清華大學

本科生专用试卷

考场纪律	一、学生应试时必须携带学生证，以备查对，学生必须按照监考教师指定的座位就坐。 二、除答卷必须用的笔、橡皮及教师指定的考试用具外，不得携带任何书籍、笔记、草稿纸等。 三、答卷时不准互借文具（包括计算器）。题纸上如有字迹不清等问题，学生应举手请监考教师解决。 四、学生应独立答卷，严禁左顾右盼、交头接耳、抄袭或看别人答卷等各种形式的作弊行为，如有违反，当场取消其考试资格，答卷作废。 五、在规定的时间内答卷，不得拖延。交卷时间到，学生须在原座位安静地等候监考教师收卷后，方可离开考场。

系别 软件学院 班号 软件02 学号 2020010108 姓名 徐浩博 成绩______

考试课程 概率论与数理统计 日期 2022 年 6 月 13 日 阅卷教师______

题 号	一	二	三	四	五	六	七	八	九	十	十一	十二	总 分
成 绩													

Tsinghua University
Undergraduate Students Examination Paper

Before you begin, please read the following instructions carefully:

1. Students need to bring valid student IDs and follow the seating arrangements.
2. Only pens, erasers, and materials specifically appointed by the lecturer are allowed in the exam. Any personal belongings such as books, notes, or scratch paper are restricted.
3. Students are not permitted to share any stationary (including calculators) with others once the exam has begun. For any questions regarding the exam paper, please raise hands to notify the examiner.
4. Students should take the exam independently, and are strictly prohibited to give or receive assistance of any kind during the exam. Any cheating, any attempt to cheat, or engaging in improper conducts, including but not limited to looking around, talking, copying other students' answers, will be subject to disqualification immediately.
5. Students are expected to stop writing immediately once the exam time is up. Before leaving the room, all students must wait in their seats for the exam paper to be collected by the examiner.

Department______ Class______ Student No.______ Name______ Score______

Course______ Date_____MM/_____DD/_____/YY Evaluated by ______

Part	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	Total Score
Score													

1. A B 独立

理由: $P(A|B) = P(A|\bar{B})$

$$\Rightarrow \frac{P(AB)}{P(B)} = \frac{P(A\bar{B})}{P(\bar{B})}$$

$$\Rightarrow P(AB)P(\bar{B}) = P(A\bar{B})P(B) \quad ①$$

而 $P(AB) = P(AB)P(\bar{B}) + P(AB)P(B)$

代入① 得 $P(AB) = P(A\bar{B})P(B) + P(AB)P(B)$

$$= (P(A\bar{B}) + P(AB))P(B)$$

$$= P(A)P(B)$$

即有 $P(AB) = P(A)P(B)$

这说明 A. B 独立

2.

(1) B_0: 第一次取出无新球　B_1: 第一次一个新　B_2: 第一次两个新

A: 第二次均新

$$P(A) = P(A|B_0)P(B_0) + P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$

$$= \frac{C_5^2}{C_9^2} \times \frac{C_4^2}{C_9^2} + \frac{C_4^1 C_5^1}{C_9^2} \times \frac{C_5^1 C_4^1}{C_9^2} + \frac{C_3^2}{C_9^2} \cdot \frac{C_5^2}{C_9^2}$$

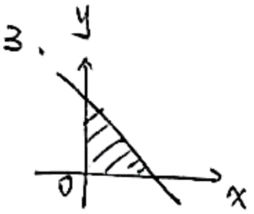
$$= \frac{60}{1296} + \frac{400}{1296} + \frac{30}{1296} = \frac{245}{648} \approx 0.378$$

(2)

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A)}$$

$$= \frac{\frac{C_3^2}{C_9^2} \times \frac{C_5^2}{C_9^2}}{\frac{245}{648}} = \frac{3}{49} \approx 0.061$$

3.



