Applied Linear Algebra
線形代数応用特論

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村田純一

Lecture Style

- Lecture in Japanese using slides in English
  - The slides in a pdf format are available at Moodle
    - Visit moodle.s.kyushu-u.ac.jp
    - Change the language to English, if necessary, by choosing 'en' at the top bar
    - Click on 'Log in' at the right top of the page
    - Enter your SSO-KID and password to log in
    - Use 'search course' to find '2019年度前期･木3･線形代数応用特論（村田　純一）' and click on the displayed course name
    - Click on 'Enroll me'（私を受講登録する） and enroll yourself in the course
    - The course contents will be displayed, in which, under the day April 11, you will find the files

Lecture Style

- Questions
  - Questions are welcome anytime during or after the class
  - You can visit my office, Room 553 on the fifth floor, or email me at murata@ees.kyushu-u.ac.jp

Evaluation

- Attendance – 20%
  - If you are going to absent from the class to attend a conference and give a presentation, let me know it **in advance.** That will be excluded when calculating the attendance rate
    学会で発表を行うために講義に出席できない場合は**事前に**連絡のこと. 出席率計算の対象から除外する
- Exam – 80%
  - The exam will be in the last week of the semester (scheduled for 18 July 2019)
  - Bound hard copies of the slides (English or Japanese or both) will be allowed in the exam
  - The questions in the exam will be printed in both Japanese and English

Evaluation

- Attendance
  - Roll calls are taken using Moodle
    - Access to the course 「2019年度前期･木3･線形代数応用特論（村田　純一）」 **through 'edunet'**, the university's LAN for education
      九大の教育用ネットワーク**edunetを使って**Moodleのコース「2019年度前期･木3･線形代数応用特論（村田　純一）」にアクセスする
    - Click “Attendance Link” on the course page
      コース内の画面の「出欠リンク」をクリック
    - **Enter the keyword of the day** and click the “Submit” button
      その日の**出欠用キーワードを入力**してから「送信」ボタンをクリック
    - Do the above in the class room **within 15 minutes into the class**
      以上を講義室内から**講義開始後15分以内**に行う

Evaluation

- Exam
  - For fair grading, it is important that all the students take the same examination on the same day
    公平な成績評価のために，全受講生が同じ日に同じ試験を受けることを重視する
    - Keep clear your calendar for the scheduled exam day (18 July 2019)
      試験予定日（2019年7月18日）に他の予定を入れない
    - If you cannot sit the exam on that day for one of the following reasons, another exam will be arranged on an appropriate day: infectious disease designated by law; bereavement of your close kin; serving as a lay judge; natural disaster or disruption of public transport
      以下のいずれかの理由で当日試験が受けられない場合は適切な日に追試験を行う：法定伝染病，近親者の死亡，裁判員としての職務，天災・交通機関停止
    - If you do not take the exam for any other reason (including attending a conference), you will have to sit another exam scheduled for January next year
      他の理由（学会参加も含む）で試験が受けられない場合は，来年1月に実施する再試験を受ける必要がある

Planned Contents

- Linear algebra revisited
  - Matrices
  - Linearity and mapping
  - Amount of information a matrix can transfer
  - Eigen values and eigen vectors
  - Jordan forms
- Some applications of linear algebra
  - Linear system control
  - Optimization
  - Classification

1. Matrices Revisited
1.1 Matrices

- A matrix consists of values arranged in rows（行）and columns（列）forming a rectangular shape:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1.1}$$

  - Each value a_{ij} is called an element, an entry or a component
  - In this lecture the elements are assumed to be real(実数):

    $$a_{ij} \in \mathbb{R},\ i = 1, \ldots, m,\ j = 1, \ldots, n,$$

    where $\mathbb{R}$ is the set of real numbers
  - The above matrix A has m rows and n columns.
    A matrix of this size is sometimes called an $m \times n$ matrix,
    and we sometimes write as

    $$A \in \mathbb{R}^{m \times n}$$

1.2 Arithmetic Operation of Matrices

- Consider the following two matrices A and B:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n},\ B = \begin{bmatrix} b_{11} & \dots & a_{1\ell} \\ \vdots & & \vdots \\ b_{k1} & \dots & b_{k\ell} \end{bmatrix} \in \mathbb{R}^{k\times \ell}$$

- Addition

  When $m = k$ and $n = \ell$,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix} \tag{1.2}$$

- Multiplication

  When $n = k$,

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & \dots & \sum_{j=1}^{n} a_{1j}b_{j\ell} \\ \vdots & & \vdots \\ \sum_{j=1}^{n} a_{mj}b_{j1} & \dots & \sum_{j=1}^{n} a_{mj}b_{j\ell} \end{bmatrix} \tag{1.3}$$

1.3 Matrices vs Scalars

- What is a matrix? For what purpose can we use a matrix?
- In order to get answers to these questions, we will first compare matrices with scalars（スカラー）

1.3 Matrices vs Scalars

A scalar

$$a \in \mathbb{R} = \mathbb{R}^{1\times 1}$$

A 1×1 'matrix'

Expanded in size

A matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

An $m \times n$ matrix

A matrix can be regarded as an extension (or expansion in size) of a scalar

What is similar and what is different between matrices and scalars?

1.3 Matrices vs Scalars

- Addition and multiplication of two scalars a and b,

  and two matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Operation	Scalars	Matrices
Addition	$a + b$	$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$
	Natural extension	
Multiplication	ab	$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$
	Non-natural extension	

$$AB \neq \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}$$

Why?

1.4 Elementary Math Problems

- To get an intuitive understanding about how matrices and their multiplication can be used, let us consider a few elementary math problems

- An elementary math problem
  - Q1: You buy five pencils. Each pencil costs 50 yen. Then how much will you pay in total?
  - A1: $50 \text{ (yen)} \times 5 \text{ (pencils)} = 250 \text{ (yen)}$

$\Downarrow$ Generalize just a little bit

$y = ax$ where a: the price of a pencil
x: the number of pencils you buy
y: the total amount you will pay

A **mapping**（写像）from a scalar x to a scalar y

(In mathematics, mapping is usually called map, while in applications the term mapping is commonly used.)

1.4 Elementary Math Problems

- A slightly complicated elementary math problem (two people buy two kinds of goods at different shops)
  - Q2: Each of you and your sister buys five pencils and two notebooks.
    You buy them at Shop A where each pencil costs 50 yen and each notebook 200 yen.
    Your sister buys at Shop B where they sell a pencil at 60 yen and a notebook at 180 yen.
    Then how much will each of you pay and how much in total will the two of you pay?

1.4 Elementary Math Problems

– A2:
You: $50 \times 5 + 200 \times 2 = 650$
Sister: $60 \times 5 + 180 \times 2 = 660$
Total: $650 + 660 = 1310$

Generalize just a little bit

$$y = y_1 + y_2$$
$$y_1 = a_{11}x_1 + a_{12}x_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2$$

where a_{11} : the price of a pencil at Shop A
a_{12} : the price of a notebook at Shop A
a_{21} : the price of a pencil at Shop B
a_{22} : the price of a notebook at Shop B
x_1 : the number of pencils each of you buys
x_2 : the number of notebooks each of you buys
y_1 : the amount you will pay
y_2 : the amount your sister will pay
y : the total payment

1.4 Elementary Math Problems

- Can we represent the relationships

$$y_1 = a_{11}x_1 + a_{12}x_2,$$
$$y_2 = a_{21}x_1 + a_{22}x_2$$

in a form similar to

$$y = ax$$

used in Q1?

Yes, we can:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\boldsymbol{y} = A\boldsymbol{x}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

A mapping from a vector $\boldsymbol{x}$ to a vector $\boldsymbol{y}$

Multiplication of a matrix and a vector can represent a mapping from a vector to a vector in a simple and beautiful form!

1.4 Elementary Math Problems

- When you and your sister buy more pencils and notebooks...

  The two of you bought $\boldsymbol{x}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, and your payment was

$$\boldsymbol{y}_1 = A\boldsymbol{x}_1 = \begin{bmatrix} 50 & 200 \\ 60 & 180 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 650 \\ 660 \end{bmatrix}$$

  - When the two of you buy, in addition to the above, $\boldsymbol{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then your payment will be

$$\boldsymbol{y}_2 = A\boldsymbol{x}_2 = \begin{bmatrix} 50 & 200 \\ 60 & 180 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 300 \\ 300 \end{bmatrix}$$

  - The total payment will be

$$\begin{aligned} \boldsymbol{y}_1 + \boldsymbol{y}_2 &= A\boldsymbol{x}_1 + A\boldsymbol{x}_2 = \begin{bmatrix} 50 & 200 \\ 60 & 180 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 50 & 200 \\ 60 & 180 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 & 200 \\ 60 & 180 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} \\ &= A(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \begin{bmatrix} 950 \\ 960 \end{bmatrix} \end{aligned}$$

Linearity（線形性）(principle of superposition（重ねの理）)!

1.4 Elementary Math Problems

- Definition of multiplication of a single matrix and a vector is somehow strange
  - But it represents a mapping simply and beautifully
- How does the multiplication of matrices enter to our stage?

