The elastodynamic Green's function for a homogeneous, isotropic, unbounded medium

Hsuty (xudy_cas@163.com)

This is my final research project (June, 2021) for the course *Advanced Methods in Applied Mathematics* in SUSTech. If you have any questions, please contact me.

关于地震学公式推导，推荐图书：

张海明, (2021), 地震学中的 Lamb 问题(上), 科学出版社

Version: 2023.3.31, First Edition

Abstract

The earthquake source provides key elements for understanding the physics of earthquake nucleation, propagation, and healing. The point force is the simplest earthquake source model. In this paper, an explicit expression is derived for a unidirectional point force located inside a homogeneous, isotropic, unbounded elastic medium. The results show that the near-field term of Greens function (attenuates as r^{-2}) is very complex, which couples the P and S waves. The far-field P wave term (attenuates as r^{-1}) has the shape of two spheres for the radiation pattern, and it has the largest amplitude in the same direction that the force acts. The far-field S wave term's (attenuates as r^{-1}) radiation pattern is similar to a doughnut without the central hole, and it has the largest amplitude normal to the direction that the force acts. A thorough studying the properties of the elastodynamic Green's function lays a solid foundation for understanding the earthquake process.

1 Introduction

Earthquakes come and go, leaving behind them trails of destruction and casualties. Although their occurrence seems a random phenomenon, it is the goal of seismologists to keep studying them in all kinds of ways to forecast them and even control them. The main contents of earthquake studies are the generation, propagation, and recording of elastic waves in the Earth and the sources that produce them (Lay and Wallace, 1995). There is a trade-off between studying the elastic waves and studying the earthquake source. In many cases, studying the elastic wave generation and propagation requires some knowledge of the earthquake source characteristics. However, the earthquake process is very complex, we start with the simplest problem, which corresponds to a spatially concentrated point force directed along with one of the coordinate axes.

The solution for the point force is very important because we can use it to build the double couple solution, which is the possible model for most earthquakes (Satuder, 1962; Burridge and Knopoff, 1964). Whats more, several natural phenomena are well explained by equivalent point-force sources, such as volcanic eruptions and landslides.

The solution, which is an exact solution for the displacement field caused by a single force in an infinite elastic medium, was first obtained by Stokes in 1849 (Lay and Wallace, 1995). In this article, we first derive the Green's function in a homogeneous, isotropic, unbounded medium. Then, we analyze the properties of Greens function, including radiation patterns and attenuation due to geometric spreading.

2 Formulation

If only infinitesimal deformations are considered, the strain in a deformable body is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1}$$

where ε_{ij} is the Cauchy strain tensor, u_i is the displacement, and a comma between the sub-indices indicates a partial derivative with respect to a space coordinate. The motion of a continuous body is determined by the body forces and tractions acting throughout a volume *V* with of an external surface *S*. According to Newton's second law, based on a Lagrangian description, we find

$$\frac{\partial}{\partial t}\int_V \rho v_i dV = \int_V f_i dV + \int_{\partial V} T_i dS, \tag{2}$$

where ρ is the density, f_i is the body force, T_i is the traction. Since the particle mass is constant in time, the left-hand side can be approximated by $\int_V \rho \frac{\partial u_i^2}{\partial t^2} dV$. Replacing the traction by Cauchy stress tensor τ_{ij} ($T_i = \tau_{ij} n_j$, where n_j is the normal to the external surface *S*), equation (2) can be written as

$$\int_V \rho \frac{\partial u_i^2}{\partial t^2} dV = \int_V f_i dV + \int_{\partial V} \tau_{ij} n_j dS. \tag{3}$$

For an elastic medium, according to Hooke's law, the stress tensor can be expressed by $\tau_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijkl}u_{k,l}$, where C_{ijkl} is the tenor of the elastic coefficients. Owing to symmetry considerations, C_{ijkl} has only 21 components are independent. Applying the Gauss's theorem, we obtain

$$\int_V \rho \frac{\partial u_i^2}{\partial t^2} dV = \int_V f_i dV + \int_V \frac{\partial}{\partial x_j}\{C_{ijkl}u_{k,l}\} dV. \tag{4}$$

For an isotropic elastic medium, the independent components of C_{ijkl} are reduced to two, λ and μ, called Lamé's coefficients, and the tenor of the elastic coefficients has the form of $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The symbol δ_{ij} is known as the Kronecker delta, which is a second-order tensor. For a homogeneous isotropic medium, λ and μ are constants. Thus, equation (4) becomes

$$\int_V \rho \frac{\partial u_i^2}{\partial t^2} dV = \int_V f_i dV + \int_V \{\lambda u_{j,ji} + \mu(u_{i,jj} + u_{j,ij})\} dV \ . \tag{5}$$

We can write the differential equation of equation (5):

$$\rho \frac{\partial u_i^2}{\partial t^2} = f_i + (\lambda + \mu)u_{j,ji} + \mu u_{i,jj}, \tag{6}$$

by using the relation $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$, equation (6) becomes

$$\rho \ddot{\boldsymbol{u}} = \boldsymbol{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}) - \mu \nabla \times (\nabla \times \boldsymbol{u}). \tag{7}$$

We have to find the solution of the equation (7). Without loss of generality, we take O as the origin of Cartesian coordinates and the x_p-axis as the body force direction. The simplest model of the earthquake source is a unidirectional point force located inside a homogeneous, isotropic, infinite elastic medium. Thus, the body force $\boldsymbol{f}$ is given by $f_i = \delta(\boldsymbol{x})\delta(t)\delta_{ip}$, and the initial conditions $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{0}$ and $\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{0}$ for $\boldsymbol{x} \neq \boldsymbol{0}$.

