**Homework 1**
**CS 259 @ SJTU**
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Due Date: May 29, 2019
Submit Date: May 30, 2019

# Problem 1: Regularized residual

### (1)

$$
\begin{aligned}
&\mathcal{L} = ||Ax - b||^2 + \lambda||x||^2 + \mu(b - Ax - r) \\
&\delta\mathcal{L} = \delta||Ax - b||^2 + \lambda * \delta||x||^2 + \delta\mu A^T(b - Ax - r) - \mu A\delta x \\
&\delta\mathcal{L} = 2 * \delta x^T(A^T Ax - A^T b - \lambda x - \mu A^T) + delta\mu(b - Ax - r) = 0
\end{aligned}
$$

$$
\begin{cases} b - Ax - r = 0 \\ A^T(Ax - b) - \lambda x = 0 \end{cases}
$$

### (2)

$$
\begin{aligned}
&b - Ax - r = 0 \\
&A^T r - \lambda x = 0 \\
&\therefore r = (\frac{1}{\lambda} AA^T + 1)^{-1} b
\end{aligned}
$$

### (3)

$$
\begin{aligned}
&\frac{1}{\lambda} AA^T + 1 = \frac{b}{r} \\
&\text{diff equation: } -\frac{1}{\lambda^2} AA^T d\lambda = -\frac{b}{r^2} \\
&\frac{dr}{d\lambda} = \frac{AA^T * r^2}{b * \lambda^2} \\
&\text{From (2): } \frac{dr}{d\lambda} = \frac{AA^T * b}{\lambda^2}(\frac{1}{\lambda} AA^T + 1)^2
\end{aligned}
$$

# Problem 2: Modified Landweber

## (1)

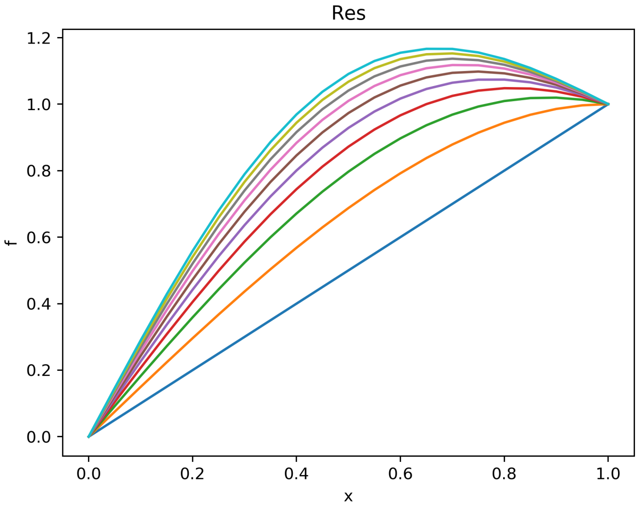
$$
\begin{aligned}
& x^{k+1} = G(x^k) = x^k - \frac{\alpha}{k+1} A^T (Ax^k - b) \\
& e^{k+1} = G(x^* + e^k) - G(x^*) \\
& \therefore e^{k+1} = x^* + e^k - \frac{\alpha}{k+1} A^T (Ax^* + Ae^k - b) - x^* \\
& \therefore e^{k+1} = (1 - \frac{A^T A}{k+1}\alpha) e^k \\
& \text{Let Constant } C = A^T A \alpha > 0 \\
& \therefore e^{k+1} = (1 - \frac{C}{k+1}) e^k < (1 - \frac{C}{k+1})^{k+1} e^0 \\
& \therefore e^k \text{ is decreasing.}
\end{aligned}
$$

## (2)

$$
\begin{aligned}
& \text{Let } x^0 = 0 \\
& x^1 = \alpha A^T b = a_1 \alpha A^T b, && a_1 = I \\
& x^2 = x^1 - \frac{1}{2}(\alpha A^T A x^1 - \alpha A^T b) \\
& x^2 = x^1 + \frac{1}{2}(-\alpha A^T A a_1 \alpha A^T b + \alpha A^T b) \\
& x^2 = x^1 + \frac{1}{2}(I - \alpha A^T A a_1)\alpha A^T b = a_1 \alpha A^T b + a_2 \alpha A^T b, && a_2 = \frac{1}{2}(I - \alpha A^T A) \\
& x^3 = (a_1 + a_2 + a_3)\alpha A^T b, a_3 = \frac{1}{3}(I - \alpha A^T A)(I - \frac{1}{2}\alpha A^T A) \\
& \quad 3 * a_3 = I - \alpha A^T A(a_1 + a_2) \\
& \qquad\quad = I - \alpha A^T A(I + \frac{1}{2}(I - \alpha A^T A)) = (I - \alpha A^T A)(I - \frac{1}{2}\alpha A^T A) \\
& \cdots \\
& x^n = (\sum_1^{n-1} a_i)\alpha A^T b, && a_n = \frac{1}{n}\prod_1^{n-1}(I - \frac{1}{i}\alpha A^T A)
\end{aligned}
$$

Let $A = U\Sigma V^T$, then $A^T A = U\Sigma^2 V^T$, then

$$
\begin{aligned}
& a_n = \frac{1}{n}\prod_0^{n-1}(I - \frac{1}{i}\alpha A^T A) \\
& \therefore x^k = \sum_1^k a_n A^T b \overset{A=U\Sigma V^T}{=\!=\!=\!=} \sum_1^k V \widetilde{\Sigma}_t^{-1} U^T b = V \sum_1^k (\widetilde{\Sigma}_t^{-1}) U^T b \\
& \therefore f_k(\sigma)^{-1} = \sum_1^k \widetilde{\sigma}_t^{-1} = \sum_1^k g(\sigma)_t, && g(\sigma)_t = \frac{1}{t}\prod_0^{t-1}(1 - \frac{1}{i}\alpha\sigma^2) * \alpha\sigma
\end{aligned}
$$




Code is in attachment.

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