HECKE ALGEBRAS: AN INCOMPLETE OVERVIEW (A DRAFT VERSION, TO BE COMPLETED)

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This is a very naive and incomplete introduction to Hecke algebras, emphasizing on their origins and motivations, and the cellularity of finite type A. For the original paper of Iwahori-Hecke algebra of finite Chevalley groups, see [Iwa64]. For more recent notes, see [Bum10, Wil03].

1. Motivation

1.1. **Group algebras and functions on groups.** Let G be a finite group. Recall that the group algebra $\mathbb{C}G$ is defined to be an associative algebra consists of formal sums

$$\mathbb{C}G = \{\sum_{g\in G} a_g g \mid a_g \in \mathbb{C}\},$$

with additions and multiplications defined by

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g,$$
$$\sum_{g\in G} a_g g \cdot \sum_{h\in G} b_h h = \sum_{g,h\in G} a_g b_h gh.$$

Denote by $\mathcal{H}$ the set of functions on G,

$$\mathcal{H} = \{f : G \to \mathbb{C}\}.$$

Clearly we may identify $\mathcal{H}$ and $\mathbb{C}G$ via the bijection $\theta : \mathcal{H} \to \mathbb{C}G$ defined by

$$\theta(f) = \sum_{g\in G} f(g) g, \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The inverse image of an element $\sum_{g\in G} a_g g$ in $\mathbb{C}G$ is the function $g \mapsto a_g$.

We can define a pointwise addition on $\mathcal{H}$,

$$(f_1 + f_2)(g) = f_1(g) + f_2(g), \quad \forall g \in G,$$

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and a convolution

$$(f_1 * f_2)(g) = \sum_{x\in G} f_1(gx^{-1})f_2(x) = \sum_{x,y\in G, xy=g} f_1(x)f_2(y), \quad \forall g \in G.$$

Lemma 1.1. *For any $f_1, f_2 \in \mathcal{H}$, we have $\theta(f_1+f_2) = \theta(f_1)+\theta(f_2)$ and $\theta(f_1*f_2) = \theta(f_1)\cdot\theta(f_2)$. Thus, $(\mathcal{H}, +, *)$ is an associative algebra over $\mathbb{C}$, and the bijection θ is an isomorphism of algebras.*

Lemma 1.2. *For any $f \in \mathcal{H}$, $g \in G$, we have*

$$\theta(f(g\cdot -)) = g^{-1}\cdot\theta(f) \quad \textit{and} \quad \theta(f(-\cdot g)) = \theta(f)\cdot g^{-1}.$$

Here $f(g\cdot -)$ denotes the function $x \mapsto f(gx)$, and similar for $f(-\cdot g)$.

Proof. $\theta(f(g\cdot -)) = \sum_{x\in G} f(gx)x = g^{-1}\sum_{x\in G} f(gx)gx = g^{-1}\sum_{x\in G} f(x)x = g^{-1}\cdot\theta(f)$. Similar for the second equation. □

1.2. **The B-bi-invariant functions on G.** Let G be a finite group as above and suppose $B \leq G$ is a subgroup. Let

$$\mathcal{H}_B = \{f : G \to \mathbb{C} \mid f(b_1 g b_2) = f(g), \forall g \in G, \forall b_1, b_2 \in B\}$$

be the set of B-bi-invariant functions on G. Then $\mathcal{H}_B$ is a linear subspace of $\mathcal{H}$, and is closed under convolution. As an algebra, $\mathcal{H}_B$ has a unity f_B with

$$f_B(g) = \begin{cases} \frac{1}{|B|}, & \text{if } g \in B; \\ 0, & \text{if } g \notin B. \end{cases}$$

Let $e_B := \theta(f_B) = \frac{1}{|B|}\sum_{b\in B} b \in \mathbb{C}G$.

Lemma 1.3.

(1) *For any $b \in B$, we have $be_B = e_B b = e_B$.*
(2) *The element e_B is an idempotent, i.e., $e_B^2 = e_B$.*
(3) *The image of $\mathcal{H}_B$ under the isomorphism $\theta : \mathcal{H} \to \mathbb{C}G$ is $e_B\mathbb{C}Ge_B$.*

Proof. The item (1) is clear, and the item (2) follows by (1).

Let $f \in \mathcal{H}_B$. By Lemma 1.2, we have

$$b\cdot\theta(f) = \theta(f(b^{-1}\cdot -)) = \theta(f), \quad \forall b \in B.$$

Thus

$$e_B\cdot\theta(f) = \frac{1}{|B|}\sum_{b\in B} b\cdot\theta(f) = \frac{1}{|B|}\sum_{b\in B}\theta(f) = \theta(f).$$

Similarly, $\theta(f) \cdot e_B = \theta(f)$. Therefore, $\theta(f) = e_B \cdot \theta(f) \cdot e_B \in e_B \mathbb{C}G e_B$.

Conversely, let $a \in e_B \mathbb{C}G e_B$ and $f = \theta^{-1}(a) \in \mathcal{H}$. By Lemma 1.2 again, we have

$$\theta(f(b_1 \cdot - \cdot b_2)) = b_1^{-1} \cdot \theta(f) \cdot b_2^{-1} = b_1^{-1} \cdot a \cdot b_2^{-1} = a = \theta(f), \quad \forall b_1, b_2 \in B.$$

The third equality uses (1) and the fact that $a \in e_B \mathbb{C}G e_B$. Since θ is an isomorphism, we have $f(b_1 \cdot - \cdot b_2) = f$ and hence $f \in \mathcal{H}_B$. □

Remark 1.4. An element $\sum_{g \in G} a_g g$ in $\theta(\mathcal{H}_B)$ is characterized by the following condition: if $x, y \in G$ lie in the same B-double coset, i.e., $BxB = ByB$, then the coefficients a_x and a_y are equal.

1.3. **G-representations with nonzero B-fixed vectors.** Let (V, ρ) be a representation of the finite group G. This means that $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism where V is a (complex) vector space. This is also equivalent to a $\mathbb{C}G$-module structure on V. Since $\theta : \mathcal{H} \to \mathbb{C}G$ is an isomorphism, we may also regard V as an $\mathcal{H}$-module,

$$f \cdot v := \theta(f)v, \quad \forall f \in \mathcal{H}, \forall v \in V.$$

Denote by V^B the subspace of B-fixed vectors,

$$V^B = \{v \in V \mid \rho(b)v = v, \forall b \in B\}.$$

Lemma 1.5. *For any $f \in \mathcal{H}_B$ and any $v \in V$, we have $f \cdot v \in V^B$.*

Proof. By Lemma 1.3(3), we have $f \cdot v = \theta(f)v \in e_B V \subseteq V^B$. □

In particular, V^B is an $\mathcal{H}_B$-module.

Theorem 1.6.

(1) *Let (V, ρ) be an irreducible representation of G such that $V^B \neq 0$. Then V^B is a simple $\mathcal{H}_B$-module.*
(2) *Let (U, σ) be another irreducible representation of G such that $U^B \neq 0$. Then $(V, \rho) \simeq (U, \sigma)$ as G-representations if and only if $V^B \simeq U^B$ as $\mathcal{H}_B$-modules.*

Corollary 1.7. *We have an injective correspondence*

$$\left\{ \begin{array}{c} \text{the isomorphism classes of} \\ \text{irreducible } G\text{-representations} \\ \text{with a nonzero } B\text{-fixed vector} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{the isomorphism classes of} \\ \text{simple } \mathcal{H}_B\text{-modules} \end{array} \right\}.$$

We will see in the next subsection that this is in fact a bijection.

Remark 1.8. We have a similar story for p-adic groups G.

1.4. **The decomposition of an induced representation.** Let $B \leq G$ be finite groups as above. Denote by $\mathbb{1}_B$ the trivial representation of B over $\mathbb{C}$, and by $\mathbb{1}_B^G$ the induced representation

$$\mathbb{1}_B^G := \mathrm{Ind}_B^G \mathbb{1}_B = \{f \in \mathcal{H} \mid f(bg) = f(g), \forall g \in G, \forall b \in B\}.$$

The action of G on $\mathbb{1}_B^G$ is defined to be

$$(g \cdot f)(x) = f(xg), \quad \forall f \in \mathbb{1}_B^G, \forall g, x \in G.$$

Recall that we have an idempotent $e_B = \frac{1}{|B|} \sum_{b \in B} b$ in $\mathbb{C}G$. We define an action ρ_r of G on the vector space $e_B \mathbb{C}G$ by

$$\rho_r(g)a = ag^{-1}, \quad \forall g \in G, \forall a \in e_B \mathbb{C}G.$$

The operation in the right hand side is the multiplication in $\mathbb{C}G$.

