Lecture 3 Duality Theory

1. Finding upper bounds on the optimal objective value of maximization problems

Consider the following maximization problem.

$$\begin{aligned} maxmize \quad & 4x_1 + \ x_2 + 5x_3 + 3x_4 \\ subject\ to \quad & x_1 - \ x_2 - \ x_3 + 3x_4 \leq 1 \\ & 5x_1 + \ x_2 + 3x_3 + 8x_4 \leq 55 \\ & -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

To get a reasonably good **lower bound** on the objective function value, we need only come up with a reasonably good feasible solution.

What is **upper bound** on the optimal objective function value?

Can we push down the lower bound any further?

Motivation of the dual problem

Rather than searching for further improvements in a haphazard way, we shall now describe the strategy in precise and general terms. We construct *linear combinations* of the constraints. That is, we multiply the first constraint by some number y_1, the second by y_2, the third by y_3, and then we add them up. (In the first case, we had $y_1 = 0$, $y_2 = 5/3$, $y_3 = 0$; in the second case, we had $y_1 = 0$, $y_2 = y_3 = 1$.) The resulting inequality reads

$$\begin{aligned} x_1 - \quad x_2 - \quad x_3 + 3x_4 &\leq 1 \\ 5x_1 + \quad x_2 + 3x_3 + 8x_4 &\leq 55 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 &\leq 3 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Of course, we want as small an upper bound on z* as we can possibly get. Thus, we are led to the following LP problem:

$$\begin{aligned}
\mathit{minimize} \quad & y_1 + 55y_2 - 3y_3 \\
\mathit{subject\ to} \quad & y_1 + 5y_2 - \ \ y_3 \geq 4 \\
& -y_1 + \ \ y_2 + 2y_3 \geq 1 \\
& -y_1 + 3y_2 + 3x_3 \geq 5 \\
& 3y_1 + 8y_2 - 5y_3 \geq 3 \\
& y_1, y_2, y_3 \geq 0.
\end{aligned}$$

2. The dual problem

This problem is called the *dual* of the original one; the original problem is called the *primal* problem. In general, the dual of the problem

$$\mathit{maxmize} \sum_{j=1}^{n} c_j x_j$$

$$\mathit{subject\ to} \ \textstyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \dots, m) \tag{2}$$

$$x_j \geq 0 \qquad (j = 1, 2, \dots, n).$$

is defined to be the problem

$$\mathit{minimize} \sum_{i=1}^{m} b_i y_i$$

$$\mathit{subject\ to} \ \textstyle\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \dots, n) \tag{3}$$

$$y_i \geq 0 \qquad (i = 1, 2, \dots, m).$$

(Note that the dual of a maximization problem is a minimization problem. Furthermore, the *m* primal constraints $\sum a_{ij} x_j \leq b_i$ are in a one-to-one correspondence with the *m* dual variable y_i; conversely, the *n* dual constraints $\sum a_{ij} y_i \geq c_j$ are in a one-to-one correspondence with the *n* primal variables x_j. The coefficient at each variable in the objective function, primal or dual, appears in the other problem as the right-hand side of the corresponding constraint.)

Recall the example

$$
\begin{aligned}
maxmize \quad & 4x_1 + x_2 + 5x_3 + 3x_4 \\
subject\ to \quad & x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

Theorem: Weak duality

For every primal feasible solution (x_1, …, x_n) and for every dual feasible solution (y_1, …, y_m), we have

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

Proof:

Theorem: Strong duality

If the primal problem has an optimal solution (x_1*, …, x_n*), then the dual problem has an optimal solution (y_1*, …, y_m*) such that

$$\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i. \tag{4}$$

Proof: