高代选讲　第十三周作业

1　$\sin z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbb{C}$

则　$\sin A = \sum_{n=0}^{\infty} (-1)^{n} \frac{A^{2n+1}}{(2n+1)!}$

$\cos z = \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n}}{(2n)!} \quad \forall z \in \mathbb{C}$

则　$\cos A = \sum_{n=0}^{\infty} (-1)^{n} \frac{A^{2n}}{(2n)!}$

$\arctan z = \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n+1}}{2n+1} \quad \forall |z|<1$

则　$\arctan A = \sum_{n=0}^{\infty} (-1)^{n} \frac{A^{2n+1}}{2n+1} \quad (\rho(A)<1)$

$\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \forall |z|<1$

则　$\ln(I-A) = -\sum_{n=1}^{\infty} \frac{A^n}{n} \quad (\rho(A)<1)$

$\sin(z+a) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z+a)^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbb{C}$

则　$\sin(A+aI) = \sum_{n=0}^{\infty} (-1)^{n} \frac{(A+aI)^{2n+1}}{(2n+1)!}$

2、

$A^2 = A \Rightarrow A^3 = A^2 \cdot A = A \cdot A = A^2 = A \Rightarrow \cdots \Rightarrow A^k = A \quad \forall k \in \mathbb{Z}_+$

① $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e = \sum_{n=0}^{\infty} \frac{1}{n!}$

而 $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=1}^{\infty} \frac{A}{n!} + I = \left(\sum_{n=1}^{\infty} \frac{1}{n!}\right) A + I = (e-1)A + I$

② $\sin x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \Rightarrow \sin 1 = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$

而 $\sin A = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{A^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{A}{(2n+1)!} = \left(\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)!}\right) A = \sin 1 \cdot A$

③ $\cos A = \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n}}{(2n)!} \Rightarrow \cos 1 = \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{(2n)!}$

而 $\cos A = \sum_{n=0}^{\infty} (-1)^{n} \frac{A^{2n}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^{n} \frac{A}{(2n)!} + I = \left(\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{(2n)!}\right) A + I = (\cos 1 - 1) A + I$

3、(a) $A = \begin{pmatrix} -5 & 1 & 4 \\ -12 & 3 & 8 \\ -6 & 1 & 5 \end{pmatrix}$　$\det(\lambda I - A) = \det\begin{pmatrix} \lambda+5 & -1 & -4 \\ 12 & \lambda-3 & -8 \\ 6 & -1 & \lambda-5 \end{pmatrix} = \det\begin{pmatrix} \lambda-1 & 0 & -\lambda+1 \\ 0 & \lambda-1 & -2\lambda+2 \\ 6 & -1 & \lambda-5 \end{pmatrix} = \begin{matrix}(\lambda-1)^2(\lambda-7) \\ +6(\lambda-1)^2\end{matrix} = (\lambda-1)^3$

$\lambda = 1$　$\lambda I - A = \begin{pmatrix} 6 & -1 & -4 \\ 12 & -2 & -8 \\ 6 & -1 & -4 \end{pmatrix}$ 则 $\dim\mathrm{Ker}(\lambda I - A) = 2$，$(\lambda I - A)^2 = 0$　$\dim\mathrm{ker}(\lambda I - A)^2 = 3$

因此 $d_1 = 1$，$d_2 = 1$，　$J(A) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\mathrm{Ker}(A - \lambda I) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} \right\}$　设 $(A - \lambda I)\alpha_1 = \begin{pmatrix} t_1 \\ 6t_1+4t_2 \\ -t_2 \end{pmatrix} \Rightarrow \begin{pmatrix} -6 & +1 & +4 \\ -12 & +2 & +8 \\ -6 & +1 & +4 \end{pmatrix} \alpha_1 = \begin{pmatrix} t_1 \\ 6t_1+4t_2 \\ -t_2 \end{pmatrix}$