It is very difficult to solve equation (7) directly. We can use Helmholtz decomposition theorem to decoupled the elastodynamic wave equation (7) into two equations, a scalar equation and a vector equation. Let $\boldsymbol{u}$ and $\boldsymbol{f}$ be represented by scalar and vector potentials (Achenbach, 1973),

$$\boldsymbol{u} = \nabla\phi + \nabla \times \boldsymbol{\Psi}, \tag{8.1}$$

$$\boldsymbol{f} = \nabla F + \nabla \times \boldsymbol{H}, \tag{8.2}$$

and the vector potentials must have zero divergence, $\nabla \cdot \boldsymbol{\Psi} = \nabla \cdot \boldsymbol{H} = 0$. Equation (7) can now be separated into two equations:

$$\alpha^2 \nabla^2 \phi + \frac{F}{\rho} = \ddot{\phi}, \tag{9.1}$$

$$\beta^2 \nabla^2 \boldsymbol{\Psi} + \frac{\boldsymbol{H}}{\rho} = \ddot{\boldsymbol{\Psi}}, \tag{9.2}$$

where $\alpha^2 = (\lambda + 2\mu)/\rho$ and $\beta^2 = \mu/\rho$ are the squared P wave velocity and the squared S wave velocity, respectively. To get the displacement $\mathbf{u}$, we need to get the expression of $\nabla\phi$ and $\nabla\times\boldsymbol{\Psi}$, which can be done by using the Fourier transform and convolution.

3 Technical Approach

3.1 Green's function for 3D wave equation

For the 3D wave equation,

$$c^2\nabla^2 g(\boldsymbol{x},t;\mathbf{0},0)+\delta(t)\delta(\boldsymbol{x})=\ddot{g}(\boldsymbol{x},t;\mathbf{0},0), \tag{10}$$

when $|\boldsymbol{x}|\to\infty$, $g(\boldsymbol{x},t;\mathbf{0},0)=0$; and $t<0$, $g(\boldsymbol{x},t;\mathbf{0},0)=\dot{g}(\boldsymbol{x},t;\mathbf{0},0)=0$. By using the Fourier transform, equation (10) becomes

$$-c^2\boldsymbol{k}^2\tilde{g}(\boldsymbol{k},w;\mathbf{0},0)+1=-w^2\tilde{g}(\boldsymbol{k},w;\mathbf{0},0)$$

then,

$$g(\boldsymbol{x},t;\mathbf{0},0)=\frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\iiint_{-\infty}^{+\infty}\frac{1}{(c\boldsymbol{k})^2-w^2}e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}+wt)}dwd\boldsymbol{k}$$

according to the residue theorem,

$$g(\boldsymbol{x},t;\mathbf{0},0)=\frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty}\frac{\sin\,(c|\boldsymbol{k}|t)}{c|\boldsymbol{k}|}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{k}$$

introduce spherical coordinates centered at $|\boldsymbol{k}|=0$, $\boldsymbol{k}\cdot\boldsymbol{x}=|\boldsymbol{k}||\boldsymbol{x}|cos\theta=k|\boldsymbol{x}|cos\theta$,

$$\begin{aligned}
g(\boldsymbol{x},t;\mathbf{0},0)&=\frac{1}{(2\pi)^3}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_0^{+\infty}\frac{\sin\,(ckt)}{ck}e^{-ik|x|cos\theta}k^2 sin\theta dk\\
&=\frac{1}{(2\pi)^2}\int_0^{+\infty}\frac{2\sin\,(ckt)\sin\,(k|\boldsymbol{x}|)}{c|\boldsymbol{x}|}dk\\
&=\frac{1}{(2\pi)^2}\int_0^{+\infty}\frac{\mathrm{cosk}(|\boldsymbol{x}|-\mathrm{ct})-\mathrm{cosk}(|\boldsymbol{x}|+\mathrm{ct})}{|\boldsymbol{x}|}dk\\
&=\frac{1}{4\pi c|\boldsymbol{x}|}\{\delta(|\boldsymbol{x}|-\mathrm{ct})+\delta(|\boldsymbol{x}|+\mathrm{ct})\}\\
&=\frac{1}{4\pi c^2}\frac{\delta\left(t-\frac{|\boldsymbol{x}|}{c}\right)+\delta\left(t+\frac{|\boldsymbol{x}|}{c}\right)}{|\boldsymbol{x}|}
\end{aligned}$$

because $|\boldsymbol{x}|>0, t>0$, the argument of the second delta function is positive and the delta function is zero. Therefore,

$$g(\boldsymbol{x},t;\boldsymbol{0},0)=\frac{1}{4\pi c^2}\frac{\delta\left(t-\frac{|\boldsymbol{x}|}{c}\right)}{|\boldsymbol{x}|}. \tag{11}$$