Lemma 1.9.

(1) *$(e_B\mathbb{C}G, \rho_r)$ is a well defined G-representation.*
(2) *The map θ defined in* (1.1) *maps $\mathbb{1}_B^G$ into $e_B\mathbb{C}G$ and induces an isomorphism of G-representations $\mathbb{1}_B^G \xrightarrow{\sim} e_B\mathbb{C}G$.*

Theorem 1.10. $\mathrm{End}_G(\mathbb{1}_B^G) \simeq \mathrm{End}_G(e_B\mathbb{C}G) \simeq e_B\mathbb{C}Ge_B \simeq \mathcal{H}_B$.

Proof. The isomorphism in the middle is given as follows. An endomorphism $\varphi \in \mathrm{End}_G(e_B\mathbb{C}G)$ corresponds to the element $\varphi(e_B)$ which is in fact an element in $e_B\mathbb{C}Ge_B$. Conversely, given an element $a \in e_B\mathbb{C}Ge_B$, we associate a with the map $\varphi : e_B\mathbb{C}G \to e_B\mathbb{C}G$ defined by $x \mapsto ax$. $\square$

Decompose $\mathbb{1}_B^G$ into a direct sum of irreducible G-representations, say

$$\mathbb{1}_B^G = \bigoplus_{i=1}^{k} (V_i, \rho_i)^{\oplus n_i},$$

where $n_i \in \mathbb{N}_{>0}$ and (V_i, ρ_i)'s are distinct irreducible G-representations. Then

$$\mathcal{H}_B \simeq \mathrm{End}_G(\mathbb{1}_B^G) \simeq \bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(\mathbb{C})$$

which is a semisimple algebra. Therefore, $\mathcal{H}_B$ has k simple modules of dimension $n_1, \ldots, n_k$ respectively.

Suppose (V,ρ) is an irreducible G-representation. By Frobenius reciprocity ($\otimes$-Hom adjunction), we have

$$\mathrm{Hom}_G(\mathbb{1}_B^G, V) \simeq \mathrm{Hom}_B(\mathbb{1}_B, V|_B).$$

The left hand side is not zero if and only if $\rho = \rho_i$ for some $i = 1, \ldots, k$. The right hand side is not zero if and only if $V^B \neq 0$. Thus, there are exactly k irreducible representations of G with nonzero B-fixed vectors. We have proved:

Theorem 1.11. *The correspondence in Corollary* 1.7 *is bijective.*

2. The Iwahori-Hecke algebra of $\mathrm{GL}_n(\mathbb{F}_q)$

In this section, let G be the finite group $\mathrm{GL}_n(\mathbb{F}_q)$, and B be the Borel subgroup $B = \{\text{upper triangular matrices in } G\}$. We consider the Hecke algebra $\mathcal{H}_B$ defined in Section 1. We shall need some results concerning the double cosets $B\backslash G/B$.

2.1. **The Bruhat decomposition.** For any element w in the symmetric group S_n, let $\dot{w} \in G$ be the permutation matrix

$$\dot{w} = \sum_{i=1}^{n} E_{w(i),i}.$$

Here $E_{i,j}$ denotes the elementary matrix with the element 1 at the (i,j)-position and 0's elsewhere. Then $w \mapsto \dot{w}$ is an injective homomorphism from S_n into G.

The well known Bruhat decomposition states that $\{\dot{w} \mid w \in S_n\}$ is a representative set for the double cosets $B\backslash G/B$.

Theorem 2.1 (Bruhat decomposition)**.** $\mathrm{GL}_n(\mathbb{F}_q) = \sqcup_{w\in S_n} B\dot{w}B.$

We shall also need to consider the cardinality of a double coset $B\dot{w}B$, and the multiplication $B\dot{x}B\dot{y}B$. Recall that S_n is generated by adjacent transpositions $(1,2), (2,3), \ldots, (n-1,n)$. We denote $s_i := (i, i+1)$ and $S := \{s_i \mid i = 1, \ldots, n-1\}$. The length function ℓ on S_n is defined to be

$$\ell(w) = \big|\{(i,j) \mid 1 \le i < j \le n, w(i) > w(j)\}\big|.$$

The number $\ell(w)$ is also equal to the least number k such that w can be written as a product of k elements in S.

Lemma 2.2.

(1) $|B\dot{w}B| = |B|q^{\ell(w)}$, $\forall w \in S_n$.

(2) *Let* $x, y \in S_n$*. If* $\ell(xy) = \ell(x) + \ell(y)$*, then* $B\dot{x}B\dot{y}B = B\dot{x}\dot{y}B$*.*

(3) *Let* $s \in S$*. Then* $B\dot{s}B\dot{s}B \subseteq B \cup B\dot{s}B$*.*

2.2. **The Iwahori-Hecke algebra $\mathcal{H}_B$.** Recall that $\mathcal{H}_B$ is defined to be the functions on G which are constant on any B-double coset,

$$\mathcal{H}_B = \{f : G \to \mathbb{C} \mid f(b_1 g b_2) = f(g), \forall g \in G, \forall b_1, b_2 \in B\}.$$

By Bruhat decomposition (see Theorem 2.1), $\mathcal{H}_B$ has a basis $\{\phi_w \in \mathcal{H}_B \mid w \in S_n\}$ where ϕ_w takes the value 1 on the double coset $B\dot{w}B$ and 0 elsewhere. The unity of $\mathcal{H}_B$ is $\frac{1}{|B|}\phi_e$, where e denotes the unity in S_n.

Proposition 2.3.

(1) *Let* $s \in S$*. Then* $\phi_s * \phi_s = |B|((q-1)\phi_s + q\phi_e)$*.*

(2) *Let* $x, y \in S_n$*. If* $\ell(xy) = \ell(x) + \ell(y)$*, then* $\phi_x * \phi_y = |B|\phi_{xy}$*.*

Proof. Suppose $x, y \in S_n$ and $\ell(xy) = \ell(x) + \ell(y)$. For $g \in G$, we have

$$\phi_x * \phi_y(g) = \sum_{a,b \in G, ab=g} \phi_x(a)\phi_y(b).$$

The term $\phi_x(a)\phi_y(b)$ is nonzero if and only if $a \in B\dot{x}B$ and $b \in B\dot{y}B$. If this happens, by Lemma 2.2 we see that $g = ab \in B\dot{x}\dot{y}B$. Thus $\phi_x * \phi_y(g) \neq 0$ only if $g \in B\dot{x}\dot{y}B$. Therefore, $\phi_x * \phi_y = \lambda\phi_{xy}$ for some $\lambda \in \mathbb{C}$.

In order to show that $\lambda = |B|$, we introduce a map ϵ on $\mathcal{H}$. For $f \in \mathcal{H}$ a function on G, we define

$$\epsilon(f) = \sum_{g \in G} f(g).$$

Then $\epsilon : \mathcal{H} \to \mathbb{C}$ is a homomorphism of algebras.

Apply ϵ on $\phi_x * \phi_y = \lambda\phi_{xy}$. The left hand side is $\epsilon(\phi_x * \phi_y) = \epsilon(\phi_x)\epsilon(\phi_y) = |B\dot{x}B||B\dot{y}B| = |B|^2 q^{\ell(x)+\ell(y)}$. The last equality uses Lemma 2.2. On the other hand, $\epsilon(\lambda\phi_{xy}) = \lambda|B|q^{\ell(xy)}$. As a result, we have $\lambda = |B|$.

Now suppose $s \in S$. For $g \in G$, we have

$$\phi_s * \phi_s(g) = \sum_{a,b \in G, ab=g} \phi_s(a)\phi_s(b).$$

The term $\phi_s(a)\phi_s(b)$ is nonzero if and only if $a, b \in B\dot{s}B$. Thus $\phi_s * \phi_s(g) \neq 0$ only if $g \in B \cup B\dot{s}B$ by Lemma 2.2. As a result $\phi_s * \phi_s = \mu\phi_s + \nu\phi_e$ for some $\mu, \nu \in \mathbb{C}$.

