Transformation Using Binary Variables

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Agenda

- Transform logical expressions
- Transform non-simultaneous constraints
- Bundle pricing problem
- Transform non-linear functions

Basic Logical Operations on Variable

- To select a subset of n projects in a manner that maximizes the total present value while satisfying the budget limitation.
- Let y_j=1 if project j is selected, and 0 otherwise.
  - Statement ***A***: project A is selected (y_A=1) or not selected (y_A=0)
  - Statement ***B***: project A is selected (y_B=1) or not selected (y_B=0)
- To obtain a correct MIP model,
  - Only linear equations/ inequalities are allowed.
  - If more than one constraint is required, these constraints have to be satisfied simultaneously.
  - Only the true value (= 1) is of interest.

Conjunction (*A* and *B*, $A \cap B$)

- The conjunction of two statements, ***A*** and ***B***, implies that both projects ***A*** and ***B*** are selected, or symbolically

$$\boxed{y_A = 1 \text{ and } y_B = 1}$$

- An alternate formulation is ________ .

Disjunction (*A* or *B*, $A \cup B$)

- The disjunction relation of two statements, ***A*** or ***B*,** implies that either ***A*** or ***B*** or both are true.
- At least one of the projects A or B must be selected.

	y_A	y_B
Project A is selected but not B	1	0
Project B is selected but not A	0	1
Both are selected	1	1

Simple Implication (If *A* Then *B*, *A*→*B*)

- If statement ***A*** is true, statement ***B*** must be true.
- If statement ***A*** is not true, statement ***B*** can be either true or false.

	y_A	y_B
Project A is selected and B is selected	1	1
Project A is not selected and project B is selected	0	1
Project A is not selected and project B is not selected	0	0

Double Implication (A If and only If B)

- Statement ***A*** implies ***B***, and ***B*** also implies ***A***.
- Project ***A*** is selected if and only if project ***B*** is selected.

	y_A	y_B
Project A is selected and B is selected	1	1
Project A is not selected and project B is not selected	0	0

Linear Expressions for Boolean Relations

Logical Relation	Linear Inequality/Equation
$y_C = y_A \cap y_B$	$y_C \le y_A$ $y_C \le y_B$ $y_C \ge y_A + y_B - 1$
$y_C = y_A \cup y_B$	$y_C \ge y_A$ $y_C \ge y_B$ $y_C \le y_A + y_B$
$y_A \rightarrow y_C$	$y_A \le y_C$
$y_C = \sim y_A$	$y_C = 1 - y_A$

Source: Applied Integer Programming: Modeling and Solution, WILEY

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  - Either/or
  - *P* out of *m* constraints
  - If/then
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Either/Or Constraints

- A decision variable may be defined by two disjunctive regions.
- For instance, either $x \le 3$ or $x \ge 10$.
- Constraints transformation:

$$\boxed{\begin{array}{l} \qquad\quad x-3 \le My \\ \text{and} \quad -x+10 \le M(1-y) \end{array}}$$

- When y=0, constraint $x \ge 10$ is always true.
- When y=1, constraint $x \le 3$ is always true.

One-Machine Scheduling Problem

- Let x_i and x_j respectively denote the start time of job *i* and job *j* to be scheduled.
- Let t_i and t_j respectively represent the known machine processing time of job *i* and job *j*.

$$\text{Either} \quad x_i + t_i \le x_j \quad \text{or} \quad x_j + t_j \le x_i$$

- Sequence constraint (if job *i* before job *j*, variable $y_{ij}=0$):

p Out of *m* Constraints Must Hold

- Consider the case where the model has a set of *m* constraints but in addition requires only some *p* out of *m* (assuming $p<m$) constraints to hold.
- Let y_i = 1 for constraint is relaxed, and 0 otherwise.

$$f_i(x) - b_i \le My_i \qquad \text{for } i = 1, 2, \ldots, m$$

and y_i is binary for all *i*.

