Homework 10 for General Physics II

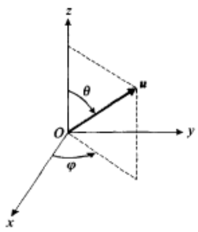
In the class, I derived the expression for the important states and operators of a spin1/2 system using results of SG experiments. In the |+Z>-|-Z> basis:

$$|+x>=\frac{\sqrt{2}}{2}|+z>+\frac{\sqrt{2}}{2}|-z>$$

$$|-x>=\frac{\sqrt{2}}{2}|+z>-\frac{\sqrt{2}}{2}|-z> \qquad S_z=\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ S_x=\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ \text{and } S_y=\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$|+y>=\frac{\sqrt{2}}{2}|+z>+\frac{\sqrt{2}}{2}i|-z>$$

$$|-y>=\frac{\sqrt{2}}{2}|+z>-\frac{\sqrt{2}}{2}i|-z>$$

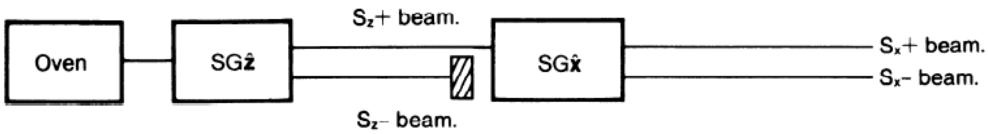


And for arbitrary direction u:

$$S_u=\sin\theta\cos\varphi(S_x)+\sin\theta\sin\varphi(S_y)+\cos\theta(S_z)=\frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}$$

All the problems below, the matrices are all shown as in |+Z>,|-Z> basis

1. Express the |+Z> and |-Z> state in terms of combination of |+x>-|-x> as well as combination of |+y>-|-y>.



2. For a SG setup shown above, two magnetic fields arranged along z and x direction. The experimental results show two electron spots, corresponding to $\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ of the spin along the x-direction. N electrons (N is large) come out of oven, half will pass and half will be blocked after SGz; What are the number of electrons appear in the final detection screen for each spot? (We may add one more SGz after the SGx and let both beams passing through the last SGz, what are the experimental observation?)

3. If we prepare a spin state of electron in forms of $|\varphi>=\frac{1}{\sqrt{6}}\begin{bmatrix}1+i\\ 2\end{bmatrix}$, now we carry out SG experiment with this input state. 1) Measure the Sz, what are the values of measurement? What are the probability for each result? What is the average of this measurement?(expectation value) 2) Answer same questions for the Sx measurement.

4. For an arbitrary direction of magnetic field in the SG experiment, say it is along direction u as shown in the figure above. The expression for Su is also provided above. 1) Find the eigenvalues and associated eigenstates of Su. (show that $|+u>=\cos(\frac{\theta}{2})e^{-i\varphi/2}|+Z>+\sin(\frac{\theta}{2})e^{i\varphi/2}|-Z>$ and $|-u>=-\sin(\frac{\theta}{2})e^{-i\varphi/2}|+Z>+\cos(\frac{\theta}{2})e^{i\varphi/2}|-Z>$
5. For a series of SG experiment, the first SG is along z and acting as a 'polarizer', i.e. prepare the spin state into |+Z>, then this |+Z> will pass the second SGu which is along the direction u as in last problem. What are the experiment results and the probability? How about we prepare a state in |+u> first and then let it pass through Sx measurement?
6. Consider a somewhat abstract two-state system, spanned by orthonormal basis |1> and |2> corresponding to some observable (say energy). The system also has other observables, known as A, B (they are observables (operator), not number), given the following measured result in (a) and (b), please try to determine the matrix form for A,B, and find out their eigenvalues as much as possible (The measuring results may not be sufficient enough for you to get all elements of the matrix).

