高等微积分第八次作业

1、 Peano余项：若f在x_0处有n阶导，则f可写作

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n) \text{ as } x\to x_0$$

其中$o((x-x_0)^n)$为某函数$\alpha(x)$满足$\lim\limits_{x\to x_0}\frac{\alpha(x)}{(x-x_0)^n}=0$.

Lagrange余项：若f在区间I上n阶可导，$\forall a,b\in I$ 有$\xi\in\mathbb{R}$严格介于a,b之间]，使

$$f(b)=f(a)+\frac{f'(a)}{1!}(b-a)+\cdots+\frac{f^{(n-1)}(x_0)}{(n-1)!}(b-a)^{n-1}+\frac{f^{(n)}(\xi)}{n!}(b-a)^n$$

2、 f在0处连续：$f(0)=\lim\limits_{x\to0}\frac{x}{e^x-1}\xlongequal{L'H\hat{o}pital}\lim\limits_{x\to0}\frac{1}{e^x}=1$

~~计算导数~~

$$f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{\frac{x}{e^x-1}-1}{x}=\lim_{x\to0}\frac{x+1-e^x}{x(e^x-1)}$$

$$\xlongequal{L'H\hat{o}pital}\lim_{x\to0}\frac{1-e^x}{e^x-1+xe^x}\xlongequal{L'H\hat{o}pital}\lim_{x\to0}\frac{-e^x}{e^x+e^x+xe^x}=-\frac{1}{2}$$

$x\neq0$时 $f'(x)=\frac{(e^x-1)-e^x x}{(e^x-1)^2}=\frac{(1-x)e^x-1}{(e^x-1)^2}$

$$f''(0)=\lim_{x\to0}\frac{f'(x)-f'(0)}{x-0}=\lim_{x\to0}\frac{\frac{(1-x)e^x-1}{(e^x-1)^2}+\frac{1}{2}}{x}=\lim_{x\to0}\frac{(1-x)e^x-1+\frac{1}{2}(e^x-1)^2}{x(e^x-1)^2}$$

$$=\lim_{x\to0}\frac{(1-x)e^x-1+\frac{1}{2}(e^x-1)^2}{x^3}\lim_{x\to0}\frac{x^2}{(e^x-1)^2}$$

$$=\lim_{x\to0}\frac{(1-x)(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3))-1+\frac{1}{2}(x+\frac{1}{2}x^2+o(x^2))(x+\frac{1}{2}x^2+o(x^2))}{x^3}$$

$$=\lim_{x\to0}\frac{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3-x-x^2-\frac{1}{2}x^3-\frac{1}{6}x^4-1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{2}x^3+o(x^3)}{x^3}$$

$$=\lim_{x\to0}\frac{\frac{1}{6}x^3-\frac{1}{24}x^4+o(x^3)}{x^3}=\frac{1}{6}$$

则 $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2}x^2+o(x^2)$

$=1-\frac{1}{2}x+\frac{1}{12}x^2+o(x^2)$

3、

$$f^{(k)}(x)=\left((1-x)^{\frac{1}{2}}\right)^{(k)}=\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)\ (1-x)^{\frac{1}{2}-k}$$

$$f^{(k)}(\xi)=\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)(1-\xi)^{\frac{1}{2}-k}$$

将$f(x)$用Lagrange余项的Taylor公式展开至2次项，$\exists\xi\in(0,\frac{1}{4})$

$$f(x)=1-\frac{1}{2}x+\frac{f^{(2)}(\xi)}{2}x^2$$

$$=1-\frac{1}{2}x-\frac{1}{8}(1-\xi)^{-\frac{3}{2}}x^2$$

$$\Rightarrow \left|f(x)-(1-\frac{1}{2}x)\right|\leq\frac{1}{8}(1-\xi)^{-\frac{3}{2}}x^2\leq\frac{1}{128}\frac{1}{(1-\xi)^{\frac{3}{2}}}\leq\frac{1}{128}\frac{1}{(1-\frac{1}{4})^{\frac{3}{2}}}$$

$$=\frac{1}{128}\frac{8}{5\sqrt{5}}=\frac{1}{80\sqrt{5}}<\frac{1}{100}$$

则 $p(x)=1-\frac{1}{2}x$ 即为所求

4、 (1)

$$F'(x)=f'(x)+\left(\frac{f^{(2)}(x)}{1!}(b-x)-f^{(1)}(x)\right)+\left(\frac{f^{(3)}(x)}{2!}(b-x)^2-\frac{f^{(2)}}{1!}(b-x)\right)$$
$$+\cdots+\left(\frac{f^{(n+1)}(x)}{n!}(b-x)^n-\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}\right)$$

$$=\frac{f^{(n+1)}(x)}{n!}(b-x)^n$$

(2) 设$g(x)=x$

由Cauchy中值定理 $\exists\xi$介于a,b之间 s.t. $\frac{F(b)-F(a)}{g(b)-g(a)}=\frac{F'(\xi)}{g'(\xi)}$

$$\Leftrightarrow\frac{F(b)-F(a)}{b-a}=\frac{f^{(n+1)}(\xi)}{n!}(b-\xi)^n\Leftrightarrow F(b)-F(a)=(b-a)\frac{f^{(n+1)}(\xi)}{n!}(b-\xi)^n \quad \#$$

5. (1) $f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + o(h^3)$

$= f(x) + f^{(1)}(x)h + \frac{f^{(2)}(x)}{2}h^2 + \frac{f^{(3)}(x)}{6}h^3 + o(h^3)$ as $h \to 0$.

