第 九 讲

下推自动机

下推自动机

- PDA介绍
- PDA的定义
- PDA的即时描述
- PDA的语言
- PDA与CFG的关系

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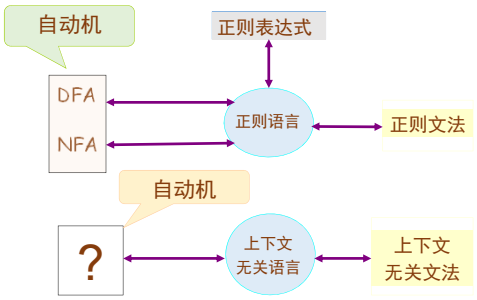
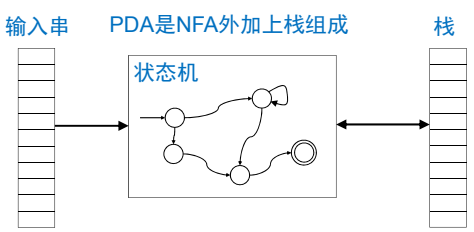
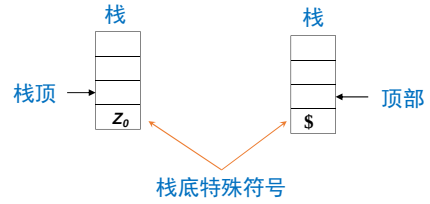


PDA介绍



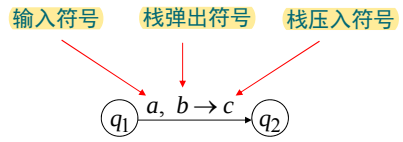
PDA介绍

初始栈底符号



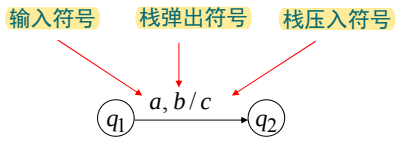
PDA介绍

状态转移

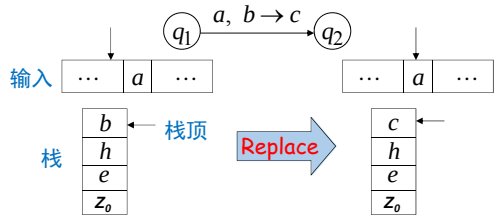


PDA介绍

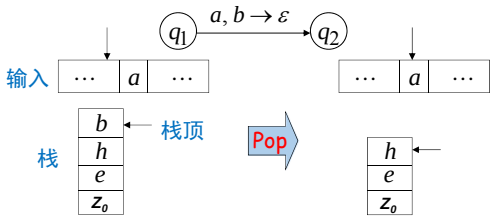
状态转移



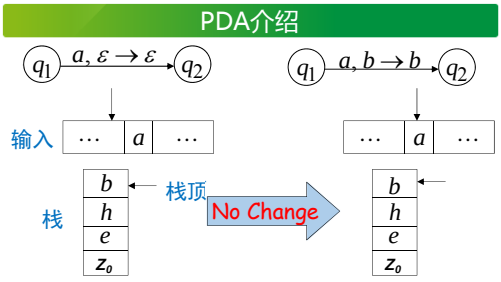
PDA介绍



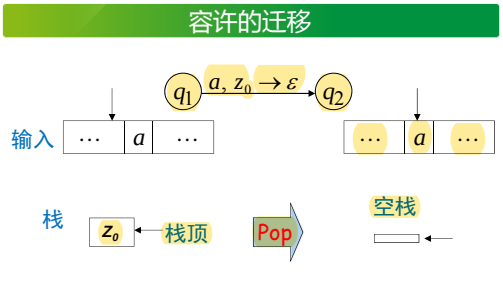
PDA介绍



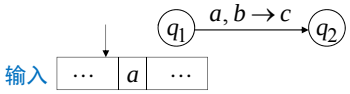
PDA介绍



容许的迁移



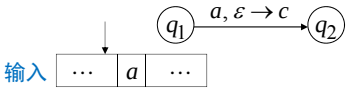
不容许的迁移



停机

自动机在q_1状态停机，且拒绝输入串

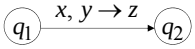
不容许的迁移

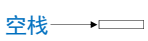


停机

自动机在q_1状态停机，且拒绝输入串

不容许的迁移

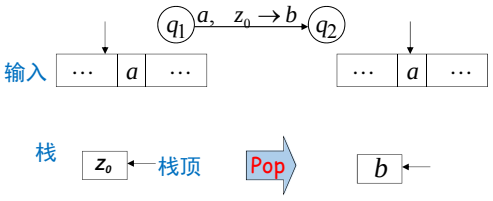




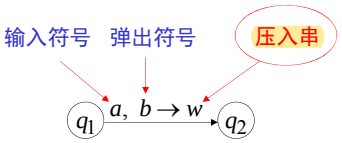
停机

当栈为空时不允许进行迁移

容许的迁移



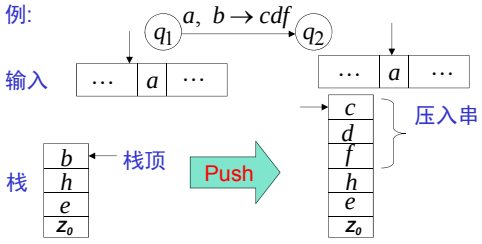
状态转移



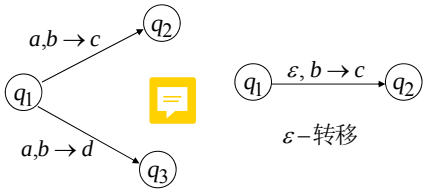
转移函数：$\delta(q_1, a, b) = \{(q_2, w)\}$

状态转移

例：

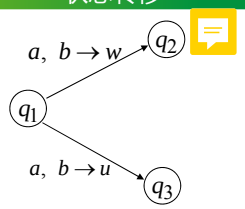


非确定PDA (NPDA)



非确定PDA (NPDA) 允许的转移

状态转移



转移函数: $\delta(q_1,a,b)=\{(q_2,w),\ (q_3,u)\}$

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PDA形式化定义

PDA为七元组 $P=(Q,\Sigma,\Gamma,\delta,q_0,Z_0,F)$.

- 有限状态集合
- 有限输入字母表
- 有限栈字符
- 转移函数
- 一个初始状态 $q_0\in Q$
- 一个栈初始符号 $Z_0\in\Gamma$
- 终态集合 $F\subseteq Q$

$\delta:\ Q\times(\Sigma\cup\{\varepsilon\})\times\Gamma\to 2^{Q\times\Gamma^*}$

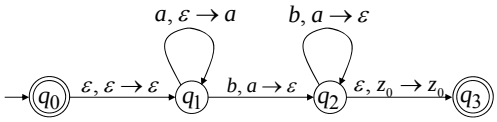
PDA形式化定义

空栈型PDA为七元组 $P=(Q,\Sigma,\Gamma,\delta,q_0,Z_0)$.

- 有限状态集合
- 有限输入字母表
- 有限栈字符
- 转移函数
- 一个初始状态 $q_0\in Q$
- 一个栈初始符号 $Z_0\in\Gamma$
- $F\subseteq Q$

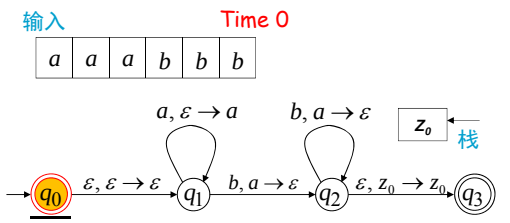
$\delta:\ Q\times(\Sigma\cup\{\varepsilon\})\times\Gamma\to 2^{Q\times\Gamma^*}$

PDA 例 1



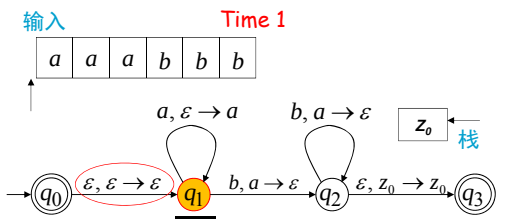
PDA 例 1

输入 Time 0



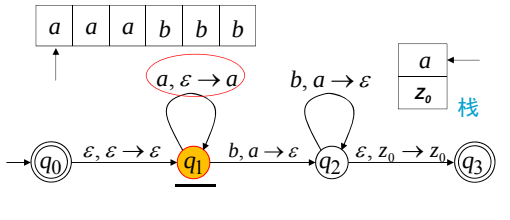
PDA 例 1

输入 Time 1



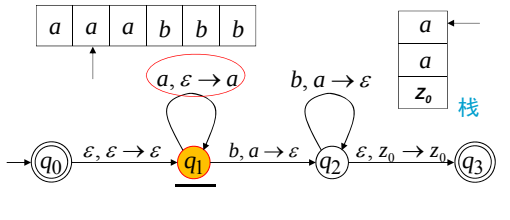
PDA 例 1

输入 Time 2



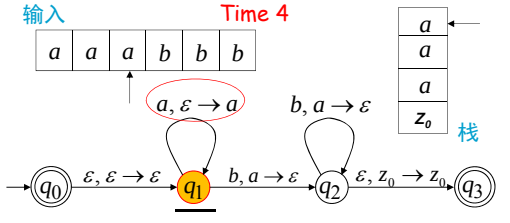
PDA 例 1

输入 Time 3



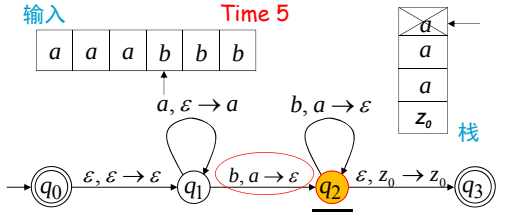
PDA 例 1

输入 Time 4



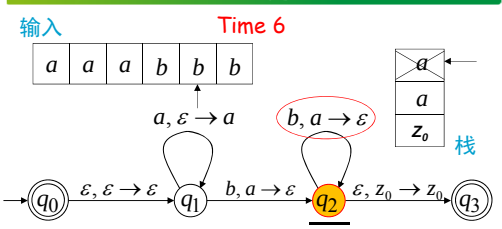
PDA 例 1

输入 Time 5



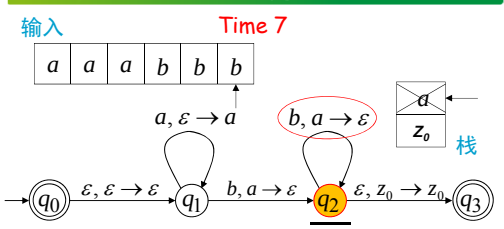
PDA 例 1

输入 Time 6



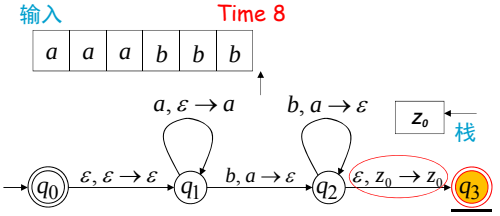
PDA 例 1

输入 Time 7



PDA 例 1

输入 Time 8



PDA 接受串

一个输入串被PDA接受，如果满足：

所有的字符串都被输入完毕

并且

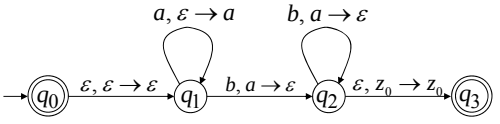
最后落在PDA的一个终态

字符串接受与否，与栈中元素无关

PDA的语言： PDA所有接受字符串构成的集合

PDA 接受串

输入串**aaabbb**会被下面的NPDA接受。



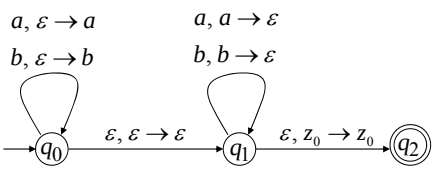
一般地，该NPDA所接受的语言是:

$$L = \{a^n b^n \mid n \geq 0\}$$

PDA 例 2

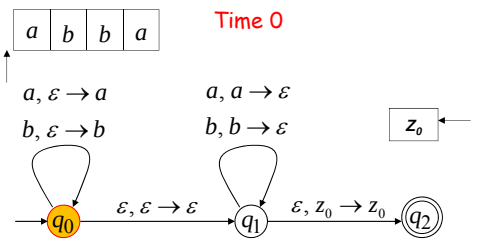
NPDA *M*

$$L(M) = \{ww^R\}$$



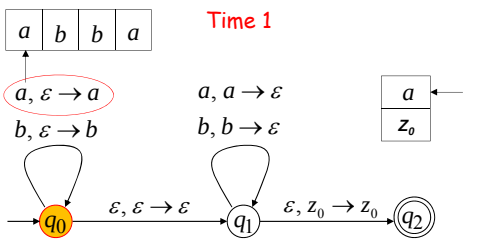
PDA 例 2

Time 0



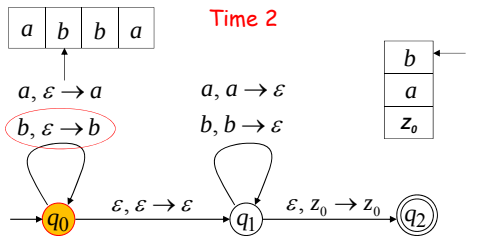
PDA 例 2

Time 1



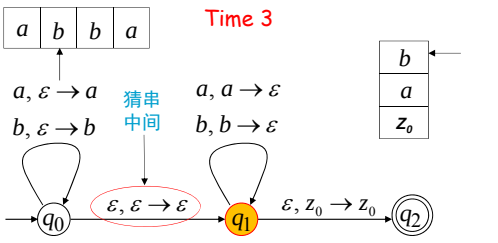
PDA 例 2

Time 2



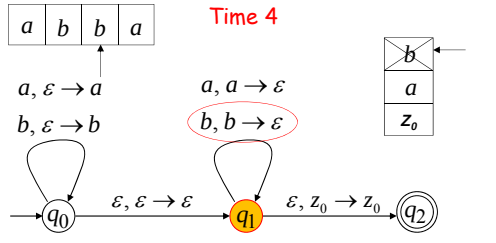
PDA 例 2

Time 3



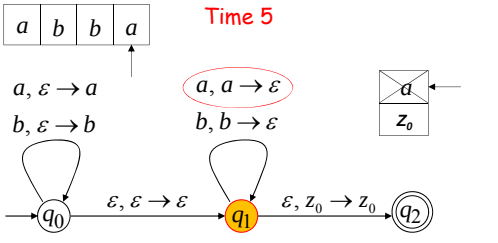
PDA 例 2

Time 4



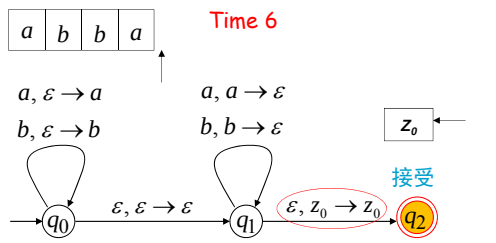
PDA 例 2

Time 5



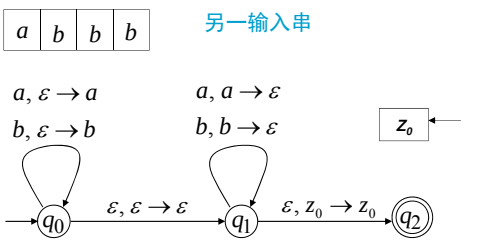
PDA 例 2

Time 6



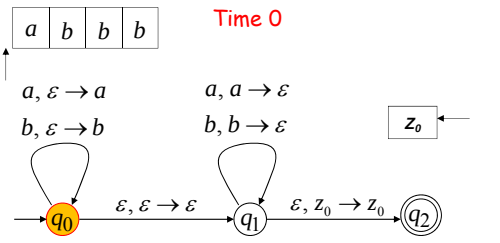
PDA 例 2

另一输入串



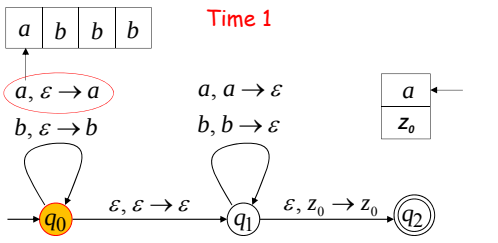
PDA 例 2

Time 0



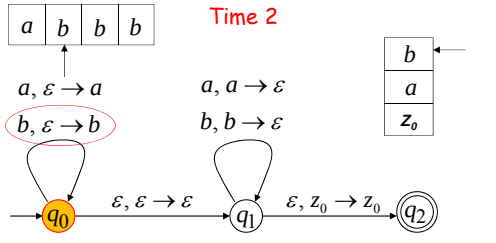
PDA 例 2

Time 1



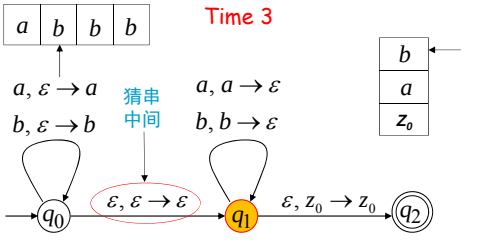
PDA 例 2

Time 2



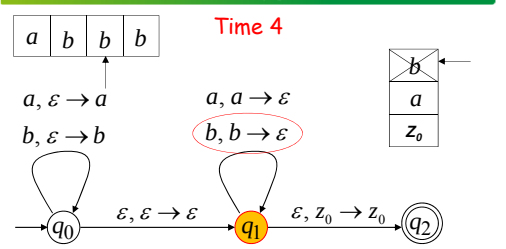
PDA 例 2

Time 3



PDA 例 2

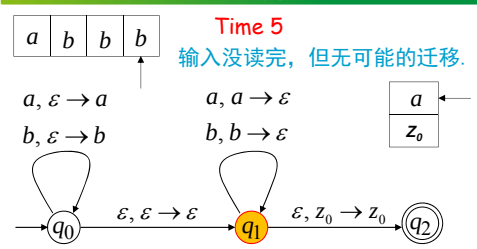
Time 4



PDA 例 2

Time 5

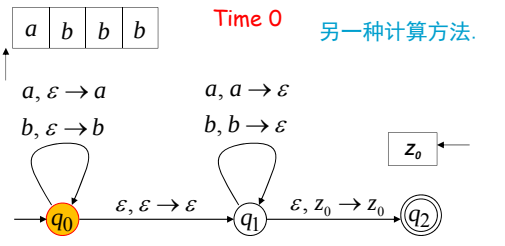
输入没读完，但无可能的迁移.



PDA 例 2

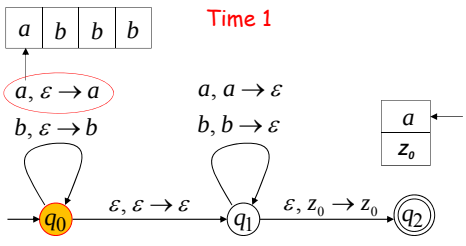
Time 0

另一种计算方法.



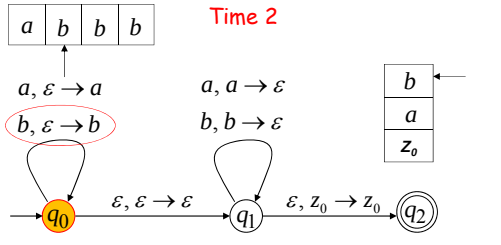
PDA 例 2

Time 1



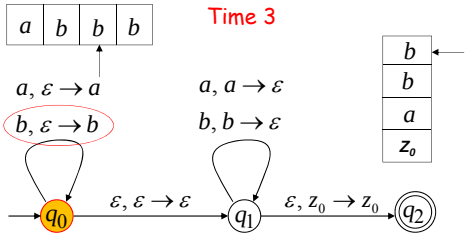
PDA 例 2

Time 2

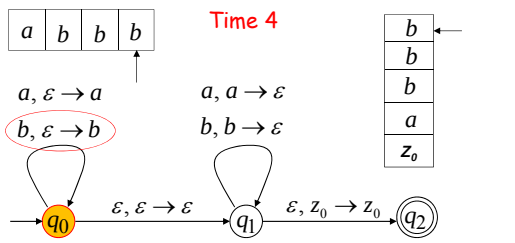


PDA 例 2

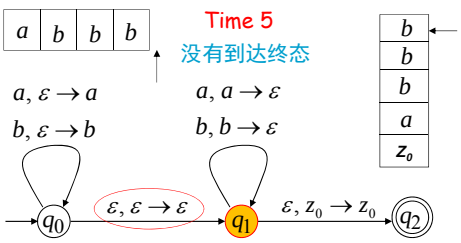
Time 3



PDA 例 2



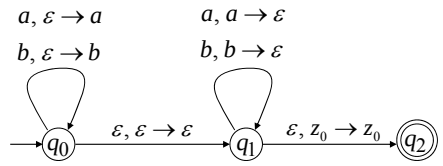
PDA 例 2



PDA 例 2

没有计算或路径接受串**abbb**

$$\Rightarrow abbb \notin L(M)$$



PDA 拒接串

一个输入串被PDA拒接，如果满足：

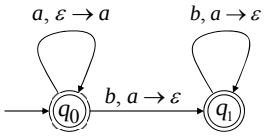
输入字符串不能读完
或者
最后不能落在PDA的终态

字符串接受与否，与栈中元素无关

PDA 例 3

设计接受如下语言的PDA：

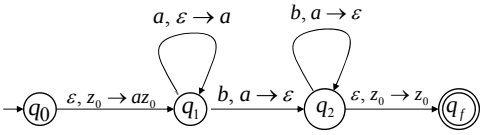
$$L(M) = \{a^n b^m \mid n \geq m\}$$



PDA 例 4

设计接受如下语言的PDA：

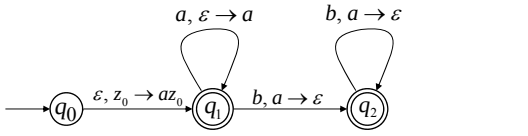
$$L(M) = \{a^{m-1} b^m \mid m \geq 1\}$$



PDA 例 5

设计接受如下语言的PDA：

$$L(M)=\{a^n b^m \mid n \geq m-1\}$$
$$=\{a^{m-1}b^m \mid m \geq 1\} \cup \{a^{m+k}b^m \mid m,k \geq 0\}$$

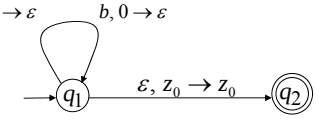


PDA 例 6

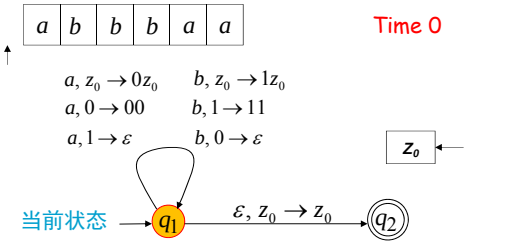
设计接受如下语言的PDA：

$$L(M)=\{w \mid n_a(w)=n_b(w)\ \}$$

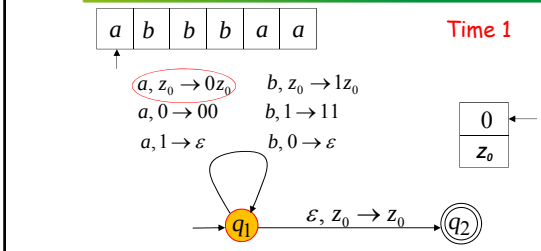
$a, z_0 \to 0z_0$　　$b, z_0 \to 1z_0$
$a, 0 \to 00$　　$b, 1 \to 11$
$a, 1 \to \varepsilon$　　$b, 0 \to \varepsilon$



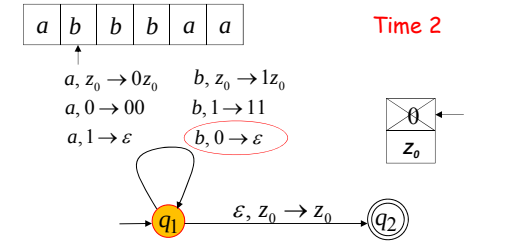
PDA 例 6



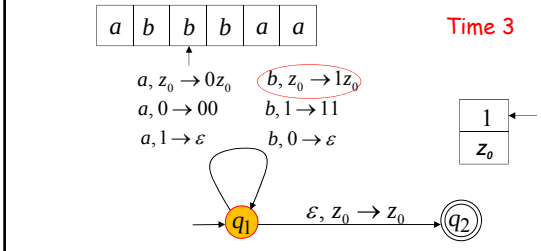
PDA 例 6



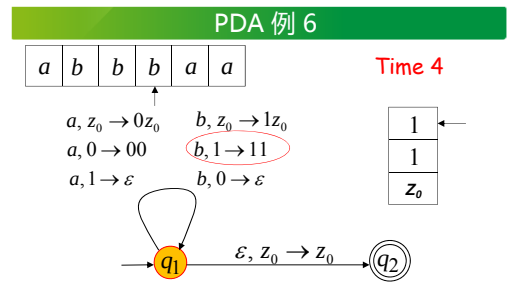
PDA 例 6



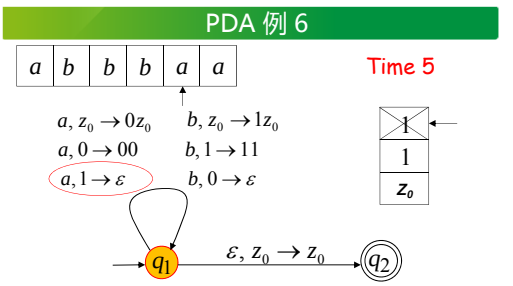
PDA 例 6



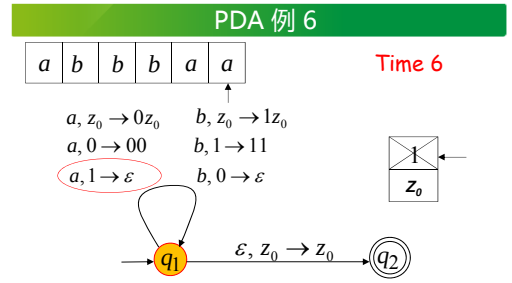
PDA 例 6



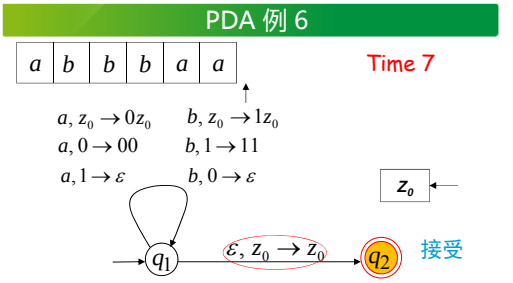
PDA 例 6



PDA 例 6



PDA 例 6



•下推自动机

- PDA简介
- PDA的定义
- PDA的即时描述
- PDA的语言
- PDA与CFG的关系

PDA 的即时描述

- 即时描述(ID)定义：

  PDA的即时描述是一个三元组

$$(q,\ w,\ \gamma)$$

当前状态　未读剩余输入　栈中当前符号

PDA 的即时描述

- 传递 ⊢ 定义：

设 *PDA P* = $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$。

$\vdash_P$ 或 ⊢ 满足：

若 $(p, \alpha) \in \delta(q, a, X)$

则 $(q, aw, X\beta) \vdash (p, w, \alpha\beta)$

其中 $p, q \in Q$, $a \in \Sigma$, $w \in \Sigma^*$, $X \in \Gamma$, $\alpha, \beta \in \Gamma^*$.

PDA 的即时描述

- ID的传递闭包

$\vdash^*_P$, 或者 $\vdash^*$定义为:

基础: 对于任意ID I, 有 $I \vdash^* I$

归纳: 如果 $I \vdash K$, $K \vdash^* J$,

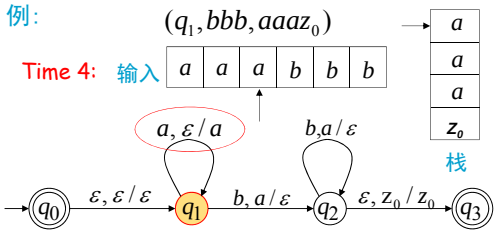
则有 $I \vdash^* J$

PDA 的即时描述

例:

$(q_1, bbb, aaaz_0)$

Time 4: 输入 a a a b b b

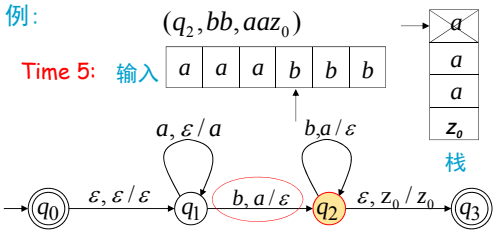


PDA 的即时描述

例:

(q_2, bb, aaz_0)

Time 5: 输入 a a a b b b



PDA 的即时描述

例:

则可以记为:

$$(q_1, bbb, aaaz_0) \vdash (q_2, bb, aaz_0)$$

Time 4 Time 5

PDA 的即时描述

定理6.5: 设 *PDA* $P=(Q,\Sigma,\Gamma,\delta,q_0,Z_0,F)$.

如果 $(q, x, \alpha) \vdash^* (p, y, \beta)$

那么 对于任意的$w \in \Sigma^*$ and $\gamma \in \Gamma^*$,

$(q, xw, \alpha\gamma) \vdash^* (p, yw, \beta\gamma)$.

证明方法：

对ID序列中传递步骤数作归纳

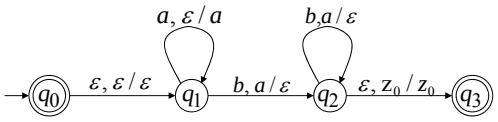
$(q, x, \alpha) \vdash^* (p, y, \beta)$.

PDA 的即时描述

例： 如下PDA接受串***aaabbb*** 的ID序列.

$(q_0,aaabbb,Z_0)$ ⊢* $(q_1,bbb,aaaZ_0)$

⊢* (q_2,ε,aaZ_0) ⊢* (q_3,ε,Z_0)



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PDA 的语言

- 终态型PDA的语言

  设 NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

  PDA(NPDA) M 的语言 $L(M)$ 定义为：

$$L(M)=\{w: w\in\Sigma^*, (q_0,w,z_0) \overset{*}{\vdash}_M (q_f,\varepsilon,u), u\in\Gamma^*\}$$

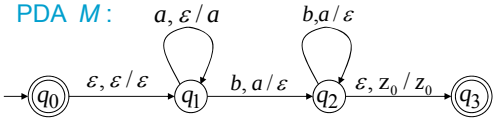
初始 ID　　最终 ID

PDA 的语言

例:　$(q_0,aaabbb,z_0) \overset{*}{\vdash} (q_3,\varepsilon,z_0)$

$aaabbb \in L(M)$

PDA M :

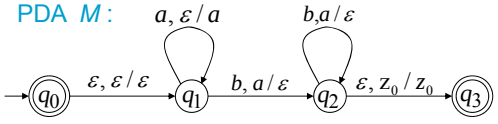


PDA 的语言

例:　$(q_0, a^nb^n,z_0) \overset{*}{\vdash} (q_3,\varepsilon,z_0)$

$a^nb^n \in L(M)$

PDA M :

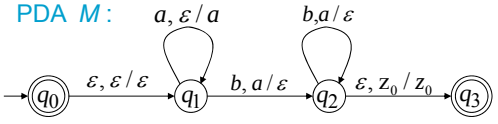


PDA 的语言

例: 则有:

$$L(M)=\{a^nb^n \mid n\geq 0\}$$

PDA M :



PDA 的语言

- **空栈型PDA的语言**

  设 NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$

  空栈型PDA M 的语言 $N(M)$ 定义为：

$$N(M) = \{w \mid (q_0, w, Z_0) \vdash^* (q, \varepsilon, \varepsilon), q \in Q\}$$

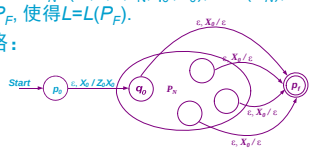
初始 ID　　最终 ID

两种 PDA 语言的关系

- 空栈型PDA到终态型PDA

  对于 PDA $P_N=(Q,\Sigma,\Gamma,\delta_N,q_0,Z_0)$, $L=L(P_N)$,存在一个 PDA P_F, 使得$L=L(P_F)$.

证明思路：



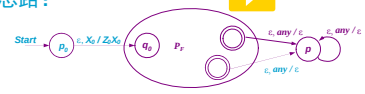
$$P_F = (Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p_f\})$$

两种 PDA 语言的关系

- 终态型PDA到空栈型PDA

  对于PDA $P_F=(Q,\Sigma,\Gamma,\delta_F,q_0,Z_0,F)$, $L=L(P_F)$, 存在一个 PDA P_N, 使得 $L=N(P_N)$.

证明思路：



$$P_N = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)$$

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PDA 与CFG

定理：

$$\{\text{上下文无关（文法）语言}\} = \{\text{PDA接受的语言}\}$$

PDA 与CFG

一方面：

$$\{\text{上下文无关（文法）语言}\} \subseteq \{\text{PDA接受的语言}\}$$

将上下文无关文法G转换为 PDA M，且:

$$L(M) = L(G)$$

PDA 与CFG

另一方面：

$$\left\{\begin{array}{l}\text{上下文无关}\\ \text{（文法）语言}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{PDA接受}\\ \text{的语言}\end{array}\right\}$$

将PDA M 转换为上下文无关文法G，且:

$$L(G)=L(M)$$

CFG 转换 PDA

- 构造方法

设 $CFG\ G=(V, T, P, S)$

构造一个空栈型 PDA:

$M=(\{q\}, T, V\cup T, \delta, q, S)$

转移函数 δ 定义如下：

(1) 对每一 $A\in V$,

$\delta(q, \varepsilon, A)=\{(q,\beta)\mid "A\to\beta"\in P\}$;

(2) 对每一 $a\in T$,

$\delta(q, a, a)=\{(q, \varepsilon)\}$.

CFG 转换 PDA

例:

对右边产生式所代表的 CFG，依上述方法构造 PDA为:

$E\to EOE \mid (E) \mid v \mid d$

$O\to + \mid *$

$(\{q\}, \{v,d,+,*,(,)\}, \{E,O,v,d,+,*,(,)\}, \delta, q, E)$

CFG 转换 PDA

例:

$E\to EOE \mid (E) \mid v \mid d$

$O\to + \mid *$

其中δ 定义为:

$\delta(q, \varepsilon, E)=\{(q, EOE), (q, (E)), (q, v), (q, d)\}$,

$\delta(q, \varepsilon, O)=\{(q, +), (q, *)\}$,

$\delta(q, v, v)=\{(q, \varepsilon)\}$, $\delta(q, d, d)=\{(q, \varepsilon)\}$

$\delta(q, +, +)=\{(q, \varepsilon)\}$, $\delta(q, *, *)=\{(q, \varepsilon)\}$

$\delta(q, (, ()=\{(q, \varepsilon)\}$, $\delta(q,),))=\{(q, \varepsilon)\}$

CFG 转换 PDA

- 定理：设 $CFG\ G$

$G=(V, T, P, S)$

上述方法构造以空栈接受方式的 $PDA\ M$

$M=(\{q\}, T, V\cup T, \delta, q, S)$

则有 $N(M)=L(G)$.

- 证明思路

需证，对任何 $w\in T^*$,

$w\in L(G) \Leftrightarrow w\in N(M)$.

CFG 转换 PDA

“⇒”

要证：若 $S\overset{*}{\underset{lm}{\Rightarrow}}w$, 则 $(q,w,S)\vdash^*(q, \varepsilon, \varepsilon)$.

先证明如下结论:

若 $A\overset{*}{\underset{lm}{\Rightarrow}}w$, 则 $(q, w, A)\vdash^*(q, \varepsilon, \varepsilon)$.

对 $A\overset{*}{\underset{lm}{\Rightarrow}}w$ 的步数 n 作归纳.

CFG 转换 PDA

基础：n=1, 必有产生式: $A \to w$

$(q, w, A) \vdash (q, w, w) \vdash^* (q, \varepsilon, \varepsilon)$.

归纳：设第一步使用产生式 $A \to X_1X_2\ldots X_m$，必有

$w=w_1w_2\ldots w_m$.

$(q,w,A) \vdash (q,w, X_1X_2\ldots X_m)$

$\vdash^* (q, w_2\ldots w_m, X_2\ldots X_m)$

$\vdash^* (q, w_3\ldots w_m, X_3\ldots X_m) \vdash^* \ldots \vdash^* (q, \varepsilon, \varepsilon)$.

CFG 转换 PDA

“$\Leftarrow$”

即证明，

对任何 $w \in T^*$, $w \in L(G) \Leftarrow w \in N(M)$.

也即对任何 $w \in T^*$,

若 $(q,w,S) \vdash^* (q, \varepsilon, \varepsilon)$, 则 $S \overset{*}{\Rightarrow} w$.

CFG 转换 PDA

先证明如下结论：

若 $(q, x, A) \vdash^* (q, \varepsilon, \varepsilon)$, 则 $A \overset{*}{\Rightarrow} x$.

对 $(q,x,A) \vdash^* (q, \varepsilon, \varepsilon)$ 的步数 n 作归纳.

基础:

n=1，必有 $x = \varepsilon$, 且 $A \to \varepsilon$ 为 G 的产生式，所以

$A \overset{*}{\Rightarrow} x$.

CFG 转换 PDA

归纳： n>1，设第一步使用产生式 $A \to X_1X_2\ldots X_m$,

将x 分为 $x=x_1x_2\ldots x_m$, 满足

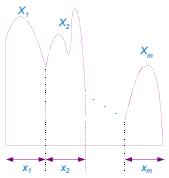
$(q, x_i, X_i) \vdash^* (q, \varepsilon, \varepsilon)$,

无论 X_i 为终结符，还是非终结符，都有

$X_i \overset{*}{\Rightarrow} x_i$.

因此, $A \Rightarrow X_1X_2\ldots X_m$,

$\overset{*}{\Rightarrow} x_1x_2\ldots x_m = x$



CFG 转换 PDA

当A取为S时

所以有结论:

对任何 $w \in T^*$,

若 $(q,w,S) \vdash^* (q, \varepsilon, \varepsilon)$, 则 $S \overset{*}{\Rightarrow} w$.

即, $w \in N(E) \Rightarrow w \in L(G)$.

PDA 转换 CFG

- 构造方法

设 *PDA M*

$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$

构造*CFG*

$G = (V, \Sigma, P, S)$

其中

$V = \{S\} \cup \{ [pXq] \mid p, q \in Q, X \in \Gamma \}$

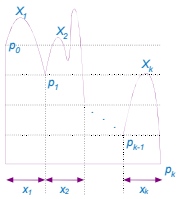
产生式集合 P 定义如下:

(1) 对$\forall\ p\in Q$, G 包含产生式
$S \to [q_0 Z_0 p]$;

(2) 变量$[pXq]$的产生式:

(i) 若 $(q, \varepsilon) \in \delta(p, a, X)$
则 $[pXq] \to a$

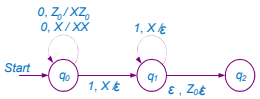
(ii) 若 $(q, X_1X_2\ldots X_k) \in \delta(p, a, X)$,
则G 包含产生式(见右图，其中$p_0=q$):
$[pXp_k] \to a[qX_1p_1][p_1X_2p_2]\ldots[p_{k-1}X_kp_k]$.



说明: $a \in \Sigma$ 或 $a = \varepsilon$; $\forall p_i \in Q$, $i = 1,2,\ldots,k$.

例

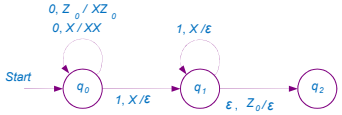
对于右图的 PDA,
构造CFG:
$G = (V, \{0,1\}, P, S)$,
其中
$V = \{S\} \cup \{[pYq] \mid p,q \in \{q_0,q_1,q_2\},\ Y \in \{Z_0, X\}\}$



产生式集合 P 定义如下:

(1) $S \to [q_0 Z_0 q_0] \mid [q_0 Z_0 q_1] \mid [q_0 Z_0 q_2]$;

(2) $(q_0, XZ_0) \in \delta(q_0, 0, Z_0)$,
$[q_0 Z_0 q_j] \to 0[q_0 X q_i][q_i Z_0 q_j]$, $i, j = 0,1,2$;



(3) $(q_0, XX) \in \delta(q_0, 0, X)$
$[q_0 X q_j] \to 0[q_0 X q_i][q_i X q_j]$, $i, j = 0,1,2$;

(4) $(q_1,\varepsilon) \in \delta(q_0, 1, X)$, $[q_0 X q_1] \to 1$;

(5) $(q_1,\varepsilon) \in \delta(q_1, 1, X)$, $[q_1 X q_1] \to 1$;

(6) $(q_2,\varepsilon) \in \delta(q_1, \varepsilon, Z_0)$, $[q_1 Z_0 q_2] \to \varepsilon$.

PDA 转换 CFG

定理：设 PDA M
$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$
依上述方法构造 CFG $G = (V, \Sigma, P, S)$
则有 $N(M) = L(G)$.

证明思路：欲证，对任何 $w \in \Sigma^*$,
$w \in N(M) \Leftrightarrow w \in L(G)$.

即证明：
$(q_0, w, Z_0) \vdash^* (p, \varepsilon, \varepsilon)$ *iff* $S \overset{*}{\Rightarrow} w$
其中 $p \in Q$.

PDA 与 CFG 的关系

CFG与PDA的等价性

课后练习

✧ **必做题：**

- **P-228 *Ex.6.1.1* (*b*)**
- **P-236 *Ex.6.2.5* (*b*)**
- **P-236 *Ex.6.2.6***
- **P-245 *Ex.6.3.2***
- **P-246 *Ex.6.3.4***
- **P-246 *!Ex.6.3.5* (*c*)**

✧ **思考题：**

- **P-236 *Ex.6.2.1* (*b*), (*c*)**
- **P-236 *!Ex.6.2.2* (*b*)**
- **P-236 *!! Ex.6.2.3* (*a*), (*b*)**
- **P-246 *Ex.6.3.7***

Thank you