10. Eigen values

- Multiplicity of eigen values and linear independence of eigen vectors
  - Characteristic polynomial（特性多項式）, characteristic equation（特性方程式）

When does $A\boldsymbol{x} = \lambda \boldsymbol{x}$ namely $(\lambda I - A)\boldsymbol{x} = \boldsymbol{0}$ hold for $\boldsymbol{x} \neq \boldsymbol{0}$?

It is when $\det(\lambda I - A) = 0$ or $\mathrm{rank}(\lambda I - A) < n$ is satisfied.

$\Bigg($ $\because$ If $\det(\lambda I - A) \neq 0$ holds, then $(\lambda I - A)^{-1}$ exists and $\boldsymbol{x} = \boldsymbol{0}$ follows. $\Bigg)$

10. Eigen values

Let us regard λ as a variable and represent it as s.

$$\psi(s) = \det(sI - A) = \det \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s - a_{nn} \end{bmatrix}$$

$$= s^n + \rho_1 s^{n-1} + \cdots + \rho_{n-1} s + \rho_n$$

. . . Characteristic polynomial（特性多項式）

$$\psi(s) = 0$$

. . . Characteristic equation（特性方程式）

This is an n-th order equation, and therefore it has n solutions including multiplicities, which can be complex.

$\Downarrow$

A has n eigen values.

10. Eigen values

– Multiplicities（重複度）

Let the distinct（相異なる）eigen values of A be $\lambda_1, \cdots, \lambda_\sigma, \sigma \le n$.

Then the characteristic polynomial $\psi(s)$ can be factorized（因数分解される）as

$$\psi(s) = (s - \lambda_1)^{m_1} (s - \lambda_2)^{m_2} \cdots (s - \lambda_\sigma)^{m_\sigma}.$$

m_i is called the algebraic multiplicity（代数的重複度）of λ_i , and satisfies the relation:

$$\sum_{i=1}^{\sigma} m_i = n.$$

The algebraic multiplicity indicates how many identical eigen values exist for the given value of λ_i .

10. Eigen values

Let us define $r_i = \mathrm{rank}(\lambda_i I - A)$. Then $r_i < n$, and $\alpha_i = n - r_i > 0$.

α_i is called the geometric multiplicity（幾何学的重複度）of λ_i ,

Unlike the algebraic multiplicity m_i , $\sum_{i=1}^{\sigma} \alpha_i$ has no relation with n. But $\alpha_i \leq m_i$ holds.

$\boldsymbol{x}_i$: an eigen vector associated with λ_i

$$\Leftrightarrow (\lambda_i I - A)\boldsymbol{x}_i = \boldsymbol{0}$$

$$\Leftrightarrow \boldsymbol{x}_i \in \mathscr{N}(\lambda_i I - A)$$

$$\dim \mathscr{N}(\lambda_i I - A) = n - \dim \mathscr{R}(\lambda_i I - A) = n - \mathrm{rank}(\lambda_i I - A)$$
$$= n - r_i = \alpha_i$$

Thus there are α_i linearly independent eigen vectors associated with the same eigen vector λ_i.

The geometric multiplicity indicates how many linearly independent eigen vectors are associated with an eigen value λ_i.

10. Eigen values

Example

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \; n = 3$$

$$\psi(s) = \det(sI - A) = \det \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & s-2 \end{bmatrix} = (s-1)^2(s-2)$$

Eigen values: $\lambda_1 = 1, \; \lambda_2 = 2$

The number of distinct eigen values $\sigma = 2$

Algebraic multiplicities

$$m_1 = 2, \; m_2 = 1$$

Geometric multiplicities

$$\operatorname{rank}(\lambda_1 I - A) = \operatorname{rank} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 1 \;\Rightarrow\; \alpha_1 = 3 - 1 = 2$$

$$\operatorname{rank}(\lambda_2 I - A) = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2 \;\Rightarrow\; \alpha_2 = 3 - 2 = 1$$

10. Eigen values

The eigen vector, $\boldsymbol{x}_1$, associated with λ_1 ?

