概统 第八次作业

习题3.3

16、(1) $\begin{cases} u = x+y \\ v = \frac{x}{x+y} \end{cases} \Rightarrow \begin{cases} x = uv \\ y = u-uv \end{cases} \Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & 1-v \\ u & -u \end{vmatrix} = -uv-(1-v)u = -u$

而 $P(x,y) = P_X(x) P_Y(y) = \begin{cases} e^{-(x+y)}, & x>0, y>0 \\ 0, & else \end{cases}$

则 $P_{U,V}(u,v) = P(x,y)|J| = \begin{cases} e^{-u}\cdot u, & u>0, 0<v<1 \\ 0, & else \end{cases}$

(2) $P_V(v) = \int_{-\infty}^{+\infty} P_{U,V}(u,v)\,du = \int_0^{+\infty} e^{-u}\cdot u\,du = -e^{-u}\cdot u\Big|_0^{+\infty} + \int_0^{+\infty} e^{-u}\,du = 1 \ (0<v<1)$

$P_U(u) = \int_{-\infty}^{+\infty} P_{U,V}(u,v)\,dv = \int_0^1 e^{-u}\cdot u\,dv = e^{-u}\cdot u \ (u>0)$

则 $P_{U,V}(u,v) = P_V(v)\cdot P_U(u)$，即 U,V 独立

17.

Pf: $\begin{cases} U = X^2+Y^2 \\ V = X/Y \end{cases} \Rightarrow \begin{cases} x = \sqrt{\frac{uv^2}{v^2+1}} \\ y = \sqrt{\frac{u}{v^2+1}} \end{cases} \Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}^{-1} = \begin{vmatrix} 2x & \frac{1}{y} \\ 2y & -\frac{x}{y^2} \end{vmatrix}^{-1} = (-2v^2-2)^{-1}$

$= -\frac{1}{2(v^2+1)}$

而 $P(x,y) = P_X(x) P_Y(y) = \frac{1}{2\pi} e^{-(\frac{x^2}{2}+\frac{y^2}{2})}$

则 $P_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}u}\cdot(-\frac{1}{2(v^2+1)}) = -\frac{e^{-\frac{1}{2}u}}{4\pi(v^2+1)} \ (u>0)$

$P_U(u) = \int_{-\infty}^{+\infty} P_{U,V}(u,v)\,dv = -\frac{e^{-\frac{1}{2}u}}{4\pi}\int_{-\infty}^{+\infty}\frac{1}{v^2+1}\,dv = -\frac{e^{-\frac{1}{2}u}}{4\pi}2\pi i\,Res[\frac{1}{v^2+1}, i]$

$= -\frac{e^{-\frac{1}{2}u}}{4\pi}2\pi i\times\frac{1}{2i} = -\frac{e^{-\frac{1}{2}u}}{4} \ (u>0)$

$P_V(v) = \int_{-\infty}^{+\infty} P_{U,V}(u,v)\,du = \int_0^{+\infty} -\frac{e^{-\frac{1}{2}u}}{4\pi(v^2+1)}\,du = -\frac{1}{4\pi(v^2+1)}\times 4 = -\frac{1}{\pi(v^2+1)}$

故 $P_{U,V}(u,v) = \frac{e^{-\frac{1}{2}u}}{4\pi(v^2+1)} \ (u>0) = P_U(u)\cdot P_V(v)$

即 U,V 相互独立

习题3.4

4、设n个点坐标为 $X_1 \dots X_n$，它们独立，

设 $K = max\{X_1 \dots X_n\}$ $L = min\{X_1 \dots X_n\}$

则 $F(k) = P(max\{X_1\dots X_n\}\le k) = k^n$

$P_K(k) = nk^{n-1}$

$F(L) = P(min\{X_1\dots X_n\}\le l) = 1-P(min\{X_1\dots X_n\}>l) = 1-(1-l)^n$

$P_L(l) = n(1-l)^{n-1}$

设 $Y = K-L$

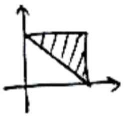
则 $EY = EK-EL$

$= \int_0^1 nk^{n-1}\cdot k\,dk - \int_0^1 n(1-l)^{n-1}\,l\,dl$

$= \frac{n}{n+1} - (-(1-l)^n l\Big|_0^1 + \int_0^1 (1-l)^n dl)$

$= \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}$

7.



$$p(x,y)=\begin{cases}2, & x\le 1,\ y\le 1,\ x+y\ge 1\\ 0\end{cases}$$

则 $E(X+Y)=\iint_{\substack{x\le 1\\ y\le 1\\ x+y\ge 1}} 2(x+y)\,dx\,dy=\int_0^1\int_{1-y}^1 2(x+y)\,dx\,dy$

$=\int_0^1 y^2+2y\ dy=\frac{1}{3}y^3+y^2\Big|_0^1=\frac{4}{3}$

$E(X+Y)^2=\iint_{\substack{x\le 1\\ y\le 1\\ x+y\ge 1}} 2(x+y)^2\,dx\,dy=\int_0^1\int_{1-y}^1 2(x+y)^2\,dx\,dy$

$=\int_0^1\left(\frac{2}{3}(1-(1-y)^3)+2(1-(1-y)^2)y+2y^3\right)dy$

$=\frac{11}{6}$

$Var(X+Y)=E(X+Y)^2-(E(X+Y))^2=\frac{1}{18}$

12.