   (a) $<1|\widehat{A}|1>=\frac{1}{2}, \quad <1|\widehat{A}^2|1>=\frac{1}{4} \quad \widehat{A}^2 \equiv \widehat{A}\widehat{A}$

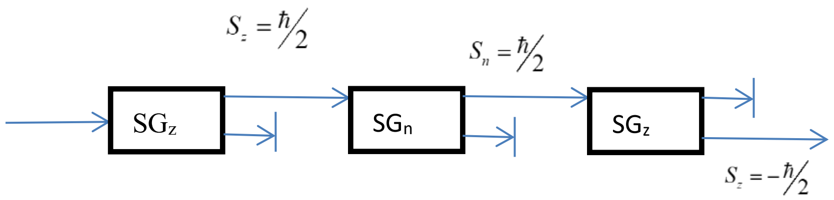
   (b) $<1|\widehat{B}|1>=1, <1|\widehat{B}^2|1>=\frac{5}{4}, <1|\widehat{B}^3|1>=\frac{7}{4}$
7. (Optional) Consider 4 Hermitian matrices in 2-D space (2X2 matrix), $I,\sigma_1,\sigma_2,\sigma_3$, where I is the identity matrix, the others satisfies a relation: $\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}$ (2 here means 2 times identity). Of course you may guessed these are Pauli matrices, but following can be derived without refer to detailed expression of the matrices, just using the relation provided above.
   (a) Prove that $Tr(\sigma_i)=0$, where Tr( ) means trace of the matrix (sum of diagonal elements). In case you forgot linear algebra, Tr(AB)=Tr(BA)
   (b) Prove that the σ matrices all have eigenvalues +1 and -1, and its determinant is -1
   (c) Show that the 4 matrices are linearly independent from the definition of linear independence. (This means any matrix in 2-D can be written as linear combination of them)

   (Of course this problem tries to refresh you with linear algebra, and rest assured I won't give such mathematical problems in final, that is why this one is optional)

8. **Prove that the position operator X in momentum representation (1-D): （If I have not covered this yet, you may leave this problem for next week）**

$$< p | X | \psi >= i\hbar \frac{d < p | \psi >}{dp_x} = i\hbar \frac{d\varphi(p)}{dp_x}$$

9. In the experiment shown below, we first select |+z> state after electron passing first SG_z device (M-field gradient along Z, the |-z> were blocked); then let it pass second SG_n whose M-field is along direction $\hat{n}$; then we select electron only in |+n> state; let them pass the 3rd SG_z and we select |-z> electrons. The second SG_n has direction along θ arbitrary, $\phi = 0$



(a) What fraction (the same meaning as probability) of particles that transmitted by the first SG will pass the last SG?
(b) What is the angle for the 2nd SG_n so to maximize the particles that will pass the last SG?
(c) If the 2nd SG_n is removed, what will be the fraction of particles that pass the last SG?

10. The state of a spin 1/2 particle is given by:

$$| \psi >= \frac{i}{\sqrt{3}} | +z > + \sqrt{\frac{2}{3}} | -z >$$

We measure S_z for particles in such state. A) What is the average (expectation value) of $<S_z>$, and what is the standard deviation (uncertainty) ΔS_z . Note the ΔS_z is defined as:

$$(\Delta S_z)^2 =< (S_z - < S_z >)^2 >=< S_z^2 > - < S_z >^2$$

B) What is the time evolution of the state if particle initially in above state is subject to a constant magnetic B field along z direction and interaction Hamiltonian is given by: $\hat{H} = \omega_0 \hat{S}_z \quad (\omega_0 = -g_s B)$

11. A study of an operator. In this problem we shall investigate an operator which may look strange (or unfamiliar) at beginning, but through a systematic study, we will start building a feeling for its physical significance. The operator is defined as:

$$\hat{R}_z(\phi) = e^{-i\phi \hat{S}_z/\hbar}$$

ϕ is an angle parameter and S_z is the spin component along z. It may appear clueless at present what this operator will do and we will see below:

(a) What is R's effect on the |+z>, |-z>?
(b) What is R's matrix form in |+z>,|-z> basis?

(c) Is R unitary, i.e. $R^{\dagger} = R^{-1}$

(d) What is R's effect on other |+x>, for simplicity, we set $\phi = \frac{\pi}{2}$ here to see this operation will change |+x> state into what state?

(e) From above, can you see what this R operator does? (hint: rotation)

(f) Puzzle: when $\phi = 2\pi$, there is some peculiarity. Prove $R_z(2\pi)|\psi> = -|\psi>$ for spin 1/2 particles.

Comment: It is weird (counter-intuitive) because rotation by 2π will not restore state but with a minus sign! This is clearly due to spin quantum number may not be integers (here is 1/2) and is a quantum effect. Though a total phase factor for a state will not affect measurement on such state, but in an interference experiment, where state with difference phase factor will show difference, and thus using such interference experiment, people can experimentally verify this quantum peculiarity for spin 1/2 particles (ref. Werner et al. *Phys. Rev. Lett.* **35**, 1053 (1975))