Let $\boldsymbol{x}_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}$. Then it must satisfy:

$$(\lambda_1 I - A)\boldsymbol{x}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -x_{13} \end{bmatrix} = \mathbf{0},$$

which results in $x_{13} = 0$ and $\boldsymbol{x}_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ 0 \end{bmatrix}$,

where x_{11} and x_{12} can be arbitrary values.

This means that there are an infinite number of eigen vectors associated with λ_1 and that, among them, we can select 2 ($= \alpha_2$) linearly independent vectors:

$$\begin{bmatrix} x_{11} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_{12} \\ 0 \end{bmatrix}, x_{11} \neq 0, x_{12} \neq 0.$$

10. Eigen values

The eigen vector, $\boldsymbol{x}_2$, associated with λ_2?

Let $\boldsymbol{x}_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}$. Then it must satisfy:

$$(\lambda_2 I - A)\boldsymbol{x}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} x_{21} \\ x_{22} \\ 0 \end{bmatrix} = \mathbf{0},$$

which results in $x_{21} = x_{22} = 0$ and $\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \\ x_{23} \end{bmatrix}$.

There is 1 $(= \alpha_2)$ linearly independent vector associated with λ_2 : $\begin{bmatrix} 0 \\ 0 \\ x_{23} \end{bmatrix}, x_{23} \neq 0$.

Note that $\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \\ x_{23} \end{bmatrix}$ is linearly independent of $\boldsymbol{x}_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ 0 \end{bmatrix}$.

10. Eigen values

– Linear independence of eigen vectors

Eigen vectors associated with distinct eigen values are linearly independent with each other.

Let us assume $c_1\boldsymbol{x}_1 + \cdots + c_\sigma \boldsymbol{x}_\sigma = \mathbf{0}$ holds for eigen vectors $\boldsymbol{x}_1. \cdots, \boldsymbol{x}_\sigma$ associated with distinct eigen values $\lambda_1, \cdots, \lambda_\sigma$.

Since

$$(\lambda_j I - A)\boldsymbol{x}_i = \lambda_j \boldsymbol{x}_i - \lambda_i \boldsymbol{x}_i = \begin{cases} \mathbf{0}, & j = i, \\ (\lambda_j - \lambda_i)\boldsymbol{x}_i \neq \mathbf{0}, & j \neq i \end{cases}$$

holds, multiplying the both hand sides of $c_1\boldsymbol{x}_1 + \cdots + c_\sigma \boldsymbol{x}_\sigma = \mathbf{0}$ by

$$(\lambda_1 I - A)\cdots(\lambda_{i-1} I - A)(\lambda_{i+1} I - A)\cdots(\lambda_\sigma I - A)$$

gives

$$c_i \underbrace{(\lambda_1 - \lambda_i)\cdots(\lambda_\sigma - \lambda_i)}_{\neq 0} \boldsymbol{x}_i = \mathbf{0}.$$

This implies $c_i = 0$ because $\boldsymbol{x}_i \neq \mathbf{0}$.

The above discussion applies to any of $i = 1, \cdots, \sigma$ and we derive

$$c_1 = \cdots = c_\sigma = 0$$

which implies $\{\boldsymbol{x}_1. \cdots, \boldsymbol{x}_\sigma\}$ are linearly independent.

10. Eigen vectors

There are α_i linearly independent eigen vectors for λ_i .

Therefore the following eigen vectors are all linearly independent with each other:

$$\underbrace{\{\underbrace{\boldsymbol{x}_{11}, \cdots, \boldsymbol{x}_{1,\alpha_1}}_{\alpha_1 \text{ eigen vectors associated with } \lambda_1}, \underbrace{\boldsymbol{x}_{21}, \cdots, \boldsymbol{x}_{2,\alpha_2}}_{\alpha_2 \text{ eigen vectors associated with } \lambda_2}, \cdots, \underbrace{\boldsymbol{x}_{\sigma,1}, \cdots, \boldsymbol{x}_{\sigma,\alpha_\sigma}}_{\alpha_\sigma \text{ eigen vectors associated with } \lambda_\sigma}\}}$$

In total, there are $\sum_{i=1}^{\sigma} \alpha_i (\leq n)$ linearly independent eigen vectors.

