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Content

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Introduction: Adjoint Method

The *adjoint method*, also known as adjoint technique or adjoint simulation technique, is a useful computerized tool for the analysis of *linear time-varying systems*.

With the adjoint method, *error budgets* or sensitivities of the LTV system due to all disturbance input terms can be automatically generated.

In this paper, new interpretations of the adjoint method are achieved by using the *adjoint definition equation* in state-space form.

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Introduction: Adjoint Concept

Suppose $\boldsymbol{G} : \mathcal{U} \mapsto \mathcal{Y}$ is a linear system, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces. The adjoint of $\boldsymbol{G}$ is the linear system $\boldsymbol{G}^* : \mathcal{Y} \mapsto \mathcal{U}$ such that

$$\langle \boldsymbol{G}u, y \rangle_{\mathcal{Y}} = \langle u, \boldsymbol{G}^* y \rangle_{\mathcal{U}} \quad \forall u \in \mathcal{U},\ \forall y \in \mathcal{Y}, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the *inner product* defined by

$$\langle f, g \rangle = \int_{t_0}^{t_\mathrm{f}} g^\mathrm{T}(t) f(t) \mathrm{d}t, \quad f, g \in \mathcal{U} \text{ or } \mathcal{Y} \tag{2}$$

$$\left\langle \begin{bmatrix} f_0 \\ f \end{bmatrix}, \begin{bmatrix} g_0 \\ g \end{bmatrix} \right\rangle = g_0^\mathrm{T} f_0 + \int_{t_0}^{t_\mathrm{f}} g^\mathrm{T}(t) f(t) \mathrm{d}t, \quad \begin{bmatrix} f_0 \\ f \end{bmatrix}, \begin{bmatrix} g_0 \\ g \end{bmatrix} \in \mathbb{R}^n \oplus \mathcal{U} \text{ or } \mathbb{R}^n \oplus \mathcal{Y} \tag{3}$$

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Adjoint Definition Equation

$$p^{\mathrm{T}}(t_{\mathrm{f}})x(t_{\mathrm{f}}) - p^{\mathrm{T}}(0)x(0) = \int_0^{t_{\mathrm{f}}} -r^{\mathrm{T}}(t)y(t) + u^{\mathrm{T}}(t)q(t)\mathrm{d}t. \tag{4}$$

Linear System

$$\boldsymbol{G} : \mathbb{R}^n \oplus \mathcal{U} \mapsto \mathbb{R}^n \oplus \mathcal{Y}; \begin{bmatrix} x_0 \\ u \end{bmatrix} \mapsto \begin{bmatrix} x_{\mathrm{f}} \\ y \end{bmatrix} \tag{5}$$

Adjoint System

$$\boldsymbol{G}^* : \mathbb{R}^n \oplus \mathcal{Y} \mapsto \mathbb{R}^n \oplus \mathcal{U}; \begin{bmatrix} p_{\mathrm{f}} \\ r \end{bmatrix} \mapsto \begin{bmatrix} p_0 \\ q \end{bmatrix} \tag{6}$$

$$\left\langle \boldsymbol{G}\begin{bmatrix} x_0 \\ u \end{bmatrix}, \begin{bmatrix} p_{\mathrm{f}} \\ r \end{bmatrix} \right\rangle_{\mathbb{R}^n \oplus \mathcal{Y}} = \left\langle \begin{bmatrix} x_0 \\ u \end{bmatrix}, \boldsymbol{G}^*\begin{bmatrix} p_{\mathrm{f}} \\ r \end{bmatrix} \right\rangle_{\mathbb{R}^n \oplus \mathcal{U}} \quad \forall \begin{bmatrix} x_0 \\ u \end{bmatrix} \in \mathbb{R}^n \oplus \mathcal{U}, \ \forall \begin{bmatrix} p_{\mathrm{f}} \\ r \end{bmatrix} \in \mathbb{R}^n \oplus \mathcal{Y}, \tag{7}$$

$$\left\langle \begin{bmatrix} f_0 \\ f \end{bmatrix}, \begin{bmatrix} g_0 \\ g \end{bmatrix} \right\rangle = g_0^{\mathrm{T}} f_0 + \int_{t_0}^{t_{\mathrm{f}}} g^{\mathrm{T}}(t)f(t)\mathrm{d}t. \tag{8}$$

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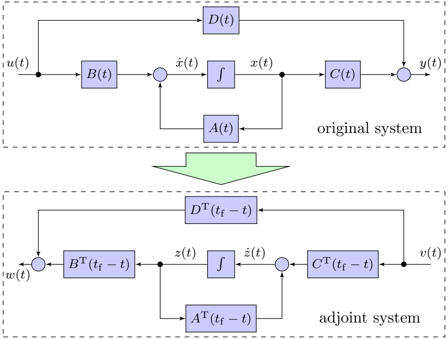
Adjoint System Summary