$\begin{pmatrix} -6 & +1 & +4 & t_1 \\ -12 & +2 & +8 & 6t_1+4t_2 \\ -6 & +1 & +4 & -t_2 \end{pmatrix} \sim \begin{pmatrix} -6 & +1 & +4 & t_1 \\ 0 & 0 & 0 & 4t_1+4t_2 \\ 0 & 0 & 0 & -t_1-t_2 \end{pmatrix}$ 则 $t_1 + t_2 = 0$ 令 $t_1 = 1, t_2 = -1$　$\alpha_1^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$　$\alpha_1^{(2)} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

因此 $A = P \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}$，其中 $P = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 6 \\ 1 & 0 & 0 \end{pmatrix}$　因此 $e^{At} = P e^{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} t} P^{-1}$

$e^{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} t} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} t & t & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} t^n & nt^n & 0 \\ 0 & t^n & 0 \\ 0 & 0 & t^n \end{pmatrix} = \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}$

则 $e^{At} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 6 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -6 & 1 & 4 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 6 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -6te^t & te^t & (4t+1)e^t \\ -6e^t & e^t & 4e^t \\ e^t & 0 & -e^t \end{pmatrix}$

$= \begin{pmatrix} (1-6t)e^t & te^t & 4te^t \\ -12te^t & (2t+1)e^t & 8te^t \\ -6te^t & te^t & (4t+1)e^t \end{pmatrix}$　　解为 $x = e^{At} x_0$，$x_0 \in \mathbb{C}^3$

(b) $A=\begin{pmatrix}1&-3&3\\-2&-6&13\\-1&-4&8\end{pmatrix}$ $\det(\lambda I-A)=\det\begin{pmatrix}\lambda-1&3&-3\\2&\lambda+6&-13\\1&4&\lambda-8\end{pmatrix}=\det\begin{pmatrix}\lambda-1&3&0\\2&\lambda+6&\lambda-7\\1&4&\lambda-4\end{pmatrix}=(\lambda-1)(\lambda^2-2\lambda+4)-3(\lambda-1)=(\lambda-1)^3$

$\lambda=1$ $A-\lambda I=\begin{pmatrix}0&-3&3\\-2&-7&13\\-1&-4&7\end{pmatrix}$ $\dim\mathrm{Ker}(A-\lambda I)=1$ $(A-\lambda I)^2=\begin{pmatrix}3&9&-18\\1&3&-6\\1&3&-6\end{pmatrix}$ $\dim\mathrm{Ker}(A-\lambda I)^2=2$

$(A-\lambda I)^3=0$ $\dim\mathrm{Ker}(A-\lambda I)^3=3$ 因此 $d_1=0$ $d_2=0$ $d_3=1$

$J(A)=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$ $\mathrm{Ker}(A-\lambda I)=\mathrm{span}\left\{\begin{pmatrix}3\\1\\1\end{pmatrix}\right\}$ 则 $\alpha^{(3)}=\begin{pmatrix}3\\1\\1\end{pmatrix}$

$(A-\lambda I)\alpha^{(2)}=\begin{pmatrix}3\\1\\1\end{pmatrix}\Rightarrow\begin{pmatrix}0&-3&3\\-2&-7&13\\-1&-4&7\end{pmatrix}\alpha^{(2)}=\begin{pmatrix}3\\1\\1\end{pmatrix}$

$$\left(\begin{array}{ccc|c}0&-3&3&3\\-2&-7&13&1\\-1&-4&17&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&4&-17&-1\\-2&-7&13&1\\0&-1&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&4&-17&-1\\0&1&-21&-1\\0&-1&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&4&-17&-1\\0&1&-21&-1\\0&0&-20&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&0&67&3\\0&1&-21&-1\\0&0&1&0\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&3\\0&1&0&-1\\0&0&1&0\end{array}\right)\Rightarrow\alpha^{(2)}=\begin{pmatrix}3\\-1\\0\end{pmatrix}$$

