1. (1) $\|X\|_1$: $\quad \mathrm{Prox}_c f = x - c\,\mathrm{Prox}_{c^{-1}} f^*(x/c)$

$= x - c\cdot P_B(x/c).$

$c\cdot P_B(x/c) = \underset{w}{\mathrm{argmin}}\left\{ \frac{1}{2}\|w-x\|_2^2 \,\middle|\, \|w\|_\infty \le c \right\}$

$\therefore c P_B(x/c) = \begin{cases} c, & x_i > c \\ x_i, & |x_i| \le c. \\ -c, & x_i < -c \end{cases}$

$\therefore \mathrm{Prox}_c f = \begin{cases} x_i + \lambda & x_i < \lambda \\ |x_i|, & |x_i| \le \lambda \\ x_i - \lambda & x_i > -\lambda \end{cases} \quad \lambda = \underset{\lambda}{\mathrm{argmax}}\{\Sigma \mathrm{Max}(x_i \lambda, 0) = 1\}.$

(2) $\|X\|_*$:

Thm : $\mathrm{prox}_c R(w) = U\,\mathrm{prox}_c\, g(\sigma(w))\, V^T \quad R = g\circ\sigma$

$\therefore \|X\|_* \Leftrightarrow \mathrm{prox}_c f(x) = U\, \underline{\mathrm{Prox}_c\, g(\sigma(x))}\, V^T$

其中 $f(x) = g(\sigma(x))$ 且.

又由 Thm : $f^* = (g\circ\sigma)^* = g^*\circ\sigma.$

$\because f(x) = \|X\|_* = \Sigma\sigma_i(x).$

$\therefore \mathrm{Prox}_c f(x) = U\cdot \mathrm{Prox}_c(\Sigma\sigma_i(x))\, V^T$

$= \sum_i U\cdot\mathrm{Prox}_c[\sigma_i(x)]\, V^T.$ 其中 $X = U\Sigma V^T.$

2. 设 $f(x) = I_L = \begin{cases} 0, & x\in L \\ \infty, & x\notin L \end{cases}$

则 $f^*(x) = I_{L^\perp}$

$P_L(x) = \mathrm{Prox}\, f(x) = \underset{x\in L.}{\mathrm{argmin}}\ \|x-v\|_2.$ ①

$P_{L^\perp}(x) = $ —— $= \underset{x\in L^\perp}{\mathrm{argmin}}\ \|x-v\|_2.$ ②

$\therefore P_L(x) + P_{L^\perp}(x) = \underset{x\in\mathbb{R}^n}{\mathrm{argmin}}\ \|x-v\|_2 = x.$ ③.

现证 $P_{L^\perp}(x) = c\cdot\mathrm{Prox}_{c^{-1}} f^*(x/c). = \underset{w}{\mathrm{argmin}}\ \|w - c\frac{x}{c}\|^2$

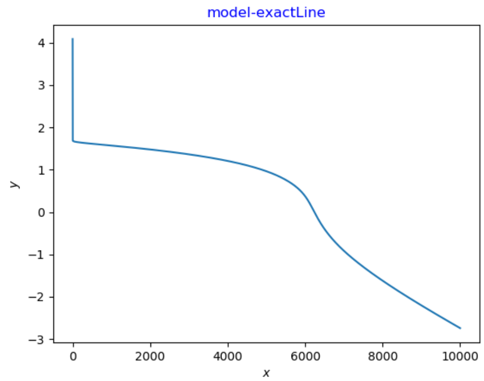
$= \underset{w^*}{\mathrm{argmin}}\ \|w^* - x\|^2.$

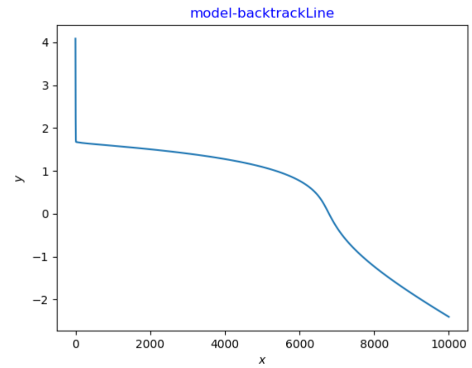
$= \underset{x}{\mathrm{argmin}}\ \|x - v\|^2.$ ④

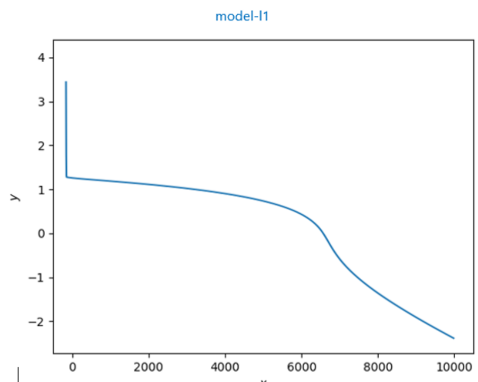
$\therefore$ ②与④同形式,

得证.

3. 代码见附件







L2 is same as bt-line