1、$f(x)=2x_1+3x_2$

$h(x)=\sqrt{x_1}+\sqrt{x_2}-5$.

$\mathcal{L}=2x_1+3x_2+\mu(\sqrt{x_1}+\sqrt{x_2}-5)$.

$$\nabla\mathcal{L}:\begin{cases}\dfrac{\partial \mathcal{L}}{\partial x_1}=2+\dfrac{\mu}{2}\dfrac{1}{\sqrt{x_1}}=0 \Rightarrow \sqrt{x_1}=-\dfrac{\mu}{4}\\[2mm] \dfrac{\partial \mathcal{L}}{\partial x_2}=3+\dfrac{\mu}{2}\dfrac{1}{\sqrt{x_2}}=0\Rightarrow \sqrt{x_2}=-\dfrac{\mu}{6}.\\[2mm] \sqrt{x_1}+\sqrt{x_2}=5 \Rightarrow \mu=-12.\end{cases}$$

$\therefore \boxed{x_1=9 \quad x_2=4 \quad \mu=-12}$

又 $f(x)$ 及 $h(x)$ is convex.

$\therefore$ it is local min.

2、$\mathcal{L}=x_1^2+x_2^2+\lambda_1(-x_2)+\lambda_2(x_1^3-x_2)+\lambda_3(x_2-x_1^3)x_1^3$

$$\nabla\mathcal{L}:\begin{cases}\dfrac{\partial \mathcal{L}}{\partial x_1}=2x_1+3x_1^2\lambda_2+3x_2x_1^2\lambda_3-6x_1^5\lambda_3.\\[2mm] \dfrac{\partial \mathcal{L}}{\partial x_2}=2x_2-\lambda_1-\lambda_2+\lambda_3x_1^3.\\ -\lambda_1x_2=0\\ \lambda_2(x_1^3-x_2)=0\\ \lambda_3(x_2-x_1^3)x_1^3=0\\ x_2\geq 0\\ x_1^3-x_2\leq 0\\ (x_1^3-x_2)x_1^3\geq 0\end{cases}$$

(2) SCQ: 若 $g_i<0 \Rightarrow x_2>0>x_1$.

$\nabla^2 g_2=\begin{pmatrix}6x_1 & 0\\ 0 & 0\end{pmatrix}\preceq 0$ not convex.

$\nabla^2 g_3=\begin{pmatrix}30x_1^4-6x_2x_1, & -3x_1^2\\ -3x_1^2, & 0\end{pmatrix}$ not convex.

$\therefore$ SCQ not holds.

(1) 1° $x_2=0$

① $x_1=0 \Rightarrow \lambda_1+\lambda_2=0$ 且 $\lambda_3\geq 0 \Rightarrow \begin{cases}\lambda_1=0\\ \lambda_2=0\\ \lambda_3\geq 0.\end{cases}$

② $x_1\neq 0$

<1> $x_1^3-x_2=0 \Rightarrow x_1=0$ 矛盾

<2> $x_1^3-x_2\neq 0 \Rightarrow x_1\neq 0$.

$\rightarrow \lambda_2=0,\ \lambda_3=0,\ \lambda_1=0.$ 无意义.

2° $x_2\neq 0. \Rightarrow \lambda_1=0.$

① $x_1=0 \Rightarrow 2x_2-\lambda_1-\lambda_2=0 \Rightarrow \lambda_2\neq 0$ 矛盾

② $x_1\neq 0$

<1> $x_2-x_1^3=0 \Rightarrow x_2=x_1^3$.

$\Rightarrow \begin{cases}2(x_1^3)-\lambda_2+\lambda_3x_1^3=0\\ 2x_1+3\lambda_2x_1^2+3\lambda_3(x_1^3)-6\lambda_3x_1^5=0\end{cases} \Rightarrow x_1+3x_1^5=0$ 矛盾

<2> $x_2-x_1^3\neq 0 \Rightarrow \lambda_2=\lambda_3=0$ 矛盾.