$(A-\lambda I)\alpha^{(3)}=\begin{pmatrix}3\\-1\\0\end{pmatrix}\Rightarrow\begin{pmatrix}0&-3&3\\-2&-7&13\\-1&-4&7\end{pmatrix}\alpha^{(1)}=\begin{pmatrix}3\\-1\\0\end{pmatrix}$

$$\left(\begin{array}{ccc|c}0&-3&3&3\\-2&-7&13&-1\\-1&-4&17&0\end{array}\right)\sim\left(\begin{array}{ccc|c}1&4&-17&0\\-2&-7&13&-1\\0&-1&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&4&-17&0\\0&1&-21&-1\\0&-1&1&1\end{array}\right)\sim\left(\begin{array}{ccc|c}1&4&-17&0\\0&1&-21&-1\\0&0&-20&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&0&67&4\\0&1&-21&-1\\0&0&-20&0\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&4\\0&1&0&-1\\0&0&1&0\end{array}\right)\Rightarrow\alpha^{(1)}=\begin{pmatrix}4\\-1\\0\end{pmatrix}$$

则 $P=\begin{pmatrix}3&3&4\\1&-1&-1\\1&0&0\end{pmatrix}$ $P^{-1}=\begin{pmatrix}0&0&1\\-1&-4&+7\\+1&3&-6\end{pmatrix}$

$A=P\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}P^{-1}\Rightarrow e^{At}=Pe^{\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}t}P^{-1}$

$$e^{\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}t}=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}t&t&0\\0&t&t\\0&0&t\end{pmatrix}^n=\sum\frac{1}{n!}\begin{pmatrix}t^n&nt^n&\frac{n(n-1)}{2}t^n\\0&t^n&nt^n\\0&0&t^n\end{pmatrix}=\begin{pmatrix}e^t&te^t&\frac12t^2e^t\\0&e^t&te^t\\0&0&e^t\end{pmatrix}$$

则 $$e^{At}=\begin{pmatrix}3&3&4\\1&-1&-1\\1&0&0\end{pmatrix}\begin{pmatrix}e^t&te^t&\frac12t^2e^t\\0&e^t&te^t\\0&0&e^t\end{pmatrix}\begin{pmatrix}0&0&1\\-1&-4&7\\1&3&-6\end{pmatrix}$$

$$=\begin{pmatrix}3&3&4\\1&-1&-1\\1&0&0\end{pmatrix}\begin{pmatrix}(\frac12t^2-t)e^t&(\frac32t^2-4t)e^t&(-3t^2+7t+1)e^t\\(t-1)e^t&(3t-4)e^t&(-6t+7)e^t\\e^t&3e^t&-6e^t\end{pmatrix}$$

$$=\begin{pmatrix}(\frac32t^2+1)e^t&(\frac92t^2-3t)e^t&(-9t^2+3t)e^t\\(\frac12t^2-2t)e^t&(\frac32t^2-7t+1)e^t&(-3t^2+13t)e^t\\(\frac12t^2-t)e^t&(\frac32t^2-4t)e^t&(-3t^2+7t+1)e^t\end{pmatrix}$$ 解为 $e^{At}x_0$，$x_0\in\mathbb{C}^3$

我们用Jordan链再解一遍此题

(a) $\lambda=1$ $\begin{pmatrix}0\\1\\0\end{pmatrix}\rightsquigarrow\begin{pmatrix}1\\2\\1\end{pmatrix}$

$\begin{pmatrix}1\\6\\0\end{pmatrix}$

$$\varphi(t)=\left(k_1\begin{pmatrix}1\\2\\1\end{pmatrix}+k_2\begin{pmatrix}1\\6\\0\end{pmatrix}+k_3\left(\begin{pmatrix}1\\2\\1\end{pmatrix}t+\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\right)e^t=\begin{pmatrix}k_1+k_2+tk_3\\2k_1+6k_2+(2t+1)k_3\\k_1+tk_3\end{pmatrix}e^t$$

