Homework for General physics II-set 5 Answers

1. Hecht, problem 11.10. (You may need the representation of the Dirac delta function and its properties, and if you use the symmetrical Fourier Transform I preferred in the class, then there will be no 2 π)

Answers:

$$\mathrm{F}\{f(x)\} = F(K) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iKx}dx \qquad (1)$$

$$\mathrm{F}\{F(K)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(K)e^{-iKT}dK \qquad (2)$$

Note here I use T instead of x for the notation of variable, because x is used in the F(K) formula (1) above, so I chose another symbol T for the dummy variable. Insert (1) into (2):

$$\mathrm{F}\{F(K)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(x)e^{-iKx}dx)e^{-iKT}dK$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} e^{-iK(x+T)}dK)\,f(x)dx$$

$$= \int_{-\infty}^{\infty}\delta(x+T)\,f(x)dx$$

The reason to choose T instead of x is to avoid common mistake of x+T=2x in above formula, T is the dummy parameter with same range as x, but when x is some value, T is not necessarily the same value!
Above I used expression of delta function:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iKx}dK \quad \text{and} \quad \delta(x) = \delta(-x)$$

Then from sampling property of delta function:

$$\mathrm{F}\{F(K)\} = \int_{-\infty}^{\infty}\delta(x+T)\,f(x)dx = f(-T)$$ since T is just a dummy variable f(-T) is just f(-x)

2. Prove that convolution defined is commutable for functions g and h, i.e.:

$$g(x)\otimes h(x) = \int_{-\infty}^{\infty} g(x')h(x-x')dx'$$

are same.

$$h(x)\otimes g(x) = \int_{-\infty}^{\infty} h(x')g(x-x')dx'$$

Answer: In the second equation, I make a substitution of variable and prove it is the same as first equation:

$T = x - x'$, then $x' = x - T$, $dT = -dx'$ and the convolution between h and g becomes:

$$h(x)\otimes g(x)=\int_{-\infty}^{\infty} h(x')g(x-x')dx'$$

please note the upper-lower limit on integration needs to be flipped,

$$=\int_{+\infty}^{-\infty} h(x-T)g(T)(-dT)$$

the x change from –infinity to +infinity, then T changes from + to -. Then:

$$h(x)\otimes g(x)=\int_{+\infty}^{-\infty} h(x-T)g(T)(-dT)=\int_{-\infty}^{\infty} h(x-T)g(T)dT$$

Since T is just a dummy variable as x', so this integration is just same as $g(x)\otimes h(x)$

3. **Prove the convolution theorem of the Fourier Transform. i.e $\mathfrak{F}\{\boldsymbol{g(x)} \circledast \boldsymbol{h(x)}\} = \boldsymbol{G(\kappa)H(\kappa)}$ (The proof is in the Hecht's book. but please try it yourself.)**

Answer:

$g(x) \circledast h(x) = f(x)$

The Fourier transform of $g(x)$is $\mathfrak{F}\{\mathrm{g(x)}\} = G(\kappa)$

The Fourier transform of $h(x)$is $\mathfrak{F}\{\mathrm{h(x)}\} = H(\kappa)$

The Fourier transform of $f(x')$is $\mathfrak{F}\{\mathrm{f(x')}\} = F(\kappa)$

$$\mathfrak{F}\{g(x) \circledast h(x)\} = \mathfrak{F}\{f(x')\} = F(\kappa) = \int_{-\infty}^{+\infty} f(x')e^{i\kappa x'}dx' = \int_{-\infty}^{+\infty} e^{i\kappa x'}dx'\,[g(x)h(x'-x)dx]$$

$$= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} h(x'-x)e^{i\kappa x'}dx'\right]g(x)dx$$

We put $\omega = x' - x$ in the inner integral, then dx' = dω.

$$\mathfrak{F}\{g(x) \circledast h(x)\} = \int_{-\infty}^{+\infty} g(x)e^{i\kappa x}dx\int_{-\infty}^{+\infty} h(\omega)e^{i\kappa\omega}d\omega = G(\kappa)H(\kappa)$$

Above is using Hecht's convention, if we use our symmetrical convention with $\frac{1}{\sqrt{2\pi}}$ in the transform and reverse transform, the convolution theorem becomes:

$$\mathfrak{F}\{\boldsymbol{g(x)} \circledast \boldsymbol{h(x)}\} = \sqrt{2\pi}\,\boldsymbol{G(\kappa)H(\kappa)}$$

4. **(Hecht's 11.22)Show:**

(1) $\mathfrak{F}\{\mathbf{f(x)cosk_0x}\} = \frac{[\mathbf{F(k-k_0)+F(k+k_0)}]}{\mathbf{2}}$

