13. Eigen values and other properties

- Several properties of a square matrix A can be found by the relation $A = T\Lambda T^{-1},\ T^{-1}AT = \Lambda$ or $A = TJT^{-1},\ T^{-1}AT = J.$

13. Eigen values and other properties

$$\det A = \det J = \begin{bmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1\end{matrix}}_{\,n_{11}\times n_{11}} & & & & \\ & \ddots & & & \\ & & \boxed{\begin{matrix}\lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1\end{matrix}}_{\,n_{1,\alpha_1}\times n_{1,\alpha_1}} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix}\lambda_\sigma & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_\sigma\end{matrix}}_{\,n_{\sigma,\alpha_\sigma}\times n_{\sigma,\alpha_\sigma}} \end{bmatrix}$$

$$= {\lambda_1}^{(n_{11}+\cdots+n_{1,\alpha_1})} \cdot {\lambda_2}^{(n_{21}+\cdots+n_{2,\alpha_2})} \cdot \cdots \cdot {\lambda_\sigma}^{(n_{\sigma 1}+\cdots+n_{\sigma,\alpha_\sigma})}$$
$$= {\lambda_1}^{m_1} \cdot {\lambda_2}^{m_2} \cdots {\lambda_\sigma}^{m_\sigma}$$

Determinant of A = the product of n eigen values of A.

$\det A = 0 \Leftrightarrow$ At least one eigen value of A is zero.

13. Eigen values and other properties

- Trace（トレース） of a square matrix A

$$\mathrm{tr}A = \sum_{i=1}^{n} a_{ii}$$: Sum of all the diagonal entries of A

$$\mathrm{tr}A = \mathrm{tr}(TJT^{-1}) = \mathrm{tr}(T^{-1}TJ) = \mathrm{tr}J$$

$$= m_1\lambda_1 + m_2\lambda_2 + \cdots + m_\sigma\lambda_\sigma$$

Trace of A is the sum of its n eigen values.

13. Eigen values and other properties

- Positive/negative definiteness of symmetrical matrices
  - A matrix A is symmetrical if $A = A^T$.
  - A symmetrical matrix A can be diagonalized as

    $$T^T A T = \Lambda, \quad A = T \Lambda T^T,$$

    where all the eigen values are real.
    Note that here $T^{-1} = T^T$.
  - Quadratic form of a symmetrical matrix A

    $$\boldsymbol{x}^T A \boldsymbol{x} = \boldsymbol{x}^T T \Lambda T^T \boldsymbol{x} = (T^T \boldsymbol{x})^T \Lambda (T^T \boldsymbol{x})$$

    Let us define $\boldsymbol{y} = T^T \boldsymbol{x}$. Then the quadratic form of A is represented as a quadratic form of Λ, as

    $$\boldsymbol{x}^T A \boldsymbol{x} = \boldsymbol{y}^T \Lambda \boldsymbol{y}.$$

13. Eigen values and other properties

- Positive/negative definiteness of symmetrical matrices

The quadratic form of a symmetrical matrix A is represented as

$$\boldsymbol{x}^T A \boldsymbol{x} = \boldsymbol{y}^T \Lambda \boldsymbol{y}.$$

when we define $\boldsymbol{y} = T^T \boldsymbol{x}$.

Since T is non-singular, T and T^T represents bijections.
So when $\boldsymbol{x}$ takes any value in $\mathbb{R}^n$, then $\boldsymbol{y}$ also takes any value in $\mathbb{R}^n$.

Therefore, A and Λ have the same positive/negative definiteness.
Also we have

$$\boldsymbol{y}^T \Lambda \boldsymbol{y} = [y_1 \ \cdots \ y_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1 {y_1}^2 + \cdots + \lambda_n {y_n}^2.$$

13. Eigen values and other properties

From the discussion on the previous page, we can conclude the following:

A is positive definite.

$\Leftrightarrow \Lambda$ is positive definite. $\Leftrightarrow\ \lambda_i > 0,\ i = 1, \cdots, n.$

A is positive semi-definite.

$\Leftrightarrow \Lambda$ is positive semi-definite. $\Leftrightarrow\ \lambda_i \geq 0,\ i = 1, \cdots, n.$

A is negative definite.

$\Leftrightarrow \Lambda$ is negative definite. $\Leftrightarrow\ \lambda_i < 0,\ i = 1, \cdots, n.$

A is negative semi-definite.

