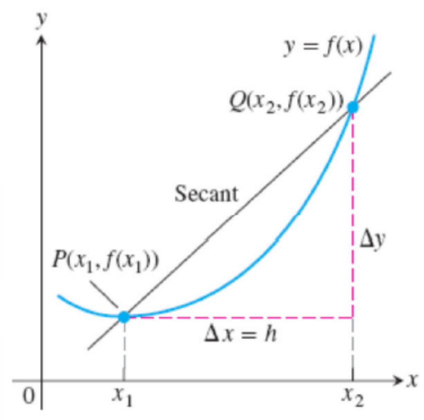
第一次习题课

1-1. Geometric and physical interpretation of derivatives

As the input x changes, the output y will also change. If the x is changed to some other value, say x + Δx , the y will change from f (x) to f (x + Δx) . **The average rate of change is defined as:**

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

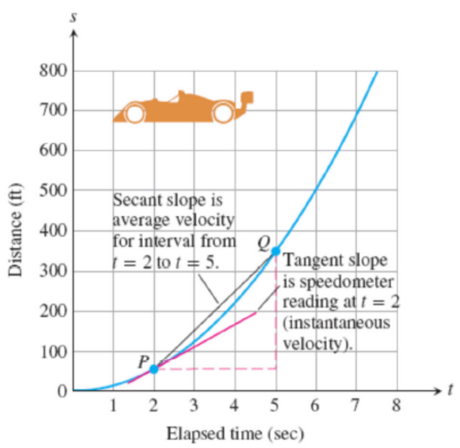


However, in many situations, the average rate of change may not be enough. We need the instantaneous rate of change. This corresponds to making the Q approaching P in the above figure, and the Δx becomes smaller, approaching 0. Under this limit, we calculate the rate of change. That is the instantaneous rate of change of function f(x) at certain point, say P.

This instantaneous rate of change is the physical interpretation of derivatives. The notation of derivatives of y = f (x) defined as:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative is also a function of the same variables. For example, if f(t) is the distance, the $\frac{df(t)}{dt}$ is the instantaneous velocity at time t, v(t).



The geometric interpretation is also clear from the figure above. As the Q approaches P, the secant line will become the **tangent line** at position P

1-2. Derivatives of basic functions

(1) Derivatives of Polynomial functions

$\frac{d}{dx}x^n = nx^{n-1}$,n is any integer

$\frac{d}{dx}x^r = rx^{r-1}$,x>0,r is any real number

(2) Derivatives of trigonometric functions

$\frac{d}{dx}sinx = cosx$.

$\frac{d}{dx}cosx = -sinx$.

(3) Derivatives of exponential functions

$\frac{d}{dx}e^x = e^x$.

$\frac{d}{dx}lnx = \frac{1}{x}$.

$\frac{\mathrm{d}}{\mathrm{dx}}a^x = (lna)a^x$.

1-3. General rules for derivative

The f(x) and g(x) are some functions of variable x. f' (x) and g' (x) are their derivatives over x. Then:

Rules of linearity

$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

$\frac{d}{dx}(cf) = c\frac{df}{dx}$, c is a constant

Production rule

$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$

Quotient rule

$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg\prime}{g^2}$

For composite functions (implicit dependence), i.e. f(u) a function of variable u, but the variable is also a function of x, u(x). We have the chain rule:

$$\frac{d}{dx}[f(u)] = \frac{df}{du}\frac{du}{dx}$$

Example 1: f(x)=tanx ,what is its derivative over x?

Example 2: f(x)=a^x , a>0, what is its derivative over x?

Example 3: f(x)=tan(1/x), what is its derivative?
Example 4: f(x)=xe^x , what is its derivative?

2-1. Antiderivative and Indefinite Integral

Antiderivative of a function f(x) is defined as:

$$\frac{\mathrm{dy}}{\mathrm{dx}} \equiv \frac{d}{dx}F(x) = F'(x) = f(x)$$

then y=F(x) is the antiderivative of the function f(x), i.e. f(x) is the derivative of F(x)

Actually the antiderivative is not a single but a group of functions, because if F(x) is an antiderivative, then F(x)+c, where c is a arbitrary constant independent of variables would also be an antiderivative of f(x), because dc/dx=0, so that d(F(x)+c)/dx=d(F(x))/dx=f(x).

We can use a symbol to present the whole group of antiderivatives:

$$\int f(x)dx \equiv F(x) + c$$

C is a constant, and this is the usual definition of **indefinite integral**.

Finding the antiderivative (or doing the indefinite integral) is much harder than finding the derivatives, though it is the reversed process of derivative. The basic rule from the definition is "**guessing**".

$$f(x) = 0, F(x) = c$$

$$f(x) = c, F(x) = \int c dx = cx$$

$$f(x) = x^n, F(x) = \int x^n dx = \frac{1}{n}x^{n+1}$$

$$f(x) = e^{ax}, F(x) = \int e^{ax} dx = \frac{1}{a}e^{ax}$$

$$f(x) = \sin(kx), F(x) = -\frac{1}{k}\cos(kx)$$

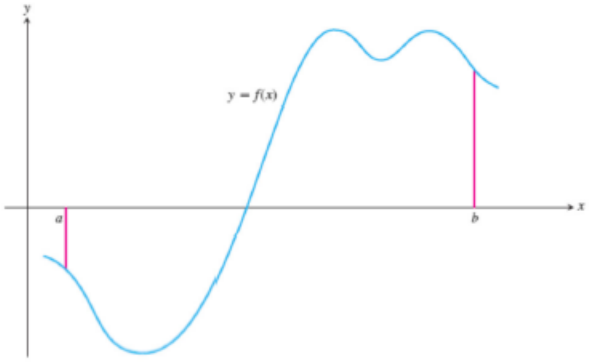
$$f(x) = \frac{1}{x}, F(x) = \int \frac{1}{x}dx = \ln|x| \quad x \neq 0$$

when x<0, introduce another variable u (u>0), and x=-u, and dx=-du. Then

$$\int \frac{1}{x}dx = \int \frac{-1}{u}(-du) = \int \frac{1}{u}du = lnu = \ln(-x) = \ln\ |x|$$