1.4 Elementary Math Problems

- Another elementary math problem (discount)
  - Q1'': You buy five pencils. Each pencil usually costs 50 yen. But the shop offers a 20% discount today. Then how much in total will you pay?
  - A1'': $5 \ (\text{pencils}) \times 50 \ (\text{yen}) = 250 \ (\text{yen})$

    $250 \ (\text{yen}) \times 0.8 = 200 \ (\text{yen})$

    or equivalently

    $5 \ (\text{pencils}) \times 50 \ (\text{yen}) \times 0.8 = 200 \ (\text{yen})$

Generalize just a little bit

$$y = \boxed{ba}x \tag{1.1}$$

Multiplication!

where a: the price of a pencil
b: discount factor = 1 – discount rate
x: the number of pencils you buy
y: the total amount you will pay

1.4 Elementary Math Problems

- Different discounts at Shop A and Shop B
  - Q2'': The situation is similar to Q2 where each of you and your sister buys five pencils and two notebooks.
    However, Shop A where you buy the goods offers a 10 % discount, and Shop B where your sister buys the goods offers a 20 % discount.
    Then how much will each of you pay and how much in total will the two of you pay?

1.4 Elementary Math Problems

– A2'':

You: $50 \times 0.9 \times 5 + 200 \times 0.9 \times 2 = 585$

Sister: $60 \times 0.8 \times 5 + 180 \times 0.8 \times 2 = 528$

Total: $585 + 528 = 1113$

Generalize just a little bit

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_1 a_{11} & b_1 a_{12} \\ b_2 a_{21} & b_2 a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = BA\boldsymbol{x}$$

Multiplication!

where a_{11} : the price of a pencil at Shop A
a_{12} : the price of a notebook at Shop A
a_{21} : the price of a pencil at Shop B
a_{22} : the price of a notebook at Shop B
b_1 : the discount factor at Shop A
b_2 : the discount factor at Shop B
x_1 : the number of pencils each of you buys
x_2 : the number of notebooks each of you buys,
y_1 : the amount you will pay
y_2 : the amount your sister will pay
y : the total payment

1.4 Elementary Math Problems

- The role of multiplication in A1'' and A2''

A1''

$y = bax$
Multiplication

$\Updownarrow$ Equivalent

η : payment before discount

$\eta = ax$: mapping from x to η

and

$y = b\eta$: mapping from η to y

Composite mapping（合成写像）
from a scalar x to a scalar y

A2''

$\boldsymbol{y} = BA\boldsymbol{x}$
Multiplication

$\Updownarrow$ Equivalent

$\boldsymbol{\eta}$: payment before discount

$\boldsymbol{\eta} = A\boldsymbol{x}$: mapping from $\boldsymbol{x}$ to $\boldsymbol{\eta}$

and

$\boldsymbol{y} = B\boldsymbol{\eta}$: mapping from $\boldsymbol{\eta}$ to $\boldsymbol{y}$

Composite mapping
from a vector $\boldsymbol{x}$ to a vector $\boldsymbol{y}$

1.4 Elementary Math Problems

- Multiplication of a matrix and a vector represents a mapping. The mapping has linearity
- Multiplication of two matrices represents a composite mapping

Matrices are closely related to mappings and linearity.

1.4 Elementary Math Problems

- Another elementary math problem (an inverse problem)
  - Q1': A stationary shop sells a pencil at 50 yen. You bought several pencils and payed 250 yen. Then how many pencils did you buy?
  - A1': $250 \text{ (yen)}/50 \text{ (yes)} = 5 \text{ (pencils)}$

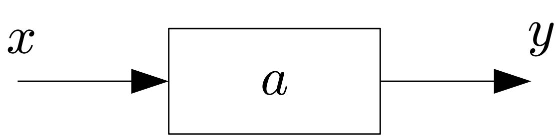
Generalize just a little bit

$x = y/a$ where a: the price of a pencil
x: the number of pencils you bought
y: the total amount you payed

1.4 Elementary Math Problems

- Problem Q1
  - Knowing the values of a and x, find the value y such that $y = ax$
  - Solution: $y = ax$
- Problem Q1'
  - Knowing the values of a and y, find the value x such that $y = ax$
  - Solution: $x = y/a$

Division



Q1

(Forward problem(順問題))

Q1'

(Inverse problem(逆問題))

The relationship among a, x and y is represented by a multiplication $y = ax$.

