《高等微积分1》第三次作业

1. $x>0$ 时 $\forall \varepsilon>0$ 取 $\delta=\frac{\varepsilon}{2}$ $\forall 0<x<\delta$ 有

$$|f(x)-0| = |f(x)| = |2x| < |2\cdot\tfrac{\varepsilon}{2}| = \varepsilon \quad 则 \lim_{x\to 0^+} f(x)=0$$

$x<0$ 时　先证 $\lim\limits_{x\to 0^-}\sin x = 0$，$\lim\limits_{x\to 0^-}\cos x = 1$

$\forall \varepsilon>0$ 取 $\delta=\varepsilon$ $\forall -\delta<x<0$ 有

$$|\sin x-0| = |\sin x| \leq |x| < \delta = \varepsilon \Rightarrow \lim_{x\to 0^-}\sin x = 0.$$

$\forall \varepsilon>0$ 取 $\delta=\sqrt{2\varepsilon}$ $\forall -\delta<x<0$ 有

$$|\cos x-1| = |-2\sin^2\tfrac{x}{2}| = 2|\sin\tfrac{x}{2}|^2 \leq 2\left(\tfrac{x}{2}\right)^2 < 2\left(\tfrac{\sqrt{2\varepsilon}}{2}\right)^2 = \varepsilon$$

则 $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(a\cos x + b\sin x) = a\lim\limits_{x\to 0^-}\cos x + b\lim\limits_{x\to 0^+}\sin x = a$

极限 $\lim\limits_{x\to 0} f(x)$ 存在 $\Leftrightarrow$ $\lim\limits_{x\to 0^+} f(x)$、$\lim\limits_{x\to 0^-} f(x)$ 均存在且相等

$\Leftrightarrow \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = a = 0 \quad \Leftrightarrow a=0$

即 $a=0$　$b\in\mathbb{R}$ 时　$\lim\limits_{x\to 0} f(x)$ 存在

2. 证明：(1) 若 $A=0$ 则 $\lim\limits_{x\to a} f(x)=0 \Rightarrow \forall\varepsilon>0$　$\varepsilon^k>0$　$\exists\delta>0$ 使 $0<|x-a|<\delta$ 时 $|f(x)|<\varepsilon^k$

则 $|\sqrt[k]{f(x)}| < \sqrt[k]{\varepsilon^k} = \varepsilon \Rightarrow \forall\varepsilon>0\ \exists\delta>0$ 使 $\forall 0<|x-a|<\delta$ 时 $|\sqrt[k]{f(x)}|<\varepsilon$

从而 $\lim\limits_{x\to a}\sqrt[k]{f(x)} = \sqrt[k]{A}$.

若 $A\neq 0$　则证引理(保号性)：$\exists\delta>0$ 使 $\forall 0<|x-a|<\delta$ 有 $f(x)\cdot A>0$.

i) 若 $A>0$ 则 取 $\varepsilon=A$ $\exists\delta>0$ 使 $\forall 0<|x-a|<\delta$ $|f(x)-A|<A$.

$\Rightarrow -A<f(x)-A<A \Rightarrow f(x)>0 \Rightarrow f(x)\cdot A>0$.

ii) 若 $A<0$ 则 取 $\varepsilon=-A>0$ $\exists\delta>0$ 使 $\forall 0<|x-a|<\delta$ $|f(x)-A|<-A$

$\Rightarrow -A<f(x)-A<-A \Rightarrow f(x)<0 \Rightarrow f(x)\cdot A>0$. 引理得证.

$$f(x)-A = \left(f(x)^{\frac{1}{k}} - A^{\frac{1}{k}}\right)\left(f(x)^{\frac{k-1}{k}} + f(x)^{\frac{k-2}{k}}A^{\frac{1}{k}} + \cdots + A^{\frac{k-1}{k}}\right)$$

$\forall\varepsilon>0$ $\exists\delta_1>0$ 使 $\forall 0<|x-a|<\delta_1$ 有 $f(x)\cdot A>0$　$\exists\delta_2$ 使 $\forall 0<|x-a|<\delta_2$ 有 $|f(x)-A|<\varepsilon|A^{\frac{k-1}{k}}|$

则取 $\delta=\min\{\delta_1,\delta_2\}$ $\forall 0<|x-a|<\delta$

$$|f(x)^{\frac{1}{k}} - A^{\frac{1}{k}}| = \frac{|f(x)-A|}{|f(x)^{\frac{k-1}{k}} + f(x)^{\frac{k-2}{k}}A^{\frac{1}{k}} + \cdots + A^{\frac{k-1}{k}}|} \leq \frac{|f(x)-A|}{|A^{\frac{k-1}{k}}|} < \frac{\varepsilon|A^{\frac{k-1}{k}}|}{|A^{\frac{k-1}{k}}|} = \varepsilon$$

其中 $f(x)^{\frac{k-1}{k}}>0$　$A^{\frac{k-1}{k}}>0$，$f(x)^{\frac{k-2}{k}}A^{\frac{1}{k}}>0$.

由极限定义　$\lim\limits_{x\to a}\sqrt[k]{f(x)} = \sqrt[k]{A}$

综上：$\lim\limits_{x\to a}\sqrt[k]{f(x)} = \sqrt[k]{A}$.

(2) 由极限定义　$\forall\varepsilon>0$ $\exists\delta_1>0$ 使 $\forall 0<|x-a|<\delta_1$ 有 $|f(x)-A|<\varepsilon|A^{\frac{k-1}{k}}|$

则取 $\delta=\delta_1$ $\forall 0<|x-a|<\delta$

$$|f(x)^{\frac{1}{k}} - A^{\frac{1}{k}}| = \frac{|f(x)-A|}{|f(x)^{\frac{k-1}{k}} + f(x)^{\frac{k-2}{k}}A^{\frac{1}{k}} + \cdots + A^{\frac{k-1}{k}}|} \leq \frac{|f(x)-A|}{|A^{\frac{k-1}{k}}|} = \varepsilon$$

由极限定义　$\lim\limits_{x\to a}\sqrt[k]{f(x)} = \sqrt[k]{A}$.

3、解：(1) $\lim\limits_{x\to1}\dfrac{x^m-1}{x^n-1} = \lim\limits_{x\to1}\dfrac{(x-1)(x^{m-1}+x^{m-2}+\cdots+x+1)}{(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)} = \lim\limits_{x\to1}\dfrac{x^{m-1}+x^{m-2}+\cdots+x+1}{x^{n-1}+x^{n-2}+\cdots+x+1}$

$= \dfrac{\lim\limits_{x\to1}(x^{m-1}+\cdots+1)}{\lim\limits_{x\to1}(x^{n-1}+\cdots+1)} = \dfrac{\lim\limits_{x\to1}x^{m-1}+\lim\limits_{x\to1}x^{m-2}+\cdots+\lim\limits_{x\to1}1}{\lim\limits_{x\to1}x^{n-1}+\lim\limits_{x\to1}x^{n-2}+\cdots+\lim\limits_{x\to1}1} = \dfrac{m}{n}$

(2) $x^n = [(x^n+p^n)-p^n] = [(x^n+p^n)^{\frac1n}-(p^n)^{\frac1n}][(x^n+p^n)^{\frac{n-1}{n}}+(x^n+p^n)^{\frac{n-2}{n}}(p^n)^{\frac1n}+\cdots+(p^n)^{\frac{n-1}{n}}]$

则 $(x^n+p^n)^{\frac1n}-p = \dfrac{x^n}{(x^n+p^n)^{\frac{n-1}{n}}+\cdots+(p^n)^{\frac{n-1}{n}}}$

则 $\lim\limits_{x\to0}\dfrac{\sqrt[n]{x^n+p^n}-p}{x^n} = \lim\limits_{x\to0}\dfrac{x^n/[(x^n+p^n)^{\frac{n-1}{n}}+\cdots+(p^n)^{\frac{n-1}{n}}]}{x^n} = \lim\limits_{x\to0}\dfrac{1}{(x^n+p^n)^{\frac{n-1}{n}}+(x^n+p^n)^{\frac{n-2}{n}}(p^n)^{\frac1n}+\cdots+(p^n)^{\frac{n-1}{n}}}$

$= \dfrac{1}{\lim\limits_{x\to0}(x^n+p^n)^{\frac{n-1}{n}}+\lim\limits_{x\to0}(x^n+p^n)^{\frac{n-2}{n}}p+\cdots+\lim\limits_{x\to0}p^{n-1}} = \dfrac{1}{\underbrace{p^{n-1}+\cdots+p^{n-1}}_{n个}} = \dfrac{1}{np^{n-1}}$

(3) 由(2) $(x^n+p^n)^{\frac1n}-p = \dfrac{x^n}{(x^n+p^n)^{\frac{n-1}{n}}+\cdots+(p^n)^{\frac{n-1}{n}}}$

$(x^n+q^n)^{\frac1n}-q = \dfrac{x^n}{(x^n+q^n)^{\frac{n-1}{n}}+\cdots+(q^n)^{\frac{n-1}{n}}}$

$\lim\limits_{x\to0}\dfrac{\sqrt[n]{x^n+p^n}-p}{\sqrt[n]{x^n+q^n}-q} = \lim\limits_{x\to0}\dfrac{(x^n+q^n)^{\frac{n-1}{n}}+\cdots+(q^n)^{\frac{n-1}{n}}}{(x^n+p^n)^{\frac{n-1}{n}}+\cdots+(p^n)^{\frac{n-1}{n}}} = \dfrac{\lim\limits_{x\to0}(x^n+q^n)^{\frac{n-1}{n}}+\lim\limits_{x\to0}(x^n+q^n)^{\frac{n-2}{n}}q+\cdots+\lim\limits_{x\to0}q^{n-1}}{\lim\limits_{x\to0}(x^n+p^n)^{\frac{n-1}{n}}+\lim\limits_{x\to0}(x^n+p^n)^{\frac{n-2}{n}}p+\cdots+\lim\limits_{x\to0}p^{n-1}}$

$= \dfrac{\underbrace{q^{n-1}+\cdots+q^{n-1}}_{n个}}{\underbrace{p^{n-1}+\cdots+p^{n-1}}_{n个}} = \dfrac{q^{n-1}}{p^{n-1}}$

(4) 设 $\arctan x = t$ $\exists\delta>0$ 使 $\forall 0<|x|<\delta$ $\arctan x \neq \lim\limits_{x\to0}\arctan x = 0$

则 $\lim\limits_{x\to0}\dfrac{\arctan x}{x} = \lim\limits_{t\to0}\dfrac{t}{\tan t}$

而 $\forall t\in[-\frac{\pi}{2},\frac{\pi}{2}]$ $|\sin t|\le|t|\le|\tan t|$ 且 $t\in(-\frac{\pi}{2},0)$ $\frac{1}{\tan t}<0$ $t\in(0,\frac{\pi}{2})$ $\frac{1}{\tan t}>0$

则 $\forall t\in(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$ 均有 $\cos t = \dfrac{\sin t}{\tan t} \le \dfrac{t}{\tan t} \le \dfrac{\tan t}{\tan t} = 1$

由夹逼定理 $\lim\limits_{t\to0}\cos t = 1 \le \lim\limits_{t\to0}\dfrac{t}{\tan t} \le 1$

则 $\lim\limits_{t\to0}\dfrac{t}{\tan t} = 1$

$\therefore \lim\limits_{x\to0}\dfrac{\arctan x}{x} = 1$

(5)

$\left.\begin{array}{l}\forall x\in(0,\frac{\pi}{2})\ 0<\sin x<x<\tan x \Rightarrow \cos x \le \dfrac{\sin x}{x} \le \dfrac{x}{x} = 1\\ \forall x\in(-\frac{\pi}{2},0)\ \tan x<x<\sin x<0 \Rightarrow \cos x \le \dfrac{\sin x}{x} \le \dfrac{x}{x} = 1\end{array}\right\}$ $\forall x\in(-\frac{\pi}{2},\frac{\pi}{2})\setminus\{0\}$ 均有 $\cos x \le \dfrac{\sin x}{x} \le 1$

由夹逼定理 $\lim\limits_{x\to0}\cos x \le \lim\limits_{x\to0}\dfrac{\sin x}{x} \le 1$

则 $\lim\limits_{x\to0}\dfrac{\sin x}{x} = 1$

$\lim\limits_{x\to0}\dfrac{\tan x-\sin x}{x^3} = \lim\limits_{x\to0}\dfrac{\sin x}{x}\lim\limits_{x\to0}\dfrac{\frac{1}{\cos x}-1}{x^2} = \lim\limits_{x\to0}\dfrac{\sin x}{x}\lim\limits_{x\to0}\dfrac{2\sin^2\frac{x}{2}}{x^2\cos x} = \lim\limits_{x\to0}\dfrac{\sin x}{x}\lim\limits_{x\to0}\dfrac{4\sin^2\frac{x}{2}}{x^2}\lim\limits_{x\to0}\dfrac{1}{2\cos x}$

$= \lim\limits_{x\to0}\dfrac{\sin x}{x}\left(\lim\limits_{x\to0}\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\lim\limits_{x\to0}\dfrac{1}{2\cos x}$

$= \lim\limits_{x\to0}\dfrac{\sin x}{x}\left(\lim\limits_{t\to0}\dfrac{\sin t}{t}\right)^2\lim\limits_{x\to0}\dfrac{1}{2\cos x}$

$= 1\times1^2\times\frac12$

$= \frac12$

4. 证明：(1) $\forall \varepsilon>0$ 设 $\delta=\min\{A(1-e^{-\varepsilon}), A(e^{\varepsilon}-1)\}>0$

$\forall -\delta<x-A<0$ 有 $0<|x-A|<A(1-e^{-\varepsilon}) \Rightarrow Ae^{-\varepsilon}<x<A$

则 $\because |\ln x-\ln A| = \ln\frac{A}{x} < \ln\frac{A}{Ae^{-\varepsilon}} = \ln e^{\varepsilon}=\varepsilon$

$\therefore \lim\limits_{x\to A^-} \ln x=\ln A$.

$\forall\ 0<x-A<\delta$ 有 $0<|x-A|<A(e^{\varepsilon}-1) \Rightarrow A<x<Ae^{\varepsilon}$

则 $|\ln x-\ln A| = \ln\frac{x}{A} < \ln\frac{Ae^{\varepsilon}}{A} = \ln e^{\varepsilon}=\varepsilon$

$\therefore \lim\limits_{x\to A^+} \ln x=\ln A$

综上：$\lim\limits_{x\to A} \ln x=\ln A$

(2)
$\forall \varepsilon>0$ 设 $\delta=\ln\frac{\varepsilon+e^c}{e^c}>0$ $\forall\ 0<x-c<\delta$

$$|e^x-e^c| = e^x-e^c < e^{\delta+c}-e^c = e^c e^{\ln\frac{\varepsilon+e^c}{e^c}}-e^c = e^c(\frac{\varepsilon+e^c}{e^c})-e^c=\varepsilon$$

则 $\lim\limits_{x\to c^+} e^x=e^c$

$\forall \varepsilon>0$ 设 $\delta=\ln\frac{e^c}{e^c-\varepsilon}>0$ $\forall -\delta<x-c<0$

$$|e^x-e^c| = e^c-e^x < e^c-e^{c-\delta} = e^c-e^{c-\ln\frac{e^c}{e^c-\varepsilon}} = e^c-e^c\frac{e^c-\varepsilon}{e^c}=\varepsilon$$

则 $\lim\limits_{x\to c^-} e^x=e^c$

综上：$\lim\limits_{x\to c} e^x=e^c$

(3)
$\because \lim\limits_{x\to x_0} u(x)=a>0$ 由保号性 $\exists B_r(x_0)\setminus$ 使 $x\in B_r(x_0)\setminus\{x\}$ $u(x)>0$ $(u(x)\neq0)$

则 $\lim\limits_{x\to x_0} v(x)\ln u(x) = \lim\limits_{x\to x_0} v(x) \lim\limits_{x\to x_0}\ln u(x) = b\lim\limits_{t\to a}\ln t = b\ln a$

则 $\lim\limits_{x\to x_0} u(x)^{v(x)} = \lim\limits_{x\to x_0} e^{v(x)\ln u(x)} = \lim\limits_{t\to b\ln a} e^t = e^{b\ln a} = a^b$

综上：$\lim\limits_{x\to x_0} u(x)^{v(x)} = a^b$

5. 解：(1) $\lim\limits_{x\to x_0}\frac{\sin f(x)}{g(x)} = \lim\limits_{x\to x_0}\frac{\sin f(x)}{f(x)}\frac{f(x)}{g(x)} = \lim\limits_{x\to x_0}\frac{\sin f(x)}{f(x)}\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = \lim\limits_{t\to0}\frac{\sin t}{t}\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = 1\cdot A=A$

(2) $\lim\limits_{x\to x_0}(1+f(x))^{\frac{1}{f(x)}} = \lim\limits_{y\to0}(1+y)^{\frac{1}{y}} = \lim\limits_{t\to\infty}(1+\frac{1}{t})^{t} = e$

设 $u(x)=(1+f(x))^{\frac{1}{f(x)}}$ $v(x)=\frac{f(x)}{g(x)}$ 由4题(3)结论 $\lim\limits_{x\to x_0}\left[(1+f(x))^{\frac{1}{f(x)}}\right]^{\frac{f(x)}{g(x)}}$

$= \lim\limits_{x\to x_0} u(x)^{v(x)} = e^A$

则 $\lim\limits_{x\to x_0}(1+f(x))^{1/g(x)} = \lim\limits_{x\to x_0}\left[(1+f(x))^{\frac{1}{f(x)}}\right]^{\frac{f(x)}{g(x)}} = e^A$.

(3) $\lim\limits_{x\to0}\frac{\sin(ax)}{\sin(bx)} = \lim\limits_{x\to0}\frac{\sin ax}{ax}\frac{bx}{\sin bx}\frac{ax}{bx} = \lim\limits_{x\to0}\frac{\sin ax}{ax}\frac{1}{\lim\limits_{x\to0}\frac{\sin bx}{bx}}\lim\limits_{x\to0}\frac{a}{b}$

$= \lim\limits_{t\to0}\frac{\sin t}{t}\frac{1}{\lim\limits_{y\to0}\frac{\sin y}{y}}\frac{a}{b} = 1\cdot1\cdot\frac{a}{b} = \frac{a}{b}$.

(4) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}} = \lim\limits_{t\to0}\frac{\cos(\frac{\pi}{2}+t)}{t} = -\lim\limits_{t\to0}\frac{\sin t}{t} = -1$

(5) $\lim\limits_{x\to0}\frac{\sin2x}{\sqrt{x+2}-\sqrt{2}} = \lim\limits_{x\to0}\frac{(\sqrt{x+2}+\sqrt{2})\sin2x}{x} = \lim\limits_{x\to0}(\sqrt{x+2}+\sqrt{2})\lim\limits_{x\to0}\frac{\sin2x}{x}$

$= \lim\limits_{x\to0}(\sqrt{x+2}+\sqrt{2})\,2\lim\limits_{x\to0}\frac{\sin2x}{2x}$

$= \lim\limits_{x\to0}(\sqrt{x+2}+\sqrt{2})\,2\lim\limits_{t\to0}\frac{\sin t}{t}$

$= 2\sqrt{2}\cdot2 = 4\sqrt{2}$.

(6) $\lim\limits_{x\to\infty}(1+\frac{k}{x})^{x} = \lim\limits_{t\to\infty}(1+\frac{k}{kt})^{kt} = \lim\limits_{t\to\infty}(1+\frac{1}{t})^{kt} = \lim\limits_{t\to\infty}[(1+\frac{1}{t})^{t}]^{k} \xlongequal{\text{由题4(3)}} e^{k}$

则 $\lim\limits_{x\to 0}(1+kx)^{\frac{1}{x}} = \lim\limits_{t\to\infty}(1+\frac{k}{t})^{t} = e^{k}$

(7) $\lim\limits_{x\to\infty}(\frac{x+a}{x-a})^{x} = \lim\limits_{t\to\infty}(\frac{t+2a}{t})^{t+a} = \lim\limits_{t\to\infty}(1+\frac{2a}{t})^{t+a} = \lim\limits_{t\to\infty}(1+\frac{2a}{t})^{t}\cdot\lim\limits_{t\to\infty}(1+\frac{2a}{t})^{a}$

$\xlongequal{\text{由题5(6)}} e^{2a}\cdot 1^{a} = e^{2a}$

(8) $\lim\limits_{x\to\infty}(1-\frac{a}{x})^{bx} = \lim\limits_{t\to\infty}(1-\frac{a}{t/b})^{t} = \lim\limits_{t\to\infty}(1+\frac{(-ab)}{t})^{t} \xlongequal{\text{由题5(6)}} e^{-ab}$

(9) 设 $f(x)=-2\sin^2 x$, $g(x)=x^2$

$\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}\sin^2 x=-2\lim\limits_{x\to 0}\sin x\cdot\lim\limits_{x\to 0}\sin x=0$

$\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{-2\sin^2 x}{x^2}=-2\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\lim\limits_{x\to 0}\frac{\sin x}{x}=-2\cdot 1\cdot 1=-2$

则 $\lim\limits_{x\to 0}(\cos 2x)^{1/x^2}=\lim\limits_{x\to 0}(1-2\sin^2 x)^{1/x^2}=\lim\limits_{x\to 0}(1+f(x))^{\frac{1}{g(x)}} \xlongequal{\text{由第(2)问}} e^{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}}=e^{-2}$

(10) 设 $f(x)=2\sin x+\cos x-1$ $\quad g(x)=x$

$\lim\limits_{x\to 0}f(x)=2\lim\limits_{x\to 0}\sin x+\lim\limits_{x\to 0}\cos x-1=0+1-1=0$

$\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{2\sin x+\cos x-1}{x}=2\lim\limits_{x\to 0}\frac{\sin x}{x}+\lim\limits_{x\to 0}\frac{\cos x-1}{x}$

$=2+\lim\limits_{x\to 0}\frac{-2\sin^2 x}{x}=2-2\lim\limits_{x\to 0}(\frac{\sin^2 x}{x^2}x)=2-2\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\lim\limits_{x\to 0}\frac{\sin x}{x}\lim\limits_{x\to 0}x=2-0=2$

则 $\lim\limits_{x\to 0}(2\sin x+\cos x)^{1/x}=\lim\limits_{x\to 0}(1+2\sin x+\cos x-1)^{1/x}$

$=\lim\limits_{x\to 0}(1+f(x))^{\frac{1}{g(x)}} \xlongequal{\text{由第(2)问}} e^{\lim\limits_{x\to 0}\frac{f(x)}{g(x)}}=e^{2}$