理科类线性代数第十二次作业

一、群、环、域

1、(a) pf： $\forall x, y \in G$

$\varphi_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \varphi_g(x)\varphi_g(y)$

$\therefore$ φ_g 是一个群同态

(b) pf： 先证单射

设 $\exists x, y, z \in G$ 且 $\varphi_g(x) = \varphi_g(y) = z$

$\Rightarrow gxg^{-1} = gyg^{-1}$

$\Rightarrow g^{-1}gxg^{-1}g = g^{-1}gyg^{-1}g$

$\Rightarrow x = y$ 则 φ_g 是单射

再证满射 $\forall x \in G$ 设 $\varphi_g(x) = y$

$\Rightarrow gxg^{-1} = y$

$\Rightarrow x = g^{-1}yg = (gg^{-1})g^{-1}yg(gg^{-1}) = g(g^{-1}g^{-1}ygg)g^{-1}$

即 $\varphi_g(g^{-1}g^{-1}ygg) = x$ 则 φ_g 是满射

$\therefore$ φ_g 是一个群同构

2、pf： $\forall a \in Ker(\varphi)$ $\forall g \in G$

有 $\varphi(gag^{-1}) = \varphi(g)\varphi(a)\varphi(g^{-1}) = \varphi(g)\cdot 0 \cdot \varphi(g^{-1}) = 0$

$\therefore gag^{-1} \in Ker(\varphi)$

即 $Ker(\varphi)$ 是一个正规子群

3、pf： 设 S_3： $1: \frac{1\ 2\ 3}{1\ 2\ 3}$ $a: \frac{1\ 2\ 3}{2\ 1\ 3}$ $b\,\frac{1\ 2\ 3}{1\ 3\ 2}$ $c\,\frac{1\ 2\ 3}{3\ 2\ 1}$ $s\,\frac{1\ 2\ 3}{2\ 3\ 1}$ $t\,\frac{1\ 2\ 3}{3\ 1\ 2}$

则径观察 $\forall x \in S_3$ $x \circ x = 1 \Rightarrow x = x^{-1}$

$bab = bs = c$ 则 a.c 共轭
$aba = at = c$ 则 b.c 共轭
$cac = ct = b$ 则 a.b 共轭
$asa = ac = t$ 则 s.t 共轭
$111 = 1$ 则 1与自身共轭

共轭类 ① a.b.c ② s.t ③ 1

4、pf： ① 先证 $ker(\varphi)$ 在加法下封闭

$\forall v, w \in ker(\varphi)$ $\varphi(v+w) = \varphi(v) + \varphi(w) = 0 \Rightarrow v+w \in ker(\varphi)$

② 再证 $\forall v \in ker(\varphi)$, $w \in R$

$\varphi(vw) = \varphi(v)\cdot\varphi(w) = 0_{R'}\cdot\varphi(w) = 0_{R'}$

$\Rightarrow vw \in ker(\varphi)$

综上：$ker(\varphi)$ 是一个理想。

5、pf： (1) 设 $a = mp + a_1$ $b = np + b_1$ $(0 \le a_1, b_1 \le p-1)$

则 $\overline{a} = a_1$，$\overline{b} = b_1$

$\overline{a+b} = \overline{(mp+a_1)+(np+b_1)} = \overline{a_1+b_1} = \overline{a_1} + \overline{b_1} = \overline{mp+a_1} + \overline{np+b_1} = \overline{a} + \overline{b}$

$\overline{ab} = \overline{(mp+a_1)(np+b_1)} = \overline{a_1b_1} = \overline{a_1}\cdot\overline{b_1} = \overline{(mp+a_1)}\cdot\overline{(mp+b_1)} = \overline{a}\cdot\overline{b}$

(2) 先证结合律 $\overline{(a+b)+c}=(\overline{a}+\overline{b})+\overline{c}=\overline{a}+(\overline{b}+\overline{c})=\overline{a+(b+c)}$

单位元 : $\overline{0}$

逆元 : $\forall\ \overline{a}\in F_p$ $\quad\overline{p-a}$ 为 $\overline{a}$ 逆元 $\quad\overline{p-a}+\overline{a}=\overline{p-a+a}=\overline{p}=\overline{0}$

交换律 : $\overline{a+b}=\overline{a}+\overline{b}=\overline{b}+\overline{a}=\overline{b+a}$

(3) i. $\forall n>p-1$ ① 若 $\overline{a^i}$ $(0\le i\le p-1)$ 有两个元素相同，设 $\overline{a^v}=\overline{a^u}$ $(v<u)$

则已存在 m,n 使 $\overline{a^m}=\overline{a^n}$ $(m<n)$

② 若 $\overline{a^i}$ $(0\le i\le p-1)$ 两两元素均不相同，则共有p个元素

F_p集合中元素共有p个，根据抽屉原理 $\exists\ m\le p-1$ 使 $\overline{a^m}=\overline{a^n}$ $(m<n)$

ii. 由 i $\exists\ m,n$ s.t $\overline{a}^m=\overline{a}^n$ $(m<n)$

则 $\underbrace{\overline{a}\,\overline{a}\cdots\overline{a}}_{m个}=\underbrace{\overline{a}\cdots\overline{a}}_{n个}$

$\Rightarrow\ \overline{1}=\underbrace{\overline{a}\cdots\overline{a}}_{(n-m)个}=\overline{a}\cdot\overline{a^{n-m-1}}$

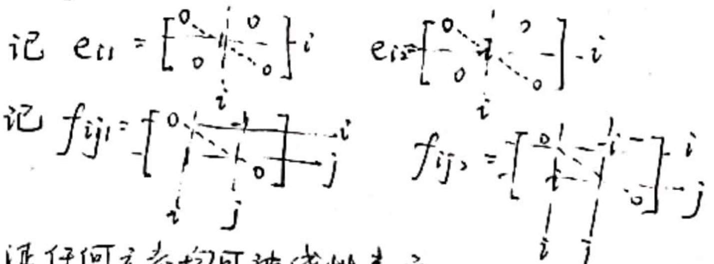
即 $\overline{a}$ 的逆元是 $\overline{a^{n-m-1}}$

iii. 设 $a=rp+a_1, b=sp+b_1,\ c=tp+c_1$ $(0\le r,s,t\le p-1)$

$(\overline{a}+\overline{b})\overline{c}=(\overline{a_1}+\overline{b_1})\overline{c_1}=\underbrace{\overline{c_1}+\cdots+\overline{c_1}}_{\overline{a_1}个\overline{c_1}}+\underbrace{\overline{c_1}+\cdots+\overline{c_1}}_{\overline{b_1}个\overline{c_1}}=\overline{a_1}\,\overline{c_1}+\overline{b_1}\,\overline{c_1}=\overline{a}\,\overline{c}+\overline{b}\,\overline{c}$ #

二、一般域上线性空间

1、找到一组元素



① 先证任何元素均可被线性表示、 设A为反厄米矩阵 .

$$A=\begin{bmatrix} a_{11}+a_{11}'i & -a_{21}+a_{21}'i & \cdots & -a_{n1}+a_{n1}'i \\ a_{21}+a_{21}'i & a_{22}+a_{22}'i & \cdots & -a_{n2}+a_{n2}'i \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+a_{n1}'i & a_{n2}+a_{n2}'i & \cdots & a_{nn}+a_{nn}'i \end{bmatrix}$$

$$=\sum_i a_{ii}e_{i1}+\sum_i a_{ii}'e_{i2}+\sum_{i<j}a_{ij}f_{ij1}+\sum_{i<j}a_{ij}'f_{ij2}$$

② 再证以上元素彼此线性无关、

设 $\sum_i(c_{i1}e_{i1}+c_{i2}e_{i2})+\sum_{i<j}(c_{ij1}f_{ij1}+c_{ij2}f_{ij2})=0$

$$\Rightarrow\begin{bmatrix} c_{11}+c_{12}i & \cdots & -c_{n1}+c_{n1}'i \\ \vdots & \ddots & \vdots \\ c_{n1}+c_{n1}'i & \cdots & c_{n1}+c_{n2}i \end{bmatrix}=0\Rightarrow\ c_{i1},c_{ij},c_{ij1},c_{ij2} 全为0$$

$\Rightarrow$ 以上元素线性无关

元素可作为一组基，基元素个数为(n^2+n)

则 V 的维数为 n^2+n .

