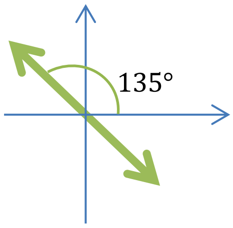
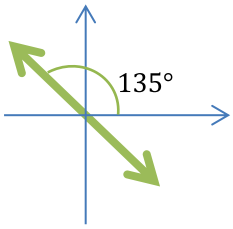
Homework for General Physics II, set6

1. (Hecht's 8.1)Describe completely the state of polarization of each of the following waves:
The trick (or key) is first express (rewrite) the wave in our standard form $\cos(kr - \omega t + \phi)$, and compare the phase difference between the two orthogonal components: (Note when expressed in this convention, $\phi > 0$ means lagging in phase)

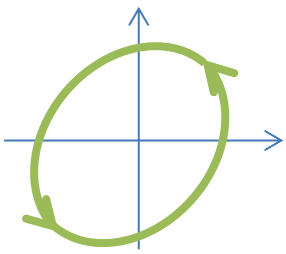
(1) $\vec{E} = \hat{\imath}E_0 cos(kz - \omega t) - \hat{\jmath}E_0 cos(kz - \omega t)$
$= \hat{\imath}E_0 cos(kz - \omega t) + \hat{\jmath}E_0 cos(kz - \omega t + \pi)$
Equal amplitudes, E_y lags E_x by π.
Therefore $\mathcal{P}$ − state at 135° or −45°.



(2) $\vec{E} = \hat{\imath}E_0 sin2\pi(z/\lambda - \nu t) - \hat{\jmath}E_0 sin2\pi(z/\lambda - \nu t)$
$= \hat{\imath}E_0 sin(kz - \omega t) - \hat{\jmath}E_0 sin(kz - \omega t)$
$= \hat{\imath}E_0 cos(kz - \omega t - \pi/2) + \hat{\jmath}E_0 cos(kz - \omega t + \pi/2)$
Equal amplitudes, E_y lags E_x by π.
Therefore $\mathfrak{p}$ − state at 135° or −45°.

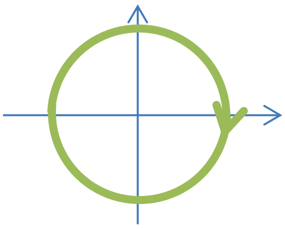


(3) $\vec{E} = \hat{\imath}E_0 sin(\omega t - kz) + \hat{\jmath}E_0 sin(\omega t - kz - \pi/4)$
$= \hat{\imath}E_0 cos(kz - \omega t + \pi/2) + \hat{\jmath}E_0 cos(kz - \omega t + 3\pi/4)$
Equal amplitudes, E_y lags E_x by $\pi/4$.
Therefore it is an ellipse tilted at +45° and right-handed .



(4) $\vec{E} = \hat{\imath}E_0 cos(\omega t - kz) + \hat{\jmath}E_0 cos(\omega t - kz + \pi/2)$
Equal amplitudes, E_y leads E_x by $\pi/2$.

Therefore it is an R – state.



2. **(Hecht's 8.6)Write an expression for an $\mathcal{R}$ – state light wave of frequency ω propagating in the positive x-direction such that at t=0 and x=0 the $\vec{E}$-field points in the negative z-direction.**

Answer:

$\vec{E} = \hat{j}E_0 sin(kx - \omega t) + \hat{k}E_0 cos(kx - \omega t)$ (Zhao Wen's)

"I would write $\vec{E} = \hat{j}E_0 \exp\left[i\left(kx - \omega t + \frac{3}{2}\pi\right)\right] + \hat{k}E_0 \exp\left[i(kx - \omega t + \pi)\right]$. Of course it is fine to write it in triangular form too." (Shuo)

3. **(Hecht's 8.8)Given that 300W/m² of light from an ordinary tungsten bulb arrives at an ideal linear polarizer. What is its radiant flux density on emerging?**

Answer:

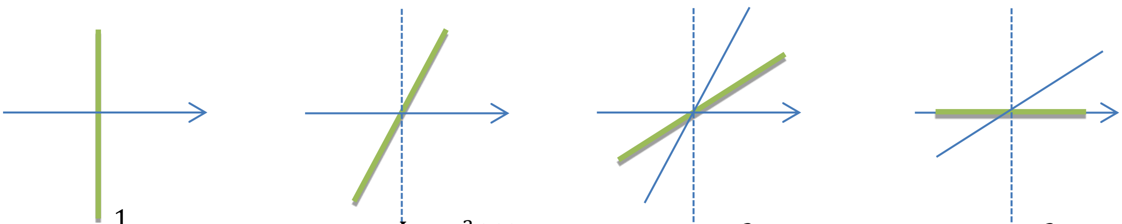
Radiant flux density: 150W/m²

The light from tungsten bulb which consists of a very large number of randomly oriented atomic emitters is natural light, which is unpolarized.

$$I = \frac{1}{2}I_0$$

4. **(Hecht's 8.19)Given that 200W/m² of randomly polarized light is incident normally on a stack of ideal linear polarizers that are positioned one behind the other with the transmission axis of the 1st vertical, the 2nd at 30°, the 3rd at 60°, and the 4th at 90°. How much light emerges?**

Answer:



$I_1 = \frac{1}{2}I_0$ $\quad I_2 = I_1 cos^2 30°$ $\quad I_3 = I_2 cos^2 30°$ $\quad I_4 = I_3 cos^2 30°$

Radiant flux density:

$$\frac{1}{2} \times 200 \times cos^6 30° = 42.1875 W/m^2$$

5. **Derive the (4.11.5) in the experimental manual for the polarization experiment. i.e. For**

the natural light input, after passing through a pair of linear polarizers whose main transmission axis (T1 along this direction) has an angle of θ. Prove:

$$\mathbf{T}_{\boldsymbol{\theta}} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{T}_1^2 + \mathbf{T}_1^2)\boldsymbol{cos^2\theta} + \mathbf{T}_1\mathbf{T}_2\boldsymbol{sin^2\theta},$$

where T is the transmission ratio of energy (I_{output}/I_{input}). T1, T2 as defined in the lab manual.

Answer:

Decompose the incident electromagnetic wave into two components, one parallel to the axis of the polarizer and the other one perpendicular.

We can obtain:

$$\mathrm{T}_\theta = \mathrm{T}_1 cos^2\theta + \mathrm{T}_2 sin^2\theta = (\mathrm{T}_1 - \mathrm{T}_2)cos^2\theta + \mathrm{T}_2$$

In the experiment, after the natural light passes the first polarizer, there are $\frac{1}{2}\mathrm{T}_1 I_0$ of the light along the axis and $\frac{1}{2}\mathrm{T}_2 I_0$ perpendicular.

According to the equation above, the total transmission intensity with respect to the second polarizer is:

$$\mathrm{I}_T = \frac{1}{2}\mathrm{T}_1 I_0(\mathrm{T}_1 cos^2\theta + \mathrm{T}_2 sin^2\theta) + \frac{1}{2}\mathrm{T}_2 I_0(\mathrm{T}_1 sin^2\theta + \mathrm{T}_2 cos^2\theta)$$

$$\Rightarrow \mathrm{T}_\theta = \frac{\mathrm{I}_T}{I_0} = \frac{1}{2}\mathrm{T}_1(\mathrm{T}_1 cos^2\theta + \mathrm{T}_2 sin^2\theta) + \frac{1}{2}\mathrm{T}_2(\mathrm{T}_1 sin^2\theta + \mathrm{T}_2 cos^2\theta)$$

$$= \frac{1}{2}(\mathrm{T}_1^2 + \mathrm{T}_1^2)cos^2\theta + \mathrm{T}_1\mathrm{T}_2 sin^2\theta$$

6. The Rs and Rp need to be calculated, where

$$R_s = r_s^{\ 2}, R_p = r_p^{\ 2}$$

Where r_s, r_p are the Fresnel coefficients which can be calculated from given condition,

$$n_i = 1, n_t = 1.5, \theta_i = 40^\circ$$

The reflected light would be partial polarized, with P, S component unequal.

$I_p^r = R_p I_p^i = \frac{1}{2} R_p I_0$ where I_p^r is the intensity of P component in reflected light; I_p^i is the P component in the incident light which is half of the total incident intensity I_0.

$$I_s^r = R_s I_s^i = \frac{1}{2} R_s I_0$$

The reflected light would be partial polarized similar in intensity distribution as an elliptical, Here I took a short cut by reading the Rs, Rp directly from Hecht's book, figure 8.35.

At 40 degree incident:

Rs=0.08, Rp=0.02 (of course you can calculate it from Fresnel equation)

The Imax= I_s^r would happen when linear polarizer along the s direction for reflected light, and Imin= I_p^r would be polarizer along p direction.

