# 《高等微积分 I》第四次作业

1、(1) $-1 \leq \sin(\pi\sqrt{n^2+1}) \leq 1$

$\Rightarrow 0 \leq \sin^2(\pi\sqrt{n^2+1}) \leq |\sin(\pi\sqrt{n^2+1})| = |\sin(\pi\sqrt{n^2+1}) - \sin(\pi n)|$

$= \left|2\cos\frac{\pi\sqrt{n^2+1}+\pi n}{2}\sin\frac{\pi\sqrt{n^2+1}-\pi n}{2}\right|$

$\leq \left|2\sin\frac{\pi\sqrt{n^2+1}-\pi n}{2}\right|$

$\leq \left|2\,\frac{\pi\sqrt{n^2+1}-\pi n}{2}\right| = \left|\frac{\pi}{\sqrt{n^2+1}+n}\right|$

而 $\lim\limits_{n\to\infty}\frac{\pi}{\sqrt{n^2+1}+n} = 0$

则根据夹逼定理 $\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+1}) = 0$ .

(2) $\forall \varepsilon > 0 \quad \exists N = [\frac{1}{8\varepsilon}]+1 \quad \forall n \geq N$

$|\sin^2(\pi\sqrt{n^2+n}) - 1| = |\sin^2(\pi\sqrt{n^2+n}) - \sin^2(\pi(n+\frac{1}{2}))|$

$= \left|\frac{1-\cos(2\pi\sqrt{n^2+n})}{2} - \frac{1-\cos(2\pi(n+\frac{1}{2}))}{2}\right|$

$= \left|\frac{\cos(2\pi(n+\frac{1}{2}))-\cos(2\pi\sqrt{n^2+n})}{2}\right|$

$= \left|\sin\frac{2\pi(n+\frac{1}{2})+2\pi\sqrt{n^2+n}}{2}\sin\frac{2\pi(n+\frac{1}{2})-2\pi\sqrt{n^2+n}}{2}\right|$

$\leq |\sin[(n+\frac{1}{2}-\sqrt{n^2+n})\pi]|$

$\leq |(n+\frac{1}{2}-\sqrt{n^2+n})\pi| = \frac{\frac{1}{4}}{n+\frac{1}{2}+\sqrt{n^2+n}} \leq \frac{\frac{1}{4}}{2n} = \frac{1}{8n} < \frac{1}{8\frac{1}{8\varepsilon}} = \varepsilon$

即 $\lim\limits_{n\to\infty}\sin^2(\pi\sqrt{n^2+n}) = 1$

2、(1) $\frac{x^2+1}{x+1} - ax - b = \frac{(1-a)x^2-(a+b)x+1-b}{x+1} = (1-a)x - (b+1) + \frac{2}{x+1}$

① $a \neq 1$ 则 $\lim\limits_{x\to+\infty}(\frac{x^2+1}{x+1}-ax-b) = \lim\limits_{x\to\infty}[(1-a)x-(b+1)+\frac{2}{x+1}]$ 极限不存在

② $a = 1$ 则 $\lim\limits_{x\to+\infty}(\frac{x^2+1}{x+1}-ax-b) = \lim\limits_{x\to\infty}[-(b+1)+\frac{2}{x+1}] = \lim\limits_{x\to\infty}\frac{2}{x+1} - b - 1 = -b-1$

而 $\lim\limits_{x\to+\infty}(\frac{x^2+1}{x+1}-ax-b) = 0$ 则 $b = -1$

综上 $a = 1,\ b = -1$ #

(2) $\sqrt{x^2-x+1} - px - q = \frac{(1-p^2)x^2-x+1}{\sqrt{x^2-x+1}+px} - q = \frac{(1-p^2)x-1+\frac{1}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+p} - q$

① $p = -1$ $\lim\limits_{x\to\infty}(\sqrt{x^2-x+1}-px-q) = \lim\limits_{x\to\infty}\frac{-1+\frac{1}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+p} = \frac{\lim(-1+\frac{1}{x})}{\lim\limits_{x\to\infty}\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+p} - q$ 而 $\lim\limits_{x\to\infty}\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+p = 1-1 = 0$

则该极限不存在

② $p = +1$ $\lim\limits_{x\to\infty}(\sqrt{x^2-x+1}-px-q) = \lim\limits_{x\to\infty}\frac{-1+\frac{1}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+p} = \frac{\lim\limits_{x\to\infty}(-1+\frac{1}{x})}{\lim\limits_{x\to\infty}\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+p} - q = \frac{-1}{1+p} - q = -\frac{1}{2} - q$

则 $-\frac{1}{2} - q = 0 \Rightarrow q = -\frac{1}{2}$

③ $p \neq \pm 1$ $\lim\limits_{x\to\infty}[(1-p^2)x-1+\frac{1}{x}]$ 不存在，则 $\lim\limits_{x\to\infty}\frac{(1-p^2)x-1+\frac{1}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+p}$ 不存在

综上： $p = 1,\ q = -\frac{1}{2}$

3、(1) 解：

$$\lim_{x\to0}\frac{\sqrt{1+f(x)}-1}{x^n}=\lim_{x\to0}\frac{\sqrt{1+f(x)}-1}{f(x)}\lim_{x\to0}\frac{f(x)}{x^n}=A\lim_{x\to0}\frac{f(x)}{f(x)(\sqrt{1+f(x)}+1)}=A\lim_{x\to0}\frac{1}{\sqrt{1+f(x)}+1}$$

而 $\lim_{x\to0}\frac{f(x)}{x^n}=A \Rightarrow \lim_{x\to0}f(x)=A\lim_{x\to0}x^n=0$ .

