高代选讲 第十四周作业

1.

(a) $\forall v = \sum_i v_i e_i \in V$

$$
\begin{aligned}
t^i(v) &= t^i\left(\sum_j v_j e_j\right) \\
&= t^i\left(\sum_j v_j \sum_k (A^{-1})_{kj} t_k\right) \\
&= \sum_j \sum_k v_j (A^{-1})_{kj}\, t^i(t_k) \\
&= \sum_j v_j (A^{-1})_{ij} \\
&= \sum_j (A^{-1})_{ij}\, e^j\left(\sum_k v_k e_k\right) \\
&= \left(\sum_j (A^{-1})_{ij} e^j\right)(v).
\end{aligned}
$$

即 $t^i = \sum_j (A^{-1})_{ij} e^j = \sum_j (A^{-1})^T{}_{ji}\, e^j$

而 $t^i = \sum_j B_{ji} e^j \quad \Rightarrow \quad B = (A^{-1})^T$.

(b) $G = A^T F A$.

$$
\begin{aligned}
g_{ij} = f(t_i, t_j) &= f\left(\sum_k (A)_{ki} e_k,\ \sum_l (A)_{lj} e_l\right) \\
&= \sum_k \sum_l (A)_{ki} (A)_{lj}\, f(e_k, e_l) \\
&= \sum_k \sum_l (A)_{ki}\, f_{kl}\, A_{lj}. \\
&= \sum_k \sum_l (A^T)_{ik}\, f_{kl}\, A_{lj}
\end{aligned}
$$

即 $G = A^T F A$.

2.

(a) 验证 Φ 是 V 上的线性映射, 即证 $\forall u, v \in V$, $\alpha, \beta \in F$ 有 $\Phi(\alpha u + \beta v) = \alpha\Phi(u) + \beta\Phi(v)$

$\Phi(\alpha u + \beta v) \in V = V^{**}$

$\forall w^* \in V^*$

$$
\begin{aligned}
\left(\Phi(\alpha u + \beta v)\right)(w^*) = \phi_{\alpha u + \beta v}(w^*) &= f(\alpha u + \beta v, w^*) \\
&= \alpha f(u, w^*) + \beta f(v, w^*) \\
&= \alpha \Phi_u(w^*) + \beta \Phi_v(w^*) \\
&= \alpha(\Phi(u))(w^*) + \beta(\Phi(v))(w^*) \\
&= (\alpha\Phi(u) + \beta\Phi(v))(w^*)
\end{aligned}
$$

即 $\Phi(\alpha u + \beta v) = \alpha\Phi(u) + \beta\Phi(v)$

这就验证了 Φ 的线性性.

(b) 可以，实际上ψ在一组基下对应的矩阵就是(1,1)张量这个双线性函数的表示矩阵

设V中一组基为$(e_1, \dots, e_n)$，ψ在其下的矩阵为A.

$$\psi(e_i) = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} \in V = V^{**}$$

$\forall\, v^* \in V^*$, $v^* = \sum_i v_i e^i$

$$v^*(\psi(e_i)) = v^*\begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} = \sum_j v_j a_{ji}$$

令$f: V^* \times V \to F$，在基下表示矩阵为A则.

$$f(e^i, e_j) = a_{ij}$$

因此
$$\begin{aligned} f(v^*, e_i) &= f(\sum_j v_j e^j, e_i) \\ &= \sum_j v_j f(e^j, e_i) \\ &= \sum_j v_j a_{ji} \end{aligned}$$

即 $v^*(\psi(e_i)) = f(\cdot, e_i)$

即 $\cdot(\psi(e_i)) = f(\cdot, e_i) \quad \forall i$.

这样我们就通过ψ构造了一个(1,1)张量F.