the laundry room is used on day $i$, a fixed cost of $f_i$ is incurred. Assume that at the beginning of a week, there are $n$ clean napkins and no dirty napkins left. Find the best laundry plan for the caterer so that the entire week's cost is minimized.

Assume that

- ✓ Number of napkins buy on day $i$ is $x_i$
- ✓ Number of napkins washed on day $i$ is $w_i$
- ✓ $y_i = 1$ means laundry room is used on day $i$ ($y_i = 0$ means laundry room is not used on day $i$).

Consider the Model below.

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{7}(f_i \cdot y_i + p \cdot x_i) \\
\text{s.t.} \quad & w_i \le u, y_i \\
& w_i = \min(u_i, d_i) \\
& w_{i-1} + x_i \ge d_i \\
& x_i = \max(0, d_i - w_{i-1}) \\
& n + x_1 \le w_1 \\
& n + x_1 \ge d_1 \\
& w_i \le M \cdot y_i \\
& f_i, y_i, x_i, p, u_i, d_i, w_i \ge 0
\end{aligned}
$$

5. Assume that you have spare saving $\$10,000$ each year. There are three investment tools you can choose from:

(1) deposit

(2) mutual funds

(3) bonds

The annual interest rate is $2\%$ if you deposit your money in a bank. If you buy mutual funds, the investment length is two years and the return is estimated to be $7\%$ after two years. If you invest in bonds, you can get $4\%$ of interest payment every year but the investment length is $4$ years. At the end of year, you will re-invest all your available money and renew your portfolio. In addition, you are advised to deposit at least $30\%$ of your available money in a bank and the amount of money invested in mutual funds not greater than twice of the amount invested in bonds throughout the entire investment period. Please formulate a mathematical model to maximize your money at the end of the fifth year. (At the beginning you already have $\$10,000$ and the investment length is five years.)

Assume that

- ✓ $x_i$ money deposit at the $i$ year
- ✓ $y_i$ money go on mutual funds at $i$ year
- ✓ $z_i$ money go on bonds at $i$ year
- ✓ $p_i$ money at $i$ year

$\max \quad p_5$

$\text{s.t.}$

$$p_1 = 10000$$
$$x_1 + y_1 + z_1 \leq 10000$$
$$\frac{x_1}{10000} \geq 0.3$$
$$\frac{y_1}{z_1} \leq 2$$
$$p_2 = 10000 + 1.02x_1 + 0.04z_1$$
$$x_2 + y_2 + z_2 \leq p_2$$
$$\frac{x_2}{p_2} \geq 0.3$$
$$\frac{y_1}{z_2} \leq 2$$
$$p_3 = 10000 + 1.02x_2 + 1.07y_1 + 0.04z_2$$
$$x_3 + y_3 + z_3 \leq p_3$$
$$\frac{x_3}{p_3} \geq 0.3$$
$$\frac{y_3}{z_3} \leq 2$$
$$p_4 = 10000 + 1.02x_3 + 1.07y_2 + 0.04z_1$$
$$x_4 + y_4 + z_4 \leq p_4$$
$$\frac{x_4}{p_4} \geq 0.3$$
$$\frac{y_4}{z_4} \leq 2$$
$$p_5 = 1.05x_5 + 1.07y_4 + 1.04z_2$$
$$x_i, y_i, z_i \geq 0$$