则 $\lim_{x\to0}\frac{\sqrt{1+f(x)}-1}{x^n}=A\frac{1}{\lim_{x\to0}\sqrt{1+f(x)}+1}=A\frac{1}{1+1}=\frac{1}{2}A$ .

(2)

$$\lim_{x\to0}\frac{\sqrt{\cos x}-\sqrt{1+\sin^2x}}{x^2}=\lim_{x\to0}\frac{\cos x-1-\sin^2x}{x^2(\sqrt{\cos x}+\sqrt{1+\sin^2x})}=\lim_{x\to0}\frac{\cos^2x+\cos x-2}{x^2(\sqrt{\cos x}+\sqrt{1+\sin^2x})}$$

$$=\lim_{x\to0}\frac{(\cos x-1)(\cos x+2)}{x^2(\sqrt{\cos x}+\sqrt{1+\sin^2x})}=\lim_{x\to0}\frac{\cos x-1}{x^2}\frac{\lim_{x\to0}(\cos x+2)}{\lim_{x\to0}(\sqrt{\cos x}+\sqrt{1+\sin^2x})}$$

$$=\lim_{x\to0}\frac{-2\sin^2\frac{x}{2}}{x^2}\frac{3}{2}=-3\lim_{t\to0}\frac{\sin^2t}{4t^2}=-\frac{3}{4}$$

4、(1) $\lim_{x\to0}\frac{1-f(x)}{x^2}=A \Rightarrow \lim_{x\to0}(1-f(x))=A\lim_{x\to0}x^2=0 \Rightarrow \lim_{x\to0}f(x)=1$

则

$$\lim_{x\to0}\frac{1-f(x)g(x)}{x^2}=\lim_{x\to0}\frac{1-f(x)g(x)}{1-f(x)}\lim_{x\to0}\frac{1-f(x)}{x^2}$$

$$=\lim_{x\to0}\left(1+f(x)\frac{1-g(x)}{1-f(x)}\right)A$$

$$=\left(1+\lim_{x\to0}f(x)\lim\frac{[1-g(x)]/x^2}{[1-f(x)]/x^2}\right)A$$

$$=\left(1+1\times\frac{B}{A}\right)A=A+B$$

(2) $\lim_{x\to0}\frac{1-f_1(x)}{x^2}=A_1 \Rightarrow \lim_{x\to0}(1-f(x))=A\lim_{x\to0}x^2=0 \Rightarrow \lim_{x\to0}f(x)=1$ .

数学归纳法，设已知 $\lim_{x\to0}\frac{1-f_1(x)\cdots f_k(n)}{x^2}=A_1+A_2+\cdots+A_k$ .

$$\lim_{x\to0}\frac{1-f_1(x)\cdots f_{k+1}(n)}{x^2}=\lim_{x\to0}\frac{1-f_1(x)\cdots f_{k+1}(x)}{1-f_{k+1}(x)}\cdot\lim_{x\to0}\frac{1-f_{k+1}(x)}{x^2}$$

$$=\lim_{x\to0}\left(1+f_{k+1}(x)\frac{1-f_1(x)f_2(x)\cdots f_k(x)}{1-f_{k+1}(x)}\right)A_{k+1}$$

$$=\left(1+\lim_{x\to0}f_{k+1}(x)\frac{\lim_{x\to0}[1-f_1(x)f_2(x)\cdots f_k(x)]/x^2}{\lim_{x\to0}[1-f(x)]/x^2}\right)A_{k+1}$$

$$=\left(1+\frac{A_1+A_2+\cdots+A_k}{A_{k+1}}\right)A_{k+1}=A_1+A_2+\cdots+A_{k+1}$$

即 $\lim_{x\to0}\frac{1-f_1(x)\cdots f_{k+1}(x)}{x^2}=A_1+\cdots+A_{k+1}$

综上：$\lim_{x\to0}\frac{1-f_1(x)\cdots f_{k+1}(x)}{x^2}=\sum_{i=1}^{n}A_i$

5、解：设 $g(x)=e^{-1/x^2}$　　$e^{\frac{1}{x^2}}\geq\frac{1}{x^2}+1 \Rightarrow \frac{1}{e^{\frac{1}{x^2}}}=e^{-1/x^2}\leq\frac{1}{\frac{1}{x^2}+1}$

