15. Differentiation of functions with respect to vectors

- A function of a vector
  - A function $f(x_1, \ldots, x_n)$ of variables $x_1, \ldots, x_n$ can be regarded as a function $f(\boldsymbol{x})$ of a vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.
  - Partial derivative of f with respective to $\boldsymbol{x}$ is defined as

$$\frac{\partial f}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix},$$

which is equal to the gradient of f, $\nabla f(\boldsymbol{x})$.

15. Differentiation of functions with respect to vectors

- A vector-valued function of a vector
  - A vector-valued function of a vector

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{bmatrix}$$

  - Partial derivative of $\boldsymbol{f}$ with respective to $\boldsymbol{x}$ is defined as

$$\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial \boldsymbol{x}} & \dots & \frac{\partial f_m}{\partial \boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

15. Differentiation of functions with respect to vectors

- Second order derivative of a function of a vector

$$\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^T = \frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{bmatrix}$$

This matrix is called Hesse matrix.

If $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is continuous, then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ and the Hesse matrix is a symmetrical matrix.

15. Differentiation of functions with respect to vectors

- Examples
  - $f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{a} = \sum_{i=1}^{n} a_i x_i,\ \boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$$\frac{\partial f}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{a}^T\boldsymbol{x} = \frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x}^T\boldsymbol{a} = \frac{\partial}{\partial \boldsymbol{x}}\sum_{i=1}^{n} a_i x_i = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \boldsymbol{a}$$

  - $\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix},\ A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$

$$\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}(A\boldsymbol{x})^T = \frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j & \dots & \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix} = A^T$$

15. Differentiation of functions with respect to vectors

- Examples
  - $f(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \; A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}$

$$\frac{\partial f}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{x}^T A \boldsymbol{x} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j + \sum_{i=1}^{n} a_{i1} x_i \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j + \sum_{i=1}^{n} a_{in} x_i \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n} a_{i1} x_i \\ \vdots \\ \sum_{i=1}^{n} a_{in} x_i \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\boldsymbol{x} + A^T \boldsymbol{x}$$

$$\left(= \frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{x}^T \underbrace{(A\boldsymbol{x})}_{\text{fixed}} + \frac{\partial}{\partial \boldsymbol{x}} \underbrace{\left(\boldsymbol{x}^T A\right)}_{\text{fixed}} \boldsymbol{x} = (A\boldsymbol{x}) + \left(\boldsymbol{x}^T A\right)^T = A\boldsymbol{x} + A^T \boldsymbol{x} \right)$$

15. Differentiation of functions with respect to vectors

$$\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^T = \frac{\partial}{\partial \boldsymbol{x}}\left(A\boldsymbol{x} + A^T\boldsymbol{x}\right)^T = \frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^T A^T + \boldsymbol{x}^T A\right) = A^T + A$$

When A is symmetrical ($A^T = A$), $\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^T = 2A$.

16. Linear system control

- Let us look at how matrices are used in system control.

16. Linear system control

- System models
  - Motion of an object is described by Newton's equation of motion:

    $$f = m\frac{d^2x}{dt^2},$$

    where x is the position of the object, m is its mass, f is the force acting on the object, and t represents time.

    When we define two variables

    $$x_1 = x,\ \ x_2 = \frac{dx}{dt},$$

    then the above second order differential equation is converted to a set of two first order differential equations:

    $$\frac{dx_1}{dt} = x_2,\ \ \frac{dx_2}{dt} = \frac{f}{m}.$$

16. Linear system control

The set of equations

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \frac{f}{m}$$

can be represented as

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f.$$

In general a linear dynamical system can be represented as

$$\frac{d}{dt}\boldsymbol{x} = A\boldsymbol{x} + B\boldsymbol{u}.$$

$\boldsymbol{u}$ is an input vector which, in the above example, is f,
and $\boldsymbol{x}$ is called a state vector.

16. Linear system control

– Discrete time model

- Let us consider discrete time:
  $k = 0,\ 1,\ 2,\ ...$

  The actual interval T between two time instances k and $k+1$ depends on the system.

  Let the value of state $\boldsymbol{x}$ at discrete time k be denoted as $\boldsymbol{x}_k$. Then we have $\boldsymbol{x}_k = \boldsymbol{x}(kT)$.