Then from definition of degree of polarization (8-30) of Hecht's:

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{R_s - R_p}{R_s + R_p} = 0.6$$

(this is just an estimation, may be off a little if you carry out strict computation on the R)

7. The answer is provided in the book

$n_o = 1.6584, \quad n_e = 1.4864$. Snell's Law:

$$\sin\theta_i = n_o \sin\theta_{to} = 0.766$$
$$\sin\theta_i = n_e \sin\theta_{te} = 0.766$$
$$\sin\theta_{to} \approx 0.463, \qquad \theta_{to} \approx 27°35'$$
$$\sin\theta_{te} \approx 0.516, \qquad \theta_{te} \approx 31°4'$$
$$\Delta\theta \approx 3°29'$$

8. **For a polarized light whose Jones vector is $\begin{bmatrix} 2 \\ -i \end{bmatrix}$, what is polarization state (including the rotation)?**

Answer:

$$\begin{bmatrix} 2 \\ -i \end{bmatrix} = \begin{bmatrix} 2 \\ e^{-i\frac{\pi}{2}} \end{bmatrix}$$

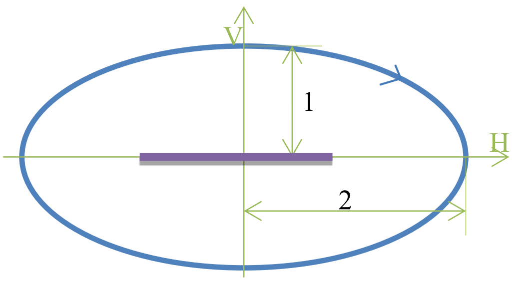
Unequal amplitudes, E_y leads E_x by $\frac{\pi}{2}$.

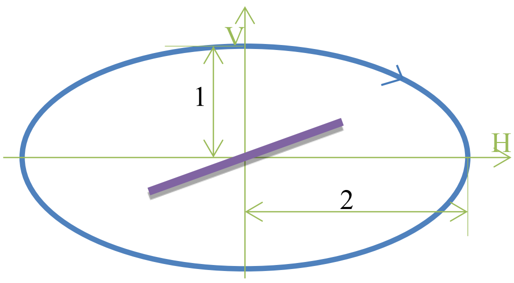
Right-elliptical polarization.

If an ideal linear polarizer is placed horizontally, the polarized light given above illuminate the linear polarizer at normal angle, what is the output energy percentage?

Answer:

$$\left(\frac{2}{\sqrt{5}}\right)^2 = \frac{4}{5}$$





If the linear polarizer is placed with +30° with respect to the horizontal direction, what is the output energy percentage?

Answer:

$$\begin{bmatrix} cos30°cos30° & sin30°cos30° \\ cos30°sin30° & sin30°sin30° \end{bmatrix}\begin{bmatrix} 2 \\ -i \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 2 \\ -i \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - i\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} - i\frac{1}{4} \end{bmatrix}$$

$$\frac{\begin{bmatrix} \frac{3}{2} - i\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} - i\frac{1}{4} \end{bmatrix}\begin{bmatrix} \frac{3}{2} + i\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2} + i\frac{1}{4} \end{bmatrix}}{(\sqrt{5})^2} = \frac{\frac{13}{4}}{5} = \frac{13}{20}$$

Or you can work this out without using matrix:

The projection along the linear polarizer direction for Horizontal component is:

$$E_1 = 2\cos 30 = \sqrt{3}$$

The projection of the Vertical component is:

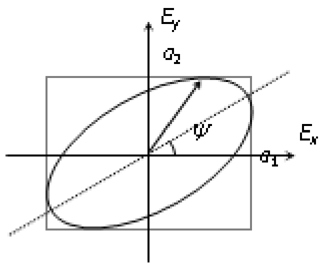
$$E_2 = e^{-i\frac{\pi}{2}} \sin 30 = -\frac{i}{2}$$

The total field along polarizer:

$$E = E_1 + E_2 = \sqrt{3} - \frac{i}{2}$$

$\frac{I}{I_0} = \frac{|E|^2}{5} = \frac{3 + 1/4}{5} = \frac{13}{20}$ same answer. Pick the method that you feel comfortable.