Let $g = e$. Then

$$\phi_s * \phi_s(e) = (\mu\phi_s + \nu\phi_e)(e)$$

$$
\begin{aligned}
&= \nu; \\
\sum_{a,b\in G, ab=e} \phi_s(a)\phi_s(b) &= \sum_{a\in G} \phi_s(a)\phi_s(a^{-1}) \\
&= \sum_{a\in B\dot{s}B} \phi_s(a)\phi_s(a^{-1}) \\
&= |B\dot{s}B| \\
&= |B|q.
\end{aligned}
$$

The last equality is due to Lemma 2.2 again. Thus $\nu = |B|q$. Apply ϵ on $\phi_s * \phi_s = \mu\phi_s + \nu\phi_e$. Then we obtain $|B|^2q^2 = \mu|B|q + \nu|B|$. Thus $\mu = |B|(q-1)$ and we are done. □

Denote $\varphi_w := \frac{1}{|B|}\phi_w$ for any $w \in S_n$. Then $\{\varphi_w \mid w \in S_n\}$ is also a basis of $\mathcal{H}_B$. Suppose $w = s_{i_1}\cdots s_{i_k}$ is a reduced expression of w, that is, $\ell(w) = k$. Then by Proposition 2.3, we have $\varphi_w = \varphi_{s_{i_1}} * \cdots * \varphi_{s_{i_k}}$. Therefore, $\mathcal{H}_B$ is generated by $\varphi_{s_1}, \ldots, \varphi_{s_{n-1}}$. By Proposition 2.3 again, these generators satisfy the following relations,

$$
\begin{gathered}
\varphi_{s_i} * \varphi_{s_i} = (q-1)\varphi_{s_i} + q\varphi_e, \quad \forall i = 1, \ldots, n-1; \\
\varphi_{s_i} * \varphi_{s_{i+1}} * \varphi_{s_i} = \varphi_{s_{i+1}} * \varphi_{s_i} * \varphi_{s_{i+1}}, \quad \forall i = 1, \ldots, n-2; \\
\varphi_{s_i} * \varphi_{s_j} = \varphi_{s_j} * \varphi_{s_i}, \quad \text{if } |i-j| > 1.
\end{gathered}
$$

In fact, these generators and relations give a presentation of $\mathcal{H}_B$ (see the next subsection).

2.3. **The Hecke algebra $\mathcal{H}_q(S_n)$.** Define $\mathcal{H}_q(S_n)$ to be the unital $\mathbb{C}$-algebra with formal generators $\{T_i \mid i = 1, \ldots, n-1\}$, subject to the following relations,

$$
\begin{gathered}
T_i^2 = (q-1)T_i + q, \quad \forall i = 1, \ldots, n-1; \\
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad \forall i = 1, \ldots, n-2; \\
T_iT_j = T_jT_i, \quad \text{if } |i-j| > 1.
\end{gathered}
$$

(The first relation is also written in the form $(T_i+1)(T_i-q) = 0$ in the literature.) Then we have a homomorphism of algebras $\pi : \mathcal{H}_q(S_n) \to \mathcal{H}_B$ defined by

$$\pi(T_i) = \varphi_{s_i}, \quad \forall i = 1, \ldots, n-1.$$

Theorem 2.4. *The homomorphism π is an isomorphism.*

3. The Hecke algebra of an arbitrary Coxeter group

3.1. **The Hecke algebra.** Let (W, S) be a Coxeter group, $S = \{s_1, \dots, s_n\}$. For $s_i, s_j \in S$, denote by m_{ij} the order of $s_i s_j$ in W. Regard q as a formal variable. Let $\mathscr{H}(W)$ be the unital $\mathbb{C}[q]$-algebra with formal generators $\{T_i \mid s_i \in S\}$, subject to the following relations,

$$T_i^2 = (q-1)T_i + q, \quad \forall i = 1, \dots, n;$$
$$\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}, \quad \forall i \neq j.$$

Lemma 3.1. *Let $w \in W$ and $s_{i_1} \cdots s_{i_k}$ be a reduced expression of w. Then the element $T_{i_1} \cdots T_{i_k} \in \mathscr{H}(W)$ is independent of the choice of the reduced expression.*

Thus, the element $T_w := T_{i_1} \cdots T_{i_k}$ is well defined.

