高代选讲 作业二

练习 8.1.1 我们验证 f 是否满足内积的三个判定标准

① 对称性 $f(\vec{a},\vec{b}) = a_1b_1 - a_1b_2 - a_2b_1 + 2a_2b_2$

$f(\vec{b},\vec{a}) = b_1a_1 - b_1a_2 - b_2a_1 + 2a_2b_2$

故 $f(\vec{a},\vec{b}) = f(\vec{b},\vec{a})$ 满足对称性

② 双线性性

设 $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $\Rightarrow c_1\vec{v}+c_2\vec{w} = \begin{bmatrix} c_1v_1+c_2w_1 \\ c_1v_2+c_2w_2 \end{bmatrix}$

$f(c_1\vec{v}+c_2\vec{w},\vec{b}) = (c_1v_1+c_2w_1)b_1 - (c_1v_1+c_2w_1)b_2 - (c_1v_2+c_2w_2)b_1 + 2(c_1v_2+c_2w_2)b_2$

$f(\vec{v},\vec{b}) = v_1b_1 - v_1b_2 - v_2b_1 + 2v_2b_2$

$f(\vec{w},\vec{b}) = w_1b_1 - w_1b_2 - w_2b_1 + 2w_2b_2$

则 $c_1f(\vec{v},\vec{b}) + c_2f(\vec{w},\vec{b}) = (c_1v_1 - c_1v_2 + c_2w_1 - c_2w_2)b_1 + (-c_1v_1 + 2c_1v_2 - c_2w_1 + 2c_2w_2)b_2$

$= f(c_1\vec{v}+c_2\vec{w},\vec{b})$

即 f 对于第一个位置有线性性

考虑到 f 有对称性，故第二个位置也有线性性

则 f 满足双线性性

③ 正定性 $f(\vec{a},\vec{a}) = a_1^2 - a_1a_2 - a_2a_1 + 2a_2^2 = a_1^2 + 2a_2^2 \geq 0$

当 $f(\vec{a},\vec{a}) = 0$ 时 $a_1 = a_2 = 0 \Rightarrow \vec{a} = 0$，则 f 有正定性

综上：f 是 R^2 上的一个内积

练习 8.1.3 即证 $\langle A,B\rangle$ 是 $R^{m\times n}$ 上的内积

① 对称性 $\langle A,B\rangle = trace(B^TA) = trace((B^TA)^T)$

$= trace(A^TB) = \langle B,A\rangle$ 满足对称性

② 双线性性 $\langle c_1A_1 + c_2A_2, B\rangle = trace(B^T(c_1A_1+c_2A_2))$

$= trace(c_1B^TA_1 + c_2B^TA_2)$

$= c_1\,trace(B^TA_1) + c_2\,trace(B^TA_2)$

$= c_1\langle A_1,B\rangle + c_2\langle A_2,B\rangle$

而 $\langle A,B\rangle$ 又有对称性，故其具有双线性性

③ 正定性 $\langle A,A\rangle = trace(A^TA)$

$= \sum_{i=1}^{n} (A^TA)_{ii}$

$= \sum_{i=1}^{n} (\sum_{j=1}^{n} (A^T)_{ij} A_{ji})$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ji}A_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ji}^2 \geq 0$

若 $\langle A,A\rangle = 0$ 则 $\forall i\forall j\ A_{ij} = 0 \Rightarrow A = 0$

即 $\langle A,B\rangle$ 有正定性

综上：$\langle A,B\rangle$ 是内积，$R^{m\times n}$ 又是线性空间

则 $R^{m\times n}$ 关于 $\langle A,B\rangle$ 构成欧氏空间

练习 8.1.5

由度量矩阵：$\langle a_1,a_1\rangle = G_{11} = 1$　$\langle a_1,a_2\rangle = G_{12} = 0$　$\langle a_1,a_3\rangle = G_{13} = 1$

$\langle a_2,a_2\rangle = G_{22} = 10$　$\langle a_2,a_3\rangle = G_{23} = -2$　$\langle a_3,a_3\rangle = G_{33} = 2$

采用G-S方法取得正交基

$$e_1' = a_1$$

$$e_2' = a_2 - \frac{\langle e_1',a_2\rangle}{\langle e_1',e_1'\rangle}e_1' = a_2 - \frac{0}{1}e_1' = a_2$$

$$e_3' = a_3 - \frac{\langle e_1',a_3\rangle}{\langle e_1',e_1'\rangle}e_1' - \frac{\langle e_2',a_3\rangle}{\langle e_2',e_2'\rangle}e_2' = a_3 - \frac{1}{1}a_1 - \frac{-2}{10}a_2 = a_3 - a_1 + \frac{1}{5}a_2$$

再标准化　$e_1 = \frac{e_1'}{\sqrt{\langle e_1',e_1'\rangle}} = a_1$

$$e_2 = \frac{e_2'}{\sqrt{\langle e_2',e_2'\rangle}} = \frac{1}{\sqrt{10}}a_2$$

$$e_3 = \frac{e_3'}{\sqrt{\langle e_3',e_3'\rangle}} = \frac{e_3'}{\sqrt{\langle -a_1+\frac{1}{5}a_2+a_3,\ -a_1+\frac{1}{5}a_2+a_3\rangle}}$$

$$= \frac{e_3'}{\sqrt{1+\frac{2}{5}+2-2-\frac{4}{5}}} = \sqrt{\frac{5}{3}}e_3'$$

$$= \frac{\sqrt{15}}{5}(a_1+\frac{1}{5}a_2+a_3)$$

则V的一组标准正交基为

$$\{a_1, \frac{\sqrt{10}}{10}a_2, \frac{\sqrt{15}}{5}(-a_1+\frac{1}{5}a_2+a_3)\}$$

练习 8.1.7

采用G-S方法取得正交基

$$e_1' = a_1 = q_1+q_5$$

$$e_2' = a_2 - \frac{\langle e_1',a_2\rangle}{\langle e_1',e_1'\rangle}e_1' = a_2 - \frac{\langle q_1+q_5, q_1-q_2+q_4\rangle}{\langle q_1+q_5, q_1+q_5\rangle}(q_1+q_5) = q_1-q_2+q_4-\frac{1}{2}q_1-\frac{1}{2}q_5$$

$$= \frac{1}{2}q_1-q_2+q_4-\frac{1}{2}q_5$$

$$e_3' = a_3 - \frac{\langle e_1',a_3\rangle}{\langle e_1',e_1'\rangle}e_1' - \frac{\langle e_2',a_3\rangle}{\langle e_2',e_2'\rangle}e_2' = a_3 - \frac{2}{2}e_1' - 0 = q_1+q_2+q_3-q_5$$

再标准化　$e_1 = \frac{e_1'}{\sqrt{\langle e_1',e_1'\rangle}} = \frac{\sqrt{2}}{2}q_1+\frac{\sqrt{2}}{2}q_5$

$$e_2 = \frac{e_2'}{\sqrt{\langle e_2',e_2'\rangle}} = \frac{\sqrt{10}}{2}(\frac{1}{2}q_1-q_2+q_4-\frac{1}{2}q_5)$$

$$= \frac{\sqrt{10}}{4}q_1-\frac{\sqrt{10}}{2}q_2+\frac{\sqrt{10}}{2}q_4-\frac{\sqrt{10}}{4}q_5$$

$$e_3 = \frac{e_3'}{\sqrt{\langle e_3',e_3'\rangle}} = \frac{1}{2}q_1+\frac{1}{2}q_2+\frac{1}{2}q_3-\frac{1}{2}q_5$$

则 $\mathrm{span}(a_1,a_2,a_3)$的一组正交基为

$$\{\frac{\sqrt{2}}{2}q_1+\frac{\sqrt{2}}{2}q_5,\ \frac{\sqrt{10}}{4}q_1-\frac{\sqrt{10}}{2}q_2+\frac{\sqrt{10}}{2}q_4-\frac{\sqrt{10}}{4}q_5,\ \frac{1}{2}q_1+\frac{1}{2}q_2+\frac{1}{2}q_3-\frac{1}{2}q_5\}$$

练习 8.1.9　设V的一组正交基为 $e_1 \dots e_n$，且 $f(e_i) = b_i$　$\forall 1\le i\le n$.

①存在：$\forall \vec{a}\in V$，设 $\vec{a} = \sum_{i=1}^n a_ie_i$，那么 $f(\vec{a}) = f(\sum_{i=1}^n a_ie_i) = \sum_{i=1}^n a_if(e_i) = \sum_{i=1}^n a_ib_i$

令 $\vec{b} = \sum_{j=1}^n b_je_j$　$\langle \vec{a},\vec{b}\rangle = \langle \sum_{i=1}^n a_ie_i, \sum_{j=1}^n b_je_j\rangle = \sum_{i=1}^n\sum_{j=1}^n a_ib_j\langle e_i,e_j\rangle$

$= \sum_{i=1}^n a_ib_i$

故 $f(\vec{a}) = \langle \vec{a},\vec{b}\rangle$，即存在向量$\vec{b}$使 $f(\vec{a}) = \langle a,b\rangle$ $\forall \vec{a}\in V$.

②唯一：设 $\vec{v}\in V$ s.t $f(\vec{a}) = \langle \vec{a},\vec{v}\rangle$，$\vec{v} = \sum_{j=1}^n v_je_j$　则 $\langle \vec{a},\vec{v}\rangle = \langle \sum_{i=1}^n a_ie_i, \sum_{j=1}^n v_je_j\rangle = \sum_{i=1}^n a_iv_i$

考虑到$\vec{a}$的任意性，$f(\vec{a}) = \langle \vec{a},\vec{v}\rangle$须保证$a_i$系数相同，故 $b_i = v_i$ $\forall i \Rightarrow \vec{v} = \vec{b}$，即$\vec{b}$唯一 □