3.2 Green's function for 3D elastic wave equation

Based on equations (9.1) and (9.2), we can get the integral representation of ϕ and $\boldsymbol{\Psi}$ by using convolution,

$$\begin{aligned}\phi(\boldsymbol{x},t)&=\frac{F}{\rho}*g=\int_{-\infty}^{+\infty}d\tau\int_V\frac{F(\boldsymbol{\xi},\tau)}{\rho}\frac{1}{4\pi\alpha^2}\frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{\alpha}\right)}{|\boldsymbol{x}-\boldsymbol{\xi}|}dV(\boldsymbol{\xi})\\&=\frac{1}{4\pi\rho\alpha^2}\int_V\frac{F\left(\boldsymbol{\xi},t-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{\alpha}\right)}{|\boldsymbol{x}-\boldsymbol{\xi}|}dV(\boldsymbol{\xi}),\end{aligned} \tag{12.1}$$

$$\begin{aligned}\boldsymbol{\Psi}(\boldsymbol{x},t)&=\frac{\boldsymbol{H}}{\rho}*g=\int_{-\infty}^{+\infty}d\tau\int_V\frac{\boldsymbol{H}(\boldsymbol{\xi},\tau)}{\rho}\frac{1}{4\pi\beta^2}\frac{\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{\beta}\right)}{|\boldsymbol{x}-\boldsymbol{\xi}|}dV(\boldsymbol{\xi})\\&=\frac{1}{4\pi\rho\beta^2}\int_V\frac{\boldsymbol{H}\left(\boldsymbol{\xi},t-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{\beta}\right)}{|\boldsymbol{x}-\boldsymbol{\xi}|}dV(\boldsymbol{\xi}).\end{aligned} \tag{12.2}$$

To get the exact expression of F and $\boldsymbol{H}$, application to equation (8.2) of the divergence and curl operations give

$$\nabla\cdot\boldsymbol{f}=\nabla\cdot\nabla F+\nabla\cdot\nabla\times\boldsymbol{H}=\nabla^2F \tag{13.1}$$

and

$$\nabla\times\boldsymbol{f}=\nabla\times\nabla F+\nabla\times\nabla\times\boldsymbol{H}=-\nabla^2\boldsymbol{H}. \tag{13.2}$$

Equations (13.1) and (13.2) have the form of Poisson's equation. For the Poisson's equation, $\nabla^2 g(\boldsymbol{x})=\delta(\boldsymbol{x})$, when $|\boldsymbol{x}|\to\infty$, $g(\boldsymbol{x})=0$, by using the Fourier transform, it becomes

$$-\boldsymbol{k}^2\tilde{g}(\boldsymbol{k})=1,$$

then,

$$g(\boldsymbol{x})=\frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty}\frac{-1}{\boldsymbol{k}^2}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{k},$$

introduce spherical coordinates centered at $|\boldsymbol{k}|=0$, $\boldsymbol{k}\cdot\boldsymbol{x}=|\boldsymbol{k}||\boldsymbol{x}|cos\theta=k|\boldsymbol{x}|cos\theta$,

$$g(\boldsymbol{x})=\frac{-1}{(2\pi)^3}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_0^{+\infty}e^{-ik|\boldsymbol{x}|cos\theta}sin\theta dk$$

$$= \frac{-1}{(2\pi)^2} \int_0^{+\infty} \frac{2\sin(k|\boldsymbol{x}|)}{k|\boldsymbol{x}|} dk$$

$$= \frac{-1}{(2\pi)^2|\boldsymbol{x}|} \int_0^{+\infty} \frac{2\sin(k|\boldsymbol{x}|)}{k|\boldsymbol{x}|} d(k|\boldsymbol{x}|)$$

$$= \frac{-1}{4\pi|\boldsymbol{x}|}$$

since, $\int_0^{+\infty} \frac{\sin(k|x|)}{k|x|} d(k|\boldsymbol{x}|) = \frac{\pi}{2}$. Therefore, the solution for the F and $\boldsymbol{H}$ can be given by,

$$F(\boldsymbol{x},t) = \nabla \cdot \boldsymbol{f}(\boldsymbol{\xi},t) * g = -\frac{1}{4\pi} \int_V \frac{\nabla \cdot \boldsymbol{f}(\boldsymbol{\xi},t)}{|\boldsymbol{x}-\boldsymbol{\xi}|} dV(\boldsymbol{\xi}), \tag{14.1}$$

$$\boldsymbol{H}(\boldsymbol{x},t) = -\nabla \times \boldsymbol{f}(\boldsymbol{\xi},t) * g = \frac{1}{4\pi} \int_V \frac{\nabla \times \boldsymbol{f}(\boldsymbol{\xi},t)}{|\boldsymbol{x}-\boldsymbol{\xi}|} dV(\boldsymbol{\xi}). \tag{14.2}$$

By substituting $f_i(\boldsymbol{\xi},t) = \delta(\boldsymbol{\xi})\delta(t)\delta_{ip}$ into equations (14.1) and (14.2), we get

$$\begin{aligned} F(\boldsymbol{x},t) &= -\frac{1}{4\pi} \int_V \frac{\nabla \cdot \{\delta(\boldsymbol{\xi})\delta(t)\delta_{ip}\}}{|\boldsymbol{x}-\boldsymbol{\xi}|} dV(\boldsymbol{\xi}) \\ &= -\frac{\delta(t)\delta_{ip}}{4\pi} \int_V \left\{ \frac{\partial}{\partial \xi_p}\left(\frac{\delta(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}\right) - \frac{\partial}{\partial \xi_p}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\right)\delta(\boldsymbol{\xi}) \right\} dV(\boldsymbol{\xi}) \\ &= -\frac{1}{4\pi} \frac{\partial}{\partial x_p} \frac{1}{|\boldsymbol{x}|} \delta(t), \end{aligned} \tag{15.1}$$