(a) $P_{X,Y}(x,y) = \begin{cases} 2, & \text{if } x,y>0,\ x+y<1 \\ 0, & \text{else} \end{cases}$

$P_X(x) = \int_0^{1-x} P_{X,Y}(x,y)\,dy = \int_0^{1-x} 2\,dy = 2-2x$

$P_y(x) = \int_0^{1-y} P_{X,Y}(x,y)\,dx = \int_0^{1-y} 2\,dx = 2-2y$

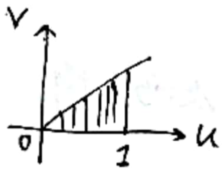
则 $P_X(\frac{1}{2})P_Y(\frac{1}{2}) = 1\times 1 \neq P_{X,Y}(X=\frac{1}{2}, y=\frac{1}{2}) = 2$

即 X, Y 不独立.

(b) $\begin{cases} U = X+Y \\ V = X \end{cases} \Rightarrow \begin{cases} X = V \\ Y = U-V \end{cases} \Rightarrow |J| = \left|\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}\right| = 1$

因此 $P_{U,V}(u,v) = P_{X,Y}(x(u,v), y(u,v)) \times |J|$

$= 2 \quad (v>0, u-v>0,\ u<1)$



$P_U(u) = \int_u^1 P_{U,V}(u,v)\,dv$

$= \int_u^1 2\,dv = 2-2u$

即 $P_Z(z) = \cancel{P_U(u)=}\ 2-2z \quad (Z = X+Y,\ 0<z<1)$

$F_Z(z) = \int_0^z (2-2z)\,dz = 2z-z^2$

因此 $F_Z(z) = \begin{cases} 0, & z<0 \\ 2z-z^2, & 0\le z\le 1 \\ 1, & z>1 \end{cases}$

(c)

$$E(Y|X=x) = \int_0^{1-x} y\, P_{Y|X}(y|x)\,dy$$

$$= \int_0^{1-x} y\, \frac{P_{X,Y}(x,y)}{P_X(x)}\,dy$$

$$= \int_0^{1-x} y\, \frac{2}{2-2x}\,dy$$

$$= \left.\frac{y^2}{2-2x}\right|_{y=0}^{1-x} = \frac{(1-x)^2}{2-2x} = \frac{1-x}{2}$$

则 $E(Y|X) = \frac{1-X}{2}$

(d) $E(X|X<Y) = \dfrac{E(X I_{X<Y})}{P(X<Y)}$

$P(X<Y) = \int_0^1 P(X<Y|Y=y)\,P_Y(y)\,dy = \int_0^1 P(X<y)(2-2y)\,dy$

$= \int_0^1 F_X(y)(2-2y)\,dy = \int_0^1 (2y-y^2)(2-2y)\,dy = \frac{1}{2}$

$$E(X I_{X<Y}) = \int_0^1 \int_0^{1-x} x I_{x<y} P_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^{\frac{1}{2}} \int_x^{1-x} 2x\, dy\, dx = \frac{1}{12}$$

则 $E(X \mid X<Y) = \dfrac{E(X I_{X<Y})}{P(X<Y)} = \dfrac{\frac{1}{12}}{\frac{1}{2}} = \dfrac{1}{6}$

4.

$X \sim \Gamma(n+1, 1)$

$EX = \dfrac{\alpha}{\lambda} = n+1$

$\mathrm{Var}\, X = \dfrac{\alpha}{\lambda^2} = n+1$

则由 Chebyshev 不等式

$$P(X > 2(n+1)) = P(X-(n+1) > n+1)$$

$$= P(|X-(n+1)| > n+1)$$

$$= P(|X-EX| > n+1)$$

$$\leq \frac{\mathrm{Var}\, X}{(n+1)^2}$$

$$= \frac{n+1}{(n+1)^2} = \frac{1}{n+1} \quad \square$$

5、

(1) $p_X(x) = \lambda e^{-\lambda x}$

$\Rightarrow F(x) = 1 - e^{-\lambda x}$

$F(\frac{1}{2}) = 1 - e^{-\frac{1}{2}\lambda} = \frac{1}{2}$

$\Rightarrow \frac{1}{2} = e^{-\frac{1}{2}\lambda}$

$\Rightarrow -\frac{1}{2}\lambda = -\ln 2$

$\Rightarrow \lambda = 2\ln 2$

(2)

$$
\begin{aligned}
E(X-c)^2 &= E(X-EX+EX-c)^2 \\
&= E[(X-EX)^2 + 2(X-EX)(EX-c) + [EX-c]^2] \\
&= E(X-EX)^2 + 2(EX-c)E(X-EX) + (EX-c)^2 \\
&= E(X-EX)^2 + (EX-c)^2 \geqslant E(X-EX)^2
\end{aligned}
$$

取等时 $c = EX$.

