7. Determinant

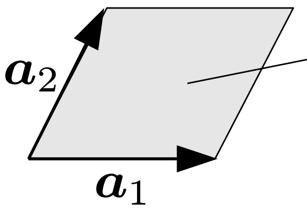
- Determinant（行列式）

Let $A = [\boldsymbol{a}_1 \ \cdots \boldsymbol{a}_n] = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \vdots \\ \tilde{\boldsymbol{a}}_n^T \end{bmatrix}$ be an $n \times n$ matrix.

The meaning of determinant of matrix$A$, $\det A$:

$|\det A|$ is equal to the volume of n-dimensional hyper parallelohedron（超平行多面体）with n edges $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ or $\tilde{\boldsymbol{a}}_1, \cdots, \tilde{\boldsymbol{a}}_n$.

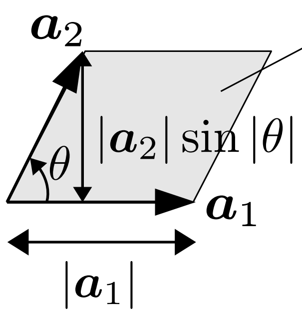
Example $n=2$ $A = [\boldsymbol{a}_1 \ \boldsymbol{a}_2]$



The area of parallelogram（平行四辺形）
$= |\det A|$

7. Determinant

$$A = [\boldsymbol{a}_1\ \boldsymbol{a}_2] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det A = a_{11}a_{22} - a_{21}a_{12}$$



The area: $|\boldsymbol{a}_1||\boldsymbol{a}_2|\sin|\theta| = |\boldsymbol{a}_1||\boldsymbol{a}_2||\sin\theta|$

$$= |\boldsymbol{a}_1||\boldsymbol{a}_2|\sqrt{1-\cos^2\theta}$$

$$= |\boldsymbol{a}_1||\boldsymbol{a}_2|\sqrt{1-\left(\frac{\boldsymbol{a}_1\cdot\boldsymbol{a}_2}{|\boldsymbol{a}_1||\boldsymbol{a}_2|}\right)^2}$$

$$= \sqrt{|\boldsymbol{a}_1|^2|\boldsymbol{a}_2|^2 - (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)^2}$$

$$= \sqrt{({a_{11}}^2 + {a_{21}}^2)({a_{12}}^2 + {a_{22}}^2) - (a_{11}a_{12} + a_{21}a_{22})^2}$$

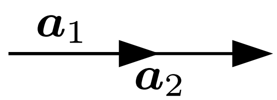
$$= \sqrt{{a_{11}}^2{a_{12}}^2 + {a_{11}}^2{a_{22}}^2 + {a_{21}}^2{a_{12}}^2 + {a_{21}}^2{a_{22}}^2 - ({a_{11}}^2{a_{12}}^2 + 2a_{11}a_{12}a_{21}a_{22} + {a_{21}}^2{a_{22}}^2)}$$

$$= \sqrt{{a_{11}}^2{a_{22}}^2 - 2a_{11}a_{12}a_{21}a_{22} + {a_{21}}^2{a_{12}}^2}$$

$$= \sqrt{(a_{11}a_{22} - a_{21}a_{12})^2} = |a_{11}a_{22} - a_{21}a_{12}|$$

7. Determinant

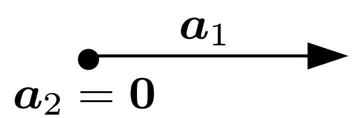
When does $\det A = 0$ happen?



$\boldsymbol{a}_2 = k\boldsymbol{a}_1$

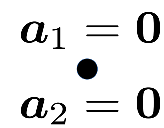
$k\boldsymbol{a}_1 - \boldsymbol{a}_2 = \mathbf{0}$

Vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ are NOT linearly independent.



$0 \cdot \boldsymbol{a}_1 - \boldsymbol{a}_2 = \mathbf{0}$

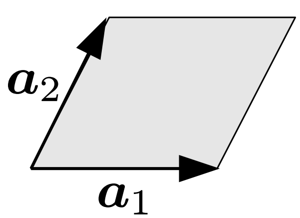
Vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ are NOT linearly independent.



$\boldsymbol{a}_1 + \boldsymbol{a}_2 = \mathbf{0}$

Vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ are NOT linearly independent.

When does $\det A \neq 0$ happen?