9. A linear polarizer whose transmission axis is along direction ψ with respect to the horizontal direction:



The elliptical polarized light with $a_1 = A, a_2 = B$ and phase $\Delta\phi = \varepsilon$, i.e. in Jones vector: $\begin{bmatrix} A \\ Be^{i\varepsilon} \end{bmatrix}$, when this light is passing the linear polarizer (dotted line in the figure), 1) what is the formula for the output intensity(I did this part in lecture)? 2) Then show that when the transmission axis is along the long principal axis of the ellipse, the outputs intensity is extreme. (the relation between principal axis direction and A, B, phase difference is given in Hecht's: pg. 328, formula 8-15: $\tan 2\alpha = \frac{2AB\cos\varepsilon}{A^2 - B^2}$)

Answer:

1) I worked it out in the lecture, basically:

$$E_{\square} = A\cos\psi + Be^{i\varepsilon}\sin\psi$$

$$I_{out} = |E_{\square}|^2 = A^2\cos^2\psi + B^2\sin^2\psi + 2AB\cos\psi\sin\psi\cos\varepsilon$$

2) Take derivative of I over ψ to find its extreme:

$$\frac{dI}{d\psi} = -2A^2\cos\psi\sin\psi + 2B^2\sin\psi\cos\psi + 2AB\cos\varepsilon\cos 2\psi = 0$$

This gives:

$$2AB\cos\varepsilon\cos 2\psi = (A^2 - B^2)\sin 2\psi$$

$$\tan 2\psi = \frac{2AB\cos\varepsilon}{A^2 - B^2}$$

This direction is exactly the direction of principal axis of the ellipse.

10. A polarization state in the H-V base is: $\begin{bmatrix} 3 \\ 2i \end{bmatrix}$, 1) find out its expression in the $|R>, |L>$ base (right and left circular polarization), i.e. finding the expansion coefficients(projection) of the polarization state in terms of circular polarization. 2) what is the expression in another $|e_1>, |e_2>$ base, where $|e_1> = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -i \end{bmatrix}, |e_2> = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ are two orthonormal elliptical base vectors, whose expression in H-V base is provided above.

Answer:

1) Let me write the input in Dirac notation as:

$|E>$ and its expression in H-V is given, to find its expression in other orthonormal base, such as |R>,|L>, i.e. |E>=a|R>+b|L>, we use dot products:

$$a = <R|E>, b = <L|E>$$

|R>,|L>'s expression in H-V are known:

$$|R> = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}, |L> = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Then:

$$a = <R|E> = \frac{1}{\sqrt{2}}[1 \ \ i]\begin{bmatrix} 3 \\ 2i \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$b = \langle L|E\rangle = \frac{1}{\sqrt{2}}[1 \quad -i]\begin{bmatrix}3\\2i\end{bmatrix} = \frac{5}{\sqrt{2}}$$

So the |E> could also be expressed as:

$$|E\rangle = \frac{1}{\sqrt{2}}|R\rangle + \frac{5}{\sqrt{2}}|L\rangle$$

2) Similarly:

$$\langle e_1|E\rangle = \frac{1}{\sqrt{5}}[2 \quad i]\begin{bmatrix}3\\2i\end{bmatrix} = \frac{4}{\sqrt{5}}$$

$$\langle e_2|E\rangle = \frac{1}{\sqrt{5}}[1 \quad \text{-}2i]\begin{bmatrix}3\\2i\end{bmatrix} = \frac{7}{\sqrt{5}}$$

You may check that the length of the vector $\langle E|E\rangle = 13$ in all expressions.

11. Using the Jones vector for a polarized light and Jones matrix for the reflection (in my note), show that if the incident light is a right-circular polarized light, at normal incidence angle (i=0) at an interface between (n_i, n_t), n_i is the refractive index of the incident media; n_t is the refractive index of the transmitted media. the polarization state of the reflected light will be left-circular polarized. Does it matter whether it is internal or external reflection? You may need the Fresnel Equations for reflection coefficients (pg. 80 of my note or pg. 114 in Hecht's and pg. 247 in Zhao's): at normal incident angle: $r_s = \frac{n_t - n_i}{n_t + n_i}, r_p = \frac{n_i - n_t}{n_t + n_i}$ s,p are two perpendicular directions defined in the notes and books)

Answer:

A right-circular polarized light:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

The reflection matrix at normal incidence:

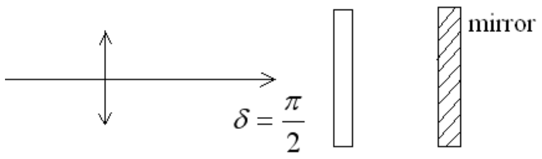
$$\begin{bmatrix}\mathrm{r} & \\ & -\mathrm{r}\end{bmatrix}$$

