Foundations

Bin Wang

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Outline

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Prerequisites

Textbook

1. CLRS, **Introduction to Algorithms (3rd edition)**, (2009), The MIT Press.

Reference

- Anany Levitin, 算法分析与设计基础, 潘彦译, (2004), 清华大学出版社
- 王晓东, 计算机算法设计与分析, 第四版, (2012), 电子工业出版社

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1. CLRS, **Introduction to Algorithms (3rd edition)**, (2009), The MIT Press.

Reference

- Donald E. Knuth(高德纳), The Art of Computer Programming (TAOCP), vol 1, 2, 3, 4A, addison-wesley publishing company.
- `http://www-cs-staff.stanford.edu/~knuth/`

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1. CLRS, **Introduction to Algorithms (3rd edition)**, (2009), The MIT Press.

Reference

- `http://en.wikipedia.org/`
- `http://www.github.com/`

Topics

Course Schedule

1. Foundations & Divide-and-Conquer.
2. Sorting and Order Statistics.
3. Dynamic Programming.
4. Greedy Algorithms.
5. Amortized Analysis, Heaps.
6. String Matching.
7. NPC, Approximation Algorithms.
8. Multithreaded Algorithms.

Policy

Grading Policy

- 平时作业(30%)
- 课堂表现(10%)
- 期末考试(60%)

Collaboration Policy

- 不能抄袭
- 引用他人成果需指明出处

Policy

Homework Policy

- 编程语言：C/C++/C #/Java/Python; 作业文档：Latex/Doc;
- 没有在规定时间内提交作业者，每迟交一天，扣10分，扣完为止;
- 交作业时漏交某些题目，每迟交一天，扣漏交题目分数的10%，扣完为止;
- 如果提交时网络学堂有故障，请在半小时内发邮件给助教，超过半小时按迟交处理。

What's algorithm?

Definition

An algorithm is any well-defined computational procedure that takes some value, or set of values, as **input** and produces some value, or set of values, as **output**. An algorithm is thus a sequence of computational steps that transform the input into the output.

What's algorithm?

Example

Sorting problem:

- **Input:** A sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$.
- **Output:** A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$.

Analysis of algorithms

Definition

The theoretical study of computer-program performance and resource usage.

What's more important than performance?

- correctness
- programmer time
- maintainability
- robustness
- user-friendliness

Analysis of algorithms

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Analysis of algorithms

Why study algorithms and performance?

- Performance often draws the line between what is feasible and what is impossible.
- Analysis of algorithms helps us to understand scalability.
- Algorithmic mathematics provides a language for talking about program behavior.
- The lessons of program performance generalize to other computing resources.

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Analysis of algorithms

Practical Use of algorithm

- The Human Genome Project has the goals of identifying all the $100,000$ genes in human DNA, determining the sequences of the 3 billion chemical base pairs that make up human DNA, storing this information in databases, and developing tools for data analysis.

Analysis of algorithms

Practical Use of algorithm

- The Internet enables people all around the world to quickly access and retrieve large amounts of information.
- Electronic commerce enables goods and services to be negotiated and exchanged electronically.

Some questions

Given a problem, can we find an algorithm to solve it?

Not always!

Hilbert's 10th Problem

What is a good algorithm?

Time is important!

Is a "good" algorithm always exist?

Not clear now!

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The problem of sorting

Input

A sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$.

Output

A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$.

Example

Input: $8, 2, 4, 9, 3, 6.$

Output: $2, 3, 4, 6, 8, 9.$

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Example

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Output: $2, 3, 4, 6, 8, 9.$

Insertion sort

```
INSERT-SORT(A)
for j = 2 to A.length
        key = A[j]
        // Insert A[j] into the sorted sequence A[1..j − 1]
        i = j − 1
        while i > 0 and A[i] > key
                A[i + 1] = A[i]
                i = i − 1
        A[i + 1] = key
```

Insertion sort

Example

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6
2	3	4	6	8	9

Insertion sort

Table: Analysis of INSERT-SORT

INSERT-SORT(A)	cost	times
for $j = 2$ **to** $A.length$	c_1	n
do $key = A[j]$	c_2	$n-1$
// Insert $A[j]$	0	0
$i = j - 1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
do $A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n}(t_j - 1)$
$i = i - 1$	c_7	$\sum_{j=2}^{n}(t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

Insertion sort

Analysis of INSERT-SORT

$$\begin{aligned} T(n) = c_1 n + c_2(n-1) + c_4(n-1) \\ + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) \\ + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) \end{aligned}$$

Insertion sort

Best case

In INSERT-SORT, the best case occurs if the array is already sorted.

$$T(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$$

The time can be expressed as $an + b$; it is thus a **linear function** of n.

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The time can be expressed as $an + b$; it is thus a **linear function** of n.

Insertion sort

Worst-case

If the array is in reverse sorted order, the worst case results.

$$T(n) = \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)n - (c_2 + c_4 + c_5 + c_8)$$

The time can be expressed as $an^2 + bn + c$; it is thus a **quadratic function** of n.

Insertion sort

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The time can be expressed as $an^2 + bn + c$; it is thus a **quadratic function** of n.

Insertion sort

Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.

Machine-independent time

Random-access machine(RAM) model

- No concurrent operations.
- Each instruction takes a constant amount of time.

Asymptotic Analysis

- Ignore machine-dependent constants.
- Look at the **growth** of $T(n)$ as $n \to \infty$.

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Θ-notation

Definition

$$\Theta(g(n)) = \{f(n) : \exists c_1, c_2, n_0 \in \mathbb{R}^+, s.t. \; \forall n \geq n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

We say that $g(n)$ is an **asymptotically tight bound** for $f(n)$. Denoted as $f(n) = \Theta(g(n))$ or $f(n) \in \Theta(g(n))$.

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Θ-notation

Example

$$\frac{1}{2}n^2 - 3n = \Theta(n^2), \quad 0.001n^3 \neq \Theta(n^2),$$

$$c_0 = \Theta(1), \quad \sum_{i=0}^{d} a_i n^i = \Theta(n^d) \quad (a_d > 0).$$

Θ-notation

Example

For all $n \geq n_0$,

$$c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2,$$

Dividing by n^2 yields,

$$c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2.$$

Choosing $c_1 = 1/14$, $c_2 = 1/2$, and $n_0 = 7$.

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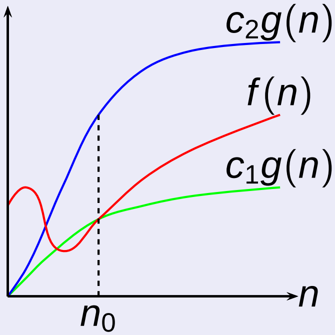
Θ-notation

$f(n) = \Theta(g(n))$



O-notation and Ω-notation

Definition

When we have only an **asymptotically upper bound**, we use O-notation.

$$O(g(n)) = \{f(n) : \exists c, n_0 \in \mathbb{R}^+, s.t.\ \forall n \geq n_0, 0 \leq f(n) \leq cg(n)\}$$

Denoted as $f(n) = O(g(n))$ or $f(n) \in O(g(n))$.

O-notation and Ω-notation

Definition

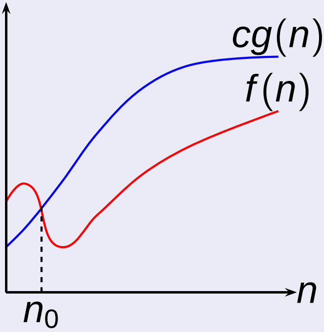
Ω-notation provides an **asymptotically lower bound**.

$$\Omega(g(n)) = \{f(n) : \exists c, n_0 \in \mathbb{R}^+, s.t. \\ \forall n \geq n_0, 0 \leq cg(n) \leq f(n)\}$$

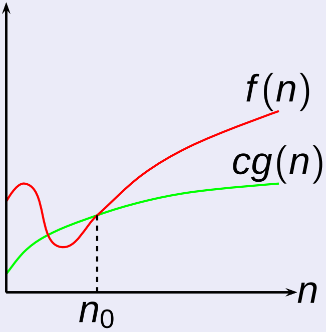
Denoted as $f(n) = \Omega(g(n))$ or $f(n) \in \Omega(g(n))$.

O-notation and Ω-notation

$f(n) = O(g(n))$



$f(n) = \Omega(g(n))$



O-notation and Ω-notation

Example

$$n = O(n^2), \quad 2n^2 = O(n^2),$$
$$2n^2 = \Omega(n), \quad 2n^2 = \Omega(n^2).$$

O-notation and Ω-notation

Theorem 3.1

For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Asymptotic notation in equations

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n) = \Theta(n^2)$$

$$\Theta(n^2) + O(n^2)$$

O-notation and Ω-notation

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Asymptotic notation in equations

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n) = \Theta(n^2)$$

$$\Theta(n^2) + O(n^2) = \Theta(n^2)$$

o-notation and ω-notation

Definition

$$\begin{aligned} o(g(n)) &= \{f(n) : \forall c > 0, \exists n_0 > 0, \\ &\quad s.t. \forall n \geq n_0, 0 \leq f(n) < cg(n)\} \end{aligned}$$

Denoted as $f(n) = o(g(n))$. Intuitively, $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

$$\begin{aligned} \omega(g(n)) &= \{f(n) : \forall c > 0, \exists n_0 > 0, \\ &\quad s.t. \forall n \geq n_0, 0 \leq cg(n) < f(n)\} \end{aligned}$$

The relation $f(n) = \omega(g(n))$ implies that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

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o-notation and ω-notation

Example

$$2n = o(n^2), \quad 2n^2 \neq o(n^2),$$
$$2n^2 = \omega(n), \quad 2n^2 \neq \omega(n^2).$$

Comparison of functions

Transitivity

$f(n) = \gamma(g(n))$ and $g(n) = \gamma(h(n))$ imply
$f(n) = \gamma(h(n))$, $\gamma = \Theta, O, \Omega, o, \omega$

Reflexivity

$f(n) = \Theta(f(n)), f(n) = O(f(n)), f(n) = \Omega(f(n))$

Comparison of functions

Symmetry

$$f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$$

Transpose symmetry

$$f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$$
$$f(n) = o(g(n)) \iff g(n) = \omega(f(n))$$

An analogy between functions and real numbers

Asymptotic Relation between functions	Relations between real numbers
$f(n) = O(g(n))$	$a \leq b$
$f(n) = \Omega(g(n))$	$a \geq b$
$f(n) = \Theta(g(n))$	$a = b$
$f(n) = o(g(n))$	$a < b$
$f(n) = \omega(g(n))$	$a > b$

History of notation

History of noation

- O-notation was presented by P. Bachmann in 1892.
- o-notation was invented by E. Landau in 1909 for his discussion of the distribution of prime numbers.
- Ω and Θ notations were advocated by D. Knuth in 1976.

Standard notations and common functions

Floors and ceilings

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

For any integer n, $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$,
and for integers $a, b > 0$
$\lceil a/b \rceil \leq (a+(b-1))/b, \lfloor a/b \rfloor \geq ((a-(b-1))/b$

Standard notations and common functions

Logarithms

For all real $a > 0$, $b > 0$, $c > 0$, and n.

$$\log_b a = \frac{1}{\log_a b}, a^{\log_b c} = c^{\log_b a}$$

$$\frac{x}{1+x} \leq \ln(1+x) \leq x$$

Standard notations and common functions

Factorials

Stirling's approximation:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

Standard notations and common functions

Factorials

Stirling's approximation:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

$n! = o(n^n), n! = \omega(2^n), \lg(n!) = \Theta(n \lg n)$

Standard notations and common functions

Functional iteration

$$f^{(i)}(n) = \begin{cases} n & i = 0 \\ f(f^{(i-1)}(n)) & i > 0 \end{cases}$$

The iterated logarithm function:

$\lg^* n = \min\{i \geq 0 : \lg^{(i)} n \leq 1\}$

Standard notations and common functions

Functional iteration

$$f^{(i)}(n) = \begin{cases} n & i = 0 \\ f(f^{(i-1)}(n)) & i > 0 \end{cases}$$

The iterated logarithm function:

$\lg^* n = \min\{i \geq 0 : \lg^{(i)} n \leq 1\}$ $\lg^* 2 = 1, \lg^* 4 = 2, \lg^* 16 = 3, \lg^* 65536 = 4, \lg^*(2^{65536}) = 5.$

Exercises

Sorting the speed of growth

$(n-2)!, 5\lg(n+100)^{10}, 2^{2n}, 0.001n^4+3n^3+1, \ln^2 n, \sqrt[3]{n}, 2^n, n!$

Which is asymptotically larger

$\lg(\lg^* n)$ or $\lg^*(\lg n)$

What is recurrences?

Fibonacci numbers

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

$$F(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F(n-1) + F(n-2) & \text{if } n > 1. \end{cases}$$

FIBONNACI(n)

1 **if** $(n = 0)$ **return** 0
2 **if** $(n = 1)$ **return** 1
3 **return** FIBONNACI($n-1$) + FIBONNACI($n-2$)

What is recurrences?

Definition

A recurrence is an equation or inequation that describes a function in terms of its value on smaller inputs.

What is recurrences?

History of recurrences

- In 1202, recurrences were studied by Leonardo Fibonacci (1170-1250).
- A. De Moivre (1667-1754) introduced the method of generating functions for solving recurrences.
- Bentley, Haken and Saxe presented the Master Theorem in 1980.

The substitution method

General method

1. **Guess** the form of the solution.
2. **Verify** by mathematical induction.

The substitution method

Example

$$T(n) = 9T(\lfloor n/3 \rfloor) + n$$

- Assume that $T(1) = \Theta(1)$
- Guess $O(n^3)$. (Prove O and Ω separately.)
- Assume that $T(k) \leq ck^3$ for $k < n$.
- Prove $T(n) \leq cn^3$ by induction.

The substitution method

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The substitution method

Example

$$
\begin{aligned}
T(n) &= 9T(n/3) + n \\
&\leq 9c(n/3)^3 + n \\
&= (c/3)n^3 + n \\
&= cn^3 - ((2c/3)n^3 - n) \\
&\qquad \nwarrow \textit{desired} - \textit{residual} \\
&\leq cn^3 \leftarrow \textit{desired}
\end{aligned}
$$

When $((2c/3)n^3 - n) \geq 0$, it is true.

The substitution method

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\end{aligned}
$$

When $((2c/3)n^3 - n) \geq 0$, it is true. ***not tight!***

The substitution method

Example

A tighter upper bound ?

Assume $T(k) \leq ck^2$ for $k < n$

$$\begin{aligned} T(n) &= 9T(n/3) + n \\ &\leq 9c(n/3)^2 + n \\ &= cn^2 + n \\ &= cn^2 - (-n) \\ &\leq cn^2 \end{aligned}$$

We can never get $-n > 0$!

The substitution method

Example

A tighter upper bound ?

Assume $T(k) \leq ck^2$ for $k < n$

$$
\begin{aligned}
T(n) &= 9T(n/3) + n \\
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&\le 9c(n/3)^2 + n \\
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&\le cn^2 \qquad\qquad \textit{Wrong}!
\end{aligned}
$$

We can never get $-n > 0$!

The substitution method

Example

A tighter upper bound !

Strengthen the inductive hypothesis:

Assume $T(k) \leq c_1k^2 - c_2k$ for $k < n$

$$\begin{aligned}
T(n) &= 9T(n/3) + n \\
&\leq 9(c_1(n/3)^2 - c_2(n/3)) + n \\
&= c_1n^2 - 3c_2n + n \\
&= (c_1n^2 - c_2n) - (2c_2n - n) \\
&\leq c_1n^2 - c_2n
\end{aligned}$$

The substitution method

Example

A tighter upper bound !

Strengthen the inductive hypothesis:

Assume $T(k) \leq c_1k^2 - c_2k$ for $k < n$

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The substitution method

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&= (c_1 n^2 - c_2 n) - (2c_2 n - n) \\
&\leq c_1 n^2 - c_2 n \qquad \text{Pick } c_2 > 1/2
\end{aligned}$$

The recursion-tree method

Definition

- A **recursion tree** models the costs of a execution of an recursive algorithm.
- Each node of a recursion tree represents the cost of a single subproblem.
- A recursion tree is good for generating a good guess, which is then verified by the substitution method.