$\therefore$ KKT点 $x^*=(0,0,0)$ 及 $\lambda=(0,0,\lambda_3)$, $\lambda_3\geq 0$

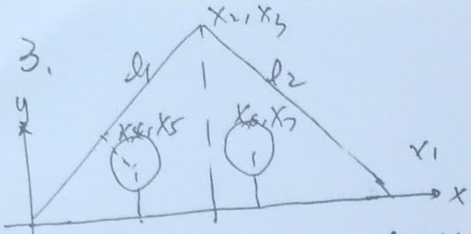
(3) $g(\lambda,\mu)=g(\lambda)=\inf\mathcal{L}$.

① $\lambda=0$ 时 $\mathcal{L}=x_1^2+x_2^2 \Rightarrow g(\lambda)=0.$

② λ 不全为0时 $\begin{cases}\lambda_3=0\ \begin{cases}\lambda_2>0 \rightarrow x_1\rightarrow -\infty\ \mathcal{L}\rightarrow -\infty.\\ \lambda_2=0 \rightarrow \lambda_1>0\ \ \mathcal{L}_{min}=\frac{1}{4}\lambda_1^2\end{cases}\\ \lambda_3>0 \Rightarrow \nabla\mathcal{L}=0 \Rightarrow x_2=x_1^2\\ \qquad \Rightarrow 取\ x_1\rightarrow -\infty,\ \mathcal{L}\rightarrow -\infty.\end{cases}$

$\therefore \boxed{g(\lambda)}=\begin{cases}0, & \lambda=0\\ \frac{1}{4}\lambda_1^2, & \lambda_2=\lambda_3=0,\ \lambda_1>0\\ -\infty, & else\end{cases}$

3.



$f = \frac{1}{2}x_1x_3$

$l_1: y = \frac{x_3}{x_2}x \Rightarrow x_3x - x_2y = 0$

$l_2: y = \frac{x_3}{x_2-x_1}(x-x_1) \Rightarrow x_3x + (x_1-x_2)y - x_3x_1 = 0.$

$$\begin{cases} x_5 \geq 1 & ① \\ x_7 \geq 1 & ② \\ x_1 \geq 0 & ③ \\ x_3 \geq 0 & ④ \\ (x_4-x_6)^2+(x_5-x_7)^2 \geq 4 & ⑤ \end{cases}$$

$$(x_4, x_5) \begin{cases} |x_3x_4 - x_2x_5|^2 \geq x_3^2 + x_2^2 & ⑥ \\ |x_3x_4 + (x_1-x_2)x_5 - x_3x_1|^2 \geq x_3^2 + (x_1-x_2)^2 & ⑦ \end{cases}$$

$$(x_6, x_7) \begin{cases} |x_3x_6 - x_2x_7|^2 \geq x_3^2 + x_2^2 & ⑧ \\ |x_3x_6 + (x_1-x_2)x_7 - x_3x_1|^2 \geq x_3^2 + (x_1-x_2)^2 & ⑨ \end{cases}$$

∴ problem: $\min_x f(x)$ with ①~⑨ (s.t.)

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2}x_1x_3 + \lambda_1(1-x_5) + \lambda_2(1-x_7) + \lambda_3(-x_1) + \lambda_4(-x_3) + \lambda_5(4-(x_4-x_6)^2+(x_5-x_7)^2) \\ & + \lambda_6(x_3^2+x_2^2-(x_3x_4-x_2x_5)^2) + \lambda_7(x_3^2+(x_1-x_2)^2-(x_3x_4+x_1x_5-x_2x_5-x_3x_1)^2) \\ & + \lambda_8(x_3^2+x_2^2-(x_3x_6-x_2x_7)^2) + \lambda_9(-(x_3x_6+x_1x_7-x_2x_7-x_3x_1)^2+x_3^2+(x_1-x_2)^2) \end{aligned}$$

$x_1 = 4+2\sqrt{2}$, $x_2 = 2+2\sqrt{2} = x_3$, $x_4 = 1+\sqrt{2}$, $x_5 = 1 = x_7$, $x_6 = 3+\sqrt{2}$.