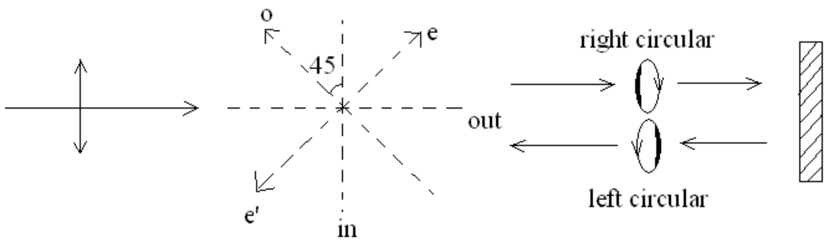
The reflected light:

$$\begin{bmatrix}\mathrm{r} & \\ & -\mathrm{r}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix} = \frac{r}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$$

Left-circular polarization no matter whether it is internal or external reflection.

12. **Use the Jones vector and matrix, work the example given in the lecture. The input is a linear polarized light, after a quarter wave plate whose fast axis (o axis) is shown in the figure and a mirror, the output is a linear light but orthogonal to the input. (verify the polarization state after each stage, for the mirror use the results of above question)**





Answer:

The linear polarized incident light: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (vertical)

The transformation matrices for wave plate, mirror, and wave plate respectively:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}, \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

The final state:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(the common phase factor does not matter, the output is Horizontal)

The output is orthogonal to the input.

Or you may chose coordinate that e(e' for reflected light case) as horizontal and o is vertical, the input would be +45 in e-o coordinates, and you can find that the output would be +45 in e'-o coordinates, which is perpendicular to the input.

13. (Hecht's 8.41)

8.41* Take two ideal Polaroids (the first with its axis vertical and the second, horizontal) and insert between them a stack of 10 half-wave plates, the first with its fast axis rotated $\pi/40$ rad from the vertical, and each subsequent one rotated $\pi/40$ rad from the previous one. Determine the ratio of the emerging to incident irradiance, showing your logic clearly.

(Understand how the half-plate 'flipped' polarization is the simplest way to solve the problem, while the matrix form here are too complicated, assume incoming lingth is natural light)

Answer:

Assuming the incoming light is a natural light with an intensity of I_0.

After the first polaroid, the intensity is halved.

Consider a polarized light passes through a half wave plate; its polarization direction is mirrored with respect to the fast axis of the plate. Keeping this in mind, one can easily acquire that the polarization is rotated by $\frac{\pi}{20}$ (vs. vertical) after the first half plate, and this is exactly the direction of second half-plate and so the output from the second half-plate is same (still $\frac{\pi}{20}$ respect to vertical). The whole process will repeat for the 3rd and 4th half-plate , the output after 4th half-plate would be $\frac{\pi}{10}$ with respect to vertical. So after 10 such plates, the output from the half-plates would be $\frac{\pi}{4}$ from the vertical.

Consequently, after the second polaroid, we have:

$$\frac{I_{out}}{I_{in}} = \frac{1}{4}$$

14. **（《光学》赵凯华 下册 pg.199 思考题 3）画出以下各种情形出射光的偏振态。（已知入射光的偏振态。）(注:按赵凯华书中定义：本题中 $\delta' = \dfrac{\pi}{2}$ 代表 o 光*落后* e 光 $\dfrac{\pi}{2}$)**