11. Matrix diagonalization

- Simple matrix（単純行列）

A matrix is said to be a simple matrix when its algebraic multiplicity m_i is equal to its geometric multiplicity α_i for $i = 1, \ldots, \sigma$.

For a simple matrix

$$\sum_{i=1}^{\sigma} \alpha_i = \sum_{i=1}^{\sigma} m_i = n$$

holds.

Therefore, it has $\sum_{i=1}^{\sigma} \alpha_i = n$ linearly independent eigen vectors:

$$\{\boldsymbol{x}_{11}, \cdots, \boldsymbol{x}_{1,\alpha_1}, \boldsymbol{x}_{21}, \cdots, \boldsymbol{x}_{2,\alpha_2}, \cdots, \boldsymbol{x}_{\sigma,1}, \cdots, \boldsymbol{x}_{\sigma,\alpha_\sigma}\}$$

In other words, the set of eigen vectors can be the basis of the n-dimensional space (the eigen vectors span the n-dimensional space).

11. Matrix diagonalization

- Diagonalization（対角化）of simple matrices

Let us define an $n \times n$ matrix T which consists of n linearly independent eigen vectors of a simple matrix A:

$$T = [\boldsymbol{x}_{11} \cdots \boldsymbol{x}_{1,\alpha_1} \cdots \boldsymbol{x}_{\sigma,1} \cdots \boldsymbol{x}_{\sigma,\alpha_\sigma}].$$

T has the full rank n, and therefore $\det T \neq 0$.

Then we have,

$$\begin{aligned} AT &= [A\boldsymbol{x}_{11} \cdots A\boldsymbol{x}_{1,\alpha_1} \cdots A\boldsymbol{x}_{\sigma,1} \cdots A\boldsymbol{x}_{\sigma,\alpha_\sigma}] \\ &= [\lambda_1\boldsymbol{x}_{11} \cdots \lambda_1\boldsymbol{x}_{1,\alpha_1} \cdots \lambda_\sigma\boldsymbol{x}_{\sigma,1} \cdots \lambda_\sigma\boldsymbol{x}_{\sigma,\alpha_\sigma}] \\ &= [\boldsymbol{x}_{11} \cdots \boldsymbol{x}_{1,\alpha_1} \cdots \boldsymbol{x}_{\sigma,1} \cdots \boldsymbol{x}_{\sigma,\alpha_\sigma}] \begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_\sigma & & \\ & 0 & & & & \ddots & \\ & & & & & & \lambda_\sigma \end{bmatrix} \\ &= T\Lambda, \end{aligned}$$

$\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_1, \cdots, \lambda_\sigma, \cdots, \lambda_\sigma)$: a diagonal matrix(対角行列).

11. Matrix diagonalization

Since $\det T \neq 0$, the inverse of T, T^{-1}, exists, and we have

$$AT = T\Lambda, \quad T^{-1}AT = \Lambda.$$

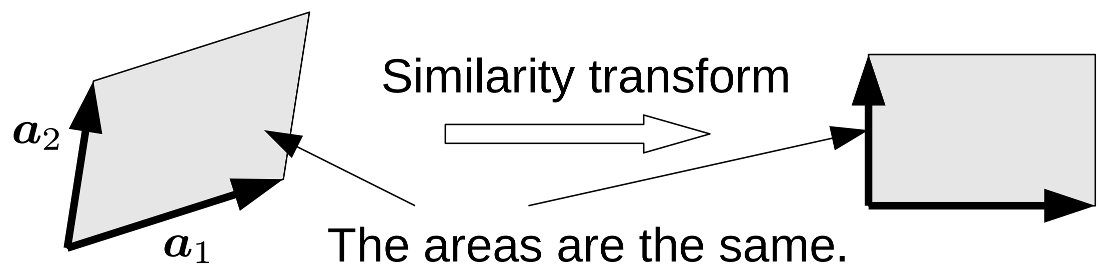
In this way, a simple matrix can be transformed into a diagonal matrix, in other words, it can be diagonalized.