Example

$$T(n) = T(\lfloor n/4 \rfloor) + T(\lfloor n/2 \rfloor) + \Theta(n^2)$$

The recursion-tree method

$$T(n) = T(\lfloor n/4 \rfloor) + T(\lfloor n/2 \rfloor) + \Theta(n^2)$$

$$T(n)$$

The recursion-tree method

$$T(n) = T(\lfloor n/4 \rfloor) + T(\lfloor n/2 \rfloor) + \Theta(n^2)$$

$$\begin{array}{ccc} & cn^2 & \\ \swarrow & & \searrow \\ T(n/4) & & T(n/2) \end{array}$$

The recursion-tree method

$$T(n) = T(\lfloor n/4 \rfloor) + T(\lfloor n/2 \rfloor) + \Theta(n^2)$$

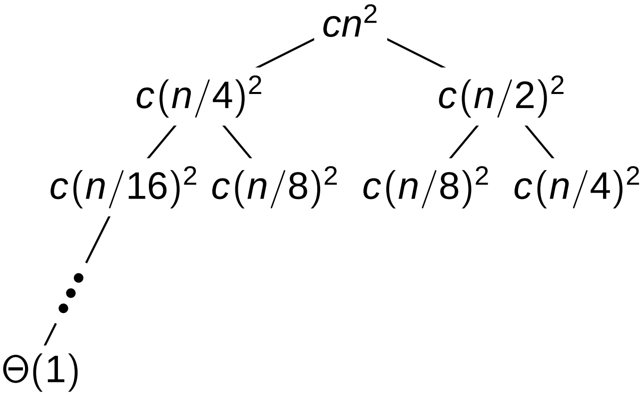
$$cn^2$$

$$c(n/4)^2 \qquad\qquad c(n/2)^2$$

$$T(n/16) \quad T(n/8) \quad T(n/8) \quad T(n/4)$$

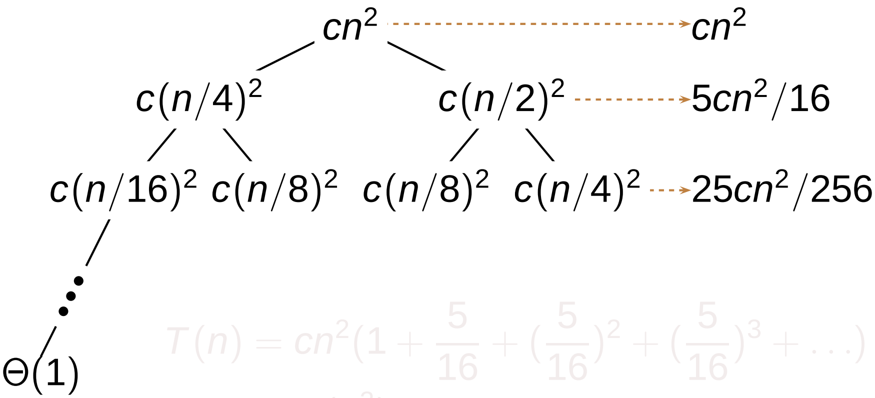
The recursion-tree method

$$T(n) = T(\lfloor n/4 \rfloor) + T(\lfloor n/2 \rfloor) + \Theta(n^2)$$



The recursion-tree method

$$T(n) = T(\lfloor n/4 \rfloor) + T(\lfloor n/2 \rfloor) + \Theta(n^2)$$



$$T(n) = cn^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \dots\right)$$
$$= \Theta(n^2)$$

The recursion-tree method

$$T(n) = T(\lfloor n/4 \rfloor) + T(\lfloor n/2 \rfloor) + \Theta(n^2)$$

$cn^2 \rightarrow cn^2$

$c(n/4)^2 \quad c(n/2)^2 \rightarrow 5cn^2/16$

$c(n/16)^2 \quad c(n/8)^2 \quad c(n/8)^2 \quad c(n/4)^2 \rightarrow 25cn^2/256$

$\vdots$

$\Theta(1)$

$$T(n) = cn^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \ldots\right)$$
$$= \Theta(n^2)$$

The master method

The master method applies to recurrences of the form

$$T(n) = aT(n/b) + f(n),$$

where $a \geq 1, b > 1$, and f is asymptotically positive.

The master method

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

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The master method

Example

- $T(n) = 9T(n/3) + n$
  We have $a = 9, b = 3, f(n) = n$, and thus we have that $n^{\log_b a} = n^{\log_3 9} = n^2$. Since $f(n) = O(n^{\log_3 9 - \epsilon})$, where $\epsilon = 1$, we can apply **case 1**. The solution is $T(n) = \Theta(n^2)$.
- $T(n) = T(2n/3) + 1$
  $a = 1, b = 3/2, f(n) = 1, f(n) = \Theta(n^{\log_b a}) = \Theta(1)$.
  Case 2 applies, $T(n) = \Theta(\lg n)$

The master method

Example

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The master method

Example

- $T(n) = 3T(n/4) + n \lg n$
  $a = 3, b = 4, f(n) = n \lg n, f(n) = \Omega(n^{\log_4 3 + \epsilon})$, where $\epsilon \approx 0.2$. For sufficiently large n,
  $af(n/b) = 3(n/4)\lg(n/4) \leq (3/4) n \lg n$ for $c = 3/4$.
  By **case 3**, $T(n) = \Theta(n \lg n)$.

The master method

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  $af(n/b) = 3(n/4)\lg(n/4) \leq (3/4)n \lg n$ for $c = 3/4$.
  By **case 3**, $T(n) = \Theta(n \lg n)$.

The master method

Is master method omnipotent?

Master Theorem fails in the following cases:

- When $f(n)$ is smaller than $n^{\log_b a}$ but not **polynomially** smaller. This is a gap between cases 1 and 2.
- When $f(n)$ is larger than $n^{\log_b a}$ but not **polynomially** larger. This is a gap between cases 2 and 3.
- When the regularity condition in case 3 fails to hold.

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- When the regularity condition in case 3 fails to hold.

The master method

Example

$$T(n) = 2T(n/2) + n \lg n$$

$a = 2, b = 2, f(n) = n \lg n$, and $n^{\log_b a} = n$.
$f(n) = n \lg n$ is asymptotically larger than n, but not **polynomially** larger. The ratio $f(n)/n = \lg n$ is asymptotically less than n^{ϵ} for any positive constant ϵ.

The master method

A more general method

In 1998, Mohamad Akra and Louay Bazzi presented a more general master method:

$$T(n) = \sum_{i=1}^{k} a_i T(\lfloor n/b_i \rfloor) + f(n)$$

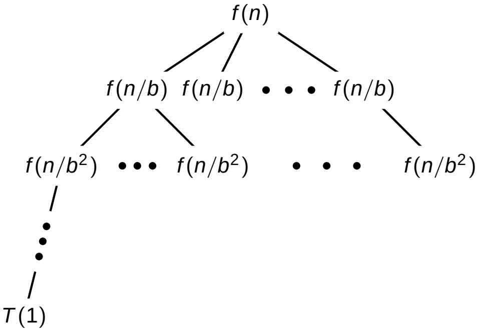
The master method

A more general method

This method would work on a recurrence such as $T(n) = T(\lfloor n/3 \rfloor) + T(\lfloor 2n/3 \rfloor) + O(n)$. We first find the value of p such that $\sum_{i=1}^{p} a_i b_i^{-p} = 1$. The solution to the recurrence is then

$$T(n) = \Theta(n^p) + \Theta(n^p \int_{n'}^{n} \frac{f(x)}{x^{p+1}} dx)$$

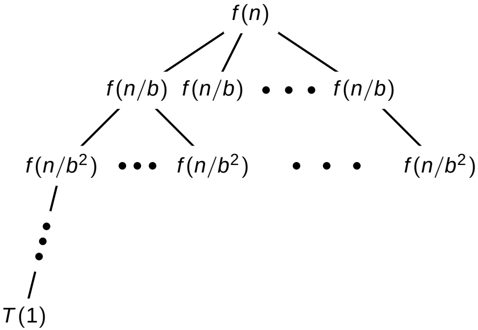
Idea of master theorem



Number of leaves

$$a^h = a^{\log_b n} = n^{\log_b a}$$

Idea of master theorem



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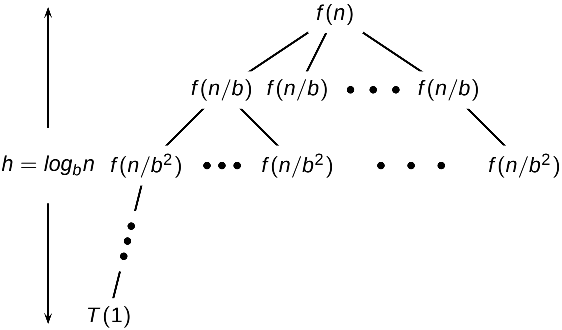

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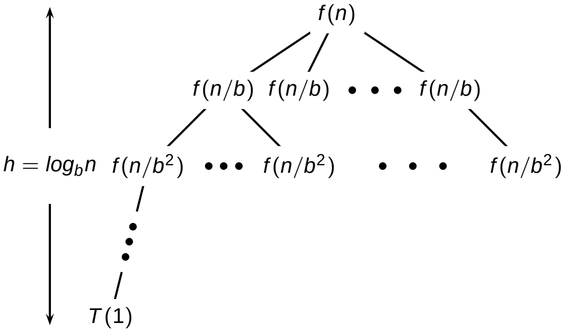

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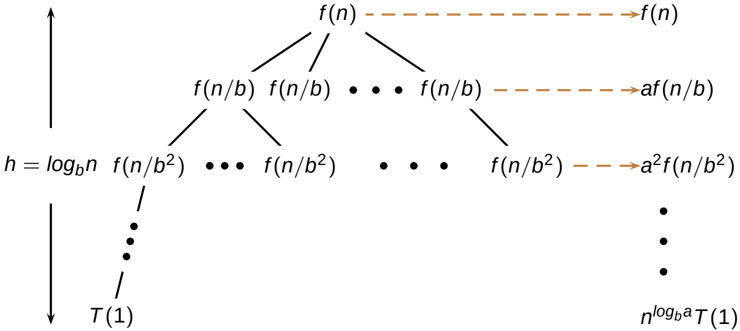

