线性代数第五次作业

1. (a) $\det\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} = a^2b^2 - a^2b^2 = 0$

$\det\begin{bmatrix} n+1 & n \\ n & n-1 \end{bmatrix} = n^2-1-n^2 = -1$

$\det\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} = (a^2+2ab+b^2)-(a^2-2ab+b^2) = 4ab$

(b) $\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} = 1\det\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = 1$

$\det\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \det\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 1\det\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2$

$\det\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = a\det\begin{bmatrix} c & a \\ a & b \end{bmatrix} - b\det\begin{bmatrix} b & c \\ a & b \end{bmatrix} + c\begin{bmatrix} b & c \\ c & a \end{bmatrix}$

$= a(bc-a^2) - b(b^2-ac) + c(ab-c^2)$

$= 3abc - a^3 - b^3 - c^3$

(c) $\det\begin{bmatrix} 1 & 5 & 6 & 8 \\ 4 & 3 & 4 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 6 & 9 \end{bmatrix} = \det\begin{bmatrix} 0 & 4 & 0 & -1 \\ 0 & -1 & -20 & -30 \\ 0 & 1 & -3 & -5 \\ 1 & 1 & 6 & 9 \end{bmatrix} = -\det\begin{bmatrix} 4 & 0 & -1 \\ -1 & -20 & -30 \\ 1 & -3 & -5 \end{bmatrix} = \det\begin{bmatrix} 4 & 0 & -1 \\ 1 & 20 & 30 \\ 1 & -3 & -5 \end{bmatrix}$

$= 4\det\begin{bmatrix} 20 & 30 \\ -3 & -5 \end{bmatrix} - \det\begin{bmatrix} 1 & 20 \\ 1 & -3 \end{bmatrix}$

$= 4\times(-10) - (-23) = -17$

2、先证充分性　$\text{rank}\,A < 2 \Rightarrow \det A = 0$

设 $A = EA'$，其中E是初等矩阵，A'是行约化阶梯形式。

$\text{ran}A < 2$　则 A 最后一行全为0

$\therefore \det A' = 0$　而 $\det A = \det E \cdot \det A'$

则 $\det A = 0$

再证必要性　$\det A = 0 \Rightarrow \text{rank}\,A < 2$

设 $A = EA'$，其中 E是初等矩阵，A'是行约化阶梯形式

$\det A = \det E \cdot \det A' = 0$

$\det E \neq 0$ 则 $\det A' = 0$

反证，假设 $\text{rank}A = 2$　则　$A' = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$　$\det A' = 1-0 = 1$ 与 $\det A' = 0$ 矛盾

则假设不成立　$\text{rank}\,A = \text{rank}\,A' < 2$

3、$\det A = \pm\det A_{11} \pm \det A_{21} \pm \det A_{31}$

而对于2×2矩阵 $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc$　考虑到 $ad=\pm1$　$bc=\pm1$　则 $\det A_{ij} = 2$；0或-2

则 $\det A = 6, 4, 2, 0, -2, -4, -6$　$\Rightarrow \det A \le 6$

①假设$\det A = 6$　则 $\det A_{11}, \det A_{21}, \det A_{31}$ 全取±2　考虑到 $\det A_{ij} = \pm2$ 充要条件是abcd中三个相等,一个不等

意味着第二、三列中 任两行元素均为三个相等另一个不等

设第一、二行满足，一、三行满足，则二、三行必有两个相等两个不等. 即A_{11}中必有一者为0

$\det A = 6$ 是不能实现的

②假设$\det A = 4$　则可以找到这样的矩阵

$\det\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \end{bmatrix} = \det\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \det\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} - \det\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2+2+0 = 4$　即行列式取的最大值为4

4、(1) 引理1：上/下三角矩阵行列式值为对角元素之积。

证：$\det\begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn}\end{bmatrix} = a_{11}\det\begin{bmatrix} a_{22} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{2n} & \cdots & a_{nn}\end{bmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$ 下三角矩阵亦同理.