即 $E(X-c)^2 \geqslant E(X-EX)^2 = \mathrm{Var}X = \frac{1}{\lambda^2} = \frac{1}{4\ln^2 2}$

因此 $\min E(X-c)^2 = \frac{1}{4\ln^2 2}$

(3) $\mathrm{Var}X = \frac{1}{\lambda^2} = \frac{1}{4\ln^2 2}$

$\sqrt{\mathrm{Var}X} = \frac{1}{2\ln 2}$

$$
\begin{aligned}
P(X > \sqrt{\mathrm{Var}X}) &= 1 - P(X \leqslant \sqrt{\mathrm{Var}X}) \\
&= 1 - P(X \leqslant \tfrac{1}{2\ln 2}) \\
&= 1 - F_X(\tfrac{1}{2\ln 2}) \\
&= 1 - (1 - e^{-2\ln 2 \cdot \frac{1}{2\ln 2}}) = e^{-1}
\end{aligned}
$$

~~(4) 设 $U = X^2 \Rightarrow X = \sqrt{U}$~~

~~$p_U(u) = p_X(u(x)) \frac{1}{2\sqrt{u}} = \lambda e^{-\lambda\sqrt{u}} \frac{1}{2\sqrt{u}}$~~

$EX_i^2 = Var X_i + (EX_i)^2 = \frac{2}{\lambda^2}$ $\quad Var \frac{1}{n}\Sigma X_i^2 = \frac{1}{n^2} Var EX_i^2 = \frac{}{n}$

(4)

$\because$ $\{X_i^2\}$ 独立同分布 $\quad Var X_i^2 = EX_i^4 - (EX_i^2)^2 < +\infty$

由辛钦大数定律 $\quad \frac{1}{n}\Sigma X_i^2 \xrightarrow{P} EX_i^2 = \frac{2}{\lambda^2}$

6. X, Z 相互独立

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X \cdot \mathrm{Var}Y}}$$

$$\Rightarrow \mathrm{Cov}(X,Y) = \sqrt{\mathrm{Var}X \cdot \mathrm{Var}Y}\ \mathrm{Corr}(X,Y)$$

$$= 3 \times 4 \times \left(-\frac{1}{2}\right) = -6$$

则 $\mathrm{Cov}(X,Z) = \mathrm{Cov}\left(X, \frac{X}{3} + \frac{Y}{2}\right)$

$$= \frac{1}{3}\mathrm{Cov}(X,X) + \frac{1}{2}\mathrm{Cov}(X,Y)$$

$$= \frac{1}{3} \times 3^2 + \frac{1}{2} \times (-6)$$

$$= 3 - 3 = 0 \quad \Rightarrow X, Z \text{ 不相关}$$

而二维正态分布有

X, Z 不相关 $\Leftrightarrow$ X, Z 独立

故 X, Z 相互独立

7、
(a)
$x>\theta$, $F(x)=\int_\theta^x e^{\theta-t}dt=-e^{\theta-t}\Big|_{t=\theta}^{x}=1-e^{\theta-x}$

则 $F(x)=\begin{cases}0, & x\le\theta\\ 1-e^{\theta-x}, & x>\theta\end{cases}$

(b) 矩估计:

$$EX=\int_\theta^{+\infty}x\,p(x)\,dx=\int_\theta^{+\infty}xe^{\theta-x}dx$$
$$=-xe^{\theta-x}\Big|_\theta^{+\infty}+\int_\theta^{+\infty}e^{\theta-x}dx$$
$$=\theta+1$$
$$\Rightarrow \theta=EX-1$$
$$\Rightarrow \hat\theta=\bar{x}-1$$

似然估计

$$L(\theta)=\prod_{i=1}^n p_i(x_i)=e^{\sum_{i=1}^n(\theta-x_i)}$$
$$\ln L(\theta)=\sum_{i=1}^n(\theta-x_i)=n\theta-\sum_{i=1}^n x_i$$
$$\frac{d\ln L(\theta)}{d\theta}=n>0 \quad 即\ L(\theta)是递增函数$$

则 而 $\theta<x_i\quad \forall\ 1\le i\le n$.

