概统 第七次作业

习题3.1

2. $P(X=Y=1)=\dfrac{\binom{4}{1}\binom{3}{1}}{\binom{8}{4}}=\dfrac{12}{70}$

$P(X=Y=2)=\dfrac{\binom{4}{2}\binom{2}{2}}{\binom{8}{4}}=\dfrac{6}{70}$

则 $P(X=Y)=P(X=Y=1)+P(X=Y=2)=\dfrac{18}{70}=\dfrac{9}{35}\approx 0.2571$

4. $P(X_1X_2=0)=1$ 说明 X_1,X_2 中有一个0是必然事件

由此. $P(X_1=X_2)=P(X_1=X_2=-1)+P(X_1=X_2=0)+P(X_1=X_2=1)$

$=P(X_1=X_2=0)$

$=P(X_1=0\wedge X_2=0)$

$=P(X_1=0)+P(X_2=0)-P(X_1=0\vee X_2=0)$ （容斥原理）

$=P(X_1=0)+P(X_2=0)-P(X_1X_2=0)$

$=0.5+0.5-1=0$

故 $P(X_1=X_2)=0$.

10. (1) $P(X>0.5,Y>0.5)=\int_{0.5}^{+\infty}\int_{0.5}^{+\infty}p(x,y)\,dy\,dx=\int_{0.5}^{1}\int_{x}^{1}p(x,y)\,dy\,dx$

$=\int_{0.5}^{1}\int_{x}^{1}6(1-y)\,dy\,dx=\int_{0.5}^{1}3-6(x-\frac{1}{2}x^2)\,dx$

$=3x-3x^2+x^3\Big|_{0.5}^{1}=1-0.875=0.125$.

(2) $P(X<0.5)=\int_{-\infty}^{+\infty}\int_{-\infty}^{0.5}p(x,y)\,dx\,dy=\int_{0}^{0.5}\int_{x}^{1}6(1-y)\,dy\,dx$

$=\int_{0}^{0.5}3-6(x-\frac{1}{2}x^2)\,dx$

$=3x-3x^2+x^3\Big|_{0}^{0.5}=0.875$.

$P(Y<0.5)=\int_{-\infty}^{0.5}\int_{-\infty}^{+\infty}p(x,y)\,dx\,dy=\int_{0}^{0.5}\int_{0}^{y}6(1-y)\,dx\,dy$

$=\int_{0}^{0.5}(6y-6y^2)\,dy=0.5$

(3) $P(X+Y<1)=\iint_{x+y<1}p(x,y)\,dx\,dy$

$=\int_{-\infty}^{+\infty}\int_{-\infty}^{1-y}p(x,y)\,dx\,dy$

$=\int_{0}^{1}\int_{0}^{1-y}p(x,y)\,I_{y>x}\,dx\,dy$

$=\int_{0}^{0.5}\int_{0}^{y}6(1-y)\,dx\,dy+\int_{0.5}^{1}\int_{0}^{1-y}6(1-y)\,dx\,dy$

$=\int_{0}^{0.5}(6y-6y^2)\,dy+\int_{0.5}^{1}6(1-y)^2\,dy=0.5+0.25=0.75$.

11. $Y\sim Exp(1)$ 即 $p_Y(y)=e^{-y}I_{y>0}$

$F_Y(y)=P(Y\le y)=\int_{-\infty}^{y}e^{-t}I_{y>0}\,dt=\int_{0}^{y}e^{-t}\,dt=1-e^{-y}$

$P(X_1=0,X_2=0)=P(Y\le 1)=1-e^{-1}$

$P(X_1=0,X_2=1)=0$

$P(X_1=1,X_2=0)=P(1<Y\le 2)=P(Y\le 2)-P(Y\le 1)=(1-e^{-2})-(1-e^{-1})=e^{-1}-e^{-2}$

$P(X_1=1,X_2=1)=P(Y>2)=1-P(Y\le 2)=e^{-2}$

则联合分布列

$X_1 \backslash X_2$	0	1
0	$1-e^{-1}$	0
1	$e^{-1}-e^{-2}$	e^{-2}

15、设这两个数的随机变量为 X、Y，$0<X,Y<1$，联合密度函数 $p(x,y)=\begin{cases}1, & 0<x<1, 0<y<1\\ 0, & else\end{cases}$

$XY\geq\frac{3}{16}$，$X+Y\leq 1$ $\Rightarrow \frac{3}{16X}\leq Y\leq 1-X$，此时 X 满足 $\frac{1}{4}\leq X\leq\frac{3}{4}$

则 $P(XY\geq\frac{3}{16}, X+Y\leq 1)=\iint_{\substack{XY\geq\frac{3}{16}\\ X+Y\leq 1}} p(x,y)\,dx\,dy$

$=\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{3}{16x}}^{1-x}1\,dy\,dx=\int_{\frac{1}{4}}^{\frac{3}{4}}(1-x-\frac{3}{16x})\,dx= x-\frac{1}{2}x^2-\frac{3}{16}\ln x\Big|_{\frac{1}{4}}^{\frac{3}{4}}$

$=\frac{1}{4}-\frac{3}{16}\ln 3$.

习题3.2

1. 联合分布列

X\Y	0	1	2
-1	0	$\frac{1}{3}$	$\frac{1}{12}$
0	$\frac{1}{6}$	0	0
1	$\frac{5}{12}$	0	0

则边际分布列

X	-1	0	1
P	$\frac{5}{12}$	$\frac{1}{6}$	$\frac{5}{12}$

Y	0	1	2
P	$\frac{7}{12}$	$\frac{1}{3}$	$\frac{1}{12}$

2、

$x>0$ 时

$F_X(x)=F(x,+\infty)=\lim_{y\to+\infty}1-e^{-\lambda_1 x}-e^{-\lambda_2 y}+e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max\{x,y\}}=1-e^{-\lambda_1 x}$

$x\leq 0$ 时

$F_X(x)=F(x,+\infty)=0$

因此 $F_X(x)=\begin{cases}1-e^{-\lambda_1 x}, & x>0\\ 0, & x\leq 0\end{cases}$

同理 $F_Y(y)=\begin{cases}1-e^{-\lambda_2 y}, & y>0\\ 0, & y\leq 0\end{cases}$

5、(1) $x>0$ 时 $p_X(x)=\int_{-\infty}^{+\infty}p_1(x,y)\,dy=\int_x^{+\infty}e^{-y}\,dy=e^{-x}$

$x\leq 0$ 时 $p_X(x)=\int_{-\infty}^{+\infty}p_1(x,y)\,dy=0$

$y>0$ 时 $p_Y(y)=\int_{-\infty}^{+\infty}p_1(x,y)\,dx=\int_0^y e^{-y}\,dx=ye^{-y}$

$y\leq 0$ 时 $p_Y(y)=\int_{-\infty}^{+\infty}p_1(x,y)\,dx=0$

综上：$p_X(x)=\begin{cases}e^{-x}, & x>0\\ 0, & x\leq 0\end{cases}$ $p_Y(y)=\begin{cases}ye^{-y}, & y>0\\ 0, & y\leq 0\end{cases}$

(2) $-1<x<1$ 时 $p_X(x)=\int_{-\infty}^{+\infty}p_2(x,y)\,dy=\int_0^{1-x^2}\frac{5}{4}(x^2+y)\,dy=\frac{5}{4}x^2(1-x^2)+\frac{5}{8}(1-x^2)^2=\frac{5}{8}(1-x^4)$

x 取其它值 $p_X(x)=\int_{-\infty}^{+\infty}p_2(x,y)\,dy=0$

$0<y<1$ 时 $p_Y(y)=\int_{-\infty}^{+\infty}p_2(x,y)\,dx=\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\frac{5}{4}(x^2+y)\,dx=\frac{5}{6}(1-y)^{\frac{3}{2}}+\frac{5}{2}y(1-y)^{1/2}=(\frac{5}{6}+\frac{5}{3}y)(1-y)^{\frac{1}{2}}$

y 取其它值 $p_Y(y)=\int_{-\infty}^{+\infty}p_2(x,y)\,dy=0$

综上：$p_X(x)=\begin{cases}\frac{5}{8}(1-x^4), & -1<x<1\\ 0, & else\end{cases}$ $p_Y(y)=\begin{cases}(\frac{5}{6}+\frac{5}{3}y)(1-y)^{\frac{1}{2}}, & 0<y<1\\ 0, & else.\end{cases}$

(3) $0<x<1$ 时 $p_X(x)=\int_{-\infty}^{+\infty}p_3(x,y)\,dy=\int_0^x\frac{1}{x}\,dy=1$

x 取其它值 $p_X(x)=\int_{-\infty}^{+\infty}p_3(x,y)\,dy=0$.

$0<y<1$ 时 $p_Y(y)=\int_{-\infty}^{+\infty}p_3(x,y)\,dx=\int_y^1\frac{1}{x}\,dx=-\ln y$

y 取其它值 $p_Y(y)=\int_{-\infty}^{+\infty}p_3(x,y)\,dy=0$.

综上：$p_X(x)=\begin{cases}1, & 0<x<1\\ 0, & else\end{cases}$ $p_Y(y)=\begin{cases}-\ln y, & 0<y<1\\ 0, & else\end{cases}$

习题 3.3

2. X,Y独立分布，则联合概率密度函数

$$p(x,y) = p_X(x)\cdot p_Y(y) = \lambda e^{-\lambda x}\cdot \mu e^{-\mu y}\cdot I_{x\geq 0} I_{y\geq 0} = \lambda\mu e^{-(\lambda x+\mu y)} I_{x,y\geq 0}$$

则 $P(X\leq Y) = \iint_{x\leq y} p(x,y)\,dx\,dy = \int_0^{+\infty}\int_0^{y} \lambda\mu e^{-(\lambda x+\mu y)}\,dx\,dy$

$$= \int_0^{+\infty} \mu e^{-\mu y}(1-e^{-\lambda y})\,dy$$

$$= \int_0^{+\infty} \mu e^{-\mu y}\,dy - \int_0^{+\infty} \mu e^{-(\mu+\lambda)y}\,dy$$

$$= 1-\frac{\mu}{\mu+\lambda} = \frac{\lambda}{\mu+\lambda}$$

则 $P(Z=1) = P(X\leq Y) = \frac{\lambda}{\mu+\lambda}$，$P(Z=0) = 1-P(Z=1) = \frac{\mu}{\mu+\lambda}$

则 Z 分布列

Z	0	1
P	$\frac{\mu}{\mu+\lambda}$	$\frac{\lambda}{\mu+\lambda}$

6. (1) $z>0$ $F_Z(z) = P(Z\leq z) = P(X+Y\leq 2z)$

$$= \iint_{x+y\leq 2z} p(x,y)\,dx\,dy = \int_0^{2z}\int_0^{2z-y} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^{2z} e^{-y}(1-e^{y-2z})\,dy$$

$$= \int_0^{2z} e^{-y}\,dy - \int_0^{2z} e^{-2z}\,dy$$

$$= 1-e^{-2z}-2ze^{-2z}$$

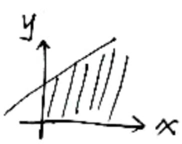
$z>0$, $p_Z(z) = F_Z'(z) = 2e^{-2z}-2e^{-2z}+4ze^{-2z} = 4ze^{-2z}$

综上： $p_Z(z) = \begin{cases} 4ze^{-2z}, & z>0 \\ 0, & z\leq 0. \end{cases}$

(2) $F_Z(z) = P(Z\leq z) = P(Y-X\leq z)$

$$= \iint_{y-x\leq z} p(x,y)\,dx\,dy$$

① $z\geq 0$ $F_Z(z) = \iint_{y-x\leq z} p(x,y)\,dx\,dy = \int_0^{+\infty}\int_0^{x+z} e^{-(x+y)}\,dy\,dx$



$$= \int_0^{+\infty} e^{-x}(1-e^{-x-z})\,dx = 1-\frac{1}{2}e^{-z}$$

② $z<0$ $F_Z(z) = \iint_{y-x\leq z} p(x,y)\,dx\,dy = \int_0^{+\infty}\int_{y-z}^{+\infty} e^{-(x+y)}\,dx\,dy$

$$= \int_0^{+\infty} e^{z-2y}\,dy = \frac{1}{2}e^{z}$$

综合①② $p_Z(z) = \begin{cases} \frac{1}{2}e^{-z}, & z\geq 0 \\ \frac{1}{2}e^{z}, & z<0 \end{cases} = \frac{1}{2}e^{-|z|}$

9. (1) $X\sim U(0,1)$, $Y\sim U(0,1)$

$z>2$: $p_Z(z) = \int_{-\infty}^{+\infty} p_X(z-y)p_Y(y)\,dy = 0$

$2\geq z>1$: $p_Z(z) = \int_{z-1}^{1} p_X(z-y)p_Y(y)\,dy = \int_{z-1}^{1} 1\,dy = 2-z$

$1\geq z>0$: $p_Z(z) = \int_0^{z} p_X(z-y)p_Y(y)\,dy = \int_0^{z} 1\,dy = z$.

$z\leq 0$: $p_Z(z) = \int_{-\infty}^{+\infty} p_X(z-y)p_Y(y)\,dy = 0$.

综上： $p_Z = \begin{cases} 2-z, & 1<z\leq 2 \\ z, & 0<z\leq 1 \\ 0, & else \end{cases}$

(2)

$p_Z(z) = \int_{-\infty}^{+\infty} p_X(x) p_Y(z-x)\,dx$ ，其中 $p_Y = e^{x-z} I_{z \geq x}$

$z \geq 1$： $p_Z(z) = \int_0^1 p_X(x) p_Y(z-x)\,dx$

$= \int_0^1 e^{x-z}\,dx = e^{-z}(e-1)$

$1 \geq z > 0$： $p_Z(z) = \int_0^z p_X(x) p_Y(z-x)\,dx$

$= \int_0^z e^{x-z}\,dx = e^{-z}(e^z - 1) = 1 - e^{-z}$

$z \leq 0$： $p_Z(z) = \int_{-\infty}^{+\infty} p_X(x) p_Y(z-x)\,dx = 0$

综上： $p_Z(z) = \begin{cases} e^{-z}(e-1), & z > 1 \\ 1 - e^{-z}, & 1 \geq z > 0 \\ 0, & \text{else} \end{cases}$.