Lecture 5 General LP Problems

1. How to handle explicit bounds on individual variables

Many linear programming problems involve explicit upper bounds on individual variables. For instance one might be confronted with the problem

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \cdots, m) \\ &&& 0 \le x_j \le u_j \qquad (j = 1, 2, \cdots, n) \end{aligned}$$

such that each u_j is a positive number. This problem may be cast in the standard form

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \cdots, m) \\ &&& x_j \le u_j \qquad (j = 1, 2, \cdots, n) \\ &&& x_j \ge 0 \qquad (j = 1, 2, \cdots, n) \end{aligned}$$

The above standard form can be solved by the simplex method. However, the upper bound or lower bound on individual variables may be viewed as some special conditions when pivoting rather than adding constraints. By doing so, we may save some computational time. Let us see the following example.

Consider

$$\begin{aligned} &\text{maximize} && 3x_1 + 5x_2 + 2x_3 \\ &\text{subject to} && x_1 + x_2 + 2x_3 \le 14 \\ &&& 2x_1 + 4x_2 + 3x_3 \le 43 \\ &&& 0 \le x_1 \le 4 \\ &&& 7 \le x_2 \le 10 \\ &&& 0 \le x_3 \le 3 \end{aligned}$$

You can convert it into the standard form

$$\begin{array}{ll} maximize & 3x_1 + 5x_2 + 2x_3 \\ subject\ to & x_1 + x_2 + 2x_3 \leq 14 \\ & 2x_1 + 4x_2 + 3x_3 \leq 43 \\ & x_1 \leq 4 \\ & -x_2 \leq -7 \\ & x_2 \leq 10 \\ & x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Is there any way to compute it efficiently?

$$\begin{array}{ll} maximize & 3x_1 + 5x_2 + 2x_3 \\ subject\ to & x_1 + x_2 + 2x_3 + x_4 = 14 \\ & 2x_1 + 4x_2 + 3x_3 + x_5 = 43 \\ & 0 \leq x_1 \leq 4 \\ & 7 \leq x_2 \leq 10 \\ & 0 \leq x_3 \leq 3 \\ & x_4, x_5 \geq 0 \end{array}$$

$$x_4 = 14 - x_1 - x_2 - 2x_3$$

$$x_5 = 43 - 2x_1 - 4x_2 - 3x_3$$

$$z = 3x_1 + 5x_2 + 2x_3$$

2. Dual-feasible dictionaries

Consider the primal problem.

$$\begin{aligned} &\mathit{maxmize} \quad 4x_1 - 13x_2 + 7x_3 \\ &\mathit{subject\ to} \quad 3x_1 + 2x_2 + 5x_3 \le 5 \\ &\qquad x_1 - 3x_2 + 2x_3 \le 3 \\ &\qquad x_1, x_2, x_3 \ge 0. \end{aligned}$$

Add the slack variables.

$$\begin{aligned} &\mathit{maxmize} \quad 4x_1 - 13x_2 + 7x_3 \\ &\mathit{subject\ to} \quad 3x_1 + 2x_2 + 5x_3 + x_4 \qquad = 5 \dots\dots\dots . y_1 \\ &\qquad x_1 - 3x_2 + 2x_3 \qquad + x_5 = 3 \dots\dots\dots . y_2 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \ge 0. \end{aligned}$$

Our objective is to point out a certain one-to-one correspondence between the dictionaries the primal and dual problems. In order to make this correspondence more transparent, we change the way to subscript the primal variables.

$$x_4 \to x_1, x_5 \to x_2, x_1 \to x_3, x_2 \to x_4, x_3 \to x_5,$$

$$\begin{aligned} &\mathit{maxmize} \quad 4x_3 - 13x_4 + 7x_5 \\ &\mathit{subject\ to} \quad 3x_3 + 2x_4 + 5x_5 \le 5 \\ &\qquad x_3 - 3x_4 + 2x_5 \le 3 \\ &\qquad x_3, x_4, x_5 \ge 0. \end{aligned}$$

$$\begin{aligned} &\mathit{maxmize} \quad 4x_3 - 13x_4 + 7x_5 \\ &\mathit{subject\ to} \quad 3x_3 + 2x_4 + 5x_5 + x_1 \qquad = 5 \dots\dots\ y_1 \\ &\qquad x_3 - 3x_4 + 2x_5 \qquad + x_2 = 3 \dots\dots y_2 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \ge 0. \end{aligned}$$

The dual problem is

$$\begin{aligned} &\mathit{minimize} \quad 5y_1 + 3y_2 \\ &\mathit{subject\ to} \quad 3y_1 + y_2 \ge 4 \\ &\qquad 2y_1 - 3y_2 \ge -13 \\ &\qquad 5y_1 + 2y_2 \ge 7 \\ &\qquad y_1, y_2 \ge 0. \end{aligned}$$

$$\begin{aligned} &\mathit{maximize} \quad -5y_1 - 3y_2 \\ &\mathit{subject\ to} \quad -3y_1 - y_2 + y_3 \qquad = -4 \\ &\qquad -2y_1 + 3y_2 \qquad + y_4 \qquad = 13 \\ &\qquad -5y_1 - 2y_2 \qquad + y_5 = -7 \\ &\qquad y_1, y_2, y_3, y_4, y_5 \ge 0. \end{aligned}$$

The initial dictionaries are

$$\begin{aligned} x_1 &= 5 - 3x_3 - 2x_4 - 5x_5 \\ x_2 &= 3 - x_3 + 3x_4 - 2x_5 \\ z &= \quad 4x_3 - 13x_4 + 7x_5 \end{aligned}$$

$$\begin{aligned} y_3 &= -4 + 3y_1 + y_2 \\ y_4 &= 13 + 2y_1 - 3y_2 \\ y_5 &= -7 + 5y_1 + 2y_2 \\ -w &= \quad -5y_1 - 3y_2 \end{aligned}$$

x_3 enters and x_2 leaves.