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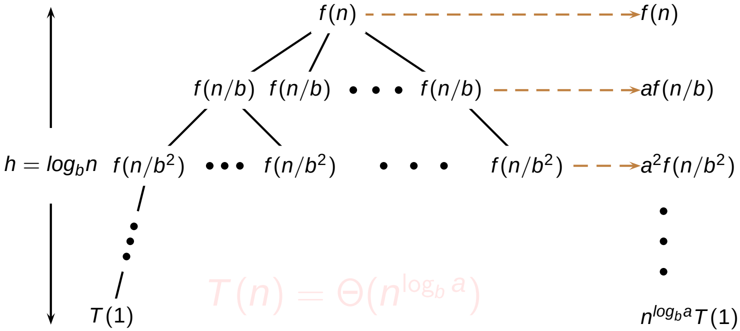
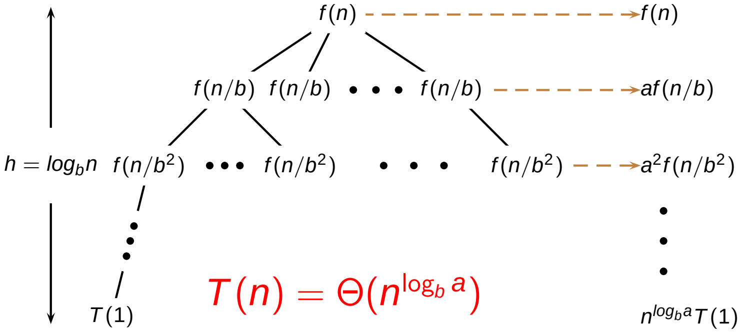


Case 1

The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

Idea of master theorem



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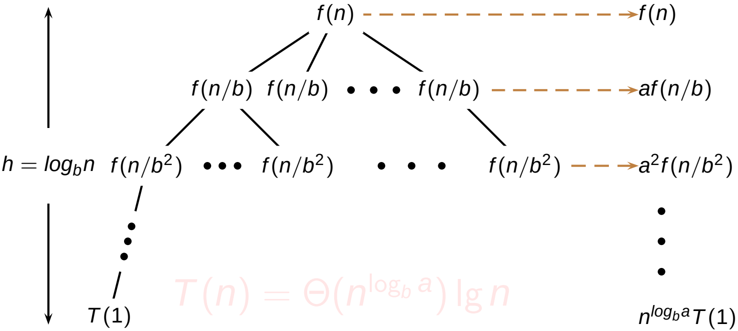
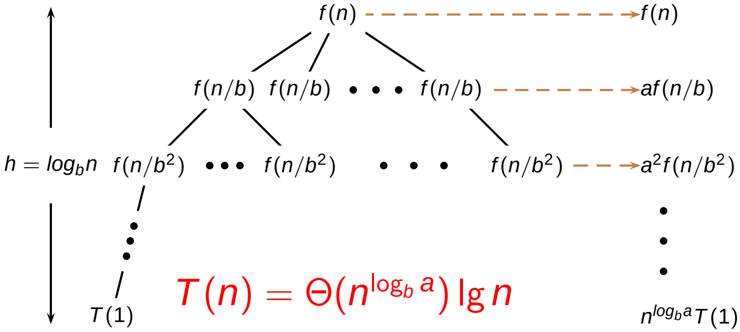


Case 2

The weight is approximately the same on each of the $\log_b n$ levels.

Idea of master theorem



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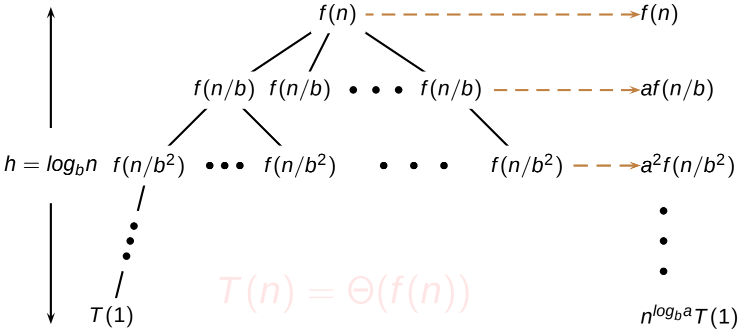
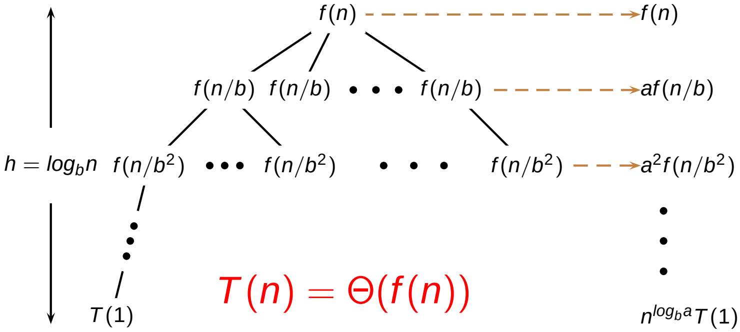


Case 3

The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

Idea of master theorem

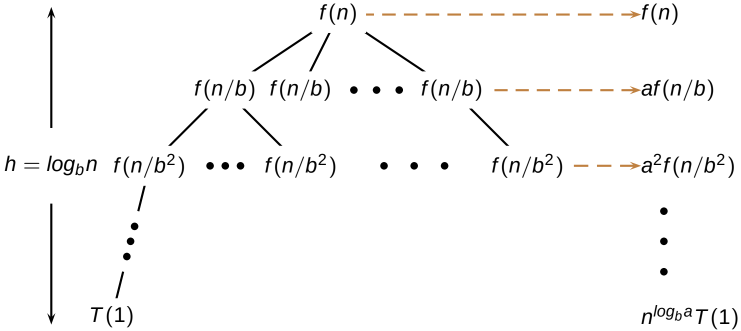


$$T(n) = \Theta(f(n))$$

Case 3

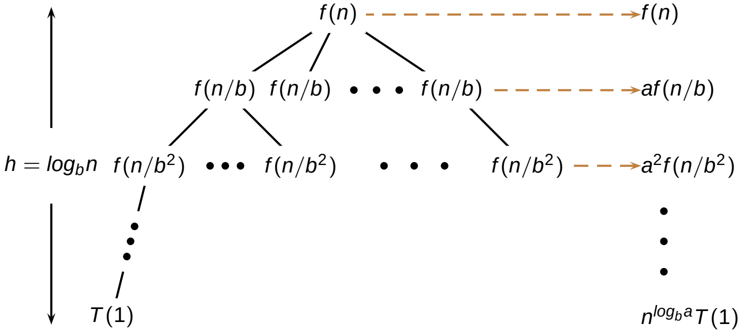
The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

Proof of master theorem



$$T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$$

Proof of master theorem



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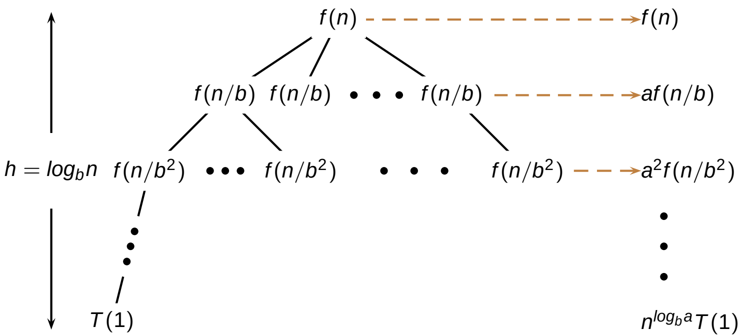

$$T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$$

Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a-\epsilon})$

Since $f(n/b^j) = O((n/b^j)^{\log_b a-\epsilon})$, then

$$g(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j)$$
$$= O\left(\sum_{j=0}^{\log_b n-1} a^j \left(\frac{n}{b^j}\right)^{\log_b a-\epsilon}\right)$$

Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$$\sum_{j=0}^{\log_b n - 1} a^j \left(\frac{n}{b^j}\right)^{\log_b a - \epsilon} = n^{\log_b a - \epsilon} \sum_{j=0}^{\log_b n - 1} \left(\frac{ab^{\epsilon}}{b^{\log_b a}}\right)^j$$

$$= n^{\log_b a - \epsilon} \sum_{j=0}^{\log_b n - 1} (b^{\epsilon})^j$$

Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$

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Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$$\sum_{j=0}^{\log_b n - 1} a^j \left(\frac{n}{b^j}\right)^{\log_b a - \epsilon} = n^{\log_b a - \epsilon}\left(\frac{b^{\epsilon \log_b n} - 1}{b^\epsilon - 1}\right)$$

$$= n^{\log_b a - \epsilon}\left(\frac{n^\epsilon - 1}{b^\epsilon - 1}\right)$$

Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a-\epsilon})$

$$\sum_{j=0}^{\log_b n-1} a^j \left(\frac{n}{b^j}\right)^{\log_b a-\epsilon} = n^{\log_b a-\epsilon}\left(\frac{b^{\epsilon\log_b n}-1}{b^\epsilon-1}\right)$$

$$= n^{\log_b a-\epsilon}\left(\frac{n^\epsilon-1}{b^\epsilon-1}\right)$$

Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$$g(n) = O\left(n^{\log_b a - \epsilon}\left(\frac{n^\epsilon - 1}{b^\epsilon - 1}\right)\right)$$

$$= O\left(n^{\log_b a - \epsilon} n^\epsilon\right)$$

$$= O(n^{\log_b a})$$

Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a-\epsilon})$

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$$= O(n^{\log_b a})$$

Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$$T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$$

$$= \Theta(n^{\log_b a}) + g(n)$$

$$= \Theta(n^{\log_b a}) + O(n^{\log_b a})$$

$$= \Theta(n^{\log_b a})$$

Proof of master theorem

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$$
\begin{aligned}
T(n) &= \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) \\
&= \Theta(n^{\log_b a}) + g(n) \\
&= \Theta(n^{\log_b a}) + O(n^{\log_b a}) \\
&= \Theta(n^{\log_b a})
\end{aligned}
$$

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Case 1: $f(n) = O(n^{\log_b a - \epsilon})$

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$$= \Theta(n^{\log_b a}) + g(n)$$

$$= \Theta(n^{\log_b a}) + O(n^{\log_b a})$$

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Proof of master theorem

Case 2: $f(n) = \Theta(n^{\log_b a})$

We have $f(n/b^j) = \Theta((n/b^j)^{\log_b a})$, then

$$g(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$$

$$= \Theta\left(\sum_{j=0}^{\log_b n - 1} a^j \left(\frac{n}{b^j}\right)^{\log_b a}\right)$$

Proof of master theorem

Case 2: $f(n) = \Theta(n^{\log_b a})$

We have $f(n/b^j) = \Theta((n/b^j)^{\log_b a})$, then

$$\sum_{j=0}^{\log_b n-1} a^j \left(\frac{n}{b^j}\right)^{\log_b a} = n^{\log_b a} \sum_{j=0}^{\log_b n-1} \left(\frac{a}{b^{\log_b a}}\right)^j$$

$$= n^{\log_b a} \sum_{j=0}^{\log_b n-1} 1$$

$$= n^{\log_b a} \log_b n$$

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Proof of master theorem

Case 2: $f(n) = \Theta(n^{\log_b a})$

$$T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$$

$$= \Theta(n^{\log_b a}) + g(n)$$

$$= \Theta(n^{\log_b a}) + \Theta(n^{\log_b a} \log_b n)$$

$$= \Theta(n^{\log_b a} \lg n)$$

Proof of master theorem

Case 2: $f(n) = \Theta(n^{\log_b a})$

$$
\begin{aligned}
T(n) &= \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) \\
&= \Theta(n^{\log_b a}) + g(n) \\
&= \Theta(n^{\log_b a}) + \Theta(n^{\log_b a} \log_b n) \\
&= \Theta(n^{\log_b a} \lg n)
\end{aligned}
$$

Proof of master theorem

Case 3: $f(n) = \Omega(n^{\log_b a+\epsilon})$

$$g(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j)$$

$$\leq \sum_{j=0}^{\log_b n-1} c^j f(n) \quad (\text{By } af(n/b) \leq cf(n))$$

$$\leq f(n)\sum_{j=0}^{\infty} c^j = f(n)\left(\frac{1}{1-c}\right)$$

Proof of master theorem

Case 3: $f(n) = \Omega(n^{\log_b a+\epsilon})$

$$
\begin{aligned}
T(n) &= \Theta(n^{\log_b a}) + g(n) \\
&= \Theta(n^{\log_b a}) + \Theta(f(n)) \\
&= \Theta(f(n))
\end{aligned}
$$

Proof of master theorem

Case 3: $f(n) = \Omega(n^{\log_b a+\epsilon})$

$$\begin{aligned} T(n) &= \Theta(n^{\log_b a}) + g(n) \\ &= \Theta(n^{\log_b a}) + \Theta(f(n)) \\ &= \Theta(f(n)) \end{aligned}$$

Changing variables

Changing variables

$$T(n) = 2T(\lfloor\sqrt{n}\rfloor) + \lg n$$

- Let $m = \lg n$, then $T(2^m) = 2T(2^{m/2}) + m$.
- Let $S(m) = T(2^m)$, then $S(m) = 2S(m/2) + m$.
- $T(n) = T(2^m) = S(m) = \Theta(m \lg m) = \Theta(\lg n \lg \lg n)$.

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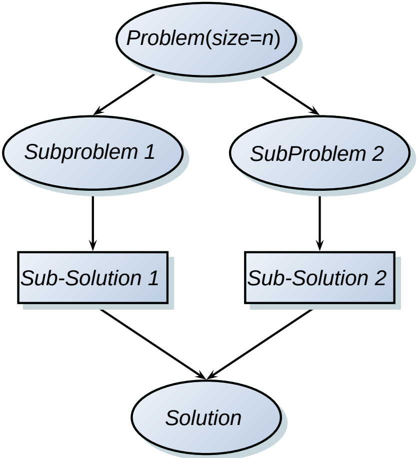
What is Divide and Conquer?



Design paradigm

1. **Divide** the problem (instance) into subproblems.
2. **Conquer** subproblems by solving them recursively.
3. **Combine** subproblems solutions.

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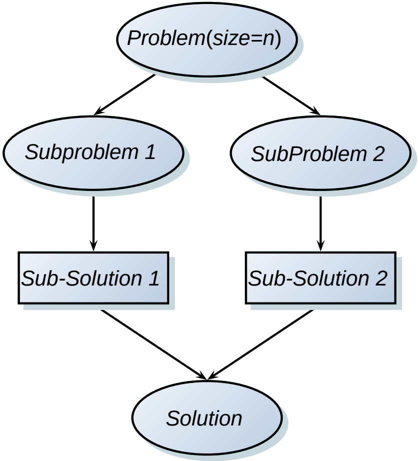

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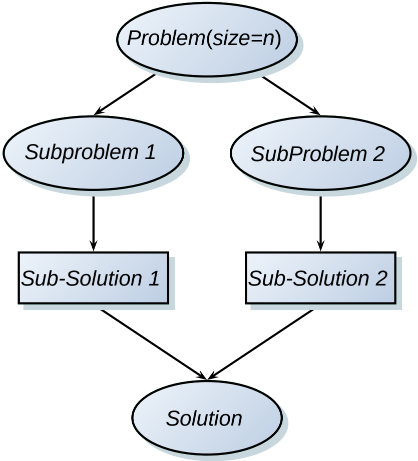

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MergeSort

Design paradigm

1. **Divide:** Trivial! We get two $n/2$-size subarrays.
2. **Conquer:** Recursively sort the two subarrays.
3. **Combine:** Linear-time merge.

MergeSort

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MergeSort

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MergeSort

```
MERGE-SORT(A,p,r)
if p < r
    q = ⌊(p + r)/2⌋
    MERGE-SORT(A,p,q)
    MERGE-SORT(A,q+1,r)
    MERGE(A,p,q,r)
```

Merging two sorted arrays

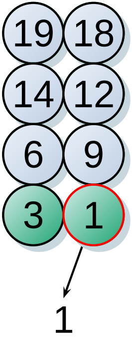
Example

$$19, 3, 6, 14, 1, 9, 18, 12$$

Merging two sorted arrays

Example

19, 3, 6, 14, 1, 9, 18, 12



Merging two sorted arrays

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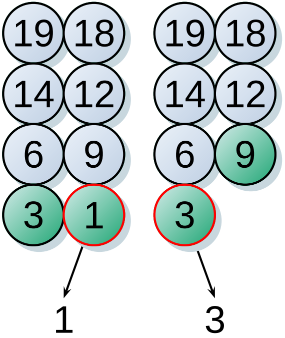
19, 3, 6, 14, 1, 9, 18, 12



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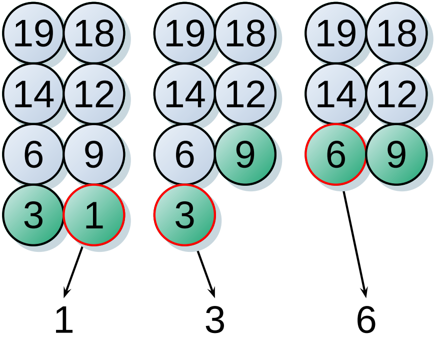

