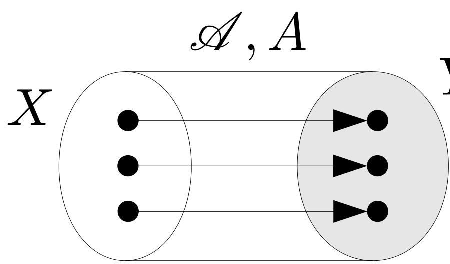
8. Inverse matrices

Linear bijection

$\mathscr{A}$: linear bijection; A : matrix representation of $\mathscr{A}$

A is a square matrix.



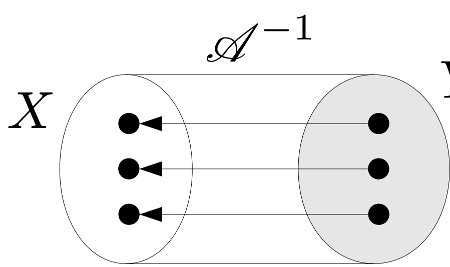
$Y = \mathscr{A}(X) = \mathscr{R}(A)$

For $\forall \boldsymbol{x} \in X$, there exist only one $\boldsymbol{y} \in Y$ such that $\boldsymbol{y}=A\boldsymbol{x}$.

For $\forall \boldsymbol{y} \in Y$, there exist only one $\boldsymbol{x} \in X$ such that $\boldsymbol{y}=A\boldsymbol{x}$.

$\Downarrow$

There exists an inverse mapping $\mathscr{A}^{-1}$ which maps $\boldsymbol{y}$ back to $\boldsymbol{x}$.



$Y = \mathscr{A}(X) = \mathscr{R}(A)$

8. Inverse matrices

$\mathscr{A}^{-1}$: inverse mapping of a linear bijection $\mathscr{A}$

Matrix representation, A^{-1}, of $\mathscr{A}^{-1}$?

For $\forall \boldsymbol{x} \in X$, the following hold

$$\boldsymbol{y} = A\boldsymbol{x}$$

$$\boldsymbol{x} = A^{-1}\boldsymbol{y}$$ (Note that we do not know what A^{-1} is like yet)

$$\Downarrow$$

$$\boldsymbol{x} = A^{-1}(A\boldsymbol{x}) = A^{-1}A\boldsymbol{x}$$

$$\Downarrow$$

$$A^{-1}A = I$$ (A^{-1} is the inverse matrix of A)

When $\mathscr{A}$ is a bijection, its inverse mapping $\mathscr{A}^{-1}$ exists.

$$\Updownarrow$$

When A is $\underset{\det A \neq 0}{\underline{\text{of full rank}}}$, its inverse matrix A^{-1} exists.

8. Inverse matrices

Matrices vs scalars

x, y : scalars; a : a scalar

Given the value of y, the value of x that satisfies $y{=}ax$ is given by,

$$x = \frac{1}{a}y = a^{-1}y,$$

where $a \neq 0$ must hold.

$\boldsymbol{x}$, $\boldsymbol{y}$: vectors; A : a matrix

Given the value of $\boldsymbol{y}$, the value of $\boldsymbol{x}$ that satisfies $\boldsymbol{y}{=}A\boldsymbol{x}$ is given by,

$$\boldsymbol{x} = A^{-1}\boldsymbol{y}$$

where $\det A \neq 0$ must hold.

8. Inverse matrices

Matrices vs scalars

A scalar a can be regarded as a 1×1 matrix and its rank can be defined as:

$$\mathrm{rank}\, a(\leq 1) = \begin{cases} 1 & a \neq 0 \\ 0 & a = 0 \end{cases}$$

$a \neq 0$, the reciprocal（逆数）of a exists.
The value of x can be determined, for a given y, such that y=ax.

a=0, the reciprocal of a does not exist.
The value of x can not be determined, for a given y, such that y=ax.

The rank of an $n \times n$ matrix A

$$\mathrm{rank}\, A(\leq n) = \begin{cases} n & \det A \neq 0 \\ \left.\begin{array}{c} n-1 \\ \vdots \\ 1 \\ 0 \end{array}\right\} & \det A = 0 \end{cases}$$

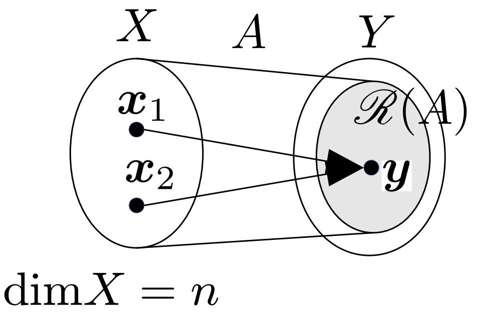
$\det A \neq 0$, the inverse of A exists.
The value of $\boldsymbol{x}$ can be determined, for a given $\boldsymbol{y}$, such that $\boldsymbol{y}$=$A\boldsymbol{x}$.

$\det A = 0$, the inverse of A does not exist.
The value of $\boldsymbol{x}$ cannot be determined, for a given $\boldsymbol{y}$, such that $\boldsymbol{y}$=$A\boldsymbol{x}$.

Any difference when $\mathrm{rank}\, A$ is 0, 1, ..., or n–1?

8. Inverse matrices

What happens when $\mathrm{rank}\, A = r \leq n-1$?



In this case, the mapping $\mathscr{A}$ is not an injection.

For a given $\boldsymbol{y} \in \mathscr{R}(A)$, there are more than one $\boldsymbol{x}$s such that $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$.
We cannot determine $\boldsymbol{x}$ uniquely.

More specifically, we can determine some part of $\boldsymbol{x}$ uniquely, but there are some degrees of freedom in the remaining part.

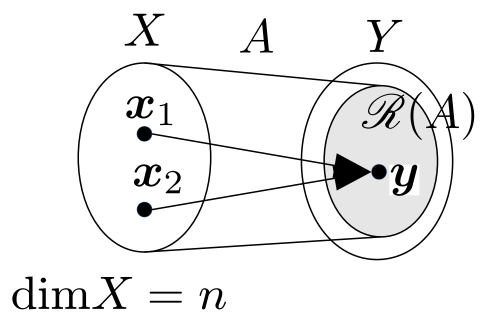
Then, how much of $\boldsymbol{x}$ can we determine uniquely?

8. Inverse matrices

Then, how much of $\boldsymbol{x}$ can we determine uniquely?

The part that cannot be uniquely determined is, in the right figure, $\boldsymbol{x}_1 - \boldsymbol{x}_2$, which is in $\mathscr{N}(A)$.

We cannot determine the part of $\boldsymbol{x}$ which is in the null space of A.



Let us suppose we found an $\boldsymbol{x}$ which satisfies $\boldsymbol{y}=A\boldsymbol{x}$ and represent it as $\bar{\boldsymbol{x}}$, i.e., $\boldsymbol{y} = A\bar{\boldsymbol{x}}$.

For $\forall \boldsymbol{x}_0 \in \mathscr{N}(A)$, $\boldsymbol{x}' = \bar{\boldsymbol{x}} + \boldsymbol{x}_0$ also satisfies $\boldsymbol{y} = A\boldsymbol{x}'$.

$$\because A\boldsymbol{x}' = A(\bar{\boldsymbol{x}} + \boldsymbol{x}_0) = A\bar{\boldsymbol{x}} + A\boldsymbol{x}_0 = \boldsymbol{y} + \boldsymbol{0} = \boldsymbol{y}$$

$$\underbrace{\dim \mathscr{N}(A)}_{\text{The part which cannot be uniquely determined from } \boldsymbol{y}.} + \underbrace{\dim \mathscr{R}(A)}_{\text{The part which can be uniquely determined from } \boldsymbol{y}.} = n = \dim X$$

8. Inverse matrices

An example: a linear differential equation.

$$\frac{dx}{dt} + x = y$$

The mapping $\mathscr{A}$ from $x(t)$ to $y(t)$ is a linear mapping. $\mathscr{A} = \left(\frac{d}{dt} + 1\right)$, $\mathscr{A}x = y$

$$\left(\begin{aligned}\mathscr{A}(ax_1 + bx_2) &= \frac{d}{dt}(ax_1 + bx_2) + (ax_1 + bx_2) = \frac{d}{dt}(ax_1) + \frac{d}{dt}(bx_2) + ax_1 + bx_2 \\ &= a\frac{dx_1}{dt} + b\frac{dx_2}{dt} + ax_1 + bx_2 = a\left(\frac{dx_1}{dt} + x_1\right) + \left(\frac{dx_2}{dt} + x_2\right) = a\mathscr{A}x_1 + b\mathscr{A}x_2\end{aligned}\right)$$

A general solution to $\frac{dx}{dt} + x = y$

The sum of a solution that satisfies $\frac{dx}{dt} + x = y$ (a particular solution) and the general solution to $\frac{dx}{dt} + x = 0$

An x that satisfies $y = \mathscr{A}x$ (which corresponds to $\bar{x}$).

An x that satisfies $\mathscr{A}x = 0$ (which belongs to $\mathscr{N}(\mathscr{A})$).

8. Inverse matrices

Example

$$\underset{A}{\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}}\underset{\boldsymbol{x}}{\begin{bmatrix}x_1\\ x_2\end{bmatrix}} = \underset{\boldsymbol{y}}{\begin{bmatrix}3\\ 6\end{bmatrix}}$$

$$n = 2,\ r = \operatorname{rank} A = 1 < 2,\ n - r = 1$$

$$\left(-2\cdot\begin{bmatrix}1\\ 2\end{bmatrix} + \begin{bmatrix}2\\ 4\end{bmatrix} = \mathbf{0},\ 0\cdot\begin{bmatrix}1\\ 2\end{bmatrix} = \mathbf{0},\ \det\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} = 1\cdot 4 - 2\cdot 2 = 0,\ \det[1] = 1\right)$$

$\bar{\boldsymbol{x}} = \begin{bmatrix}1\\ 1\end{bmatrix}$ satisfies $A\bar{\boldsymbol{x}} = \boldsymbol{y}$.

$A\boldsymbol{x} = \mathbf{0}$ holds when $x_1 + 2x_2 = 0$.

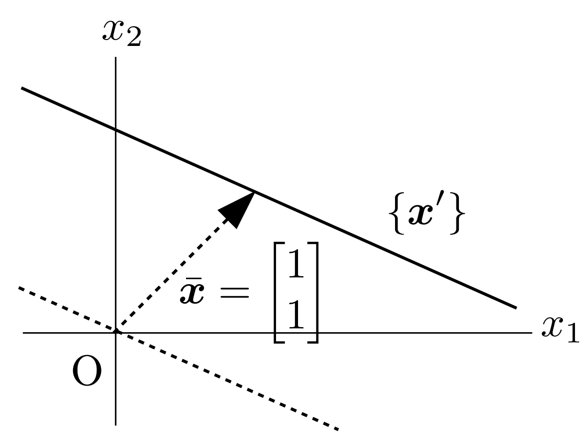
Therefore, $\forall \boldsymbol{x}_0 \in \mathscr{N}(A)$ can be represented as $\boldsymbol{x}_0 = \begin{bmatrix}-2\alpha\\ \alpha\end{bmatrix}$,

where α is an arbitrary real number.

If we define $\boldsymbol{x}' = \bar{\boldsymbol{x}} + \boldsymbol{x}_0 = \begin{bmatrix}1-2\alpha\\ 1+\alpha\end{bmatrix}$, then

$$A\boldsymbol{x}' = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}1-2\alpha\\ 1+\alpha\end{bmatrix}$$

$$= \begin{bmatrix}(1-2\alpha)+2(1+\alpha)\\ 2(1-2\alpha)+4(1+\alpha)\end{bmatrix} = \begin{bmatrix}3\\ 6\end{bmatrix} = \boldsymbol{y}.$$



$\mathscr{N}(A) = \{\boldsymbol{x}_n\}\ :\ x_1 + 2x_2 = 0$

The dimension is $n-r=1$.

8. Inverse matrices

When an $n \times n$ matrix A is not of full rank ($\mathrm{rank}\ A < n$), we cannot uniquely determine $\boldsymbol{x}$ such that $\boldsymbol{y} = A\boldsymbol{x}$ for a given $\boldsymbol{y}$ (there are infinitely many $\boldsymbol{x}$s).

How can we determine it uniquely?

For a linear differential equation $\frac{dx}{dt} + x = y$, we give initial conditions to determine the solution uniquely.

Let us give other conditions than the value of $\boldsymbol{y}$.

8. Inverse matrices

Example

Suppose we impose another condition, in addition to $\boldsymbol{y} = A\boldsymbol{x}$,

that $\|\boldsymbol{x}\| = \sqrt{\sum_{i=1}^{n} {x_i}^2}$ should be minimized.

In the example on page 90,

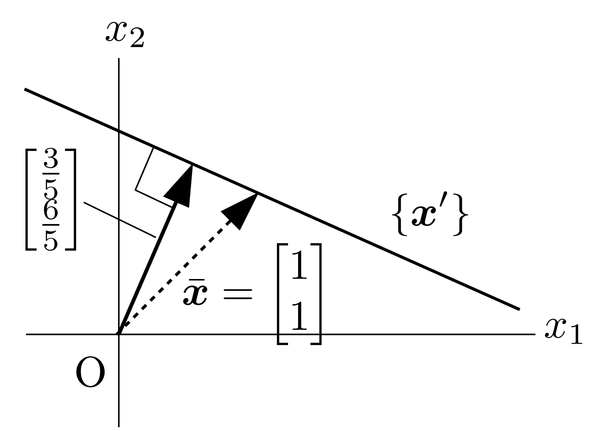
$$\|\boldsymbol{x}'\| = \left\| \begin{bmatrix} 1-2\alpha \\ 1+\alpha \end{bmatrix} \right\| = \sqrt{(1-2\alpha)^2 + (1+\alpha)^2} = \sqrt{5\alpha^2 - 2\alpha + 2} = \sqrt{5\left(\alpha - \frac{1}{5}\right)^2 + \frac{9}{5}},$$

which takes the minimum value when

$$\alpha = \frac{1}{5},$$

and we obtain the unique value of $\boldsymbol{x}'$ as

$$\boldsymbol{x}' = \begin{bmatrix} \frac{3}{5} \\ \frac{6}{5} \end{bmatrix}.$$



8. Inverse matrices

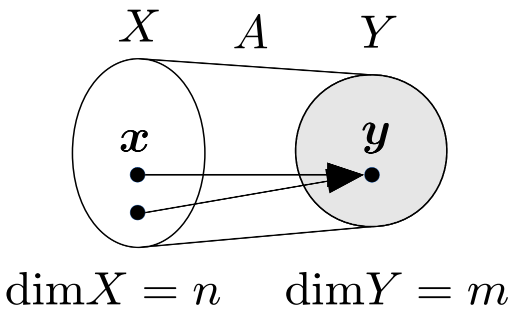
When a matrix A is not a square matrix, can we define a matrix similar to the inverse matrix?

Let us suppose we have a non-square matrix $A \in \mathbb{R}^{m \times n},\ m \neq n$.

8. Inverse matrices

- Suppose A is of row full rank, i.e., $\mathrm{rank} A = m < n$.

$$A = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{mn} & \cdots & \cdots & a_{mn} \end{bmatrix}$$



The mapping represented by the matrix is a surjection but NOT an injection.

For any given $\boldsymbol{y} \in Y$, there are infinitely many $\boldsymbol{x}$s that satisfy $\boldsymbol{y} = A\boldsymbol{x}$.

In order to determine $\boldsymbol{x}$ uniquely, we need additional conditions just as discussed on pages 91 and 92.

8. Inverse matrices

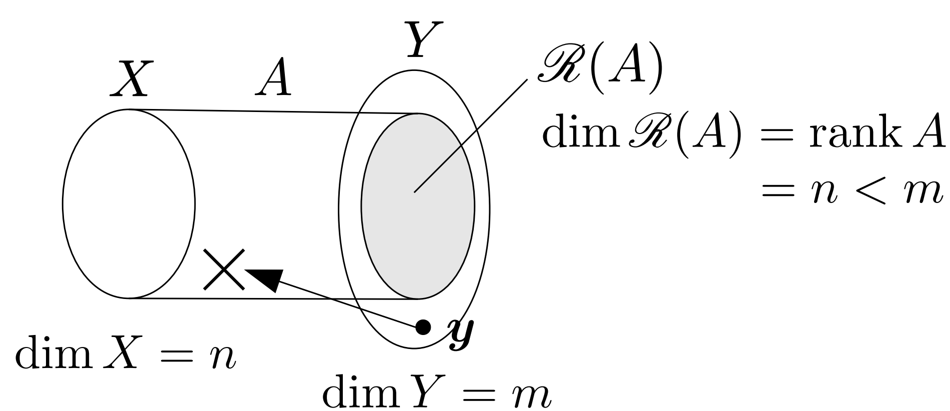
- Suppose A is of column full rank, i.e., $\mathrm{rank} A = n < m$.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{mn} & \cdots & a_{mn} \end{bmatrix}$$

Since A is not of row full rank, the mapping represented by A is not a surjection.