解答：

已知入射光的偏振态，求通过波晶片后出射光的偏振态，可归结为如下步骤：

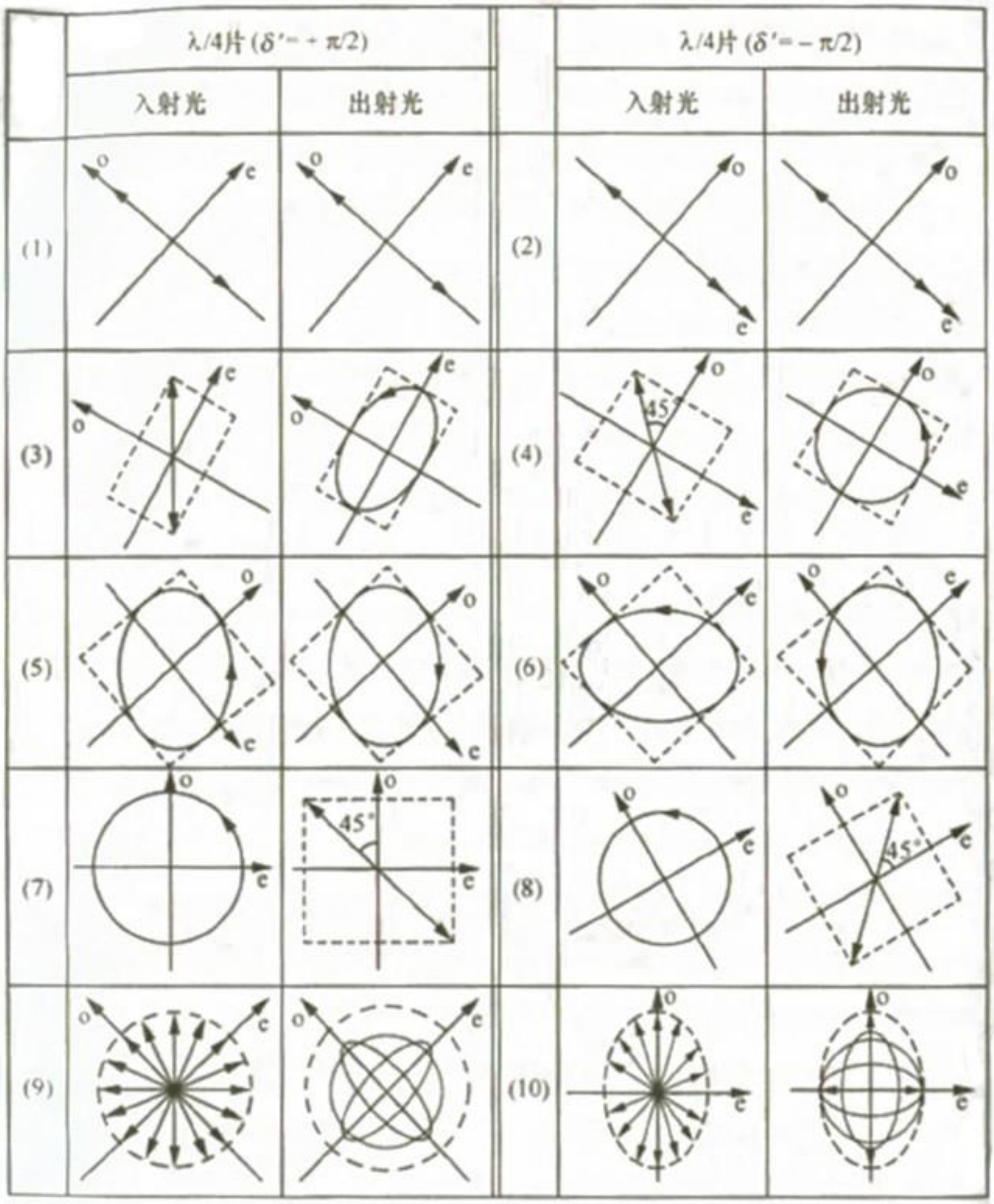
(1) 将入射光的电矢量沿波晶片的 e 轴和 o 轴分解，并分别求出其振幅 A_e 和 A_o 及入射点 o 振动相对于 e 振动的相位差 δ_{in}。

(2) 求出出射光两分量的振幅及相位差 δ_{out}。在忽略反射、吸收等损耗的情况下，出射光的振幅仍为 A_e 和 A_o，从而电矢量端点的轨迹与长、宽为 $2A_e$、$2A_o$ 的矩形内切，矩形的各边分别与 e、o 轴平行。出射光 o 分量相对于 e 分量的相位差：

$\delta_{out}=\delta_{in}+\delta'$

其中 $\delta' = \frac{2\pi}{\lambda}(n_e - n_o)d$，是光在波晶片体内传播引起的 e,o 分量间的相位差。

(3) 根据出射光 o 分量与 e 分量的振幅关系和相位关系，合成出射光的偏振态。



(1) 入射光只有 o 分量，故出射光仍为只有 o 分量的线偏振光，且振幅大小不变。

(2) 入射光只有 e 分量，故出射光仍为只有 e 分量的线偏振光，且振幅大小不变。

(3) 入射光为第一、三象限内的线偏振光，$\delta_{in}=0$，$\delta'=-\pi/2$。故$\delta_{out}=-\pi/2$；出射光为左旋正椭圆偏振光。

(4) 入射光为第二、四象限内的线偏振光，且偏振方向与 o 轴成 45º 角，$\delta_{in}=\pi$，$\delta'=\pi/2$。故$\delta_{out}=3\pi/2$；出射光为左旋圆偏振光。

(5) 入射光为长轴在第二、四象限内的左旋斜椭圆偏振光，δ_{in}为第三象限内的某个角度，$\delta'=\pi/2$，故 δ_{out}为第二象限内的某个角度；出射光为右旋斜椭圆偏振光。

(6) 入射光为第二、四象限内的左旋椭圆偏振光，δ_{in} 为第三象限内的某个角度，$\delta'=\pi/2$，故 δ_{out}为第四象限内的某个角度；出射光为第一、三象限内的左旋椭圆偏振光。

(7) 入射光为左旋圆偏振光，$\delta_{in}=-\pi/2$，故$\delta_{out}=-\pi$，出射光为第二、四象限内的线偏振

光，且振动方向与 o 轴成 45º 角。

(8) 入射光为左旋圆偏振光，$\delta_{\rm in} = -\pi/2$，故$\delta_{\rm out} = 0$，出射光为第一、三象限内的线偏振光，且振动方向与 e 轴成 45º 角。

(9) 入射光为自然光，可看成是大量取向轴对称分布、振幅大小相等、相位无关联的线偏振光的集合。经波晶片后，各线偏振光分别转化为沿 o、e 轴的线偏振光，左、右旋圆偏振光以及长短轴比例不同的左、右旋正椭圆偏振光。简言之，出射光是各种长短轴比例的且大量相位无关联的左、右旋正椭圆偏振光（圆偏振光、线偏振光是椭圆偏振光的特例）的集合。从宏观上看，出射光的偏振态仍具有轴对称性。尽管细微偏振结构与入射光不同，但宏观结果是相同的，因此出射光仍可认为是自然光。

(10) 入射光为部分偏振光，它是大量相位无关联的线偏振光的集合；它们的振幅大小不同，不具有轴对称性。出射光为线偏振光、圆偏振光和长短轴比例各不相同的椭圆偏振光的集合；他们在相位上也是无关联的。与(9)不同的是，这时出射光在宏观上仍无轴对称性，仍是部分偏振光。

15． In class, I had outlined two methods to find out a right circular polarizer's matrix form in H-V basis (From its effect on the H,V basis vector or from point of view of a projector). Here is another way to do the same problem from point of view of basis transform: For a circular polarizer in the R, L basis, i.e. the unit vectors form the orthonormal basis are $|e1\rangle = |R\rangle, |e2\rangle = |L\rangle$, R, L stands for right and left circular polarized light, its matrix form is simple. 1) Write out right circular polarizer under this basis. (this is easy) 2) Starting from above, find out the matrix form of right circular polarizer under H-V basis, where H, V stands for horizontal and vertical linear polarized light. (You need to find out the transform matrix between the R-L basis and H-V basis，the class lecture using the physical rotation as an example, this transform of basis can be treated in the similar fashion). To test your result (may not be same with Hecht's book), work it on an arbitrary polarized light, to see whether the output is indeed right circular polarized.

Answer：(A detailed answer with different methods can also be found in the supplementary material under name of "Matrix representation and Dirac Notation")

(1) In the |R>, |L> basis, where |R> is expressed as $\begin{bmatrix}1\\0\end{bmatrix}$, |L> is $\begin{bmatrix}0\\1\end{bmatrix}$, the right circular polarizer R is simple. (The matrix form is exactly same as a H linear polarizer in the H-V basis). In R-L basis, the R's matrix form is:

$$R_{R-L} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

(2) In the H-V basis, R's form will be different. The determination of R in H-V basis can be treated as a Unitary Transformation. i.e. Change basis from R-L to that of H-V, while the form of R is known under R-L basis, its form under H-V basis can be determined if we know the relations between the basis. The most important relations among the different basis are simple here, since we already know the |R>,|L> expression in |H>,|V>:

$$|R> = \frac{\sqrt{2}}{2}(|H> - i|V>)$$

$$|L> = \frac{\sqrt{2}}{2}(|H> + i|V>)$$

Then we can claculate terms like <H|R>, <H|L>....