(2) $\lim\limits_{h\to 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3}$

$$= \lim_{h\to 0} \frac{\begin{array}{l} f(x) + 3f^{(1)}(x)h + \frac{9}{2}f^{(2)}(x)h^2 + \frac{9}{2}f^{(3)}(x)h^3 - 3f(x) - 6f^{(1)}(x) - 6f^{(2)}(x) - 4f^{(3)}(x) \\ \quad + 3f(x) + 3f^{(1)}(x)h + \frac{3}{2}f^{(2)}(x)h^2 + \frac{1}{2}f^{(3)}(x)h^3 - f(x) + o(h^3) \end{array}}{h^3}$$

$$= \lim_{h\to 0} \frac{f^{(3)}(x)h^3}{h^3} = f^{(3)}(x)$$

6. (1) $f'(x) = \frac{\frac{1}{x}x^{\alpha} - \alpha x^{\alpha-1}\ln x}{x^{2\alpha}} = \frac{x^{\alpha-1} - \alpha x^{\alpha-1}\ln x}{x^{2\alpha}} = \frac{1-\alpha\ln x}{x^{\alpha+1}}$

则 在 $(0, e^{\frac{1}{\alpha}}]$ $f'(x) \leqslant 0$ $\Rightarrow$ $\forall x \in (0, e^{\frac{1}{\alpha}}]$ 由 Lagrange Thm，$\exists \xi \in (x, e^{\frac{1}{\alpha}})$

使 $f'(\xi) = \frac{f(e^{\frac{1}{\alpha}}) - f(x)}{e^{\frac{1}{\alpha}} - x} \leqslant 0$

$\Rightarrow f(x) \leqslant f(e^{\frac{1}{\alpha}})$.

同理 $\forall x \in [e^{\frac{1}{\alpha}}, +\infty)$ $f(x) \leqslant f(e^{\frac{1}{\alpha}})$

$\therefore$ f 在 $(0,+\infty)$ 最大值为 $f(e^{\frac{1}{\alpha}}) = \frac{1}{\alpha e}$

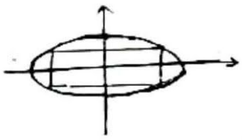
(2) 设 $f(x) = \sqrt[x]{x} = x^{1/x}$ $(x \geqslant 1)$

$f'(x) = (e^{\frac{1}{x}\ln x})' = x^{1/x}\left[-\frac{1}{x^2}\ln x + \frac{1}{x^2}\right] = \frac{x^{1/x}}{x^2}(1-\ln x)$

则 在 $[1,e]$ 上 $f'(x) \geqslant 0 \Rightarrow f(x)\uparrow \Rightarrow \forall x \in [1,2]$ $f(x) \leqslant f(2)$

$[e,+\infty)$ 上 $f'(x) \leqslant 0 \Rightarrow f(x)\downarrow \Rightarrow \forall x \in [3,+\infty)$ $f(x) \leqslant f(3)$

$\therefore \max\limits_{n\geqslant 1} \sqrt[n]{n} = \max\{f(2), f(3)\} = \max\{\sqrt{2}, \sqrt[3]{3}\} = \sqrt[3]{3}$.

7. 

设内接的点在第一象限为 (x,y) $(0<x<a,\ 0<y<b)$

则 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = b\sqrt{1-\frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2}$

设 $C(x) = 4x + 4y = 4\left(\frac{b}{a}\sqrt{a^2-x^2} + x\right)$.

$C'(x) = 4\left(\frac{b}{a}\frac{-x}{\sqrt{a^2-x^2}} + 1\right) = 4\,\frac{-\frac{b}{a}x + \sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}$

$\forall x \in (0, \frac{a^2}{\sqrt{a^2+b^2}})$ $C'(x) \geqslant 0 \Rightarrow C(x)\uparrow$ $C(x) \leqslant C(\frac{a^2}{\sqrt{a^2+b^2}})$

$x \in (\frac{a^2}{\sqrt{a^2+b^2}}, a)$ $C'(x) \leqslant 0 \Rightarrow C(x)\downarrow$ $C(x) \leqslant C(\frac{a^2}{\sqrt{a^2+b^2}})$

$\therefore \forall x \in (0,a)$ $C(x) \leqslant C(\frac{a^2}{\sqrt{a^2+b^2}}) = 4\sqrt{a^2+b^2}$

设 $S(x) = 4xy = 4\frac{b}{a}\sqrt{a^2-x^2}\cdot x = 4\frac{b}{a}\sqrt{a^2x^2-x^4}$

$= 4\frac{b}{a}\sqrt{-(x^2-\frac{a^2}{2})^2 + \frac{a^4}{4}}$

$\leqslant 4\frac{b}{a}\sqrt{\frac{a^4}{4}} = 2ab$.

取等时 $x = \frac{\sqrt{2}}{2}a$，可以取到

综上：面积最大值 $2ab$，周长最大值 $4\sqrt{a^2+b^2}$