# Homework requirements

- Digital format (can be typeset or photos) is preferred
- Submit by next lecture
- Each homework 10 points; 1 point deducted for each day of delay

# Contact information

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# Homework (3)

1. Compute the subgradients of $f(\mathbf{x}) = \frac{1}{2}x_1^2 + |x_2|$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_\infty$, and $\|\mathbf{X}\|_{2,1}$.
2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that $\partial f : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone mapping, i.e.,

$$\langle \mathbf{g}_1 - \mathbf{g}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq 0, \quad \forall \mathbf{g}_i \in \partial f(\mathbf{x}_i), i = 1, 2. \tag{2}$$

Further, if $f$ is $\mu$-strongly convex, then the above inequality can be strengthened as

$$\langle \mathbf{g}_1 - \mathbf{g}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq \mu \|\mathbf{x}_1 - \mathbf{x}_2\|^2, \quad \forall \mathbf{g}_i \in \partial f(\mathbf{x}_i), i = 1, 2. \tag{3}$$

# Homework (3)

3. Show that $f(\mathbf{x}) = \sum_{i=1}^{r} \alpha_i x_{[i]}$ is a convex function of $\mathbf{x}$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the $i$th largest component of $\mathbf{x}$.

4. Show that $f(\mathbf{x}) = -\log\left(-\log\left(\sum_{i=1}^{m} e^{\mathbf{a}_i^T\mathbf{x}+b_i}\right)\right)$ is convex on $\mathbf{dom}\, f = \{\mathbf{x} \mid \sum_{i=1}^{m} e^{\mathbf{a}_i^T\mathbf{x}+b_i} < 1\}$.

5. Prove that $f_k(\mathbf{X}) = \sum\limits_{i=1}^{k} \sigma_i(\mathbf{X})$ is a convex function of $\mathbf{X}$ for all $k = 1, \cdots, \mathrm{rank}(\mathbf{X})$.

6. Show that the Minkowski function:

$$M_C(\mathbf{x}) = \inf\{t > 0 \mid t^{-1}\mathbf{x} \in C\},$$

is convex, where $C$ is a convex set. Find its conjugate function.

# Homework (9)

7. Derive the conjugates of the following functions.

   (a) Sum of largest elements. $f(\mathbf{x}) = \sum_{i=1}^{r} x_{[i]}$ *on* $\mathbb{R}^n$.

   (b) Piecewise-linear function on $\mathbb{R}$. $f(x) = \max_{i=1,\ldots,n}(a_i x + b_i)$ *on* $\mathbb{R}$. *You can assume that the* $a_i$ *are sorted in increasing order, i.e.,* $a_1 \leq \ldots \leq a_m$, *and that none of the functions* $a_i x + b_i$ *is redundant, i.e., for each* $k$ *there is at least one* $x$ *with* $f(x) = a_k x + b_k$.

   (c) Power function. $f(x) = x^p$ *on* $\mathbb{R}_{++}$, *where* $p > 1$. *Repeat for* $p < 0$.

   (d) Geometric mean. $f(\mathbf{x}) = -(\prod x_i)^{1/n}$ *on* $\mathbb{R}^n_{++}$.

   (e) Negative generalized logarithm for second-order cone. $f(\mathbf{x}, t) = -\log(t^2 - \mathbf{x}^T\mathbf{x})$ *on* $\{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|\mathbf{x}\|_2 < t\}$.

Note: Computing a conjugate function needs to specify its domain as well.

# Homework (3)

8. Find the proximal mappings of the following functions.

   (a) $\|\mathbf{x}\|_2$.

   (b) $\|\mathbf{x}\|_\infty$.

   (c) $\|\mathbf{X}\|_2$.

9. Find the projection onto the second order cone.

10. Let $d(\mathbf{x})$ be the distance from $\mathbf{x}$ to a closed convex set $\mathcal{C}$: $d(\mathbf{x}) = \inf\limits_{\mathbf{y}\in\mathcal{C}} \|\mathbf{x}-\mathbf{y}\|_2$.

Find the proximal mappings of $d(\mathbf{x})$ and $\frac{1}{2}d^2(\mathbf{x})$.

11. Prove that:

   (a) If $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{a}^T\mathbf{x}$, then $P_c f(\mathbf{x}) = P_c g(\mathbf{x} - c\mathbf{a})$.

   (b) If $f(\mathbf{x}) = g(\mathbf{x}) + \frac{1}{2\mu}\|\mathbf{x}-\mathbf{a}\|^2$, then $P_c f(\mathbf{x}) = P_\lambda g(\lambda(c^{-1}\mathbf{x} + \mu^{-1}\mathbf{a}))$, where $\lambda^{-1} = \mu^{-1} + c^{-1}$.