概统 第十次作业

习题4.2

1. $\varphi(t)=\sum e^{itx}p(X=x)= e^0\times 0.4+e^{it}\times 0.3+e^{2it}\times 0.2+e^{3it}\times 0.1$

$= 0.4+0.3e^{it}+0.2e^{2it}+0.1e^{3it}$

2. $\varphi(t)=\sum_{x=1}^{\infty}e^{itx}p(X=x)=\sum_{x=1}^{\infty}e^{itx}(1-p)^{x-1}p = \frac{p}{1-p}\sum_{x=1}^{\infty}[e^{it}(1-p)]^x=\frac{p}{1-p}\frac{e^{it}(1-p)}{1-e^{it}(1-p)}$

$= p\frac{e^{it}}{1-e^{it}(1-p)}=\frac{p}{1-p}\left(1-\frac{1}{1-e^{it}(1-p)}\right)$

$\varphi(t)=-\frac{p}{1-p}+\frac{p}{1-p}\left(1+(1-p)e^{it}+(1-p)^2e^{2it}+\cdots\right)$

$=-\frac{p}{1-p}+\frac{p}{1-p}\left(1+(1-p)(1+it-\frac{1}{2}t^2+o(t^2))+(1-p)^2(1+2it-\frac{4}{2}t^2+o(t^2))+\cdots\right)$

$=\frac{p}{1-p}\sum_{k=1}^{\infty}\left[(1-p)^k(1+ikt-\frac{1}{2}k^2t^2+o(t^2))\right]$

$=\frac{p}{1-p}\left[\frac{1-p}{p}+\frac{1-p}{p^2}it-\frac{p^2-3p+2}{2p^3}t^2+o(t^2)\right]$

则 $iEX=\frac{i}{p}$, $-\frac{1}{2}EX^2=\frac{p-2}{2p^2}$

$\Rightarrow \underline{EX=\frac{1}{p}}$ # $EX^2=\frac{2-p}{p^2}$

则 $\mathrm{Var}X=EX^2-(EX)^2=\frac{2-p}{p^2}-\frac{1}{p^2}=\frac{1-p}{p^2}$

4. (1) $F_1(x)=\frac{a}{2}\int_{-\infty}^{x}e^{-a|t|}dt \qquad p(x)=F_1'(x)=\frac{a}{2}e^{-a|x|}$

$\varphi(t)=\int_{-\infty}^{+\infty}e^{itx}p(x)dx=\frac{a}{2}\int_{-\infty}^{+\infty}e^{itx-a|x|}dx=\frac{a}{2}\left(\int_{-\infty}^{0}e^{(a+it)x}dx+\int_{0}^{+\infty}e^{(-a+it)x}dx\right)$

$=\frac{a}{2}\left(\frac{1}{a+it}-\frac{1}{-a+it}\right)=\frac{a^2}{a^2+t^2}$

$\varphi(t)=\frac{1}{1+(t/a)^2}=1-(\frac{t}{a})^2+o(t^2)$

则 $iEX=0 \quad -\frac{1}{2}EX^2=-\frac{1}{a^2} \Rightarrow EX=0,\ EX^2=\frac{2}{a^2} \Rightarrow \mathrm{Var}X=\frac{2}{a^2}$

(2) $F_2(x)=\frac{a}{\pi}\int_{-\infty}^{x}\frac{1}{t^2+a^2}dt \qquad p(x)=F_2'(x)=\frac{a}{\pi}\frac{1}{x^2+a^2}$

$t\geq 0,\ \varphi(t)=\int_{-\infty}^{+\infty}e^{itx}p(x)dx=\frac{a}{\pi}\int_{-\infty}^{+\infty}e^{itx}\frac{1}{x^2+a^2}dx=\frac{a}{\pi}2\pi i\,\mathrm{Res}\left[\frac{e^{itx}}{x^2+a^2},ai\right]$

$=2ai\frac{e^{-at}}{2ai}=e^{-at}$

而 $t<0$ 时 $\varphi(-t)=\overline{\varphi(-t)}=e^{at}$

则 $\varphi(t)=e^{-a|t|}$ $t=0$ 处不可导，故期望、方差不存在

5. $\varphi(t)=e^{i\mu t-\frac{\sigma^2t^2}{2}}$

$\varphi'(t)=(i\mu-\sigma^2t)e^{i\mu t-\frac{\sigma^2t^2}{2}} \qquad EX=\frac{1}{i}\varphi'(0)=\mu$