Answer:

$$\mathfrak{F}\{f(x)cosk_0x\} = \frac{1}{\sqrt{2\pi}}\int f(x)cosk_0e^{-ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int f(x)\frac{e^{ik_0x}+e^{-ik_0x}}{2}e^{-i\kappa x}\,d\kappa$$

$$= \frac{1}{\sqrt{2\pi}}\int f(x)\frac{e^{-i(k-k_0)x}+e^{-i(k+k_0)x}}{2}d\kappa = \frac{[F(k-k_0)+F(k+k_0)]}{2}$$

I used phase shift property of the FT.

(2) $\mathfrak{F}\{\mathbf{f(x)sink_0x}\} = \frac{[\mathbf{F(k-k_0)-F(k+k_0)}]}{\mathbf{2i}}$

Answer:

$$\mathfrak{F}\{f(x)sink_0x\} = \frac{1}{\sqrt{2\pi}}\int f(x)sink_0e^{-ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int f(x)\frac{e^{ik_0x}-e^{-ik_0x}}{2i}e^{-i\kappa x}\,d\kappa$$
$$= \frac{1}{\sqrt{2\pi}}\int f(x)\frac{e^{-i(k-k_0)x}-e^{-i(k+k_0)x}}{2i}\,d\kappa = \frac{[F(k-k_0)-F(k+k_0)]}{2i}$$

5. **(Hecht's 11.25)Given the function $\mathrm{f(x)} = \delta(x+3)+\delta(x-2)+\delta(x-5)$, convolve it with the arbitrary function $\mathrm{h(x)}$.**

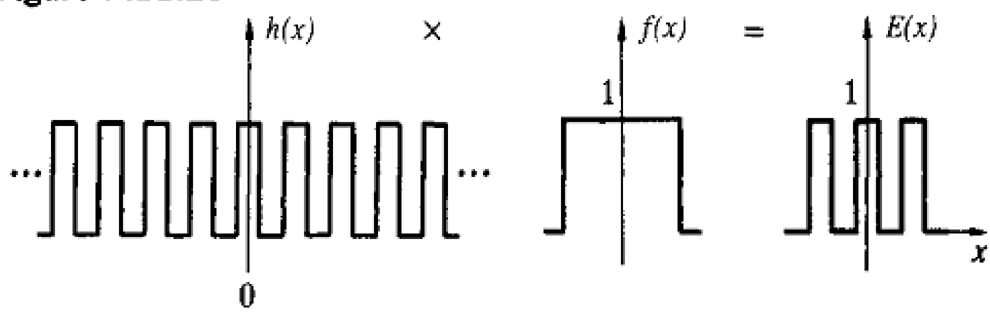
Answer:

$\mathbf{f(x)} \circledast \mathbf{h(x)} = [\delta(x+3)+\delta(x-2)+\delta(x-5)] \circledast \mathbf{h(x)} = h(x+3)+h(x-2)+h(x-5)$

Convolution between a shifted delta function with h(x) is just shift of the function h.

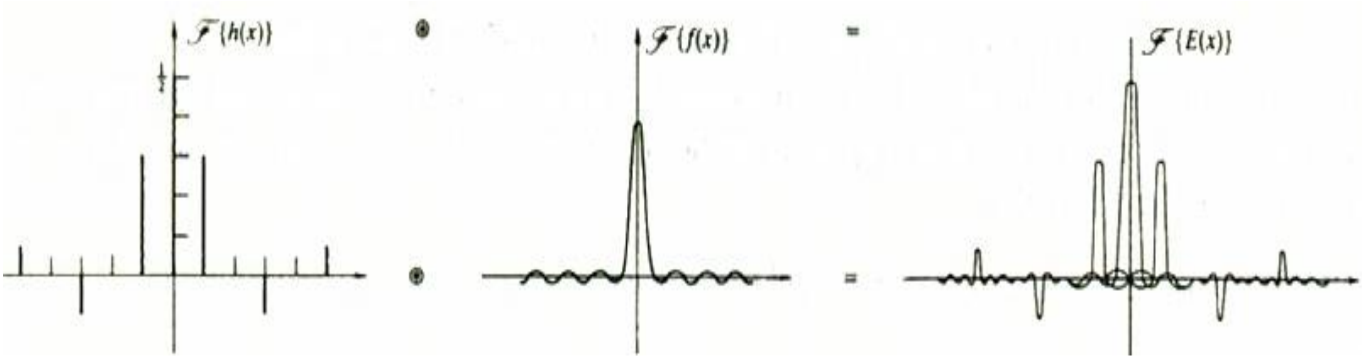
6. **(Hecht's 11.28)In one dimension, the electric field across an illuminated aperture consisting of several opaque bars forming a grating. Considering it to be created by taking the product of a periodic rectangular wave $h(x)$ and a unit rectangular function $f(x)$, sketch the resulting electric field in the Fraunhofer region.**

Figure P.11.28



(It is effectively a 3 slits grating, you may compare the Fourier result with the multi-slits' result, they are the same.)

