HW3

Problem 1 逻辑

1-1 ① 任取 $s \in \{\varphi \wedge p\}$，并设 $(s,s') \in [\![st_1]\!]$

由 $s \in \{\varphi \wedge p\}$ 知 $[\![\varphi \wedge p]\!]_s = true$

由布尔表达式语义：$[\![p]\!]_s = true$，$[\![\varphi]\!]_s = true$

因而 $s \in \{\varphi\}$.

再由分支语句语义

$$[\![if\ (p)\ \{st_1\}\ else\ \{st_2\}]\!] = \{(s,s'') \mid \text{或 } [\![p]\!]_s = true \text{ 且 } (s,s'') \in [\![st_1]\!] \text{ 或 } [\![p]\!]_s = false \text{ 且 } (s,s'') \in [\![st_2]\!]\}$$

已知合 $[\![p]\!]_s = true$，$(s,s') \in [\![st_1]\!]$

可知 $(s,s') \in [\![if\ (p)\ \{st_1\}\ else\ \{st_2\}]\!]$

而 $s \in \{\varphi\}$，且 $\{\varphi\}\ if\ (p)\ \{st_1\}\ else\ \{st_2\}\ \{\psi\}$ 是有效式，

故 $s' \in \{\psi\}$

得证：$\forall s \in \{\varphi \wedge p\}$ 若 $(s,s'') \in [\![st_1]\!]$

则 $s' \in \{\psi\}$，这说明

$\{\varphi \wedge p\}\ st_1\ \{\psi\}$ 是有效式

② 任取 $s \in \{\varphi \wedge \neg p\}$，并设 $(s,s') \in [\![st_2]\!]$

由 $s \in \{\varphi \wedge \neg p\}$ 知 $[\![\varphi \wedge p]\!]_s = true$

由布尔表达式语义：$[\![\neg p]\!]_s = true$，$[\![\varphi]\!]_s = true$

因而 $[\![p]\!] = false$，$s \in \{\varphi\}$

再由分支语句语义

$$[\![if\ (p)\ \{st_1\}\ else\ \{st_2\}]\!] = \{(s,s'') \mid \text{或 } [\![p]\!]_s = true \text{ 且 } (s,s'') \in [\![st_1]\!] \text{ 或 } [\![p]\!]_s = false \text{ 且 } (s,s'') \in [\![st_2]\!]\}$$

已知合 $[\![p]\!]_s = false$，$(s,s') \in [\![st_2]\!]$

可知 $(s,s') \in [\![if\ (p)\ \{st_1\}\ else\ \{st_2\}]\!]$

由 $s \in \{\varphi\}$，且 $\{\varphi\}$ if (p) $\{st_1\}$ else $\{st_2\}$ $\{\psi\}$ 是有效式，

故 $s' \in \{\psi\}$

得证：$\forall s \in \{\varphi \wedge p\}$ 若 $(s, s') \in [\![st_2]\!]$

则 $s' \in \{\psi\}$，这说明

$\{\varphi \wedge \neg p\}\ st_2\ \{\psi\}$ 是有效式

综合①②：$\{\varphi \wedge p\}\ st_1\ \{\psi\}$ 和 $\{\varphi \wedge \neg p\}\ st_2\ \{\psi\}$ 都是有效式

1-2

$$\dfrac{\dfrac{\dfrac{}{\{x-1 \geq 0\}\ x := x-1\ \{x \geq 0\}}\text{赋值}}{\{x > 0\}\ x := x-1\ \{x \geq 0\}}\text{前提加强}}{\{x \geq 0\}\ \text{while}\ (x>0)\ \{x := x-1\}\ \{x \geq 0 \wedge \neg(x>0)\}}\text{循环不变式}$$

