《高等微积分2》第十一周作业

1. 解: $D=\{(x,y)\mid x>0,\ y>0,\ y\le \frac{b}{x},\ y\ge \frac{a}{x},\ y\le dx,\ y\ge cx\}$

$$\text{Area}(D)=\iint_D 1\,dx\,dy=\int_{x>0}dx\int_{\max\{\frac{a}{x},cx\}}^{\min\{\frac{b}{x},dx\}}dy$$

$$=\int_{\sqrt{\frac{a}{d}}}^{\sqrt{\frac{b}{c}}}dx\left(\min\{\tfrac{b}{x},dx\}-\max\{\tfrac{a}{x},cx\}\right)$$

① $\sqrt{\frac{b}{d}}\le\sqrt{\frac{a}{c}}$

$$\text{Area}(D)=\int_{\sqrt{\frac{a}{d}}}^{\sqrt{\frac{b}{d}}}dx\left(dx-\frac{a}{x}\right)+\int_{\sqrt{\frac{b}{d}}}^{\sqrt{\frac{a}{c}}}dx\left(\frac{b}{x}-\frac{a}{x}\right)+\int_{\sqrt{\frac{a}{c}}}^{\sqrt{\frac{b}{c}}}dx\left(\frac{b}{x}-cx\right)$$

$$=\left(\frac12dx^2-a\ln x\right)\Big|_{\sqrt{\frac{a}{d}}}^{\sqrt{\frac{b}{d}}}+(b-a)\ln x\Big|_{\sqrt{\frac{b}{d}}}^{\sqrt{\frac{a}{c}}}+\left(b\ln x-\frac12cx^2\right)\Big|_{\sqrt{\frac{a}{c}}}^{\sqrt{\frac{b}{c}}}$$

$$=\frac12b-\frac12a-\frac12a\ln\frac{b}{d}+\frac12a\ln\frac{a}{d}+\frac12(b-a)\ln\frac{a}{c}-\frac12(b-a)\ln\frac{b}{d}$$

$$+\frac12b\ln\frac{b}{c}-\frac12b\ln\frac{a}{c}-\frac12b+\frac12a=\frac12(b-a)\ln\frac{d}{c}$$

② $\sqrt{\frac{b}{d}}\ge\sqrt{\frac{a}{c}}$

$$\text{Area}(D)=\int_{\sqrt{\frac{a}{d}}}^{\sqrt{\frac{a}{c}}}dx\left(dx-\frac{a}{x}\right)+\int_{\sqrt{\frac{a}{c}}}^{\sqrt{\frac{b}{d}}}dx\,(dx-cx)+\int_{\sqrt{\frac{b}{d}}}^{\sqrt{\frac{b}{c}}}dx\left(\frac{b}{x}-cx\right)$$

$$=\left(\frac12dx^2-a\ln x\right)\Big|_{\sqrt{\frac{a}{d}}}^{\sqrt{\frac{a}{c}}}+\frac12(d-c)x^2\Big|_{\sqrt{\frac{a}{c}}}^{\sqrt{\frac{b}{d}}}+\left(b\ln x-\frac12cx^2\right)\Big|_{\sqrt{\frac{b}{d}}}^{\sqrt{\frac{b}{c}}}$$

$$=\frac12\frac{ad}{c}-\frac12a-\frac12a\ln\frac{a}{c}+\frac12a\ln\frac{a}{d}+\frac12(d-c)\left(\frac{b}{d}-\frac{a}{c}\right)+\frac12b\ln\frac{b}{c}-\frac12b\ln\frac{b}{d}$$

$$-\frac12b+\frac12\frac{bc}{d}=\frac12(b-a)\ln\frac{d}{c}$$

综上：D的面积为 $\frac12(b-a)\ln\frac{d}{c}$

2. $D=\{(x,y)\mid x>0,\ y>0,\ y\ge\sqrt{1-\frac{x^2}{4}},\ y\le\sqrt{4-\frac{x^2}{4}},\ y\le\sqrt{x^2-1},\ y\ge\sqrt{x^2-4}\}$

$\sqrt{1-\frac{x^2}{4}}\le\sqrt{x^2-1}\Rightarrow x\ge\sqrt{\frac85}$ $\quad\sqrt{4-\frac{x^2}{4}}\ge\sqrt{x^2-4}\Rightarrow x\le\sqrt{\frac{32}{5}}$. 即 $\sqrt{\frac85}\le x\le\sqrt{\frac{32}{5}}$

$$\min\{\sqrt{4-\tfrac{x^2}{4}},\sqrt{x^2-1}\}=\begin{cases}\sqrt{4-\frac{x^2}{4}}, & x\ge2\\ \sqrt{x^2-1}, & x\le2\end{cases}$$

$$\max\{\sqrt{1-\tfrac{x^2}{4}},\sqrt{x^2-4}\}=\begin{cases}\sqrt{x^2-4}, & x\ge2\\ \sqrt{1-\frac{x^2}{4}}, & x\le2\end{cases}$$

$$\iint_D\frac{xy}{x^2-y^2}dx\,dy=\int_{\sqrt{\frac85}}^{\sqrt{\frac{32}{5}}}dx\int_{\max\{\sqrt{1-\frac{x^2}{4}},\sqrt{x^2-4}\}}^{\min\{\sqrt{4-\frac{x^2}{4}},\sqrt{x^2-1}\}}dy\,\frac{xy}{x^2-y^2}$$

$$=\int_{\sqrt{\frac85}}^{\sqrt{\frac{32}{5}}}dx\left(-\frac12x\right)\ln(x^2-y^2)\Big|_{\max\{\sqrt{1-\frac{x^2}{4}},\sqrt{x^2-4}\}}^{\min\{\sqrt{4-\frac{x^2}{4}},\sqrt{x^2-1}\}}$$

$$=\int_{\sqrt{\frac85}}^{2}dx\left(-\frac12x\right)\ln\frac{4}{5x^2-4}+\int_2^{\sqrt{\frac{32}{5}}}dx\left(-\frac12x\right)\ln\left(\frac{5x^2}{16}-1\right)$$

$$=\frac15\int_{\sqrt{\frac85}}^{2}d\left(\frac{5x^2-4}{4}\right)\ln\frac{5x^2-4}{4}-\frac45\int_2^{\sqrt{\frac{32}{5}}}d\left(\frac{5x^2}{16}-1\right)\ln\left(\frac{5x^2}{16}-1\right)$$

$$=\frac15(x\ln x-x)\Big|_1^4-\frac45(x\ln x-x)\Big|_{\frac14}^{1}$$

$$=\frac85\ln2-\frac45+\frac15+\frac45+\frac15\ln\frac14-\frac15=\frac65\ln2$$

3. $\iiint_V z\,dx\,dy\,dz = \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} dx\,dy \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} z\,dz$

$= \frac{1}{2}\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} dx\,dy \left[\frac{1}{2}z^2\right]\Big|_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$

$= \frac{1}{2}\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} dx\,dy\ \frac{1}{2}c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$

$= \frac{c^2}{2}\int_{|x|\le a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dy$

$= \frac{c^2}{2}\int_{-a}^{a} dx \left[\left(1-\frac{x^2}{a^2}\right)y - \frac{y^3}{3b^2}\right]\Big|_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}$

$= \frac{c^2}{2}\int_{-a}^{a} dx \left[2b\sqrt{1-\frac{x^2}{a^2}}\left(1-\frac{x^2}{a^2}\right) - \frac{2}{3}b\sqrt{1-\frac{x^2}{a^2}}\left(1-\frac{x^2}{a^2}\right)\right]$

$= \frac{2c^2b}{3}\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2}dx$

$\underline{\underline{设\ a\cos\theta = x}}\ \frac{2bc^2}{3}\int_\pi^0 -a\sin^4\theta\,d\theta = \frac{2abc^2}{3}\int_0^\pi \sin^4\theta\,d\theta$

$= \frac{2abc^2}{3}\int_0^\pi \frac{1-2\cos 2\theta+\cos^2 2\theta}{4}d\theta = \frac{2abc^2}{3}\frac{\pi}{4} - \frac{abc^2}{3}\int_0^\pi \cos 2\theta\,d\theta + \frac{2abc^2}{3}\int_0^\pi \frac{1+\cos 4\theta}{8}d\theta$