17^2

2、 (a) $p=5$ $\left[\begin{array}{cc|c} 6 & -3 & 3 \\ 2 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 14 & 0 & 7 \\ 2 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 0 & 2 \\ 2 & 1 & 1 \end{array}\right]$

$$\begin{cases} 4x_1 = 2 \\ 2x_1 + x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 0 \end{cases}$$

$p=11$ $\left[\begin{array}{cc|c} 6 & -3 & 3 \\ 2 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 14 & 0 & 7 \\ 2 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 0 & 7 \\ 2 & 6 & 1 \end{array}\right]$

$$\begin{cases} 3x_1 = 7 \\ 2x_1 + 6x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 6 \\ x_2 = 0 \end{cases}$$

$p=17$ $\left[\begin{array}{cc|c} 6 & -3 & 3 \\ 2 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{ccc} 14 & 0 & 7 \\ 2 & 6 & 1 \end{array}\right]$

$$\begin{cases} 14x_1 = 7 \\ 2x_1 + 6x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 9 \\ x_2 = 0 \end{cases}$$

(b) $p=7$ $\left[\begin{array}{cc|c} 6 & -3 & 3 \\ 2 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 14 & 0 & 7 \\ 2 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 6 & 1 \\ 0 & 0 & 0 \end{array}\right]$

即 $2x_1 + 6x_2 = 1$

$\begin{cases} x_1 = 0 \\ x_2 = 6 \end{cases}$ $\begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases}$ $\begin{cases} x_1 = 2 \\ x_2 = 3 \end{cases}$ $\begin{cases} x_1 = 3 \\ x_2 = 5 \end{cases}$ $\begin{cases} x_1 = 4 \\ x_2 = 0 \end{cases}$

$\begin{cases} x_1 = 5 \\ x_2 = 2 \end{cases}$ $\begin{cases} x_1 = 6 \\ x_2 = 4 \end{cases}$

3、 pf: 设 $\ker f$ 的一组基为 $\{e_1 \dots e_s\}$，将其扩充为 V_1 的一组基 $\{e_1 \dots e_s, e_{s+1} \dots e_n\}$

设 $W = \text{span}\{e_{s+1}, \dots e_n\}$ 而 $\ker(f) = \text{span}\{e_1, \dots e_s\}$

$\forall v \in V_1, w \in W$ $e_1 \dots e_n$ 线性无关 则 $v + w = 0$ 时

$c_1 e_1 + \cdots + c_n e_n = 0 \Rightarrow v = w = 0$ 即可表法唯一 月

因此 W 与 V_1 和为直和 $\dim W + \dim V_1 = n$

$\Rightarrow W \oplus V_1 = \mathbb{R}^n$

考虑 $f_{s+1} = f(e_{s+1}) \cdots f_n = f(e_n)$

$f_{s+1} \cdots f_n$ 线性无关，因为若 $c_{s+1} f_{s+1} + \cdots + c_n f_n = 0$

$\Rightarrow f(c_{s+1} e_{s+1} + \cdots + c_n e_n) = 0$

$\Rightarrow c_{s+1} e_{s+1} + \cdots + c_n e_n \in \operatorname{Ker} f$

而 $W \cap V = \{0\}$ $\therefore c_{s+1} e_{s+1} + \cdots + c_n e_n = 0$

$c_{s+1} = \cdots = c_n = 0$

$\therefore f_{s+1} \cdots f_n$ 线性无关

而 $\text{span}\{f_{s+1}, \dots, f_n\} = \operatorname{Im} f$

$\Rightarrow \dim \operatorname{Im} f = n - s$

而 $\dim \ker f = s$

$\therefore \dim \ker f + \dim \operatorname{Im} f = n$

4. a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -2 & 0 & 1 & -2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & 0 & 0 & 2 \\ 0 & -6 & 0 & -4 \\ 0 & 0 & 3 & -4 \end{array}\right]$

则 $\begin{cases} 6x_1 = 2 \\ -6x_2 = -4 \\ 3x_3 = -4 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{3} \\ x_2 = \frac{2}{3} \\ x_3 = -\frac{4}{3} \end{cases}$

b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

则 $\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$

c) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} -2 & 0 & 1 & -2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$ 无解

d) $\left[\begin{array}{ccc|c} 6 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & 0 & 0 & 2 \\ 0 & -6 & 0 & -4 \\ 0 & 0 & 3 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 3 \end{array}\right]$

则 $\begin{cases} 6x_1 = 2 \\ x_2 = 3 \\ 3x_3 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = 6 \\ x_2 = 3 \\ x_3 = 1 \end{cases}$