A constant/coefficient
(pencil price)

A variable
(the number of pencils)

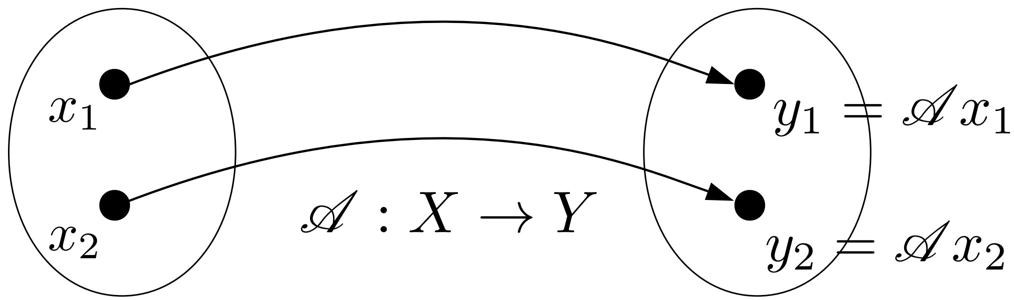
How and when can we define a division by a matrix?

1.4 Elementary Math Problems

- We have found matrices are closely related to mappings and linearity
- Now we look into linearity and mapping
- Then we will come back to multiplication of matrices and division by a matrix

2. Linearity
2.1 Linearity

- **Principle of superposition** holds
  - Linear mapping, transform（変換）, function, operator（作用素）$\mathscr{A}$



Set（集合）X　　　　Set Y

$\mathscr{A}$ is linear $\Leftrightarrow \begin{cases} \mathscr{A}(x_1 + x_2) = \mathscr{A} x_1 + \mathscr{A} x_2 \\ \mathscr{A}(c_1 x_1) = c_1(\mathscr{A} x_1) \end{cases}$

$$\Leftrightarrow \mathscr{A}(c_1 x_1 + c_2 x_2) = c_1(\mathscr{A} x_1) + c_2(\mathscr{A} x_2) \qquad (2.1)$$

c_1, c_2 : constants

(Here, we assume that $c_1 x_1 + c_2 x_2 \in X$ and $c_1(\mathscr{A} x_1) + c_2(\mathscr{A} x_2) \in Y$.
We will discuss this later.)

Example: $\mathscr{A} = 2\times,\ \text{i.e.},\ y = 2 \times x = 2x$

2.1 Linearity

– Linearity in equations（方程式）

Linear equations（線形方程式）: unknown variables appear only as first order terns（未知変数が1次の項としてのみ現れる）.

$$ax + by = r \quad \text{or} \quad \begin{bmatrix} a & b \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Unknown variables}} = r$$

$$a(t)\frac{d}{dt}x(t) + b(t)x(t) = f(t) \quad \text{or} \quad \begin{bmatrix} a(t) & b(t) \end{bmatrix} \underbrace{\begin{bmatrix} \frac{d}{dt}x(t) \\ x(t) \end{bmatrix}}_{\text{Unknown variables}} = f(t)$$

2.1 Linearity

– Principle of superposition in linear equations

Let $\mathscr{A}$ be a mapping from the known term to the solution.

$$\mathscr{A} : r \mapsto \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{for} \quad [a\ b]\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{r}_{\text{Known term}}$$

Suppose $\mathscr{A} r_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, i.e. $[a\ b]\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = r_1$, and

$\mathscr{A} r_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, i.e. $[a\ b]\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = r_2$, then

$$[a\ b]\left(c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = c_1[a\ b]\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2[a\ b]\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = c_1 r_1 + c_2 r_2$$

and therefore

$$\mathscr{A}(c_1 r_1 + c_2 r_2) = c_1(\mathscr{A} r_1) + c_2(\mathscr{A} r_2).$$

2.1 Linearity

$\mathscr{A} : f \mapsto x$ for $\begin{bmatrix} a(t) & b(t) \end{bmatrix} \begin{bmatrix} \frac{d}{dt} x(t) \\ x(t) \end{bmatrix} = \underline{f(t)}$

Known term

Example

How do you solve $x(t) + \dfrac{d}{dt} x(t) = \sin t$?

2.1 Linearity

- Examples of linear and non-linear mappings/operators

$$\mathscr{A}x = \frac{d}{dt}x$$

$$\mathscr{A}(c_1x_1 + c_2x_2) = \frac{d}{dt}(c_1x_1 + c_2x_2) = c_1\frac{dx_1}{dt} + c_2\frac{dx_2}{dt}$$

Linear

$$\mathscr{A}x = x^2$$

$$\begin{aligned}\mathscr{A}(c_1x_1 + c_2x_2) &= (c_1x_1 + c_2x_2)^2 \\ &= {c_1}^2{x_1}^2 + 2c_1c_2x_1x_2 + {c_2}^2{x_2}^2 \\ &\neq c_1{x_1}^2 + c_2{x_2}^2\end{aligned}$$