  The derivative of $\boldsymbol{x}(t)$ with respective t is approximated as

$$\frac{d}{dt}\boldsymbol{x}(t) \simeq \frac{\boldsymbol{x}((k+1)T) - \boldsymbol{x}(kT)}{T} = \frac{\boldsymbol{x}_{k+1} - \boldsymbol{x}_k}{T}.$$

  This is called forward Euler approximation（前進オイラー近似）.

16. Linear system control

- Equation representing a discrete time system

  Using the forward Euler approximation, we can approximate the equation

  $$\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t).$$

  as

  $$\frac{\boldsymbol{x}_{k+1} - \boldsymbol{x}_k}{T} \simeq A\boldsymbol{x}_k + B\boldsymbol{u}_k,$$

  or

  $$\boldsymbol{x}_{k+1} \simeq (AT + I)\boldsymbol{x}_k + BT\boldsymbol{u}_k.$$

  Therefore in discrete time, a linear dynamical system can be represented in the form of

  $$\boldsymbol{x}_{k+1} = F\boldsymbol{x}_k + G\boldsymbol{u}_k,$$

  which is a difference equation（差分方程式）.

16. Linear system control

– The above discussion is based on Euler approximation and therefore is not exact.
If we want to derive an exact discrete-time representation of a linear dynamical system from its continuous-time representation, then we have to start with the solution of the continuous-time differential equation $\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$.

The solution is

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau,$$

where

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots .$$

16. Linear system control

Then we have

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= \boldsymbol{x}((k+1)T) \\
&= e^{A(k+1)T}\boldsymbol{x}(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}B\boldsymbol{u}(\tau)d\tau \\
&= e^{AT}e^{AkT}\boldsymbol{x}(0) + \int_0^{kT} e^{AT}e^{A(kT-\tau)}B\boldsymbol{u}(\tau)d\tau \\
&\qquad\qquad + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}B\boldsymbol{u}(\tau)d\tau \\
&= e^{AT}\left(e^{AkT}\boldsymbol{x}(0) + \int_0^{kT} e^{A(kT-\tau)}B\boldsymbol{u}(\tau)d\tau\right) \\
&\qquad\qquad + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}B\boldsymbol{u}(\tau)d\tau \\
&= e^{AT}\boldsymbol{x}_k + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}B\boldsymbol{u}(\tau)d\tau
\end{aligned}
$$

16. Linear system control

When $\boldsymbol{u}(t)$ is constant for $kT \leq t \leq (k+1)T$, then we denote its value by $\boldsymbol{u}_k$ and obtain

$$\begin{aligned}\boldsymbol{x}_{k+1} &= e^{AT}\boldsymbol{x}_k + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B d\tau \boldsymbol{u}_k \\ &= e^{AT}\boldsymbol{x}_k + \left(\int_0^T e^{As} B ds \right) \boldsymbol{u}_k\end{aligned}$$

and derive $F = e^{AT}$ and $G = \displaystyle\int_0^T e^{As} B ds$.

16. Linear system control

Since

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots.$$

we have

$$F = e^{AT} = I + AT + \frac{1}{2!}A^2T^2 + \frac{1}{3!}A^3T^3 + \ldots,$$

$$G = \int_0^T e^{As}Bds = \int_0^T \left(I + As + \frac{1}{2!}A^2s^2 + \frac{1}{3!}A^3s^3 + \ldots\right)Bds$$
$$= \left(IT + \frac{1}{2}AT^2 + \frac{1}{3!}A^2T^3 + \frac{1}{4!}A^3T^4 + \ldots\right)B.$$

If we truncate the above series at the first order term of T, then

$$F \simeq I + AT,\ G \simeq BT,$$

which are equal to those derived on page 159.