11. Matrix diagonalization

- Similarity transform（相似変換）

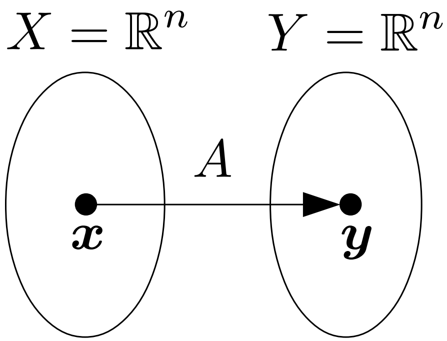
The operation of $T^{-1}AT$ on a matrix A, namely multiplying a non-singular matrix（正則行列）T from the right side and its inverse from the left side, is called a similarity transform, and A is said to be similar to the matrix $T^{-1}AT$.

$$\det(T^{-1}AT) = \det(T^{-1}) \cdot \det A \cdot \det T = \frac{1}{\det T} \cdot \det A \cdot \det T = \det A$$



11. Matrix diagonalization

– A matrix as a mapping and its similarity transform



Let us consider two n-dimensional linear spaces $X = \mathbb{R}^n$ and $Y = \mathbb{R}^n$, and a mapping represented by an $n \times n$ square matrix A, $\boldsymbol{y}$=$A\boldsymbol{x}$, where $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$.

Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are represented using the natural basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ where

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \boldsymbol{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

That is,

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n = \begin{bmatrix} \boldsymbol{e}_1 & \dots & \boldsymbol{e}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = I \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boldsymbol{y} = y_1\boldsymbol{e}_1 + \cdots + y_n\boldsymbol{e}_n = I \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

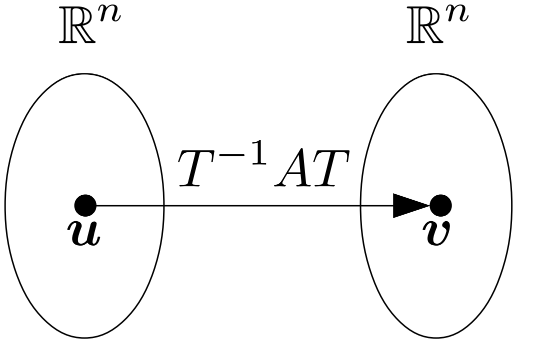
11. Matrix diagonalization

If we define $\boldsymbol{u} = T^{-1}\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{v} = T^{-1}\boldsymbol{y} \in \mathbb{R}^n$ then we have

$\boldsymbol{x} = T\boldsymbol{u}, \boldsymbol{y} = T\boldsymbol{v}$ and therefore,

$$(T\boldsymbol{v}) = A(T\boldsymbol{u})$$

$$\boldsymbol{v} = T^{-1}AT\boldsymbol{u}.$$



The matrix $T^{-1}AT$ represents a mapping from $\boldsymbol{u}$ to $\boldsymbol{v}$: $\boldsymbol{v} = (T^{-1}AT)\boldsymbol{u}$.

11. Matrix diagonalization

Relation between $\boldsymbol{x}$, $\boldsymbol{y}$, A and $\boldsymbol{u}$, $\boldsymbol{v}$, $T^{-1}AT$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{e}_1 & \dots & \boldsymbol{e}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = I \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = T\boldsymbol{u} = \begin{bmatrix} \boldsymbol{t}_1 & \dots & \boldsymbol{t}_n \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