Original Linear System

$$\left.\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t),\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}\right\}$$

Adjoint System

$$\left.\begin{aligned}\dot{z}(t) &= A^{\mathrm{T}}(t_{\mathrm{f}}-t)z(t) + C^{\mathrm{T}}(t_{\mathrm{f}}-t)v(t),\\ w(t) &= B^{\mathrm{T}}(t_{\mathrm{f}}-t)z(t) + D^{\mathrm{T}}(t_{\mathrm{f}}-t)v(t)\end{aligned}\right\}$$



$$z^{\mathrm{T}}(0)x(t_{\mathrm{f}}) - z^{\mathrm{T}}(t_{\mathrm{f}})x(0) = \int_0^{t_{\mathrm{f}}} -v^{\mathrm{T}}(t_{\mathrm{f}}-t)y(t) + u^{\mathrm{T}}(t)w(t_{\mathrm{f}}-t)\mathrm{d}t \tag{9}$$

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Conclusion

- The interpretation of adjoint simulation was achieved in a new and universal way, by using the adjoint definition equation in state-space form;
- The adjoint technique in covariance analysis was also derived;
- The adjoint concept is fundamental in mathematics, and naturally arises or is widely used in optimization, optimal control, stability analysis, navigation problems, circuit analysis, etc.

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Thank You!

Questions?

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Appendix

Basic Definitions and Properties of LTV

Consider the linear system

$$\begin{aligned} \boldsymbol{G} &: \mathcal{U} \mapsto \mathcal{Y} \\ &: u \mapsto y = \boldsymbol{G}u, \end{aligned} \tag{10}$$

described by state-space equations

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \in \mathbb{R}^n \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned} \tag{11}$$

We are primarily concerned with the linear system $\boldsymbol{G}$ in the finite-horizon case. In (10), $\mathcal{U}$ and $\mathcal{Y}$ are finite-horizon Lebesgue 2-spaces $\mathcal{L}_2[t_0, t_f]$, and $u \in \mathcal{U}$, $y \in \mathcal{Y}$. In (11), $u(t) \in \mathbb{R}^m$ is the input vector, $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the output vector; $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are continuous real matrix valued functions of time with appropriate dimensions.
Let $\Phi(t, \tau)$ be the *state transition matrix* associated with system (11), which has the following properties

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, \tau) = A(t)\Phi(t, \tau), \ \Phi(\tau, \tau) = I, \tag{12a}$$

$$\Phi^{-1}(t, \tau) = \Phi(\tau, t), \tag{12b}$$

in which I denotes the identity matrix. Then the solution to (11) is

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\mathrm{d}\tau, \tag{13a}$$

$$y(t) = C(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)\mathrm{d}\tau + D(t)u(t). \tag{13b}$$

In (13b), the first term is known as the *zero-input response*, the other terms the *zero-state response*. The zero-state response may be represented by the integral operator

$$y(t) = \int_{t_0}^{t_f} H(t, \tau)u(\tau)\mathrm{d}\tau, \tag{14}$$

in which

Construction Rules for Adjoint System

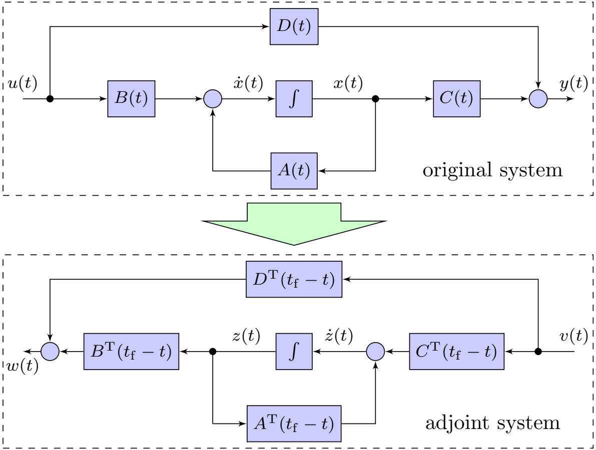

Figure 1: Construction of adjoint system from original system

The block diagrams of the orignal system and the adjoint system are shown in Figure. 1:, which can be used to explain the general construction rules [Laning; Zarchan; Yanushevsky] of an adjoint system from the orignal system:

1. Replace t by $t_{\mathrm{f}} - t$ in the arguments of all variable coefficients where t_{f} is the final time.
2. Reverse all signal flow, redefine branch points (•) as sum points (◯), and vice versa.

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