Branch and Bound

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Branch and Bound Method

- Traditional approach to solving integer programming problems.
- Based on principle that total set of feasible solutions can be partitioned into smaller subsets of solutions.
- Smaller subsets evaluated until best solution is found.
- Method is a tedious and complex mathematical process.
- See Module C – "Integer Programming: the Branch and Bound Method" for detailed description of method.

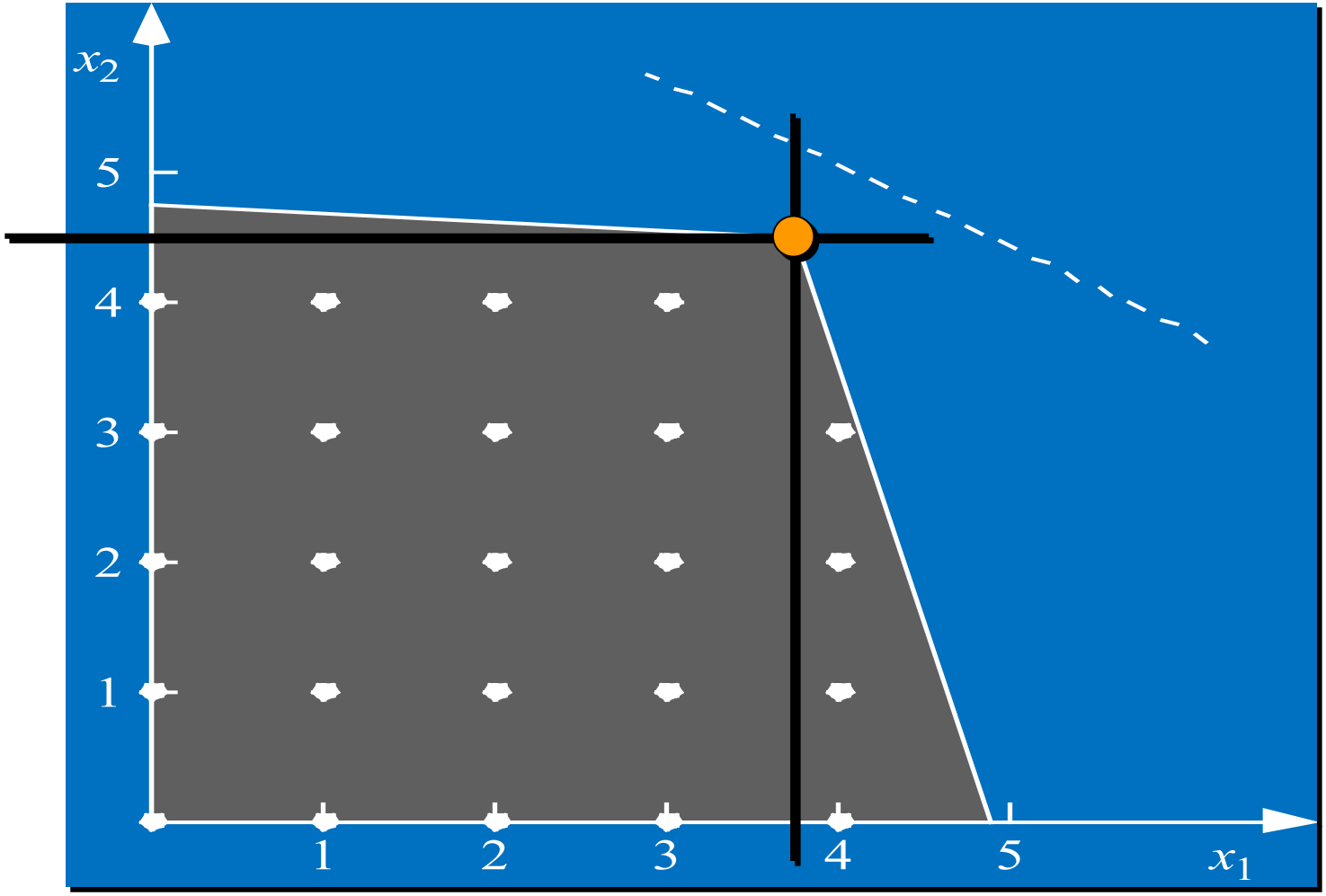
Branch and Bound

Problems are solved as LP. If not integers, one of them is chosen and 2 new constraints are added.

