概统　第十三次作业

习题 5.1　1. 研究的总体是该地区所有观众是否观看该频道（观看过为1，否则为0
样本是电话访查的观众是否观看该频道（观看过为1，否则为0

习题 5.3
5.　设容量为n, m 的样本分别是 $x_1 \dots x_n$ 和 $x_{n+1} \cdots x_{n+m}$.

则 i. $$\frac{n\bar{x}_1+m\bar{x}_2}{n+m} = \frac{n\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)+m\left(\frac{1}{m}\sum_{i=n+1}^{n+m}x_i\right)}{n+m}$$

$$= \frac{\sum_{i=1}^{n}x_i + \sum_{i=n+1}^{n+m}x_i}{n+m} = \frac{1}{n+m}\sum_{i=1}^{n+m}x_i = \bar{x}$$

ii. $$\frac{(n-1)s_1^2+(m-1)s_2^2}{n+m-1} + \frac{nm(\bar{x}_1-\bar{x}_2)^2}{(n+m)(n+m-1)}$$

$$= \frac{(n-1)\left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x}_1)^2\right)+(m-1)\left(\frac{1}{m-1}\sum_{i=n+1}^{n+m}(x_i-\bar{x}_2)^2\right)}{n+m-1} + \frac{nm(\bar{x}_1-\bar{x}_2)^2}{(n+m)(n+m-1)}$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x}_1)^2+\sum_{i=n+1}^{n+m}(x_i-\bar{x}_2)^2}{n+m-1} + \frac{nm(\bar{x}_1-\bar{x}_2)^2}{(n+m)(n+m-1)}$$

$$= \frac{\sum_{i=1}^{n}x_i^2-2\left(\sum_{i=1}^{n}x_i\right)\bar{x}_1+n\bar{x}_1^2+\sum_{i=n+1}^{n+m}x_i^2-2\left(\sum_{i=1}^{n}x_2\right)\bar{x}_2+m\bar{x}_2^2}{n+m-1} + \frac{nm(\bar{x}_1-\bar{x}_2)^2}{(n+m)(n+m-1)}$$

$$= \frac{\sum_{i=1}^{n+m}x_i^2-n\bar{x}_1^2-m\bar{x}_2^2}{n+m-1} + \frac{nm(\bar{x}_1-\bar{x}_2)^2}{(n+m)(n+m-1)}$$

$$= \frac{\sum_{i=1}^{n+m}x_i^2}{n+m-1} + \frac{nm\bar{x}_1^2-n(n+m)\bar{x}_1^2+nm\bar{x}_2^2-m(n+m)\bar{x}_2^2-2nm\bar{x}_1\bar{x}_2}{(n+m)(n+m-1)}$$

$$= \frac{1}{n+m-1}\sum_{i=1}^{n+m}x_i^2 + \frac{-n^2\bar{x}_1^2-m^2\bar{x}_2^2-2nm\bar{x}_1\bar{x}_2}{(n+m)(n+m-1)}$$

$$= \frac{1}{n+m-1}\sum_{i=1}^{n+m}x_i^2 - \frac{(n\bar{x}_1+m\bar{x}_2)^2}{(n+m)(n+m-1)}$$

$$= \frac{1}{n+m-1}\sum_{i=1}^{n+m}x_i^2 - \frac{n+m}{n+m-1}\left(\frac{n\bar{x}_1+m\bar{x}_2}{n+m}\right)^2$$

$$= \frac{1}{n+m-1}\left(\sum_{i=1}^{n+m}x_i^2-(n+m)\bar{x}^2\right)$$

$$= \frac{1}{n+m-1}\left(\sum_{i=1}^{n+m}x_i^2-2\left(\sum_{i=1}^{n+m}x_i\right)\bar{x}+(n+m)\bar{x}^2\right)$$

$$= \frac{1}{n+m-1}\sum_{i=1}^{n+m}(x_i-\bar{x})^2 = s^2 \quad \square$$

26. $F(x) = \int_0^x p(t)\,dt = \int_0^x 6t(1-t)\,dt = 3t^2-2t^3\Big|_0^x = 3x^2-2x^3$

$$P_{(5)}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1}(1-F(x))^{n-k} p(x)$$

$$= \frac{9!}{(4!)^2}(3x^2-2x^3)^4(1-3x^2+2x^3)^4\, 6x(1-x)$$

$$= 3780\, x(1-x)(3x^2-2x^3)^4(1-3x^2+2x^3)^4$$

由书上定理 $n\to\infty$ 而 $F(x)=3x^2-2x^3=0.5 \Rightarrow x_{0.5}=0.5$, $p(x_{0.5})=1.5$.

$$m_{0.5} \sim N\left(x_{0.5}, \frac{1}{4n\cdot p^2(x_{0.5})}\right) = N\left(0.5, \frac{1}{81}\right).$$

习题 6.1

5. $E\bar{x} = E\left(\frac{1}{n}\sum_{i=1}^n x_i\right) = \frac{1}{n}\sum_{i=1}^n Ex_i = \frac{1}{n}\sum_{i=1}^n\left(\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x\,dx\right) = \frac{1}{n}\sum_{i=1}^n(\theta) = \frac{1}{n}n\theta = \theta$

故 $\bar{x}$ 是 θ 的无偏估计

$$F_{x_{(n)}}(x) = F(x_{(n)}\le x) = F(x_1\le n, \dots x_n\le n) = (F(x))^n.$$

$$\Rightarrow p_{x_{(n)}}(x) = n(F(x))^{n-1}p(x).$$

$$F_{x_{(1)}}(x) = F(x_{(1)}\le x) = F(\overline{x_{(1)}\ge x}) = 1-F(x_{(1)}\ge x) = 1-F(x_1\ge x \dots x_n\ge x)$$

$$= 1-(1-F(x))^n$$

$$\Rightarrow p_{x_{(1)}}(x) = n(1-F(x))^n p(x).$$

而 $F(x) = \int_{\theta-\frac{1}{2}}^x p(t)\,dt = x-(\theta-\frac{1}{2})$

故 $$EX_{(n)} = \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} xn(F(x))^{n-1}p(x)\,dx = n\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x(x-(\theta-\tfrac{1}{2}))^{n-1}\cdot dx$$

$$= x(x-(\theta-\tfrac{1}{2}))^n\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} - \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}(x-(\theta-\tfrac{1}{2}))^n\,dx$$

$$= \theta+\frac{1}{2}-\frac{1}{n+1}$$

$$EX_{(1)} = \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} xn(1-F(x))^n p(x)\,dx = n\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x(1-x+(\theta-\tfrac{1}{2}))^{n-1}dx$$

$$= -x(1-x+\theta-\tfrac{1}{2}))^n\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} + \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}(1-x+(\theta-\tfrac{1}{2}))^n dx$$

$$= -(\theta-\tfrac{1}{2}) + \frac{1}{n+1}$$

则 $E\left(\frac{1}{2}(X_{(1)}+X_{(n)})\right) = \frac{1}{2}\left(\theta+\frac{1}{2}-\frac{1}{n+1}-(\theta-\frac{1}{2})+\frac{1}{n+1}\right) = \theta$

故 $\frac{1}{2}(X_{(1)}+X_{(n)})$ 是 θ 的无偏估计

$$EX_{(n)}^2 = n\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x^2(x-(\theta-\tfrac{1}{2}))^{n-1}dx = x^2(x-(\theta-\tfrac{1}{2}))^n\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} - 2\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x(x-(\theta-\tfrac{1}{2}))^n$$

$$= (\theta+\tfrac{1}{2})^2 - \frac{2}{n+1}\left(\theta+\frac{1}{2}-\frac{1}{n+2}\right)$$

$$\mathrm{Var}X_{(n)} = EX_{(n)}^2-(EX_{(n)})^2 = (\theta+\tfrac{1}{2})^2-\frac{2}{n+1}(\theta+\tfrac{1}{2})+\frac{2}{(n+1)(n+2)}-(\theta+\tfrac{1}{2})^2+\frac{2(\theta+\frac{1}{2})}{n+1}-\frac{1}{(n+1)^2}$$

$$= \frac{1}{n+1}\left(\frac{2}{n+2}-\frac{1}{n+1}\right) = \frac{n}{(n+1)^2(n+2)}$$