KKT: $\nabla\mathcal{L} = 0$; $\lambda_i g_i(x) = 0$; $\lambda_i \geq 0$.

$\Rightarrow$ ③, ④ $\Rightarrow \lambda_3 = \lambda_4 = 0.$

$\Rightarrow$ ⑦, ⑧ $\Rightarrow \lambda_7 = \lambda_8 = 0$

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2}x_1x_3 + \lambda_1(1-x_5) + \lambda_2(1-x_7) + \lambda_5(4-(x_4-x_6)^2+(x_5-x_7)^2) \\ & + \lambda_6(x_3^2+x_2^2-(x_3x_4-x_2x_5)^2) + \lambda_9(x_3^2+(x_1-x_2)^2-(x_3x_6+x_1x_7-x_2x_7-x_3x_1)^2) \end{aligned}$$

$\nabla\mathcal{L} = 0 \Big|_{x=(x^*)}$ $\quad \mathcal{L} =$

$$\begin{aligned} \mathcal{L} = {} & (\lambda_1+\lambda_2+4\lambda_5) + \lambda_9x_1^2 + \tfrac{1}{2}x_1x_3 + \lambda_6x_2^2 - 2\lambda_9x_1x_2 + \lambda_9x_2^2 + \lambda_6x_3^2 + \lambda_9x_3^2 \\ & - \lambda_5x_4^2 + 2\lambda_5x_4x_6 + \lambda_5x_5^2 - \lambda_1x_5 - 2\lambda_5x_5x_7 - \lambda_5x_6^2 - \lambda_2x_7 + \lambda_5x_7^2 \\ & - \lambda_6x_3^2x_4^2 - \lambda_6x_2^2x_5^2 - \lambda_9(x_3^2x_6^2 + x_1^2x_7^2 + x_2^2x_7^2 + x_1^2x_3^2) \\ & + 2\lambda_6x_2x_3x_4x_5 - 2\lambda_9x_3x_6x_1x_7 + 2\lambda_9x_2x_3x_6x_7 + 2\lambda_9x_3^2x_1x_6 + 2\lambda_9x_1x_2x_7^2 + 2\lambda_9x_1^2x_3x_7 \\ & - 2\lambda_9x_1x_2x_3x_7 \end{aligned}$$

由 $\nabla\mathcal{L}\Big|_{x=(x^*)} = 0$.

代入可解得

$\lambda = (2+\sqrt{2}, 2+\sqrt{2}, 0, 0, \frac{2+\sqrt{2}}{4}, -\frac{1+\sqrt{2}}{4}, 0, 0, -\frac{1+\sqrt{2}}{4}).$

$$\boxed{\begin{cases} \lambda_3 = \lambda_4 = \lambda_7 = \lambda_8 = 0 \\ \lambda_1 = \lambda_2 = 2+\sqrt{2} \\ \lambda_5 = (2+\sqrt{2})/4 \\ \lambda_6 = \lambda_9 = \frac{2+\sqrt{2}}{4(4+3\sqrt{2})} = -\frac{1+\sqrt{2}}{4} \end{cases}}$$

4. $f(x) = (x_1-2)^2 + (x_2-3)^2$.

$\mathcal{L} = (x_1-2)^2 + (x_2-3)^2 + \lambda_1(x_1+x_2) + \lambda_2(x_1^2-4)$

(1) f, g_1, g_2 is convex $\Rightarrow$ convex prob.

取 $x_1 = 1$, $x_2 = -2 \Rightarrow g_1, g_2 < 0 \Rightarrow$ SCQ 满足

无限制 ① $\begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{matrix} x_1 = 2 \\ x_2 = 3 \end{matrix}$ 即 $\boxed{x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$

(2) ② $\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - 4 + \lambda_1 \\ \frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 - 6 + \lambda_1 \\ \lambda_1(x_1+x_2) = 0 \\ g_1(x) \le 0. \end{cases} \Rightarrow \begin{matrix} 1^\circ\ \lambda_1 = 0\ \text{矛盾} \\ 2^\circ\ x_1 + x_2 = 0 \Rightarrow \boxed{\lambda_1 = 5\quad x = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}} \end{matrix}$

只有 $g_1 \le 0$ 限制

③ $\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - 4 + 2\lambda_2 x_1 \\ \frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 - 6 = 0 \quad \boxed{x_2 = 3} \\ \lambda_2 g_2 = \lambda_2(x_1^2 - 4) = 0 \\ \cancel{\lambda_2 \ge 0},\ x_1^2 - 4 \le 0. \end{cases}$

$\Rightarrow$ $1^\circ\ \lambda_2 = 0$ 矛盾 $(g_1 > 0)$.

故 $x_1 = \pm 2$.

$\therefore x_1 = -2$, $\lambda_2 = -2$

$\therefore \boxed{\lambda_2 = -5\quad x = \begin{pmatrix} 3 \\ -2 \end{pmatrix}}$

(3) dual.

$\mathcal{L} =$ 上式.

$\nabla \mathcal{L} = 0 \Rightarrow \begin{cases} 2x_1 - 4 + \lambda_1 + 2\lambda_2 x_1 = 0 \\ 2x_2 - 6 + \lambda_1 = 0. \end{cases}$

$\Rightarrow \begin{cases} x_1 = \frac{4-\lambda_1}{2(1+\lambda_2)} \\ x_2 = \frac{6-\lambda_1}{2} \end{cases}$

$g(\lambda) = \inf \mathcal{L} = \mathcal{L}(x^*, \lambda)$, x^* 满足上式, $\operatorname{dom} g = \left\{ (\lambda_1, \lambda_2) \,\middle|\, \begin{matrix} \frac{4-\lambda_1}{1+\lambda_2} + (6-\lambda_1) \le 0,\ \left|\frac{4-\lambda_1}{1+\lambda_2}\right| \le 2 \\ \lambda_1 \ge 0,\ \lambda_2 \ge 0 \end{matrix} \right\}$

5. $\min \frac{1}{2}\|w\|^2$, s.t $g_i = 1 - y_i(\langle w, x_i\rangle + \beta) \le 0$ $= g_i$

(1) SCQ: g_i is affine.

$\therefore$ 对 g_i $\exists\, \varepsilon_i > 0$, 使 $g_i - \varepsilon_i \le 0$.

$\therefore$ $g_i < 0$ 成立 $\Rightarrow$ SCQ holds.

explain: 存在一个~~超平面~~使数据集 well-separated.

(2) $\mathcal{L} = \frac{1}{2}\|w\|^2 + \sum \lambda_i (1 - y_i(w^T x + \beta))$

$\nabla \mathcal{L}(w, \lambda) = w - \sum \lambda_i y_i x_i = 0 \Rightarrow w_0 = \sum \lambda_i y_i x_i$

$g(\lambda) = \inf \mathcal{L} = \mathcal{L}(w_0) = \frac{1}{2}(\sum \lambda_i y_i x_i)^2 + \sum \lambda_i (1 - y_i[(\sum \lambda_i y_i x_i)^T x_i + \beta])$

$= \frac{1}{2}\sum\sum \lambda_i \lambda_j (y_i x_i)^T y_j x_j + \sum \lambda_i - (\sum \lambda_i y_i)\beta - \sum \lambda_i y_i [\sum \lambda_j (y_j x_j)^T] x_i$

$= \frac{1}{2}\sum\sum \lambda_i \lambda_j y_i y_j x_i^T x_j + \sum \lambda_i - \sum\sum \lambda_i \lambda_j y_i x_j^T y_j^T x_i$

$= \sum \lambda_i - \frac{1}{2}\sum\sum \lambda_i \lambda_j y_i y_j x_i^T x_j$

$\operatorname{dom} g = \{(\lambda_i) \mid \sum \lambda_i y_i = 0\}$.

