Homework requirements

- Digital format (can be typeset or photos) is preferred
- Submit by next lecture
- Each homework 10 points; 1 point deducted for each day of delay

Contact information

- TA:
  - 陈程：optinml2019@163.com
- zlin@pku.edu.cn
- http://www.cis.pku.edu.cn/faculty/vision/zlin/zlin.htm

Homework (2)

1. Which of the following sets are convex?

(a) A slab, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n |\ \alpha \leq \mathbf{a}^T\mathbf{x} \leq \beta\}$.

(b) A wedge, i.e., $\{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{a}_1^T\mathbf{x} \leq b_1,\ \mathbf{a}_2^T\mathbf{x} \leq b_2\}$.

(c) The set of points closer to a given point than a given set, i.e., $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S\}$ where $S \subseteq \mathbb{R}^n$.

(d) The set of points closer to one set than another, i.e., $\{\mathbf{x} | \mathbf{dist}(\mathbf{x}, S) \leq \mathbf{dist}(\mathbf{x}, T)\}$, where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|_2 |\ \mathbf{z} \in S\}.$$

(e) The set $\{\mathbf{x} | \mathbf{x} + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

(f) The set of points whose distance to $\mathbf{a}$ does not exceed a fixed fraction θ of the distance to $\mathbf{b}$, i.e., the set $\{\mathbf{x} | \|\mathbf{x} - \mathbf{a}\|_2 \leq \theta\|\mathbf{x} - \mathbf{b}\|_2\}$ ($\mathbf{a} \neq \mathbf{b}$ and $0 \leq \theta \leq 1$).

Homework (2)

2. Judge which of the following functions are convex, concave, strictly convex, strictly concave, strongly convex or concave, and find their moduli if they are strongly convex or concave.

(a) $f(x) = e^x - 1$ on $\mathbb{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

(f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 < \alpha < 1$ on $\mathbb{R}^2_{++}$.

3. Prove that $f : \mathbb{R}^n \to \mathbb{R}$ is convex (resp., strictly convex, strongly convex) iff for every $\mathbf{x} \neq \mathbf{y} \in \text{dom } f$, the function $g(t) = f(t\mathbf{x} + (1-t)\mathbf{y})$ is a convex (resp., stictly convex, strongly convex) function on $[0, 1]$.

Homework (2)

4. Suppose $p < 1, p \neq 0$. Show that the function

$$f(\mathbf{x}) = \left(\sum_{i=1}^{n} \mathbf{x}_i^p\right)^{1/p}$$

with $\operatorname{dom} f = \mathbb{R}^n_{++}$ is concave.

5. Prove that if f is a convex function, then for all $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, and a_1, a_2 and $a_3 \in (0, 1)$ such that $a_1 + a_2 + a_3 = 1$, we have

$$\langle \nabla f(\mathbf{x}_3), a_1\mathbf{x}_1 + a_2\mathbf{x}_2 - \overset{1-a_3}{\cancel{a_3}}\mathbf{x}_3 \rangle \leq a_1 f(\mathbf{x}_1) + a_2 f(\mathbf{x}_2) - \overset{1-a_3}{\cancel{a_3}} f(\mathbf{x}_3).$$

6. Prove that for convex f, the Bregman distance $B_f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle$ is convex in $\mathbf{x}$ but not necessarily in $\mathbf{y}$.

7. Compute the subgradients of $f(\mathbf{x}) = \frac{1}{2}x_1^2 + |x_2|$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_\infty$, and $\|\mathbf{X}\|_{2,1}$.