高等微积分2第九周作业

1. (1) $\frac{\partial f}{\partial x} = \cos x \sin y \qquad \frac{\partial f}{\partial y} = (2+\sin x)\cos y$

若 (x,y) 为临界点，则满足临界点方程

$$\begin{cases} \frac{\partial f}{\partial x} = \cos x \sin y = 0 \\ \frac{\partial f}{\partial y} = (2+\sin x)\cos y = 0 \end{cases}$$

① $\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ 或 $\frac{3}{2}\pi$ 此时也要满足 $\cos y = 0 \Rightarrow y = \frac{\pi}{2}$ 或 $\frac{3}{2}\pi$

② $\sin y = 0 \Rightarrow y = \pi$ 此时 $(2+\sin x)\cos y = -(2+\sin x) \neq 0$ 故此种情况无临界点

∴ 临界点有四个 $(\frac{\pi}{2},\frac{\pi}{2})$ $(\frac{\pi}{2},\frac{3\pi}{2})$ $(\frac{3\pi}{2},\frac{\pi}{2})$ 和 $(\frac{3\pi}{2},\frac{3\pi}{2})$

(2) $\frac{\partial^2 f}{\partial x^2} = -\sin x \sin y \qquad \frac{\partial^2 f}{\partial x \partial y} = \cos x \cos y \qquad \frac{\partial^2 f}{\partial y^2} = -(2+\sin x)\sin y$

i) $(\frac{\pi}{2},\frac{\pi}{2})$. $Hf(x_0) = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$ 为负定矩阵，故 $(\frac{\pi}{2},\frac{\pi}{2})$ 为极大值点

ii) $(\frac{\pi}{2},\frac{3\pi}{2})$ $Hf(x_0) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ 为正定矩阵，故 $(\frac{\pi}{2},\frac{3\pi}{2})$ 为极小值点

iii) $(\frac{3\pi}{2},\frac{\pi}{2})$ $Hf(x_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 为不定矩阵，故不为极值点

iv) $(\frac{3\pi}{2},\frac{3\pi}{2})$ $Hf(x_0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ 为不定矩阵，故不为极值点

2. (1) $B = \overline{B_0}(1)$ 为闭球，显然紧致且有界

$\Rightarrow$ 根据最值Thm，连续函数 f 在 B 上有最大值.

(2) 设 $P(x_0, y_0)$ 为最大值

① $P \in B$ 内部 $\quad \frac{\partial f}{\partial x} = 1+yz \quad \frac{\partial f}{\partial y} = 1+xz \quad \frac{\partial f}{\partial z} = 1+xy$

P 满足临界点方程 $\begin{cases} 1+yz=0 & (1) \\ 1+xz=0 & (2) \\ 1+xy=0 & (3) \end{cases}$

(3)代入 $x\cdot$(1)式 $x+z(-1)=0 \Rightarrow x=z$ 同理 $x=y=z \Rightarrow 1+x^2=0$ 在 $\mathbb{R}$ 上无解

② $P \in \partial B$ 即约束为 $x^2+y^2+z^2=1$

设辅助函数为 $F(x,y,z) = f - \lambda(x^2+y^2+z^2-1) = x+y+z+xyz-\lambda(x^2+y^2+z^2-1)$

$$\begin{cases} \frac{\partial F}{\partial x} = 1+yz-2\lambda x = 0 & ① \\ \frac{\partial F}{\partial y} = 1+xz-2\lambda y = 0 & ② \\ \frac{\partial F}{\partial z} = 1+xy-2\lambda z = 0 & ③ \\ \frac{\partial F}{\partial \lambda} = -(x^2+y^2+z^2-1) = 0 & ④ \end{cases}$$

x, y, z 均不为0

不妨设 $y=0$ $\quad 1+xz=0 \Rightarrow xy=-1$

$\Rightarrow |x||y| = 1 \leq \frac{x^2+y^2}{2} \Rightarrow x^2+y^2 \geq 2$ 与④矛盾

故 x, y, z 均不为0

②代入①. $(x-y)(1+xz+yz)=0$

同理 $(x-z)(1+yx+yz)=0$

$(y-z)(1+xy+xz)=0$

由(1)式 $x=y$ 或 $1+xz+yz=0$

若 $1+xz+yz=0$ $\quad z(x+y)=-1 \Rightarrow |z||x+y|=1$

$\Rightarrow 1 \leq \frac{|x+y|^2+|z|^2}{2} \leq \frac{(|x|+|y|)^2+|z|^2}{2} \leq \frac{2|x|^2+2|y|^2+|z|^2}{2}$ 这与④矛盾