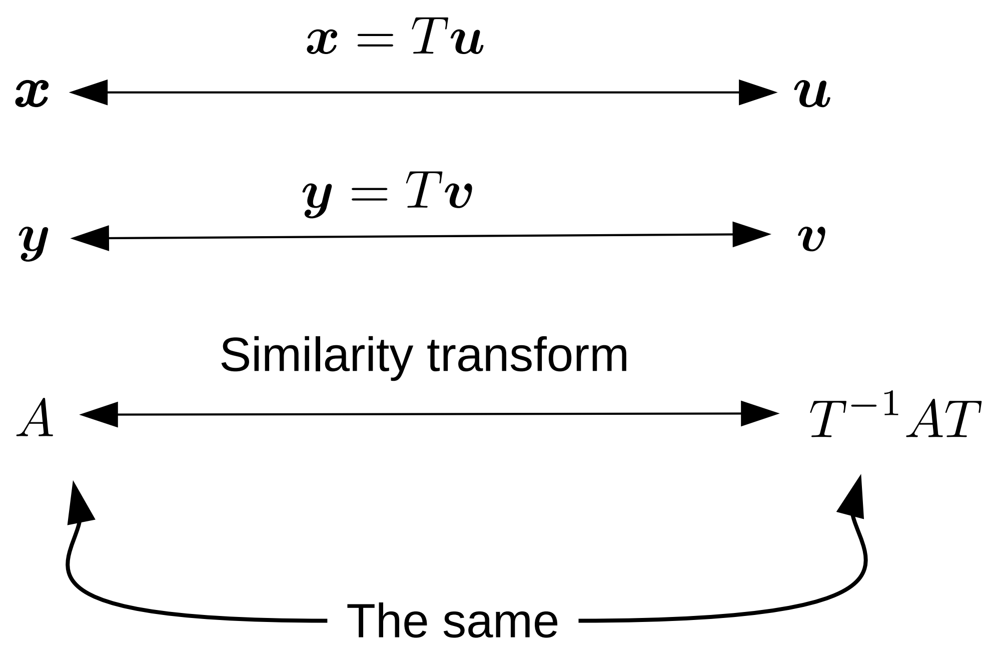
$\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the basis of $\mathbb{R}^n$ and $\boldsymbol{x}$ is a set of coordinates of $\boldsymbol{x}$. $\{\boldsymbol{t}_1, \dots, \boldsymbol{t}_n\}$ is another basis of $\mathbb{R}^n$ and $\boldsymbol{u}$ is another set of coordinates of $\boldsymbol{x}$.

$\boldsymbol{x}$ and $\boldsymbol{u}$ represent the same vector but with different bases.

Similarly, $\boldsymbol{y}$ and $\boldsymbol{v}$ represent the same vector but with different bases.

Also, A and $T^{-1}AT$ represent the same mapping between the same linear spaces but having different bases.

11. Matrix diagonalization



Basis: $\{e_1, \ldots, e_n\}$ Basis: $\{t_1, \ldots, t_n\}$

Diagonalization of a matrix A uses an A-specific basis T and re-represent the same mapping in a simpler form (a diagonal matrix).

12. Jordan form

- When A is not a simple matrix, it cannot be diagonalized.
  - One of the reasons why non-simple matrices cannot be diagonalized is that they do not have the enough number of linearly independent eigen vectors to form the matrix T.
    - We need n linearly independent eigen vectors to form the $n \times n$ matrix T.
    - A non-simple matrix has $\sum_{i=1}^{\sigma} \alpha_i$ linearly independent eigen vectors.

    But, since $\sum_{i=1}^{\sigma} \alpha_i < \sum_{i=1}^{\sigma} m_i = n$, we need $m_i - \alpha_i$ more linearly independent eigen vectors (or substitutes) for each λ_i.

12. Jordan form

- Generalized eigen vectors（一般化固有ベクトル）

For a matrix A and its eigen value λ_i , a vector $\boldsymbol{x} \neq \mathbf{0}$ that satisfies

$$(\lambda_i I - A)^k \boldsymbol{x} = \mathbf{0}, \text{ for some integer } k \geq 2$$

is called a generalized eigen vector.

Note that when $k{=}1$, $\boldsymbol{x}$ is an ordinary eigen vector.

12. Jordan form

- How to find generalized eigen vectors

Let us start with $\boldsymbol{x}_{i1}$ among the eigen vectors $\boldsymbol{x}_{i1}, \cdots, \boldsymbol{x}_{i,\alpha_i}$ associated with the eigen value λ_i.

The first generalized eigen vector $\boldsymbol{x}_{i1}^1$: $\boldsymbol{x}_{i1}^1 = \boldsymbol{x}_{i1}$ (the eigen vector itself)

Then $(\lambda_i I - A)\boldsymbol{x}_{i1}^1 = \boldsymbol{0}$, $\boldsymbol{x}_{i1}^1 \neq \boldsymbol{0}$ holds.

