概统 第九次作业

2. $p_{X,Y}(x,y) = P(X=x, Y=y) = p^2(1-p)^{y-2} \quad (0<x<y)$

$p_Y(y) = \sum_{x=1}^{y-1} p_{X,Y}(x,y) = p^2(1-p)^{y-2}(y-1)$

则 $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{p^2(1-p)^{y-2}}{p^2(1-p)^{y-2}(y-1)} = \frac{1}{y-1} \quad (0<x<y)$

$p_{Y|X}(y|x) = \frac{p_{X|Y}(x|y)\,p_Y(y)}{\sum_{t=x+1}^{\infty} p_{X|Y}(x|t)\,p_Y(t)} = \frac{\frac{1}{y-1}\times p^2(1-p)^{y-2}\times(y-1)}{\sum_{t=x+1}^{\infty} p^2(1-p)^{t-2}} = \frac{p^2(1-p)^{y-2}}{p^2\frac{1}{p}(1-p)^{x-1}} = p(1-p)^{y-x-1} \quad (0<x<y)$

则联合分布 $F_{X,Y}(x,y) = P(X\le x, Y\le y) = \sum_{s=2}^{y}\sum_{t=1}^{s-1} p_{X,Y}(t,s) = \sum_{s=2}^{y}\sum_{t=1}^{s-1} p^2(1-p)^{s-2}$

$= \sum_{s=2}^{y}(s-1)p^2(1-p)^{s-2} = 1-(1+py-p)(1-p)^{y-1}$

条件分布 $F_{X|Y}(x|y) = \sum_{t=1}^{x} p_{X|Y}(t|y) = \frac{x}{y-1} \quad (0<x<y)$

$F_{Y|X}(y|x) = \sum_{t=x+1}^{y} p_{Y|X}(x|t) = \sum_{t=x+1}^{y} p(1-p)^{t-x-1} = 1-(1-p)^{y-x-1} \quad (0<x<y)$

6. $p_Y(y) = \int_{-\infty}^{+\infty} p(x,y)\,dx = \int_{|y|}^{1} 1\,dx = 1-|y| \quad (-1<y<1)$

则 $p(x|y) = \frac{p(x,y)}{p_Y(y)} = \frac{1}{1-|y|} \quad (|y|<x,\ 0<x<1)$

则 $p(x|y) = \begin{cases} \frac{1}{1-|y|}, & |y|<x,\ 0<x<1 \\ 0, & \text{else} \end{cases}$

7. $p_X(x) = \int_{-\infty}^{+\infty} p(x,y)\,dy = \int_{x^2}^{1}\frac{21}{4}x^2y\,dy = \frac{21}{8}x^2(1-x^4)$

则 $p(y|x) = \frac{p(x,y)}{p_X(x)} = \frac{\frac{21}{4}x^2y}{\frac{21}{8}x^2(1-x^4)} = \frac{2y}{1-x^4}$

$p(y|X=0.5) = \frac{32}{15}y \quad (0.25\le y\le 1)$

$P(Y\ge 0.75|X=0.5) = \int_{0.75}^{1}\frac{32}{15}y = \frac{7}{15}$

8. $p(x,y) = p(x|y)\,p_Y(y) = \begin{cases} 15x^2y, & 0<x<y<1 \\ 0, & \text{else} \end{cases}$

$p_X(x) = \int_{-\infty}^{+\infty} p(x,y)\,dy = \int_{x}^{1} 15x^2y\,dy = \frac{15}{2}x^2(1-x^2)$

$P(X>0.5) = \int_{0.5}^{1} p_X(x)\,dx = \int_{0.5}^{1}\frac{15}{2}x^2(1-x^2) = \frac{15}{2}\left(\frac{1}{3}x^3-\frac{1}{5}x^5\right)\Big|_{0.5}^{1} = \frac{47}{64}$

9. $X\sim U(1,2) \quad Y|X\sim Exp(x)$.