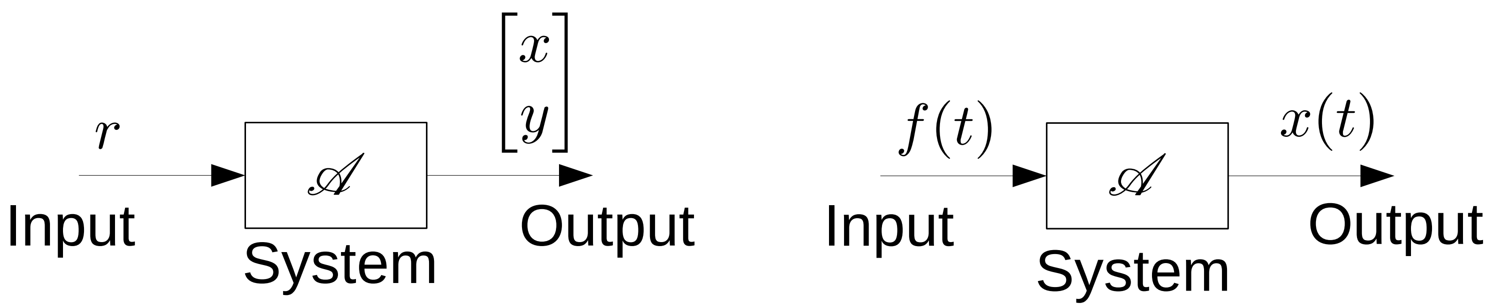
Non-linear

2.2 Linear systems

In linear equations,

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = r \quad \text{and} \quad \begin{bmatrix} a(t) & b(t) \end{bmatrix} \begin{bmatrix} \frac{d}{dt}x(t) \\ x(t) \end{bmatrix} = f(t)$$

the known terms (r and $f(t)$) can be regarded as **inputs**, and the solutions (x, y and $x(t)$) can be regarded as corresponding **outputs**.



- Linear systems
  - Principle of superposition holds between the input and the output

2.3 Linear space

- Linear spaces
  - Let F be a set of constants that is
    closed under addition, subtraction, multiplication and division（加減乗除について閉じている）, and
    has the zero element（零元）, 0, $\left(c+0=0+c=c \text{ for } c \in F \right)$
    and the multiplicative identity（乗法単位元）, 1
    Then F is called a field (体) $\left(c \cdot 1 = 1 \cdot c = c \text{ for } c \in F \right)$

  - Let X be a set where
    addition is defined between any two elements in X, and
    multiplication by any element in F is defined for any element in X

  - If, $c_1 x_1 + c_2 x_2 \in X$ holds for $\forall x_1 \in X, \forall x_2 \in X$ and $\forall c_1 \in F, \forall c_2 \in F$,
    then (F,X) is called a **linear space**（線形空間） or X is said to be a linear space on F

Elements in F: **scalars**;   elements in X: **vectors**

2.3 Linear space

– When we defined linearity on page 25, we assumed that

$$c_1x_1 + c_2x_2$$

exists in X

– In linear spaces $c_1x_1 + c_2x_2$ always exists.

$$\Downarrow$$

A linear space is the basis
to define linear mappings and linear equations.

2.3 Linear space

- Examples of linear spaces
  - $F : \mathbb{R}, \quad X : \mathbb{R}^n$ $\qquad \mathbb{R}$: set of real numbers

    $\boldsymbol{x} \in X$ is a real vector having n components.

    F contains the zero element 0,
    and therefore X contains the zero vector $\mathbf{0}$.

    (For $\boldsymbol{x} \in X$, $0 \cdot \boldsymbol{x} = \mathbf{0} \in X$)

  - $F : \mathbb{R}, \quad X$: set of continuous functions of t

    F contains the zero element 0,
    and therefore X contains the function which always takes the value 0.

Linear spaces always contain their own zero element.

3. Mapping
3.1 Mapping

- Mapping

X, Y: sets

A mapping $\mathscr{A}$ relates an element of Y, $y \in Y$, to any element of X, $x \in X$

$$\mathscr{A} : x \mapsto y \; (y = \mathscr{A} x), \quad \mathscr{A} : X \to Y$$

$\mathscr{A}$: mapping or transform（変換）, especially when $Y = \mathbb{R}$, function（関数）

X: domain（定義域）of $\mathscr{A}$

$\mathscr{A}(X)$ (or $\mathscr{R}(\mathscr{A})$) $= \{y | y = \mathscr{A} x, x \in X\}$: range（値域）of $\mathscr{A}$