6. strong duality : ① f, g_i is convex

~~② slater conditions.~~

dual problem : $\max G(\lambda, \mu)$, s.t. $\lambda \geq 0$, $G > -\infty$.

(a) $\min (x^2+1) = f(x)$ $\quad g_1 = x^2 - 6x + 8 \leq 0$.

$$\mathcal{L}(x, \lambda) = x^2+1 + \lambda(x^2-6x+8)$$

dual func G: $\nabla_x \mathcal{L} = 2x + \lambda(2x-6) = 0 \Rightarrow x = \frac{3\lambda}{1+\lambda}$.

$$G(\lambda) = (1+\lambda)\left(\frac{3\lambda}{1+\lambda}\right)^2 - 6\lambda \frac{3\lambda}{1+\lambda} + 8\lambda + 1$$

$$G(\lambda) = \boxed{-\frac{9\lambda^2}{1+\lambda} + 8\lambda + 1}$$

$\therefore$ ① dual problem : $\max G(\lambda)$ s.t. $\lambda \geq 0$

② 而 $f(x) = x^2+1$ is convex $g_1(x) = x^2-6x+8$ is convex $\Rightarrow$ hold strong dualities.

(b) $\min c^T x$ s.t. $g_1(x) \leq 0$ $(g_1 = f)$

$$\mathcal{L} = c^T x + \lambda \cdot g_1(x)$$

dual func : ~~$\nabla_x \mathcal{L} = c + \lambda \cdot \nabla_x g_1(x) = 0$~~

~~$\Rightarrow \lambda \nabla_x g_1(x) = -c \Rightarrow \nabla g_1(x) = -\frac{c}{\lambda}$.~~

$\therefore \boxed{G(\lambda)} = \inf \mathcal{L} = -\sup(-c^T x - \lambda f(x)) = -\lambda \sup\left(-\frac{c^T}{\lambda} x - f(x)\right) = \boxed{-\lambda \cdot f^*\left(-\frac{c}{\lambda}\right)}$

$\therefore$ ① dual problem : $\max G(\lambda)$ s.t. $\lambda \geq 0$.

② $c^T x$ is affine, 若 $f(x)$ is convex $\Rightarrow$ strong duality.

若 $f(x)$ is not convex $\Rightarrow$ no strong duality.

(c) $\min \frac{1}{n} \sum \phi_i(y_i) + \frac{1}{2}\mu_0 \|x\|^2$ s.t. $y_i = a_i^T x \Rightarrow a_i^T x - y_i = 0$

$$\mathcal{L} = \frac{1}{n} \sum \phi_i(y_i) + \frac{1}{2}\mu_0 \|x\|^2 + \sum \mu_i (a_i^T x - y_i).$$

$$g(\mu) = \inf \mathcal{L} = \frac{1}{n} \sum \inf \phi_i(y_i) + \sum \mu_i \inf (a_i^T x - y_i) + \frac{\mu_0}{2} \inf \|x\|^2.$$

~~$= -\frac{1}{n}\sum \sup \phi_i^* - \frac{\mu_0}{2}\sup\left(-\sum \mu_i (a_i^T x - y_i) - \|x\|^2\right)$~~

$$= -\frac{1}{n} \sup_y \left(\sum \phi_i(y_i) + \sum \mu_i y_i\right) + -\frac{\mu_0}{2} \sup_x \left(\frac{-2\sum \mu_i a_i^T x}{\mu_0} - \|x\|^2\right)$$

$$\therefore \boxed{g(\mu)} = -\frac{1}{n} \sum \phi_i^*(n \cdot \mu_i) + -\frac{\mu_0}{2} f^*\left(-\frac{2}{\mu_0} \sum \mu_i a_i\right)$$

其中 f^* 为 $\|\cdot\|_2$ 的dual func, ϕ_i^* 为 ϕ_i dual func.

$\therefore$ 当 F is convex $\Rightarrow$ strong duality

not convex $\Rightarrow$ no strong.