2021 叶俊随统期中.

一.

$P(A^c)P(B^c)=\frac{1}{9}$

$P(A)\,P(B^c)=P(A^c)\,P(B)$

$[1-P(A^c)]P(B^c)=P(A^c)[1-P(B^c)]$

$P_{A^c}=P_{B^c}=\frac{1}{3}$ $\qquad P_A=P_B=\frac{2}{3}$

(1)

$P(A|X=1)=\frac{P(X=1|A)P(A)}{P(X=1)}=\frac{P_B\cdot P_A}{P_{AB}}=1$

$P_A=P(A|X=1)\cdot P(X=1)+P(A|X=-1)\cdot P(X=-1)$

$P(A|X=-1)=\frac{2}{5}$

(2)

$EX=1\cdot\frac{4}{9}+(-1)\cdot\frac{5}{9}=-\frac{1}{9}$

$DX=EX^2-(EX)^2=1-\frac{1}{81}=\frac{80}{81}$

二.

(1) $X,Y\sim\begin{pmatrix}1 & 0\\ p & 1-p\end{pmatrix}$

$P(Z=1)=P(X=1,Y=0)+P(X=0,Y=1)$

$=2p(1-p)$

$P(Z=0)=P(X=1,Y=1)+P(X=0,Y=0)=(1-p)^2+p^2$

$EZ=2p(1-p)$

$EZ^2=2p(1-p)$

$DZ=EZ^2-(EZ)^2=2p(1-p)[1-2p(1-p)]$

$X=0\Rightarrow Z=\begin{cases}1 & Y=1\\ 0 & 其他\end{cases}$

$X=1\Rightarrow Z=\begin{cases}1 & Y=0\\ 0 & 其他\end{cases}$

XZ	0	1
P	$1-p(1-p)$	$p(1-p)$

$EX=p$

$DX=p(1-p)$

(2) $r_{XZ}=\frac{Cov(X,Z)}{\sqrt{DXDZ}}$

$Cov(X,Z)=E(XZ)-E(X)E(Z)$

$=p(1-p)-p\cdot 2p(1-p).$

$=p-p^2-2p^2+2p^3=2p^3-3p^2+p$

$r_{X,Z}=\frac{(p-2p^2)(1-p)}{\sqrt{p(1-p)}\sqrt{2p(1-p)[1-2p(1-p)]}}=\frac{1-2p}{\sqrt{2-4p(1-p)}}$

(3) 相互独立:

$X\backslash Z$	0	1	
0	$(1-p)^2$	$(1-p)p$	$1-p$
1	p^2	$p(1-p)$	p
	$(1-p)^2+p^2$	$2p(1-p)$	

$(1-p)[(1-p)^2+p^2]=(1-p)^2$

$p[(1-p)^2+p^2]=p^2$

$p\cdot 2p(1-p)=p(1-p)$

$(1-p)\cdot 2p(1-p)=p(1-p)$

$\Rightarrow p=\frac{1}{2}.$

三.

(1)

$$P_n = P_{n-1} \cdot \frac{1}{2} + (1-P_{n-1}) \cdot \frac{3}{4}.$$

(2)

$$P(X=k) = \begin{cases} \frac{1}{2} & k=1 \\ (\frac{1}{2}) \cdot (\frac{1}{4})^{k-2} \cdot \frac{3}{4} & k \geq 2 \end{cases}$$

(3)

$$M_X(u) = E(e^{uX}) = \sum_{k=1}^{+\infty} e^{uk} \cdot P(X=k).$$

$$= \frac{1}{2}e^u + \sum_{k=2}^{+\infty} e^{uk} (\tfrac{1}{2}) \cdot (\tfrac{1}{4})^{k-2} \cdot \frac{3}{4}.$$

$$= \frac{1}{2}e^u + \frac{3}{8}e^{2u} \sum_{k=0}^{+\infty} (\tfrac{1}{4}e^u)^k$$

$$= \frac{1}{2}e^u + \frac{3}{8} \cdot e^{2u} \cdot \frac{1}{1-\frac{1}{4}e^u} = \frac{2e^u+e^{2u}}{4-e^u}$$

$$M_X'(u) = \frac{(2e^u+2e^{2u})(4-e^u) - (2e^u+e^{2u})(-e^u)}{(4-e^u)^2}$$

$$= \frac{8e^u+8e^{2u}-e^{2u}}{(4-e^u)^2}$$

$$EX = M_X'(0) = \frac{5}{3}$$

四.

(1)

$$A_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$E(A_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{2}$$

$$D(A_n) = \sum_{i=1}^{n} \frac{1}{n^2} DX_i = \frac{9}{4n}$$

(2). $A_3 = \frac{1}{3}\sum_{i=1}^{3} X_i$

$$P(A_3=1) = P\left(\sum_{i=1}^{3} X_i = 3\right)$$

$$= C_3^1 (\tfrac{1}{2})^3 = \frac{3}{8}$$

(3).

$$Cov(A_k, A_m) = E(A_k \cdot A_m) - E(A_k) \cdot E(A_m)$$

$$= E(A_k \cdot A_m) - \frac{1}{4}$$

$$= E\left(\frac{1}{k}\sum_{i=1}^{k} X_i \cdot \frac{1}{m}\sum_{i=1}^{m} X_i\right) - \frac{1}{4}$$

$$= \frac{1}{mk}\left[E\left(\sum_{i=1}^{k} X_i\right)^2 + E\sum_{i=k+1}^{m} X_i \cdot E\sum_{i=1}^{k} X_i\right] - \frac{1}{4}$$

$$= \frac{1}{mk}\left[\frac{9}{4}k + (\tfrac{k}{2})^2 + \frac{m-k}{2} \cdot \frac{k}{2}\right] - \frac{1}{4}$$

$$= \frac{9}{4m}$$

五.