同理: $\mathrm{Var}X_{(1)} = \frac{n}{(n+1)^2(n+2)}$

另一方面 $\mathrm{Var}\,\bar{x} = \mathrm{Var}\frac{\sum x_i}{n} = \frac{1}{n}\mathrm{Var}X = \frac{1}{n}\frac{1}{12(b-a)^2} = \frac{1}{12n}$

则 $\mathrm{Var}\frac{1}{2}(X_{(1)}+X_{(n)}) = \frac{1}{4}(\mathrm{Var}X_{(1)}+\mathrm{Var}X_{(n)}+2\mathrm{Cov}(X_{(1)},X_{(n)})) \le \frac{1}{4}(\mathrm{Var}X_{(1)}+\mathrm{Var}X_{(n)}+2\sqrt{\mathrm{Var}X_{(1)}\mathrm{Var}X_{(n)}})$

$$= \frac{1}{4}\times 4\frac{n}{(n+1)^2(n+2)} = \frac{n}{(n+1)^2(n+2)}$$

而 $\frac{1}{12} < \frac{n^2}{(n+1)^2(n+2)}$ 在 $n\geq 2$ 时成立（$\frac{n^2}{(n+1)^2(n+2)}$ 递增）

故 $Var\,\frac{1}{2}(X_{(1)}+X_{(n)}) \leq \frac{n}{(n+1)^2(n+2)} < \frac{1}{12n} = Var\,\bar{x}$ 在 $n\geq 2$ 成立.

即 $n\geq 2$ 时 $\frac{1}{2}(x_{(1)}+x_{(n)})$ 更有效

习题 6.2

3. (1) $EX = \sum_{k=0}^{N-1} kP(X=k) = \sum_{k=0}^{N-1} k\frac{1}{N} = \frac{(N-1)N}{2}N = \frac{N-1}{2}$

故 $\hat{N} = 2\bar{x}+1$

(2) $EX = \sum_{k=2}^{\infty} kP(X=k) = \sum_{k=2}^{\infty} k(k-1)\theta^2(1-\theta)^{k-2}$

$= \theta^2 \sum_{k=2}^{\infty} k(k-1)(1-\theta)^{k-2}$

$= \theta^2 \frac{d}{d\theta^2}\left(\sum_{k=0}^{\infty}(1-\theta)^k\right)$

$= \theta^2 \frac{d}{d\theta}\left(\frac{1}{1-(1-\theta)}\right) = \theta^2 \times \frac{2}{\theta^3} = \frac{2}{\theta}$

$\Rightarrow \hat{\theta} = \frac{2}{\bar{x}}$

6. (1) 设有 n 个错字. 每个错字被甲发现的概率为 P_1，乙为 P_2，两人共同为 P.

则 $\hat{P}_1 = \frac{a}{n}$，$\hat{P}_2 = \frac{b}{n}$，$\hat{P} = \frac{c}{n}$

而 $P = P_1P_2 \Rightarrow \hat{P} = \hat{P}_1\hat{P}_2$

$\Rightarrow \frac{c}{n} = \frac{ab}{n^2} \Rightarrow \hat{n} = \frac{ab}{c}$

(2) 未发现的 $\hat{n}-a-b+c = \frac{ab}{c}-a-b+c$

习题 6.3

4. 设10个石子中有石灰石的数目为 X

$P(X=k;p) = \binom{10}{k}p^k(1-p)^{10-k}$

$L(p) = \prod_{i=1}^{n} p(x_i;p) = \prod_{i=1}^{n}\binom{10}{k}p^{x_i}(1-p)^{10-x_i}$

$\ln L(p) = \sum_{i=1}^{n}\ln\binom{10}{k}p^{x_i}(1-p)^{10-x_i} = n\ln\binom{10}{k} + \left(\sum_{i=1}^{n}x_i\right)\ln p + \left(\sum_{i=1}^{n}10-x_i\right)\ln(1-p)$

$\frac{\partial \ln L(p)}{\partial p} = \frac{\sum_{i=1}^{n}x_i}{p} - \frac{\sum_{i=1}^{n}(10-x_i)}{1-p} = 0 \Rightarrow \frac{1-p}{p} = \frac{\sum(10-x_i)}{\sum x_i} \Rightarrow \frac{1}{p} = \frac{\sum(10-x_i)+\sum x_i}{\sum x_i}$