(b) $\lambda=1$ $\begin{pmatrix}4\\-1\\0\end{pmatrix}\rightsquigarrow\begin{pmatrix}3\\-1\\0\end{pmatrix}\rightsquigarrow\begin{pmatrix}3\\1\\1\end{pmatrix}$

$$\varphi(t)=\left(k_1\begin{pmatrix}3\\1\\1\end{pmatrix}+k_2\left(\begin{pmatrix}3\\1\\1\end{pmatrix}t+\begin{pmatrix}3\\-1\\0\end{pmatrix}\right)+k_3\left(\frac12\begin{pmatrix}3\\1\\1\end{pmatrix}t^2+\begin{pmatrix}3\\-1\\0\end{pmatrix}t+\begin{pmatrix}4\\-1\\0\end{pmatrix}\right)\right)e^t$$

$$=\begin{pmatrix}3k_1+(3t+3)k_2+(\frac32t^2+3t+4)k_3\\k_1+(t-1)k_2+(\frac12t^2-t-1)k_3\\k_1+tk_3+\frac12t^2k_3\end{pmatrix}e^t$$

这两种方法答案是一致的

4. 设 $e_1 \dots e_n$ 是 V 的一组基

先证 $e^1 \dots e^n$ 张成 V^*　　　　($e^i \in V^*$ s.t. $e^i(e_j) = \delta_{ij}$ $\forall j$).

$\forall f \in V^*$ 设 $f(e_j) = c_j \in F$

则 $\forall v \in V$, 设 $v = \sum_{j=1}^{n} a_j e_j$

则 $f(v) = f(\sum_{j=1}^{n} a_j e_j) = \sum_{j=1}^{n} a_j f(e_j)$

$= \sum_{j=1}^{n} a_j c_j = \sum_{j=1}^{n} \left(c_j e^j (\sum_{i=1}^{n} a_i e_i) \right)$

$= \sum_{j=1}^{n} c_j e^j (v)$.

故 $\forall v \in V$, $f(v) = (\sum_{j=1}^{n} c_j e^j)(v)$

即 f 可表示为 $\{e^i\}$ 的线性组合, 即 $e^1 \dots e^n$ 张成 V^*

再证 $e^1 \dots e^n$ 线性无关

假设 $\sum_{i=1}^{n} c_i e^i = 0$

则 $\forall 1 \le j \le i$　$\sum_{i=1}^{n} c_i e^i (e_j) = c_j = 0(e_j) = 0$

则 $c_1 = \dots = c_n = 0$, 因此 $\{e^i\}$ 线性无关

综上: $e^1 \dots e^n$ 是 V^* 一组基　$\dim V^* = n$

5. $\ln(e^A e^B) = (e^A e^B - I) - \frac{1}{2}(e^A e^B - I)^2 + \frac{1}{3}(e^A e^B - I)^3 - \cdots$

$e^A e^B - I = (\sum_{n=0}^{\infty} \frac{1}{n!} A^n)(\sum_{n=0}^{\infty} \frac{1}{n!} B^n) - I$

$= (I + A + \frac{1}{2}A^2)(I + B + \frac{1}{2}B^2) - I + o$

$= B + \frac{1}{2}B^2 + A + AB + \frac{1}{2}A^2 + o$　(o 为高阶项)

则 $\ln(e^A e^B) = (B + \frac{1}{2}B^2 + A + AB + \frac{1}{2}A^2) - \frac{1}{2}(B + \frac{1}{2}B^2 + A + AB + \frac{1}{2}A^2)^2 + o$

$= B + \frac{1}{2}B^2 + A + AB + \frac{1}{2}A^2 - \frac{1}{2}(B^2 + A^2 + BA + AB) + o$

$= A + B + \frac{1}{2}AB - \frac{1}{2}BA + o$

因此 $\ln(e^A e^B)$ 前四项可写为 $A + B + \frac{1}{2}AB - \frac{1}{2}BA$.