Merging two sorted arrays

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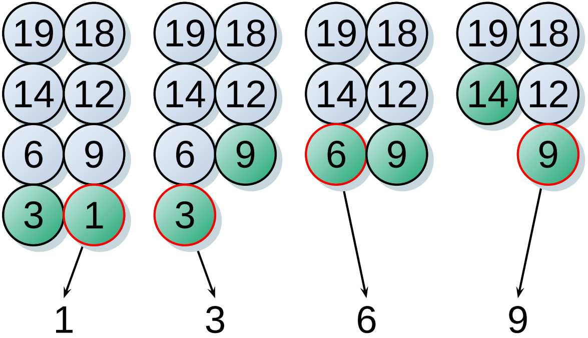

Merging two sorted arrays

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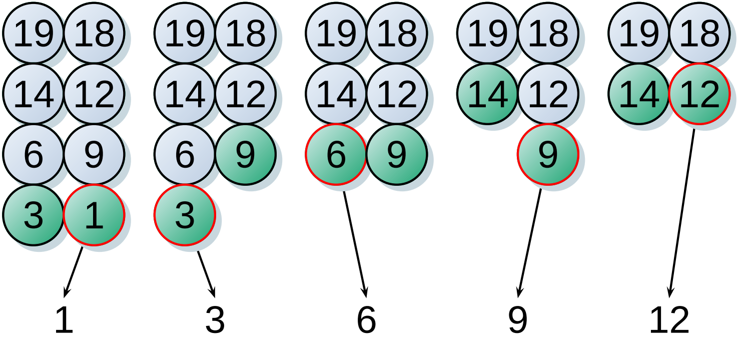

Analysis of MergeSort

Analysis paradigm

1. **Divide:** $D(n) = \Theta(1)$.
2. **Conquer:** Two subarrays = $2T(n/2)$.
3. **Combine:** $C(n) = \Theta(n)$.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) + \Theta(1) & \text{if } n > 1. \end{cases}$$

Master theorem, Case 2: $\Theta(n^{\log_b a}) = \Theta(n)$

$$T(n) = \Theta(n \lg n)$$

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Binary search

Design paradigm

1. Divide: Trivial! Check the middle element.
2. Conquer: Recursively search one subarray.
3. Combine: Return the position.

Example

Search 10 in the following array:

1	4	7	9	10	15	20

Binary search

Design paradigm

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Example

Search 10 in the following array:

1 4 7 9 $\boxed{10}$ 15 20

Analysis of Binary search

Analysis paradigm

1. **Divide:** $D(n) = \Theta(1)$.
2. **Conquer:** Only search one subarray = $T(n/2)$.
3. **Combine:** $\Theta(1)$.

$$T(n) = 1T(n/2) + \Theta(1)$$

Master theorem, Case 2: $\Theta(n^{\log_b a}) = \Theta(1)$

$$T(n) = \Theta(\lg n)$$

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Fibonacci numbers

Naive recursive algorithm

FIBONNACI(n)

if $(n = 0)$ **return** 0 ;
if $(n = 1)$ **return** 1 ;
return FIBONNACI$(n - 1)$
+ FIBONNACI$(n - 2)$;

Fibonacci numbers

Naive recursive algorithm

$$T(n) = T(n-1) + T(n-2)$$

$$T(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$T(n) = \Omega(\phi^n), \phi = (1+\sqrt{5})/2$$

Fibonacci numbers

Naive recursive algorithm

$$T(n) = T(n-1) + T(n-2)$$

$$T(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$T(n) = \Omega(\phi^n), \phi = (1+\sqrt{5})/2$$

Fibonacci numbers

Naive recursive squaring

$F_n = \phi^n/\sqrt{5}$ rounded to the nearest integer.

- $T(n) = T(n/2) + \Theta(1) \Longrightarrow T(n) = \Theta(\lg n)$.
- ***Unreliable!***

Bottom-up

- Compute $F_0, F_1, \ldots, F_n$.
- $T(n) = \Theta(n)$.

Fibonacci numbers

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Bottom-up

- Compute $F_0, F_1, \ldots, F_n$.
- $T(n) = \Theta(n)$.

Fibonacci numbers

Recursive squaring

Theorem:

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

Fibonacci numbers

Proof.

$$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1$$

Inductive step($n \geq 2$) :

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$\square$

Fibonacci numbers

Proof.

$$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1$$

Inductive step($n \geq 2$) :

$$\begin{aligned} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} &= \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

□

Fibonacci numbers

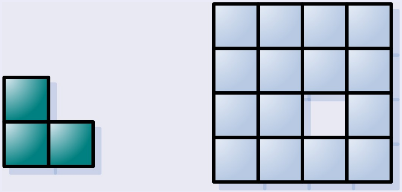
Recursive squaring

1. **Divide:** $n/2$
2. **Conquer:** Calculate $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n/2}$
3. **Combine:** $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n/2} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n/2}$

$T(n) = T(n/2) + \Theta(1) \Longrightarrow T(n) = \Theta(\lg n)$.

Exercises

1. Triomino拼图，用一个L型瓦片（含三个方块）覆盖一个缺少了一个方块的$2^n \times 2^n$的棋盘。设计此问题的分治算法并分析复杂度。



Big integers multiplication

A simple example

$$
\begin{aligned}
23 * 14 &= (2 \cdot 10^1 + 3 \cdot 10^0) \\
&\quad * (1 \cdot 10^1 + 4 \cdot 10^0) \\
&= (2 * 1)10^2 + (3 * 1 + 2 * 4)10^1 \\
&\quad + (3 * 4)10^0
\end{aligned}
$$

$$
\begin{aligned}
(3 * 1 + 2 * 4) &= (2 + 3) * (1 + 4) \\
&\quad - (2 * 1) - (3 * 4)
\end{aligned}
$$

Big integers multiplication

A simple example

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&\qquad + (3 * 4)10^0
\end{aligned}
$$

$$
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(3 * 1 + 2 * 4) &= (2 + 3) * (1 + 4) \\
&\qquad - (2 * 1) - (3 * 4)
\end{aligned}
$$

Big integers multiplication

A general example

$$
\begin{aligned}
c = a * b &= (a_1 10^{n/2} + a_0) * (b_1 10^{n/2} + b_0) \\
&= (a_1 * b_1) 10^n + (a_1 * b_0 + a_0 * b_1) 10^{n/2} \\
&\quad + (a_0 * b_0) \\
&= c_2 10^n + c_1 10^{n/2} + c_0
\end{aligned}
$$

$$c_1 = (a_1 + a_0) * (b_1 + b_0) - (c_2 + c_0)$$

Big integers multiplication

A general example

$$
\begin{aligned}
c = a * b &= (a_1 10^{n/2} + a_0) * (b_1 10^{n/2} + b_0) \\
&= (a_1 * b_1) 10^n + (a_1 * b_0 + a_0 * b_1) 10^{n/2} \\
&\qquad + (a_0 * b_0) \\
&= c_2 10^n + c_1 10^{n/2} + c_0 \\
c_1 &= (a_1 + a_0) * (b_1 + b_0) - (c_2 + c_0)
\end{aligned}
$$

Big integers multiplication

Analysis

1. **Divide:** $D(n) = \Theta(n)$.
2. **Conquer:** $3T(n/2)$.
3. **Combine:** $C(n) = \Theta(n)$.

Master theorem

$$T(n) = 3T(n/2) + \Theta(n)$$

Case 1: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{1.585})$

Matrix multiplication

Matrix multiplication

- **Input:** $A = [a_{ij}], B = [b_{ij}]$.
- **Output:** $C = [c_{ij}] = A \cdot B \quad i, j = 1, 2, \cdots, n$.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

$$\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

Matrix multiplication

MATRIX-MULTIPLY(A, B)

```
n = A.rows
let C be an n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

$T(n) = \Theta(n^3)$

Matrix multiplication

MATRIX-MULTIPLY(A, B)