(1) $X \sim B(n,p)$.

$X_1 \mid X_1+X_2=n$

$P(X_1=k \mid X_1+X_2=n)$

$$=\frac{P(X_1=k,\ X_2=n-k)}{P(X_1+X_2=n)}$$

$$=\frac{C_{n_1}^{k}\cdot C_{n_2}^{n-k}\ p^{n}\ q^{n_1+n_2-n}}{C_{n_1+n_2}^{n}\ p^{n}\ q^{n_1+n_2-n}}$$

$$=\frac{C_{n_1}^{k}\ C_{n_2}^{n-k}}{C_{n_1+n_2}^{n}} \rightarrow \text{超几何.}$$

$(k\leqslant n_1$ 且 $k\leqslant n$ 且 $k\geqslant n-n_2)$ 否则 $P=0$

$C_{n_1}^{k}\ p^{k}\ q^{n_1-k}$

$C_{n_2}^{n-k}\ p^{n-k}\ q^{n_2-n+k}$

$X_1 \sim n_1$次

$X_2 \sim n_2$次.

$X_1+X_2 \sim n_1+n_2$次

$E(X_1 \mid X_1+X_2)$

$$=\sum^{r} k\,P(X_1=k \mid X_1+X_2=n)$$

$$=\sum_{0}^{r} k\cdot\frac{C_{n_1}^{k}\ C_{n_2}^{n-k}}{C_{n_1+n_2}^{n}}$$

$$=\sum_{1}^{r}\frac{n_1\cdot C_{n_1-1}^{k-1}\cdot C_{n_2}^{n-k}}{\frac{n_1+n_2}{n}\cdot C_{n_1+n_2-1}^{n-1}}$$

$$=\frac{n\cdot n_1}{n_1+n_2}$$

(2).

$X_i \sim Ge(p)$

$P(X_i=k)=pq^{k-1}$

$P(X_1=k \mid X_1+X_2=n)$

$$=\frac{P(X_1=k)\ P(X_2=n-k)}{P(X_1+X_2=n)}$$

$$=\frac{pq^{k-1}\cdot pq^{n-k-1}}{C_{n-1}^{1}\ p^{2}\ q^{n-2}}$$

$$=\frac{1}{n-1} \quad \text{均匀分布.}$$

$E(X_1 \mid X_1+X_2=n)$

$$=\sum_{k=1}^{n-1} k\,P(X_1=k \mid X_1+X_2=n)$$

$$=\frac{1}{n-1}\sum_{k=1}^{n-1} k$$

$$=\frac{n}{2}$$

$$E(X_1 \mid X_1+X_2)=\frac{X_1+X_2}{2}$$

六. (1)

男: $A_t \sim P_o(t)$

女: $B_t \sim P_o(2t)$

可加性: $A_t+B_t \sim P_o(3t)$.

$E(A_t \mid A_t+B_t=4)$

$$=\sum_{k=0}^{4} k\frac{P(A_t=k,\ B_t=4-k)}{P(A_t+B_t=4)}$$

$$=\sum_{k=0}^{4} k\cdot\frac{\frac{t^k}{k!}\cancel{e^{-t}}\cdot\frac{(2t)^{4-k}}{(4-k)!}\cdot\cancel{e^{-2t}}}{\frac{(3t)^4}{4!}\ \cancel{e^{-3t}}}$$

$$=\frac{2^4\cdot 4}{3^4}\sum_{k=1}^{4}\frac{C_3^{k-1}}{2^k}$$

$$=\frac{4}{3}$$

$\rightarrow$ 随机分流不变性 $\frac{4}{3}$.

(2)

$$P(B_{\frac{1}{2}t}=3 \mid A_t+B_t=4)$$

$$=\frac{P(B_{\frac{1}{2}t}=3,\ A_t+B_t=4)}{P(A_t+B_t=4)}$$

$$=\frac{P(B_{\frac{1}{2}t}=3)\ P(A_t+B_t-B_{\frac{t}{2}}=1)}{P(A_t+B_t=4)}$$

$$=\frac{P(B_{\frac{1}{2}t}=3)\ \sum_{i=0}^{1} P(A_t+B_t-B_{\frac{t}{2}}=1 \mid A_t=i)\cdot P(A_t=i)}{P(A_t+B_t=4)}$$

$$=\frac{\frac{t^3}{3!}e^{-t}\cdot\left(\frac{t}{1!}e^{-t}\cdot\frac{t^0}{1}e^{-t}+\frac{t^0}{1}e^{-t}\cdot\frac{t}{1}e^{-t}\right)}{\frac{(3t)^4}{4!}e^{-3t}}=\frac{8}{81}$$