The second generalized eigen vector $\boldsymbol{x}_{i1}^2$: $(\lambda_i I - A)\boldsymbol{x}_{i1}^2 = -\boldsymbol{x}_{i1}^1$, $\boldsymbol{x}_{i1}^2 \neq \boldsymbol{0}$

Then $(\lambda_i I - A)^2\boldsymbol{x}_{i1}^2 = -(\lambda_i I - A)\boldsymbol{x}_{i1}^1 = \boldsymbol{0}$ holds, which implies that $\boldsymbol{x}_{i1}^2$ is a generalized eigen vector.

The third generalized eigen vector $\boldsymbol{x}_{i1}^3$: $(\lambda_i I - A)\boldsymbol{x}_{i1}^3 = -\boldsymbol{x}_{i1}^2$, $\boldsymbol{x}_{i1}^3 \neq \boldsymbol{0}$

Then $(\lambda_i I - A)^3\boldsymbol{x}_{i1}^3 = -(\lambda_i I - A)^2\boldsymbol{x}_{i1}^2 = \boldsymbol{0}$ holds, which implies that $\boldsymbol{x}_{i1}^3$ is a generalized eigen vector.

$\vdots$

12. Jordan form

The n_{i1}-th generalized eigen vector $\boldsymbol{x}_{i1}^{n_{i1}}$: $(\lambda_i I - A)\boldsymbol{x}_{i1}^{n_{i1}} = -\boldsymbol{x}_{i1}^{n_{i1}-1}$, $\boldsymbol{x}_{i1}^{n_{i1}} \neq \mathbf{0}$

Then $(\lambda_i I - A)^{n_{i1}}\boldsymbol{x}_{i1}^{n_{i1}} = \mathbf{0}$ holds, which implies that $\boldsymbol{x}_{i1}^{n_{i1}}$ is a generalized eigen vector.

Here n_{i1} is the minimum k such that $\boldsymbol{x}_{i1}^{k+1}$ defined by $(\lambda_i I - A)\boldsymbol{x}_{i1}^{k+1} = -\boldsymbol{x}_{i1}^{k}$ is not linearly independent of the generalized eigen vectors found so far, $\boldsymbol{x}_{i1}^{1}, \cdots, \boldsymbol{x}_{i1}^{k}$.

As a result, we can find n_{i1} linearly independent generalized eigen vectors starting with the ordinary eigen vector $\boldsymbol{x}_{i1}^{1} = \boldsymbol{x}_{i1}$ associated with the eigen value λ_i .

12. Jordan form

Similarly, we can find generalized eigen vectors starting with each of the other ordinary eigen vectors $\boldsymbol{x}_{i2}, \cdots, \boldsymbol{x}_{i,\alpha_i}$ associated with the eigen value λ_i.

The numbers of generalized eigen vectors are $n_{i2}, \cdots, n_{i,\alpha_i}$,

respectively, and $\displaystyle\sum_{j=1}^{\alpha_i} n_{ij} = m_i$.

Moreover the m_i generalized eigen vectors

$$\{\boldsymbol{x}_{i1}^{1}, \cdots, \boldsymbol{x}_{i1}^{n_{i1}}, . \cdots, \boldsymbol{x}_{i,\alpha_i}^{1}, \cdots, \boldsymbol{x}_{i,\alpha_i}^{n_{i,\alpha_i}}\}$$

are linearly independent with each other. Note that the ordinary eigen vectors are included in the above.

12. Jordan form

The above discussion applies to each of the distinct eigen values $\lambda_1, \cdots, \lambda_\sigma$.

So, we obtain the set of n generalized eigen vectors

$$\{\boldsymbol{x}_{11}^1, \cdots, \boldsymbol{x}_{1,\alpha_1}^{n_{1,\alpha_1}}, \cdots, \boldsymbol{x}_{\sigma 1}^1, \cdots, \boldsymbol{x}_{\sigma,\alpha_\sigma}^{n_{\sigma,\alpha_\sigma}}\}$$

and they are linearly independent with each other.