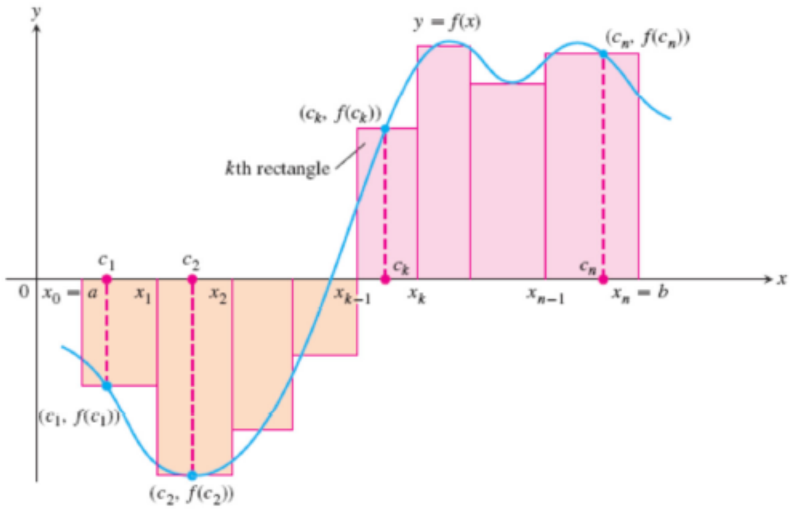
2-2. Definite Integral and Its Geometric Interpretation

The geometric interpretation is the area 'under' the curve, or more precisely for single variable integration, the area enclosed by the curve and the x-axis, the sign is positive for area above the axis, negative for area below the axis (refer to figure below).



For a general function, we can divide the total closed interval [a,b] into smaller subintervals || Pi ||= Δxi , pick a xi= ci within each interval, and Use f(ci)Δxi to approximate the area under the curve within that interval, then the total area between [a,b] would be approximated by Riemann sum:

$$F_{[a,b]} = \sum_{k=1}^{n} f(c_k)\Delta x_k$$



Of course how good this approximation would depend on how you choose the subintervals and xi=ci. But as ||Pi||= Δxi approaches 0, the choice would not matter, and the sum would approach to a fixed value which is the area under the curve. At this limit (||Pi||→ 0), the sum becomes what we called **definite integral:**

$$F_{[a,b]} = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)dx$$

The definite integral is to treat the problems like areas under a curve etc. For example, what Is the area enclosed by a circle? The area enclosed by x axis and curves like y=2x^2-8. If we know the density distribution, then we can obtain the mass of the object with certain shape.

If we have a v(t) curve, then the area enclosed by the curve and the t-axis will be the distance traveled within that time interval.

However, these properties and definition won' t help us too much in the calculation of definite integrals, it is usually hard to get definite integral from the limiting case of the Riemann sum.

2-3 Fundamental Theorem of Calculus and Calculation of Definite Integral

The fundamental theorem states that:
If f(x) is a continuous function for every point in interval [a,b], and F(x) is the antiderivative of f(x) in [a,b], then:

$$\int_a^b f(x)dx = F(a) - F(b)$$

Take the example of the total distance traveled between during a time period. We could calculate the distance by integrating the velocity function over time by definite integral, or if we know the function of location with time (which is the antiderivative of velocity, or velocity is the derivative of it, same thing), then we can simply calculate the distance between the initial and final time by direct subtraction.

Problem 9 in HW1：

(a) Exponential decay or growth: If the initial value (t=0)of some material (you money in bank; the C14 isotope in fossil, the mass of a rain droplet during condensation, you name it) is A0, and its changing with time at a constant rate kA, k is some proportional constant, i.e. dA/dt=kA, find the number of material after some time T, what is the time that the number dropped (this may require k > or < 0?)by a factor of e, and this time is usually called lifetime.
(b) The cooling of an egg. The temperature change can be approximated by heat dissipation and we shall approximate the dissipation is proportional to temperature difference between object and a sink (the sink' s temperature is kept at constant), i.e. dT/dt=-k(T-T0) where T0 is the temperature of the sink. For an egg boiled to 98 C, and was put into a water bath sink of 18C. After 5 minutes the egg was cooled to 38 C (the sink is always at 18C) and how much more time it will take for the egg to reach 20C?
(c) Draining a tank. The draining of the water tank depends on the height of the water level, and the change of volume of water in the cylinder is, $dV/dt = -\frac{1}{2}\sqrt{x}$ the radius of water is 5ft and the height is 16 ft initially, how long it will take to drain all the water?
(d) Bullet in air: The wind resistance that the bullet feels is proportional to its velocity, i.e. F=-kv (K is a constant, minus is because it is against bullet' s motion). If the initial velocity of the bullet is V0, find the velocity relation over time, i.e. after time t how fast the bullet will travel; and how far the bullet had traveled after time t? What happened if the resistance force is $F = -kv - lv^2$where l is another positive constant?