Chapter 1

Vector Spaces

Definition 1.1. *A **vector space** over a field $\mathbb{K}$ is a set V together with two operations, addition and scalar multiplication, that satisfy the following eight axioms:*

(V1) $(\forall a, b \in V)(a + b = b + a)$.

(V2) $(\forall a, b, c \in V)((a + b) + c = a + (b + c))$.

(V3) $(\exists 0 \in V)(\forall a \in V)(0 + a = a)$.

(V4) $(\forall a \in V)(\exists - a \in V)(a + -a = 0)$.

(V5) $(\forall a \in V)(1 \cdot a = a)$.

(V6) $(\forall a \in V, \forall k, l \in \mathbb{K})((k \cdot l) \cdot a = k \cdot (l \cdot a))$.

(V7) $(\forall a, b \in V, \forall k \in \mathbb{K})(k \cdot (a + b) = k \cdot a + k \cdot b)$.

(V8) $(\forall a \in V, \forall k, l \in \mathbb{K})((k + l) \cdot a = k \cdot a + l \cdot a)$.

Example 1.1 (**Easy**)**.** *The set of real numbers $\mathbb{R}$ is a vector space over itself, with usual addition and multiplication as the operations.*

Example 1.2 (**Medium**)**.** *The set of all polynomials with real coefficients, $P(\mathbb{R})$, is a vector space over the real numbers with polynomial addition and scalar multiplication defined as usual.*

Example 1.3 (Hard). *The set of all continuous real-valued functions defined on a closed interval $[a, b]$, denoted by $C([a, b])$, is a vector space over the real numbers. The operations are function addition and scalar multiplication defined as follows: for $f, g \in C([a, b])$ and $c \in \mathbb{R}$,*

$$(f + g)(x) = f(x) + g(x), \forall x \in [a, b],$$
$$(c \cdot f)(x) = c \cdot f(x), \forall x \in [a, b].$$

Proposition 1.1. *Given a field $\mathbb{K}$, all vector spaces of dimension n over $\mathbb{K}$ are isomorphic.*

Proof. Let V and W be vector spaces of dimension n over $\mathbb{K}$. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be bases for V and W, respectively. Define a function $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_n w_n,$$

for $a_1, \ldots, a_n \in \mathbb{K}$. One can verify that T is a linear transformation. Since every vector in each vector space can be expressed as a linear combination of its basis vectors, T is surjective. Since the coefficients a_i are unique for each vector in the basis, T is also injective. Hence, T is a bijective linear transformation, i.e., an isomorphism, so $V \cong W$. Thus, all vector spaces of dimension n over $\mathbb{K}$ are isomorphic. $\square$