Branch and Bound Enumeration

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Course No: 546 U6110

Agenda

- Geometric Interpretation
- Enumeration Tree
- A Variation of the Procedures
- Zero-One Problem
- Node Selection and Branching Rules.

Introduction

- Consider the following MILP problem:

(MILP) maximize z,

$$
\begin{aligned}
\text{Subject to}\quad & c^T x + d^T y - z = 0, \\
& A\,x + D\,y \le b, \\
& x \ge 0, \text{integer}, \\
& y \ge 0,
\end{aligned}
$$

where A is an m×n and D is an m×n’ matrix.

Geometric Interpretations (1/3)

Define

S={(x, y, z): $c^T x + d^T y - z = 0,\ A x + D y \leq b, x \geq 0, y \geq 0\}$.

(LP^0): maximize z,

subject to (x, y , z)∈ S.

x^ℓ : vector x with ℓ $(1 \leq \ell \leq n)$ of its components fixed at nonnegative integer value.

$$s^\ell = \{(x^\ell, y, z) : c^T x + d^T y - z = 0, A x^\ell + D y \leq b, x^\ell \geq 0, y \geq 0\}.$$

(LP^ℓ): maximize z,

subject to (x^ℓ, y , z)∈ S^ℓ.

Geometric Interpretations (2/3)

$z^{\ell} =$ ______________ for any MILP solution found from the point x^{ℓ}.

Initially, z^{ℓ} = optimal solution to LP^{ℓ}.

$z(\ell, k, t)$=optimal solution to LP^{ℓ} with the additional constraint $x_k = t$.

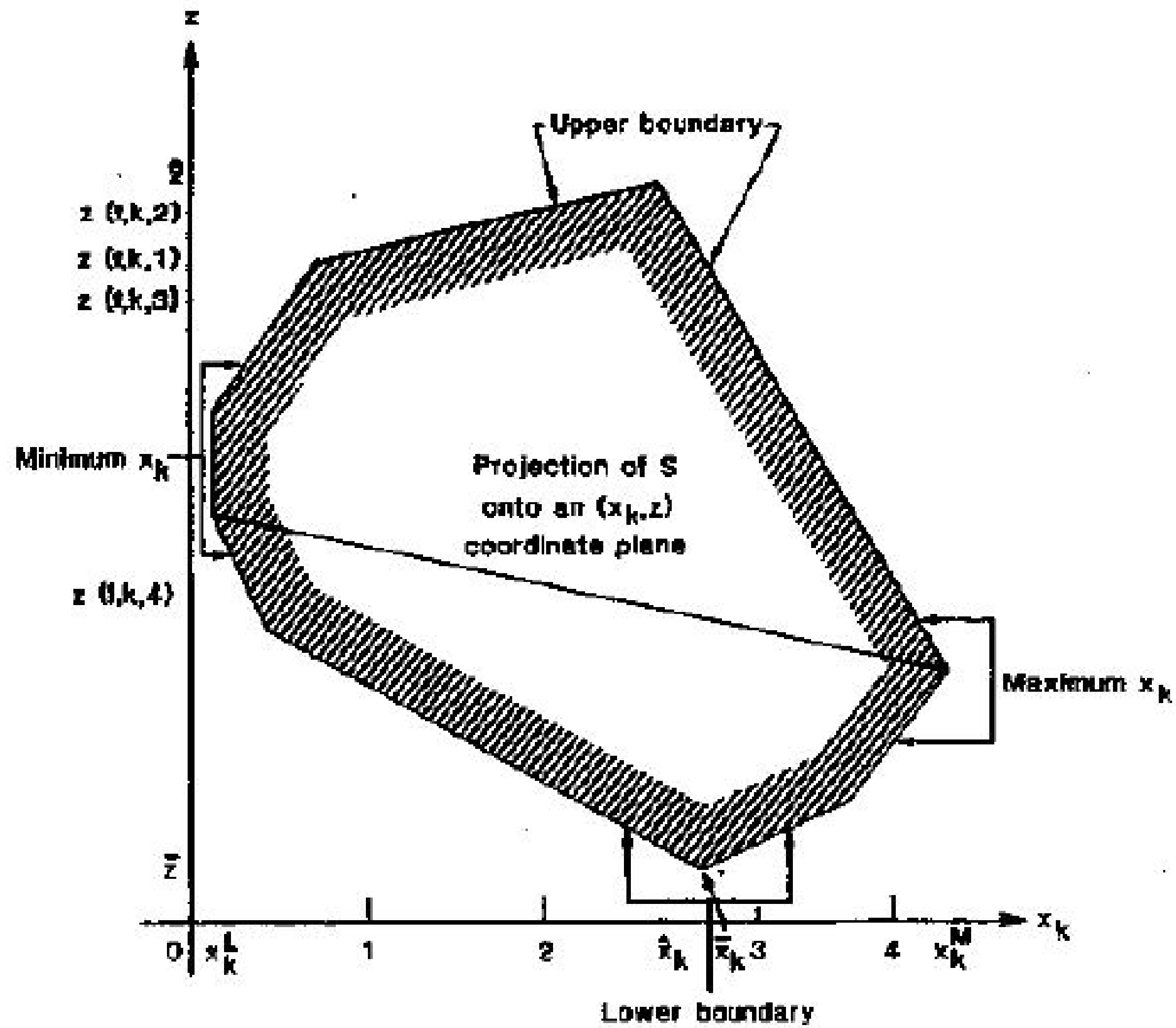
x_k^M = maximum value of x_k when $(x^{\ell}, y, z) \in S^{\ell}$.

z^M = objective function value at x_k^M.

x_k^L = minimum value of x_k when $(x^{\ell}, y, z) \in S^{\ell}$.

z^L = objective function value at x_k^L.

Geometric Interpretations (3/3)



Theorem

- If S^l in not empty, it defines a convex set in the n+n'+1 space. Moreover, the projection S^l onto any of the $(n-l)(x_k, z)$-coordinate planes, where x_k is a free variable, is a convex polygon whose upper(lower) boundary is a concave (convex) function of the x_k axis.

Remarks (1/2)

1. $S^{\ell} = \emptyset \Leftrightarrow LP^{\ell}$ is ____________

2. $\hat{z} = z^{\ell}$, optimal solution to LP^{ℓ}; it is an upper bound for any MILP solution derived from point .

3. $\bar{z}$ is the solution of $\{\min z: (x^{\ell}, y, z) \epsilon S^{\ell}\}$.

4. The boundary of the polygon to the right of the line segment defined by $\overline{x_k}, \bar{z}$ and $(\widehat{x_k}, \hat{z})$ may be traced by solving the LP $\{\max x_k: (x^{\ell}, y, z) \epsilon S^{\ell}\}$, for all fixed values of z between $\bar{z}$ and $\hat{z}$.

Remarks (2/2)

5. If $\{\max z : (x^{\ell}, y, z) \in S^{\ell}\}$ is unbounded, then $(\widehat{x_k}, \hat{z})$ is at ______ and the polygon is unbounded from above.

6. Since the upper boundary of S^{ℓ} is concave, any MILP solution derived from Point x^{ℓ} cannot exceed

$$z^{\ell} = \max \{z(\ell, k, \underline{\hspace{2em}}), z(\ell, k, \underline{\hspace{3em}})\}.$$

7. If LP^{ℓ} has no feasible solution when , $x_k \geq \lfloor \widehat{x_k} \rfloor + I_1$ or $x_k \leq \lfloor \widehat{x_k} \rfloor - I_2$ then $\lfloor \widehat{x_k} \rfloor + I_1$ and $\lfloor \widehat{x_k} \rfloor - I_2$ are upper and lower bounds, respectively , on the variable x_k for any MILP solution derived from point x^{ℓ}.

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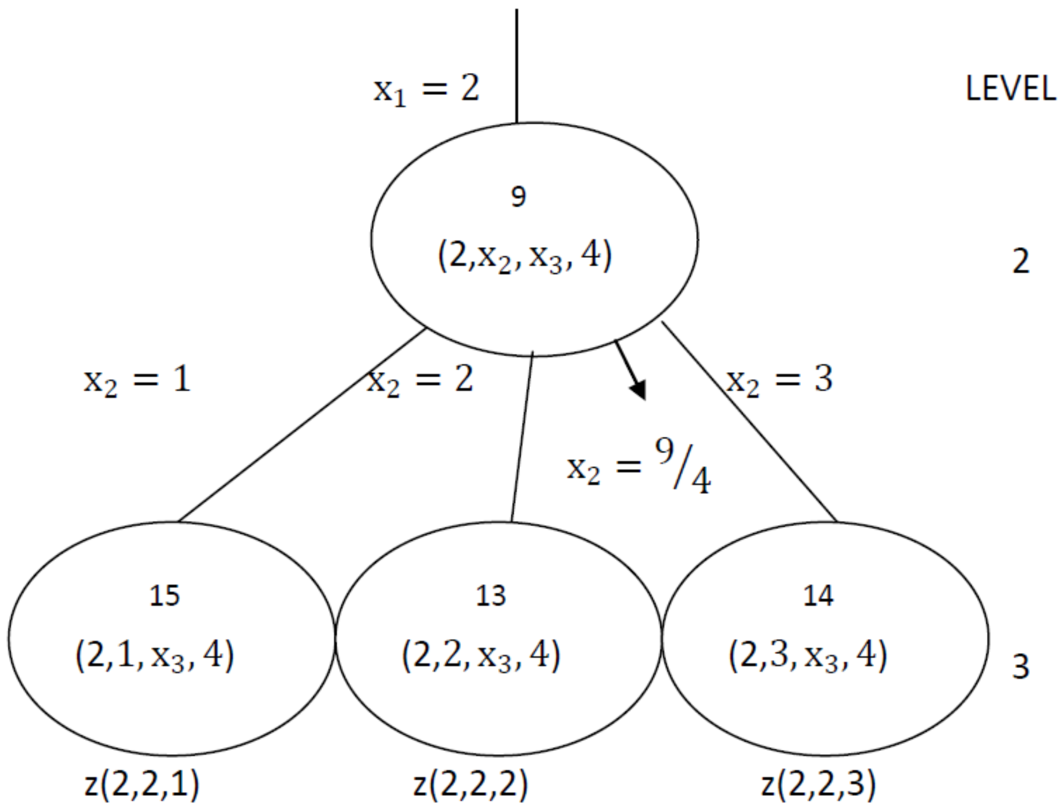
TREE ALGORITHM (Land and Doig (1960))

A node corresponds to a point x^{ℓ}.

A branch links node x^{ℓ} with node $x^{\ell+1}$.



x_2 is free in node 9, but it is set to 2,3, and 1 in nodes 13,14, and 15, respectively.

Enumeration Tree (1/3)

- Dangling node: a node which lacks emanating branches.
- Level number : number of fixed variables.

- Solving LP^2 in node 9, we get $x_2 = 9/4$. Then x_2 is set to $\lfloor 9/4 \rfloor$ and $\lfloor 9/4 \rfloor + 1$ to create nodes 13 and 14, respectively.

- Solving LP^3 in nodes 13 and 14, we get objective values $z(2,2,2)$ and $z(2,2,3)$ respectively. Then $z^2 = \max\ \{z(2,2,2)\ ,\ z(2,2,3)\ \}$, where z^2 is an upper bound to any MILP solutions derived from node 9.

Enumeration Tree (2/3)

- Assume we set $z^2=z(2,2,2)$. By fixing $x_2=1$, we create node15. If node 13 is eliminated, then

  $$z^2=\max\{z(2,2,1), z(2,2,3)\}$$

- Note also that, if the current best feasible solution, z^*, is such that $z^* \geq z(2,2,2)$, then none of the nodes emanating from node 9 can be a candidate to improve the current best solution.
- If $z^* \geq z(2,2,3)$, then all nodes to the right of node 14 may be implicitly enumerated.
- If $z^* \geq z(2,2,3)$, $z^* \geq z(2,2,1)$, and $z^* \leq z(2,2,2)$, then only node 13 needs to be examined at level 3.

Enumeration Tree (3/3)

- Only nodes at which the LP solution exceeds z^* should be explicitly examined and used to create other nodes(these nodes remain in the list of dangling nodes).
- If an LP at a node is infeasible, then all the LP's at nodes either to the left or to the right are infeasible.
- If an LP at a node gives a feasible MILP solution with an objective function value better than z^*, z^* is updated to the value.
- At any step, the node with the largest LP solution from the list of dangling nodes should be selected first (this is one criterion).
- A newly created node is labeled dangling if its LP solution exceeds z^*.
- If the list of dangling nodes is empty, either the optimum has been found(z^*)or the problem is infeasible.

BASIC ALGORITHM (MILP maximization problem) (1/2)

- Step 1.(Initialization) Set the initial solution to the MILP, z*,to -∞ or some predetermined value (i.e., a known lower bound). The initial node with all variables free is x^0.Solve LP^0.If it is infeasible, so is the MILP and terminate. If it has a feasible MILP solution, the MILP is solved and terminate. Otherwise, set $x^l=x^0$ and go to step 2.
- Step 2. (Branching) From the solution of LP^l, $(\hat{x}, \hat{y}, \hat{z})$, select a variable x_k with a noninteger value. Define two nodes from x^l by fixing x_k to $\lfloor x_k \rfloor$ and $\lfloor x_k \rfloor+1$. At each new node, solve the subproblem $LP^{(l+1)}$ and, if the LP solution exceeds z*, label the corresponding node dangling. Check these nodes for an improved solution to the MILP. If one is found, update the value of the best MILP solution, z*,and drop the dangling nodes with an LP solution not exceeding the new value of z*. Go to step3.

BASIC ALGORITHM (MILP maximization problem) (2/2)

- Step 3. (Termination test) If the list of dangling nodes is empty, either the optimal solution to the MILP has been found, $\mathbf{z}^*$, or the problem is infeasible, and terminate. Otherwise, go to step 4.
- Step 4.(Bounding) Select the dangling node x^l with the largest LP solution(break ties arbitrarily).Suppose the point $x^{(l-1)}$ defines the selected point x^l by setting x_k=t. Set $z^{(l-1)}$, an upper bound to any feasible MILP solution found from $x^{(l-1)}$,equal to z(l-1, k ,t) at x^l. Create a node to the immediately right or left of x^l so that if another dangling node created from $x^{(l-1)}$ is eventually selected, a new(not higher) value for $z^{(l-1)}$ may be found. Remove x^l from the list of dangling nodes and go to step2.

Example (1/4)

$$\text{Max } 78x_1 + 77x_2 + 90x_3 + 97y_1 + 31y_2$$

$$\begin{aligned}
\text{S.t.} \quad & 11x_1 + 4x_2 - 41x_3 + 44y_1 + 7y_2 \le 82, \\
& -87x_1 + 33x_2 + 24x_3 + 14y_1 - 13y_2 \le 77, \\
& 61x_1 + 69x_2 + 69x_3 - 57y_1 + 23y_2 \le 87, \\
& x_1, x_2, x_3 \ge 0 \text{ and are integer; } y_1, y_2 \ge 0.
\end{aligned}$$

Example (2/4)

Step 1: $z_* = -\infty$.

$x^0 = (x_1, x_2, x_3)$.

Solution of LP^0: $z = 1{,}171.63, x_1 = 1.5027, x_2 = 0, x_3 = 5.0452,$

$y_1 = 6.1892, \qquad y_2 = 0.$

Step 2: x_1 is set to $\lfloor 1.5027 \rfloor = 1$ to define node 2 with $x^1 = (1, x_2, x_3)$.

Solution of LP^1 at node 2: $z(0,1,,1) = 1{,}056.358, x_1 = 1, x_2 = 0, x_3 = 4.4278, y_1 = 5.5032, y_2 = 1.4855.$

x_1 is set to $\lfloor 1.5027 \rfloor + 1 = 2$ to define node 3 with $x^1 = (2, x_2, x_3)$.

Solution of LP^1 at node 3: $z(0,1,2) = 773.432$.

Nodes 2 and 3 become dangling nodes.

Example (3/4)

Step 3 and 4:Node 2 is the dangling node with the largest LP solution.

$z^0 = \max\{1{,}056.358, 773.432\} = 1{,}056.358$.

Node 4 with $x^1 = (0, x_2, x_3)$to the left of node 2 is created.

Solution of LP^1 at node 4: $z(0,1,0) = 827.04$.

Node 4 becomes a dangling node, and node 2 is removed form the list of dangling nodes.

Step 2: At node 2, $x_3 = 4.4278$. So, x_3 is set to 4 and5 to create nodes 5 and 6, respectively.

Solution of LP^2 at node 5: $z(1,3,4) = 991.6577, x_1 = 1, x_2 = 0.2849, x_3 = 4, y_1 = 5.1498, y_2 = 1.0384$.

Solution of LP^2 at node 6: Infeasible.

Node 5 becomes a dangling node.

Example (4/4)

Steps 3 and 4: At node 2, $z^1 = \max\{991.6577\} = 991.677$.

List of dangling nodes $= \{3,4,5\}$.

Node 5 is the dangling node with the largest LP solution.

Node 7 with $x^2 = (1, x_2, 3)$to the left of node 5 is created.

Soltuion of LP^2 at node 7: $z(1,3,3) = 840.3124$.

Node 7 becomes a dangling node, and node 5 is removed from the list of dangling nodes.

Step 2: At node 5, $x_2 = 0.2849$. So, x_2 is set to 0 and 1 to create nodes 8 and 9.

Solution of LP^3 at node 8: $z(2,2,0) = 982.4933, x_1 = 1, x_2 = 0, x_3 = 4, y_1 = 5.0709, y_2 = 1.6974$.

Solution of LP^3 at node 9: Infeasible.

Node 8 yields an MILP solution; hence $z^* = 982.4933$.

$z^* = 982.4933$ exceeds the solution of all dangling nodes. So, we drop all dangling nodes.

Step 3: There are no dangling nodes and the process terminates.

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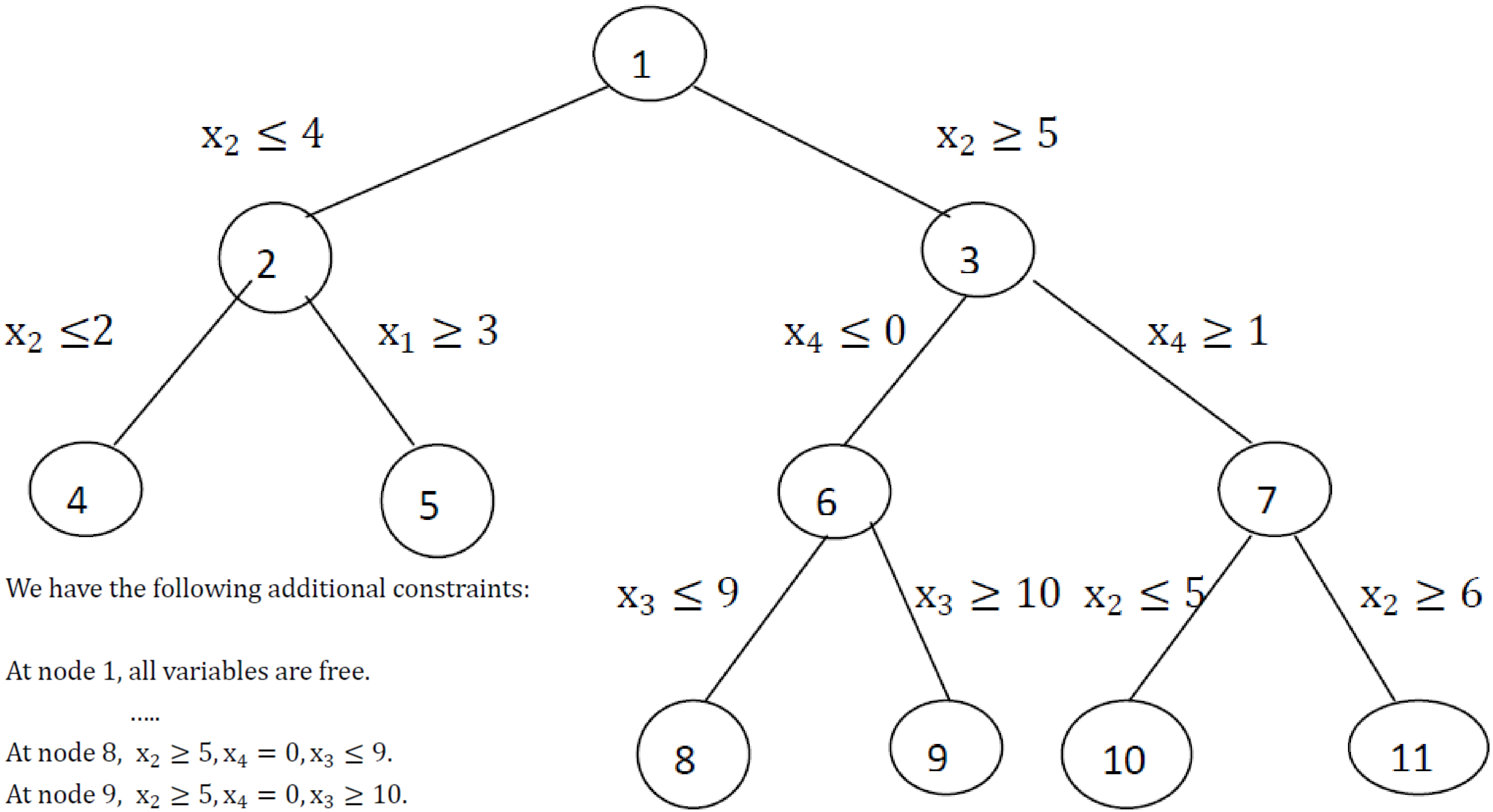
TREE ALGORITHM(Dakin(1965))

- The first algorithm may involve excessive storage requirements because the list of dangling nodes may become very large. In the second algorithm only two nodes are created from each dangling node . If in the optimal LP solution at a dangling node $x_k = t$, where t is not integer, the first node is created with the additional constraint $x_k \leq \lfloor t \rfloor$, and the second node is defined with the constraint $x_k \geq \lfloor t \rfloor + 1$.

Dakin Algorithm (1/2)



We have the following additional constraints:

At node 1, all variables are free.

.....

At node 8, $x_2 \ge 5, x_4 = 0, x_3 \le 9$.

At node 9, $x_2 \ge 5, x_4 = 0, x_3 \ge 10$.

At node 10, $x_2 = 5, x_4 \ge 1$.

At node 11, $x_2 \ge 6, x_4 \ge 1$.

Dakin Algorithm (2/2)

- A node i is labeled dangling if its LP is feasible, exceeds the current best solution to the MILP($\mathbf{z}^*$) and does not solve the MILP.
- We select the dangling node with the largest optimal LP solution to define two nodes at the next level.
- If an improved MILP solution is found, $\mathbf{z}^*$ is updated,and those dangling nodes with an LP solution not exceeding $\mathbf{z}^*$ are dropped from the list of dangling nodes.
- When the list of dangling node is empty, the algorithm terminates.
- The algorithm converges since in the worst case nodes would be created until the permitted range of all integer variables is reduced to zero, in which case they must take an integer value.

Example (1/3)

$$\text{maximize } z = -7x_1 - 3x_2 - 4x_3,$$

$$\begin{aligned} \text{subject to } \; x_1 + 2x_2 + 3x_3 &\geq 8 \implies x_1 + 2x_2 + 3x_3 - x_4 \qquad = 8, \\ 3x_1 + x_2 + x_3 \quad &\geq 5 \implies 3x_1 + x_2 + x_3 \qquad - x_5 = 5, \end{aligned}$$

$$x_1, x_2, x_3 \geq 0, \text{integer},$$

Example (2/3)

Initialization: $z^* = -\infty$.

Node 1: $x^0 = (x_1, x_2, x_3, x_4, x_5)^T$

LP solution: $z = -\frac{71}{5}$, $x_1 = \frac{2}{5}$, $x_2 = \frac{19}{5}$, $x_3 = x_4 = x_5 = 0$.

Nodes 2 and 3 are created with $x_2 \leq \left\lfloor \frac{19}{5} \right\rfloor = 3$ and $x_2 \geq \left\lfloor \frac{19}{5} \right\rfloor + 1 = 4$, respectively.

Node2: $x^1 = (x_1, x_2 \leq 3, x_3, x_4, x_5)^T$

LP solution: $z = -\frac{29}{2}$, $x_1 = \frac{1}{2}$, $x_2 = 3$, $x_3 = \frac{1}{2}, x_4 = x_5 = 0$.

Node3: $x^1 = (x_1, x_2 \geq 4, x_3, x_4, x_5)^T$

LP solution: $z = -\frac{43}{3}$, $x_1 = \frac{1}{3}$, $x_2 = 4$, $x_3 = 0, x_4 = \frac{1}{3}, x_5 = 0$.

Example (3/3)

At node 1, $z^0 = \max\left\{-\frac{29}{2}, -\frac{43}{3}\right\} = -\frac{43}{3}$.

Since this is an all integer problem, $z^0 = \left\lfloor -\frac{43}{3} \right\rfloor = -15$.

Set of dangling nodes={2,3}.

Nodes 3 has the largest optimal LP solution.

Nodes 4 and 5 are created with $x_1 \leq \left\lfloor \frac{1}{3} \right\rfloor = 0$ and $x_1 \geq \left\lfloor \frac{1}{3} \right\rfloor + 1 = 1$, respectively.

Node 4: $x^2 = (x_1 = 0, x_2 \geq 4, x_3, x_4, x_5)^T$

LP solution: z=-15, $x_1 = 0, x_2 = 5, x_3 = 0, x_4 = 2,\ x_5 = 0$.

This is an MILP solution. Then, $z^* = -15$.

Note that

$z^0 = z^* = -15 \ \Rightarrow$ The solution at node 4 is an optimal MILP solution.

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1-0 Problem

- Now consider an MILP problem where all integer variable are either 0 or 1.
- For any integer variable x_i,

$$x_i \leq 0 \Rightarrow x_i = 0,$$

$$x_i \geq 1 \Rightarrow x_i = 1.$$

- This implies that the Land and Doig algorithm is exactly the same as the Dakin algorithm

Example

$$\text{Max } 12x_1 + 16x_2 + 22x_3 + 8x_4$$

$$\text{S.t.} \quad 4x_1 + 5x_2 + 7x_3 + 3x_4 \le 14,$$

$$x_i \text{ are binary.}$$

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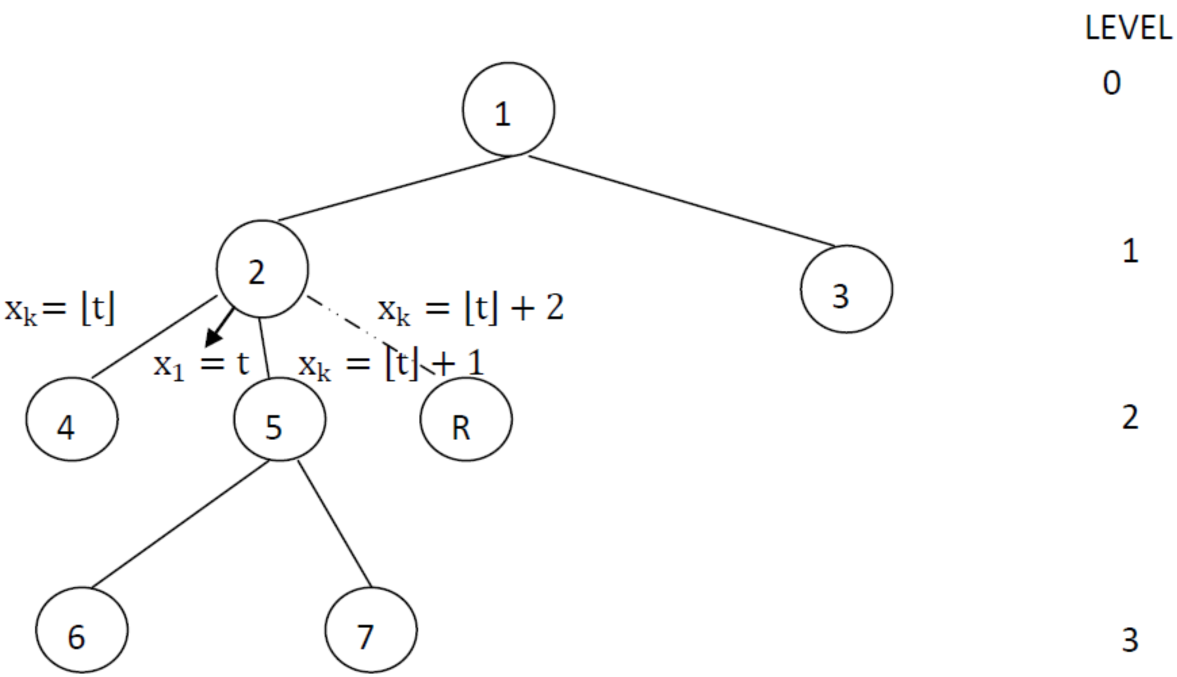
Node Selection (1/3)

- The dangling node with the largest optimal LP solution is selected to create new nodes. The intend is to quickly find a "good " MILP solution. The problem is that this rule may require excessive computer storage.
- Select the most recently created dangling node with the largest optimal LP solution.

Node Selection (2/3)



- Node 2 has an optimal LP solution larger than of node 3.
- Node 5 has an optimal LP solution larger than of node 4.
- It is possible that the optimal LP solution at node 5 is larger than of node 3.

Node Selection (3/3)

- When this forward process stops, say at level 3, the process reverts back to some level ℓ, here level 2, and node R is created and its LP relaxation is solved. Then, the node with the largest LP solution at this level is selected. Note that with this rule the size of the list of dangling nodes becomes ______.
- The problem is that it may take ______ to find a "good " MILP solution.

