MATH 22 Section 010
Exponential & Logarithmic Functions

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Disclaimer: I made this brief review for students to understand the relation between exponential and logarithmic functions. Only reading this is NOT enough for the exams.

1 Basics

Suppose $a > 0$ and $a \neq 1$. Then

- $f(x) = a^x$ is the inverse of $g(x) = \log_a(x)$, that is, $g(x) = f^{-1}(x)$
  - One can check the inverse by using $f(f^{-1}(x)) = x = f^{-1}(f(x))$.
  - See Quiz 8 for finding inverse and verifying inverse. One should know how to find inverse functions.
  - See Problem 1 & 2 below.
- $f(x) = a^x$ has domain $(-\infty, \infty)$, so the range of $f^{-1}(x) = \log_a(x)$ is $(-\infty, \infty)$.
  - This is because $Dom(f) = Range(f^{-1})$.
- $f(x) = a^x$ has range $(0, \infty)$, so the domain of $f^{-1}(x) = \log_a(x)$ is $(0, \infty)$.
  - This is because $Range(f) = Dom(f^{-1})$.
  - See Problem 3 below.
- $f(x) = a^x$ has an horizontal asymptote $y = 0$.
- $g(x) = \log_a x$ has an vertical asymptote $x = 0$.

2 Properties

Suppose the base be the natural exponent e here. One may replace e with any other number except 0 and 1.

Proposition 2.1. *For any $a, b, r \in (-\infty, \infty)$ and let $m = e^a$, $n = e^b$, we have*

$$e^0 = 1 \iff \ln 1 = 0$$
$$e^1 = e \iff \ln e = 1$$
$$e^{a+b} = e^a e^b \iff \ln mn = \ln m + \ln n$$
$$e^{a-b} = \frac{e^a}{e^b} \iff \ln \frac{m}{n} = \ln m - \ln n$$
$$(e^a)^r = e^{ar} \iff \ln(m^r) = r \ln m$$

Note that $m > 0$ and $n > 0$.

See Problem 4, 5 & 6 for the application of Prop 2.1.

3 Problems

Problem 1 (Solving $a^x = k$) **Key: Take $\log_a$ on both sides.**

(a) How to solve $2^x = \frac{1}{16}$?

(b) How to solve $3^x = 81$?

Problem 2 (Solving $\log_a x = k$) **Key: Take a^x on both sides.**

(a) Solve $\log_7 x = 2$.

(b) Solve $\log_2(4x) = 10$.

Problem 3 (Find the domain of $\log_a g(x)$) **Key: Solve $g(x) > 0$**

(a) Find the domain of $\log(10x)$.
We know $10x > 0 \Longrightarrow x > 0$. So the domain is $(0, \infty)$.

(b) Find the domain of $\ln(20 + 5x)$.
Note that $\ln x = \log_e x$ is also a log function. We have

$$20 + 5x > 0 \Longrightarrow 5x > -20 \Longrightarrow x > -4$$

So the domain is $(-4, \infty)$. Note that the vertical asymptote is at $20 + 5x = 0$, which is $x = -4$.

Problem 4 Solve the logarithmic equations. **Key: Product Rule of Logarithms**

(a)

$$\log x + \log 2x = 2$$

(b)

$$\log x + \log(2x+5) = \log 7$$

Problem 5 Solve the exponential equations.

(a)

$$e^{x^2+2x+1} = 1$$

(b)

$$e^{2t+3} = e^{t-1}$$

Problem 6 (Power Rule)

(a) Find the value of $\log_3 \left(\log_3 3^{3^{20}}\right)$.