$\varphi^{(2)}(t)=[(i\mu-\sigma^2t)^2-\sigma^2]e^{i\mu t-\frac{\sigma^2t^2}{2}} \qquad EX^2=\frac{1}{-1}\varphi''(0)=\mu^2+\sigma^2$

$\varphi^{(3)}(t)=\left([(i\mu-\sigma^2t)^2-\sigma^2](i\mu-\sigma^2t)-2(i\mu-\sigma^2t)\sigma^2\right)e^{i\mu t-\frac{\sigma^2t^2}{2}}$

$\varphi^{(4)}(t)=\left(\left([(i\mu-\sigma^2t)^2-\sigma^2](i\mu-\sigma^2t)-2(i\mu-\sigma^2t)\sigma^2\right)(i\mu-\sigma^2t)-3\sigma^2(i\mu-\sigma^2t)^2+3\sigma^4\right)e^{i\mu t-\frac{\sigma^2t^2}{2}}$

$\varphi^{(3)}(0)=\left(i\mu(-\mu^2-\sigma^2)-2i\mu\sigma^2\right)=(\mu^3+3\mu\sigma^2)(-i) \qquad EX^3=\frac{1}{-i}\varphi^{(3)}(0)=\mu^3+3\mu\sigma^2$

$\varphi^{(4)}(0)=\mu^2(\mu^2+3\sigma^2)+3\mu^2\sigma^2+3\sigma^4 \qquad EX^4=\varphi^{(4)}(0)=\mu^4+6\mu^2\sigma^2+3\sigma^4$

$=\mu^4+6\mu^2\sigma^2+3\sigma^4$

$$E(X-EX)^3 = EX^3 - 3E(X^2)E(X) - 3EX(E(X))^2 + (EX)^3 = 0$$
$$E(X-EX)^4 = EX^4 - 4EX^3EX + 6EX^2(EX)^2 - 4EX(EX)^3 + (EX)^4 = 3\sigma^4$$

10、 $\varphi_{X_i}(t) = \frac{\lambda}{\lambda - it}$

$$\varphi_Y(t) = \varphi_{\sum X_i}(t) = \prod_{i=1}^{n} \varphi_{X_i}(t) = \left(\frac{\lambda}{\lambda - it}\right)^n$$

$$p_Y(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\left(\frac{\lambda}{\lambda-it}\right)^n dt$$
$$= \frac{1}{2\pi}\left[-\frac{i}{n-1} e^{-itx}\frac{\lambda^n}{(\lambda-it)^{n-1}}\Big|_{-\infty}^{+\infty} + \frac{x}{n-1}\int_{-\infty}^{+\infty} e^{-itx}\frac{\lambda^n}{(\lambda-it)^{n-1}}dt\right]$$
$$= \frac{1}{2\pi}\left[\frac{x}{n-1}\int_{-\infty}^{+\infty} e^{-itx}\frac{\lambda^n}{(\lambda-it)^{n-1}}dt\right]$$
$$= \frac{1}{2\pi}\frac{x^2}{(n-1)(n-2)}\int_{-\infty}^{+\infty}\frac{\lambda^n}{(\lambda-it)^{n-2}}dt$$
$$= \cdots$$
$$= \frac{1}{2\pi}\frac{x^{n-1}}{(n-1)\cdots 1}\int_{-\infty}^{+\infty}\frac{\lambda^n}{\lambda-it}dt$$
$$= \frac{1}{2\pi}\frac{(\lambda x)^{n-1}}{(n-1)!}\int_{-\infty}^{+\infty}\varphi_{X_i}(t)dt$$
$$= \frac{1}{2\pi}\frac{(\lambda x)^{n-1}}{(n-1)!}2\pi\lambda e^{-\lambda x} \quad (x \geq 0)$$
$$= \frac{\lambda^n x^{n-1}}{\Gamma(n)}e^{-\lambda x} \quad (x \geq 0)$$

