9. Positive/negative definite matrices

- Quadratic forms

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$ (an $n \times n$ matrix)

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$ (an n vector)

$\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$ is called a quadratic form（二次形式）.

With the matrix A and the vector $\boldsymbol{x}$, the quadratic form can be expressed as $\boldsymbol{x}^T A \boldsymbol{x}$.

9. Positive/negative definite matrices

Let A be a real symmetric matrix, i.e., $A^T = A$.

For $\forall \boldsymbol{x} \neq \mathbf{0}$,

If $\boldsymbol{x}^T A \boldsymbol{x} > 0$, A is said to be a positive definite matrix（正定値行列）, and sometimes expressed as $A > 0$;

If $\boldsymbol{x}^T A \boldsymbol{x} \geq 0$, A is said to be a positive semi-definite matrix（準正定値行列）, and sometimes expressed as $A \geq 0$;

If $\boldsymbol{x}^T A \boldsymbol{x} < 0$, A is said to be a negative definite matrix（負定値行列）, and sometimes expressed as $A < 0$;

If $\boldsymbol{x}^T A \boldsymbol{x} \leq 0$, A is said to be a negative semi-definite matrix（準負定値行列）, and sometimes expressed as $A \leq 0$.

9. Positive/negative definite matrices

For a scalar a,

$$x^T a x = xax = ax^2 > 0 \Leftrightarrow a > 0$$

$$x^T a x = ax^2 \geq 0 \Leftrightarrow a \geq 0$$

$$x^T a x = ax^2 < 0 \Leftrightarrow a < 0$$

$$x^T a x = ax^2 \leq 0 \Leftrightarrow a \leq 0$$

$$\forall x \neq 0$$

Positive (negative) definiteness of a matrix corresponds to positiveness (negativeness) of a scalar, but is not the same.

9. Positive/negative definite matrices

The statement “if $A > 0$ does not hold, then $A \leq 0$ ” is not correct.

There is a symmetrical matrix A which satisfies
$\boldsymbol{x}_1^T A \boldsymbol{x}_1 \leq 0$ for a vector $\boldsymbol{x}_1$,
but for another vector $\boldsymbol{x}_2$, $\boldsymbol{x}_2^T A \boldsymbol{x}_2 > 0$.

Example

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Not necessarily all symmetrical matrices are classified into the four groups mentioned on page 101.

10. Eigen values

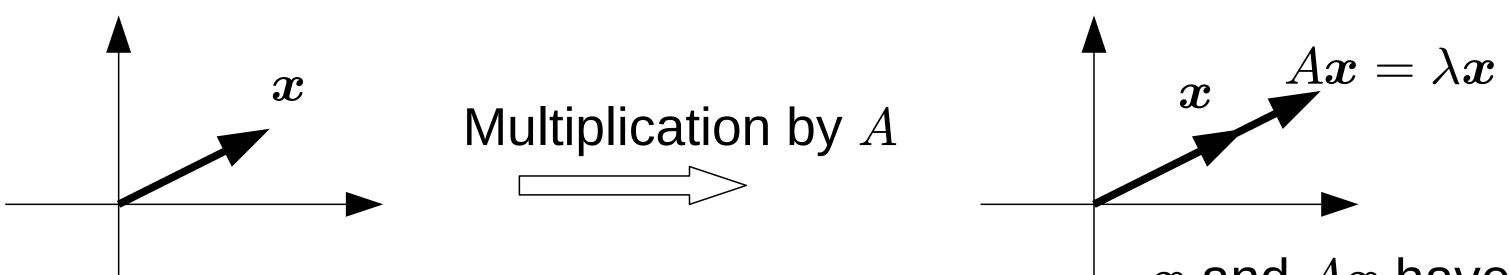
- So far, we have been focusing on the amount of information (the dimensions of linear spaces), and have investigated,
  - whether or not the information is lost by mapping (multiplication by a matrix);
  - whether or not we can transfer information from $\boldsymbol{x} \in X$ to $\boldsymbol{y} \in Y$ by $\boldsymbol{y} = A\boldsymbol{x}$, i.e., whether or not we can obtain any value of $\boldsymbol{y}$ by setting the value of $\boldsymbol{x}$ appropriately;
  - whether or not we can retrieve the value of $\boldsymbol{x}$ from $\boldsymbol{y}$ such that $\boldsymbol{y} = A\boldsymbol{x}$; and
  - the relation between the above and the features (rank and determinant) of matrices.
- Mapping changes the values of vectors (not only the amount of information they carry). In the following, we are going to investigate these changes and their relation to the features (eigen values and eigen vectors) of matrices.

10. Eigen values

We assume that, throughout in this section, A is an $n \times n$ matrix.

- Eigen values（固有値）and eigen vectors（固有ベクトル）

When a vector $\boldsymbol{x} \neq \mathbf{0}$ exists such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for a scalar λ, λ is called an eigen value of A, and $\boldsymbol{x}$ is called an eigen vector associated with λ.



$\boldsymbol{x}$ and $A\boldsymbol{x}$ have the same direction when λ is real.