$0<y<1$, $F_Y(y)=p(\max\{X_1\dots X_5\}\le y)=p(X_1\le y\cdots X_5\le y)$

$=p(X_1\le y)\cdots p(X_5\le y)$

$=p^5(X_1\le y)$

$=(\int_0^y 2x\,dx)^5=(y^2)^5=y^{10}$

$y\ge 1$时, $F_Y(y)=1$

$y\le 0$时, $F_Y(y)=0$

则 $p_Y(y)=\begin{cases}10y^9, & 0<y<1\\ 0, & \text{else}\end{cases}$

$EY=\int_{-\infty}^{+\infty} y\,p_Y(y)=\int_0^1 10y^{10}dy=\frac{10}{11}$

$EY^2=\int_{-\infty}^{+\infty} y^2 p_Y(y)=\int_0^1 10y^{11}dy=\frac{5}{6}$

则 $Var\,Y=EY^2-(EY)^2=\frac{5}{6}-\frac{100}{121}=\frac{605-600}{726}=\frac{5}{726}$.

16.

$X+Y=\begin{cases}2, & u\ge 1\\ 0, & -1\le u<1\\ -2, & u<-1\end{cases}$ 设 $Z=X+Y$ 则 $Z=\begin{cases}2, & u\ge 1\\ 0, & -1\le u<1\\ -2, & u<-1\end{cases}$

则 $E(X+Y)=2\cdot p(u\ge 1)+0\cdot p(-1\le u<1)+-2\cdot p(u<-1)$

$=2\times\frac{1}{4}-2\times\frac{1}{4}=0$

$E(X+Y)^2=4\cdot p(u\ge 1)+0\cdot p(-1\le u<1)+4\cdot p(u<-1)$

$=4\times\frac{1}{4}+4\times\frac{1}{4}=2$

$Var(X+Y)=E(X+Y)^2-(E(X+Y))^2=2$

23、 $X, Y \sim b(n, \frac{1}{2})$

$EX = np = \frac{1}{2}n$, $EY = np = \frac{1}{2}n$

$E(XY) = E(X(n-X)) = E(nX - X^2) = nEX - EX^2$

$= \frac{1}{2}n^2 - (VarX + (EX)^2) = \frac{1}{2}n^2 - (np(1-p) + \frac{1}{4}n^2)$

$= \frac{1}{2}n^2 - (\frac{1}{4}n + \frac{1}{4}n^2)$

$= \frac{1}{4}n^2 - \frac{1}{4}n$

则 $Cov(X,Y) = E(XY) - EXEY = \frac{1}{4}n^2 - \frac{1}{4}n - \frac{1}{4}n^2 = -\frac{1}{4}n$.

$Corr = \frac{Cov(X,Y)}{\sqrt{VarX \cdot VarY}} = \frac{Cov(X,Y)}{\sqrt{\frac{1}{4}n \cdot \frac{1}{4}n}} = \frac{-\frac{1}{4}n}{\frac{1}{4}n \cdot \frac{1}{4}n} = -1$.

25、 设 $X_i = \begin{cases} 1, & 第i个人抽中自己礼品 \\ 0, & else \end{cases}$

则 $EX_i = \frac{1}{n}$, $Var X_i = EX_i^2 - (EX_i)^2 = \frac{1}{n} - \frac{1}{n^2}$

因此 $EX = EX_1 + \cdots + EX_n = n \times \frac{1}{n} = 1$.

设 $i \neq j$ $Cov(X_i, X_j) = E(X_iX_j) - EX_iEX_j$

$= \frac{(n-2)!}{n!} - \frac{1}{n^2} = \frac{1}{n(n-1)} - \frac{1}{n^2} = \frac{1}{n^2(n-1)}$

因此 $VarX = Var(X_1 + \cdots + X_n)$

$= VarX_1 + \cdots + VarX_n + 2\sum_{1\le i<j\le n} Cov(X_i, X_j)$.

$= 1 - \frac{1}{n} + 2 \times \frac{(n-1)n}{2} \times \frac{1}{n^2(n-1)}$

$= 1 - \frac{1}{n} + \frac{1}{n} = 1$

27、 $P_X(x) = \int_{-\infty}^{+\infty} p(x,y)\,dy = \int_{-x}^{x} 1\,dy = 2x \quad (0<x<1)$.

故 $P_X(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & else \end{cases}$

$1>y\geq 0$, $P_Y(y) = \int_{-\infty}^{+\infty} p(x,y)dx = \int_{y}^{1} 1\,dy = 1-y$.