12. Jordan form

- Jordan form

Now we have n linearly independent generalized eigen vectors, and form the matrix T as

$$T = [\boldsymbol{x}_{11}^1 \cdots \boldsymbol{x}_{1,\alpha_1}^{n_{1,\alpha_1}} \cdots \boldsymbol{x}_{\sigma 1}^1 \cdots \boldsymbol{x}_{\sigma,\alpha_\sigma}^{n_{\sigma,\alpha_\sigma}}].$$

Since we have used the relationship

$$(\lambda_i I - A)\boldsymbol{x}_{ij}^k = -\boldsymbol{x}_{ij}^{k-1},\ k = 2, 3, \cdots$$

to find the generalized eigen vectors, the relation

$$A\boldsymbol{x}_{ij}^k = \lambda_i \boldsymbol{x}_{ij}^k + \boldsymbol{x}_{ij}^{k-1}$$

holds.

Therefore we have

$$\begin{aligned} AT &= [A\boldsymbol{x}_{11}^1 A\boldsymbol{x}_{11}^2 \cdots A\boldsymbol{x}_{11}^{n_{11}} \cdots A\boldsymbol{x}_{\sigma,\alpha_\sigma}^1 A\boldsymbol{x}_{\sigma,\alpha_\sigma}^2 \cdots A\boldsymbol{x}_{\sigma,\alpha_\sigma}^{n_{\sigma,\alpha_\sigma}}] \\ &= [\lambda_1 \boldsymbol{x}_{11}^1\ \lambda_1 \boldsymbol{x}_{11}^2 + \boldsymbol{x}_{11}^1\ \cdots\ \lambda_1 \boldsymbol{x}_{11}^{n_{11}} + \boldsymbol{x}_{11}^{n_{11}-1}\ \cdots \\ &\qquad \lambda_\sigma \boldsymbol{x}_{\sigma,\alpha_\sigma}^1\ \lambda_\sigma \boldsymbol{x}_{\sigma,\alpha_\sigma}^2 + \boldsymbol{x}_{\sigma,\alpha_\sigma}^1 \cdots\ \lambda_\sigma \boldsymbol{x}_{\sigma,\alpha_\sigma}^{n_{\sigma,\alpha_\sigma}} + \boldsymbol{x}_{\sigma,\alpha_\sigma}^{n_{\sigma,\alpha_\sigma}-1}]. \end{aligned}$$

12. Jordan form

$$\therefore\ AT = [\boldsymbol{x}_{11}^{1}\ \boldsymbol{x}_{11}^{2}\ \cdots\ \boldsymbol{x}_{11}^{n_{11}}\ \cdots\ \boldsymbol{x}_{\sigma,\alpha_\sigma}^{1}\ \boldsymbol{x}_{\sigma,\alpha_\sigma}^{2}\ \cdots\ \boldsymbol{x}_{\sigma,\alpha_\sigma}^{n_{\sigma,\alpha_\sigma}}]$$

$$\times \begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{matrix}}\ n_{11} \times n_{11} & & & & \mathbf{0} \\ & \ddots & & & \\ & & \boxed{\begin{matrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{matrix}}\ n_{1,\alpha_1} \times n_{1,\alpha_1} & & \\ & & & \ddots & \\ \mathbf{0} & & & & n_{\sigma,\alpha_\sigma} \times n_{\sigma,\alpha_\sigma}\ \boxed{\begin{matrix} \lambda_\sigma & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_\sigma \end{matrix}} \end{bmatrix}$$

$$= TJ$$

$\therefore T^{-1}AT = J$, J is called the Jordan form of matrix A.

12. Jordan form

When A is a simple matrix,

A is similar to the diagonal matrix Λ , i.e.,

$$A = T\Lambda T^{-1},\ T^{-1}AT = \Lambda.$$

The properties of A can be found by looking into Λ.

When A is NOT a simple matrix,

A is similar to the Jordan form J, i.e.,

$$A = TJT^{-1},\ T^{-1}AT = J.$$

The properties of A can be found by looking into J.

A Jordan form has entries whose values are 1.
This is because we obtained the generalized eigen vectors based on the relation

$$(\lambda_i I - A)\boldsymbol{x}_{ij}^{k} = -\boldsymbol{x}_{ij}^{k-1} = (-1)\cdot \boldsymbol{x}_{ij}^{k-1}.$$

This "-1" can be any constant, and therefore the values "1" that appear in the Jordan form can also be any constant.