(b) Expand the expression

$$\ln \frac{\sqrt{x+9}e^9}{(x-1)^3\sqrt{x+7}}$$

4 Solution

Problem 1 (Solving $a^x = k$)

(a) We know 2^x is the inverse of $\log_2 x$. That is, $\log_2 2^x = x$. So we can take $\log_2$ on both sides.

$$\log_2 2^x = \log_2 \frac{1}{16}$$

$$\Longrightarrow x = \log_2 \frac{1}{16} = \log_2 2^{-4} = -4$$

(b) Similarly, we can take $\log_3$ on both sides.

$$\log_3 3^x = \log_3 81$$

$$\Longrightarrow x = \log_3 3^4 = 4$$

Problem 2 (Solving $\log_a x = k$)

(a) Solve $\log_7 x = 2$.
We know 7^x is the inverse of $\log_7 x$, that is, $7^{\log_7 x} = x$. So

$$7^{\log_7 x} = 7^2$$

$$\implies x = 49$$

(b) Solve $\log_2(4x) = 10$.
Similarly, we have $2^{\log_2(4x)} = 4x$. So

$$2^{\log_2(4x)} = 2^{10}$$

$$\implies 4x = 2^{10} = 1024 \implies x = 256$$

Problem 3 (Find the domain of $\log_a g(x)$)

(a) We know $10x > 0 \implies x > 0$. So the domain is $(0, \infty)$.

(b) Note that $\ln x = \log_e x$ is also a log function. We have

$$20 + 5x > 0 \implies 5x > -20 \implies x > -4$$

So the domain is $(-4, \infty)$. Note that the vertical asymptote is at $20 + 5x = 0$, which is $x = -4$.

Problem 4 (Product Rule)

(a)

$$\begin{aligned} \log x + \log 2x = \log(2x^2) = 2 \\ \implies 10^{\log(2x^2)} = 10^2 \implies 2x^2 = 100 \\ \implies x^2 = 50 \implies x = \pm 5\sqrt{2} \end{aligned}$$

Note that $\log(-5\sqrt{2})$ is undefined, so $x = 5\sqrt{2}$.

(b)

$$\begin{aligned} \log x + \log(2x+5) = \log\left(x(2x+5)\right) = \log 7 \\ \implies 10^{\log(x(2x+5))} = 10^{\log 7} \\ \implies (x(2x+5)) = 7 \\ \implies 2x^2 + 5x - 7 = 0 \\ \implies (2x+7)(x-1) = 0 \\ \implies x = \frac{-7}{2}, 1 \end{aligned}$$

Note that $\log(\frac{-7}{2})$ is undefined, so $x = 1$. ($\log 1$ and $\log(2+5)$ are both defined)

Problem 5 Solve the exponential equations.

(a) We take log on both sides.

$$
\begin{aligned}
\ln\left(e^{x^2+2x+1}\right) &= \ln 1 \\
\implies x^2+2x+1 &= 0 \quad \text{since } \ln 1 = 0 \\
\implies (x+1)^2 &= 0 \\
\implies x &= -1
\end{aligned}
$$

(b) We take log on both sides.

$$
\begin{aligned}
\ln\left(e^{2t+3}\right) &= \ln\left(e^{t-1}\right) \\
\implies 2t+3 &= t-1 \\
\implies t &= -4
\end{aligned}
$$

Problem 6 (Power Rule)

(a)

$$
\log_3\left(\log_3 3^{3^{20}}\right) = \log_3\left(3^{20}\log_3 3\right) = \log_3 3^{20} = 20\log_3 3 = 20
$$

(b)

$$
\begin{aligned}
\ln\frac{\sqrt{x+9}e^9}{(x-1)^3\sqrt{x+7}} &= \ln\left(\sqrt{x+9}e^9\right) - \ln\left((x-1)^3\sqrt{x+7}\right) \\
&= \ln\sqrt{x+9} + \ln e^9 - \left(\ln(x-1)^3 + \ln\sqrt{x+7}\right) \\
&= \ln((x+9)^{\frac{1}{2}} + \ln e^9 - \left(\ln(x-1)^3 + \ln(x+7)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\ln(x+9) + 9 - 3\ln(x-1) - \frac{1}{2}\ln(x+7)
\end{aligned}
$$