If/Then Constraints

- If constraint ***A*** holds, constraint ***B*** must hold.
- If constraint ***A*** **doesn't** hold, constraint ***B*** can be either true or false (be relaxed).
- If ***A*** then ***B*** is equivalent to the logical statement ~***A*** $\cup$ ***B***.

y_A	y_B	$y_{A \rightarrow} y_B$	$\sim y_A \cup y_B$
1	1	1	1
0	1	1	1
0	0	1	1
1	0	0	0

If/Then Constraints (con't)

- We have two constraints: $f_1(x) - b_1 < 0$ and $f_2(x) - b_2 \le 0$.
- $\sim \boldsymbol{A} \cup \boldsymbol{B}$: either or $f_2(x) - b_2 \le 0$.
- Constraint $\boldsymbol{A}$ is satisfied, only when y is 1. That is, constraint $\boldsymbol{B}$ must be satisfied.
- When y = 0, constraint A can't be satisfied.
- Thus, A=1 and B=0 never be happened.

Example - If/Then

- If $x_1=1$, then $x_2=x_3=x_4=0$
- Because all variables are binary, the following can be obtained

- $\sim A \cup B$: $x_1 \leq 0$ and $x_2+x_3+x_4 \leq 0$
- **Then,** $\boxed{\begin{array}{l} x_1 \leq My \\ x_2 + x_3 + x_4 \leq M(1-y) \end{array}}$

- $x_1=1$ then $y=1$, $\boxed{x_1 \leq 3, x_2 + x_3 + x_4 \leq 0}$
- $x_1=0$, x_2, x_3, x_4 are unrestricted. $\boxed{x_1 \leq 0, x_2 + x_3 + x_4 \leq 3}$

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Bundle Pricing Problem

- Products or services are offered by comprising multiple components. All components may also be purchased individually.
  - A software package can compose of several selected modules depending on the requests of users.
  - A fast food restaurant provides combo meals, each of which may consist of a burger, fries, and a soft drink.
  - A travel agency offers products of airfare, rental car and hotel.

Economic Model

- The objective for a company is to set prices for all products and bundles so that the total profit is maximized.
- A reservation price is defined as the maximum price of a customer is willing to pay for a product.
- When the reservation price is higher than the price, the customer will purchase the product.
- A customer will choose the product that maximizes the difference between the reservation price and the price, called surplus.
- For example, a burger and fries are sold at $50 and $30 respectively. The combo meal of these two is priced at $75.

Example (1/3)

- Assume that the size of each customer segment can be forecasted accurately as the average reservation price for each product option.
- Each customer only buy one product, either a computer, a monitor or the bundle.

		Maximum Price Customer is Willing to Pay		
Customer Segment	Expected Size (in 10,000)	Computer Only	LCD Monitor Only	Both
Home	5	600	350	850
Government and educational	15	700	350	1000
Small firms	8	650	300	900
Medium/large firms	12	700	300	900

Source: Applied Integer Programming: Modeling and Solution, WILEY

Example (2/3)

- Input parameters: n_i= size of customer segment i, r_{ij} = reservation price of customer segment i for bundle j.
- Decision variable: x_j = price of bundle j.
- State variable: y_{ij} =1 if customer segment i purchase bundle j. s_i=consumer surplus achieved by customer segment i.
- The objective is to maximize the total revenue:

Example (3/3)

- Each customer buy one bundle:

$$\sum_{j} y_{ij} = 1 \ \forall i$$

- Customer segment *i* will buy bundle *j* when the surplus of bundle *j* is maximum (for each *j*):

$$s_i \geq r_{ij} - x_j \text{ where } s_i =$$

- Constraints on variables:

$$x_j \geq 0, y_{ij} \text{ is binary.}$$

Linearization

$$\text{Max} \quad \sum_i n_i \sum_j x_j y_{ij}$$

$$\text{S.t.} \quad \sum_j y_{ij} = 1 \ \forall i$$

$$\sum_j (r_{ij} y_{ij} - x_j y_{ij}) + x_j \geq r_{ij}$$

$$x_j \geq 0, \ y_{ij} \ \text{is binary.}$$

- This is a nonlinear model.
- Replace $x_j y_{ij}$ by z_{ij}.
- y_{ij} is binary, so $z_{ij} \leq x_j$.
- When $y_{ij} = 1$, $z_{ij} = x_j$