$= \frac{\pi}{4}abc^2$

4. 设 $\begin{cases} u = x+y \\ v = y+z \\ w = x+z \end{cases} \Rightarrow \begin{cases} x = \frac{u-v+w}{2} \\ y = \frac{u+v-w}{2} \\ z = \frac{-u+v+w}{2} \end{cases} \Rightarrow \Phi(u,v,w) = \left(\frac{u-v+w}{2}, \frac{u+v-w}{2}, \frac{-u+v+w}{2}\right)$

是双射且 C^1 光滑

$J_\Phi = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix} = \begin{pmatrix} \frac12 & \frac12 & -\frac12 \\ -\frac12 & \frac12 & \frac12 \\ \frac12 & -\frac12 & \frac12 \end{pmatrix} \Rightarrow \det J_\Phi = \frac{1}{2}$

$V = \{(x,y,z) \mid x^2+y^2+z^2+xy+xz+yz\le 1\} \Rightarrow \{(x,y,z) \mid (x+y)^2+(y+z)^2+(x+z)^2\le 2\}$

$V' = \{(u,v,w) \mid u^2+v^2+w^2\le 2\} \Rightarrow \Phi(V') = V$

则 $\iiint_V 1\,dx\,dy\,dz = \iiint_{V'} 1\,|\det J_\Phi|\,du\,dv\,dw = \frac{1}{2}\iiint_{V'} 1\,du\,dv\,dw$

$= \frac{1}{2}\mathrm{vol}(\overline{B_r}(\sqrt{2})) = \frac{1}{2}\,\frac{4}{3}\pi\, 2\sqrt{2} = \frac{4\sqrt{2}}{3}\pi$

5. (1) pf: 由 Fubini $\iint_{[a,x_0]\times[c,d]} \frac{\partial f}{\partial x}dx\,dy = \int_a^{x_0}dx\int_c^d \frac{\partial f}{\partial x}dy$

$= \int_c^d dy\int_a^{x_0}\frac{\partial f(x,y)}{\partial x}dx = \int_c^d dy\,(f(x_0,y)-f(a,y))$

因此 $\int_c^d f(x_0,y)\,dy = \int_c^d f(a,y)\,dy + \int_a^{x_0}dx\int_c^d \frac{\partial f}{\partial x}dy$ 得证

(2) pf: 由上问 $\int_c^d f(x_0,y)\,dy - \int_c^d f(a,y)\,dy = \int_a^{x_0}dx\int_c^d\frac{\partial f}{\partial x}dy$

设 $h(x) = \int_c^d \frac{\partial f}{\partial y}dy$ 则

$g(x_0) - g(a) = \int_a^{x_0}dx\,h(x)$

$\Rightarrow g(x) - g(a) = \int_a^x h(t)\,dt$

求导: $g'(x) = h(x) = \int_c^d \frac{\partial f}{\partial y}dy$, 得证

6. 设 $S(t)=\{(1-t)(u,v,0)+t(0,0,1)\mid (u,v,0)\in D,\ t\in[0,1]\}$

$=\{((1-t)u,\ (1-t)v,\ t)\mid (u,v,0)\in D,\ t\in[0,1]\}$

设 $\Phi(u,v,t)=((1-t)u,\ (1-t)v,\ t)$

$J_\Phi=\begin{pmatrix} x_u & y_u & z_u\\ x_v & y_v & z_v\\ x_t & y_t & z_t\end{pmatrix}=\begin{pmatrix}1-t & 0 & 0\\ 0 & 1-t & 0\\ -u & -v & 1\end{pmatrix}$ $\det J_\Phi=(1-t)^2$

设 $V'=\{(u,v,t)\mid (u,v,0)\in D,\ t\in[0,\delta]\}$ D有界，$D\in[-x_0,x_0]\times[-y_0,y_0]$

$\Phi(V')=\{(1-t)(u,v,0)+t(0,0,1)\mid (u,v,0)\in D,\ t\in[0,\delta]\}$

$\iiint_{\Phi(V')} 1\,dx\,dy\,dz=\iiint_{V'} 1\,|\det J_\Phi|\,du\,dv\,dt$

$\left(\chi_D=\begin{cases}1, & \vec{x}\in V'\\ 0, & \vec{x}\notin V'\end{cases}\right)$ $=\iiint_{V'} 1\,(1-t)^2\,du\,dv\,dt$

$=\iiint_{[-x_0,x_0]\times[-y_0,y_0]\times[0,\delta]}\chi_D(1-t)^2\,du\,dv\,dt$

$=\int_0^\delta(1-t)^2dt\iint_{[-x_0,x_0]\times[-y_0,y_0]}\chi_D\,du\,dv$

$=\int_0^\delta S(1-t)^2dt=-\frac{1}{3}\int_0^\delta S(1-t)^2d(1-t)$

$=-\frac{1}{3}\left[S(1-t)^3\right]\Big|_0^\delta=-\frac{1}{3}\left[S(1-\delta)^3-S\right]$

$vol(V)=\iiint_V 1\,dx\,dy\,dz=\lim_{\delta\to1^-}\iiint_{\Phi(V')}1\,dx\,dy\,dz=\lim_{\delta\to1^-}-\frac{1}{3}\left[S(1-\delta)^3-S\right]=\frac{1}{3}S$

7. $\iiint_Q x^ay^bz^c\,dx\,dy\,dz=\iint_{\substack{x,y\ge0\\x+y\le1}}x^ay^b\,dx\,dy\int_0^{1-x-y}z^c\,dz$

$=\frac{1}{c+1}\iint_{\substack{x,y\ge0\\x+y\le1}}x^ay^b\,dx\,dy\,(1-x-y)^{c+1}$

$=\frac{1}{c+1}\int_0^1x^a\,dx\int_0^{1-x}(1-x-y)^{c+1}y^b\,dy$

$\int_0^{1-x}(1-x-y)^{c+b+1-n}y^n\,dy=\left[\frac{1}{c+2}(1-x-y)^{c+2}y^n\right]\Big|_0^{1-x}-\frac{n}{c+b+2-n}\int_0^{1-x}(1-x-y)^{c+b+2-n}y^{n-1}\,dy$

$=-\frac{n}{c+b+2-n}\int_0^{1-x}(1-x-y)^{c+b+2-n}y^{n-1}\,dy$

设 $I_n=\int_0^{1-x}(1-x-y)^{c+b+1-n}y^n\,dy$

则 $I_n=-\frac{n}{c+b+2-n}I_{n-1}$

$I_b=\left(-\frac{b}{c+2}\right)\left(-\frac{b-1}{c+3}\right)\cdots\left(-\frac{1}{c+b+1}\right)I_0$

$=\frac{b!\,(c+1)!}{(c+b+1)!}(-1)^bI_0$

而 $I_0=\int_0^{1-x}(1-x-y)^{c+b+1}dy=(-1)\int_0^{1-x}(1-x-y)^{c+b+1}d(1-x-y)$

$=-1\left[(1-x-y)^{c+b+2}\frac{1}{c+b+2}\right]\Big|_0^{1-x}$

$=\frac{(1-x)^{c+b+2}}{c+b+2}$

则 $I_n=\frac{b!\,(c+1)!}{(c+b+2)!}(-1)^b(1-x)^{c+b+2}$

$\iiint_Q x^ay^bz^c\,dx\,dy\,dz=\frac{1}{c+1}\frac{b!\,(c+1)!}{(c+b+2)!}(-1)^b\int_0^1x^a(1-x)^{c+b+2}dx$

$=\frac{b!\,c!}{(c+b+2)!}(-1)^b\int_0^1x^a(1-x)^{c+b+2}dx$

同理 $\int_0^1x^a(1-x)^{c+b+2}dx=\frac{a!\,(c+b+2)!}{(a+b+c+3)!}(-1)^a$,

则 $\iiint_Q x^ay^bz^c\,dx\,dy\,dz=\frac{a!\,b!\,c!}{(a+b+c+3)!}(-1)^{a+b}$