Therefore, for $\boldsymbol{y}$ which is in Y but not in $\mathscr{R}(A)$, there is no $\boldsymbol{x}$ such that $\boldsymbol{y} = A\boldsymbol{x}$.

Then, what is the best we can do?



8. Inverse matrices

– Now let us start with $\boldsymbol{y} = A\boldsymbol{x}$ where A is of column full rank.

$$\boldsymbol{y} = A\boldsymbol{x}$$

We multiply the both hand sides by A^T from the left.

$$A^T\boldsymbol{y} = A^T A\boldsymbol{x}$$

When A is of full column rank, the inverse matrix of $A^T A$ exists.
(In general, $\mathrm{rank} A = \mathrm{rank} A^T A = \mathrm{rank} A A^T$.)

$$(A^T A)^{-1} A^T \boldsymbol{y} = (A^T A)^{-1} A^T A \boldsymbol{x} = \boldsymbol{x}$$

The matrix $A^\dagger = (A^T A)^{-1} A^T$ has a property similar to the inverse matrix:

$$A^\dagger A = (A^T A)^{-1} A^T A = I,$$

but

$$A A^\dagger = A(A^T A)^{-1} A^T \neq I.$$

$A^\dagger$ is called a pseudo-inverse matrix（疑似逆行列）of A.

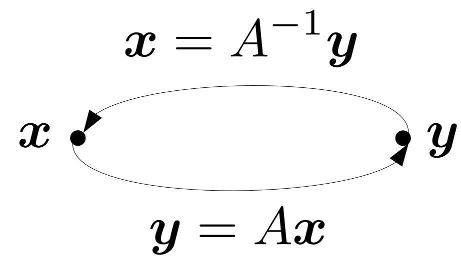
8. Inverse matrices

- Pseudo-inverse matrices

Suppose A has its inverse matrix A^{-1} .

If we define $\boldsymbol{x}$ as $\boldsymbol{x} = A^{-1}\boldsymbol{y}$, then we have

$$A\boldsymbol{x} = A(A^{-1}\boldsymbol{y}) = \boldsymbol{y}.$$



Suppose A does not have its inverse matrix A^{-1} but has the pseudo-inverse matrix $A^{\dagger}$.

If we define $\hat{\boldsymbol{x}}$ as $\hat{\boldsymbol{x}} = A^{\dagger}\boldsymbol{y} = (A^T A)^{-1}A^T\boldsymbol{y}$, then is

$$\hat{\boldsymbol{y}} = A\hat{\boldsymbol{x}} = A(A^T A)^{-1}A^T\boldsymbol{y}$$

equal to $\boldsymbol{y}$? If not, what is the difference between $\hat{\boldsymbol{y}}$ and $\boldsymbol{y}$?

The answer is that

$$\hat{\boldsymbol{y}} \neq \boldsymbol{y},$$

and that $\boldsymbol{y} - \hat{\boldsymbol{y}}$ is orthogonal（直交している）to $\hat{\boldsymbol{y}}$.