Theorem 3.2. *The $\mathbb{C}[q]$-algebra $\mathscr{H}(W)$ has a basis $\{T_w \mid w \in W\}$. The multiplication is described as follows. For any $w \in W$ and $s \in S$,*

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) = \ell(w) + 1; \\ (q-1)T_w + qT_{ws}, & \text{if } \ell(ws) = \ell(w) - 1. \end{cases}$$

The multiplication $T_s T_w$ is similar.

3.2. **The Kazhdan-Lusztig basis.** In order to introduce the Kazhdan-Lusztig basis, it is necessary to involve the square root $q^{\frac{1}{2}}$ of q and its inverse $q^{-\frac{1}{2}}$. Let $\mathcal{H}(W) := \mathscr{H}(W) \otimes_{\mathbb{C}[q]} \mathbb{C}[q^{\pm\frac{1}{2}}]$ be the $\mathbb{C}[q^{\pm\frac{1}{2}}]$-algebra obtained from a base change of $\mathscr{H}(W)$. Then $\mathcal{H}(W)$ is a free $\mathbb{C}[q^{\pm\frac{1}{2}}]$-module with a basis $\{T_w \mid w \in W\}$.

Note that any T_i, and hence any T_w, is invertible in $\mathcal{H}(W)$. There is a $\mathbb{C}$-algebra involution $\bar{\cdot}$ on $\mathcal{H}(W)$ defined as follows,

$$\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}; \quad \overline{T_w} = T_{w^{-1}}^{-1}, \quad \forall w \in W.$$

(Note that T_s is invertible in $\mathcal{H}(W)$ for any $s \in S$, $T_s^{-1} = q^{-1}T_s + q^{-1} - 1$. Thus T_w is invertible in $\mathcal{H}(W)$ for any $w \in W$.)

Theorem 3.3. *For any $w \in W$, there is a unique element $C_w \in \mathcal{H}(W)$ such that*

$$\overline{C_w} = C_w$$

and

$$C_w = \sum_{y \le w} (-1)^{\ell(w)+\ell(y)} q^{\frac{\ell(w)-2\ell(y)}{2}} \overline{P_{y,w}} T_y,$$

where $P_{y,w} \in \mathbb{Z}[q]$ *is a polynomial in* q*, such that* $\deg P_{y,w} \le \frac{1}{2}(\ell(w) - \ell(y) - 1)$ *for* $y < w$*, and* $P_{w,w} = 1$*.*

Corollary 3.4. *The set of elements* $\{C_w \mid w \in W\}$ *forms a* $\mathbb{C}[q^{\pm\frac{1}{2}}]$*-basis of* $\mathcal{H}(W)$*.*

Proof. This is because the transition matrix from the basis $\{q^{-\frac{\ell(w)}{2}} T_w \mid w \in W\}$ to the set of elements $\{C_w \mid w \in W\}$ is "upper triangular" (with respect to the Bruhat order $\le$) with all "diagonal entries" being 1. □

3.3. **The Kazhdan-Lusztig cells.** Since $\{C_w \mid w \in W\}$ is a basis of $\mathcal{H}(W)$, any element $h \in \mathcal{H}$ can be uniquely written as a linear combination of the elements C_w, say $h = \sum_{w \in W} a_w C_w$, $a_w \in \mathbb{C}[q^{\pm\frac{1}{2}}]$. We say that the basis element C_w appears in h if $a_w \neq 0$.

Definition 3.5. Let $y, w \in W$.

(1) If C_y appears in hC_w for some $h \in \mathcal{H}(W)$, then we write $y \underset{\mathrm{L}}{\leqslant} w$.
(2) If C_y appears in $C_w h$ for some $h \in \mathcal{H}(W)$, then we write $y \underset{\mathrm{R}}{\leqslant} w$.
(3) If C_y appears in $h_1 C_w h_2$ for some $h_1, h_2 \in \mathcal{H}(W)$, then we write $y \underset{\mathrm{LR}}{\leqslant} w$.

