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1. Let z^* be the optimal objective function value of

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \cdots, m) \\ & x_j \geq 0 \quad (i = 1, 2, \cdots, n) \end{aligned}$$

and let $y_1^* \cdots y_m^*$ be any optimal solution of the dual problem. Prove that

$$\sum_{j=1}^{n} c_j x_j \leq z^* \sum_{i=1}^{m} y_i^* t_i$$

for every feasible solution $x_1^* \cdots x_n^*$ of

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i + t_i \quad (i = 1, 2, \cdots, m) \\ & x_j \geq 0 \quad (i = 1, 2, \cdots, n) \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{n} c_j x_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_j \\ &\leq \sum_{i=1}^{m} (b_i + t_i) y_i^* = \sum_{i=1}^{m} b_i y_i^* + \sum_{i=1}^{m} t_i y_i^* \end{aligned}$$

Since $y_1^* \cdots y_m^*$ are the optimal solution of the dual problem, the strong duality holds.

$$\Longrightarrow z^* = \sum_{i=1}^{m} b_i y_i^*$$

$$\Longrightarrow \sum_{i=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i^* + \sum_{i=1}^{m} t_i y_i^* = z^* + \sum_{i=1}^{m} t_i y_i^*$$

2. For each of the two problems below, use the complementary slackness in the preview section to check the optimality of the proposed solution.

(a)

$$
\begin{array}{rrrrrrrrrrrl}
\max & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & & \\
\text{s.t.} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq & 4 \\
& 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq & 3 \\
& 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq & 5 \\
& 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 1 \\
& & & & & \multicolumn{3}{c}{x_1, x_2, x_3, x_4, x_5} & & & \geq & 0
\end{array}
$$

Proposed solution: $x_1^* = 0,\ x_2^* = \frac{4}{3},\ x_3^* = \frac{2}{3},\ x_4^* = \frac{5}{3},\ x_5^* = 0$

(b)

$$
\begin{array}{rrrrrrrrrrrrrl}
\max & 4x_1 & + & 5x_2 & + & x_3 & + & 3x_4 & - & 5x_5 & + & 8x_6 & & \\
\text{s.t.} & x_1 & & & - & 4x_3 & + & 3x_4 & + & x_5 & + & x_6 & \leq & 1 \\
& 5x_1 & + & 3x_2 & + & x_3 & & & - & 5x_5 & + & 3x_6 & \leq & 4 \\
& 4x_1 & + & 5x_2 & - & 3x_3 & + & 3x_4 & - & 4x_5 & + & x_6 & \leq & 4 \\
& & - & x_2 & & & + & 2x_4 & + & x_5 & - & 5x_6 & \leq & 5 \\
& -2x_1 & + & x_2 & + & x_3 & + & x_4 & + & 2x_5 & + & 2x_6 & \leq & 7 \\
& 2x_1 & - & 3x_2 & + & 2x_3 & - & x_4 & + & 4x_5 & + & 5x_6 & \leq & 5 \\
& & & & & \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & & & \geq & 0
\end{array}
$$

Proposed solution: $x_1^* = 0,\ x_2^* = 0,\ x_3^* = \frac{5}{2},\ x_4^* = \frac{7}{2},\ x_5^* = 0,\ x_6^* = \frac{1}{2}$

(a) $x_1^* = 0,\ x_2^* = \frac{4}{3},\ x_3^* = \frac{2}{3},\ x_4^* = \frac{5}{3},\ x_5^* = 0 \implies y_6 = 0,\ y_7 = 0,\ y_8 = 0$

$$
\begin{array}{rcrcrcrcrcrcrl}
(0) & + & 3(\frac{4}{3}) & + & 5(\frac{2}{3}) & - & 2(\frac{5}{3}) & + & 2(0) & = & 4 & & & \\
4(0) & + & 2(\frac{4}{3}) & - & 2(\frac{2}{3}) & + & (\frac{5}{3}) & + & 2(0) & = & 3 & & & \\
2(0) & + & 4(\frac{4}{3}) & + & 4(\frac{2}{3}) & - & 2(\frac{5}{3}) & + & 2(0) & = & 2 & \leq & 5 & \implies y_3 = 0 \\
3(0) & + & (\frac{4}{3}) & + & 2(\frac{2}{3}) & - & (\frac{5}{3}) & + & 2(0) & = & 0 & \leq & 2 &
\end{array}
$$

Then consider the dual form:

$$
\begin{array}{rcrcrcrcrl}
1y_1 & + & 4y_2 & + & \cancel{2y_3} & + & 3y_4 & \geq & 7 & \quad (1) \\
3y_1 & + & 2y_2 & + & \cancel{4y_3} & + & y_4 & = & 6 & \quad (2) \\
5y_1 & - & 2y_2 & + & \cancel{4y_3} & + & 2y_4 & = & 5 & \quad (3) \\
-2y_1 & + & y_2 & - & \cancel{2y_3} & - & y_4 & = & -2 & \quad (4) \\
2y_1 & + & y_2 & + & \cancel{5y_3} & - & 2y_4 & \geq & 3 & \quad (5)
\end{array}
$$

We can obtain $y_1 = 1,\ y_2 = 1$ and $y_4 = 1$ from equation $(2)(3)(4)$. Since the solution could not match equation (5), the proposed solution is not the optimal solution.

(b) $x_1^* = 0,\ x_2^* = 0,\ x_3^* = \frac{5}{2},\ x_4^* = \frac{7}{2},\ x_5^* = 0,\ x_6^* = \frac{1}{2} \implies y_9 = 0,\ y_{10} = 0,\ y_{12} = 0$

$$
\begin{array}{rcrcrcrcrcrcl}
(0) & & & - & 4(\frac{5}{2}) & + & 3(\frac{7}{2}) & + & (0) & + & 4(\frac{1}{2}) & = & 1 \\
5(0) & + & 3(0) & + & (\frac{5}{2}) & + & & - & 5(0) & + & 3(\frac{1}{2}) & = & 4 \\
4(0) & + & 5(0) & - & 3(\frac{5}{2}) & & 3(\frac{7}{2}) & - & 4(0) & + & 2(\frac{1}{2}) & = & \frac{7}{2} \leq 4 \implies y_3 = 0 \\
& - & (0) & & & + & 2(\frac{7}{2}) & + & (0) & - & 0(\frac{1}{2}) & = & \frac{9}{2} \leq 5 \implies y_4 = 0 \\
-2(0) & + & (0) & + & (\frac{5}{2}) & + & (\frac{7}{2}) & + & 2(0) & + & 0(\frac{1}{2}) & = & 7 \\
2(0) & - & 3(0) & + & 2(\frac{5}{2}) & - & (\frac{7}{2}) & + & 4(0) & + & 0(\frac{1}{2}) & = & 4 \leq 5 \implies y_6 = 0
\end{array}
$$

Then consider the dual form:

$$
\begin{array}{rcrcrcrcrcrcrl}
1y_1 & + & 5y_2 & + & \cancel{4y_3} & & & - & 2y_5 & + & \cancel{2y_6} & \geq & 4 & (1) \\
 & & 3y_2 & + & \cancel{5y_3} & - & \cancel{y_4} & + & y_5 & - & \cancel{3y_6} & \geq & 5 & (2) \\
-4y_1 & + & y_2 & - & \cancel{3y_3} & & & + & y_5 & + & \cancel{2y_6} & = & 1 & (3) \\
3y_1 & & & + & \cancel{3y_3} & + & \cancel{2y_4} & + & y_5 & - & \cancel{y_6} & = & 3 & (4) \\
-y_1 & - & 5y_2 & - & \cancel{4y_3} & + & \cancel{y_4} & + & 2y_5 & + & \cancel{4y_6} & \geq & -5 & (5) \\
1y_1 & + & 3y_2 & + & \cancel{y_3} & + & \cancel{2y_4} & + & 2y_5 & + & \cancel{5y_6} & = & 8 & (6)
\end{array}
$$

We can obtain $y_1 = \frac{1}{2},\ y_2 = \frac{3}{2}$ and $y_3 = \frac{3}{2}$ from equation $(3)(4)(6)$. After back substitution and check in another equation. The proposed solution $x_1^* = 0,\ x_2^* = 0,\ x_3^* = \frac{5}{2},\ x_4^* = \frac{7}{2},\ x_5^* = 0$ and $x_6^* = \frac{1}{2}$ is the optimal solution.