# Homework 13 for GPII

1. Griffiths' 2.29

Analyze the odd bound state wave functions for the finite potential well. Derive the transcendental equation for the allowed energies, and solve it graphically. Analyzing the two limiting cases (wide-deep well (meaning $Z_0$ big) and narrow-shallow well (small $Z_0$)) and is there always an odd bound state?

2. Griffiths P2.41

A particle of mass m in the harmonic potential starts out in the state:

$$\psi(x,0) = A(1-2\sqrt{\frac{m\omega}{\hbar}}x)^2 e^{-\frac{m\omega}{2\hbar}x^2}$$ for some constant A.

(a) What is the expectation value of energy?

(b) At some later time T, the wave function is:

$$\psi(x,T) = B(1+2\sqrt{\frac{m\omega}{\hbar}}x)^2 e^{-\frac{m\omega}{2\hbar}x^2}$$

What is the smallest possible value of T?

3. Griffiths P2.42

Find the allowed energies of the half harmonic oscillator:

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2, & x > 0 \\ \infty & x < 0 \end{cases}$$

Hint: This requires some thought on even-odd parity but very little computation.

4. Find the energy levels and wave functions of the two harmonic oscillators with $m_1$ and $m_2$, have identical frequency $\omega$ when isolated. Now suppose they are coupled by extra interaction potential $\frac{1}{2}K(\hat{x}_1 - \hat{x}_2)^2$. (Hint: think how you treat this in classical mechanics would help)

5. For the ground state harmonic oscillator, we know it satisfy the lower limit (wave function is a Gaussian) of uncertainty relation: $\Delta x \Delta p = \frac{\hbar}{2}$, where $\Delta A \equiv (<\hat{A}^2> - <\hat{A}>^2)^{1/2}$, prove this is consistent with the virial theorem (for harmonic potential) that <T>=<V>.

(Following problems 6, 7 are optional)

6. Hellmann-Feynman Theorem ( Griffiths P6.32)

For a system with Hamiltonian H, H is an operator depends on some parameter $\lambda$, say $\hat{H} = \hat{H}(\lambda)$. The $\lambda$ is pretty general, for example the harmonic oscillator $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2$ the mass m, the frequency $\omega$, x and even $\hbar$ can all be considered as parameters. The energy (eigenvalue of H) and eigenfunction may generally depend on $\lambda$, i.e. $E_n(\lambda)$ and $\psi_n(\lambda)$. The Hellmann-Feynman theorem states:

$$\frac{\partial E_n}{\partial \lambda} = <\frac{\partial H}{\partial \lambda}>_{\psi_n} = <\psi_n | \frac{\partial H}{\partial \lambda} | \psi_n>

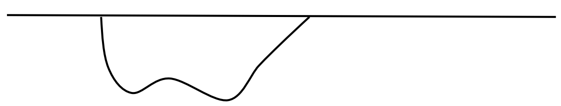
(a) Prove the Hellmann-Feynman theorem.

(b) It is useful in quick evaluation on expectation values or to see dependence of energy on parameters. For instance, apply it to harmonic oscillator: (1) using $\lambda = \hbar$, what is the expression for <T>? (2) using $\lambda = \omega$, what is the expression for <V>? (3) using $\lambda = m$, what is the relation?

7. (Extra problem applying Hellman-Feynman)

(a) Using Hellman-Feynman to prove that if we have two potential $V_1(x)$ and $V_2(x)$, where $V_2(x) > V_1(x)$ for all x; then eigen-energies for these two potential have relation: $E_{2n} > E_{1n}$.

(b) For a 1-D arbitrary potential (continuous) well as sketched in figure;



There is always at least one bound state inside the well.

8. For the 1-D mass=m harmonic oscillator with $V = \frac{1}{2}kX^2$; we set $k = m\omega^2$ as usual. The oscillator is in a general quantum state $|\psi>$ ( it may not be an eigenstate of energy but a superposition of them). Please use the Ehrenfest theorem proving the following relation about the expectation value between <x> and <p>:

$$< X > (t) = x_0 cos\omega t + \frac{p_0}{m\omega} sin\omega t$$

$$< P > (t) = p_0 cos\omega t - m\omega x_0 sin\omega t$$

Where $< X > (t), < P > (t)$ is the average of X, P at time t; $x_0 =< X > (t = 0)$; $p_0 =< P > (t = 0)$ :the initial average of x and p.

9. One basic Practice on the eigenfunctions of harmonic oscillator

(Adapted from Griffiths' 2.10, 2.16)

Let $\psi_{0,} \psi_1, \psi_2, \psi_6$ are eigenfunctions for energy n=0,1,2, 6

1) Please write out the expression of these eigenfunctions in terms of x with the following procedures: First use $\xi = \sqrt{\frac{m\omega}{\hbar}} x$; and write out the Hermite polynomial $H(\xi)$ with the highest non-zero coefficient be $2^n$; and find the other coefficient with recursion formula (note no need for recursion formula for n=0 and 1). Then times the gaussian term$e^{-\xi^2/2}$ and normalization factor (you may directly calculate the normalization factor using the general formula Griffiths 2.85); finally replace the $\xi$ with x.
2) Confirm that among $\psi_{0,} \psi_1, \psi_2$, they are orthogonal. (applying the even odd parity, you only need to calculate one integral and you may need the definite integration result: $\int_0^\infty x^{2n}\, e^{-ax^2} dx = \frac{(2n-1)!!}{2^{n+1}a^n} \sqrt{\frac{\pi}{a}}\ for\ a > 0$; please google or baidu for the double factorial (双阶乘) yourself)