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{x},t) &= \frac{1}{4\pi} \int_V \frac{\nabla \times \{\delta(\boldsymbol{\xi})\delta(t)\delta_{kp}\}}{|\boldsymbol{x}-\boldsymbol{\xi}|} dV(\boldsymbol{\xi}) \\ &= \frac{\boldsymbol{e}_i}{4\pi} \varepsilon_{jki} \int_V \frac{\frac{\partial}{\partial \xi_j}\{\delta(\boldsymbol{\xi})\delta(t)\delta_{kp}\}}{|\boldsymbol{x}-\boldsymbol{\xi}|} dV(\boldsymbol{\xi}) \\ &= \frac{\delta(t)\delta_{kp}\boldsymbol{e}_i}{4\pi} \varepsilon_{jki} \int_V \left\{ \frac{\partial}{\partial \xi_j}\left(\frac{\delta(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}\right) - \frac{\partial}{\partial \xi_j}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\right) \right\} dV(\boldsymbol{\xi}) \\ &= -\frac{\delta(t)\boldsymbol{e}_i}{4\pi} \varepsilon_{ipj} \frac{\partial}{\partial x_j} \frac{1}{|\boldsymbol{x}|}, \end{aligned} \tag{15.2}$$

where the symbol ε_{jki} is known as the Levi-Civita symbol. Substituting equations (15.1) and (15.2) into equations (12.1) and (12.2) yields

$$\phi(\boldsymbol{x},t) = -\frac{1}{(4\pi\alpha)^2\rho}\int\limits_V \frac{\delta\left(t-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{\alpha}\right)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\frac{\partial}{\partial\xi_p}\frac{1}{|\boldsymbol{\xi}|}dV(\boldsymbol{\xi}), \tag{16.1}$$

$$\boldsymbol{\Psi}(\boldsymbol{x},t) = -\frac{\varepsilon_{ipj}\boldsymbol{e}_i}{(4\pi\beta)^2\rho}\int\limits_V \frac{\delta\left(t-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{\beta}\right)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\frac{\partial}{\partial\xi_j}\frac{1}{|\boldsymbol{\xi}|}dV(\boldsymbol{\xi}), \tag{16.2}$$

Let $\boldsymbol{x}$ be the center of the spherical coordinate system (Figure 1), $R = |\boldsymbol{x}-\boldsymbol{\xi}|$, then

$$\int\limits_V \frac{\delta\left(t-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c}\right)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\frac{\partial}{\partial\xi_j}\frac{1}{|\boldsymbol{\xi}|}dV(\boldsymbol{\xi}) = \int_0^{+\infty}\iint\limits_{|x-\xi|=R}\frac{\delta\left(t-\frac{R}{c}\right)}{R}\frac{\partial}{\partial\xi_j}\frac{1}{|\boldsymbol{\xi}|}dS(\boldsymbol{\xi})\,dR$$

$$= -\int_0^{+\infty}\lim_{|\eta|\to 0}\iint\limits_{|x-\xi|=R}\frac{\delta\left(t-\frac{R}{c}\right)}{R}\frac{\partial}{\partial\eta_j}\frac{1}{|\boldsymbol{\xi}-\boldsymbol{\eta}|}dS(\boldsymbol{\xi})\,dR$$

$$= -\int_0^{+\infty}\frac{\delta\left(t-\frac{R}{c}\right)}{R}\lim_{|\eta|\to 0}\frac{\partial}{\partial\eta_j}\iint\limits_{|x-\xi|=R}\frac{1}{|\boldsymbol{\xi}-\boldsymbol{\eta}|}dS(\boldsymbol{\xi})\,dR$$

let $|\boldsymbol{\xi}-\boldsymbol{\eta}| = R'$, then

$$\iint\limits_{|x-\xi|=R}\frac{1}{|\boldsymbol{\xi}-\boldsymbol{\eta}|}dS(\boldsymbol{\xi}) = \int_0^{2\pi}d\varphi\int_0^{\pi}\frac{R^2 sin\theta}{R'}d\theta = \int_0^{\pi}\frac{2\pi R^2 sin\theta}{R'}d\theta$$

According to the law of cosines,

$$R'^2 = r^2 + R^2 - 2rRcos\theta \Rightarrow R'dR' = Rrsin\theta d\theta$$

then,

$$\int_0^{\pi}\frac{2\pi R^2 sin\theta}{R'}d\theta = \int_{\theta=0}^{\theta=\pi}\frac{2\pi R}{r}dR' = \frac{2\pi R}{r}(R+r-|r-R|)$$

$$= \begin{cases}\frac{4\pi R^2}{r}, r > R\\ 4\pi R, r < R\end{cases} = 4\pi R + 4\pi R^2\left(\frac{1}{r}-\frac{1}{R}\right)H(r-R)$$

where $H(r-R)$ is the Heaviside function, notice $\boldsymbol{\eta}$ is dependent of R, we get

$$\lim_{|\eta|\to 0}\frac{\partial}{\partial\eta_j}\iint\limits_{|x-\xi|=R}\frac{1}{|\boldsymbol{\xi}-\boldsymbol{\eta}|}dS(\boldsymbol{\xi})$$

$$= \lim_{|\eta|\to 0}\frac{\partial}{\partial\eta_j}\left\{4\pi R + 4\pi R^2\left(\frac{1}{r}-\frac{1}{R}\right)H(r-R)\right\}$$

$$= \lim_{|\eta| \to 0} 4\pi R^2 \frac{\partial}{\partial \eta_j} \frac{1}{r} H(r - R)$$

$$= \lim_{|\eta| \to 0} 4\pi R^2 \frac{\partial}{\partial \eta_j} \frac{1}{|\boldsymbol{x} - \boldsymbol{\eta}|} H(|\boldsymbol{x} - \boldsymbol{\eta}| - R)$$

$$= -4\pi R^2 \frac{\partial}{\partial x_j} \frac{1}{|\boldsymbol{x}|} H(|\boldsymbol{x}| - R)$$

thus,

$$\int_V \frac{\delta\left(t - \frac{|\boldsymbol{x} - \boldsymbol{\xi}|}{c}\right)}{|\boldsymbol{x} - \boldsymbol{\xi}|} \frac{\partial}{\partial \xi_j} \frac{1}{|\boldsymbol{\xi}|} dV(\boldsymbol{\xi}) = \int_0^{|x|} \frac{\delta\left(t - \frac{R}{c}\right)}{R} 4\pi R^2 \frac{\partial}{\partial x_j} \frac{1}{|\boldsymbol{x}|} dR$$