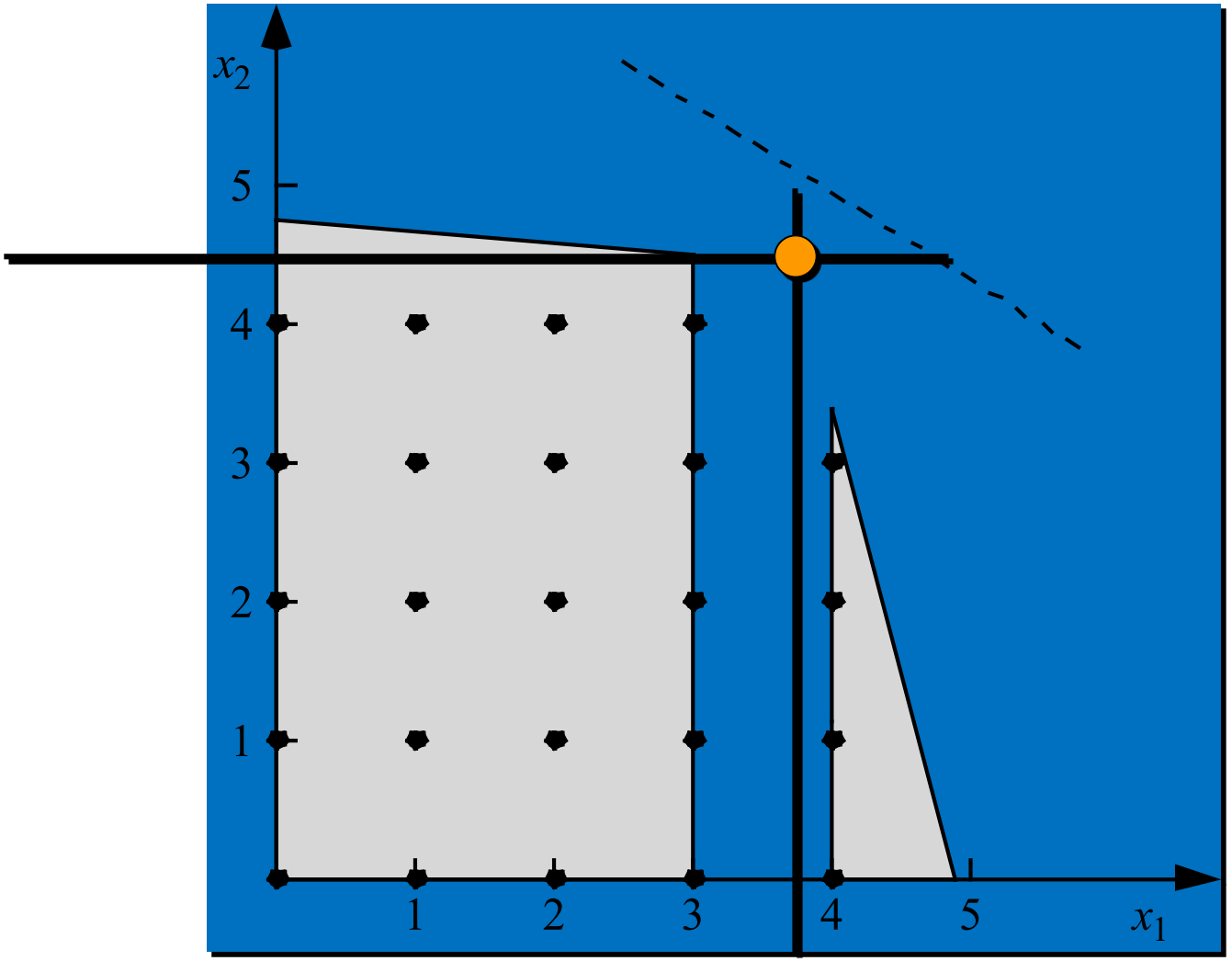
$$x_i \leq \lfloor v \rfloor \text{ and } x_i \geq \lfloor v \rfloor + 1$$

Proceeds until optimal.

