1. Convex set:

(1) slab (2) wedge (3) ~~dis closer than given point.~~ dis of fixed point closer than given set.

(4) $\{\|x-a\|_2 \le \theta\|x-b\|_2\}$ <def of a unit ball>

2. (a) $e^x - 1$: ~~convex~~ strictly convex.

(b) $x_1 x_2$: ~~Strongly convex. $\mu \in (0,1)$~~ neither convex nor concave.

(c) $\frac{1}{x_1 x_2}$: strictly convex

(d) $\frac{x_1}{x_2}$: ~~strictly convex~~ neither convex nor concave.

(e) $\frac{x_1^2}{x_2}$ $(x_1 \geq 0)$: convex

(f) $x_1^\alpha x_2^{1-\alpha}$ $(\alpha \in (0,1))$: ~~strictly~~ concave. $\begin{pmatrix} \alpha(\alpha-1)\cdot x_1^{\alpha-2} x_2^{1-\alpha}, & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha}, & -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-1} \end{pmatrix}$

3. Proof: f is convex $\Leftrightarrow g(t) = f(tx+(1-t)y)$ is convex, $\forall x, y \in D$, $t \in [0,1]$.

"$\Rightarrow$" 设 $t_1, t_2 \in [0,1]$, $p \in [0,1]$. (先证 convex)

$g(pt_1 + (1-p)t_2) = f(pt_1 x + (1-p)t_2 x + [1-pt_1 - (1-p)t_2] y)$.

$= f([pt_1 x + (p-pt_1)y] + [t_2(1-p)x + (1-t_2-t_2p+pt_2)y])$

$= f(p[t_1x+(1-t_1)y] + (1-p)[t_2x+(1-t_2)y])$

$\therefore g(pt_1+(1-p)t_2) \le pf(t_1x+(1-t_1)y) + (1-p)f(t_2x+(1-t_2)y) = pg(t_1)+(1-p)g(t_2)$ #

"$\Leftarrow$" 设 $t_1, t_2 \in [0,1]$, $p \in [0,1]$

$\underline{g(pt_1+(1-p)t_2)} \le pg(t_1) + (1-p)g(t_2)$.

$\Updownarrow$

$f(p[t_1x+(1-t_1)y] + (1-p)[t_2x+(1-t_2)y]) \le pf(t_1x+(1-t_1)y) + (1-p)f(t_2x+(1-t_2)y)$.

设 $x^* = t_1x+(1-t_1)y \in D$.

$y^* = t_2x+(1-t_2)y \in D$ $\Rightarrow f(px^*+(1-p)y^*) \le pf(x^*) + (1-p)f(y^*)$ #.

赵同理

4. $f(x) = \|x\|_p \quad p<1$ 且 $p\neq 0$.

$\nabla f(x) = \|x\|_p^{1-p}(x_1^{p-1}, \dots, x_n^{p-1})$

$\nabla^2 f(x) = (1-p)\cdot(x_i^{p-1} x_j^{p-1})_{ij} \cdot \|x\|_p^{1-2p} + (p-1)\,diag(x_i^{p-2})\,\|x\|_p^{1-p}$.

设 $z = \begin{pmatrix} x_1^p \\ \vdots \\ x_n^p \end{pmatrix} \quad A = \begin{pmatrix} \frac{1}{x_1} & & \\ & \ddots & \\ & & \frac{1}{x_n} \end{pmatrix}$.

$\therefore \nabla^2 f(x) = \dfrac{(1-p)\cdot f(x)}{|1\cdot z|^2} \cdot A^T(zz^T)A - \dfrac{(1-p)f}{1\cdot z} \cdot A^T \cdot diag(z) \cdot A$

欲证凹 $\Rightarrow \nabla^2 f(x) \preceq 0 \Rightarrow Y^T \nabla^2 f(x) Y \leq 0. \quad \forall Y$

$\therefore Y^T \nabla^2 f\, Y = \boxed{\dfrac{(1-p)f}{(1\cdot z)^2}} \cdot Y^T A^T (zz^T - diag(z)\|1\cdot z\|_1)AY$

$\overset{\text{or}}{\sim} (Ay)^T[zz^T - \|z\|_1 diag(z)](Ay)$ 设 $p = AY$.

$\therefore \overset{\text{设}p=AY}{=} \underbrace{p^T zz^T p}_{(z^T p)^T z^T p} - \|z\|_1 \cdot p^T diag(z) p. = \left((\Sigma p_i z_i)^2 - (\Sigma z_i)(\Sigma p_i^2 z_i) \right)$

又 再令 $m_i = p_i^2 z_i \quad n_i = z_i$ 即 $\Sigma(\sqrt{m_i n_i}) - (\Sigma m_i)(\Sigma n_i)$

由 Cauchy inequality

$\therefore$ 上式 $\leq 0 \Rightarrow Y^T \nabla^2 f Y \leq 0 \Rightarrow$ #

5. 即 $\langle \nabla f(x_3), a_1(x_1 - x_3) + a_2(x_2 - x_3) \rangle = a_1 \langle \nabla f(x_3), x_1 - x_3 \rangle + a_2 \langle \nabla f(x_3), x_2 - x_3 \rangle$

$\nabla f(x_3) = \dfrac{f(x_i) - f(x_3)}{x_i - x_3} \Rightarrow \leq a_1 \cdot [f(x_1) - f(x_3)] + a_2[f(x_2) - f(x_3)]$

$= a_1 f(x_1) + a_2 f(x_2) - (1-a_3) f(x_3)$ #

6. ① convex in x

$B\langle x,y\rangle = f(x) - f(y) - \langle \nabla f(y), x-y \rangle$

fixed y, 故 $f(y)$ 及 $\langle \nabla f(y), x-y \rangle$ 为线性. is convex and concave

而 $f(x)$ is convex

$\therefore B\langle x,y \rangle$ is convex

② not in y.

设 $f(x) = e^x$ convex

$B = (y - 1 - x_0)e^y + e^{x_0}$

$\nabla^2 B = (y - x_0 + 1)e^y$ 故不确定.

7. $f(x) = \frac{1}{2}x_1^2 + |x_2|$: $\partial f = \begin{cases} \{(m,n) \mid n \in [-1,1], m = 0\}, & x_2 = 0 \text{时} \\ \{(m,n) \mid m = x_1, n = sgn(x_2)\} & x_2 \neq 0 \text{时} \end{cases}$

$f(x) = \|x\|_2$: $\partial f =$

$f(x) = \|x\|_\infty$:

$f(x) = \|X\|_{2,1}$: