高等微积分I第十次作业

1. 解：(1) $\int \arcsin x\,dx = \int x'\arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx = x\arcsin x + \int\frac{d(1-x^2)}{2\sqrt{1-x^2}}$

$= x\arcsin x + \sqrt{1-x^2} + C$

(2) $\int \frac{x^2}{\sqrt{a^2-x^2}}dx \xlongequal{x=a\cos\theta} \int \frac{a^2\cos^2\theta}{a\sin\theta}\, d(a\cos\theta) = -\int a^2\cos^2\theta\, d\theta = -\int a^2 \frac{1+\cos2\theta}{2}d\theta$

$= -\frac{a^2\theta}{2} - \frac12 a^2\int\cos2\theta\, d\theta = -\frac{a^2}{2}\theta - \frac14 a^2\sin2\theta + C$

$\xlongequal{\theta=\arccos\frac{x}{a}} -\frac{a^2}{2}\arccos\frac{x}{a} - \frac12 a^2 \sin(\arccos\frac{x}{a})\frac{x}{a} + C$

$= -\frac{a^2}{2}\arccos\frac xa - \frac12 ax\sqrt{1-\frac{x^2}{a^2}} + C$

$= -\frac{a^2}{2}\arccos\frac xa - \frac12 x\sqrt{a^2-x^2}+C$

(3) $\int\frac{x+1}{\sqrt{x^2-4x}}dx = \int \frac{x-2+3}{\sqrt{(x-2)^2-4}}dx \xlongequal{t=x-2} \int\frac{t+3}{\sqrt{t^2-4}}dt = \int\frac{t}{\sqrt{t^2-4}}dt + \int\frac{3}{\sqrt{t^2-4}}dt$

而 $\int\frac{t}{\sqrt{t^2-4}}dt = \int\frac{d(t^2-4)}{2\sqrt{t^2-4}} = \sqrt{t^2-4}+C_1$

$\int\frac{3}{\sqrt{t^2-4}}dt \xlongequal{t=2\sec\theta} \int\frac{3}{\tan\theta}\frac{\sin\theta}{\cos^2\theta}d\theta = \int\frac{3}{\cos\theta}d\theta$

$=\int\frac{3\,d\sin\theta}{(1-\sin^2\theta)} \xlongequal{y=\sin\theta}\int\frac{3\,dy}{(1-y^2)} = \int\frac32\left(\frac{1}{1+y}+\frac{1}{1-y}\right)dy = \frac32\ln\left|\frac{1+y}{1-y}\right|+C_2$

$=\frac32\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|+C_2 = \frac32\ln\left|\frac{t+\sqrt{t^2-4}}{t-\sqrt{t^2-4}}\right|+C_2$

$\int\frac{x+1}{\sqrt{x^2-4x}} = \sqrt{t^2-4}+\frac32\ln\left|\frac{t+\sqrt{t^2-4}}{t-\sqrt{t^2-4}}\right|+C = \sqrt{x^2-4x}+\frac32\ln\left|\frac{x-2+\sqrt{x^2-4x}}{x-2-\sqrt{x^2-4x}}\right|+C$

(4) $\int\frac{1}{x^3+1}dx = \int\left(\frac{1}{3(x+1)}+\frac{-x+2}{3(x^2-x+1)}\right)dx = \frac13\ln|x+1| - \frac16\int\frac{x-\frac12}{(x-\frac12)^2+\frac34}dx + \int\frac{\frac12}{(x-\frac12)^2+\frac34}dx$

$=\frac13\ln|x+1| - \frac16\ln|x^2-x+1| + \frac12\frac{2}{\sqrt3}\arctan\left(\frac{2}{\sqrt3}(x-\frac12)\right)+C$

$=\frac13\ln|x+1|-\frac16\ln|x^2-x+1|+\frac{1}{\sqrt3}\arctan\left(\frac{2}{\sqrt3}x-\frac{1}{\sqrt3}\right)+C$

(5) $\int\frac{\sqrt x}{(1+x)^2}dx = \int -\sqrt x\, d\left(\frac{1}{1+x}\right) = -\frac{\sqrt x}{1+x} + \int\frac{1}{2(1+x)\sqrt x}dx \xlongequal{t=\sqrt x} -\frac{\sqrt x}{1+x}+\int\frac{1}{1+t^2}dt$