Branching Rules (1/5)

- After selecting a dangling node, which variable with a noninteger value do we choose to define two new nodes?
- The one with the larges fractional component.
- Round each integer variable that is not integer to the next lower and higher integers, solve the subproblems at the pair of nodes, and select the variable that produces the node with the largest LP solution. Although this rule may decrease the number of nodes, it requires more computational effort.

Branching Rules (2/5)

- From the optimal tableau, it is possible to derive a not-satisfied constraint for each variable at a noninteger value. Then, by performing one dual simplex pivot iteration, we may select a variable according to the size of the decrease in objective function value.

Assume $x_k = n_k + f_k$,

Where $n_k = \lfloor x_k \rfloor$ and $f_k = x_k - n_k$.

Then, $x_k \leq n_k \implies x_k^D = -x_k + n_k \geq 0$,

And $x_k \geq n_k + 1 \implies x_k^U = x_k - n_k - 1 \geq 0$.

Branching Rules (3/5)

$$\text{If } x_k = (n_k + f_k) + \sum_{j=1}^{n} a_{kj} \left(-x_{J(j)}\right),$$

$$\text{Then } x_k^D = -f_k + \sum_{j=1}^{n} -a_{kj} \left(-x_{J(j)}\right),$$

$$\text{and } x_k^U = (f_k - 1) + \sum_{j=1}^{n} a_{kj} \left(-x_{J(j)}\right).$$

After we pivot in row x_k^D, the objective function value becomes

$$a'_{0,0} = a_{0,0} - f_k \frac{a_{0,q}}{a_{k,q}}, \quad \text{where } \frac{a_{0,q}}{a_{k,q}} = \min\left\{\frac{a_{0,j}}{a_{k,j}} : -a_{k,j} < 0\right\},$$

And, if we pivot in row x_k^U,

$$a'_{0,0} = a_{0,0} - (f_k - 1) \frac{a_{0,r}}{a_{k,r}}, \text{where } \frac{a_{0,r}}{-a_{k,r}} = \min\left\{\frac{a_{0,j}}{-a_{k,j}} : a_{k,j} < 0\right\},$$

Branching Rules (4/5)

	1	$-x_{J(1)} \cdots - x_{J(G)} \cdots - x_{J(q)} \cdots - x_{J(r)} \cdots - x_{J(n)}$
Z=	$a_{0,0}$	$a_{0,1} \quad \cdots \quad a_{0,G} \quad \cdots \quad a_{0,q} \quad \ldots \quad a_{0,r} \quad \ldots \quad a_{0,n}$
x_k	$n_k + f_k$	$a_{k,l} \quad \cdots \quad a_{k,G} \quad \cdots \quad a_{k,q} \quad \cdots \quad a_{k,r} \quad \cdots \quad a_{k,n}$
$X_k^D =$	$-f_k$	$-a_{k,l} \qquad (-a_{k,q}) \qquad -a_{k,n}$
$X_k^U =$	$f_k - 1$	$a_{k,l} \quad \cdots \quad \cdots \quad \cdots \quad (a_{k,r}) \quad \ldots \quad a_{k,n}$

$$P_k^D = f_k \frac{a_{0,q}}{a_{k,q}} \geq 0, \text{and } P_k^U = (f_k - 1) \frac{a_{0,r}}{a_{k,r}} \geq 0 \,.$$

Branching Rules (5/5)

- Since x_k is integer, $a_{0,0} - \min\{p_k^D, p_k^u\}$ is an upper bound to any MILP solution from the selected node. Therefore, the following condition must be satisfied to be able to improve the current best solution:

$$a_{0,0} - \max_{f_j > 0} \{\min\{P_j^D, P_j^U\}\} > z^*.$$

- If the MILP solution can be improved, the variable with the smallest penalty can be chosen to create a "good" MILP solution. If we select the variable with the largest penalty, then the current node will be eliminated as quickly as possible.

Example

	1	$(-x_3)$	$(-x_4)$	$(-x_5)$
x_0	$-71/5$	$3/5$	$2/5$	$11/5$
x_1	$2/5$	$-1/5$	$1/5$	$-2/5$
x_2	$19/5$	$8/5$	$-3/5$	$1/5$
x_3	0	-1	0	0
x_4	0	0	-1	0
x_5	0	0	0	-1

Questions

- Final project
  - Submit one page to describe your project on 5/4.