Theorem 3.6. *The relations* $\underset{\mathrm{L}}{\leqslant}, \underset{\mathrm{R}}{\leqslant}, \underset{\mathrm{LR}}{\leqslant}$ *are pre-orders on* W*.*

Remark 3.7. Definition 3.5 is quite different from the original definition in [KL79]. The proof of the equivalence of the two definitions, as well as the proof of Theorem 3.6, uses the validity of the Kazhdan-Lusztig positivity conjecture, which is proved in [EW14] using nontrivial tools.

The pre-orders $\underset{\mathrm{L}}{\leqslant}, \underset{\mathrm{R}}{\leqslant}, \underset{\mathrm{LR}}{\leqslant}$ generate equivalence relations on W:

Definition 3.8. Let $y, w \in W$.

(1) If $y \underset{\mathrm{L}}{\leqslant} w$ and $w \underset{\mathrm{L}}{\leqslant} y$, then we write $y \underset{\mathrm{L}}{\sim} w$. We define the relations $\underset{\mathrm{R}}{\sim}$ and $\underset{\mathrm{LR}}{\sim}$ in a similar way.
(2) The relation $\underset{\mathrm{L}}{\sim}$ is an equivalence relation on W. An equivalence class is called a left cell of W. Similarly we define right cells and two-sided cells.

Obviously, if $y \underset{\mathrm{L}}{\leqslant} w$ or $y \underset{\mathrm{R}}{\leqslant} w$, then we have $y \underset{\mathrm{LR}}{\leqslant} w$. Thus, any two-sided cell is a disjoint union of some left cells, as well as a disjoint union of some right cells.

3.4. **Cell representations.** Let Γ be a left cell. For $y \in W$, if there exists some element $w \in \Gamma$ such that $y \underset{\mathrm{L}}{\leqslant} w$, then we simply write $y \underset{\mathrm{L}}{\leqslant} \Gamma$.

Denote by $\mathcal{H}(\underset{\mathrm{L}}{\leqslant} \Gamma)$ the $\mathbb{C}[q^{\pm\frac{1}{2}}]$-submodule of $\mathcal{H}(W)$ spanned by $\{C_y \mid y \underset{\mathrm{L}}{\leqslant} \Gamma\}$, and by $\mathcal{H}(\underset{\mathrm{L}}{<} \Gamma)$ the $\mathbb{C}[q^{\pm\frac{1}{2}}]$-submodule spanned by $\{C_y \mid y \underset{\mathrm{L}}{\leqslant} \Gamma, y \notin \Gamma\}$. Then $\mathcal{H}(\underset{\mathrm{L}}{\leqslant} \Gamma)$ and $\mathcal{H}(\underset{\mathrm{L}}{<} \Gamma)$ are both left ideals of $\mathcal{H}(W)$, and thus left $\mathcal{H}(W)$-modules. The quotient $E(\Gamma) := \mathcal{H}(\underset{\mathrm{L}}{\leqslant} \Gamma)/\mathcal{H}(\underset{\mathrm{L}}{<} \Gamma)$ is called a cell representation (or a cell module) of $\mathcal{H}(W)$.

Similarly, for a right cell Γ' or a two-sided cell $\mathcal{C}$, we can define cell representations $E(\Gamma') := \mathcal{H}(\underset{\mathrm{R}}{\leqslant} \Gamma')/\mathcal{H}(\underset{\mathrm{R}}{<} \Gamma')$ and $E(\mathcal{C}) := \mathcal{H}(\underset{\mathrm{LR}}{\leqslant} \mathcal{C})/\mathcal{H}(\underset{\mathrm{LR}}{<} \mathcal{C})$ which is a right $\mathcal{H}(W)$-module and an $\mathcal{H}(W)$-bi-module respectively.