$= -\frac{\sqrt x}{1+x}+\arctan t + C = -\frac{\sqrt x}{1+x}+\arctan\sqrt x + C$

2. 解：(1) $\int_1^2 x\ln^2x\, dx = \int_1^2\left(\frac12x^2\right)'\ln^2x\, dx = \frac12x^2\ln^2x\Big|_1^2 - \frac12\int_1^2 x^2\, 2\ln x\,\frac1x dx = 2\ln^22 - \int_1^2 x\ln x\, dx$

$=2\ln^22 - \int_1^2\left(\frac12x^2\right)'\ln x\, dx = 2\ln^22 - \frac12x^2\ln x\Big|_1^2 + \int_1^2\frac12 x\, dx$

$=2\ln^22 - 2\ln2 + \frac14x^2\Big|_1^2 = 2\ln^22-2\ln2+\frac34$

(2) $\int_0^{\frac\pi4}\frac{\tan x}{\cos^2x}dx = \int_0^{\frac\pi4}\tan x\, d(\tan x) = \int_0^1 t\, dt = \frac12t^2\Big|_0^1 = \frac12$

(3) $\int_0^{\frac\pi2}\frac{1}{1+a\cos x}dx \xlongequal{x=2\arctan t}\int_0^1\frac{1}{1+a\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt = \int_0^1\frac{2\,dt}{1+t^2+a-at^2} = \frac{2}{1-a}\int_0^1\frac{dt}{t^2+\frac{1+a}{1-a}}$

$=\frac{2}{1-a}\sqrt{\frac{1-a}{1+a}}\arctan\left(\sqrt{\frac{1-a}{1+a}}\,t\right)\Big|_0^1 = 2\sqrt{\frac{1}{1-a^2}}\arctan\sqrt{\frac{1-a}{1+a}}$

3. 解：$\int_0^\pi\frac{(\cos x-a)\sin x}{(1+a^2-2a\cos x)^{3/2}}dx = -\int_0^\pi\frac{\cos x-a}{(1+a^2-2a\cos x)^{3/2}}d\cos x = -\int_1^{-1}\frac{t-a}{(1+a^2-2at)^{3/2}}dt = \int_{-1}^1\frac{t-a}{(1+a^2-2at)^{3/2}}dt$

$=-\frac{1}{2a}\left[\int_{-1}^1\frac{-2at+2a^2}{(1+a^2-2at)^{3/2}}dt\right] = -\frac1{2a}\left[\int_{-1}^1\frac{-2at+a^2+1}{(1+a^2-2at)^{3/2}}dt+\int_{-1}^1\frac{a^2-1}{(1+a^2-2at)^{3/2}}dt\right]$

$=-\frac1{2a}\left[\int_{-1}^1\frac{1}{(1+a^2-2at)^{1/2}}dt+\int_{-1}^1\frac{a^2-1}{(1+a^2-2at)^{3/2}}dt\right] = \frac{1}{4a^2}\int_{-1}^1\frac{d(1+a^2-2at)}{(1+a^2-2at)^{1/2}}+\frac{a^2-1}{4a^2}\int_{-1}^1\frac{d(1+a^2-2at)}{(1+a^2-2at)^{3/2}}$

$=\frac{1}{4a^2}\,2(1+a^2-2at)^{1/2}\Big|_{-1}^1 - \frac{a^2-1}{4a^2}\,2(1+a^2-2at)^{-1/2}\Big|_{-1}^1 = \frac{1}{2a^2}(|a-1|-|a+1|) - \frac{a^2-1}{2a^2}\left(\frac{1}{|a-1|}-\frac{1}{|a+1|}\right)$

4. (1) $\int \frac{dx}{a+\sin x} \xlongequal{x=2\arctan t} \int \frac{2\frac{1}{1+t^2}}{a+\frac{2t}{1+t^2}}dt = \int \frac{2}{(1+t^2)a+2t}dt = \int \frac{2\,dt}{(\sqrt{a}t+\frac{1}{\sqrt{a}})^2+a-\frac{1}{a}}$

$= \frac{2}{\sqrt{a}} \int \frac{d(\sqrt{a}t+\frac{1}{\sqrt{a}})}{(\sqrt{a}t+\frac{1}{\sqrt{a}})^2+a-\frac{1}{a}} = \frac{2}{\sqrt{a}} \frac{1}{\sqrt{a-\frac{1}{a}}} \arctan \frac{\sqrt{a}t+\frac{1}{\sqrt{a}}}{\sqrt{a-\frac{1}{a}}} + C$

$= \frac{2}{\sqrt{a^2-1}} \arctan \frac{at+1}{\sqrt{a^2-1}} + C = \frac{2}{\sqrt{a^2-1}} \arctan \frac{a\tan\frac{x}{2}+1}{\sqrt{a^2-1}} + C$