these terms will form what is called Transform Matrix T:

$$T_{from\ basis1-2..\rightarrow basis1'-2'..} = \begin{bmatrix} <1'|1> & <1'|2> & ... \\ <2'|1> & <2'|2> & ... \\ . & & \\ . & & \end{bmatrix}$$

In this 2-D case, $T_{R-L\rightarrow H-V} = \begin{bmatrix} <H|R> & <H|L> \\ <V|R> & <V|L> \end{bmatrix}$

Knowing the transform matrix T, the matrix form of the operation in new basis cab be determined by:

$R_{H-V} = TR_{R-L}T^{\dagger}$ where $T^{\dagger}$ (pronounced as T dagger) is the transposed and complex-conjugate of T

$$<H|R> = \frac{\sqrt{2}}{2}; <H|L> = \frac{\sqrt{2}}{2}; <V|R> = -\frac{\sqrt{2}}{2}i; <V|L> = \frac{\sqrt{2}}{2}i$$

$T = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ and $T^{\dagger} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$

(In physical transformation which do not change the magnitude of the vector, such transfromation is called Unitary transformation, which has the property $T^{\dagger} = T^{-1}$)

$$R_{H-V} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

Of course the above expression can be more readily proved use the projector, since R=|R><R|, inserting the H_V expression for |R> will give the same result.

For arbitrary polarization $|P> = \begin{bmatrix} A \\ B \end{bmatrix}$, after passing through right-circular polarizer:

$$R|P> = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A+iB \\ -iA+B \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A+iB \\ -i(A+iB) \end{bmatrix} = \frac{1}{2}(A+iB)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The output is exactly a right circular polarized light expressed in H-V basis. (its amplitude is going to be changed depending on the exact form of A,B)

16. For an optical element whose matrix form in H-V base is:

$\hat{O} = \frac{1}{5}\begin{bmatrix} 4 & -2i \\ 2i & 1 \end{bmatrix}$, find out its eigenvalues and associated eigenvectors (normalized); can you tell the action of this O on any polarization input?

Answer:

Standard method for this 2X2 matrix eigenvalue problem, the secular (characteristic) equation is:

$\begin{vmatrix} 4-\lambda & -2i \\ 2i & 1-\lambda \end{vmatrix} = 0$ (Note: Here I neglect 1/5 the common factor, I will find out the eigenvalue of the above equation, then times 1/5 later)

You may directly expand the determinant and work out the quadratic equation for λ, but here I used another property for eigenvalues (especially useful in 2 by 2 matrix), i.e their sum equals the trace and their product equals the determinant of the original matrix:

$$\lambda_1\lambda_2 = \det(\hat{O}) = 0$$

$$\lambda_1 + \lambda_2 = trace(\hat{O}) = 5$$

So $\lambda_1 = 5, \lambda_2 = 0$

Now times the common factor 1/5, and the eigenvalues for the $\hat{O}$ is 1 and 0.

For the eigenvector, I just use $\begin{bmatrix} 4 & -2i \\ 2i & 1 \end{bmatrix}$ (since will normalize later so this matrix eigenvector is same as $\hat{O}$)

For $\lambda_1 = 5$, its associated eigenvector satisfies:

$\begin{bmatrix} 4-5 & -2i \\ 2i & 1-5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$ of course this is:

$$\begin{bmatrix} -1 & -2i \\ 2i & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

Gauss elimination (of course this simple one you can directly find relation between a and b)

$$\begin{bmatrix} 1 & -2i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

There is a free variable (expected from definition of eigenvector, any vector along this direction is an eigenvector), and can be fixed by normalization. I set the free variable as b=1 (if b=0, a would be 0 too, it is a trivial one) for non-trivial vector, then from:

$-a - 2ib = 0,\ a = -2i$

So $|\lambda_1'>=\begin{bmatrix} -2i \\ 1 \end{bmatrix}$, normalized one will be: $|\lambda_1>=\frac{1}{\sqrt{5}}\begin{bmatrix} -2i \\ 1 \end{bmatrix}=\frac{-i}{\sqrt{5}}\begin{bmatrix} 2 \\ i \end{bmatrix}$, this is the eigenvector with eigenvalue 1.

Similarly, we can find eigenvector with eigenvalue 0:

$$|\lambda_2>=\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

We see that the eigenvectors are two elliptical polarized light with eigenvalues 1and 0, so the O is an elliptical polarizer.

Recommended problems in Zhao's book: Questions 2,6, problems 4,8,9 on pg 180-182; Question 2, problems 1,5 on pg 186-187;