Introduction to Optimization

Homework #4 – Due Wednesday, November 22

1. Let z^* be the optimal objective function value of

$$maxmize \sum_{j=1}^{n} c_j x_j$$

$$subject\ to\ \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \dots, m)$$

$$x_j \ge 0 \qquad (j = 1, 2, \dots, n).$$

and let $y_1^* \cdots y_m^*$ be any optimal solution of the dual problem. Prove that

$$\sum_{j=1}^{n} c_j x_j \le z^* + \sum_{i=1}^{m} y_i^* t_i$$

for every feasible solution $x_1 \cdots x_n$ of

$$maxmize \sum_{j=1}^{n} c_j x_j$$

$$subject\ to\ \sum_{j=1}^{n} a_{ij} x_j \le b_i + t_i \qquad (i = 1, 2, \dots, m)$$

$$x_j \ge 0 \qquad (j = 1, 2, \dots, n).$$

Preview section: Complementary Slackness

Theorem：Let $x_1^* \cdots x_n^*$ be a feasible solution of the primal problem and $y_1^* \cdots y_m^*$ be a feasible solution of the dual problem. Both solutions are optimal if and only if

$$x_{n+i}^* \cdot y_i^* = 0,\ i = 1, \cdots, m \qquad (1)$$

$$x_j^* \cdot y_{m+j}^* = 0,\ j = 1, \cdots, n \qquad (2)$$

(1) tells us that either the i^{th} inequality in the primal problem holds at equality or the i^{th} dual variable is equal to 0.

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \ \ or\ y_i^* = 0, i = 1, \cdots, m$$

(2) tells us that either the i^{th} inequality in the dual problem holds at equality or the j^{th} primal variable is equal to 0.

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \ \ or\ x_j^* = 0, j = 1, \cdots, n$$

2. For each of the two problems below, use the complementary slackness in the preview section to check the optimality of the proposed solution.

(a). Maximize $7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5$

subject to
$$\begin{aligned} x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 &\le 4 \\ 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 &\le 3 \\ 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 &\le 5 \\ 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 &\le 1 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0. \end{aligned}$$

Proposed solution: $x_1^* = 0, x_2^* = \frac{4}{3}, x_3^* = \frac{2}{3}, x_4^* = \frac{5}{3}, x_5^* = 0.$

(b). Maximize $4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$

subject to
$$\begin{aligned} x_1 - 4x_3 + 3x_4 + x_5 + x_6 &\le 1 \\ 5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 &\le 4 \\ 4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 &\le 4 \\ - x_2 + 2x_4 + x_5 - 5x_6 &\le 5 \\ -2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 &\le 7 \\ 2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 &\le 5 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\ge 0. \end{aligned}$$

Proposed solution: $x_1 = 0, x_2 = 0, x_3 = \frac{5}{2}, x_4 = \frac{7}{2}, x_5 = 0, x_6 = \frac{1}{2}.$

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