10. Eigen values

Example

Eigen values of matrix $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ are $\lambda_1 = \dfrac{1}{2},\ \lambda_2 = \dfrac{3}{2}$, and

examples of their corresponding eigen vectors are

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Verification

$$A\boldsymbol{x}_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$$

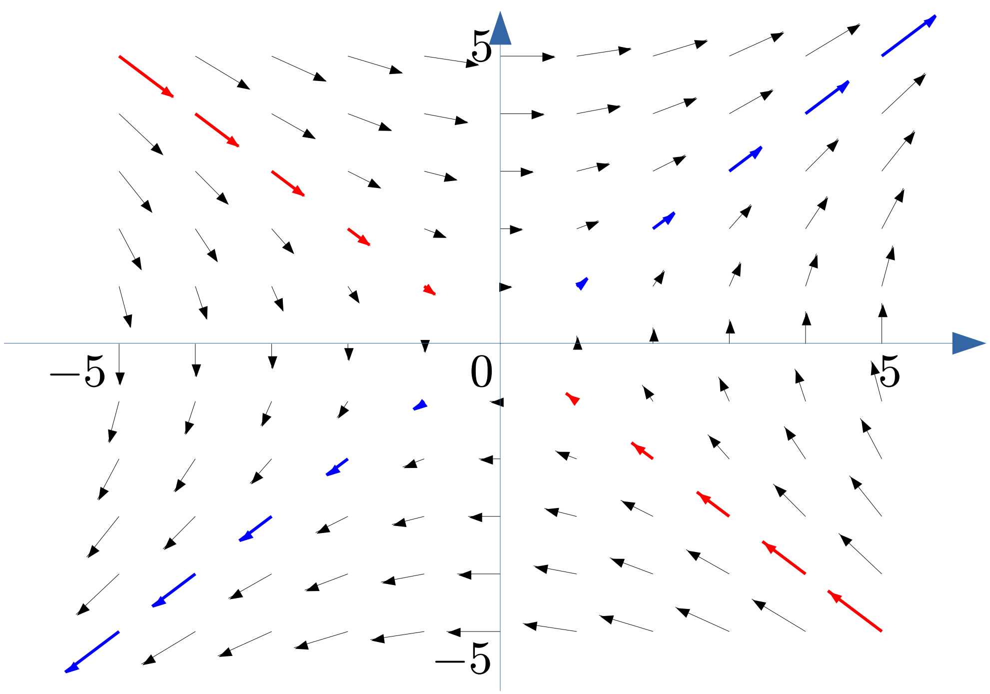
$$A\boldsymbol{x}_2 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = \frac{3}{2}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$$

10. Eigen values

How does a mapping represented by matrix A change the vectors?
In other words, where does vector $\boldsymbol{x}$ go by $A\boldsymbol{x}$?

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \quad \lambda_1 = \frac{1}{2},\ \lambda_2 = \frac{3}{2} \quad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A matrix with real eigen values.



Tails of arrows: $\boldsymbol{x}$, heads of arrows: $A\boldsymbol{x}$

When vector $\boldsymbol{x}$ is either in the direction of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$,

i.e., $\boldsymbol{x} = \alpha \boldsymbol{x}_1$, then $A\boldsymbol{x}$ moves away from the origin: $A(\alpha \boldsymbol{x}_1) = \frac{3}{2}(\alpha \boldsymbol{x}_1)$.

When vector $\boldsymbol{x}$ is either in the direction of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

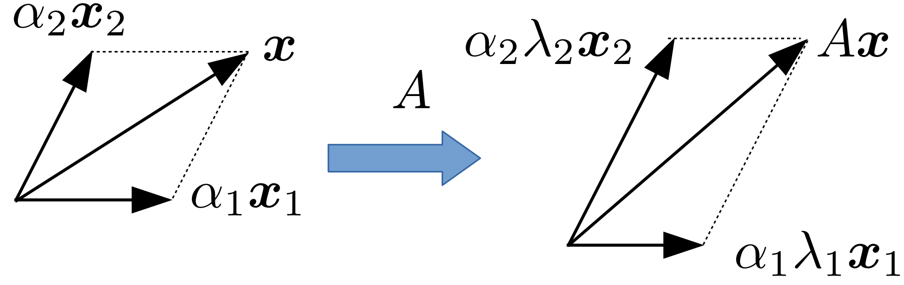
i.e., $\boldsymbol{x} = \alpha \boldsymbol{x}_2$, then $A\boldsymbol{x}$ moves towards the origin: $A(\alpha \boldsymbol{x}_2) = \frac{1}{2}(\alpha \boldsymbol{x}_2)$.

10. Eigen values

When eigen vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are linearly independent with each other, any vector $\boldsymbol{x}$ can be represented as $\boldsymbol{x} = \alpha_1 \boldsymbol{x}_1 + \cdots + \alpha_n \boldsymbol{x}_n$, and then,

$$\begin{aligned} A\boldsymbol{x} &= A(\alpha_1 \boldsymbol{x}_1 + \cdots + \alpha_n \boldsymbol{x}_n) = \alpha_1 A\boldsymbol{x}_1 + \cdots + \alpha_n A\boldsymbol{x}_n \\ &= \alpha_1 \lambda_1 \boldsymbol{x}_1 + \cdots + \alpha_n \lambda_n \boldsymbol{x}_n. \end{aligned}$$

If $|\lambda_\ell| \geq |\lambda_i|,\ i = 1, \ldots, n,\ i \neq \ell$, then $A^k \boldsymbol{x}$ approaches a vector having the same direction as $\boldsymbol{x}_\ell$.



$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ \frac{13}{8} \end{bmatrix} = \begin{bmatrix} 1.75 \\ 1.625 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{4} \\ \frac{13}{8} \end{bmatrix} = \begin{bmatrix} \frac{41}{16} \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 2.5625 \\ 2.5 \end{bmatrix}$$

10. Eigen values

When eigen values are complex numbers（複素数）, eigen vectors are in general complex vectors.

Example

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \ \lambda_1 = 1 + i, \ \lambda_2 = 1 - i, \ \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \ \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

We cannot draw a diagram as the one on page 107.