理科线性代数第七次作业

一、正定矩阵

1、pf：反证，假设正定矩阵 A 不可逆

则 $Ax=0$ 有非零解

$\Rightarrow x^TAx=0$ 与正定矩阵性质 $\forall x\neq 0$ 有 $x^TAx>0$ 矛盾

综上：正定矩阵可逆 #

2、pf：$\forall x\neq 0 \quad Ax\neq 0$

$\Rightarrow \|Ax\|^2\neq 0$

$\Rightarrow \|Ax\|>0$

$\Rightarrow (Ax)^TAx = x^TA^TAx = x^T(A^TA)x>0$ 而 $(A^TA)^T = A^T A^{TT} = A^TA$

$\therefore A^TA$ 是正定矩阵 #

3、pf：对于 $A^{-1}-I$ 的特征方程 $\lambda x = (A^{-1}-I)x$

$\Rightarrow A(A^{-1}-I)x = \lambda Ax$

$\Rightarrow (I-A)x = \lambda Ax$

$\Rightarrow x^T(I-A)x = \lambda x^TAx$

而 $\forall x\neq 0 \quad x^T(I-A)x>0$

$x^TAx>0$

则 $\lambda = \dfrac{x^T(I-A)x}{x^TAx}>0$

$\therefore A^{-1}-I$ 的所有特征值均为正

又 $(A^{-1}-I)^T = (A^{-1})^T - I^T = (A^T)^{-1}-I = A^{-1}-I$ 为对称矩阵

$\therefore A^{-1}-I$ 是对称矩阵 #

4. (a) pf：$\forall x\neq 0 \quad x^TSx>0$

不妨取 $x=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ $\quad x^TSx = S_{11}>0$ #

(b) pf：设 $S=\begin{bmatrix}a_{11} & a_{21} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \ddots\end{bmatrix}$ 则 $E_1=\begin{bmatrix}1 & 0 & \cdots & 0\\ -\frac{a_{21}}{a_{11}} & 1 & & \vdots \\ \vdots & & \ddots & \\ \frac{a_{n1}}{a_{11}} & \cdots & & 1\end{bmatrix}$ $S'=\begin{bmatrix}a_{11} & a_{21} & \cdots & a_{n1}\\ 0 & a_{22}-\frac{a_{21}^2}{a_{11}} & & a_{n2}-\frac{a_{n1}a_{21}}{a_{11}}\\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}-\frac{a_{n1}a_{21}}{a_{11}} & \cdots & a_{nn}-\frac{a_{n1}^2}{a_{11}}\end{bmatrix}$

$S = E_1^{-1}S'$

$\Rightarrow x^TSx = x^TE_1^{-1}S'x>0$ 不妨令 $x=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$

$x^TE_1S'x = [0\ 1\ 0\cdots 0]\begin{bmatrix}1 & 0 & \cdots\\ +\frac{a_{21}}{a_{11}} & 1 & \\ \vdots & & \ddots\end{bmatrix}\begin{bmatrix}a_{11} & a_{21} & \cdots\\ 0 & a_{22}-\frac{a_{21}^2}{a_{11}} & \\ \vdots & & \ddots\end{bmatrix}\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$

$= [+\frac{a_{21}}{a_{11}}\ 1\ 0\cdots 0]\begin{bmatrix}a_{21}\\ a_{22}-\frac{a_{21}^2}{a_{11}}\\ \vdots \\ a_{n2}-\frac{a_{n1}a_{21}}{a_{11}}\end{bmatrix}$

$= +\frac{a_{21}^2}{a_{11}} + a_{22} - \frac{a_{21}^2}{a_{11}} = a_{22}>0 \Rightarrow S_{22}>0$ #

(c) pf：$\forall 1\leq k\leq n$.

下证第 k 个主元也为正 设 $S=\begin{bmatrix}a_{11} & a_{21} & \cdots\\ a_{21} & a_{22} & \\ \vdots & & \ddots\end{bmatrix}$

$$E_{k-1}=\begin{array}{c} \downarrow \text{第}(k-1)\text{列} \\ \begin{bmatrix}1 & & & & 0\\ * & 1 & \ddots & & \\ \vdots & \vdots & & 1 & \\ \vdots & \vdots & & * & \ddots \\ * & * & \cdots & * & 1\end{bmatrix}\end{array}$$

$$S_{k-1}=\begin{bmatrix}a_{11} & * & \cdots & * & * & \cdots\\ 0 & a_{22} & & \vdots & \vdots & \cdots\\ \vdots & \vdots & \ddots & c_{(k-1)(k-1)} & & \\ \vdots & \vdots & & 0 & a_{kk} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & 0 & * & \end{bmatrix}$$