$0>y\geq -1$, $P_Y(y) = \int_{-\infty}^{+\infty} p(x,y)dx = \int_{-y}^{1} 1\,dy = 1+y$

故 $P_Y(y) = \begin{cases} 1-y, & 0\le y<1 \\ 1+y, & -1<y<0 \\ 0, & else \end{cases}$

$E(X) = \int_{-\infty}^{+\infty} x\,p_X(x)dx = \int_0^1 2x^2 dx = \frac{2}{3}$

$E(Y) = \int_{-\infty}^{+\infty} y\,P_Y(y)dy = \int_{-1}^{0}(y + y^2)dy + \int_0^1 (y - y^2)dy = 0$

$E(XY) = \iint_{\mathbb{R}^2} xy\,p(x,y)dxdy = \iint_{\substack{0<x<1\\0\le y<1}} xy(2x)(1-y)\,dxdy + \iint_{\substack{0<x<1\\-1<y<0}} xy(2x)(1+y)\,dxdy$

$= \frac{1}{9} - \frac{1}{9} = 0$.

$Cov(X,Y) = E(XY) - EXEY = 0$.

28. $p_X(x) = \int_{-\infty}^{+\infty} p(x,y)\,dy = \int_0^x 3x\,dy = 3x^2 \quad (0<x<1)$.

$p_Y(y) = \int_{-\infty}^{+\infty} p(x,y)\,dx = \int_y^1 3x\,dx = \frac{3}{2} - \frac{3}{2}y^2 \quad (0<y<1)$.

$E(X) = \int_{-\infty}^{+\infty} x\, p_X(x) = \int_0^1 3x^3\,dx = \frac{3}{4}$

$E(Y) = \int_{-\infty}^{+\infty} y\, p_Y(y) = \int_0^1 \left(\frac{3}{2}y - \frac{3}{2}y^3\right) dy = \frac{3}{4} - \frac{3}{8} = \frac{3}{8}$.

$$E(XY) = \iint_{\mathbb{R}^2} xy\, p(x,y)\,dx\,dy = \int_0^1 \int_y^1 3x^2 y\,dx\,dy = \int_0^1 (1-y^3)y\,dy = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

则 $Cov(X,Y) = E(XY) - EXEY = \frac{3}{10} - \frac{9}{32} = \frac{48-45}{160} = \frac{3}{160}$.

而 $E(X^2) = \int_{-\infty}^{+\infty} x^2 p_X(x) = \int_0^1 3x^4\,dx = \frac{3}{5}$

$E(Y^2) = \int_{-\infty}^{+\infty} y^2 p_Y(y) = \int_0^1 \left(\frac{3}{2}y^2 - \frac{3}{2}y^4\right) dy = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}$

$Var(X^2) = E(X^2) - (E(X))^2 = \frac{3}{5} - \frac{9}{16} = \frac{96-90}{160} = \frac{6}{160}$

$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{1}{5} - \frac{9}{64} = \frac{64-45}{320} = \frac{19}{320}$

$$Corr = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{\frac{3}{160}}{\sqrt{\frac{6}{160} \times \frac{19}{320}}} = \frac{3}{\sqrt{57}}$$

44. (1) 根据全概率公式 $F_Z(z) = P(X\cdot Y \le z)$

$$= P(Y=-1)\,P(X\cdot Y \le z \wedge Y=-1) + P(Y=1)\,P(X\cdot Y \le z \wedge Y=1)$$
$$= \frac{1}{2}\left(P(X \ge -z) + P(X \le z)\right)$$
$$= \frac{1}{2}\left(1 - P(X<-z) + P(X \le z)\right)$$
$$= \frac{1}{2}\left(1 - F_X(-z) + F_X(z)\right) = F_X(z).$$

$X \sim N(0,1)$

则 $Z \sim N(0,1)$.

(2) $EX = 0$, $EZ = 0$,

$E(X\cdot Z) = E(X^2\cdot Y) = EX^2 \cdot EY = 0$

则 $Cov(X,Z) = E(X\cdot Z) - EXEZ = 0$, ∴ X, Z 不相关

但 $F_Z(1) = F_X(1)$

$$F_{X,Z}(1,1) = P(X\le 1,\ XY \le 1)$$
$$= \frac{1}{2}P(X\le 1 \wedge XY\le 1 \mid Y=1) + \frac{1}{2}P(X\le 1 \wedge XY \le 1 \mid Y=-1)$$
$$= \frac{1}{2}P(X\le 1) + \frac{1}{2}P(-1\le X\le 1) = \frac{1}{2}P(X\le 1) + \frac{1}{2}P(X\le 1) - \frac{1}{2}P(X\le -1)$$
$$= P(X\le 1) - \frac{1}{2}P(X\le -1)$$
$$= F_X(1) - \frac{1}{2}F_X(-1) = \frac{3}{2}F_X(1) - \frac{1}{2}$$

设 $F_X(1)F_Z(1) = F_X^2(1) = F_{X,Z}(1,1) = \frac{3}{2}F_X(1) - \frac{1}{2}$

则 $F_X(1) = \frac{1}{2}$ 或 1 显然矛盾

因此 $F_X(1)F_Z(1) \ne F_{X,Z}(1,1)$，即 X, Z 不独立

综上：X, Z 不相关且不独立