~~C_1: closed; open; bounded (finite); compact.~~

1.

	open	close	bounded	compact	inter	closure	boundary
ϕ	V	V	V	V	ϕ	ϕ	ϕ
$\mathbb{R}^n$	V	V	×	×	$\mathbb{R}^n$	$\mathbb{R}^n$	ϕ
$[0,1)\cup(2,3)\cup(4,5]$	×	×	V	×	$(0,1)\cup(2,3)\cup(4,5)$	$[0,1]\cup[2,3]\cup[4,5]$	$\{0,1,2,3,4,5\}$
$\{(x,y)^T \mid x\geq 0, y>0\}$	×	×	×	×	$(0,+\infty)\times(0,+\infty)$	$[0,+\infty)\times[0,+\infty)$	$\{(x,y)^T \mid x=0$ 或 $y=0\}$
$\mathbb{Z}$ (under $\mathbb{R}^n$)	×	×	×	×	ϕ	$\mathbb{Z}$	$\mathbb{Z}$
$\{\frac{1}{x} \mid x\in\mathbb{Z}\}$	×	×	V	×	ϕ	$\{0\}\cup\{\frac{1}{x}\}$	$\{0\}\cup\{\frac{1}{x}\}$
$\{(\frac{1}{k}, \sin k)^T \mid k\in\mathbb{Z}\}$	×	×	V	×	ϕ	normal $\cup\{x=0, y\in[-1,1]\}$	normal $\cup\{x=0, y\in[-1,1]\}$

2.

	(Q-linear) rate of convergence	rate-constant	x^*
$x_k=\frac{1}{2^k}$	1	$\frac{1}{2}$	0
$1+\frac{5}{10^{2k}}$	1	$\frac{1}{100}$	1
2^{-2^k}	2	1	0
3^{-k^2}	1 ~~1/9~~ (R-linear)	1/9	0
$\begin{cases}1-2^{-2^k} & \text{odd}\\ 1+2^{-k} & \text{even}\end{cases}$	1 (R-linear)	$\frac{1}{2}$	1

$e_k=\begin{cases}-2^{-2^k} & \text{odd}\\ 2^{-k} & \text{even}\end{cases}$

3.

(1) $f(x)=\sqrt[p]{\sum x_i^p}$

$= \|x\|_p$

$\frac{\partial f(x)}{\partial x_1} = x_1^{p-1} \cdot (\sum x_i^p)^{\frac{1}{p}-1} =$

① $\nabla f(x) = (\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n})$

$= (x^{p-1}) \cdot \frac{1}{\|x\|_{\frac{p}{p-1}}}$

$(x^{p-1}) = (x_1^{p-1}, \cdots, x_n^{p-1})$.

Answer

②

Similarly:

$H_{ij} = \frac{\partial f_i}{\partial x_j} = \frac{\partial (x_i^{p-1} \cdot \|x\|_{p/p-1})}{\partial x_j}$

$= x_i^{p-1} \cdot \frac{\partial (\sum x_t^p)^{\frac{1}{p}-1}}{\partial x_j} = x_i^{p-1} \cdot (\sum x_t^p)^{\frac{1}{p}-2} (p-1) x_j^{p-1} \quad (i \neq j)$

$= H_{ii} + x_i^{p-2} \cdot (\sum x_t^p)^{1/p-1} (p-1) \quad (i=j)$.

$\therefore H = H_1 + H_2$, $H_1 = (p-1) \cdot \|X\|_{p-1} \cdot diag(x_i^{p-2})$, $H_2 = (p-1) \cdot \|X\|_{p-1} \cdot (x_i^{p-1} x_j^{p-1})$

X^*

$X^* = (x_i^{p-1} x_j^{p-1})$

Answer

(2) $f(x) = (a^T x)(b^T x)$

$x, b, a \in \mathbb{R}^{n\times 1}$

令 $g = a^T x \quad h = b^T x$

① $\nabla f(x) = \nabla g \cdot h + g \nabla h$

$= \nabla(a^T x) \cdot b^T x + a^T x \cdot \nabla(b^T x)$

$= a(b^T x) + (a^T x) b$

$= a \cdot b^T x + b \cdot a^T x$.

