高等微积分2第七周作业

1、(1) 设 $F(x,y,z)=x^3+y^3+z^3+xyz-4$

则 $\frac{\partial F}{\partial z}=F_z=3z^2+xy$

$\frac{\partial F}{\partial z}\Big|_{(1,1,1)}=F_z(1,1,1)=3+1=4\neq 0$ 则 $(1,1,1)$ 附近 z 可表示为 x,y 隐函数

(2) 由(1)设在 $(1,1,1)$ 附近 z 可表示为隐函数 $z=z(x,y)$

则 F 为 $F(x,y,z)=F(x,y,z(x,y))=0$ ① $\forall (x,y)\in$ 某 $(1,1)$ 邻域 U

在 U 内 对①式求导 $\begin{cases}\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0 & ②\\ \frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0 & ③\end{cases}$

而 $\frac{\partial F}{\partial x}\Big|_{(1,1,1)}=3x^2+yz=4$ $\quad\frac{\partial F}{\partial y}\Big|_{(1,1,1)}=3y^2+xz=4$

$\frac{\partial F}{\partial z}\Big|_{(1,1,1)}=3z^2+xy=4$

因此 $\begin{cases}4+4\frac{\partial z}{\partial x}\Big|_{(1,1)}=0\\ 4+4\frac{\partial z}{\partial y}\Big|_{(1,1)}=0\end{cases}\Rightarrow \underline{\frac{\partial z}{\partial y}\Big|_{(1,1)}=\frac{\partial z}{\partial x}\Big|_{(1,1)}=-1}$

(3) 在 U 内 对②式 x 求偏导 $\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial z\partial x}\frac{\partial z}{\partial x}+\frac{\partial^2 F}{\partial x\partial z}\frac{\partial z}{\partial x}+\frac{\partial^2 F}{\partial z^2}\left(\frac{\partial z}{\partial x}\right)^2+\frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x^2}=0$

$\frac{\partial^2 F}{\partial x^2}\Big|_{(1,1,1)}=6x=6$ $\quad\frac{\partial^2 F}{\partial z\partial x}\Big|_{(1,1,1)}=y=1$ $\quad\frac{\partial^2 F}{\partial z^2}\Big|_{(1,1,1)}=6z=6$

则有 $6+(-1)+(-1)+6+4\frac{\partial^2 z}{\partial x^2}\Big|_{(1,1,1)}=0\Rightarrow \frac{\partial^2 z}{\partial x^2}\Big|_{(1,1)}=-\frac{5}{2}$

对②式 y 求偏导 $\frac{\partial^2 F}{\partial y\partial x}+\frac{\partial^2 F}{\partial z\partial x}\frac{\partial z}{\partial y}+\frac{\partial^2 F}{\partial y\partial z}\frac{\partial z}{\partial x}+\frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial y\partial x}=0$

$\frac{\partial^2 F}{\partial y\partial x}\Big|_{(1,1,1)}=z=1$ $\quad\frac{\partial^2 F}{\partial y\partial z}\Big|_{(1,1,1)}=x=1$

则有 $1+(-1)+(-1)+(+6)+4\frac{\partial^2 z}{\partial y\partial x}=0\Rightarrow \frac{\partial^2 z}{\partial y\partial x}\Big|_{(1,1)}=-\frac{5}{4}$

对③式 y 求偏导 $\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z\partial y}\frac{\partial z}{\partial y}+\frac{\partial^2 F}{\partial y\partial z}\frac{\partial z}{\partial y}+\frac{\partial^2 F}{\partial z^2}\left(\frac{\partial z}{\partial y}\right)^2+\frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial y^2}=0$

则有 $6+(-1)+(-1)+6+4\frac{\partial^2 z}{\partial y^2}\Big|_{(1,1,1)}=0\Rightarrow \frac{\partial^2 z}{\partial y^2}\Big|_{(1,1)}=-\frac{5}{2}$

z 在 $(1,1)$ 附近 Taylor 公式可表示为

$$z(1+\Delta x,1+\Delta y)=z(1,1)+\frac{\partial z}{\partial x}\Big|_{(1,1)}\Delta x+\frac{\partial z}{\partial y}\Big|_{(1,1)}\Delta y+\frac{1}{2}\left(\frac{\partial^2 z}{\partial x^2}\Delta x^2+2\frac{\partial^2 z}{\partial x\partial y}\Delta x\Delta y+\frac{\partial^2 z}{\partial y^2}\Delta y^2\right)+o(\Delta x^2+\Delta y^2)$$

$$=1-\Delta x-\Delta y-\frac{5}{4}\Delta x^2-\frac{5}{4}\Delta x\Delta y-\frac{5}{4}\Delta y^2+o(\Delta x^2+\Delta y^2)$$

2、(1) $g(x)=g(x,y(x))=0\Rightarrow \frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}y'(x)=0\Rightarrow y'(x)=-\frac{g'_x(x,y(x))}{g'_y(x,y(x))}$

$h(x)=f(x,y(x))\Rightarrow h'(x)=f'_x(x,y(x))+f'_y(x,y(x))\,y'(x)=f'_x(x,y(x))-\frac{f'_y(x,y(x))g'_x(x,y(x))}{g'_y(x,y(x))}$

(2) $h''(x)=(h'(x))'=f''_{xx}+f''_{xy}y'(x)+\frac{-(f''_{yx}g'_x+f''_{yy}g'_x y'+f'_y g''_{xx}+f'_y g''_{xy}y')g'_y+(g''_{yx}+g''_{yy}y')f'_y g'_x}{(g'_y)^2}$

$$=\frac{f''_{xx}g'^2_y-f''_{xy}g'_x g'_y-f''_{yx}g'_x g'_y+f''_{yy}g'^2_x-f'_y g''_{xx}g'_y+f'_y g''_{xy}g'_x+f'_y g''_{yx}g'_x-f'_y g''_{yy}\frac{g'^2_x}{g'_y}}{g'^2_y}$$

$$=\frac{f''_{xx}g'^2_y-2f''_{xy}g'_x g'_y+f''_{yy}g'^2_x-f'_y g''_{xx}g'_y+2f'_y g''_{xy}g'_x-f'_y g''_{yy}\frac{g'^2_x}{g'_y}}{g'^2_y}$$

3\. (1) $\frac{\partial\phi_1}{\partial x}=\frac{\partial(x)}{\partial x}=1$ $\quad \frac{\partial\phi_2}{\partial x}=\frac{\partial\left(\frac{\partial L(x,v)}{\partial v}\right)}{\partial x}=\frac{\partial^2 L(x,v)}{\partial x\partial v}$

$\frac{\partial\phi_1}{\partial v}=\frac{\partial(x)}{\partial v}=0$ $\quad \frac{\partial\phi_2}{\partial v}=\frac{\partial\left(\frac{\partial L(x,v)}{\partial v}\right)}{\partial v}=\frac{\partial^2 L(x,v)}{\partial v^2}$

则 $J(\phi)_{(x,v)}=\begin{pmatrix}\frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_1}{\partial v}\\ \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_2}{\partial v}\end{pmatrix}_{(x,v)}=\begin{pmatrix}1 & 0\\ \frac{\partial^2 L}{\partial x\partial v}(x,v) & \frac{\partial^2 L}{\partial v^2}(x,v)\end{pmatrix}$

(2) $\det J(\phi)_{(x,v)}=\frac{\partial^2 L}{\partial v^2}(x,v)\neq 0$ 则 $J(\phi)_{(x,v)}$ 可逆

根据反函数定理 ϕ 在 (x,v) 附近有 C' 的逆映射

(3) $J^{-1}(\phi)_{(q,p)}=J(\phi^{-1})_{(q,p)}=\begin{pmatrix}\frac{\partial\phi_1}{\partial q} & \frac{\partial\phi_1}{\partial p}\\ \frac{\partial\phi_2}{\partial q} & \frac{\partial\phi_2}{\partial p}\end{pmatrix}=\frac{1}{\frac{\partial^2 L}{\partial v^2}(x,v)}\begin{pmatrix}\frac{\partial^2 L}{\partial v^2}(x,v) & 0\\ -\frac{\partial^2 L}{\partial x\partial v}(x,v) & 1\end{pmatrix}$