(2) $\int_0^{2\pi} \frac{dx}{a^2-\sin^2 x} = \frac{1}{2a}\int_0^{2\pi}\left(\frac{1}{a+\sin x}+\frac{1}{a-\sin x}\right)dx = \frac{1}{2a}\left[\int_0^{2\pi}\frac{1}{a+\sin x}dx + \int_0^{2\pi}\frac{d(-\pi+x)}{a+\sin(-\pi+x)}dx\right]$

$= \frac{1}{2a}\left[\int_0^{2\pi}\frac{1}{a+\sin x}dx + \int_{-\pi}^{\pi}\frac{1}{a+\sin x}dx\right]$

设 $F_1(x) = \begin{cases} \frac{2}{\sqrt{a^2-1}}\arctan\frac{a\tan\frac{x}{2}+1}{\sqrt{a^2-1}}, & x\in(-\pi,\pi) \\ -\frac{\pi}{\sqrt{a^2-1}}, & x=-\pi \\ \frac{\pi}{\sqrt{a^2-1}}, & x=\pi \end{cases}$

则 $F_1(x)\in C([-\pi,\pi])$，且 $[-\pi,\pi]$ 上 $F_1'(x)=\frac{1}{a+\sin x}$

由 N-L 公式 $\int_{-\pi}^{\pi}\frac{1}{a+\sin x}dx = F_1(x)\Big|_{-\pi}^{\pi} = \frac{2\pi}{\sqrt{a^2-1}}$

$\int_0^{\pi}\frac{1}{a+\sin x}dx = F_1(x)\Big|_0^{\pi} = \frac{\pi}{\sqrt{a^2-1}} - F_1(0)$

设 $F_2(x) = \begin{cases} \frac{2}{\sqrt{a^2-1}}\arctan\frac{a\tan\frac{x}{2}+1}{\sqrt{a^2-1}}, & x\in(\pi,2\pi] \\ -\frac{\pi}{\sqrt{a^2-1}}, & x=\pi \end{cases}$

则 $F_2(x)\in C([\pi,2\pi])$ 且 $[\pi,2\pi]$ 上 $F_2'(x)=\frac{1}{a+\sin x}$

由 N-L 公式 $\int_{\pi}^{2\pi}\frac{1}{a+\sin x}dx = F_2(2\pi)+\frac{\pi}{\sqrt{a^2-1}}$

则 $\int_0^{2\pi}\frac{dx}{a^2-\sin^2x} = \frac{1}{2a}\left[\int_0^{\pi}\frac{1}{a+\sin x}dx+\int_{\pi}^{2\pi}\frac{1}{a+\sin x}dx+\int_{-\pi}^{\pi}\frac{1}{a+\sin x}dx\right]$

$= \frac{1}{2a}\left[\frac{\pi}{\sqrt{a^2-1}} - F_1(0) + F_2(2\pi) + \frac{\pi}{\sqrt{a^2-1}} + \frac{2\pi}{\sqrt{a^2-1}}\right] = \frac{2\pi}{a\sqrt{a^2-1}}$

5. (1) $\int \frac{1}{x^4+1}dx = \int\left(\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} - \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}\right)dx$

$= \frac{1}{2\sqrt{2}}\left[\int\frac{\frac{1}{2}d(x^2+\sqrt{2}x+1)}{x^2+\sqrt{2}x+1} + \int\frac{\frac{\sqrt{2}}{2}dx}{x^2+\sqrt{2}x+1} - \int\frac{\frac{1}{2}d(x^2-\sqrt{2}x+1)}{x^2-\sqrt{2}x+1} + \int\frac{\frac{\sqrt{2}}{2}dx}{x^2-\sqrt{2}x+1}\right]$

$= \frac{1}{2\sqrt{2}}\left[\frac{1}{2}\ln(x^2+\sqrt{2}x+1) - \frac{1}{2}\ln(x^2-\sqrt{2}x+1) + \frac{\sqrt{2}}{2}\int\frac{dx}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{\sqrt{2}}{2}\int\frac{dx}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right]$

$= \frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{4}\sqrt{2}\arctan\sqrt{2}(x+\frac{\sqrt{2}}{2}) + \frac{1}{4}\sqrt{2}\arctan\sqrt{2}(x-\frac{\sqrt{2}}{2}) + C$

(2) $\int_0^A \frac{1}{x^4+1}dx = \left[\frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x+1) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x-1)\right]\Big|_0^A$

$= \frac{1}{4\sqrt{2}}\ln\frac{A^2+\sqrt{2}A+1}{A^2-\sqrt{2}A+1} + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}A+1) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}A-1)$

$\lim\limits_{A\to+\infty}\frac{1}{4\sqrt{2}}\ln\frac{A^2+\sqrt{2}A+1}{A^2-\sqrt{2}A+1} = \frac{1}{4\sqrt{2}}\lim\limits_{A\to+\infty}\ln\frac{1+\frac{\sqrt{2}}{A}+\frac{1}{A^2}}{1-\frac{\sqrt{2}}{A}+\frac{1}{A^2}} = 0$

$\lim\limits_{A\to+\infty}\frac{\sqrt{2}}{4}\arctan(\sqrt{2}A-1) = \lim\limits_{A\to+\infty}\frac{\sqrt{2}}{4}\arctan(\sqrt{2}A+1) = \frac{\sqrt{2}}{8}\pi$

$\therefore \int_0^{+\infty}\frac{1}{x^4+1}dx = \lim\limits_{A\to+\infty}\int_0^A\frac{1}{x^4+1}dx = \frac{\sqrt{2}}{4}\pi$

6、(1) 设 $F(x)=\int_0^x f(t)\,dt$

$$\int_0^1 f(nx)dx=\frac{1}{n}\int_0^n f(x)dx=\frac{1}{n}F(n)$$

$$\lim_{n\to\infty}\int_0^1 f(nx)dx=\lim_{n\to\infty}\frac{F(n)}{n}=\lim_{x\to+\infty}\frac{F(x)}{x}=\lim_{x\to+\infty}\frac{F'(x)}{x'}=\lim_{x\to+\infty}f(x)=L$$

(2) 设 $C=\int_0^T h(t)dt$

$$\int_0^T g(x)h(nx)dx \overset{x=\frac{t}{n}}{=\!=} \frac{1}{n}\int_0^{nT} g(\tfrac{t}{n})h(t)dt=\frac{1}{n}\sum_{i=1}^{n}\int_{(i-1)T}^{iT} g(\tfrac{t}{n})h(t)\,dt \quad ①$$

$\forall\varepsilon>0$ $g(x)$在$[0,T]$上连续，则在$[0,T]$上一致连续、$\forall\varepsilon'=\frac{\varepsilon}{C}$ $\exists\delta>0$ $\forall |x-y|<\delta$ 有 $|g(x)-g(y)|<\varepsilon'$

取 $N=\frac{T}{\delta}$ $n\geq N$ 时 $\frac{T}{n}\leq\delta$

$$\left|\frac{1}{n}\sum_{i=1}^{n}\int_{(i-1)T}^{iT} g(\tfrac{t}{n})h(t)dt-\frac{1}{n}\sum_{i=1}^{n}\int_{(i-1)T}^{iT} g(\tfrac{iT}{n})h(t)dt\right|$$

$$=\left|\frac{1}{n}\sum_{i=1}^{n}\left[\int_{(i-1)T}^{iT}\left(g(\tfrac{t}{n})-g(\tfrac{iT}{n})\right)h(t)dt\right]\right|\leq\left|\frac{1}{n}\sum_{i=1}^{n}\int_{(i-1)T}^{iT}\varepsilon'\;h(t)dt\right|$$

$$=\left|\frac{\varepsilon'}{n}\sum_{i=1}^{n}\int_0^T h(t)dt\right|$$

$$=\left|\frac{\varepsilon'}{n}nC\right|=\varepsilon$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int_{(i-1)T}^{iT} g(\tfrac{t}{n})h(t)dt=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int_{(i-1)T}^{iT} g(\tfrac{iT}{n})h(t)dt \quad ②$$

$$=C\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}g(\tfrac{iT}{n})$$

而对于 $g(x)$ 在 $[0,T]$ 上，

取划分 $x_i=\frac{iT}{n}$

Riemann 和 $\int_0^T g(x)dx=\lim_{\frac{T}{n}\to 0}\sum_{i=1}^{n}g(\tfrac{iT}{n})\left(\tfrac{iT}{n}-\tfrac{i-1}{n}T\right)$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}g(\tfrac{iT}{n})\cdot\frac{T}{n} \Rightarrow \frac{1}{T}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}g(\tfrac{iT}{n}) \quad ③$$

结合①~③ $\lim_{n\to\infty}\int_0^T g(x)h(nx)dx=\frac{1}{T}\left(\int_0^T g(x)dx\right)\left(\int_0^T h(x)dx\right)$