即 $\theta<\min\{x_i\}$

因此 $\hat\theta=\min\{x_i\mid i=1\dots n\}=x_{(1)}$.

(c) 矩估计 $E\hat\theta=E\bar{x}-1=EX_i-1=\theta+1-1=\theta$, 故无偏

$$Var\hat\theta=Var(\bar{x}-1)=Var\,\bar{x}$$
$$=\frac{1}{n^2}Var(X_1+\cdots X_n)=\frac{1}{n^2}\sum_{i=1}^n Var X_i$$

而 $EX^2=\int_\theta^{+\infty}x^2p(x)dx=\int_\theta^{+\infty}x^2e^{\theta-x}dx=-x^2e^{\theta-x}\Big|_\theta^{+\infty}+2\int_\theta^{+\infty}xe^{\theta-x}dx$
$$=\theta^2+2\theta+2$$

$$Var X=EX^2-(EX)^2=2\theta+2$$

因此 $Var\hat\theta=\frac{1}{n^2}\sum Var\,x_i=\frac{1}{n^2}n(2\theta+2)=\frac{2\theta+2}{n}\to 0\quad(n\to\infty)$

由定理6.2.1(书本上) $\bar{x}-1$ 是相合估计

即矩估计既满足相合性又满足无偏性

似然估计 $\hat{\theta}_n = x_{(1)}$ ， $p_{x_{(1)}}(x) = n(1-F(x))^{n-1} p(x)$

$$= n e^{(n-1)(\theta-x)} e^{\theta-x}$$

$$= n e^{n(\theta-x)} \quad (x>\theta)$$

则 $E\hat{\theta}_n = E x_{(1)} = \int_\theta^{+\infty} x\, p_{x_{(1)}}(x)\,dx$

$$= \int_\theta^{+\infty} n x e^{n(\theta-x)}\,dx$$

$$= -x e^{n(\theta-x)}\Big|_{x=\theta}^{+\infty} + \int_\theta^{+\infty} e^{n(\theta-x)}\,dx$$

$$= \theta - \frac{1}{n} e^{n(\theta-x)}\Big|_\theta^{+\infty}$$

$$= \theta + \frac{1}{n}$$

，即 $x_{(1)}$ 不是无偏估计

而 $\lim\limits_{n\to+\infty} E\hat{\theta}_n = \theta$.

$$E\hat{\theta}_n^2 = E x_{(1)}^2 = \int_\theta^{+\infty} x^2 p_{x_{(1)}}(x)\,dx$$

$$= \int_\theta^{+\infty} n x^2 e^{n(\theta-x)}\,dx$$

$$= -x^2 e^{n(\theta-x)}\Big|_\theta^{+\infty} + \int_\theta^{+\infty} 2x e^{n(\theta-x)}\,dx$$

$$= \theta^2 + \frac{2}{n} E x_{(1)}$$

$$= \theta^2 + \frac{2}{n}\left(\theta + \frac{1}{n}\right)$$

$$Var\,\hat{\theta}_n = E\hat{\theta}_n^2 - (E\hat{\theta}_n)^2$$

$$= \theta^2 + \frac{2}{n}\theta + \frac{2}{n^2} - \theta^2 - \frac{2}{n}\theta - \frac{1}{n^2}$$

$$= \frac{1}{n^2} \to 0 .$$

则 $\lim\limits_{n\to+\infty} Var\,\hat{\theta}_n = 0$.

由定理 6.2.1（书本上） $\hat{\theta} = x_{(1)}$ 是相合估计

综上：似然估计不满足无偏性

但满足相合性