16. Linear system control

- Reachability
  - Let us consider a discrete-time linear dynamical system. If, for any destination state $\boldsymbol{x}_D$, there is a sequence of inputs $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ such that the state transitions from $\mathbf{0}$ to $\boldsymbol{x}_D$ in a finite time N, the system is said to be reachable（可到達）.

$$
\begin{aligned}
\boldsymbol{x}_0 &= \mathbf{0}, \\
\boldsymbol{x}_1 &= F\boldsymbol{x}_0 + G\boldsymbol{u}_0 = G\boldsymbol{u}_0, \\
\boldsymbol{x}_2 &= F\boldsymbol{x}_1 + G\boldsymbol{u}_1 = FG\boldsymbol{u}_0 + G\boldsymbol{u}_1, \\
\boldsymbol{x}_3 &= F\boldsymbol{x}_2 + G\boldsymbol{u}_2 = F(FG\boldsymbol{u}_0 + G\boldsymbol{u}_1) + G\boldsymbol{u}_2 \\
&\qquad\qquad\qquad\;\; = F^2G\boldsymbol{u}_0 + FG\boldsymbol{u}_1 + G\boldsymbol{u}_2, \\
&\qquad\qquad\qquad\;\; \vdots \\
\boldsymbol{x}_{N+1} &= F^N G\boldsymbol{u}_0 + F^{N-1}G\boldsymbol{u}_1 + \cdots + G\boldsymbol{u}_N.
\end{aligned}
$$

16. Linear system control

So the question is if there are a finite N and a vector

$$\boldsymbol{U}_N = \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_N \end{bmatrix}$$

such that

$$\boldsymbol{x}_D = \boldsymbol{x}_{N+1} = \begin{bmatrix} F^N G & F^{N-1} G & \cdots & G \end{bmatrix} \boldsymbol{U}_N,$$

for any $\boldsymbol{x}_D$.

Such $\boldsymbol{U}_N$ exists when the mapping represented by $\begin{bmatrix} F^N G & F^{N-1} G & \cdots & G \end{bmatrix}$ is a surjection, i.e., the matrix has a row full rank.

16. Linear system control

Suppose that the dimension of state vector $\boldsymbol{x}$ is n and that the dimension of the input vector $\boldsymbol{u}$ is p.
Then $F \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times p}$, and therefore

$$\begin{bmatrix} F^N G & F^{N-1}G & \cdots & G \end{bmatrix} \in \mathbb{R}^{n\times(N+1)p}.$$

So the system is reachable when all the n row vectors of $\begin{bmatrix} F^N G & F^{N-1}G & \cdots & G \end{bmatrix}$ are linearly independent with each other.

The matrix $\begin{bmatrix} F^N G & F^{N-1}G & \cdots & G \end{bmatrix}$ is called a reachability matrix.

We just need consider N such that $(N+1)p \geq n$.

16. Linear system control

- Examples
  - $$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

    $$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ n = 2,\ p = 1.$$

    The minimum N that satisfies $(N+1)p \geq n$ is 1.

    The reachability matrix is $\begin{bmatrix} FG & G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, which is of row full rank, and therefore the system is reachable.

    For $\boldsymbol{x}_D = \begin{bmatrix} x_{D,1} \\ x_{D,2} \end{bmatrix}$, from the relation $\boldsymbol{x}_D = \begin{bmatrix} FG & G \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$, we can derive the appropriate input sequence as

    $$u_0 = x_{D,1},\ u_1 = x_{D,2} - x_{D,1}.$$

16. Linear system control

- $\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k$

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ G = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ n = 2,\ p = 1.$$

The minimum N that satisfies $(N+1)p \geq n$ is 1.

The reachability matrix is $\begin{bmatrix} FG & G \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, which is NOT of row full rank, and therefore the system is NOT reachable.

16. Linear system control

- Output equation
  - Let $\boldsymbol{y}$ be the output vector from the system. The output is measurement of the states (mixtures of the states) and is represented by

$$\boldsymbol{y}_k = H\boldsymbol{x}_k.$$

16. Linear system control

- Observability
  - Suppose we have a sequence of inputs $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1}$ and the derived sequence of outputs $\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_N$ from the system in response to the inputs. Can we know the initial state $\boldsymbol{x}_0$ from the input and output sequences? If we can, we say the system is observable（可観測）.