Answer:



The FT of h(x) which is actually a Fourier expansion of periodic function h; and the expansion C_m is shown in the first figure. It can be treated as $C_m\delta(K-mK_0)$, $K_0=\frac{2\pi}{d}$, d is the period and d=2a, a is the width of each opening. When they convolve with sinc function of F(f(x)), the result is just shifted sinc functions with height modified by C_m.

Also notice the missing order since d=2a, so the 2, 4, 6...orders would be missing, i.e. $C_{2,4,6..}=0$.

7. **(Hecht's 11.35)Consider the function in Fig. 11.35 as a cosine carrier multiplied by an exponential envelope. Use the frequency convolution theorem to evaluate its Fourier transform.**

(This is basically using the results in Zhao's book; table V-3, from results 2 and 9 to prove result 8. The convolution theorem in the class is that if the original function is the convolution of two functions, then its Fourier Transform is the product of individual transform; here the situation is reversed, the original function is the product, you may verify that the transform will be the convolution of the individual transforms)

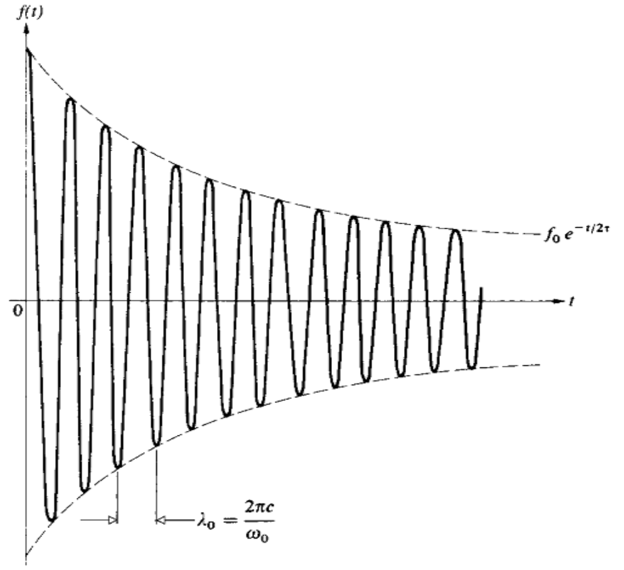


Figure 11.35 A damped harmonic wave.

Answer:

$$f(t) = cos\omega_0 t$$

$$g(t) = e^{-\frac{t}{2\tau}}, (t > 0)$$

$$h(t) = f(t) \cdot g(t)$$

$$\mathfrak{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int cos\omega_0 t e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} e^{-i\omega t}\, dt$$

$$= \frac{2\pi}{2\sqrt{2\pi}} \int \frac{1}{2\pi} \left[e^{-i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}\right] dt$$

$$= \sqrt{\frac{\pi}{2}} [\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$

$$\mathfrak{F}\{g(t)\} = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{t}{2\tau}} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int e^{-\left(\frac{1}{2\tau}+i\omega\right)t} dt = -\frac{1}{\sqrt{2\pi}} \left(\frac{1}{2\tau} + i\omega\right)^{-1}$$ t is from 0 to infinite.

$$\mathfrak{F}\{h(t)\} = \mathfrak{F}\{f(t)\} \circledast \mathfrak{F}\{g(t)\} = -\frac{1}{2}[\left(\frac{1}{2\tau} + i(\omega+\omega_0)\right)^{-1} + \left(\frac{1}{2\tau} + i(\omega-\omega_0)\right)^{-1}]$$

8. **For a 4F system, the focal lengths are 10cm, the diameters of the lenses are 5 cm, and the wavelength of light in use is 500 nm, what is the minimum structure of the object that can be resolved in the image formed by such system? (I.e. the smallest linear feature of the object that can be resolved on the image.)**