故 $Y \sim Ga(n, \lambda)$

11、(1) 我们已在4(2)中计算了 $X' \sim Ch(a, 1)$， $\varphi_{X'} = e^{-a|t|}$

此处 $X = X' + \mu$， $\lambda = a$，故 $\varphi_X(t) = \varphi_{X'+\mu}(t) = e^{i\mu}e^{-\lambda|t|} = \exp\{i\mu t - \lambda|t|$

设 $X_i \sim Ch(\lambda_i, \mu_i)$ 且 $X_1 \dots X_n$ 独立

$$\varphi_{X_1+\dots+X_n}(t) = \prod_{i=1}^{n}\varphi_{X_i}(t) = \left(\exp\{i\mu_i t - \lambda_i|t|\}\right)^n = \exp\{i\sum\mu_i t - \sum\lambda_i|t|\}$$

由唯一性定理 $X_1 + \dots + X_n \sim Ch(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\lambda_i)$，即 Ch分布有可加性

(2) $$\varphi_{X+Y}(t) = Ee^{it(X+Y)} = Ee^{2itX} = \varphi_X(2t) = \exp\{-\lambda|2t|\} = (\exp\{-\lambda|t|\})^2$$
$$= Ee^{itX}\cdot Ee^{itY} = \varphi_X(t)\varphi_Y(t)$$

而 $X = Y$，X, Y 明显不独立

(3) 由(1) $\varphi_{X_1+\dots+X_n}(t) = \exp\{i\sum\mu_i t - \sum\lambda_i|t|\} = \exp\{in\mu t - n\lambda|t|\}$

$$\varphi_{\frac{X_1+\dots+X_n}{n}}(t) = \varphi_{X_1+\dots+X_n}\left(\frac{1}{n}t\right) = \exp\{i\mu t - \lambda|t|\} = \varphi_{X_1}(t)$$

则 $\frac{X_1+X_2+\dots+X_n}{n}$ 与 X_1 特征函数相同

由唯一性定理，二者同分布。

12. 先证 $p(x)$ 关于原点对称 $\Rightarrow$ 特征函数是实偶函数

$p(x)=p(-x)$ 则 $\varphi(t)=\int_{-\infty}^{+\infty}e^{itx}p(x)\,dx=\int_{-\infty}^{+\infty}e^{itx}p(-x)\,dx$

$$=-\int_{-\infty}^{+\infty}e^{i(-t)(-x)}p(-x)\,d(-x)$$

$$=-\int_{+\infty}^{-\infty}e^{i(-t)x}p(x)\,dx$$

$$=\int_{-\infty}^{+\infty}e^{i(-t)x}p(x)\,dx=\varphi(-t)$$

考虑到 $\varphi(-t)=\overline{\varphi(t)}=\varphi(t)$ 故 $\varphi(t)$ 是实偶函数.

再证特征函数是实偶函数 $\Rightarrow$ $p(x)$ 关于原点对称.

$\varphi(-t)=\overline{\varphi(t)}=\varphi(t)$ 则

$$p(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-itx}\varphi(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-itx}\overline{\varphi(t)}\,dt$$

$$=\frac{1}{2\pi}\overline{\int_{-\infty}^{+\infty}e^{itx}\varphi(t)\,dt}=\frac{1}{2\pi}\overline{\int_{-\infty}^{+\infty}e^{it(-x)}\varphi(t)\,dt}=\frac{1}{2\pi}\overline{p(x)2\pi}=p(-x)$$

故 $p(x)$ 关于原点对称.

13. $\varphi_{X_i}(t)=e^{i\mu t-\frac{\sigma^2t^2}{2}}$

则 $\varphi_{\sum X_i}(t)=\prod_{i=1}^{n}\varphi_{X_i}(t)=\prod_{i=1}^{n}e^{i\mu t-\frac{\sigma^2t^2}{2}}=e^{in\mu t-\frac{n\sigma^2t^2}{2}}$

$\varphi_{\overline{X}}(t)=\varphi_{\frac{\sum X_i}{n}}(t)=\varphi_{\sum X_i}(\frac{1}{n}t)=e^{in\mu\frac{1}{n}t-\frac{n\sigma^2\frac{t^2}{n^2}}{2}}=e^{i\mu t-\frac{\sigma^2t^2}{2n}}$