Let $R = \tau c, dR = c d\tau$, we can get

$$\int_V \frac{\delta\left(t - \frac{|\boldsymbol{x} - \boldsymbol{\xi}|}{c}\right)}{|\boldsymbol{x} - \boldsymbol{\xi}|} \frac{\partial}{\partial \xi_j} \frac{1}{|\boldsymbol{\xi}|} dV(\boldsymbol{\xi}) = \int_0^{\frac{|x|}{c}} \frac{\delta(t - \tau)}{\tau} 4\pi (\tau c)^2 \frac{\partial}{\partial x_j} \frac{1}{|\boldsymbol{x}|} d\tau. \tag{17}$$

Substituting equation (17) into equations (16.1) and (16.2) yields

$$\phi(\boldsymbol{x}, t) = -\frac{1}{4\pi\rho} \frac{\partial}{\partial x_p} \frac{1}{|\boldsymbol{x}|} \int_0^{\frac{|x|}{\alpha}} \tau \delta(t - \tau)\, d\tau, \tag{18.1}$$

$$\boldsymbol{\Psi}(\boldsymbol{x}, t) = -\frac{\varepsilon_{ipj} \boldsymbol{e}_i}{4\pi\rho} \frac{\partial}{\partial x_j} \frac{1}{|\boldsymbol{x}|} \int_0^{\frac{|x|}{\beta}} \tau \delta(t - \tau)\, d\tau. \tag{18.2}$$

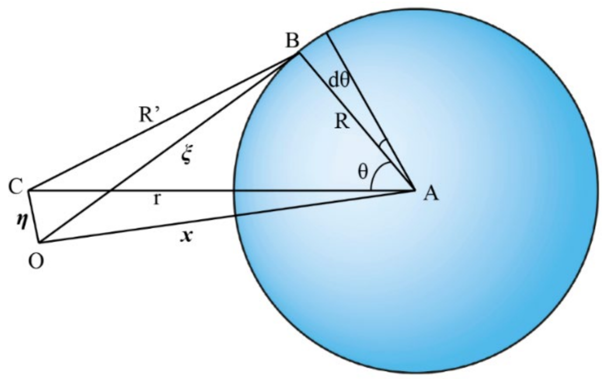

Figure 1 Geometry for the integration. Vector $\boldsymbol{\xi}$ and $\boldsymbol{x}$ correspond to the receiver and source locations, respectively.

By using integration by parts,

$$I(|\boldsymbol{x}|, c) = \int_0^{\frac{|\boldsymbol{x}|}{c}} \tau\delta(t-\tau)\,d\tau = -\int_0^{\frac{|\boldsymbol{x}|}{c}} \tau\,dH(t-\tau)$$

$$= -\left\{ \tau H(t-\tau)\Big|_{\tau=0}^{\tau=\frac{|\boldsymbol{x}|}{c}} - \int_0^{\frac{|\boldsymbol{x}|}{c}} H(t-\tau)d\tau \right\}$$

$$= -\frac{|\boldsymbol{x}|}{c} H\left(t - \frac{|\boldsymbol{x}|}{c}\right) - (t-\tau)H(t-\tau)\Big|_{\tau=0}^{\tau=\frac{|\boldsymbol{x}|}{c}} = t\left\{H(t) - H(t - \frac{|\boldsymbol{x}|}{c})\right\}$$

Based on equation (8.1),

$$\boldsymbol{G}(\boldsymbol{x}, t) = \boldsymbol{u} = \nabla\phi + \nabla \times \boldsymbol{\Psi} = \frac{\partial\phi}{\partial x_n}\boldsymbol{e}_n + \varepsilon_{kin}\frac{\partial\Psi_i}{\partial x_k}\boldsymbol{e}_n$$

then,

$$G_{np}(\boldsymbol{x}, t; \boldsymbol{0}, 0) = \frac{\partial\phi}{\partial x_n} + \varepsilon_{kin}\frac{\partial\Psi_i}{\partial x_k}$$

$$= \frac{\partial}{\partial x_n}\left\{-\frac{1}{4\pi\rho}\frac{\partial}{\partial x_p}\frac{1}{|\boldsymbol{x}|}I(|\boldsymbol{x}|, \alpha)\right\} + \frac{\partial}{\partial x_k}\left\{-\frac{\varepsilon_{kin}\varepsilon_{ipj}}{4\pi\rho}\frac{\partial}{\partial x_j}\frac{1}{|\boldsymbol{x}|}I(|\boldsymbol{x}|, \beta)\right\}$$

$$= -\frac{1}{4\pi\rho}\frac{\partial^2 \frac{1}{|\boldsymbol{x}|}}{\partial x_n \partial x_p}I(|\boldsymbol{x}|, \alpha) - \frac{1}{4\pi\rho}\frac{\partial \frac{1}{|\boldsymbol{x}|}}{\partial x_p}\frac{\partial I(|\boldsymbol{x}|, \alpha)}{\partial x_n}$$