因此 $x=y$ 同理 $y=z$ 有 $x=y=z$，代入④

$3x^2=1 \Rightarrow x=\pm\frac{\sqrt{3}}{3}$ 即 $x=y=z=\pm\frac{\sqrt{3}}{3}$

$(x,y,z) = (\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3})$ 时 $f = \frac{10\sqrt{3}}{9}$

$(x,y,z) = (-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3})$ 时 $f = -\frac{10\sqrt{3}}{9}$

综上：f 在 B 上最大值为 $\frac{10\sqrt{3}}{9}$

3、 设 Lagrange 辅助函数为 $F(x,y,z,\mu,\lambda) = xyz - \mu(x+y+z-1) - \lambda(x^2+y^2+z^2-1)$

$$\begin{cases} \frac{\partial F}{\partial x} = yz - \mu - 2\lambda x = 0 & ① \\ \frac{\partial F}{\partial y} = xz - \mu - 2\lambda y = 0 & ② \\ \frac{\partial F}{\partial z} = xy - \mu - 2\lambda z = 0 & ③ \\ \frac{\partial F}{\partial \mu} = -(x+y+z-1) = 0 & ④ \\ \frac{\partial F}{\partial \lambda} = -(x^2+y^2+z^2-1) = 0 & ⑤ \end{cases}$$

②－① 得

$(x-y)z + 2\lambda(x-y) = 0$

$\Rightarrow (2\lambda+z)(x-y) = 0$ ⑥

同理 $(2\lambda+x)(y-z) = 0$ ⑦

$(2\lambda+y)(x-z) = 0$ ⑧

设 $x=y=z$ 则 由④ $x=y=z=\frac{1}{3}$ 与⑤不符

故 $2\lambda+z=0$，$2\lambda+x=0$，$2\lambda+y=0$ 有一个必成立

不妨设 $2\lambda+x=0$ 则 由⑧ $2\lambda+z=0$ 与 $x-y=0$ 必有一式成立

若 $2\lambda+z=0 \Leftrightarrow x=z$，若 $x-y=0 \Leftrightarrow x=y$

即⑥说明 y, z 必有一个等于 x，不妨设 $x=y$

代入④⑤ $\begin{cases} 2x+z=1 \\ 2x^2+z^2=1 \end{cases} \Rightarrow 2x^2+(1-2x)^2=1 \Rightarrow 6x^2-4x=0 \Rightarrow x=0$ 或 $\frac{2}{3}$

i) 若 $x=y=0$ 则 $z=1$，此时 $\mu=0$ $\lambda=0$，$f=0$

ii) 若 $x=y=\frac{2}{3}$ 则 $z=-\frac{1}{3}$ 此时 $\mu=\frac{2}{9}$ $\lambda=-\frac{1}{3}$，$f=-\frac{4}{27}$

综上 $f=xyz$ 最小值为 $-\frac{4}{27}$

4、若在 S 约束下求 f 最大值，则先判断是否是奇异点，设 $g=\sum_{i=1}^{n} x_i^2-1$

$P(x_1, x_2 \dots x_n)$ 是奇异点则 $\frac{\partial g}{\partial x_i}\Big|_P = 2x_i = 0 \quad \forall 1\le i\le n$，此点显然不满足约束

则 $\left(\frac{\partial g_i}{\partial x_j}\right)_{1\le j\le n} \in M_{1\times n}$ 在 S 上处处满秩，则处处为光滑点

设 Lagrange 辅助函数为 $F(x_1 \dots x_n, \lambda) = f - \lambda' g$

则 $(\bar{x}_1 \dots \bar{x}_n)$ 满足 F 的临界点方程

$$\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} - 2\lambda' x_i = 0 \quad \forall 1\le i\le n$$

令 $\lambda = 2\lambda'$ 则 $\frac{\partial f}{\partial x_i} = \lambda x_i \quad \forall 1\le i\le n$

此即 $\left(\frac{\partial f}{\partial x_1} \dots \frac{\partial f}{\partial x_n}\right) = \lambda(\bar{x}_1 \dots \bar{x}_n)$

5. $h(t) = f(x_1(t), \dots, x_n(t))$

(1) $h'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1(t) \dots x_n(t))\, x_i'(t)$

$h''(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(x_1(t) \dots x_n(t))\, x_i'(t) x_j'(t) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1(t) \dots x_n(t))\, x_i''(t)$

(2) 若 p 是 f 在约束 $g=0$ 下的条件极值点，且 p 是 $g=0$ 的光滑点，那么 $\exists \lambda \in \mathbb{R}$ 使 $(x_1(0) \dots x_n(0), \lambda)$ 是 $F(x_1 \dots x_n, \lambda) = f(x_1 \dots x_n) - \lambda g(x_1 \dots x_n)$ 的临界点（也即 $\frac{\partial f}{\partial x_i}\Big|_p - \lambda \frac{\partial g}{\partial x_i}\Big|_p = 0 \quad \forall 1\le i\le n$ 成立且 $g(x_1(0) \dots x_n(0)) = 0$）

(3) $\because g(x_1(t) \dots x_n(t)) = 0 \Rightarrow \frac{\partial g}{\partial x_i}\Big|_{(x_1(t) \dots x_n(t))} x_i'(t) = 0$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g}{\partial x_j \partial x_i}(x_1(t) \dots x_n(t))\, x_i'(t) x_j'(t) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x_1(t) \dots x_n(t))\, x_i''(t) = 0$$

由 (2) $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_p = \lambda \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\Big|_p$

则
$$\begin{aligned} \sum\sum \frac{\partial^2 F}{\partial x_i \partial x_j}\Big|_p x_i'(0) x_j'(0) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p x_i'(0) x_j'(0) - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda \frac{\partial^2 g}{\partial x_i \partial x_j}\Big|_p x_i'(0) x_j'(0) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p x_i'(0) x_j'(0) + \sum_{i=1}^{n} \lambda \frac{\partial g}{\partial x_i}\Big|_p x_i''(0) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p x_i'(0) x_j'(0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_p x_i''(0) \\ &= h''(0) \end{aligned}$$

6、证明 $\exists \lambda \geq 0$ s.t $\begin{cases} f_x(x_0,y_0)-\lambda g_x(x_0,y_0)=0 \\ f_y(x_0,y_0)-\lambda g_y(x_0,y_0)=0 \end{cases}$ 即证 $\exists \lambda \geq 0$ s.t. $\nabla f(x_0,y_0)=\lambda \nabla g(x_0,y_0)$

首先，(x_0,y_0) 是 $g=0$ 约束下 f 极值点且不为奇异点

因此 (x_0,y_0) 是 Lagrange 辅助函数 $F=f-\lambda g$ 的临界点

即 $\begin{cases} F_x=f_x-\lambda g_x=0 & ① \\ F_y=f_y-\lambda g_y=0 & ② \\ F_\lambda=-g=0 & ③ \end{cases}$ 由①② $\nabla f(x_0,y_0)=\lambda \nabla g(x_0,y_0)$.

下只需证 $\lambda \geq 0$ 设 $\vec{x_0}=(x_0,y_0)$

$\lim\limits_{t\to 0^+} \dfrac{g(\vec{x_0}+t\nabla g(\vec{x_0}))-g(\vec{x_0})}{t} = \nabla_{\nabla g(\vec{x_0})} g(\vec{x_0}) = \nabla g(\vec{x_0})\cdot \nabla g(\vec{x_0}) = |\nabla g(\vec{x_0})|^2 \geq 0$

该说明 $\exists \delta$ s.t $\forall 0<t<\delta$ 有 $g(\vec{x_0}+t\nabla g(\vec{x_0}))-g(\vec{x_0})\geq 0$

因而 $\forall 0<t<\delta$ $\vec{x_0}+t\nabla g(\vec{x_0})\in D$ $\therefore f(\vec{x_0}+t\nabla g(\vec{x_0}))\geqslant f(\vec{x_0})$

所以 $\lim\limits_{t\to 0^+} \dfrac{f(\vec{x_0}+t\nabla g(\vec{x_0}))-f(\vec{x_0})}{t} = \nabla_{\nabla g(\vec{x_0})} f(\vec{x_0}) = \nabla g(\vec{x_0})\cdot \nabla f(\vec{x_0})$

$= \lambda |\nabla g(\vec{x_0})|^2 \geq 0$

$\therefore \lambda \geq 0$ 成立 □