$\forall\varepsilon>0$ 取 $\delta=\sqrt{\varepsilon}>0$ $\forall\, 0<x<\delta$ 有

$$e^{-1/x^2}\leq\frac{1}{\frac{1}{x^2}+1}=\frac{x^2}{x^2+1}<x^2<\varepsilon$$

即 $\lim_{x\to0^+}e^{-1/x^2}=0$

同理 $\lim_{x\to0^-}e^{-1/x^2}=0$

则 $\lim_{x\to0^+}f(x)=\lim_{x\to0^-}f(x)=f(0)=0$

$\therefore$ $f$ 在 $x=0$ 处连续

6. ① $x<-1$ 时 $f(x)=\lim\limits_{n\to\infty}\dfrac{1+\frac{1}{x^{2n+1}}}{1-\frac{1}{x^n}+\frac{1}{x^{2n}}}=\dfrac{\lim\limits_{n\to\infty}(1+\frac{1}{x^{2n+1}})}{\lim\limits_{n\to\infty}(1-\frac{1}{x^n}+\frac{1}{x^{2n}})}=\dfrac{1}{1}=1$

② $x=-1$ 时 $f(x)=\lim\limits_{n\to\infty}\dfrac{(-1)^{2n+1}+1}{(-1)^{2n+1}-(-1)^{n+1}+(-1)}=\dfrac{0}{-1}=0$.

③ $-1<x<0$ 时 $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n+1}+1}{x(x^{2n}-x^n+1)}=\dfrac{1+\lim\limits_{n\to\infty}x^{2n+1}}{x\lim\limits_{n\to\infty}(x^{2n}-x^n+1)}=\dfrac{1+0}{x(0-0+1)}=\dfrac{1}{x}$

易知 $\forall M>0$ 取 $\delta=\frac{1}{M}$ $\forall -\delta<x<0$ 有

$$f(x)=\frac{1}{x}<-M$$

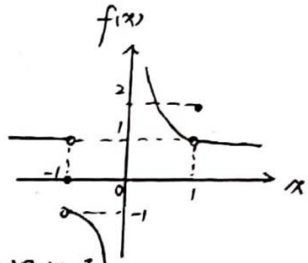
则 $\lim\limits_{x\to0^-}f(x)=-\infty$

④ $0<x<1$ 时 $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n+1}+1}{x(x^{2n}-x^n+1)}=\dfrac{1+\lim\limits_{n\to\infty}x^{2n+1}}{x\lim\limits_{n\to\infty}(x^{2n}-x^n+1)}=\dfrac{1+0}{x(0-0+1)}=\dfrac{1}{x}$

同理 $\lim\limits_{x\to0^+}f(x)=+\infty$

⑤ $x=+1$ 时 $f(x)=\lim\limits_{n\to\infty}\dfrac{1^{2n+1}+1}{1^{2n+1}-1^{n+1}+1}=2$.

⑥ $x>1$ 时 $f(x)=\lim\limits_{n\to\infty}\dfrac{1+\frac{1}{x^{2n+1}}}{1-\frac{1}{x^n}+\frac{1}{x^{2n}}}=\dfrac{\lim\limits_{n\to\infty}(1+\frac{1}{x^{2n+1}})}{\lim\limits_{n\to\infty}(1-\frac{1}{x^n}+\frac{1}{x^{2n}})}=\dfrac{1}{1}=1$




综上：$f(x)$ 有二个间断点，现分别讨论.

i) $x=-1$ $\lim\limits_{x\to(-1)^-}f(x)=\lim\limits_{x\to(-1)^-}1=1$

$\lim\limits_{x\to(-1)^+}f(x)=\lim\limits_{x\to(-1)^+}\dfrac{1}{x}=-1$ 而 $f(1)=0$ 则 $x=-1$ 为跳跃间断点

ii) $x=+1$ $\lim\limits_{x\to1^-}f(x)=\lim\limits_{x\to1^-}\dfrac{1}{x}=1$

$\lim\limits_{x\to1^+}f(x)=\lim\limits_{x\to1^+}1=1$ 而 $f(1)=2$ 则 $x=1$ 为可去间断点

7. 证明: $f\in C([a,b])$ 由最值定理

$\exists\, x_1\in[a,b]$ s.t. $\forall x\in[a,b]$ 有 $f(x_1)\le f(x)$. 令 $f(x_1)=m$

$\exists\, x_2\in[a,b]$ s.t. $\forall x\in[a,b]$ 有 $f(x)\le f(x_2)$ 令 $f(x_2)=M$

则 $f([a,b])\subset[m,M]$

下证 $[m,M]\subset f([a,b])$

即证 $\forall y\in[m,M]$ 均有 $y\in f([a,b])$

对此, 设 $g(x)=f(x)-y$ 易知 $g(x)\in C([a,b])$

则 $g(x_1)=f(x_1)-y=m-y<0$

$g(x_2)=f(x_2)-y=M-y>0$

由介值定理 $\exists\, x_0\in(x_1,x_2)$ s.t. $g(x)=0$.

$\therefore g(x_0)=0 \Rightarrow f(x_0)=y$

则 $\exists\, x_0\in[a,b]$ 使 $y=f(x_0)\in f([a,b])$ □