How Integer Programs are Solved: Original Graph



After Branch and Bound on x_1



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Step 1: Solve problem using LP. If solution is integer—finished. Otherwise, next.

Step 2: Branch on non-integer variable from step 1. Split problem into two pieces: integer above, and integer below.

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Step 3: Create nodes of these branches and solve the new LP problems.

Step 4:

a) Infeasible, terminate branch;

b) Feasible, not integer, back to Step 2;

c) Feasible and integer, go to Step 5.

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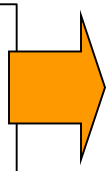
Step 5: Check branches.

1) The feasible solution is a lower bound of the optimum (Max problem).
2) If the feasible solution is better than the LP solution of a node, the branch of that node is fathomed.
3) If there are no remaining branches, the feasible solution is the solution to the problem.

First Branch

Original Problem

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$



$x_1 = 3.75$
$x_2 = 1.50$
$Z = 35.25$

Sub-problem A

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$
$$x_1 \ge 4$$

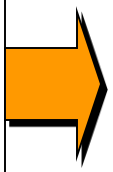
Sub-problem B

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$
$$x_1 \le 3$$

Second Branch

Sub-problem A

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$
$$x_1 \ge 4$$



$$x_1 = 4$$
$$x_2 = 1.20$$
$$Z = 35.2$$

Sub-problem C

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$
$$x_1 \ge 4$$
$$x_2 \ge 2$$

Sub-problem D

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$
$$x_1 \ge 4$$
$$x_2 \le 1$$

Third Branch

Sub-problem B

$$\begin{aligned} \text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \le 3 \end{aligned}$$

$$x_1 = 3$$
$$x_2 = 2$$
$$Z = 33$$

Integer solution $\Rightarrow$
No more branch is needed along this sub-problem.

Fourth Branch

Sub-problem C

$$\begin{aligned}\text{Max} \quad & 7x_1 + 6x_2 \\ \text{st.} \quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \ge 4 \\ & x_2 \ge 2\end{aligned}$$

No feasible solution

No feasible solution ⇒

No more branch is needed along this sub-problem.

Fifth Branch

Sub-problem D

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$
$$x_1 \ge 4$$
$$x_2 \le 1$$

$$x_1 = 4.16$$
$$x_2 = 1$$
$$Z = 35.12$$

Sub-problem E

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$
$$x_1 \ge 4,\ x_1 \le 4$$
$$x_2 \le 1$$

Sub-problem F

$$\text{Max} \quad 7x_1 + 6x_2$$
$$\text{st.} \quad 2x_1 + 3x_2 \le 12$$
$$6x_1 + 5x_2 \le 30$$
$$x_1 \ge 4,\ x_1 \ge 5$$
$$x_2 \le 1$$

Sixth Branch

Sub-problem E

$$\begin{aligned}\text{Max}\quad & 7x_1 + 6x_2 \\ \text{st.}\quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \ge 4,\ x_1 \le 4 \\ & x_2 \le 1\end{aligned}$$

$$x_1 = 4,\quad x_2 = 1,\quad Z = 34$$

Sub-problem F

$$\begin{aligned}\text{Max}\quad & 7x_1 + 6x_2 \\ \text{st.}\quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \ge 4,\ x_1 \ge 5 \\ & x_2 \le 1\end{aligned}$$

$$x_1 = 5,\quad x_2 = 0,\quad Z = 35$$

Integer solutions $\Rightarrow$

No more branch is needed along this sub-problem.

Branch & Bound - Overall