则 $\alpha^T S \alpha = \alpha^T E_{k-1} S_{k-1} \alpha > 0$

不妨设 $\alpha = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} k \end{array}$ $\quad \alpha^T E_{k-1} S_{k-1} \alpha = [0 \cdots \underset{k}{1} \cdots 0] \begin{bmatrix} * & & & 0 \\ \vdots & \ddots & & \\ \vdots & & 1 & \\ * & * \cdots & 0 & \ddots \\ & & & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & & & * \\ & a_{22}' & & \\ & & a_{kk}^{(k-1)} & \\ 0 & 0 & * & \ddots \\ & & * & a_{nn}^{(k-1)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} k$

$$= [-\frac{a_{k1}}{a_{11}} \; -\frac{a_{k2}'}{a_{22}'} \; -\frac{a_{k3}''}{a_{33}''} \cdots \underset{k}{1} \; 0 \cdots 0] \begin{bmatrix} a_{k1} \\ a_{k2}' \\ a_{k3}'' \\ \vdots \\ a_{kk}^{(k-1)} \\ * \\ \vdots \end{bmatrix} = -\frac{a_{k1}^2}{a_{11}} - \frac{a_{k2}'^2}{a_{22}'} - \cdots + a_{kk}^{(k-1)} > 0$$

而 $-\frac{a_{k1}^2}{a_{11}} < 0 \cdots$ 　则 $a_{kk}^{(k-1)} > 0$ 　即 $n=k$ 时也成立.

则 S 的所有主元都是正的 #

(d) 由 (c) 的证明 　$S = LU'$ 其中 L 为初等变换矩阵. U' 对角元均为正，不妨 $U' = \begin{bmatrix} u_1 & & & * \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{bmatrix}$

则令 $D = \begin{bmatrix} \frac{1}{u_1} & & & 0 \\ & \frac{1}{u_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{u_n} \end{bmatrix}$

则 $U' = \begin{bmatrix} u_1 & & * \\ & u_2 & \\ 0 & & \ddots \; u_n \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{u_1} & & 0 \\ & \frac{1}{u_2} & \\ 0 & & \ddots \; \frac{1}{u_n} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}}_{U}$

$\therefore S = LU' = LDU$ 可以 LDU 分解

(e) 归纳假设 k 阶对称矩阵可以写成 $ES'E^T$ 形式，其中 E 为初等行变换矩阵，S' 为对角矩阵形式，E^T 为初等列变换矩阵

对于 k 阶. $S = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}$ 　则根据假设 $\begin{bmatrix} a_{22} & \cdots & a_{k2} \\ \vdots & \ddots & \vdots \\ a_{k2} & \cdots & a_{nk} \end{bmatrix} = E_{k-1} S' F_{k-1}$

$$= \underbrace{\begin{bmatrix} 1 & & & 0 \\ \frac{a_{21}}{a_{11}} & 1 & & \\ \vdots & & \ddots & \\ \frac{a_{k1}}{a_{11}} & 0 & \cdots & 1 \end{bmatrix}}_{E_k'} \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & a_{k2} & \cdots & a_{kk} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \frac{a_{21}}{a_{11}} & \cdots & \frac{a_{k1}}{a_{11}} \\ & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}}_{E_k'^T}$$

$$= \underbrace{E_k' \begin{bmatrix} 1 & 0 \\ 0 & E_{k-1} \end{bmatrix}}_{E_k} \begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \; a_{kk} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & E_{k-1}^T \end{bmatrix} E_k'^T}_{E_k^T}$$

E_k、F_k 分别为初等行、列变换矩阵

则归纳成立. $S = ES'E^T$，其中 S' 对角元素为主元，全正.

$\forall x \neq 0$, $\alpha^T S \alpha = \alpha^T E S' E^T \alpha$
$= (E^T\alpha)^T S' (E^T \alpha)$

设 $E^T \alpha = y$ ，E^T 可逆则 $y \neq 0$.

$\therefore \alpha^T S x = y^T S' y = [y_1 \cdots y_n] \begin{bmatrix} a_{11}' & & 0 \\ & a_{22}' & \\ 0 & & \ddots \; a_{nn}' \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$
$= \sum y_i^2 a_{ii}' > 0$

$\therefore$ S 为正定矩阵

5. pf: 首先，将 A 通过消元得到 A'，则 A' 是上三角矩阵，$(A')_{ij}=a_{ij}$，det 在变换中值不变

必要性：采用数学归纳法，$n=1$ 时 $\det_1 = a_{11} > 0$ （$\det_i$ 为 i 阶顺序主子式）

假设 $n=k-1$ 时 $\forall 1\le i\le k-1$ 有 $a_{ii}>0$

则 $\det_k = a_{11}\cdot a_{22}\cdots a_{kk} > 0$ 而 $a_{ii}>0 \ \forall 1\le i\le k-1$

则 $a_{kk}>0$

综上：每一个主元值大于0，由4题，A为正定矩阵

充分性：A 为正定矩阵，则 $\det_k = a_{11}\cdot a_{22}\cdots a_{kk} > 0$

因此任一阶顺序主子式均大于0 #

6. (a) $S=\begin{bmatrix}5&2&-1\\2&1&-1\\-1&-1&\lambda\end{bmatrix}$

$$f = x_2^2 + 2x_2(2x_1 - x_3) + 5x_1^2 + \lambda x_3^2 - 2x_1x_3$$
$$= (x_2 + 2x_1 - x_3)^2 + x_1^2 + (\lambda-1)x_3^2 + 2x_1x_3$$

则 $x_1^2 + 2x_1x_3 + (\lambda-1)x_3^2 > 0$

$\Delta = 4 - 4(\lambda-1) = 8-4\lambda < 0 \Rightarrow \lambda > 2$

(b) $S=\begin{bmatrix}2&\lambda&1\\\lambda&1&0\\1&0&3\end{bmatrix}$

$$f = x_2^2 + 2\lambda x_1x_2 + \lambda^2x_1^2 + (2-\lambda^2)x_1^2 + 2x_1x_3 + 3x_3^2$$
$$= (x_2+\lambda x_1)^2 + 3x_3^2 + 2x_1x_3 + (2-\lambda^2)x_1^2$$

则 $3x_3^2 + 2x_1x_3 + (2-\lambda^2)x_1^2 > 0$

$\Delta = 4 - 4\times 3(2-\lambda^2) < 0 \Rightarrow \lambda^2 < \frac{5}{3} \Rightarrow -\frac{\sqrt{15}}{3} < \lambda < \frac{\sqrt{15}}{3}$

(c) $S=\begin{bmatrix}1&\lambda&-1\\\lambda&1&2\\-1&2&5\end{bmatrix}$

$$f = x_1^2 + 2(\lambda x_2 - x_3)x_1 + (\lambda x_2 - x_3)^2 + x_2^2 + 5x_3^2 + 4x_2x_3 - \lambda^2x_2^2 - x_3^2 + 2\lambda x_2x_3$$
$$= (x_1+\lambda x_2 - x_3)^2 + 4x_3^2 + (2\lambda+4)x_2x_3 + (1-\lambda^2)x_2^2$$

则 $4x_3^2 + (2\lambda+4)x_2x_3 + (1-\lambda^2)x_2^2 > 0$

$\Delta = (2\lambda+4)^2 - 16(1-\lambda^2) < 0 \Rightarrow 5\lambda^2 + 4\lambda < 0 \Rightarrow -\frac{4}{5} < \lambda < 0$

(d) $S=\begin{bmatrix}1&\lambda&5\\\lambda&4&3\\5&3&1\end{bmatrix}$

$$f = x_1^2 + 2(\lambda x_2 + 5x_3)x_1 + (\lambda x_2 + 5x_3)^2 + 4x_2^2 + x_3^2 + 6x_2x_3 - \lambda^2x_2^2 - 10\lambda x_2x_3 - 25x_3^2$$
$$= (x_1+\lambda x_2+5x_3)^2 + (4-\lambda^2)x_2^2 + (6-10\lambda)x_2x_3 - 24x_3^2$$

则 $-24x_3^2 + (6-10\lambda)x_2x_3 + (4-\lambda^2)x_2^2 > 0$

这显然不可能，因此不存在这样的 λ

(e) $S=\begin{bmatrix}2&\lambda&3\\\lambda&2&1\\3&1&1\end{bmatrix}$

$$f = x_3^2 + 2(3x_1+x_2)x_3 + (3x_1+x_2)^2 + 2x_1^2 + 2x_2^2 + 2\lambda x_1x_2 - 9x_1^2 - 6x_1x_2 - x_2^2$$
$$= (x_3+3x_1+x_2)^2 - 7x_1^2 + (2\lambda-6)x_1x_2 + x_2^2$$

则 $-7x_1^2 + (2\lambda-6)x_1x_2 + x_2^2 > 0$

这显然不可能，因此不存在这样的 λ #

7. (a) 设 S 的特征值 $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$

$\exists P$ 使 $S = P^T\Lambda P$，其中 $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2 \ldots \lambda_n)$，P 为正交矩阵 $P^T = [v_1 \ldots v_n]$

$v_1 \cdots v_n$ 对应 $\lambda_1 \ldots \lambda_n$ 的特征向量

不妨令 $y = \frac{x}{\|x\|}$ 则 $\frac{x^TSx}{x^Tx} = y^TSy = y^TP^T\Lambda Py = (Py)^T\Lambda Py$

$$= [y_1' \cdots y_n'] \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} \begin{bmatrix}y_1'\\ \vdots \\ y_n'\end{bmatrix}$$

$$= \lambda_1 y_1'^2 + \cdots + \lambda_n y_n'^2 \le \lambda_1(y_1'^2 + \cdots + y_n'^2) = \lambda_1\|y\|^2 = \lambda_1$$

取等时 $y_1' = 1$，$y_2' = \cdots = y_n' = 0$ $\Rightarrow Py = \begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} \Rightarrow y = P^T\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} = v_1 \Rightarrow x = cy = cv_1$

综上：$\frac{x^TSx}{x^Tx}$ 最大值为 S 的最大特征值，对应的 x 为最大特征值对应的特征向量 #

(b) 设 $y_2=\frac{x_2}{\|x_2\|}$ $\quad \frac{x_2^TSx_2}{x_2^Tx_2}=y_2^TSy_2=y_2^TP\Lambda Py_2=(Py_2)^T\Lambda(Py_2)$

考虑到 $x_1^Tx_2=0 \Rightarrow y_2^Tx_1=0 \Rightarrow y_2P^TPx_1=0 \Rightarrow (Py_2)^TPx_1=0$

而 $Px_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ 则 $Py_2=\begin{bmatrix}0\\ *\\\vdots\\ *\end{bmatrix}$

因此 $\frac{x_2^TSx_2}{x_2^Tx_2}=[0\ y_2''\cdots y_n'']\begin{bmatrix}\lambda_1&&\\&\ddots&\\&&\lambda_n\end{bmatrix}\begin{bmatrix}0\\y_2''\\\vdots\\y_n''\end{bmatrix}$

$=\lambda_2y_2''^2+\cdots+\lambda_ny_n''^2\le\lambda_2(y_2''^2+\cdots+y_n''^2)=\lambda_2\|y_2\|^2=\lambda_2$.

取等，此时 $y_2''=1,\ y_3''=\cdots=y_n''=0$ $\quad Py_2=\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}\Rightarrow y_2=P^T\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}=v_2$.

综上：$\frac{x^TSx}{x^Tx}$最大值为S的次大特征值（若最大特征值代数重数≥2则仍为最大特征值）

x_2为取次大特征值时找到的垂直于x_1的特征向量 #

2、特征值，特征向量的一个应用

1. pf: S的特征方程 $Sx=\lambda x$ ① $(x\neq0)$

取共轭 $S\bar{x}=\bar{\lambda}\bar{x}$ ②

① $\times\bar{x}^T$ $\quad \bar{x}^TSx=\lambda\bar{x}^Tx$ ③

② $\times x^T$ $\quad x^TS\bar{x}=\bar{\lambda}x^T\bar{x}$

取转置 $\bar{x}^TS^Tx=\bar{x}^TSx=\bar{\lambda}\bar{x}^Tx$ ④

③-④ $\quad (\lambda-\bar{\lambda})\bar{x}^Tx=0$ $\quad$ 而 $\bar{x}^Tx=\|x\|^2\neq0$.

$\therefore\lambda=\bar{\lambda}\Rightarrow$ 对称矩阵特征值都是实数 #.

2、pf:设这n个线性无关特征向量属于V，$\dim V=n$.

由基定理，n个特征向量构成V的一组基.

由Gram-Schmit方法，一组基可以构造出一组正交归一基.