$\Rightarrow p = \frac{\sum x_i}{\sum 10} = \frac{\sum x_i}{10n} = \frac{1}{10}\bar{x}$

数据中 $\bar{x} = \frac{1+12+21+92+130+126+84+24+9+0}{100} = 4.99$

故 $p = \frac{1}{10}\bar{x} = 0.499$

8. (1) $L(\theta)=\prod_{i=1}^{n}p(x_i;\theta)=\prod_{i=1}^{n}e^{-(x_i-\theta)}=e^{-\sum_{i=1}^{n}(x_i-\theta)}$

$\ln L(\theta)=-\sum_{i=1}^{n}(x_i-\theta)=-\sum_{i=1}^{n}x_i+n\theta$

$\frac{\partial \ln L(\theta)}{\partial\theta}=\theta>0$

故 $\ln L(\theta)$ 单调递增 $\theta<x_i \Rightarrow \theta<\min\{x_i\}$

$L(\hat\theta)=L(\min\{x_i\})=\sup L(\theta)$

则 $\hat\theta=\min\{x_i\}$

$P(|\min\{x_i\}-\theta|\geq\varepsilon)=P(x_1-\theta\geq\varepsilon,\dots,x_n-\theta\geq\varepsilon)$

$=(P(x_i\geq\theta+\varepsilon))^n=(1-P(x_i\leq\theta+\varepsilon))^n$

$=\left(1-\int_{\theta}^{\theta+\varepsilon}e^{-(x-\theta)}dx\right)^n$

$=(e^{-\varepsilon})^n=e^{-\varepsilon n}$

$\lim_{n\to\infty}P(|\min\{x_i\}-\theta|\geq\varepsilon)=\lim_{n\to\infty}e^{-\varepsilon n}=0$. 故 $\min\{x_i\}\xrightarrow{P}\theta$ 满足相合性

$p_{\min\{x_i\}}(x)=p_{x_{(1)}}(x)=\frac{n!}{(n-1)!}(1-F(x))^{n-1}p(x)=n(e^{-(x-\theta)})^{n-1}e^{-(x-\theta)}=ne^{-n(x-\theta)}$

$E\min\{x_i\}=\int_{\theta}^{+\infty}nx\,e^{-n(x-\theta)}dx=-xe^{-n(x-\theta)}\Big|_{\theta}^{+\infty}+\int_{\theta}^{+\infty}e^{-n(x-\theta)}dx$

$=\theta+(-\frac{1}{n})e^{-n(x-\theta)}\Big|_{\theta}^{+\infty}=\theta+\frac{1}{n}$

则它并不是无偏估计

(2) $EX=\int_{\theta}^{+\infty}xe^{-(x-\theta)}dx=\int_{0}^{+\infty}(x+\theta)e^{-x}dx=\int_{0}^{+\infty}xe^{-x}dx+\int_{0}^{+\infty}\theta e^{-x}dx$

$=1+\theta(-e^{-x})\Big|_{0}^{+\infty}=1+\theta$.

则 $\hat\theta=\bar{x}-1$

$E(\hat\theta_n)=E(\bar{x}-1)=E(\bar{x})-1=EX-1=\theta$

$Var(\hat\theta_n)=Var(\bar{x})=Var(\frac{1}{n}\sum x_i)=\frac{1}{n^2}\sum_{i=1}^{n}Var(x_i)=\frac{1}{n}Var(X)$

$EX^2=\int_{\theta}^{+\infty}x^2e^{-(x-\theta)}dx=-x^2e^{-(x-\theta)}\Big|_{\theta}^{+\infty}+2\int_{\theta}^{+\infty}xe^{-(x-\theta)}dx$

$=\theta^2+2\theta+2$

$Var X=EX^2-(EX)^2=\theta^2+2\theta+2-\theta^2-2\theta-1=1$

则 $Var(\hat\theta_n)=\frac{1}{n}\to 0\ (n\to\infty)$ 故满足相合性

$E(\hat\theta)=E(\bar{x}-1)=E(\bar{x})-1=EX-1=1+\theta-1=\theta$ 满足无偏性