$$\boldsymbol{x}_0 = \boldsymbol{x}_0,\ \boldsymbol{y}_0 = H\boldsymbol{x}_0,$$

$$\boldsymbol{x}_1 = F\boldsymbol{x}_0 + G\boldsymbol{u}_0,\ \boldsymbol{y}_1 = HF\boldsymbol{x}_0 + HG\boldsymbol{u}_0,$$

$$\boldsymbol{x}_2 = F\boldsymbol{x}_1 + G\boldsymbol{u}_1 = F^2\boldsymbol{x}_0 + FG\boldsymbol{u}_0 + G\boldsymbol{u}_1,$$

$$\boldsymbol{y}_2 = HF^2\boldsymbol{x}_0 + HFG\boldsymbol{u}_0 + HG\boldsymbol{u}_1,$$

$$\boldsymbol{x}_3 = F\boldsymbol{x}_2 + G\boldsymbol{u}_2 = F^3\boldsymbol{x}_0 + F^2G\boldsymbol{u}_0 + FG\boldsymbol{u}_1 + G\boldsymbol{u}_2,$$

$$\boldsymbol{y}_3 = HF^3\boldsymbol{x}_0 + HF^2G\boldsymbol{u}_0 + HFG\boldsymbol{u}_1 + HG\boldsymbol{u}_2,$$

$$\vdots$$

$$\boldsymbol{x}_N = F^N\boldsymbol{x}_0 + F^{N-1}G\boldsymbol{u}_0 + \cdots + G\boldsymbol{u}_{N-1}.$$

$$\boldsymbol{y}_N = HF^N\boldsymbol{x}_0 + HF^{N-1}G\boldsymbol{u}_0 + \cdots + HG\boldsymbol{u}_{N-1}.$$

16. Linear system control

- The relationships between the initial states and the input and output sequences are

$$\begin{aligned} \boldsymbol{y}_0 &= H\boldsymbol{x}_0, \\ \boldsymbol{y}_1 - HG\boldsymbol{u}_0 &= HF\boldsymbol{x}_0, \\ \boldsymbol{y}_2 - HFG\boldsymbol{u}_0 - HG\boldsymbol{u}_1 &= HF^2\boldsymbol{x}_0, \\ &\vdots \\ \boldsymbol{y}_N - HF^{N-1}G\boldsymbol{u}_0 - \cdots - HG\boldsymbol{u}_{N-1} &= HF^N\boldsymbol{x}_0. \end{aligned}$$

These are expressed as

$$\begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{y}_1 - HG\boldsymbol{u}_0 \\ \vdots \\ \boldsymbol{y}_N - HF^{N-1}G\boldsymbol{u}_0 - \cdots - HG\boldsymbol{u}_{N-1} \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^N \end{bmatrix} \boldsymbol{x}_0.$$

16. Linear system control

- From the equation

$$\begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{y}_1 - HG\boldsymbol{u}_0 \\ \vdots \\ \boldsymbol{y}_N - HF^{N-1}G\boldsymbol{u}_0 - \cdots - HG\boldsymbol{u}_{N-1} \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^N \end{bmatrix} \boldsymbol{x}_0,$$

we know we can uniquely determine $\boldsymbol{x}_0$ when the matrix

$$\begin{bmatrix} H \\ HF \\ \vdots \\ HF^N \end{bmatrix}$$

represents an injection, i.e., the matrix has a column full rank.

16. Linear system control

Suppose the dimension of output vector $\boldsymbol{y}$ is q.
Then $H \in \mathbb{R}^{q \times n}$, and therefore

$$\begin{bmatrix} H \\ HF \\ \vdots \\ HF^N \end{bmatrix} \in \mathbb{R}^{(N+1)q \times n}.$$

So the system is observable when all the n column vectors of $\begin{bmatrix} H \\ HF \\ \vdots \\ HF^N \end{bmatrix}$ are linearly independent with each other.

The matrix $\begin{bmatrix} H \\ HF \\ \vdots \\ HF^N \end{bmatrix}$ is called an observability matrix.

We just need consider N such that $(N+1)q \geq n$.

16. Linear system control

– Examples

- $\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k,\ y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}$

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 \end{bmatrix},\ n = 2,\ q = 1.$$

The minimum N that satisfies $(N+1)q \geq n$ is 1.

The observability matrix is $\begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, which is

of full column rank, and therefore the system is observable.

16. Linear system control

- $\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k, \ y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}$

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ H = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The observability matrix is $\begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, which is NOT of full column rank, and therefore the system is NOT observable.