② $\nabla^2 f(x) = \frac{\partial \nabla f(x)}{\partial x}$

$= (a \cdot b^T)^T + (b \cdot a^T)^T$

$= b a^T + a b^T$

第三题三个小问，每个小问两个答案分别用小圈 1,2 标出

(3) $f(X) = \frac{1}{2}\|AX - b\|_2^2$

$= \frac{1}{2}(AX-b)^T(AX-b)$

$= \frac{1}{2}(b^T b + x^T A^T A X - 2(A^T b)^T x)$

① $\nabla f = \frac{1}{2} \cdot (-2A^T b + A^T A x + (A^T A)^T x)$

$= -A^T b + A^T A X$

$= A^T (AX - b)$.

② $\nabla^2 f = \frac{\partial(\nabla f)}{\partial x} = A^T A$ (obviously).

4. dual. $\|z\|^* = \sup\{z^T x \mid \|x\| \leq 1\}$.

注：$\|X\|_{M/n}$ 代表 $\|X\|_M$

4.

dual norm $\|X\|^* = \sup\{X^TY \mid \|Y\|_M \le 1\}$

$\|X\|_M = \sqrt{X^TMX}$, $M \succ 0$

假设 $Y = \frac{(M^{-1}X)}{\|M^{-1}X\|_M}$

下证① Y 满足 $\|Y\| \le 1$

② Y 是满足 $\|Y\| \le 1$ 中取得sup 的 Y

① $Y^TMY = \frac{X^TM^{-T}MM^{-1}X}{(X^TM^{-T}MM^{-1}X)} = 1 \le 1$ 得证

$M^{-T} = M^{-1}$

$X^* = (X_i X_j^{P-1})_{ij}$.

② 对 $f = \frac{X^TY}{\|Y\|_M}$ 求导

$\frac{\partial f}{\partial Y} = 0 \Rightarrow Y = \frac{M^{-1}X}{\|M^{-1}X\|_M}$ 取得极值

综上 $\|X\|^* = X^T \cdot Y = X^TM^{-1}X$

$= \|X\|_{M^{-1}}$

5. 不妨取 λ.

有 $(A+B)^{-1}AX = \lambda \cdot X$

$\therefore \ AX = (A+B)\lambda X$

$\therefore \ X^TAX = \lambda X^T(A+B)X$

$\therefore \ (1-\lambda)X^TAX = \lambda X^TBX$

又 $\because$ A p.d ; B p.s.d.

$\therefore \ 1-\lambda > 0$; $\lambda \ge 0$ 用

6. $\text{cond}(A) = \frac{\sigma_{max}}{\sigma_{min}}$

$AA^T = \begin{pmatrix} 1 & 6 & 15 \\ 6 & 16 & 30 \\ 15 & 30 & 81 \end{pmatrix}$

$\Rightarrow \sigma = \sqrt{\lambda(AA^T)} \approx \begin{pmatrix} 14.12 \\ -0.6 \\ -0.47 \end{pmatrix}$

$\therefore \text{cond}(A) \approx 31.55$.

7. $X = (X_{ij})_{ij}$.

$\langle A(X), y \rangle = \langle X, A^*(y) \rangle$

$\because A(X) = x_{11} + x_{12} - x_{31} + 2x_{33} \in \mathbb{R}$

$\therefore y \in \mathbb{R}$

$\therefore \langle X, A^*(y) \rangle$ while $A^*(y): \mathbb{R} \to \mathbb{R}^{3\times 3}$

$A^*(1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{pmatrix}$

$A^*(y) = \begin{pmatrix} y & y & 0 \\ 0 & 0 & 0 \\ -y & 0 & 2y \end{pmatrix}$.