$$\dfrac{\dfrac{\dfrac{}{\{x-1 \geq 0\}\ x := x-1\ \{x \geq 0\}}\text{赋值}}{\{x \geq 0 \wedge x > 0\}\ x := x-1\ \{x \geq 0\}}\text{前提加强}}{\{x \geq 0\}\ \text{while}\ (x>0)\ \{x := x-1\}\ \{x = 0\}}\text{循环不变式}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{x-1 \geq 0\}\ x := x-1\ \{x \geq 0\}}\text{赋值}}{\{x > 0\}\ x := x-1\ \{x \geq 0\}}\text{前提加强}}{\{x > 0\}\ x := x-1;\ \text{while}\ (x>0)\ \{x := x-1\}\ \{x = 0\}}\text{顺序}}{\{x > 0\}\ x := x-1;\ \text{while}\ (x>0)\ \{x := x-1\}\ \{\varphi \wedge x \leq 0\}}\text{弱化}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{\varphi \wedge x > 0\}\ x := x-1;\ \text{while}\ (x>0)\ \{x := x-1\}\ \{\varphi \wedge x \leq 0\}}\quad \dfrac{}{\{\varphi \wedge x \leq 0\}\ \text{skip}\ \{\varphi \wedge x \leq 0\}}\text{空语句}}{\{\varphi\}\ \text{if}\ (x>0)\ \{x := x-1;\ \text{while}\ (x>0)\ \{x := x-1\}\}\ \text{else skip}\ \{\varphi \wedge x \leq 0\}}\text{分支}}{\{\varphi\}\ \ \text{while}\ (x>0)\ \{x := x-1\}\ \{\varphi \wedge x \leq 0\}}\text{循环不变式}$$

其中 φ：$\exists t.\ x = 3t$

Problem 2 证明

2-1 等价

① $[\![?p]\!] = \{(s,s) \mid s \models p\}$

② $[\![\text{if } (p) \text{ skip else } ?\text{false}]\!]$

$$= \{(s,s') \mid ([\![p]\!]_s = \text{true} \wedge (s,s') \in [\![\text{skip}]\!]) \vee ([\![p]\!]_s = \text{false} \wedge (s,s') \in [\![?\text{false}]\!])\}$$

$$= \{(s,s') \mid ([\![p]\!]_s = \text{true} \wedge s' = s) \vee ([\![p]\!]_s = \text{false} \wedge [\![\text{false}]\!]_s = \text{true} \wedge s = s')\}$$

$$= \{(s,s') \mid ([\![p]\!]_s = \text{true} \wedge s' = s) \vee \text{false}\}$$

$$= \{(s,s) \mid [\![p]\!]_s = \text{true}\}$$

$$= \{(s,s) \mid s \models p\}$$

所以 $[\![?p]\!] = [\![\text{if } (p) \text{ skip else } ?\text{false}]\!]$

同步 = 等语义等价

2-2 推导树

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{\varphi'\}\ ?\neg p\ \{\varphi' \wedge \neg p\}} \quad \{\varphi' \wedge \neg p\}\ st\ \{\varphi'\}}{\{\varphi'\}\ ?\neg p; st\ \{\varphi'\}}}{\{\varphi'\}\ (?\neg p; st)^*\ \{\varphi'\}}}{\dfrac{\{\varphi\}\ st\ \{\varphi'\} \quad \{\varphi\}\ st; (?\neg p; st)^*\ \{\varphi'\}}{\{\varphi\}\ st; (?\neg p; st)^*; ?p\ \{\varphi' \wedge p\}} \quad \dfrac{}{\{\varphi'\}\ ?p\ \{\varphi' \wedge p\}}}$$

$$\equiv \dfrac{\{\varphi\}\ st; (?\neg p; st)^*; ?p\ \{\varphi' \wedge p\}}{\{\varphi\}\ \text{repeat } st \text{ until } (p)\ \{\varphi' \wedge p\}}$$

同步

$$\dfrac{\{\varphi\}\ st\ \{\varphi'\} \quad \{\varphi' \wedge \neg p\}\ st\ \{\varphi'\}}{\{\varphi\}\ \text{repeat } st \text{ until } (p)\ \{\varphi' \wedge p\}}$$ 是可靠的

Problem 3 数组

3-1

1. $\exists k.\forall i,j.\ (0<k<|a| \wedge 0\le i<k \wedge k\le j<|a|) \rightarrow (a[i]<a[j])$

2. $\forall i,v.\ (a\langle i \triangleleft v\rangle = b\langle i\triangleleft v\rangle) \rightarrow (\forall j.\ (0\le j<|a| \wedge a[j]\ne b[j]) \rightarrow (i=j))$

