概统 第十一周作业

习题 3.4

31. $Var\,Y = Var(aX_1+bX_2) = a^2 Var X_1 + b^2 Var X_2 = (a^2+b^2)\sigma^2$

$Var\,Z = Var(aX_1-bX_2) = a^2 Var X_1 + b^2 Var X_2 = (a^2+b^2)\sigma^2$.

$Cov(Y,Z) = Cov(aX_1+bX_2,\ aX_1-bX_2)$

$= a^2 Cov(X_1,X_1) - b^2 Cov(X_2,X_2)$

$= (a^2-b^2)\sigma^2$

$r(Y,Z) = \dfrac{Cov(Y,Z)}{\sqrt{VarY,\ VarZ}} = \dfrac{a^2-b^2}{a^2+b^2}$

32. (1) $E(\max\{X,Y\}) = E\left(\dfrac{X+Y+|X-Y|}{2}\right) = \frac{1}{2}E(|X-Y|)$

令 $Z=X-Y$ $\quad EZ = EX-EY=0 \quad Var Z = Var X + Var Y - 2Cov(X,Y) = 2-2\rho$

因此 $Z\sim(0,\ 2-2\rho)$

$E(|X-Y|) = \int_{-\infty}^{+\infty}|z|\,p(z)\,dz = \int_{-\infty}^{+\infty}|z|\frac{1}{\sqrt{4\pi(1-\rho)}}e^{-\frac{z^2}{4(1-\rho)}}dz$

$= 2\int_0^{+\infty} z\frac{1}{\sqrt{4\pi(1-\rho)}}e^{-\frac{z^2}{4(1-\rho)}}dz = \frac{1}{\sqrt{4\pi(1-\rho)}}4(1-\rho)\int_0^{+\infty}e^{-\frac{z^2}{4(1-\rho)}}d\left(\frac{z^2}{4(1-\rho)}\right)$

$= \frac{2\sqrt{1-\rho}}{\sqrt{\pi}}\times 1 = 2\sqrt{\frac{1-\rho}{\pi}}$

$E(\max\{X,Y\}) = \frac{1}{2}E(|X-Y|) = \sqrt{\frac{1-\rho}{\pi}}$.

(2) $Cov(X-Y,\ XY) = Cov(X,XY) - Cov(Y,XY)$

$= EX^2Y - EXEXY - EXY^2 - EYEXY$

$= EX^2Y - EXY^2 = 0$ (积分函数是奇函数)

因此 $Corr(X-Y,XY)=0$.

38. $Cov(X_1,X_2) = Cov\left(-\frac{bX_2+cX_3}{a},\ X_2\right) = -\frac{b}{a}Cov(X_2,X_2) - \frac{c}{a}Cov(X_2,X_3)$

即 $Cov(X_1,X_2) = -\frac{b}{a}\sigma^2 - \frac{c}{a}Cov(X_2,X_3)$

$\Rightarrow\ a\,Cov(X_1,X_2) + c\,Cov(X_2,X_3) = -b\sigma^2$

同理: $a\,Cov(X_1,X_3) + b\,Cov(X_2,X_3) = -c\sigma^2$

$b\,Cov(X_1,X_2) + c\,Cov(X_1,X_3) = -a\sigma^2$

解得 $\rho_{12} = \dfrac{Cov(X_1,X_2)}{\sigma^2} = \dfrac{a^2-b^2-c^2}{2ab}$

$\rho_{13} = \dfrac{b^2-a^2-c^2}{2ac}$

$\rho_{23} = \dfrac{a^2-b^2-c^2}{2bc}$

42. $Cov(\vec{X})$ 是半正定的 $\Rightarrow$ 特征值均为非负的 $\Rightarrow \det Cov(\vec{X}) \geq 0$.

$\Rightarrow \det\begin{pmatrix} Var X_1 & Cov(X_1,X_2) & Cov(X_1,X_3) \\ Cov(X_1,X_2) & Var X_2 & Cov(X_2,X_3) \\ Cov(X_1,X_3) & Cov(X_2,X_3) & Var X_3 \end{pmatrix} \geq 0$

$\Rightarrow Var X_1 Var X_2 Var X_3 - Var X_1 Cov^2(X_2,X_3) - Var X_2 Cov^2(X_1,X_3) - Var X_3 Cov^2(X_1,X_2) + 2\,Cov(X_1,X_2)\,Cov(X_1,X_3)\,Cov(X_2,X_3) \geq 0$

$\Rightarrow 1 + \dfrac{Cov(X_1,X_2)\,Cov(X_1,X_3)\,Cov(X_2,X_3)}{Var X_1\ Var X_2\ Var X_3} = \dfrac{Cov^2(X_2,X_3)}{Var X_2\ Var X_3} + \dfrac{Cov^2(X_1,X_3)}{Var X_1\ Var X_3} + \dfrac{Cov(X_1,X_2)}{Var X_1\ Var X_2}$.

$\Rightarrow 1 + 2\rho_{12}\rho_{13}\rho_{23} \geq \rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2$

43. Y_1, Y_2, Y_3 两两不相关

$\Leftrightarrow$ $Cov(Y_1, Y_2) = Cov(X_1+X_2, X_2+X_3) = Cov(X_1,X_2) + Cov(X_1,X_3) + Cov(X_2,X_3) + VarX_2$

$= \sum_{i<j} Cov(X_i, X_j) + \sigma^2 = 0$.

$Cov(Y_1, Y_3) = \sum_{i<j} Cov(X_i, X_j) + \sigma^2 = 0$.

$Cov(Y_2, Y_3) = \sum_{i<j} Cov(X_i, X_j) + \sigma^2 = 0$.

$\Leftrightarrow$ $\sum_{i<j} Cov(X_i, X_j) + \sigma^2 = 0$

$\Leftrightarrow$ $\sum_{i<3} \frac{Cov(X_i, X_j)}{\sigma^2} = -1$

$\Leftrightarrow$ $\rho_{12} + \rho_{13} + \rho_{23} = -1$ $\square$.

习题 4.1

1. $\{|X-Y| \geq \varepsilon\} = \{|(X-X_n)-(Y-Y_n)| \geq \varepsilon\}$

$\subseteq \{|X-X_n| \geq \frac{\varepsilon}{2}\} \cup \{|Y-Y_n| \geq \frac{\varepsilon}{2}\}$

$\Rightarrow P(|X-Y| \geq \varepsilon) \leq P(|X-X_n| \geq \frac{\varepsilon}{2}) + P(|Y-Y_n| \geq \frac{\varepsilon}{2}) \to 0 \quad (n \to \infty)$

因此 $P(|X-Y| \geq \varepsilon) = 0$

$P(X \neq Y) = P(\bigcup_{n=1}^{\infty} |X-Y| > \frac{1}{n}) = \sum_{n=1}^{\infty} P(|X-Y| > \frac{1}{n}) = 0$

$\Rightarrow P(X=Y) = 1$

6. (1) $\{D(x+n)\}$

$F(x) = \lim_{n\to\infty} D(x+n) = D(+\infty) = 1$

故 $F(-\infty) = 1$，F 不是分布函数

(2) $\{D(x+\frac{1}{n})\}$

$F(x) = \lim_{n\to\infty} D(x+\frac{1}{n}) = D(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases}$ 是分布函数

(3) $\{D(x-\frac{1}{n})\}$

$F(x) = \lim_{n\to\infty} D(x-\frac{1}{n}) = D(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x>0 \end{cases}$ 无右连续性，不是分布函数.

13.

$0<\varepsilon<\beta$ 时 $P(|Y_n-\beta| \geq \varepsilon) = P\{|\max\{X_1 \cdots X_n\} - \beta| \geq \varepsilon\}$

$= P\{\beta - \max\{X_1 \ldots X_n\} \geq \varepsilon\}$

$= P\{\beta - X_1 \geq \varepsilon, \beta - X_2 \geq \varepsilon, \ldots, \beta - X_n \geq \varepsilon\}$

$= P(\beta-X_1 \geq \varepsilon) P(\beta-X_2 \geq \varepsilon) \cdots P(\beta - X_n \geq \varepsilon)$

$= [P(X_1 \leq \beta-\varepsilon)]^n = (\frac{\beta-\varepsilon}{\beta})^n \to 0 \quad (n\to\infty)$

$\varepsilon \geq \beta$ 时 $P(|Y_n-\beta| \geq \varepsilon) = [P(X_1 \leq \beta-\varepsilon)]^n \leq [P(X_1 \leq 0)]^n = 0^n = 0$

因此 $Y_n \xrightarrow{P} \beta$