Linearization-MILP

$$\text{Max} \quad \sum_i n_i \sum_j z_{ij}$$

$$\text{S.t.} \quad \sum_j y_{ij} = 1 \ \forall i$$

$$\sum_j (r_{ij} y_{ij} - z_{ij}) + x_j \geq r_{ij}$$

$$z_{ij} \leq x_j$$

$$z_{ij} \geq x_j - (1 - y_{ij}) M_j, \ (M_j \text{ is an upper bound on } x_j)$$

$$z_{ij} \geq 0, x_j \geq 0, y_{ij} \text{ is binary.}$$

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  - Non-binary to binary variables
  - 0-1 polynomial functions

Transform Non-Binary Variables

- If a variable with an upper bound, it can be represented by a set of binary variables.
- If a variable with lower and upper bounds, it can be converted in the similar way.

$$l \leq x \leq u$$

- If a variable may only take one value in a set with all discrete elements, the variable can be expressed as:

$$x = \{1, 3, 5, 7, 9\}$$

0-1 Polynomial Functions

- Consider a simple quadratic function in which the variables must be 0 or 1,

$$f(y_1, y_2, \ldots, y_n) = \sum_j y_j^2 + \sum_{j \neq k} y_j y_k$$

y_j	y^2_j
1	1
0	0

Simply replace y_j^2 with y_j.

y_j	y_k	$y_{jk}=y_jy_k$	$2y_{jk}$	$y_j + y_k$	$y_{jk}+1$
0	0	0	0	0	1
0	1	0	0	1	1
1	0	0	0	1	1
1	1	1	2	2	2

$$2y_{jk} \leq y_j + y_k \leq y_{jk} + 1$$
$$y_j, y_k, y_{jk} = 0 \text{ or } 1$$

A 0-1 Cubic Functions

- Similarly, for a function with product terms of 3 binary variables:

Combination						
y_i	y_j	y_k	y_{ijk}	$3y_{ijk}$	$y_i + y_j + y_k$	$y_{ijk} + 2$
0	0	0	0	0	0	2
0	0	1	0	0	1	2
0	1	0	0	0	1	2
0	1	1	0	0	2	2
1	0	0	0	0	1	2
1	0	1	0	0	2	2
1	1	0	0	0	2	2
1	1	1	1	3	3	3

$$3y_{ijk} \le y_i + y_j + y_k \le y_{ijk} + 2$$
$$y_i, y_j, y_k, y_{ijk} = 0 \text{ or } 1$$

Example (1/2)

$$\begin{aligned} \text{Maximize} \quad & 2y_1y_2y_3^2 + y_1^2y_2 \\ \text{subject to} \quad & 12y_1 + 7y_2^2y_3 - 3y_1y_3 \le 16 \\ & y_1, y_2, y_3 = 0 \text{ or } 1 \end{aligned}$$

- The conversion procedure is as follows:
  - 1. Drop all positive exponents from the problem. Since $y^n = y$ for any binary y and $n > 0$, we can drop all positive exponents.
  - 2. Replace each product term with a new binary variable.

- Let $y_{123} = y_1 y_2 y_3$, $y_{12} = y_1 y_2$, $y_{23} = y_2 y_3$, and $y_{13} = y_1 y_3$.

Example (2/2)

$$
\begin{array}{ll}
\text{Maximize} & 2y_{123} + y_{12} \\
\text{subject to} & 12y_1 + 7y_{23} - 3y_{13} \leq 16 \\
 & \left\{ \begin{array}{l} y_1 + \quad y_2 + \quad y_3 \geq 3y_{123} \\ y_1 + \quad y_2 + \quad y_3 \leq y_{123} + 2 \end{array} \right.
\end{array}
$$

All variables are binary.