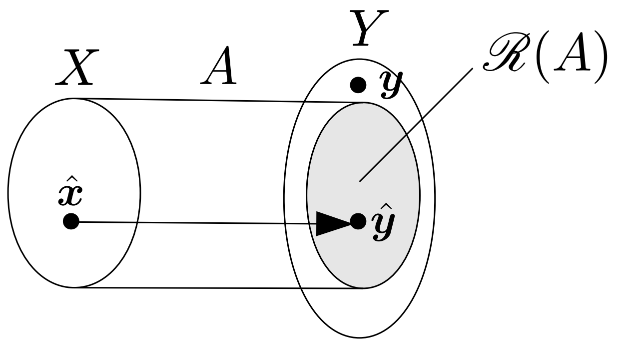
8. Inverse matrices

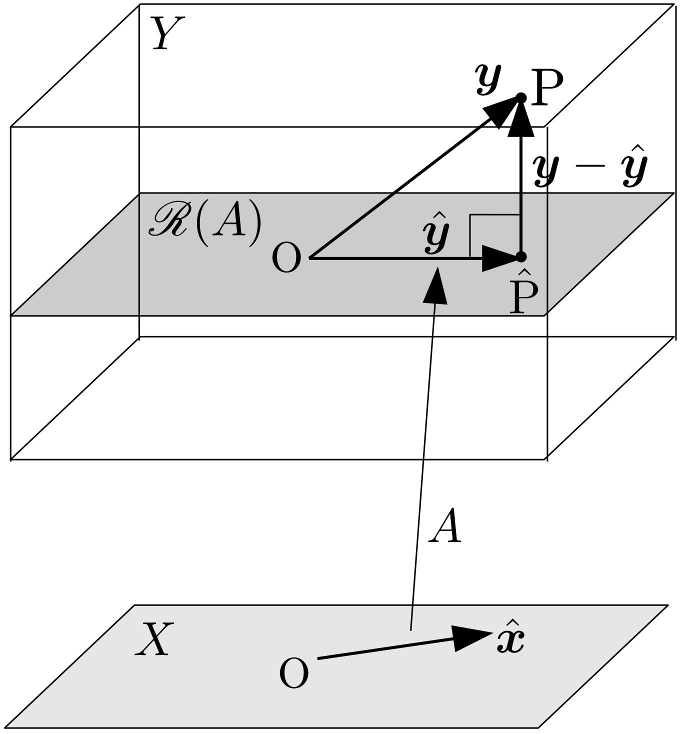
The inner product of $\boldsymbol{y} - \hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{y}}$:

$$\begin{aligned}
(\boldsymbol{y} - \hat{\boldsymbol{y}})^T \hat{\boldsymbol{y}} &= \left[\boldsymbol{y} - \{A(A^T A)^{-1} A^T \boldsymbol{y}\}\right]^T \{A(A^T A)^{-1} A^T \boldsymbol{y}\} \\
&= \left[\{I - A(A^T A)^{-1} A^T\}\boldsymbol{y}\right]^T A(A^T A)^{-1} A^T \boldsymbol{y} \\
&= \boldsymbol{y}^T \{I - A(A^T A)^{-1} A^T\}^T A(A^T A)^{-1} A^T \boldsymbol{y} \\
&= \boldsymbol{y}^T \{I - A(A^T A)^{-1} A^T\} A(A^T A)^{-1} A^T \boldsymbol{y} \\
&= \boldsymbol{y}^T \{A(A^T A)^{-1} A^T - A(A^T A)^{-1} A^T A(A^T A)^{-1} A^T\} \boldsymbol{y} \\
&= \boldsymbol{y}^T \{A(A^T A)^{-1} A^T - A(A^T A)^{-1} A^T\} \boldsymbol{y} \\
&= \boldsymbol{y}^T \, 0 \, \boldsymbol{y} \\
&= 0
\end{aligned}$$

8. Inverse matrices

$\hat{\boldsymbol{y}} = A\hat{\boldsymbol{x}} \in \mathscr{R}(A)$ but not necessarily $\boldsymbol{y} \in \mathscr{R}(A)$.





$\boldsymbol{y} - \hat{\boldsymbol{y}}$ is orthogonal to $\hat{\boldsymbol{y}}$.

In the left figure, point $\hat{\mathrm{P}}$ is the foot on $\mathscr{R}(A)$ of the perpendicular line（垂線の足）from point P.

Point $\hat{\mathrm{P}}$ is the closest point on $\mathscr{R}(A)$ to P.

Vector $\hat{\boldsymbol{y}}$ is the closet vector on $\mathscr{R}(A)$ to vector $\boldsymbol{y}$ (the best approximator of $\boldsymbol{y}$).

$\hat{\boldsymbol{x}}$ is the value of $\boldsymbol{x}$ that minimizes

$$(\boldsymbol{y} - A\boldsymbol{x})^T(\boldsymbol{y} - A\boldsymbol{x}).$$

… Least squares method（最小2乗法）