3-2

while-stat : while $(i<n)$ { if $(m<a[i])$ { $m:=a[i]$ } else { skip } ; $i:=i+1$ }

if-stat : if $(m<a[i])$ { $m:=a[i]$ } else { skip }

φ : $\forall k.\ 0\le k<i \rightarrow m\ge a[k]$

φ' : $\forall k.\ 0\le k<i+1 \rightarrow m\ge a[k]$

即证 $\{m<a[0] \wedge i=0\}$ while-stat $\{\forall k.\ (0\le k<n \rightarrow m\ge a[k])\}$

~~是~~ 有效式

则 ~~有推导树~~

赋值:
$$\frac{}{\{\varphi' \wedge i+1\le n\}\ i:=i+1\ \{\varphi\wedge i\le n\}}$$

顺序:
$$\frac{\{\varphi\wedge i<n\}\ \text{if-stat}\ \{\varphi'\wedge i+1\le n\} \qquad \{\varphi' \wedge i+1\le n\}\ i:=i+1\ \{\varphi\wedge i\le n\}}{\{\varphi\wedge i<n\}\ \text{if-stat};\ i:=i+1\ \{\varphi\wedge i\le n\}}$$

前提加强:
$$\frac{\{\varphi\wedge i<n\}\ \text{if-stat};\ i:=i+1\ \{\varphi\wedge i\le n\}}{\{\varphi\wedge i\le n\wedge i<n\}\ \text{if-stat};\ i:=i+1\ \{\varphi\wedge i\le n\}}$$

循环不:
$$\frac{\{\varphi\wedge i\le n\wedge i<n\}\ \text{if-stat};\ i:=i+1\ \{\varphi\wedge i\le n\}}{\{\varphi\wedge i\le n\}\ \text{while-stat}\ \{\varphi\wedge i\le n\wedge \neg(i<n)\}}$$

后置弱化:
$$\frac{\{\varphi\wedge i\le n\}\ \text{while-stat}\ \{\varphi\wedge i\le n\wedge \neg(i<n)\}}{\{\varphi\wedge i\le n\}\ \text{while-stat}\ \{\forall k.\ (0\le k<n\rightarrow m\ge a[k])\}}$$

前提加强:
$$\frac{\{\varphi\wedge i\le n\}\ \text{while-stat}\ \{\forall k.\ (0\le k<n\rightarrow m\ge a[k])\}}{\{m<a[0]\wedge i=0\}\ \text{while-stat}\ \{\forall k.\ (0\le k<n\rightarrow m\ge a[k])\}}$$

$$\frac{\{\varphi \wedge i<n \wedge m<a[i]\}\ m:=a[i]\ \{\varphi' \wedge i+1 \le n\} \qquad \{\varphi \wedge i<n \wedge m \ge a[i]\}\ skip\ \{\varphi' \wedge i+1 \le n\}}{\{\varphi \wedge i<n\}\ \text{if-stat}\ \{\varphi' \wedge i+1 \le n\}}$$

分支

右边

空语句

$$\frac{\{(\forall k\ 0<k<i \to m \ge a[k]) \wedge i \le n\}\ skip\ \{(\forall k\ 0<k<i \to m \ge a[k]) \wedge i \le n\}}{\{(\forall k.\ 0 \le k<i \to m \ge a[k]) \wedge i \le n\}\ skip\ \{\varphi' \wedge i+1 \le n\}}$$

结论弱化

前提加强 $\{\varphi \wedge i<n \wedge m \ge a[i]\}\ skip\ \{\varphi' \wedge i+1 \le n\}$

左边

赋值

$$\frac{\{(\forall k.\ 0 \le k \le i \to a[i] \ge a[k]) \wedge i<n\}\ m:=a[i]\ \{(\forall k\ 0 \le k \le i \to a[i] \ge a[k]) \wedge i<n\}}{\{(\forall k.\ 0 \le k \le i \to a[i] \ge a[k]) \wedge i<n\}\ m:=a[i]\ \{\varphi' \wedge i+1 \le n\}}$$

结论弱化

前提加强 $\{\varphi \wedge i<n \wedge m<a[i]\}\ m:=a[i]\ \{\varphi' \wedge i+1 \le n\}$

综上：所给出的程序三元组是对的