```
n = A.rows
let C be an n × n matrix
for i = 1 to n
        for j = 1 to n
                c_ij = 0
                for k = 1 to n
                        c_ij = c_ij + a_ik · b_kj
return C
```

$$T(n) = \Theta(n^3)$$

Matrix multiplication

Idea of Divide and Conquer

Divide a $n \times n$ matrix multiplication into 2×2 $(n/2) \times (n/2)$ submatrix multiplication.

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\begin{aligned} r &= ae + bg, & s &= af + bh, \\ t &= ce + dg, & u &= cf + dh. \end{aligned}$$

Matrix multiplication

Analysis

$$T(n) = \quad T(n/2) +$$

Master theorem

$$f(n) = \Theta(n^2) = O(n^{\log_b a - \epsilon}) = O(n^{3-\epsilon})$$

Case 1: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^3)$

No improvment !?

Matrix multiplication

Analysis

$$T(n) = 8T(n/2) +$$

Master theorem

$$f(n) = \Theta(n^2) = O(n^{\log_b a - \epsilon}) = O(n^{3-\epsilon})$$

Case 1: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^3)$

No improvment !?

Matrix multiplication

Analysis

$$T(n) = 8T(n/2) + \Theta(n^2)$$

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Case 1: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^3)$

No improvment !?

Matrix multiplication

Strassen's idea

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

7 mults, 18 adds/subs

Matrix multiplication

Strassen's idea

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7 mults, 18 adds/subs

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Strassen's idea

$$P_1 = a \cdot (f - h)$$
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$$P_3 = (c + d) \cdot e$$
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$$P_5 = (a + d) \cdot (e + h)$$
$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e + f)$$

$$\begin{aligned} r &= P_5 + P_4 - P_2 + P_6 \\ &= (a + d)(e + h) \\ &\quad + d(g - e) - (a + b)h \\ &\quad (b - d)(g + h) \\ &= ae + ah + de + dh \\ &\quad + dg - de - ah - bh \\ &\quad - bg + bh - dg - dh \\ &= ae + bg \end{aligned}$$

Matrix multiplication

Strassen's Divide and Conquer

1. **Divide:** Partition A and B into $(n/2) \times (n/2)$ submatrices.
2. **Conquer:** Perform 7 multiplications of $(n/2) \times (n/2)$ submatrices recursively.
3. **Combine:** Form C using $+$ and $-$ on $(n/2) \times (n/2)$ submatrices.

Matrix multiplication

Master theorem

$$T(n) = 7T(n/2) + \Theta(n^2)$$

$$f(n) = \Theta(n^2) = O(n^{\log_b a - \epsilon}) \approx O(n^{2.81-\epsilon})$$

Case 1: $T(n) = \Theta(n^{\log_b a}) \approx \Theta(n^{2.81})$

Matrix multiplication

Master theorem

$$T(n) = 7T(n/2) + \Theta(n^2)$$

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Case 1: $T(n) = \Theta(n^{\log_b a}) \approx \Theta(n^{2.81})$

Matrix multiplication

Discussion

- The “crossover point” of Strassen’s algorithm on various systems ranging from $n = 400$ to $n = 2150$.
- In 1971, Hopcroft and Kerr proved that 7 multiplications is the best for 2×2 partition.
- **Best to date** (of theoretical interest only): $\Theta(n^{2.376\ldots})$.

Matrix multiplication

Discussion

- Strassen's algorithm is often not the method of choice for matrix multiplication:
  - The constant factor hidden in the running time is larger than the simple procedure.
  - For sparse matrices, we have better algorithms.
  - Strassen's algorithm is not quite numerically stable.
  - It uses too much memories.

Finding the closest pair of points

Problem

Given a set P of $n \geq 2$ points, we now consider the problem of finding the closest pair of points in the set.
Closest refers to the usual euclidean distance: the distance between points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Brute-force algorithm

Simply look at all $\Theta(n^2)$ pairs of points.

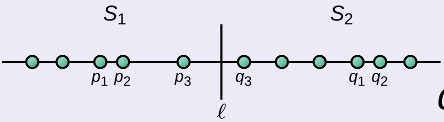
Finding the closest pair of points

Divide-and-conquer algorithm

1. **Divide:** Find a vertical line l that bisects the point set P into two sets $|P_L| = \lceil |P|/2 \rceil$, $|P_R| = \lfloor |P|/2 \rfloor$.
2. **Conquer:** Find the closest pairs of points in P_L and P_R.
3. **Combine:** How?

Finding the closest pair of points

One dimension example



$$S_1 = \{x \in S \mid x \leq m\}$$
$$S_2 = \{x \in S \mid x > m\}$$

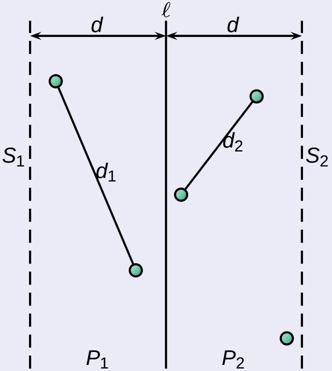
$$d = \min\{|p_1 - p_2|, \ |q_1 - q_2|\}$$
$$d_{\min} = \min\{d, |p_3 - q_3|\}$$

Finding the closest pair of points

Two dimension example

$$S_1 = \{p \in S \mid x(p) \le m\}$$
$$S_2 = \{p \in S \mid x(p) > m\}$$

$$d = \min\{d_1, d_2\}$$



Finding the closest pair of points

Two dimension example

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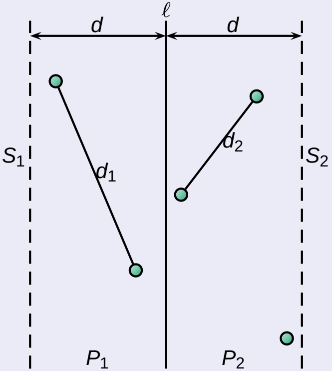
$d_{\min} = \min\{d,$ closest pair in ℓ neighborhood $\}$



Finding the closest pair of points

Two dimension example

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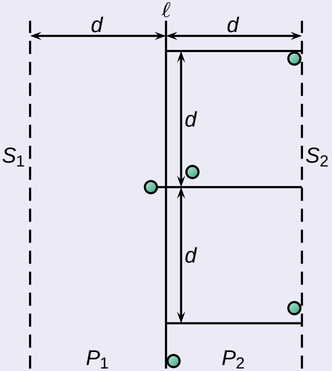
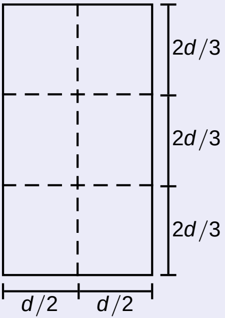


Finding the closest pair of points

How many points in the region?

6 is the maximum!

$$(x(u) - x(v))^2 + (y(u) - y(v))^2 \le (d/2)^2 + (2d/3)^2 = 25d^2/36$$



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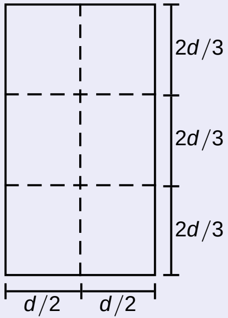
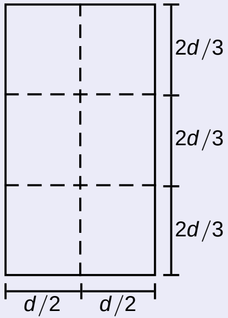


Finding the closest pair of points

How many points in the region?

6 is the maximum!

$$(x(u) - x(v))^2 + (y(u) - y(v))^2 \leq (d/2)^2 + (2d/3)^2 = 25d^2/36 \leq d^2$$

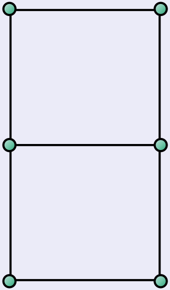


Finding the closest pair of points

How many points in the region?

6 is the maximum!

$$(x(u) - x(v))^2 + (y(u) - y(v))^2$$
$$\leq (d/2)^2 + (2d/3)^2$$
$$= 25d^2/36$$
$$\leq d^2$$



Finding the closest pair of points

Analysis

1. **Divide:** $D(n) = \Theta(n)$.
2. **Conquer:** $2T(n/2)$.
3. **Combine:** $C(n) = \Theta(n)$.

Master theorem

$$T(n) = 2T(n/2) + \Theta(n)$$

Case 2: $T(n) = \Theta(n \lg n)$

Hiring Problem

Hiring Problem

Suppose that you need to hire a new office assistant from n candidates. After the interview, you must decide whether to hire him (her) or not. If you hire somebody, you have to pay some money.

Hiring Problem

```
HIRE-ASSISTANT(n)
best = 0
for i = 1 to n
        interview candidate i
        if candidate i is better than
                        candidate best
                best = i
                hire candidate i
```

Hiring Problem

Worst-case analysis

- We actually hire every candidate that we interview.
- If every hiring cost is c_h, the total hiring cost is $O(nc_h)$.

Hiring Problem

Probabilistic analysis

- ***Why?*** More practical!
- In order to perform a probabilistic analysis, we must use knowledge of, or make assumption about, the distribution of the inputs.
- We must have greater control over the order in which we interview the candidates.

Review of probability knowledge

Expectation

The ***expected value(expectation)*** of a discrete random variable X is

$$E[X] = \sum_{x} x Pr\{X = x\}.$$

Its variance is

$$V[X] = E[X - E[X]]^2 = E[X^2] - [E[X]]^2.$$

Review of probability knowledge

Expectation

Some properties:

$$E[X+Y] = E[X] + E[Y] \qquad E[aX] = aE[X]$$
$$E[XY] = E[X]E[Y] \qquad V[aX] = a^2V[X]$$
$$V[X+Y] = V[X] + V[Y]$$

Review of probability knowledge

Conditional probability

The ***conditional probability*** of an event A given that another event B occurs is defined to be

$$Pr\{A|B\} = \frac{Pr\{A \cap B\}}{Pr\{B\}}$$

Hence we have $Pr\{A \cap B\} = Pr\{A|B\}Pr\{B\}$

Indicator Random Variable

Definition

Give a sample space S and an event A, the ***indicator random variable*** $I\{A\}$ associated with event A is defined as

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

Lemma 5.1

Given a sample space S and an event A in the sample space S, let $X_A = I\{A\}$. Then $E[X_A] = Pr\{A\}$.

Indicator Random Variable

Definition

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Lemma 5.1

Given a sample space S and an event A in the sample space S , let $X_A = I\{A\}$. Then $E[X_A] = Pr\{A\}$.

Indicator Random Variable

Flip a coin

$$E[X_H] = E[I\{Y = H\}] = Pr\{Y = H\} = 1/2.$$

Flip n coins

$$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/2 = n/2$$

Indicator Random Variable

Flip a coin

$$\begin{aligned} E[X_H] &= E[I\{Y = H\}] \\ &= Pr\{Y = H\} = 1/2. \end{aligned}$$

Flip n coins

$$\begin{aligned} E[X] = E[\sum_{i=1}^{n} X_i] &= \sum_{i=1}^{n} E[X_i] \\ &= \sum_{i=1}^{n} 1/2 = n/2 \end{aligned}$$

Analysis of the hiring problem

Definition

Let X be the number of hired persons and let X_i be

$$X_i = I\{\text{candidate } i \text{ is hired}\} = \begin{cases} 1 & \text{if candidate } i \text{ is hired} \\ 0 & \text{if candidate } i \text{ is not hired} \end{cases}$$

and

$$X = X_1 + X_2 + \cdots + X_n$$

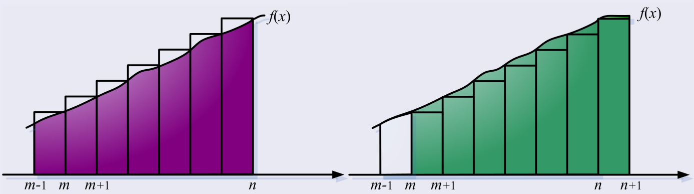
Analysis of the hiring problem

Hired expectation

$$
\begin{aligned}
E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} E[X_i] \\
&= \sum_{i=1}^{n} 1/i
\end{aligned}
$$

Analysis of the hiring problem

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$$



Analysis of the hiring problem

Hired expectation

When $f(k)$ is a monotonically decreasing function:

$$\int_{m}^{n+1} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$$

$$\ln(n+1) = \int_{1}^{n+1} \frac{dx}{x} \leq \sum_{k=1}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{dx}{x} + 1 = \ln n + 1$$

Analysis of the hiring problem

Hired expectation

When $f(k)$ is a monotonically decreasing function:

$$\int_{m}^{n+1} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)dx$$

$$\ln(n+1) = \int_{1}^{n+1} \frac{dx}{x} \le \sum_{k=1}^{n} \frac{1}{k} \le \int_{1}^{n} \frac{dx}{x} + 1 = \ln n + 1$$

Analysis of the hiring problem

Hired expectation

$$
\begin{aligned}
E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/i \\
&= \ln n + O(1)
\end{aligned}
$$

Analysis of the hiring problem

Lemma 5.2

Assuming that the candidates are presented in a random order, algorithm HIRE-ASSISTANT has a total hiring cost of $O(c_h \ln n)$.

Randomized algorithms

RANDOMIZED-HIRE-ASSISTANT(n)

```
randomly permute the list of candidates
best = 0
for i = 1 to n
        interview candidate i
        if candidate i is better than
                candidate best
            best = i
            hire candidate i
```

Randomized algorithms

Lemma 5.3

The expected hiring cost of the procedure RANDOMIZED-HIRE-ASSISTANT is $O(c_h \ln n)$.

Randomly permuting arrays

PERMUTE-BY-SORTING(A)

```
n = A.length
let P[1..n] be a new array
for i = 1 to n
        P[i] = RANDOM(1, n^3)
sort A, using P as sort keys.
```

Example

Let $A = (1, 2, 3, 4)$ and choose random priorities $P = (10, 2, 57, 21)$, then the new A is $(2, 1, 4, 3)$.

Uniform Random Permutation

Lemma 5.4

Procedure PERMUTE-BY-SORTING produces ***a uniform random permutation*** of the input, assuming that all priorities are distinct.

Uniform Random Permutation

Proof.

$$
\begin{aligned}
& Pr\{X_1 \cap X_2 \cap \cdots \cap X_{n-1} \cap X_n\} \\
=& Pr\{X_1\} \cdot Pr\{X_2 | X_1\} \cdots Pr\{X_n | X_{n-1} \cap \cdots \cap X_1\} \\
=& \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right) \cdots \left(\frac{1}{2}\right)\left(\frac{1}{1}\right) \\
=& \frac{1}{n!}
\end{aligned}
$$

$\square$

A Better Random Permutation

RANDOMIZE-IN-PLACE(A)

$n = A.length$
for $i = 1$ **to** n
    swap $A[i] \leftrightarrow A[\text{RANDOM}(i, n)]$

Lemma 5.5

Procedure RANDOMIZE-IN-PLACE computes a uniform random permutation.

A Better Random Permutation

Proof.

We use the following loop invariant:
Just prior to the ith iteration of the **for** loop of lines $2-3$, for each possible $i-1$-permutation, the subarray $A[1..i-1]$ contains this $i-1$-permutation with probability $(n-i+1)!/n!$. □

The on-line hiring problem

ON-LINE-MAXIMUM(k, n)

$bestscore = -\infty$
for $i = 1$ **to** k
  if $score(i) > bestscore$
    $bestscore = score(i)$
for $i = k + 1$ **to** n
  if $score(i) > bestscore$
    return i
return n

The on-line hiring problem

Analysis

$$Pr\{S\} = \sum_{i=k+1}^{n} Pr\{S_i\}$$

In order to succeed when the best-qualified applicant is the ith one, two things must happen.

The on-line hiring problem

Analysis

- The best-qualified applicant must be in position i, an event which we denote by B_i.
- The algorithm must ***not*** select any of the applicants in positions $k+1$ through $i-1$. We use O_i to denote the event.

The on-line hiring problem

Analysis

$$Pr\{S_i\} = Pr\{Bi \cap O_i\} = Pr\{B_i\}Pr\{O_i\}.$$

$$Pr\{S\} = \sum_{i=k+1}^{n} Pr\{S_i\} = \sum_{i=k+1}^{n} \frac{k}{n(i-1)}$$

$$= \frac{k}{n} \sum_{i=k+1}^{n} \frac{1}{i-1} = \frac{k}{n} \sum_{i=k}^{n-1} \frac{1}{i}$$

The on-line hiring problem

Analysis

We have

$$\int_{k}^{n} \frac{1}{x} dx \leq \sum_{i=k}^{n-1} \frac{1}{i} \leq \int_{k-1}^{n-1} \frac{1}{x} dx$$

Such that

$$\frac{k}{n}(\ln n - \ln k) \leq Pr\{S\} \leq \frac{k}{n}(\ln(n-1) - \ln(k-1))$$

The on-line hiring problem

Analysis

We have

$$\int_{k}^{n} \frac{1}{x} dx \leq \sum_{i=k}^{n-1} \frac{1}{i} \leq \int_{k-1}^{n-1} \frac{1}{x} dx$$

Such that

$$\frac{k}{n}(\ln n - \ln k) \leq Pr\{S\} \leq \frac{k}{n}(\ln(n-1) - \ln(k-1))$$

The on-line hiring problem

Analysis

We get

$$\frac{d(\frac{k}{n}(\ln n - \ln k))}{dk} = \frac{1}{n}(\ln n - \ln k - 1).$$

When $\frac{1}{n}(\ln n - \ln k - 1) = 0$, $Pr\{S\}$ is maximized. Thus if we implement our strategy with $k = n/e$, we will succeed in hiring our best-qualified applicant with the probability at least $1/e$.

The on-line hiring problem

Analysis

We get

$$\frac{d(\frac{k}{n}(\ln n - \ln k))}{dk} = \frac{1}{n}(\ln n - \ln k - 1).$$

When $\frac{1}{n}(\ln n - \ln k - 1) = 0$, $Pr\{S\}$ is maximized. Thus if we implement our strategy with $k = n/e$, we will succeed in hiring our best-qualified applicant with the probability at least $1/e$.