$$- \frac{\varepsilon_{kin}\varepsilon_{ipj}}{4\pi\rho}\frac{\partial^2 \frac{1}{|\boldsymbol{x}|}}{\partial x_k \partial x_j}I(|\boldsymbol{x}|, \beta) - \frac{\varepsilon_{kin}\varepsilon_{ipj}}{4\pi\rho}\frac{\partial \frac{1}{|\boldsymbol{x}|}}{\partial x_j}\frac{\partial I(|\boldsymbol{x}|, \beta)}{\partial x_k}$$

where G_{np} means the force is applied in the p-direction, and the displacement is in the n-direction. The derivation can be calculated by

$$r = |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad \frac{\partial}{\partial x_p}|\boldsymbol{x}| = \frac{x_p}{r} = \gamma_p, \qquad \frac{\partial}{\partial x_p}\frac{1}{|\boldsymbol{x}|} = -\frac{\gamma_p}{r^2},$$

$$\frac{\partial^2 \frac{1}{|\boldsymbol{x}|}}{\partial x_n \partial x_p} = \frac{1}{r^3}(3\gamma_n\gamma_p - \delta_{np}),$$

$$\frac{\partial I(|\boldsymbol{x}|, \alpha)}{\partial x_n} = \frac{\partial\left\{tH(t) - tH(t - \frac{|\boldsymbol{x}|}{\alpha})\right\}}{\partial x_n} = \frac{t\gamma_n}{\alpha}\delta\left(t - \frac{|\boldsymbol{x}|}{\alpha}\right) = \frac{r\gamma_n}{\alpha^2}\delta\left(t - \frac{r}{\alpha}\right),$$

where γ_p is the direction cosines. With these expressions, and $\varepsilon_{kin}\varepsilon_{ipj} = \delta_{np}\delta_{jk} - \delta_{nj}\delta_{pk}$, we obtain

$$G_{np}(\boldsymbol{x}, t; \mathbf{0}, 0) = \frac{3\gamma_n\gamma_p - \delta_{np}}{4\pi\rho r^3} t\left\{H\left(t - \frac{r}{\alpha}\right) - H\left(t - \frac{r}{\beta}\right)\right\}$$
$$+ \frac{\gamma_n\gamma_p}{4\pi\rho\alpha^2 r}\delta\left(t - \frac{r}{\alpha}\right) + \frac{\delta_{np} - \gamma_n\gamma_p}{4\pi\rho\beta^2 r}\delta\left(t - \frac{r}{\beta}\right). \quad (19)$$

Equation (19) is the Green's function for a homogeneous, isotropic, unbounded elastic medium. For the right-hand side of equation (19), the first term depends on r^{-2}, which means that it will dominate for small values of r ($\lambda \gg r$). For this reason, it is known as the near-field term. The last two terms, on the other hand, depend on r^{-1}, which means that they will dominate for large values of r ($r \gg \lambda$). Therefore, they are known as the far-field terms. This equation constitutes the basic building blocks for seismic source studies, and we next discuss it in detail.

4 Results

4.1 Properties of the elastic wave fields

For the far-field term G_{np}^{FP},

$$G_{np}^{FP}(\boldsymbol{x}, t; \mathbf{0}, 0) = \frac{\gamma_n\gamma_p}{4\pi\rho\alpha^2 r}\delta\left(t - \frac{r}{\alpha}\right), \quad (20.1)$$

1) the factor $\delta\left(t - \frac{r}{\alpha}\right)$ indicates wave propagation with P wave velocity α, and the waveform can only be observed when $t = \frac{r}{\alpha}$;

2) attenuates as r^{-1};

3) particles move ($\gamma_n\gamma_p$) in the same direction that the wave is moving in (γ_n), because γ_p is fixed ($\gamma_n\gamma_p \propto \gamma_n$). The term G_{np}^{FP} corresponds to far-field P-wave motion (Figure 2).

For the far-field term G_{np}^{FS},

$$G_{np}^{FS}(\boldsymbol{x}, t; \mathbf{0}, 0) = \frac{\delta_{np} - \gamma_n\gamma_p}{4\pi\rho\beta^2 r}\delta\left(t - \frac{r}{\beta}\right). \quad (20.2)$$

1) the factor $\delta\left(t - \frac{r}{\beta}\right)$ indicates wave propagation with S wave velocity β, and the waveform can only be observed when $t = \frac{r}{\beta}$;

2) attenuates as r^{-1};

3) particles move $(\delta_{np} - \gamma_n \gamma_p)$ perpendicular to the direction of wave propagation (γ_n), as can be seen by computing the scalar product:

$$\gamma_n(\delta_{np} - \gamma_n \gamma_p) = 0,$$

here the property $\gamma_n \gamma_n = 1$ was used. The term G_{np}^{FS} corresponds to far-field S-wave motion (Figure 2).

For the near-field term G_{np}^{N},

$$G_{np}^{N}(\boldsymbol{x}, t; \boldsymbol{0}, 0) = \frac{3\gamma_n \gamma_p - \delta_{np}}{4\pi\rho r^3} t \left\{ H\left(t - \frac{r}{\alpha}\right) - H\left(t - \frac{r}{\beta}\right) \right\}, \tag{20.3}$$

1) the factor $t\left\{ H\left(t - \frac{r}{\alpha}\right) - H\left(t - \frac{r}{\beta}\right) \right\}$ shows that the wave amplitude increases linearly with time within $\frac{r}{\alpha} < t < \frac{r}{\beta}$;

2) attenuates as r^{-2}, because $t = \frac{r}{c}$;

3) particles move $(3\gamma_n \gamma_p - \delta_{np})$ with a combination of the P wave and S wave motions, as can be seen by writing the vectorial factor as $\gamma_n \gamma_p - \delta_{np} + 2\gamma_n \gamma_p$.

