Homework 10 for General Physics II

Here are more practices on the application of linear algebra in the quantum postulates.

(To simplify the typing, I use capital letter such as H, X for operator, skipping the "hat" on top of them)

1. For a Hermite operator H (here H is just a general operator), it has eigenvalues of E_n; and corresponding eigenvectors $|\varphi_n>$. Assume the states $|\varphi_n>$ forms a discrete orthonormal basis. The operator $U(m,n)$ is defined by: $U(m,n) = |\varphi_m><\varphi_n|$
   a) Find the expression of the adjoint $U^\dagger(m,n)$ of $U(m,n)$.
   b) Calculate the commutator: $[H, U(m,n)]$.
   c) Prove the relation: $U(m,n)U^\dagger(p,q) = \delta_{nq}U(m,p)$
   d) Calculate the $Tr\{U(m,n)\}$, the trace of the U(m,n).
   e) Let A be another operator, and its matrix element in the $|\varphi_n>$ basis is give by: $A_{mn} =<\varphi_m|A|\varphi_n>$; prove that $A = \sum_{m,n} A_{mn}U(m,n)$

2. In a 2-D vector space spanned by orthonormal base vectors |1> and |2>; an operator with matrix form in such basis is: $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$:
   a) Is σ_y Hermitian? Calculate its eigenvalues and eigenvectors (eigenvectors expressed in standard form with normalization and first component real)
   b) Write out the projector matrices of these eigenvectors, i.e. $\rho_1 = |\varphi_1><\varphi_1|$; where $|\varphi_1>$ is the first eigenvector; similar for ρ_2; confirm the completeness (also called "closure" relation), i.e. $\sum_j \rho_j = I$

3. In a 3-D space spanned by three orthonormal base vectors: $|u_1>; |u_2> and |u_3>$;the kets $|\psi_0> and\ |\psi_1>$ are:

   $$|\psi_0> = \frac{1}{\sqrt{2}}|u_1> + \frac{i}{2}|u_2> + \frac{1}{2}|u_3>$$

   $$|\psi_1> = \frac{1}{\sqrt{3}}|u_1> + \frac{i}{\sqrt{3}}|u_2>$$

   a) Are these kets normalized, if not make it/them normalized. (Using the normalized form for the rest of computation)
   b) Calculate the matrix form in the |u> basis of the projectors $\rho_0\ and\ \rho_1$; they project any vector to the state $|\psi_0>\ and\ |\psi_1>$ respectively. Verify these matrices are Hermitian as all projectors should be.

4. Consider the Hamiltonian H of a particle moving along a line, it is defined by:

   $H = \frac{P^2}{2m} + V(X)$, X, P are operators for position and momentum and they satisfy commutation : $[X, P] = i\hbar$. Let $|\varphi_n>$ be the eigenvectors of H: $H|\varphi_n> = E_n|\varphi_n>$; here n is a discrete index.
   a) Show that: $<\varphi_n|P|\varphi_{n'}> = \alpha <\varphi_n|X|\varphi_{n'}>$; where α is a coefficient which depends on the difference between $E_n\ and\ E_{n'}$; Find out the expression for α. (Hint: you may start from the commutation relation: [X, H])

b) From above, deduce the following equation:

$$\sum_{n'}(E_n - E_{n'})^2 | < \varphi_n | X | \varphi_{n'} > |^2 = \frac{\hbar^2}{m^2} < \varphi_n | P^2 | \varphi_n >$$

5. In a 3-D state space spanned by orthonormal base set: |1>, |2> and |3> (in this order); two operators H and B are given their matrix form:

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \; B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Where ω_0 *and* b are real constants.

a) Are H and B Hermitian?

b) Show that H and B commute, i.e. HB=BH or [H,B]=0; Find out their eigenvalues and eigenvectors respectively; Find out a basis of eigenvectors **common** to H and B.

c) In the eigenvalue case, those of H would be in form of n$\hbar\omega_0$; n is called quantum number of H (n is unitless); those of B would be in from of Lb; L is the unitless quantum number for B; what are the values of n, L?

d) In this part we will investigate the trouble caused by degeneracy, i.e. multiple eigenvectors correspond to one eigenvalue. The eigenvectors of observables are obviously important, since they form the base vectors. In the case of non-degenerate situation, there is a 1 to 1 correspondence between eigenvalue and its eigenvector, i.e. knowing the eigenvalue or quantum number (say measurement give the result of some eigenvalue and cause the state collapse to the eigenstate), we know the unique eigenstate (up to a common phase factor). However, such confidence will not be justified in the situation of degeneracy, as in the last question, you will see that if we know the energy $E = -\hbar\omega_0$,(or n=-1) we cannot fix the state. To restore the 1 to 1 correspondence (i.e. knowing the measurement result, we are confident to say we know the state) between eigenvalue and eigenstate, commutable observables will be used as in this example (since commutable observables can be determined simultaneously).

Question: First let me explain the symbol: For a set of operator/operators, {H} means we only know the eigenvalues of H (or quantum number n); {B} means we only know the eigenvalues of B (or quantum numbers of L); {H,B} means we know the (n,L); {H^2,B} means we know the eigenvalues of H^2 and B. Of the above 4 sets of operators: {H}; {B}; {H,B}; {H^2,B}, which set can uniquely specify the eigenvector? [1](That set of commutable observables then will be called Complete Set of Commutable Observables, shorthand as C.S.C.O)

6. For a particle moves in a 1-D line, its wave function is given by:

[1] Let me explain this a bit more; for example in case of {H}, we can get n=1 or n=-1; if n=1, the state may collapse to some state, and let me specify that state as |n=1>, the question is does this |n=1> uniquely correspond to the eigen-base vector you determined in b)? same goes to |n=-1>; and the other sets of observables. Thus the importance of CSCO is that we can use quantum numbers of these observables to uniquely specify the eigen-base vectors.

$\psi(x) = N\frac{e^{ip_0x/\hbar}}{\sqrt{x^2+a^2}}$; $p_0; a$ are real constants and N is the normalization factor.

a) Determine the N so that $\psi(x)$ is normalized.

b) The position of the particle is measured. What is the probability of finding the position of the particle between $-\frac{a}{\sqrt{3}}\ and + \frac{a}{\sqrt{3}}$?

c) Calculate the mean value (means average) of momentum for a particle in state $\psi(x)$.

7. For a particle with mass=m, moves in 1-D line, with potential energy $V(X) = -fX$; f is a positive real number, X is position operator.

   a) Write the Ehrenfest theorem for the mean value of position X and momentum P of the particle. Integrate these equations (i.e. find out expression of how <X> , <P> change with time, you may specify symbols for integration constant); you may see the resemblance to classical mechanics.

   b) Show that the standard deviation ΔP (also known as root-mean-square deviation; defined as: $(\Delta P)^2 =< P^2 > -< P >^2$) does not vary with time.