引理2：分块倍乘矩阵 $S(A)$：$\det(S(A)) = \det(A)$

$$\det(S_{ij}(A)) = \det\begin{array}{c} \begin{bmatrix} 1 & \cdots & & \cdots & 0 \\ \vdots & & A & & \vdots \\ 0 & \cdots & & \cdots & 1 \end{bmatrix} i \\ j \end{array} = \det\begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = \det[A] = \det(A).$$

分块消元矩阵 $E_{ij}(A)$：$\det(E_{ij}(A)) = 1$

$$\det(E_{ij}(A)) = \det\begin{bmatrix} 1 & & & \\ & 1 & & \\ & A & 1 & \\ & & & 1 \end{bmatrix} = 1 \quad \text{(上/下三角矩阵)}$$

证明：

$$\det M = \det(A^{-1}A)\det(M) = \det(A^{-1})\det\begin{bmatrix} I & 0 \\ 0 & A\end{bmatrix}\det\begin{bmatrix} A & B \\ C & D\end{bmatrix}$$

$$= \det(A^{-1})\det\begin{bmatrix} A & B \\ AC & AD\end{bmatrix} = \det(A^{-1})\det\begin{bmatrix} I & 0 \\ -C & I\end{bmatrix}\det\begin{bmatrix} A & B \\ AC & AD\end{bmatrix}$$

$$= \det(A^{-1})\det\begin{bmatrix} A & B \\ 0 & AD-CB\end{bmatrix} = \det(A^{-1})\det\begin{bmatrix} A & B \\ 0 & AD-CB\end{bmatrix}$$

$$= \det\begin{bmatrix} A^{-1} & 0 \\ 0 & I\end{bmatrix}\det\begin{bmatrix} A & B \\ 0 & AD-CB\end{bmatrix} = \det\begin{bmatrix} I & A^{-1}B \\ 0 & AD-CB\end{bmatrix}$$

$$\xrightarrow[\text{连续递归}]{\text{对第一列展开}} \det[AD-CB]$$

(2) 反例 [第1题第(c)问]

$M = \begin{bmatrix} 1 & 5 & 6 & 8 \\ 4 & 3 & 4 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 6 & 9 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 5 \\ 4 & 3\end{bmatrix}$ $\quad B = \begin{bmatrix} 6 & 8 \\ 4 & 6\end{bmatrix}$

$C = \begin{bmatrix} 1 & 2 \\ 1 & 1\end{bmatrix}$ $\quad D = \begin{bmatrix} 3 & 4 \\ 6 & 9\end{bmatrix}$

有 $AC = \begin{bmatrix} 6 & 7 \\ 7 & 11\end{bmatrix} \neq CA = \begin{bmatrix} 9 & 11 \\ 5 & 7\end{bmatrix}$ 且 A 可逆

$\det M = -17$

而 $AD - CB = \begin{bmatrix} 33 & 49 \\ 30 & 43\end{bmatrix} - \begin{bmatrix} 14 & 20 \\ 10 & 14\end{bmatrix} = \begin{bmatrix} 19 & 29 \\ 20 & 29\end{bmatrix}$

$\det(AD-CB) = -29$

$\therefore \det M \neq \det(AD-CB)$

5、(a)

$$|M| = \begin{vmatrix} A & B \\ C & D\end{vmatrix} = \begin{vmatrix} \begin{matrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm}\end{matrix} & B \\ 0 & C \end{vmatrix} = \sum_{i=1}^{m}(-1)^{i+1}a_{i1}\begin{vmatrix} A_{i1} & B' \\ 0 & C\end{vmatrix} = \sum_{\substack{i=1 \\ (i\neq j)}}^{m}\sum_{j=1}^{m}(-1)^{i+j}a_{i1}a_{j2}\begin{vmatrix} (A_{i1})_{j2} & B'' \\ 0 & C\end{vmatrix}$$

$$= \cdots = \left(\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_m=1}^{m}(-1)^{\sum_{\alpha=1}^{m} i_\alpha}\prod_{\alpha=1}^{m}a_{i_\alpha,\alpha}\right)\det C = |A||C|$$