则 $\frac{\partial v(q,p)}{\partial q}=-\frac{\frac{\partial^2 L(x,v)}{\partial x\partial v}}{\frac{\partial^2 L(x,v)}{\partial v^2}}$ $\quad \frac{\partial v(q,p)}{\partial p}=\frac{1}{\frac{\partial^2 L(x,v)}{\partial v^2}}$ $\quad \frac{\partial x(q,p)}{\partial q}=1$ $\quad \frac{\partial x(q,p)}{\partial p}=0$

则 $\frac{\partial H}{\partial q}=\frac{\partial}{\partial q}\big(p\cdot v(q,p)-L(x(q,p),v(q,p))\big)$

$=p\frac{\partial v(q,p)}{\partial q}-\frac{\partial L}{\partial x}(x(q,p),v(q,p))\frac{\partial x(q,p)}{\partial q}-\frac{\partial L}{\partial v}(x(q,p),v(q,p))\frac{\partial v(q,p)}{\partial q}$

$=-p\frac{\frac{\partial^2 L(x,v)}{\partial x\partial v}}{\frac{\partial^2 L(x,v)}{\partial v^2}}-\frac{\partial L}{\partial x}(x,v)+\frac{\partial L}{\partial v}(x,v)\frac{\frac{\partial^2 L(x,v)}{\partial x\partial v}}{\frac{\partial^2 L(x,v)}{\partial v^2}}=-\frac{\partial L}{\partial x}(x,v)=-L_1'(\phi^{-1}(q,p))$

$\frac{\partial H}{\partial p}=\frac{\partial}{\partial p}\big(p\cdot v(q,p)-L(x(q,p),v(q,p))\big)$

$=v(q,p)+p\cdot\frac{\partial v(q,p)}{\partial p}-\frac{\partial L}{\partial x}(x,v)\frac{\partial x}{\partial p}(q,p)-\frac{\partial L}{\partial v}(x,v)\frac{\partial v}{\partial p}(q,p)$

$=v(q,p)+\frac{p}{\frac{\partial^2 L(x,v)}{\partial v^2}}-\frac{\partial L}{\partial v}(x,v)\frac{1}{\frac{\partial^2 L(x,v)}{\partial v^2}}=v(q,p)$

(4) 由 $\phi(x,v)=\left(x,\frac{\partial L(x,v)}{\partial v}\right)$
$\phi^{-1}(q,p)=(x,v)$ $\Rightarrow x(t)=q(t),\ p(t)=\frac{\partial L(x(t),v(t))}{\partial v}$

Hamilton 方程①式 $\frac{dq(t)}{dt}=\frac{\partial H(q(t),p(t))}{\partial p}$ 由(3)题

$\Leftrightarrow \frac{dx(t)}{dt}=v(t)$ 即为 Euler-Lagrange 方程①式

Hamilton 方程②式 $\frac{dp(t)}{dt}=-\frac{\partial H(q(t),p(t))}{\partial q}$ 由(3)题

$\Leftrightarrow \frac{d}{dt}\frac{\partial L(x(t),v(t))}{\partial v}=\frac{\partial L(x(t),v(t))}{\partial x}$ 即为 Euler-Lagrange 方程②式

4\. (1) $f: U\to V$ 是 C' 的且在 x_0 点 $J_f(x_0)$ 可逆

则根据反函数 Thm f 在 x_0 点某邻域 X 有 C' 的反函数 g，设 $f(X)=Y$

X 是开集则 Y 也为开集 $\quad \forall y\in Y$ 设 $f^{-1}(y)=x_1$，$g(y)=x_2$，$x_1,x_2\in X$

$\Rightarrow f(x_1)=f(x_2)$ 而 f 在 X 内为单射，则 $x_1=x_2$，从而 $\forall y\in Y$ $f^{-1}(y)=g(y)$

g 在 Y 内是 C' 的 从而 f^{-1} 在 Y 内是 C' 的 $\Rightarrow f^{-1}$ 在 $f(x_0)$ 处可微

(2) $\forall y\in V$，设 $f^{-1}(y)=x\in U$，则 $J_f(x)$ 可逆 由(1)题 f^{-1} 在 $f(x_0)=y$ 某开邻域 Y 内

是 C' 的，$\forall y$ 均成立 $\Rightarrow f^{-1}$ 在 V 内是 C' 的