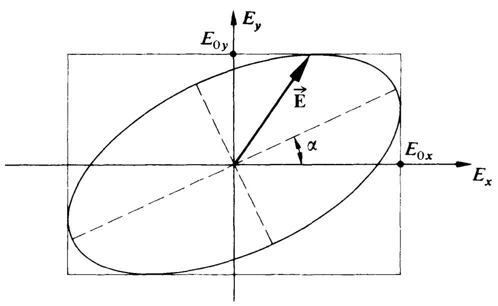
Answer:

$$\frac{d}{F} \ge \frac{\lambda}{R} \Rightarrow d_{min} = \frac{2\lambda}{D} F = \frac{2 \times 500 \times 10^{-9}}{5 \times 10^{-2}} \times 10 \times 10^{-2} = 2000nm$$

9. (Optional) This problem is designed to help you knowing the ellipse (for the elliptical polarized light)

For a elliptical polarization given by: $\begin{bmatrix} A \\ Be^{i\varphi} \end{bmatrix}$, as derived in the Hecht's book (Pg. 328, relation 8.14) The H-V component of Electric field amplitude E_x, E_y satisfy:

$\frac{E_x^2}{A^2} + \frac{E_y^2}{B^2} - 2\left(\frac{E_x}{A}\right)\left(\frac{E_y}{B}\right) cos\varphi = sin^2\varphi$ This is a tilted ellipse and in this problem you are going to find the angle of the major axis of the ellipse (Hecht's relation 8.15) and also the short-long axis length of the ellipse.



1) Start writing the relation in standard polynomial form of the conic section curves: Let x, y stand for E_x, E_y to

save some writing. Rewrite the relation $\frac{E_x^2}{A^2}+\frac{E_y^2}{B^2}-2\left(\frac{E_x}{A}\right)\left(\frac{E_y}{B}\right)cos\varphi = sin^2\varphi$ into forms of $ax^2+bxy+cy^2+dx+ey+f=0$. What is the coefficient of a,b,c,d,e,f in terms of A,B,φ?

2) Suppose we now rotate the coordinate axis by angle α; find out the expression of the ellipse $ax^2+bxy+cy^2+dx+ey+f=0$ in terms of new coordinate x', y'. i.e. In the new coordinate the expression would be: $a'x'^2+b'x'y'+c'y'^2+d'x+e'y+f'=0$; what are the relations between the a',b',c'… with a,b,c.
3) To find out the angle of α, i.e. in the new coordinate, the ellipse is would be un-tilted, that requires the term b'xy (cross term of xy) should be zero, meaning b'=0, this will fixed the angle α, express it in terms of A,B,φ (You will get Hecht's 8.15)
4) Also knowing the form of ellipse in x'-y' coordinate, you can express the half-long axis (call it a") of the ellipse in terms of A,B,φ

Answers:

1) Just by comparison, $a=\frac{1}{A^2}; b=-2\frac{cos\varphi}{AB}; c=\frac{1}{B^2}, f=-sin^2\varphi$ and d,e=0. Actually since d,e,f are not important for the tilting of the ellipse (f affects how big the ellipse is, d,e term affect the translation of the ellipse), I shall concentrate if a,b,c terms.
2) For the rotated basis, we know the transformation relation: $\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}cos\alpha & sin\alpha\\-sin\alpha & cos\alpha\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$; or $x'=xcos\alpha+ysin\alpha$; $y'=x(-sin\alpha)+ycos\alpha$; or $x=x'cos\alpha-y'sin\alpha; y=x'sin\alpha+y'cos\alpha$ansform); we can replace x, y with x',y' in the original $ax^2+bxy+cy^2$ to find out a', b'….: (Here I skipped half-hour work of substitution):
$a'=acos^2\alpha+bcos\alpha sin\alpha+csin^2\alpha$;
$b'=bcos2\alpha+(c-a)sin2\alpha$;
$c'=asin^2\alpha-bsin\alpha cos\alpha+ccos^2\alpha$;
$d'=dcos\alpha+esin\alpha; e'=-dsin\alpha+ecos\alpha; f'=f$ (d', e' are 0 here since d,e are 0)
3) To have a standard ellipse in x'-y', we required b'=0, that gives:
$bcos2\alpha+(c-a)sin2\alpha=0$; then
$tan2\alpha=-\frac{b}{c-a}=\frac{-2^{cos\varphi}/_{AB}}{1/_{A^2}-1/_{B^2}}=\frac{2ABcos\varphi}{A^2-B^2}$ This is exactly hecht's 8-15.
4) Knowing now the $a'x'^2+c'y'^2+f=0$; the half long axis a", which corresponds to : $\frac{x'^2}{a"^2}+\frac{y'^2}{b"^2}=1$ is straightforward: $a"^2=-\frac{f}{a'}$ you may throw A,B,φ now into the a' and f. (a quite complicated formula which is not very useful in application)