Neither $\boldsymbol{a}_1$ nor $\boldsymbol{a}_2$ is a zero vector, and they have different directions.

There is no k such that $\boldsymbol{a}_1 = k\boldsymbol{a}_2$.

Vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ ARE linearly independent.

7. Determinant

In general

$$\det A \neq 0 \ \Leftrightarrow \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}, \{\tilde{\boldsymbol{a}}_1, \cdots, \tilde{\boldsymbol{a}}_n\} \text{ : linearly independent}$$

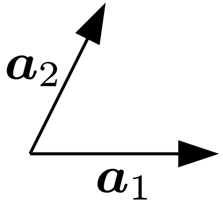
$$\Leftrightarrow \mathrm{rank} A = n$$

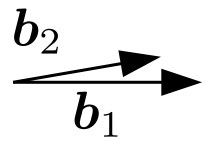
$$\det A = 0 \ \Leftrightarrow \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}, \{\tilde{\boldsymbol{a}}_1, \cdots, \tilde{\boldsymbol{a}}_n\} \text{ : NOT linearly independent}$$

$$\Leftrightarrow \mathrm{rank} A < n$$

However $\det A \begin{cases} = 0 \\ \neq 0 \end{cases}$ only shows $\mathrm{rank} A \begin{cases} < n \\ = n \end{cases}$.

Still, unlike the rank, the determinant takes a real value.
And therefore it is useful to judge which of the matrices A and B is closer to non-full rank by looking at their determinants, e.g. $\det A$=10 and $\det B$=0.1.





7. Determinant

Let us consider a linear mapping represented by a 2×2 matrix A: $\boldsymbol{y} = A\boldsymbol{x}$.

Suppose that a vector $\boldsymbol{x}_1$ is mapped to a vector $\boldsymbol{y}_1$, and another vector $\boldsymbol{x}_2$ is mapped to another vector $\boldsymbol{y}_2$: $\boldsymbol{y}_1 = A\boldsymbol{x}_1,\ \boldsymbol{y}_2 = A\boldsymbol{x}_2.$

These can be expressed using matrices as,

$$\begin{bmatrix}\boldsymbol{y}_1 & \boldsymbol{y}_2\end{bmatrix} = A\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix}.$$

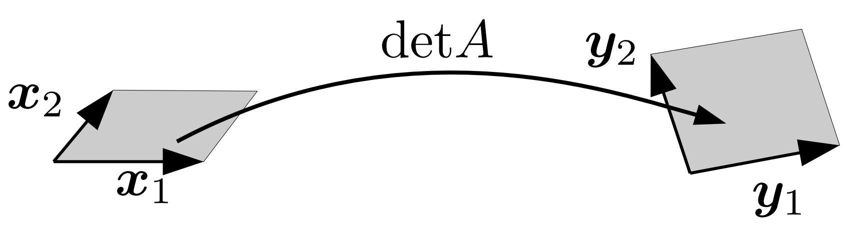
Here, all the matrices $\begin{bmatrix}\boldsymbol{y}_1 & \boldsymbol{y}_2\end{bmatrix}$, A and $\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix}$ are square.

Then we have,

$$\det\begin{bmatrix}\boldsymbol{y}_1 & \boldsymbol{y}_2\end{bmatrix} = \det\left(A\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix}\right) = \det A \cdot \det\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix},$$

$$\det A = \frac{\det\begin{bmatrix}\boldsymbol{y}_1 & \boldsymbol{y}_2\end{bmatrix}}{\det\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix}}.$$

So, the determinant of a matrix indicates how many times larger the area spanned by the mapped vectors $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ is than the original one.



7. Determinant

The relation between the rank and the determinant of a matrix

$\mathrm{rank}A$ is the maximum size (the number of rows, the number of columns) of the minors（小行列式）of A whose value are not zero.

An example for intuitive understanding: a diagonal matrix（対角行列）

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

A minor（whose size is $(n–2)\times(n–2)$）

$$= a_3 \times a_4 \times \cdots \times a_n \neq 0$$

$$\Downarrow$$

$$\mathrm{rank}A = n - 2$$

$$A\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ a_3 x_3 \\ \vdots \\ a_n x_n \end{bmatrix}$$

The matrix A can transfer only $n–2$ elements $x_3, \cdots, x_n$ out

of the n elements of the vector $\boldsymbol{x}$.