Exercise 3.7, Question 11 Solution

Summary of Results

(a) Regression of y onto x (without intercept)

- Coefficient estimate: $\hat{\beta}_{yx} = 1.9762$
- Standard error: ≈ 0.117
- **t-statistic**: ≈ 16.898
- p-value: < 0.0001
- R^2 (uncentered): 0.743

(b) Regression of x onto y (without intercept)

- Coefficient estimate: $\hat{\beta}_{xy} = 0.3757$
- Standard error: ≈ 0.022
- **t-statistic**: ≈ 16.898
- p-value: < 0.0001
- R^2 (uncentered): 0.743

(c) Relationship between (a) and (b)

- The t-statistics are **identical** in both regressions: $t \approx 16.898$.
- The slope coefficients are not reciprocals, due to differences in variance between x and y.
- This symmetry in t-statistics is expected in regressions **without intercept**.

(d) Algebraic Form of the t-Statistic

In a simple linear regression without intercept, the slope estimate is:

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$$

The standard error of $\hat{\beta}$ is:

$$SE(\hat{\beta}) = \sqrt{\frac{\sum (y_i - x_i \hat{\beta})^2}{(n-1)\sum x_i^2}}$$

The t-statistic for testing $H_0 : \beta = 0$ is:

$$t = \frac{\hat{\beta}}{SE(\hat{\beta})}$$

An alternative algebraic form of the t-statistic is:

$$t = \frac{\sqrt{n-1}\sum_{i=1}^{n} x_i y_i}{\sqrt{\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2}}$$

This expression shows the symmetry between x and y.

(e) Equality of t-Statistics

From the expression above, it is clear that the t-statistic depends only on the sums:

$$\sum x_i y_i, \quad \sum x_i^2, \quad \sum y_i^2$$

This symmetry implies that the t-statistic for the regression of y on x is the same as for the regression of x on y, when both are performed **without an intercept**. Our computations confirm this: both regressions yielded $t \approx 16.898$.

(f) Regression with Intercept

When an intercept is included, the symmetry breaks. The regression line no longer passes through the origin, and the standard error and estimated variance change. As a result:

- The t-statistics for y on x and x on y **differ**.
- The R^2 and coefficient values also generally change.

Summary Table

Case	**Slope** $(\hat{\beta})$	**t-statistic**	**Same t?**
$y \sim x$ (no intercept)	1.9762	16.898	Yes
$x \sim y$ (no intercept)	0.3757	16.898	Yes
$y \sim x$ (with intercept)	≈ 2.0	$\neq 16.898$	No
$x \sim y$ (with intercept)	≈ 0.5	$\neq 16.898$	No