4. The cellularity of $\mathcal{H}(S_n)$

4.1. **Some notions.** Let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ be a partition of n. That means, $\lambda_1 \geq \cdots \geq \lambda_l > 0$ are integers and their sum $\sum_i \lambda_i$ equals to n. We say that n is the size of λ, and l is the length of λ. The partition λ can be visualized via the Young diagram: we draw l rows of boxes such that the i-th row has λ_i boxes (the Young diagram is also denoted by the symbol λ). For example, the partition $(4, 2, 2, 1) \vdash 9$ corresponds to the following Young diagram,

$$(4,2,2,1) = \ydiagram{4,2,2,1}$$

A Young tableau of shape λ is the Young diagram λ filled with some positive integers without repetition, such that the entries increase from left to right along each row and from top to bottom along each column. For example, the following is a Young tableau of shape $(4, 2, 2, 1)$,

$$\begin{ytableau} 3 & 5 & 6 & 7 \\ 4 & 10 \\ 9 & 11 \\ 12 \end{ytableau}$$

while none of the following three is a Young tableau,

3	3	6	7
4	10		
9	11		
12			

5	3	6	7
4	10		
9	11		
12			

4	5	6	7
3	10		
9	11		
12			

If the numbers filled in the Young tableau are exactly $1, \dots, n$, where n is the size of λ, then the Young tableau is called standard. For example, the following is a standard Young tableau,

1	3	4	9
2	6		
5	7		
8			

If a (standard) Young tableau T is of shape $\lambda \vdash n$, we also say n is the size of T.

In the rest of this section, we fix a natural number n, and denote by Λ the set of partitions of n. For $\lambda \in \Lambda$, let $M(\lambda)$ be the set of standard Young tableaux of shape λ.

4.2. **The Robinson-Schensted correspondence.** Given arbitrarily an element $w \in S_n$, there is a algorithm, called the row bumping algorithm, yielding a pair of standard Young tableaux of the same shape, say, $(P(w), Q(w))$. Here $P(w)$ and $Q(w)$ are two standard Young tableaux of size n, determined by w and the row bumping algorithm. We refer the reader to [Wil03] for details of the algorithm.

Theorem 4.1 (The Robinson-Schensted correspondence)**.** *There is a one-to-one correspondence between elements of S_n and the set*

$$\{(P, Q) \mid P, Q \in M(\lambda) \text{ for some } \lambda \in \Lambda\}.$$

The correspondence is given by the row bumping algorithm $w \mapsto (P(w), Q(w))$.

The Robinson-Schensted correspondence has the following properties.

Proposition 4.2. *Let $w, y \in S_n$.*

(1) *We have $P(w) = Q(w^{-1})$, $Q(w) = P(w^{-1})$, i.e., $w^{-1} \mapsto (Q(w), P(w))$.*
(2) *$w \underset{\mathrm{L}}{\sim} y$ if and only if $Q(w) = Q(y)$.*
(3) *$w \underset{\mathrm{R}}{\sim} y$ if and only if $P(w) = P(y)$.*
(4) *$w \underset{\mathrm{LR}}{\sim} y$ if and only if $P(w)$ and $P(y)$ have the same shape.*

By these properties, Λ can be equipped with a partial order $\underset{\mathrm{LR}}{\leqslant}$. For $\lambda, \mu \in \Lambda$, we write $\lambda \underset{\mathrm{LR}}{\leqslant} \mu$ if there exist $y, w \in S_n$ such that $y \underset{\mathrm{LR}}{\leqslant} w$ and $P(y), P(w)$ are of shapes λ, μ respectively.

Question 4.3. Can we interpret the relation $y \underset{\mathrm{L}}{\leqslant} w$ into a description of $Q(w)$ and $Q(y)$? How about the relation $y \underset{\mathrm{LR}}{\leqslant} w$ and the shapes of $P(y)$ and $P(w)$?

4.3. **The cellularity of $\mathcal{H}(S_n)$.** Now we write $C^\lambda_{P,Q} := C_w$, where $P = P(w)$ and $Q = Q(w)$ for $w \in S_n$, and λ is their shape.

Theorem 4.4.

(1) *The set $\{C^\lambda_{P,Q} \mid \lambda \in \Lambda, \text{and } P, Q \in M(\lambda)\}$ is a $\mathbb{C}[q^{\pm\frac{1}{2}}]$-basis of $\mathcal{H}(S_n)$.*

(2) *The $\mathbb{C}[q^{\pm\frac{1}{2}}]$-linear map $(-)^*$ defined by $(C^\lambda_{P,Q})^* = C^\lambda_{Q,P}$ is an anti automorphism of $\mathcal{H}(S_n)$.*

(3) *Let $\lambda \in \Lambda$ and $P, Q \in M(\lambda)$. