第三次习题课

1， partial derivative

1)definition and its geometry meaning

To answer the question: as the variable changes, how the function changes? We obtain the concept of derivative of single value function, namely

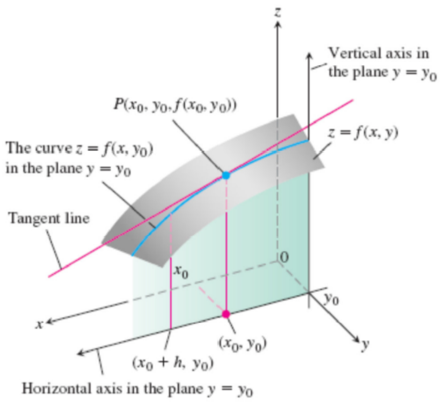
$$For\ y=f(x);\ dy=[\frac{d}{dx}f(x)]dx=f'(x)dx$$

Where f '(x) is the derivative over x, it is the rate of change of f over x; its geometric meaning is the slope of the tangent line of the curve y=f(x), passing through (x,f(x)).

Here for multi-variable function f(x,y), the change of f would be related to changes of both x and y. The rates of changes are defined through the partial derivatives:

$$\frac{\partial f}{\partial x}|_{x_0,y_0}=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

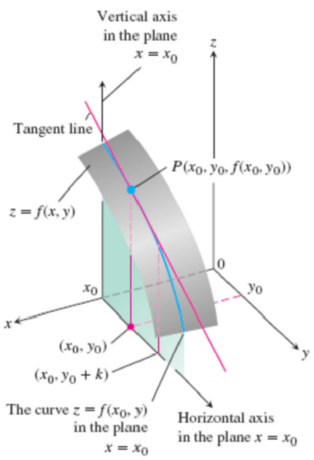
Noticed that from its definition, it is a derivative at point (x0,y0), holding the y constant (y does not change) and varies the x.



It is the slope of a tangent line. This tangent line is tangent to a curve which is the cut of y=y0 (a planes parallel to x-z) plane with the f(x,y) surface, and the tangent line passes through (x0,y0)

Similarly the partial derivative over y can be defined as:

$$\frac{\partial f}{\partial y}|_{x_0,y_0}=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$



$$\frac{\partial f}{\partial x} \equiv \frac{\partial f}{\partial x}\Big|_{\text{constant y}} \equiv \left(\frac{\partial f}{\partial x}\right)_y \equiv f_x$$
$$\frac{\partial f}{\partial y} \equiv \frac{\partial f}{\partial y}\Big|_{\text{constant x}} \equiv \left(\frac{\partial f}{\partial y}\right)_x \equiv f_y$$

2)second order partial derivatives

The 1st order partial derivative of f(x,y) is generally a function of (x,y) too, so we can take the further partial derivative over the 1st order partial derivative and this will give us 2nd order partial derivatives.

$$\frac{\partial f_x}{\partial x} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x^2} \equiv f_{xx}$$
$$\frac{\partial f_x}{\partial y} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv f_{xy}$$

$$\frac{\partial f_y}{\partial x} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y \partial x} \equiv f_{yx}$$
$$\frac{\partial f_y}{\partial y} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y^2} \equiv f_{yy}$$

EXAMPLE 9 Finding Second-Order Partial Derivatives

If $f(x, y) = x \cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) = \cos y + ye^x$$

So

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x) = -x \sin y + e^x$$

So

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

In the example, you will notice that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy} = f_{yx}$$

The relation holds for regular functions we shall encounter in physics.

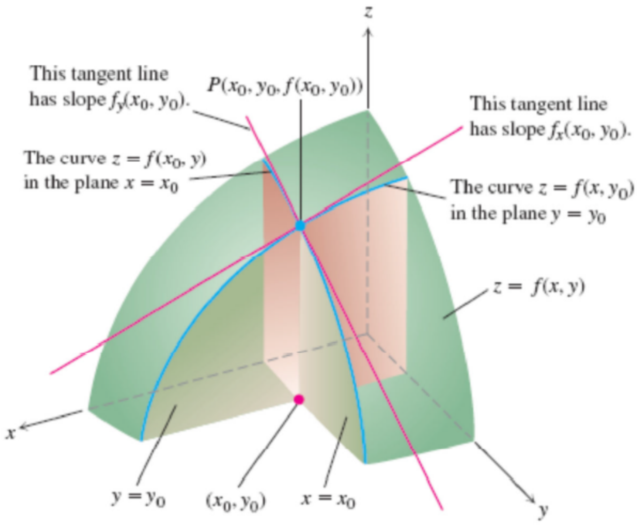
2. Total Differentials

1)defination

This is to say if we have an infinitesimal change of variables from (x0, y0) to (x0+ dx, y0+ dy) , the change of function is given by

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

It is called **total differential** of the function.



we are using the tangent plane at (x0 , y0) to approximate the original function surface. The tangent plane is formed by the two tangent lines at (x0, y0) that defined the partial derivatives. The exact change of function is of course along the surface, but for small Δx , Δy ,the tangent plane is very close to the surface.

2)chain rules

(1) f (x, y), x = x(t) and y = y(t)

The function only explicitly depends on x and y. However the x, y each is a function of some other variable t. Now the question is what is the derivative of the function over t

$$df = f_x dx + f_y dy$$

But now $dx = (\frac{dx}{dt})dt = x'dt, dy = (\frac{dy}{dt})dt = y'dt$, so:

$$df = (f_x x' + f_y y')dt$$

This is the differential of f with change of dt, then:

$$\frac{df}{dt} = f_x x' + f_y y' = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

There are cases in physics, a function not only explicitly depends on x, y, but also on t, and the x and y depend on t too. In such case, the derivative of the function over t is

$$\frac{d}{dt}f(x,y,t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}\frac{dt}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}$$

(2) f (x, y), x = x(s ,t) and y = y(s ,t)

$$df = f_x dx + f_y dy$$

$$dx = \frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt; \ dy = \frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt$$

Then:

$$df = (\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s})ds + (\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t})dt$$

This is the total differential expressed in the form that s, t are variables, and from this we can find partial derivatives of f over s

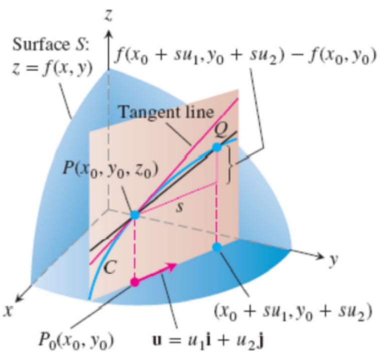
$$\frac{\partial f}{\partial s}\Big|_{\text{constant t}} \equiv \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial f}{\partial t}\Big|_{\text{constant s}} \equiv \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

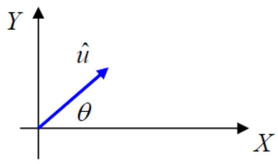
we discussed a technique of implicit derivative. Basically the y=f(x) is not given explicitly,
the x, y relation are provided by an equation g(x, y)=0

$$dg(x, y) = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0$$
$$\frac{dy}{dx} = -\frac{\partial g / \partial x}{\partial g / \partial y}$$

3. Gradient Vector and Directional Derivative

There are many ways to cut the surface of f(x,y) at (x0 , y0) , such as the figure below shows. The plane is not x = x0 or y = y0 as before, but a plane contain the point (x0, y0) and a direction vector u. θ is the angle between the direction of u and x-axis. Now if we move along the u a small distance s, what is the change of function?





This is the meaning of directional derivative, the rate of change of function along any direction specified by u .

$$\left(\frac{df}{ds}\right)_{\hat{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$
$$\left(\frac{df}{ds}\right)_{\hat{u}, P_0} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

It is very useful to introduce a new vector, the gradient vector whose definition is

$$\nabla f \equiv \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Because of the definition of gradient and directional derivative we have

$$df = \nabla f \cdot d\vec{s}$$

$$df = \nabla f \cdot d\vec{s} = |\nabla f||ds|\cos\alpha$$

1) Clearly at α = 0, for same |ds|, df has the largest positive value. This is when the direction of change overlaps with the direction of gradient, The gradient is point toward the direction of fastest change of the function over same small change of variables.
2) if the displacement is along a level curve where f(x, y)=c. then clearly df=0. This means α=π/2 and ∇f is perpendicular to the to the level curve, i.e. the curve with f(x, y)=c.

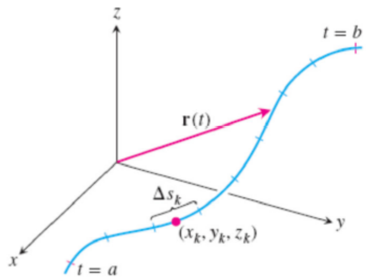
4.Path (line) integral and curl

1)line integral

The line integral is an integration, but in this integration, the x,y,z' s are not free to vary, they have to stay on a curve in 2-D or 3-D space. For example, the curve is a wire, and its density may change along the wire,

so to get the total mass of the wire, we may need to compute the integral:

$$M = \int_C \rho(x,y,z)ds$$



If The curve is defined by parametric equations: x = x(t), y = y(t), z = z(t) , where t is the parameter

Because for two dimensional curve $\mathrm{ds} = \sqrt{dx^2+dy^2}$, we generalize this to three dimension, $\mathrm{ds} = \sqrt{dx^2+dy^2+dz^2}$, so for a function f(x ,y, z), where x = x(t), y = y(t), z = z(t),we have

$$\int_{t=a}^{t=b} f[x(t),y(t),z(t)][(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2]^{\frac{1}{2}}dt$$

Example：$\int \frac{z^2}{x^2+y^2}ds$, on the curve x=acost, y=asint, z=bt, 0≤ t ≤ 2π

We have $\mathrm{ds} = \sqrt{dx^2+dy^2} = \sqrt{a^2+b^2}dt$, so$\int \frac{z^2}{x^2+y^2}ds = (\frac{b}{a})^2\sqrt{a^2+b^2}\int_0^{2\pi} t^2\,dt = \frac{8}{3}(\frac{b}{a})^2\sqrt{a^2+b^2}\pi^3$

3) Work as Line Integral

The work done by a force in physics is defined as:

$$W = \int_C \vec{F}\cdot d\vec{r}$$

$$\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$$

$$d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

Combine these we have

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C Mdx + Ndy + Pdz$$

Example :

例 1 计算第二型曲线积分：

$$\int_K xy\mathrm{d}x, \quad \int_L xy\mathrm{d}x, \quad \int_M xy\mathrm{d}x,$$

其中 K 是从(0,0)到(1,1)的直线段，L 是从(0,0)到(1,1)的抛物线 $y=x^2$ 的一段，M 是由从(0,0)到(1,0)的一段直线 M_1 和从(1,0)到(1,1)的直线段 M_2 连成的有向折线.

解 K 和 L 的方程分别是 $y=x$ 和 $y=x^2$，x 从 0 变到 1，因此

$$\int_K xy\mathrm{d}x = \int_0^1 x^2\mathrm{d}x = \frac{1}{3},$$

$$\int_L xy\mathrm{d}x = \int_0^1 x \cdot x^2\mathrm{d}x = \int_0^1 x^3\mathrm{d}x = \frac{1}{4}.$$

最后

$$\int_M xy\mathrm{d}x = \int_{M_1} xy\mathrm{d}x + \int_{M_2} xy\mathrm{d}x.$$

在 M_1 上 $y=0$，而在 M_2 上 $x=1$，所以 $\mathrm{d}x=0$，总起来有

$$\int_M xy\mathrm{d}x = 0.$$ □

4) green theorem and curl

Green Theorem states that if the vector field F, is defined and differentiable everywhere in the domain enclosed by the loop, then the loop line integral equals to an area integral.

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy \equiv \iint_R (N_x - M_y)dxdy$$

The Nx － My also defines a physical quantity of the vector field in 2-D, it is called curl of the field

$$curl(\vec{F}) \equiv N_x - M_y$$

Now let’ s look for the special case of conservative force. We see that

$N_x = M_y$, then $\oint_c \vec{F} \cdot d\vec{r} = 0$

Gradient theorem allows us to show that the work will be path independent if the force is gradient of potential function.

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = \int_c df = f(b) - f(a)$$

Now Green theorem shows that if the curl of the force is zero, work done along a loop will be zero. Because of

$\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ or $f_{xy} = f_{yx}$

If we know F is gradient of f, we can immediately know curl(F) is equal to zero

In 3-D case, the Green theorem in 2-D is extended to

$$\vec{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k} \equiv < M, N.P >$$

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_S curl(\vec{F}) \cdot \hat{n} dA = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dA \qquad (94)$$

Where

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = (P_y - N_z)\hat{i} + (M_z - P_x)\hat{j} + (N_x - M_y)\hat{k}$$