高等微积分2 第四周作业

1\. (1) $\frac{\partial z}{\partial x} = y + f(\frac{y}{x}) + x(-\frac{y}{x^2})f'(\frac{y}{x}) = y + f(\frac{y}{x}) - \frac{y}{x}f'(\frac{y}{x})$

$\frac{\partial z}{\partial y} = x + x(\frac{1}{x})f'(\frac{y}{x}) = x + f'(\frac{y}{x})$

则 $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + xf(\frac{y}{x}) - yf'(\frac{y}{x}) + xy + yf'(\frac{y}{x}) = 2xy + xf(\frac{y}{x})$

(2) $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial x}\Big|_{\vec{x}} \cdot 0 - \frac{x}{y^2}\frac{\partial f}{\partial y}\Big|_{\vec{x}}$.

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) = \frac{2x}{y^3}\frac{\partial f}{\partial y}\Big|_{\vec{x}} - \frac{x}{y^2}\left[\frac{\partial^2 f}{\partial x \partial y}\Big|_{\vec{x}} 0 + \frac{\partial^2 f}{\partial y^2}\Big|_{\vec{x}}(-\frac{x}{y^2})\right] = \frac{2x}{y^3}\frac{\partial f}{\partial y}\Big|_{\vec{x}} + \frac{x^2}{y^4}\frac{\partial^2 f}{\partial y^2}\Big|_{\vec{x}}$

(3) $\frac{\partial z}{\partial y} = xf'(\frac{y}{x})(\frac{1}{x}) + g(\frac{x}{y}) + y(-\frac{x}{y^2})g'(\frac{x}{y}) = f'(\frac{y}{x}) + g(\frac{x}{y}) - \frac{x}{y}g'(\frac{x}{y})$

$\frac{\partial^2 z}{\partial x \partial y} = (-\frac{y}{x^2})f''(\frac{y}{x}) + \frac{1}{y}g'(\frac{x}{y}) - \frac{1}{y}g'(\frac{x}{y}) - \frac{x}{y^2}g''(\frac{x}{y}) = -\frac{y}{x^2}f''(\frac{y}{x}) - \frac{x}{y^2}g''(\frac{x}{y})$.

(4) $\frac{\partial z}{\partial x} = -\frac{y}{f^2(x^2-y^2)}2xf'(x^2-y^2) = -2xy\frac{f'(x^2-y^2)}{f^2(x^2-y^2)}$

$\frac{\partial z}{\partial y} = \frac{1}{f(x^2-y^2)} - \frac{y}{f^2(x^2-y^2)}(-2y)f'(x^2-y^2) = \frac{1}{f(x^2-y^2)} + 2y^2\frac{f'(x^2-y^2)}{f^2(x^2-y^2)}$

则 $\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = -2y\frac{f'(x^2-y^2)}{f^2(x^2-y^2)} + \frac{1}{y}\frac{1}{f(x^2-y^2)} + 2y\frac{f'(x^2-y^2)}{f^2(x^2-y^2)} = \frac{1}{yf(x^2-y^2)}$

(5) $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x}\Big|_{\vec{x}} + y\frac{\partial f}{\partial y}\Big|_{\vec{x}} + yz\frac{\partial f}{\partial z}\Big|_{\vec{x}}$

$\frac{\partial u}{\partial y} = x\frac{\partial f}{\partial y}\Big|_{\vec{x}} + xz\frac{\partial f}{\partial z}\Big|_{\vec{x}}$

$\frac{\partial u}{\partial z} = xy\frac{\partial f}{\partial z}\Big|_{\vec{x}}$

2\. 设 $w = F(u^2-x^2, u^2-y^2, u^2-z^2)$

$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial x}\Big|_{\vec{x}}(-2x) = -2x\frac{\partial F}{\partial x}\Big|_{\vec{x}}$

$\frac{\partial w}{\partial y} = \frac{\partial F}{\partial y}\Big|_{\vec{x}}(-2y) = -2y\frac{\partial F}{\partial y}\Big|_{\vec{x}}$.