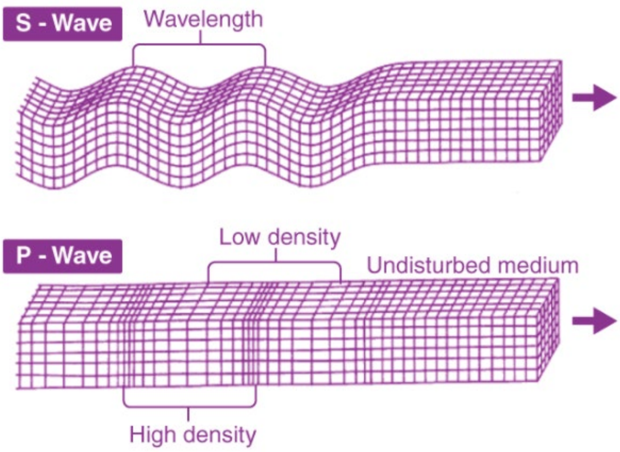

Figure 2 A harmonic plane S wave (top) and P wave (bottom) traveling horizontally across the page. (source: https://byjus.com/physics/p-wave/).

4.2 Radiation patterns of elastic waves

In this section, we will discuss the radiation patterns, which gives the dependence of the displacement on the source-receiver direction. We assume that ρ, α, β and r are constants, and the point force is in the x_1-direction. We ignore all the constant terms. Then, from equations (20.1) and (20.2), we get

$$G_{n1}^{P} = \gamma_n \gamma_1, \; G_{n1}^{S} = \delta_{n1} - \gamma_n \gamma_1$$

and

$$G_{n1}^{N} = 3\gamma_n\gamma_1 - \delta_{n1}.$$

In the spherical coordinates (Figure 3), the direction cosines are given by

$$\gamma_1 = sin\theta cos\varphi,\ \gamma_2 = sin\theta sin\varphi,\ \gamma_3 = cos\theta.$$

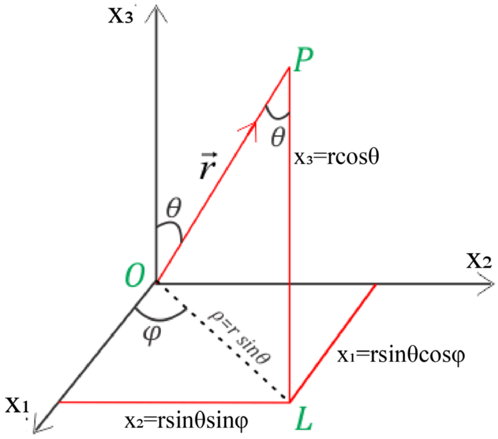

Figure 3 Spherical polar coordinates

With these expressions, we obtain:

$$G_{11}^{P} = (sin\theta cos\varphi)^2, G_{21}^{P} = \frac{1}{2}sin2\varphi(sin\theta)^2, G_{31}^{P} = \frac{1}{2}sin2\theta cos\varphi,$$

$$G_{11}^{S} = 1-(sin\theta cos\varphi)^2, G_{21}^{S} = -\frac{1}{2}sin2\varphi(sin\theta)^2, G_{31}^{S} = -\frac{1}{2}sin2\theta cos\varphi\ ,$$

and

$$G_{11}^{N} = 3(sin\theta cos\varphi)^2 - 1, G_{21}^{N} = \frac{3}{2}sin2\varphi(sin\theta)^2, G_{31}^{N} = \frac{3}{2}sin2\theta cos\varphi.$$

For the amplitudes, both the near-field and far-field radiation patterns are symmetric about the x_1-axis (Figure 4). The P-wave amplitude is the largest in the direction of the force, while the S-wave amplitude is the largest normal to the force. The far-field P wave radiation pattern has the shape of two spheres (Figure 4a), and in the $x_2 = 0$ plane the motion is zero. The far-field S wave radiation pattern resembles a doughnut without the central hole (Figure 4b). Pay attention to that along the point force direction (x_1-axis), there is no S-wave motion. The near-field radiation pattern is more complicated and can be considered as a combination of the P-wave and S-wave motions (Figure 4c).

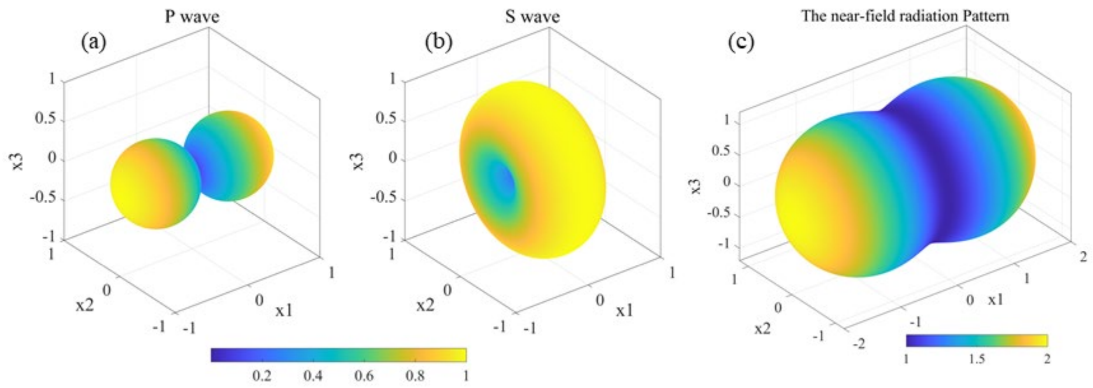

Figure 4 The far-field (a, b) and near-field (c) radiation patterns generated by the point force in the x_1-direction.