∴这些特征向量可以构造一组正交归一基 #

3. pf: $x=a_1x_1+\cdots+a_nx_n$

$x^T=a_1x_1^T+\cdots+a_nx_n^T$

$\Rightarrow x^Tx=(a_1x_1^T+\cdots+a_nx_n^T)(a_1x_1+\cdots+a_nx_n)$,

$=\sum_{1\le i\le n,1\le j\le n}a_ia_j\,x_i^Tx_j$

考虑到 $i=j$ 时 $x_i^Tx_j=1$ $\quad i\neq j$ $\ x_i^Tx_j=0$.

则 $x^Tx=\sum_{1\le i\le n}a_i^2x_i^Tx_i=a_1^2+\cdots+a_n^2=1$ #

4. pf: $\overline{\sigma_M^2}=\overline{M^2}-\bar{M}^2$

$\Leftarrow x^T(M-\bar{M}I)^2x=x^TM^2x-x^TMx\,x^TMx$

$\Leftarrow x^TM^2x-2x^TM\bar{M}x+x^T\bar{M}\bar{M}x=x^TM^2x-x^TMx\,x^TMx$

$\Leftarrow -2\bar{M}^2+\bar{M}^2(x^Tx)=-\bar{M}^2$

当 $x^Tx=\|x\|=1$ 时 $\quad -2\bar{M}^2+\bar{M}^2(x^Tx)=-2\bar{M}^2+\bar{M}^2=-\bar{M}^2$ 得证.

解: $\overline{M^2}=\bar{M}^2\Rightarrow x^TM^2x=x^TMx\,x^TMx$

若x是M的一个特征向量则有 $\lambda x=Mx$

$(x^TM^T)(Mx)=x^T(Mx)x^T(Mx)$

$\Rightarrow\lambda^2x^Tx=\lambda^2x^Tx\,x^Tx$ $\quad$ 则条件为x为M归一的特征向量 #

$\Rightarrow\|x\|^2=\|x\|^4$.

X

3. pf: $\|(P+taQ)x\|^2 = x^T(P+taQ)^T(P+taQ)x = x^T(P^T+taQ^T)(P+taQ)x$

$= x^T(P+taQ)^2x = x^T(P^2+ta^2Q^2+taPQ+taQP)x$

$= x^T(P^2+ta^2Q^2+taI)x = x^T(P^2+Q^2)x+tax^Tx$

$= x^T(P^2+ta^2Q^2)x + a \geq 0$

$\Rightarrow a^2x^TQ^2x + a + x^TP^2x \geq 0 \Rightarrow \Delta = 1-4(x^TQ^2x)(x^TP^2x) \leq 0 \Rightarrow (x^TQ^2x)(x^TP^2x) \geq \frac{1}{4}$

而 $\sigma_P\sigma_Q = \sqrt{\sigma_P^2}\cdot\sqrt{\sigma_Q^2} = \sqrt{P^2}\sqrt{Q^2} = \sqrt{x^TQ^2x}\sqrt{x^TP^2x} \geq \sqrt{\frac{1}{4}} = \frac{1}{2}$ #

3、奇异值分解

(1) $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ $A^TA = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$

$|\lambda I - A^TA| = \begin{vmatrix} \lambda-8 & -2 \\ -2 & \lambda-5 \end{vmatrix} = (\lambda-8)(\lambda-5)-4$

$= \lambda^2-13\lambda+36 = (\lambda-4)(\lambda-9)$

则 $\lambda_1 = 9,\ \lambda_2 = 4 \Rightarrow \sigma_1 = 3,\ \sigma_2 = 2$

$\sigma_1 = 3$ 时 $[\lambda I - A^TA] = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$ 取特征向量 $v_1 = \begin{bmatrix} \frac{2}{\sqrt5} \\ \frac{1}{\sqrt5} \end{bmatrix}$

$u_1 = \frac{Av_1}{|Av_1|} = \frac{1}{3}\begin{bmatrix} \frac{3}{\sqrt5} \\ \frac{6}{\sqrt5} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} \end{bmatrix}$

$\sigma_2 = 2$ 时 $\lambda I - A^TA = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$ 取特征向量 $v_2 = \begin{bmatrix} -\frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} \end{bmatrix}$

$u_2 = \frac{Av_2}{|Av_2|} = \frac{1}{2}\begin{bmatrix} -\frac{4}{\sqrt5} \\ \frac{2}{\sqrt5} \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt5} \\ \frac{1}{\sqrt5} \end{bmatrix}$

则 $A = \underbrace{\begin{bmatrix} -\frac{1}{\sqrt5} & -\frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{bmatrix}}_{V^T}$

(2) $A = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}$ $A^TA = \begin{bmatrix} 7 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}$

$|\lambda I - A^TA| = \begin{vmatrix} \lambda-74 & -32 \\ -32 & \lambda-26 \end{vmatrix} = (\lambda-74)(\lambda-26)-1024$

$= \lambda-100\lambda+900 = (\lambda-90)(\lambda-10)$

则 $\lambda_1 = 90$ $\lambda_2 = 10$ $\Rightarrow \sigma_1 = 3\sqrt{10},\ \sigma_2 = \sqrt{10}$

$\sigma_1 = 3\sqrt{10}$ 时 $\lambda I - A^TA = \begin{bmatrix} 16 & -32 \\ -32 & 64 \end{bmatrix}$ 取特征向量 $v_1 = \begin{bmatrix} \frac{2}{\sqrt5} \\ \frac{1}{\sqrt5} \end{bmatrix}$

$u_1 = \frac{Av_1}{|Av_1|} = \frac{1}{3\sqrt{10}}\begin{bmatrix} \frac{15}{\sqrt5} \\ 0 \\ \frac{15}{\sqrt5} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt2}{2} \\ 0 \\ \frac{\sqrt2}{2} \end{bmatrix}$

$\sigma_2 = \sqrt{10}$ 时 $\lambda I - A^TA = \begin{bmatrix} -64 & -32 \\ -32 & -16 \end{bmatrix}$ 取特征向量 $v_2 = \begin{bmatrix} -\frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} \end{bmatrix}$

$u_2 = \frac{Av_2}{|Av_2|} = \frac{1}{\sqrt{10}}\begin{bmatrix} -\frac{5}{\sqrt5} \\ 0 \\ \frac{5}{\sqrt5} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt2}{2} \\ 0 \\ \frac{\sqrt2}{2} \end{bmatrix}$ 扩充 $u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

则 $A = \underbrace{\begin{bmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{bmatrix}}_{V^T}$

(3) $A=\begin{bmatrix}4 & -2\\ 2 & -1\\ 0 & 0\end{bmatrix}$ $\quad A^TA=\begin{bmatrix}4 & 2 & 0\\ -2 & -1 & 0\end{bmatrix}\begin{bmatrix}4 & -2\\ 2 & -1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}20 & -10\\ -10 & 5\end{bmatrix}$

$|\lambda I-A^TA|=\begin{vmatrix}\lambda-20 & 10\\ 10 & \lambda-5\end{vmatrix}=(\lambda-20)(\lambda-5)-100=\lambda^2-25\lambda=\lambda(\lambda-25)$.

则 $\lambda_1=25\quad \lambda_2=0\quad\Rightarrow \sigma_1=5,\ \sigma_2=0$

$\sigma_1=5$ 时 $[\lambda I-A^TA]=\begin{bmatrix}5 & 10\\ 10 & 20\end{bmatrix}$ 取特征向量 $v_1=\begin{bmatrix}\frac{2}{\sqrt5}\\ -\frac{1}{\sqrt5}\end{bmatrix}$

$\sigma_2=0$ 时 $\lambda I-A^TA=\begin{bmatrix}-20 & 10\\ 10 & -5\end{bmatrix}$ 取特征向量 $v_2=\begin{bmatrix}\frac{1}{\sqrt5}\\ \frac{2}{\sqrt5}\end{bmatrix}$

$u_1=\frac{Av_1}{|Av_1|}=\frac{1}{5}\begin{bmatrix}\frac{10}{\sqrt5}\\ \frac{5}{\sqrt5}\\ 0\end{bmatrix}=\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\\ 0\end{bmatrix}$

将 u 扩充 $u_2=\begin{bmatrix}-\frac{1}{\sqrt5}\\ \frac{2}{\sqrt5}\\ 0\end{bmatrix}\quad u_3=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

则 $A=\underbrace{\begin{bmatrix}\frac{2}{\sqrt5} & -\frac{1}{\sqrt5} & 0\\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} & 0\\ 0 & 0 & 1\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}5 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}\frac{2}{\sqrt5} & -\frac{1}{\sqrt5}\\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{bmatrix}}_{V^T}$