$p(y|x) = xe^{-xy} \quad (y>0)$

则 $p(x,y) = p(y|x)\,p(x) = xe^{-xy} \quad (y>0,\ 1<x<2)$

$F(XY\le t) = \iint_{\substack{xy\le t \\ y>0 \\ 1<x<2}} p(x,y)\,dxdy = \iint_{\substack{1<x<2 \\ 0<y\le\frac{t}{x}}} xe^{-xy}\,dxdy = \int_1^2\int_0^{\frac{t}{x}} xe^{-xy}\,dy\,dx$

$= \int_1^2(1-e^{-t})\,dx = 1-e^{-t} \quad (t\ge 0)$ 则 $p_{XY}(t) = F'(XY\le t) = e^{-t} \quad (t\ge 0)$

因此 $XY\sim Exp(1)$

10. $P(Y=2) = 0.01+0.03+0.05+0.05+0.05+0.06 = 0.25$

$$P(X|Y=2) = \frac{P(X=x,Y=2)}{P(Y=2)} = 4P(X=x,Y=2).$$

因此 $E(X|Y=2) = \sum_{x=0}^{5} 4x\,P(X=x,Y=2)$

$$= 4(0\times0.01+1\times0.03+2\times0.05+3\times0.05+4\times0.05+5\times0.06) = 3.12$$

$P(X=0) = 0+0.01+0.01+0.01 = 0.03$.

$$P(Y=y|X=0) = \frac{P(Y=y,X=0)}{P(X=0)} = \frac{100}{3}P(Y=y,X=0)$$

因此 $E(Y|X=0) = \sum_{y=0}^{3} \frac{100}{3}\,y\,P(Y=y,X=0)$

$$= \frac{100}{3}(0\times0+1\times0.01+2\times0.01+3\times0.01) = 2$$

16. $X, Y \sim Exp(\lambda)$ $\quad p_X(x) = \lambda e^{-\lambda x}$ $\quad p_Y(y) = \lambda e^{-\lambda y}$

$$E(Z) = E(E(Z|X)) = \int_{-\infty}^{+\infty} E(Z|X)\,p_X(x)\,dx$$

$$E(Z|X=x) = \int_0^{+\infty} z\,p_Y(Y=y|X=x)\,dy = \int_0^{+\infty} z\,p_Y(Y=y)\,dy$$

$$= \int_0^x (3x+1)\,p_Y(y)\,dy + \int_x^{+\infty}(6y)\,p_Y(y)\,dy$$

$$= \int_0^x (3x+1)\lambda e^{-\lambda y}\,dy + \int_x^{+\infty}(6y)\lambda e^{-\lambda y}\,dy$$

$$= -(3x+1)e^{-\lambda y}\Big|_0^x - 6ye^{-\lambda y}\Big|_x^{+\infty} + \int_x^{+\infty} 6e^{-\lambda y}\,dy$$

$$= 3x+1-(3x+1)e^{-\lambda x} + 6xe^{-\lambda x} - \frac{6}{\lambda}e^{-\lambda y}\Big|_x^{+\infty}$$

$$= 3x+1+\left(3x-1+\frac{6}{\lambda}\right)e^{-\lambda x}$$

$$E(Z) = \int_0^{+\infty}\left[3x+1+\left(3x-1+\frac{6}{\lambda}\right)e^{-\lambda x}\right]\lambda e^{-\lambda x}\,dx$$

$$= \int_0^{+\infty}\lambda e^{-\lambda x}dx + \int_0^{+\infty}3\lambda x\,e^{-2\lambda x} + \int_0^{+\infty}(6-\lambda)e^{-2\lambda x}\,dx + \int_0^{+\infty}3x\lambda e^{-\lambda x}\,dx$$

$$= 1 + \frac{3}{2}\cdot\frac{1}{2\lambda} + \frac{6-\lambda}{2\lambda} + \frac{3}{\lambda} = \frac{1}{2} + \frac{27}{4\lambda}$$

18、设 $\sum_{i=1}^{N} X_i = X$

$$EX = E(E(X|N)) = \sum E(X|N)P(N) = \sum E\left(\sum_{i=1}^{N}X_i\right)P(N) = \sum N E(X_1)P(N) = E(X_1)E(N)$$

$$EX^2 = E(E(X^2|N)) = \sum E(X^2|N)P(N) = \sum E\left(\sum_{i=1}^{N}X_i\right)^2 P(N)$$

$$= \sum\left(NEX_1^2 + N(N-1)(EX_1)^2\right)P(N)$$

$$= \sum NEX_1^2P(N) + \sum N(N-1)(EX_1)^2P(N)$$

$$= E(N)E(X_1^2) + E(N(N-1))(EX_1)^2.$$

$$Var\,X = EX^2 - (EX)^2 = E(N)E(X_1^2) + E(N(N-1))(EX_1)^2 - (EX_1)^2(EN)^2.$$

$$= \left(E(N)E(X_1^2) - E(N)(EX_1)^2\right) + \left(E(N(N-1))(EX_1)^2 + EN(EX_1)^2 + (EN)^2(EX_1)^2\right)$$

$$= E(N)Var(X_1) + Var(N)\cdot(EX_1)^2$$