(b) 同理：$\begin{vmatrix} A & 0 \\ C & D\end{vmatrix} = |A||D|$

a) 由4题引理2：消元矩阵行列式为1

$$|M| = \begin{vmatrix} A & B \\ C & D\end{vmatrix} = \begin{vmatrix} I & -BD^{-1} \\ 0 & I\end{vmatrix}\begin{vmatrix} A & B \\ C & D\end{vmatrix} = \begin{vmatrix} A-BD^{-1}C & 0 \\ C & D\end{vmatrix} = |D||A-BD^{-1}C|$$

b)

$$|M| = \begin{vmatrix} A & B \\ C & D\end{vmatrix} = \begin{vmatrix} I & 0 \\ -CA^{-1} & I\end{vmatrix}\begin{vmatrix} A & B \\ C & D\end{vmatrix} = \begin{vmatrix} A & B \\ 0 & D-CA^{-1}B\end{vmatrix} = |A||D-CA^{-1}B|$$

6.

(1) $(M)_{11} = +[7] = +7$ $(M)_{12} = -[6] = -6$

$(M)_{21} = -[5] = -5$ $(M)_{22} = +[7] = 7$

$M = \begin{bmatrix}7 & -6\\ -5 & 4\end{bmatrix}$ $A^* = M^T = \begin{bmatrix}7 & -5\\ -6 & 4\end{bmatrix}$

$\det A = 28 - 30 = -2$

则 $A^{-1} = \frac{A^*}{\det A} = \begin{bmatrix}-\frac{7}{2} & \frac{5}{2}\\ 3 & -2\end{bmatrix}$

(2) $(M)_{11} = +\begin{vmatrix}7 & 5\\ 6 & 8\end{vmatrix} = 26$ $(M)_{12} = -\begin{vmatrix}6 & 5\\ 4 & 8\end{vmatrix} = -28$ $(M)_{13} = +\begin{vmatrix}6 & 7\\ 4 & 6\end{vmatrix} = 8$

$(M)_{21} = -\begin{vmatrix}5 & 6\\ 6 & 8\end{vmatrix} = -4$ $(M)_{22} = +\begin{vmatrix}4 & 6\\ 4 & 8\end{vmatrix} = 8$ $(M)_{23} = -\begin{vmatrix}4 & 5\\ 4 & 6\end{vmatrix} = -4$

$(M)_{31} = +\begin{vmatrix}5 & 6\\ 7 & 5\end{vmatrix} = -17$ $(M)_{32} = -\begin{vmatrix}4 & 6\\ 6 & 5\end{vmatrix} = 16$ $(M)_{33} = +\begin{vmatrix}4 & 5\\ 6 & 7\end{vmatrix} = -2$

$M = \begin{vmatrix}26 & -28 & 8\\ -4 & 8 & -4\\ -17 & 16 & -2\end{vmatrix}$ $A^* = \begin{vmatrix}26 & -4 & -17\\ -28 & 8 & 16\\ 8 & -4 & -2\end{vmatrix}$

$\det A = 4\begin{vmatrix}7 & 5\\ 6 & 8\end{vmatrix} - 6\begin{vmatrix}5 & 6\\ 6 & 8\end{vmatrix} + 4\begin{vmatrix}5 & 6\\ 7 & 5\end{vmatrix} = 104 - 24 - 68 = 12$

则 $A^{-1} = \frac{A^*}{\det A} = \begin{bmatrix}\frac{13}{6} & -\frac{1}{3} & -\frac{17}{12}\\ -\frac{7}{3} & \frac{2}{3} & \frac{4}{3}\\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{6}\end{bmatrix}$

7. (1) $\det A = 5 - 12 = -7$

$\det B_1 = \begin{vmatrix}2 & 3\\ 3 & 5\end{vmatrix} = 1$ $x_1 = \frac{\det B_1}{\det A} = -\frac{1}{7}$

$\det B_2 = \begin{vmatrix}1 & 2\\ 4 & 3\end{vmatrix} = -5$ $x_2 = \frac{\det B_2}{\det A} = \frac{5}{7}$

则 $x = \begin{bmatrix}-\frac{1}{7}\\ \frac{5}{7}\end{bmatrix}$

(2) $\det A = \begin{vmatrix}2 & 3 & 11 & 5\\ 1 & 1 & 5 & 2\\ 2 & 1 & 3 & 2\\ 1 & 1 & 3 & 4\end{vmatrix} = \begin{vmatrix}0 & 1 & 5 & -3\\ 0 & 0 & 2 & -2\\ 0 & -1 & -3 & -6\\ 1 & 1 & 3 & 4\end{vmatrix} = -\begin{vmatrix}1 & 5 & -3\\ 0 & 2 & -2\\ -1 & -3 & -6\end{vmatrix} = \begin{vmatrix}1 & 5 & -3\\ 0 & 2 & -2\\ 1 & 3 & 6\end{vmatrix} = \begin{vmatrix}2 & -2\\ 3 & 6\end{vmatrix} + \begin{vmatrix}5 & -3\\ 2 & -2\end{vmatrix}$
$= 18 - 4 = 14$

$\det B_1 = \begin{vmatrix}2 & 3 & 11 & 5\\ 1 & 1 & 5 & 2\\ -3 & 1 & 3 & 2\\ -3 & 1 & 3 & 4\end{vmatrix} = \begin{vmatrix}0 & 1 & 1 & 1\\ 1 & 1 & 5 & 2\\ 0 & 4 & 18 & 8\\ 0 & 4 & 18 & 10\end{vmatrix} = -\begin{vmatrix}1 & 1 & 1\\ 4 & 18 & 8\\ 4 & 18 & 10\end{vmatrix} = -\begin{vmatrix}1 & 1 & 1\\ 4 & 18 & 8\\ 0 & 0 & 2\end{vmatrix} = -2\begin{vmatrix}1 & 1\\ 4 & 18\end{vmatrix} = -28$

$x_1 = \frac{\det B_1}{\det A} = -2$

$\det B_2 = \begin{vmatrix}2 & 2 & 11 & 5\\ 1 & 1 & 5 & 2\\ 2 & -3 & 3 & 2\\ 1 & -3 & 3 & 4\end{vmatrix} = \begin{vmatrix}0 & 0 & 1 & 1\\ 1 & 1 & 5 & 2\\ 0 & -5 & -7 & -2\\ 0 & -4 & -2 & 2\end{vmatrix} = -\begin{vmatrix}0 & 1 & 1\\ -5 & -7 & -2\\ -4 & -2 & 2\end{vmatrix} = \begin{vmatrix}0 & 1 & 1\\ 5 & 7 & 2\\ -4 & -2 & 2\end{vmatrix} = -\begin{vmatrix}5 & 2\\ -4 & 2\end{vmatrix} + \begin{vmatrix}5 & 7\\ -4 & -2\end{vmatrix} = 0$

$x_2 = \frac{\det B_2}{\det A} = 0$

$\det B_3 = \begin{vmatrix}2 & 3 & 2 & 5\\ 1 & 1 & 1 & 2\\ 2 & 1 & -3 & 2\\ 1 & 1 & -3 & 4\end{vmatrix} = \begin{vmatrix}0 & 1 & 0 & 1\\ 1 & 1 & 1 & 2\\ 0 & -1 & -5 & -2\\ 0 & 0 & -4 & 2\end{vmatrix} = -\begin{vmatrix}1 & 0 & 1\\ -1 & -5 & -2\\ 0 & -4 & 2\end{vmatrix} = \begin{vmatrix}1 & 0 & 1\\ 1 & 5 & 2\\ 0 & -4 & 2\end{vmatrix} = \begin{vmatrix}5 & 2\\ -4 & 2\end{vmatrix} + \begin{vmatrix}1 & 5\\ 0 & -4\end{vmatrix} = 14$

$x_3 = \frac{\det B_3}{\det A} = 1$

$\det B_4 = \begin{vmatrix}2 & 3 & 11 & 2\\ 1 & 1 & 5 & 1\\ 2 & 1 & 3 & -3\\ 1 & 1 & 3 & -3\end{vmatrix} = \begin{vmatrix}0 & 1 & 1 & 0\\ 1 & 1 & 5 & 1\\ 0 & -1 & -7 & -5\\ 0 & 0 & -2 & -4\end{vmatrix} = -\begin{vmatrix}1 & 1 & 0\\ -1 & -7 & -5\\ 0 & -2 & -4\end{vmatrix} = \begin{vmatrix}1 & 1 & 0\\ 1 & 7 & 5\\ 0 & -2 & 4\end{vmatrix} = \begin{vmatrix}7 & 5\\ -2 & -4\end{vmatrix} - \begin{vmatrix}1 & 0\\ 2 & 4\end{vmatrix} = -14$

$x_3 = \frac{\det B_4}{\det A} = -1$

则 $x = \begin{bmatrix}-2\\ 0\\ 1\\ -1\end{bmatrix}$

8、

$$A=\begin{bmatrix}6&4&2\\2&2&2\\2&0&2\end{bmatrix}$$

$(M)_{11}=+\begin{vmatrix}2&2\\0&2\end{vmatrix}=4$ $(M)_{12}=-\begin{vmatrix}2&2\\2&2\end{vmatrix}=0$ $(M)_{13}=+\begin{vmatrix}2&2\\2&0\end{vmatrix}=-4$

$(M)_{21}=-\begin{vmatrix}4&2\\0&2\end{vmatrix}=-8$ $(M)_{22}=+\begin{vmatrix}6&2\\2&2\end{vmatrix}=8$ $(M)_{23}=-\begin{vmatrix}6&4\\2&0\end{vmatrix}=8$

$(M)_{31}=+\begin{vmatrix}4&2\\2&2\end{vmatrix}=4$ $(M)_{32}=-\begin{vmatrix}6&2\\2&2\end{vmatrix}=-8$ $(M)_{33}=+\begin{vmatrix}6&4\\2&2\end{vmatrix}=4$

$$M=\begin{bmatrix}4&0&-4\\-8&8&8\\4&-8&4\end{bmatrix}\qquad A^*=M^T=\begin{bmatrix}4&-8&4\\0&8&-8\\-4&8&4\end{bmatrix}$$

$(M')_{11}=+\begin{vmatrix}8&-8\\8&4\end{vmatrix}=96$ $(M')_{12}=-\begin{vmatrix}0&-8\\-4&4\end{vmatrix}=32$ $(M')_{13}=+\begin{vmatrix}0&8\\-4&8\end{vmatrix}=32$

$(M')_{21}=-\begin{vmatrix}-8&4\\8&4\end{vmatrix}=+64$ $(M')_{22}=+\begin{vmatrix}4&4\\-4&4\end{vmatrix}=32$ $(M')_{23}=-\begin{vmatrix}4&-8\\-4&8\end{vmatrix}=0$

$(M')_{31}=+\begin{vmatrix}-8&4\\8&-8\end{vmatrix}=32$ $(M')_{32}=-\begin{vmatrix}4&4\\0&-8\end{vmatrix}=32$ $(M')_{33}=+\begin{vmatrix}4&-8\\0&8\end{vmatrix}=32$

$$M'=\begin{bmatrix}96&32&32\\+64&32&0\\32&32&32\end{bmatrix}\qquad (A^*)^*=M'^T=\begin{bmatrix}96&+64&32\\32&32&32\\32&0&32\end{bmatrix}$$

9. (a) 证明：

设 $A^*A=C$

当 $i=j$ 时 $C_{ii}=\sum_{k=1}^{n}(A^*)_{ik}a_{ki}=\sum_{k=1}^{n}(M)_{ki}a_{ki}=\sum_{k=1}^{n}(-1)^{k+i}\det A_{ki}\cdot a_{ki}=|A|$

当 $i\neq j$ 时 $C_{ij}=\sum_{k=1}^{n}(A^*)_{ik}a_{kj}=\sum_{k=1}^{n}(M)_{ki}a_{kj}=\sum_{k=1}^{n}(-1)^{k+i}\det A_{ki}\cdot a_{kj}$

而 $\det A_{ki}=\sum_{l=1}^{n}a_{lj}(-1)^{syn}\det(A_{ki})_{lj}$ （$l\neq k$，其中 $k>l$ 与 $k<l$ 符号相反）

$\therefore C_{ij}=\sum_{k=1}^{n}\sum_{l=1}^{n}(-1)^{k+i}(-1)^{syn}a_{kj}a_{lj}\det(A_{ki})_{lj}$ （$l\neq k$，其中 $k>l$ 与 $k<l$ 符号相反）

则对于某对 k 与 l 时符号相反则相互抵消

$C_{ij}=0$

综上：$A^*A=|A|I_{n\times n}$

同理 $AA^*=|A|I_{n\times n}$

$\therefore A^*A=AA^*=|A|I_{n\times n}$

(b) 证明：先证 A 可逆 $\Rightarrow$ A^* 可逆

由(a) $AA^*=A^*A=|A|I_{n\times n}$ 且 $|A|\neq 0$

$\left(\frac{A}{|A|}\right)A^*=A^*\left(\frac{A}{|A|}\right)=I_{n\times n}$

$\therefore \frac{A}{|A|}=(A^*)^{-1}$ 则 A^* 可逆

再证 A^* 可逆 $\Rightarrow$ A 可逆

反证法，假设 A 不可逆，则 $|A|=0$ $AA^*=A^*A=0$

$\Rightarrow A=0\cdot(A^*)^{-1}=0$

$\Rightarrow A^*=0$ $\Rightarrow$ A^* 不可逆与条件矛盾！

$\therefore$ A 可逆

(c) 证明：由(a) $AA^*=|A|I_{n\times n}$

对两边取行列式 $|A||A^*|=\begin{vmatrix}|A|&&0\\&\ddots&\\0&&|A|\end{vmatrix}=|A|\begin{vmatrix}1&&0\\&|A|\ddots&\\0&&|A|\end{vmatrix}=\cdots=|A|^n$

$\Rightarrow |A^*|=|A|^{n-1}$

(d)

$AA^* = |A| I_{n\times n}$

$\Rightarrow A^* = |A| A^{-1}$

类似地有 $A^*(A^*)^* = |A^*| I_{n\times n}$

$= |A|^{n-1} I_{n\times n}$.

由(b) A可逆则 A^*可逆

$(A^*)^* = |A|^{n-1}(A^*)^{-1} = |A|^{n-1}(|A|A^{-1})^{-1} = |A|^{n-2} A$

10、证明: ①性质1: 单位矩阵 $|I_{n\times n}| = 1$

对第i列 仅有 $a_{ii}=1$ 其余为0.

$|I_{n\times n}| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}|A_{ij}| = a_{ii}|A_{ii}| = a_{ii}|I_{(n-1)\times(n-1)}| = |I_{(n-1)\times(n-1)}| = \cdots = |I_{1\times 1}| = 1$

对第i行同理

②性质2: 对于行来说, 行列式的作用是线性的

假设 $D = [cA_k + c'B_k]$ $A = [A_k]$ $B = [B_k]$ 即证 $|D| = c|A| + c'|B|$

对第i列展开 $|D| = \sum_{\nu=1}^{n} d_{\nu i}(-1)^{\nu+i}|D_{\nu i}|$ 即证$\forall \nu$ 有 $d_{\nu i}(-1)^{\nu+i}|D_{\nu i}| = c(a_{\nu i}(-1)^{\nu+i}|A_{\nu i}|) + c'(b_{\nu i}(-1)^{\nu+i}|B_{\nu i}|)$

$\nu = k$ 时 $\begin{cases} d_{\nu i} = ca_{\nu i} + c'b_{\nu i} \\ |D_{\nu i}| = |A_{\nu i}| = |B_{\nu i}| \end{cases}$

$\Rightarrow d_{\nu i}(-1)^{\nu+i}|D_{\nu i}| = c(a_{\nu i}(-1)^{\nu+i}|A_{\nu i}|) + c'(b_{\nu i}(-1)^{\nu+i}|B_{\nu i}|)$

$\nu \neq k$ 时 $d_{\nu i} = a_{\nu i} = b_{\nu i}$

欲证 $d_{\nu i}(-1)^{\nu+i}|D_{\nu i}| = c(a_{\nu i}(-1)^{\nu+i}|A_{\nu i}|) + c'(b_{\nu i}(-1)^{\nu+i}|B_{\nu i}|)$

即证 $|D_{\nu i}| = c|A_{\nu i}| + c'|B_{\nu i}|$

如此将矩阵成功降阶, 改为证一个低阶矩阵是否满足线性关系

可知经有限次降阶后必有$\nu = k$, 因此综上 $|D_{\nu i}| = c|A_{\nu i}| + c'|B_{\nu i}|$ 得证.

则 $|D| = c|A| + c'|B|$ 得证

对第i行展开 同理分情况讨论, 若 $i = k$ 则

$\begin{cases} d_{i\nu} = ca_{i\nu} + c'b_{i\nu} \\ |D_{i\nu}| = |A_{i\nu}| = |B_{i\nu}| \end{cases}$ 显然有 $d_{\nu i}(-1)^{\nu+i}|D_{\nu i}| = c(a_{\nu i}(-1)^{\nu+i}|A_{\nu i}|) + c'(b_{\nu i}(-1)^{\nu+i}|B_{\nu i}|)$

$\Rightarrow |D| = c|A| + c'|B|$

若 $i \neq k$ 则 $d_{\nu i} = a_{\nu i} = b_{\nu i}$

即证 $|D_{\nu i}| = c|A_{\nu i}| + c'|B_{\nu i}|$

同理经有限次降阶, 当 $i = k$ 时得证

③性质3: 若两行相等则行列式为0

设 $D = \begin{bmatrix} w_p \\ w_q \end{bmatrix}$ 且 $w_p = w_q$, 即证 $|D| = 0$

对第i行展开 $|D| = \sum_{\nu=1}^{n} d_{i\nu}(-1)^{\nu+i}|D_{i\nu}|$

→当 $i = p$ 或 q 不妨设 $i = p$ 则 $|D| = \sum_{\nu=1}^{n} d_{p\nu}(-1)^{\nu+i}|D_{p\nu}|$

$= \sum_{\nu=1}^{n} d_{q\nu}(-1)^{\nu+i}|D_{p\nu}|$

而与9(a)类似地证明: 某行的元素与异行的代数余子式相乘值为0

$\left(\begin{array}{l} |D_{p\nu}| = \sum_{\alpha=1}^{n} d_{q\alpha}(-1)^{syn}\det(D_{p\nu})_{q\alpha} \quad (\alpha \neq \nu, \alpha > \nu \text{与} \alpha < \nu\ (-1)^{syn} \text{符号相反}) \\ |D| = \sum_{\nu=1}^{n}\sum_{\alpha=1}^{n} d_{q\nu} d_{q\alpha}(-1)^{\nu+i}(-1)^{syn}\det(D_{p\nu})_{q\alpha} = 0 \end{array} \right)$

$\therefore |D| = 0$

→当 $i \neq p$ 或 q 只需证明 $|D_{i\nu}| = 0$

如此将矩阵降阶, 改为证一个低阶矩阵是否在相同条件下满足 $|D| = 0$.

而经有限次降阶, 在二阶时 $\begin{vmatrix} a & a \\ b & b \end{vmatrix} = 0$ 显然成立, 经递归反推 $|D| = 0$ 也成立

对第i列展开：$|D|=\sum_{\nu=1}^{n} d_{\nu i}(-1)^{\nu+i}|D_{\nu i}|$

当 $\nu\neq p$ 或 q 时，由前所证，继续对任一行展开均有 $|D_{\nu i}|=0$，因此 $|D|=0$.

当 $\nu=p$ 或 q 时 只需证 $d_{pi}(-1)^{p+i}|D_{pi}|+d_{qi}(-1)^{q+i}|D_{qi}|=0$. 假设 $p<q$.

$$|D_{pi}|=\sum_{\alpha=1}^{n}(-1)^{q+i+1}d_{q\alpha}|(D_{pi})_{q\alpha}|$$

$$|D_{qi}|=\sum_{\alpha=1}^{n}(-1)^{p+i}d_{p\alpha}|(D_{qi})_{p\alpha}|$$

$$\therefore d_{pi}(-1)^{p+i}|D_{pi}|+d_{qi}(-1)^{q+i}|D_{qi}|=\sum_{\alpha=1}^{n}\left[(-1)^{p+q+2i+1}d_{pi}d_{q\alpha}|(D_{pi})_{q\alpha}|+(-1)^{p+q+2i}d_{qi}d_{p\alpha}|(D_{qi})_{p\alpha}|\right]$$

又 $\because d_{pi}=d_{qi}$，$d_{q\alpha}=d_{p\alpha}$　$|(D_{pi})_{q\alpha}|=|(D_{qi})_{p\alpha}|$

$\therefore d_{pi}(-1)^{p+i}|D_{pi}|+d_{qi}(-1)^{q+i}|D_{qi}|=0$

因此 $|D|=0$ 得证

综合以上：行列式用任何行或任何列展开的公式满足行列式函数的三个性质

11. 证明：由9题(a)　$AA^*=A^*A=|A|I_{n\times n}$.

A不可逆则 $|A|=0 \Rightarrow AA^*=0$

则 $\mathrm{Col}\,A^*\subset \mathrm{Nul}\,A$

$\Rightarrow \mathrm{rank}\,A^*\leq \dim\mathrm{Nul}\,A=n-\mathrm{rank}\,A$

$\Rightarrow \mathrm{rank}\,A+\mathrm{rank}\,A^*\leq n$

而A不可逆 $\Rightarrow$ A不满秩　$\mathrm{rank}\,A\leq n-1$

$\Rightarrow \mathrm{rank}\,A^*\leq n-(n-1)=1$

则 A^*的秩为0或1

12. 解

$$\lambda I-A=\begin{bmatrix}\lambda-2&1&0&0\\1&\lambda-2&1&0\\0&1&\lambda-2&1\\0&0&1&\lambda-2\end{bmatrix}$$

$$\det(\lambda I-A)=\begin{vmatrix}\lambda-2&1&0&0\\1&\lambda-2&1&0\\0&1&\lambda-2&1\\0&0&1&\lambda-2\end{vmatrix}=\begin{vmatrix}0&1-(\lambda-2)^2&2-\lambda&\\1&\lambda-2&1&\\&1&\lambda-2&1\\&&1&\lambda-2\end{vmatrix}$$

$$=-1\begin{vmatrix}1-(\lambda-2)^2&2-\lambda&\\1&\lambda-2&1\\&1&\lambda-2\end{vmatrix}=\begin{vmatrix}(\lambda-2)^2-1&\lambda-2&\\1&\lambda-2&1\\&1&\lambda-2\end{vmatrix}$$

$$=[(\lambda-2)^2-1]\begin{vmatrix}\lambda-2&1\\1&\lambda-2\end{vmatrix}-\begin{vmatrix}\lambda-2&0\\1&\lambda-2\end{vmatrix}$$

$$=[(\lambda-2)^2-1]^2-(\lambda-2)^2$$

$$=[(\lambda-2)^2+(\lambda-2)-1][(\lambda-2)^2-(\lambda-2)-1]$$