$\frac{\partial w}{\partial z} = \frac{\partial F}{\partial z}\Big|_{\vec{x}}(-2z) = -2z\frac{\partial F}{\partial z}\Big|_{\vec{x}}$.

$\frac{\partial w}{\partial u} = 2u\frac{\partial F}{\partial x}\Big|_{\vec{x}} + 2u\frac{\partial F}{\partial y}\Big|_{\vec{x}} + 2u\frac{\partial F}{\partial z}\Big|_{\vec{x}} = 2u(\frac{\partial F}{\partial x}\Big|_{\vec{x}} + \frac{\partial F}{\partial y}\Big|_{\vec{x}} + \frac{\partial F}{\partial z}\Big|_{\vec{x}})$.

3\. (1) $\frac{\partial Q}{\partial x_i} = 2A_{ii}x_i + \sum_{\substack{j=1 \\ (j\neq i)}}^{n} A_{ij}x_j + \sum_{\substack{j=1 \\ (j\neq i)}}^{n} A_{ji}x_i = 2\sum_{j=1}^{n} A_{ij}x_j$

则 $dQ = \sum_{i=1}^{n}\frac{\partial Q}{\partial x_i}dx_i = \sum_{i=1}^{n}(2\sum_{j=1}^{n}A_{ij}x_j dx_i) = 2\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}x_j dx_i$

(2) $\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i}\Big|_{\vec{x}} e^{-\frac{1}{2}Q(x_1\cdots x_n)} + f(x_1\cdots x_n)\, e^{-\frac{1}{2}Q(x_1\cdots x_n)}(-\frac{1}{2}\frac{\partial Q}{\partial x_i})$

$= (\frac{\partial f}{\partial x_i}\Big|_{\vec{x}} - \sum_{j=1}^{n}A_{ij}x_j f(x_1\dots x_n))\, e^{-\frac{1}{2}Q(x_1\dots x_n)}$.

4\. (1) $g'(t) = x\frac{\partial f}{\partial x}\Big|_{(tx,ty,tz)} + y\frac{\partial f}{\partial y}\Big|_{(tx,ty,tz)} + z\frac{\partial f}{\partial z}\Big|_{(tx,ty,tz)}$ ①

(2) pf: 由① $\int_0^1 g'(t)dt = \int_0^1 xf_x(tx,ty,tz)dt + \int_0^1 yf_y(tx,ty,tz)dt + \int_0^1 zf_z(tx,ty,tz)dt$.

由 Newton-Leibniz $\int_0^1 g'(t)dt = g(1) - g(0) = f(x,y,z) - f(0,0,0)$

故有 $f(x,y,z) = f(0,0,0) + x\int_0^1 f_x(tx,ty,tz)dt + y\int_0^1 f_y(tx,ty,tz)dt + z\int_0^1 f_z(tx,ty,tz)dt$

(3) 设 $g(t) = f(tx, ty, tz)$ 由(1) $g'(t) = x f_x(tx, ty, tz) + y f_y(tx, ty, tz) + z f_z(tx, ty, tz)$

则 $h'(t) = \dfrac{g'(t)}{t^n} - n\dfrac{f(tx, ty, tz)}{t^{n+1}}$

$= \dfrac{1}{t^n}\left[x f_x(tx, ty, tz) + y f_y(tx, ty, tz) + z f_z(tx, ty, tz) - f(tx, ty, tz)\right]$

$= \dfrac{1}{t^n}\left[\dfrac{n}{t} f(tx, ty, tz) - \dfrac{n}{t} f(tx, ty, tz)\right] = 0$

(4) 由(3) $h'(t) = 0$ 且 $h(t)$ 在 $t>0$ 连续

则 $h(t)$ 是常函数

$\forall t>0$ $h(t) = h(1) = f(x, y, z)$

而 $h(t) = \dfrac{f(tx, ty, tz)}{t^n}$

$\Rightarrow f(x, y, z) = \dfrac{f(tx, ty, tz)}{t^n}$ $\forall t>0$.

$\Rightarrow f(tx, ty, tz) = t^n f(x, y, z)$ $\forall t>0$.