$\Leftrightarrow \Lambda$ is negative semi-definite. $\Leftrightarrow\ \lambda_i \leq 0,\ i = 1, \cdots, n.$

14. Norm and singular values

- Norm of vectors
  - When a function $\|x\|$ of a vector $\boldsymbol{x}$ satisfies the following conditions, it is called norm（ノルム）of vector $\boldsymbol{x}$. The norm represents “magnitude” or “length” of the vector.
    - For $\boldsymbol{x} \neq \boldsymbol{0}$, $\|\boldsymbol{x}\| > 0$. $\|\boldsymbol{x}\| = 0$ holds if and only if $\boldsymbol{x} = \boldsymbol{0}$.
    - For any vector $\boldsymbol{x}$ and any scalar α, $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$.
    - For any vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (triangular inequality（三角不等式）)
- For a real number $p(\geq 1)$, $\|\boldsymbol{x}\|_p$ below is a norm of a vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

Examples

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \quad \|\boldsymbol{x}\|_\infty = \max_i |x_i|$$

14. Norm and singular values

- Norm of matrices
  - Norm of matrices can be defined just like the norm of vectors. However, it is more convenient to define the norm for matrices regarding them as representations of mappings.

An $m \times n$ matrix A maps an n-dimensional vector $\boldsymbol{x}$ to an m-dimensional vector $\boldsymbol{y}$, by $\boldsymbol{y} = A\boldsymbol{x}$.

The norm of A can be determined how it enlarges or shrinks the vector by the mapping, namely, the ratio of the norms $\dfrac{\|\boldsymbol{y}\|}{\|\boldsymbol{x}\|}$.

The above ratio is not constant but depends on $\boldsymbol{x}$. So, we define the norm of A as follows:

$$\max_{\|\boldsymbol{x}\| \le 1} \|\boldsymbol{y}\| = \max_{\|\boldsymbol{x}\| = 1} \|\boldsymbol{y}\| = \max_{\boldsymbol{x} \ne \boldsymbol{0}} \frac{\|\boldsymbol{y}\|}{\|\boldsymbol{x}\|} = \max_{\boldsymbol{x} \ne \boldsymbol{0}} \frac{\|A\boldsymbol{x}\|}{\|\boldsymbol{x}\|},$$

which is called induced norm（誘導ノルム）of matrix A.

14. Norm and singular values

– Examples of induced norm

Vector norm		Induced matrix norm
$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} \|x_i\|$	$\Rightarrow$	$\|A\| = \max_j \sum_{i=1}^{m} \|a_{ij}\|$
$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n} \|x_i\|^2}$	$\Rightarrow$	$\|A\| = \sqrt{\lambda_{\max}(A^T A)}$
		$\lambda_{\max}$ is the maximum eigen value.
$\|\boldsymbol{x}\|_\infty = \max_i \|x_i\|$	$\Rightarrow$	$\|A\| = \max_i \sum_{j=1}^{n} \|a_{ij}\|$

14. Norm and singular values

- Eigen values are defined for square matrices.
  Then can we define similar values for non-square (rectangular) matrices?
  Yes, we can define singular values for rectangular matrices.

14. Norm and singular values

- Matrix $A^T A$ and singular values
  - Let A be an $m \times n$ matrix.

$A^T A$ is symmetrical and positive semi-definite.

$$\left(\begin{array}{l} (A^T A)^T = A^T (A^T)^T = A^T A \\ \text{For any } \boldsymbol{x} \neq \boldsymbol{0},\ \boldsymbol{x}^T (A^T A) \boldsymbol{x} = (A\boldsymbol{x})^T (A\boldsymbol{x}) = \|A\boldsymbol{x}\|_2{}^2 \geq 0. \end{array} \right)$$

So its all eigen values are non-negative

$$\lambda_i(A^T A) \geq 0,\ i = 1, \cdots, n,$$

and we can define

$$\sigma_i = \sqrt{\lambda_i(A^T A)},\ i = 1, \ldots, n,$$

which are called singular values（特異値）of matrix A.

14. Norm and singular values

The matrix norm induced from $\|\boldsymbol{x}\|_2$ is defined as

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)},$$

and now can be written as

$$\|A\| = \sigma_{\max}(A).$$

14. Norm and singular values

- Singular values
  - Let us define a matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} (m \geq n) \text{ or } \begin{bmatrix} \sigma_1 & & 0 & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ 0 & & \sigma_m & 0 & \dots & 0 \end{bmatrix} (m \leq n).$$

Note that $\sigma_{m+1} = \cdots = \sigma_n = 0 \; (m \leq n)$.

Then there are orthogonal matrices（直交行列）U and V such that

$$A = U \Sigma V,$$

where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ and

$$U^T U = U U^T = I, \; V^T V = V V^T = I.$$

14. Norm and singular values

$A = U\Sigma V$ is called singular value decomposition（特異値分解）of A.

When $\mathrm{rank} A = r$, then $n - r$ singular values out of $\sigma_1, \cdots, \sigma_n$ are zero.