第五章习题解答

5. 证明：因z_0是$f(z)$的零点，又已知$f(z) \not\equiv 0$, 知存在$m \in N, \phi(z)$在z_0解析，$\phi(z_0) \neq 0$, $f(z) = (z - z_0)^m \phi(z)$. 类似的有$n \in N, \psi(z)$在$z_0$解析，$\psi(z_0) \neq 0$, $g(z) = (z - z_0)^n \psi(z)$.

(a). 若$m < n$, 则有

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\phi(z)}{(z - z_0)^{n-m}\psi(z)} = \infty.$$

$$\lim_{z \to z_0} \frac{f'(z)}{g'(z)} = \lim_{z \to z_0} \frac{m(z-z_0)^{m-1}\phi(z)+(z-z_0)^m\phi'(z)}{n(z-z_0)^{n-1}\psi(z)+(z-z_0)^n\psi'(z)}$$

$$= \lim_{z \to z_0} \frac{m\phi(z)+o(z-z_0)}{(z-z_0)^{n-m}(n\psi(z)+o(z-z_0))} = \infty.$$

即等式两边均等于∞.

(b). 若$m > n$. 由上面的分析可得$\lim_{z \to z_0} \frac{g(z)}{f(z)} = \lim_{z \to z_0} \frac{g'(z)}{f'(z)} = \infty$, 这等价于

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)} = 0.$$

(c). 若$m = n$. 由上面的分析可得：

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\phi(z)}{\psi(z)} = \frac{\phi(z_0)}{\psi(z_0)}.$$

$$\lim_{z \to z_0} \frac{f'(z)}{g'(z)} = \lim_{z \to z_0} \frac{m(z - z_0)^{m-1}\phi(z) + (z - z_0)^m\phi'(z)}{m(z - z_0)^{m-1}\psi(z) + (z - z_0)^m\psi'(z)} = \frac{\phi(z_0)}{\psi(z_0)}.$$

即等式两边均等于$\frac{\phi(z_0)}{\psi(z_0)}$.

9(3). 易见当$m \leq 0$时，被积函数处处解析，积分为零. 当$m \geq 1$时，由$\cos z$的Taylor级数得被积函数为

$$f(z) = \frac{1 - \cos z}{z^m} = \frac{1 - \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k}}{(2k)!}}{z^m} = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1} z^{2k-m}}{(2k)!}.$$

从而有

$$\oint_{|z|=\frac{3}{2}} f(z)dz = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{(2k)!} \oint_{|z|=\frac{3}{2}} z^{2k-m}dz.$$

因当$p \geq 0$ 和$p = -2q, q \in N$时，有$\oint_{|z|=\frac{3}{2}} z^p dz = 0$, 知当$m = 1$或为偶数时，积分为零。

当$m = 2n+1, n \in N$时，积分为

$$\frac{(-1)^{n-1}2\pi i}{(2n)!} = \frac{(-1)^{\frac{m-3}{2}}2\pi i}{(m-1)!}.$$

9(6). 记

$$I = \oint_{|z|=1} \frac{1}{(z-a)^n(z-b)^n}dz.$$

(A). $|a| < |b| < 1$. 当$n > 1$时，在闭曲线$|z| = 1$内被积函数有两个相异n阶极点，故积分为零。

当$n = 1$时，

$$I = 2\pi i \left[Res[\tfrac{1}{(z-a)(z-b)}, a] + Res[\tfrac{1}{(z-a)(z-b)}, b] \right]$$

$$= 2\pi i \left[\tfrac{1}{a-b} + \tfrac{1}{b-a} \right] = 0.$$

(B). $|a| < 1 < |b|$. 由Laurant级数（见后面的附注）：

$$\frac{1}{(1+x)^n} = \sum_{k=0}^{+\infty}(-1)^k \frac{(k+n-1)(k+n-2)\cdots(k+1)}{(n-1)!}x^k, \ \ |x| < 1,$$

可得当$|z-a| < |b-a|$时，

$$\tfrac{1}{(z-a)^n(z-b)^n} = \tfrac{1}{(z-a)^n} \tfrac{1}{(a-b)^n} \tfrac{1}{\left(1+\frac{z-a}{a-b}\right)^n}$$

$$= \tfrac{1}{(z-a)^n} \tfrac{1}{(a-b)^n} \sum_{k=0}^{+\infty}(-1)^k \tfrac{(k+n-1)(k+n-2)\cdots(k+1)}{(a-b)^k(n-1)!}(z-a)^k$$

$$= \sum_{k=0}^{+\infty}(-1)^k \tfrac{(k+n-1)(k+n-2)\cdots(k+1)}{(a-b)^{n+k}(n-1)!}(z-a)^{k-n}.$$

由此式可得$(z-a)^{-1}$的系数为

$$(-1)^{n-1} \frac{(2n-2)!}{(a-b)^{2n-1}((n-1)!)^2}.$$

因而得

$$I = (-1)^{n-1} \frac{(2n-2)!}{(a-b)^{2n-1}((n-1)!)^2}2\pi i.$$

(C). $1 < |a| < |b|$. 这时被积函数在闭曲线内和闭曲线上无奇点，故$I = 0$.

12(2). 令$t = \frac{1}{z}$, 则原积分变为

$$\oint_{|t|=\frac{1}{2}} \frac{e^t}{t^4(1+t)}dt.$$

2

在$t = 0$附近可得Laurant级数：

$$\frac{e^t}{t^4(1+t)} = \frac{1}{t^4} + \frac{1}{2t^2} - \frac{1}{3t} + O(1).$$

由此可得原积分为$2\pi i \left(\frac{-1}{3} \right) = -\frac{2\pi i}{3}$.

12(3). 当$n = 1$时，

$$I_1 = \oint_{|z|=r>1} \frac{z^2}{1+z} dz = \oint_{|z|=r>1} (z-1)dz + \oint_{|z|=r>1} \frac{1}{1+z} dz = \oint_{|z|=r>1} \frac{1}{1+z} dz = 2\pi i.$$

当$n > 1$时，由$z_k^n = -1$, 得$-z_k = \frac{1}{z_k^{n-1}}, k = 1, 2, \cdots, n,$及

$$\begin{aligned}
I_n &= \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \oint_{|z|=r>1} (z^n - 1)dz + \oint_{|z|=r>1} \frac{1}{1+z^n} dz = \oint_{|z|=r>1} \frac{1}{1+z^n} dz \\
&= 2\pi i \sum_{k=1}^n Res\left[\frac{1}{1+z^n}, z_k \right] = 2\pi i \sum_{k=1}^n \frac{1}{nz_k^{n-1}} \\
&= \frac{-2\pi i}{n} \sum_{k=1}^n z_k = 0.
\end{aligned}$$

故得$I_1 = 2\pi i, I_n = 0, n > 1$.（其中等式$\sum_{k=1}^n z_k = 0$可由根与系数的关系得到。）

13(4). 易得方程$z^4 + 1 = 0$在上复平面的两个根分别为$z_1 = e^{\frac{\pi}{4}}, z_2 = e^{\frac{3\pi}{4}}$. 记

$$I = \int_0^{+\infty} \frac{x^2}{1+x^4} dx,$$

则

$$\begin{aligned}
2I &= \int_{-\infty}^{+\infty} \frac{x^2}{1+x^4} dx = 2\pi i \left(Res\left[\frac{z^2}{1+z^4}, z_1 \right] + Res\left[\frac{z^2}{1+z^4}, z_2 \right] \right) \\
&= 2\pi i \left(\frac{z_1^2}{4z_1^3} + \frac{z_2^2}{4z_2^3} \right) = \frac{\pi i}{2} \left(\frac{1}{z_1} + \frac{1}{z_2} \right) \\
&= \frac{\pi i}{2} \left(e^{-\frac{\pi i}{4}} + e^{-\frac{3\pi i}{4}} \right) = \frac{\pi i}{2} \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} \right) \\
&= \frac{\pi i}{2} \left(-2i\sin\frac{\pi}{4} \right) = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

故得

$$I = \frac{\pi}{2\sqrt{2}}.$$

13(6). 由

$$\int_{-\infty}^{+\infty} \frac{xe^{ix}}{1+x^2} dx = 2\pi i Res\left[\frac{ze^{iz}}{1+z^2}, i \right] = 2\pi i \frac{ie^{-1}}{2i} = \frac{\pi i}{e}.$$

得

$$\int_{-\infty}^{+\infty} \frac{x\sin x}{1+x^2} dx = Im\left(\int_{-\infty}^{+\infty} \frac{xe^{ix}}{1+x^2} dx \right) = \frac{\pi}{e}.$$

附注：当$|x| < 1$时，令

$$f(x) = \frac{1}{1+x} = \sum_{m=0}^{+\infty}(-1)^m x^m$$

则可得

$$f^{(n-1)}(x) \quad = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} = \sum_{m=n-1}^{+\infty}(-1)^m m(m-1)(m-2)\cdots(m-n+2)x^{m-n+1}$$

$$= \sum_{k=0}^{+\infty}(-1)^{k+n-1}(k+n-1)(k+n-2)\cdots(k+1)x^k.$$

由此即得当$|x| < 1$时，

$$\frac{1}{(1+x)^n} = \frac{(-1)^{n-1}}{(n-1)!}f^{(n-1)}(x) = \sum_{k=0}^{+\infty}\frac{(-1)^k(k+n-1)(k+n-2)\cdots(k+1)}{(n-1)!}x^k.$$