$$
\begin{aligned}
x_1 &= -\ 4 + 3x_2 - 11x_4 + x_5 \\
x_3 &= \ \ 3 - x_2 + 3x_4 - 2x_5 \\
z &= 12 - 4x_2 - x_4 - x_5
\end{aligned}
$$

By complementary slackness, $x_i y_i = 0$ for all $i = 1, \ldots 5$. y_3 leaves and y_2 enters. We have

$$
\begin{aligned}
y_2 &= \ \ 4 - 3y_1 + y_3 \\
y_4 &= \ \ 1 + 11y_1 - 3y_3 \\
y_5 &= \ \ 1 - y_1 + 2y_3 \\
-w &= -12 + 4y_1 - 3y_3
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
x_3 &= -\ 5 - 2x_1 + 5x_2 - 19x_4 \\
x_5 &= \ \ 4 + x_1 - 3x_2 + 11x_4 \\
z &= \ \ 8 - x_1 - x_2 - 12x_4
\end{aligned}
\quad \text{goes with} \quad
\begin{aligned}
y_1 &= \ \ 1 + 2y_3 - y_5 \\
y_2 &= \ \ 1 - 5y_3 + 3y_5 \\
y_4 &= 12 + 19y_3 - 11y_5 \\
-w &= -8 + 5y_3 - 4y_5
\end{aligned}
$$

In each of these pairs, the primal dictionary is a mirror image of the dual dictionary. The general representation is

$$
\begin{aligned}
x_r &= \overline{b_r} - \sum_{s \in N} \overline{a_{rs}}\, x_s \quad (r \in B) \\
z &= \bar{d} - \sum_{s \in N} \overline{c_s}\, x_s
\end{aligned}
$$

The corresponding dual dictionary is

$$
\begin{aligned}
y_s &= -\overline{c_s} + \sum_{r \in B} \overline{a_{rs}}\, y_r \quad (s \in N) \\
-w &= -\bar{d} - \sum_{r \in B} \overline{b_r}\, y_r
\end{aligned}
$$

The primal dictionary is feasible if $\overline{b_r} \geq 0$ for all $r \in \boldsymbol{B}$. Now we refer to the dual dictionary as dual-feasible if the corresponding dual dictionary is feasible. Thus, the dual dictionary is feasible if and only if $\overline{c_s} \leq 0$ for all $s \in \boldsymbol{N}$.

3. The dual simplex method

We have observed that solving an LP problem by the simplex method, we obtain a solution of its dual as a by-product. Vice versa, solving the dual we also solve the primal. This observation is useful for solving problems such as

$$
\begin{aligned}
&maxmize & -4x_1 - 8x_2 - 9x_3 & & \\
&subject\ to & 2x_1 - x_2 - x_3 &\le 1 & \\
& & 3x_1 - 4x_2 + x_3 &\le 3 & \qquad (1)\\
& & -5x_1 \qquad - 2x_3 &\le -8 & \\
& & x_1, x_2, x_3 &\ge 0. &
\end{aligned}
$$

The initial dictionary is

$$
\begin{aligned}
x_4 &= 1 - 2x_1 + x_2 + x_3 \\
x_5 &= 3 - 3x_1 + 4x_2 - x_3 \\
x_6 &= -8 + 5x_1 \qquad + 2x_3 \\
z &= \quad -4x_1 - 8x_2 - 9x_3
\end{aligned}
$$

Since this problem does not have feasible origin, the routine approach calls for the two-phase method. Nevertheless, we can avoid the two-phase method as soon as we realize that the dual of (1) does have feasible origin.

$$
\begin{aligned}
&minimize & y_1 + 3y_2 - 8y_3 & & \\
&subject\ to & 2y_1 + 3y_2 - 5y_3 &\ge -4 & \\
& & -y_1 - 4y_2 \qquad &\ge -8 & \qquad (2)\\
& & -y_1 + y_2 - 2y_3 &\ge -9 & \\
& & y_1, y_2, y_3 &\ge 0. &
\end{aligned}
$$

Hence we may simply solve the dual and then read the optimal primal solution of the final table for the dual. In this section, we shall discuss the *dual simplex method* of solving the dual without actually doing so. We shall first describe it as a mirror image of the simplex method and then we shall illustrate it on the presented example. We note that the dual simplex method is nothing but a disguised simplex method working on the dual. The dual simplex method produces a sequence of dual feasible dictionaries;

as soon as it finds one which is also primal feasible, the method terminates. In each iteration of the simplex method, we first choose the entering variable and then determine the leaving variable. For the entering variable, we may choose any nonbasic variable with a positive coefficient in the z-row. Then we determine the leaving variable so as to preserve primal feasibility in our next dictionary. On the other hand, in each iteration of the dual simplex method, we first choose the leaving variable and then determine the entering variable. For the leaving variable, we may choose any basic variable whose current value is most negative. Then we shall determine the entering variable so as to preserve dual feasibility in our next dictionary.

The standard form of the dual problem is

$$
\begin{aligned}
&maximize && -\ y_1 - 3y_2 + 8y_3 \\
&subject\ to && -2y_1 - 3y_2 + 5y_3 \le 4 \\
& && y_1 + 4y_2 \le 8 \\
& && y_1 - y_2 + 2y_3 \le 9 \\
& && y_1, y_2, y_3 \ge 0.
\end{aligned}
$$