4.3 The geometric spreading of elastic waves

In this section, we mainly discuss the attenuation characteristics of the near-field and far-field elastic waves due to geometric spreading. Geometric spreading is simply the energy density decrease as the elastic wavefront expands. Based on the mean density and mean elastic wave velocities of the crust (Turcotte and Schubert, 2014), we set the related parameters (Table 1).

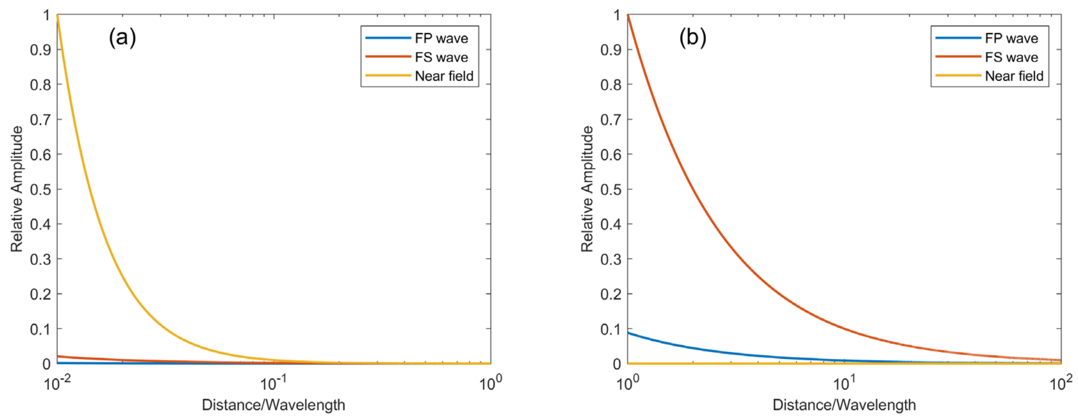

Figure 5 Attenuation of elastic waves due to geometric spreading, the relative amplitude is normalized with the maximum near-field wave motion (a) and the maximum far-field S wave motion (b), respectively.

When the source-receiver distance is less than one-tenth of the wavelength, the near-field term dominates (Figure 5a). However, the near-field term decays very rapidly. The far-field S-wave motion dominates at distances greater than one wavelength (Figure 5b). For this reason, we can find that the first arriving wave (Primary wave) has a relatively smaller amplitude and the later arriving wave (Secondary wave) has a relatively larger amplitude at a distant seismogram (Figure 6). In a word, in a homogeneous, isotropic medium, the geometric spreading of the body wave is proportional to the reciprocal of the distance between source and receiver.

Table 1 Some parameters for calculating the geometric spreading of elastic waves

P wave velocity	5.55 km/s	Frequency	1 Hz
S wave velocity	3.25 km/s	θ	30°
Density	$2.6 \times 10^3\ kg/m^3$	φ	60°

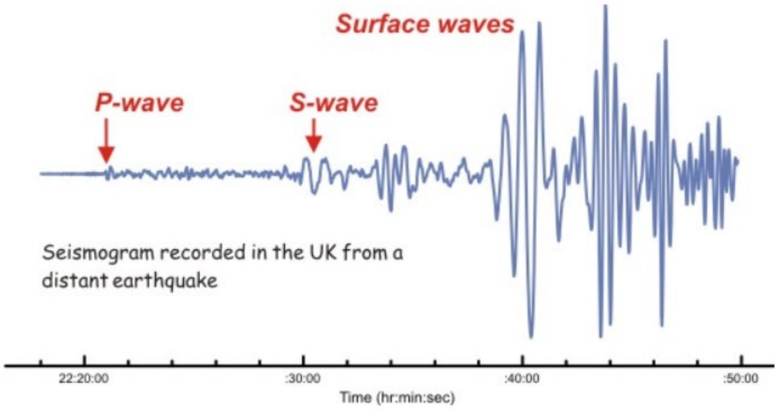

Figure 6 A seismogram showing the P wave and S wave. (source: http://earthquakes.bgs.ac.uk/)

5 Summary

In this paper, based on Fourier transform, convolution and generalized function, we obtain the elastodynamic Green's function in a homogeneous, isotropic, unbounded medium. The near-field term of Greens function decays rapidly, which is proportional to r^{-2}, and the far-field term decays proportional to r^{-1}. Then, we analyze the radiation patterns of the far-field and near-field terms. They are symmetrical along with the axis on which the point force acts. Specifically, the near-field radiation pattern is more complex, including P-wave and S-wave motions. The far-field P wave has the largest amplitude in the same direction that the force acts, while the far-field S wave has the largest amplitude perpendicular to the direction that the force acts.

References

Achenbach, J., 1973. Wave propagation in elastic solids, North-Holland, Amsterdam.

Burridge, R. and L. Knopoff, 1964. Body force equivalents for seismic dislocations, Bull. Seism. Soc. Am. 54, 1875–1888.

Lay, Thorne; Wallace, Terry C. 1995. Modern Global Seismology. London: Academic Press.

Stauder, W., 1962. The focal mechanism of earthquakes, in H. Landsberg and J. Van Mieghem, eds., Advances in Geophysics vol 9, Academic, New York, 1–76.

Turcotte, D., and Schubert, G. 2014. *Geodynamics* (3rd ed.). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511843877.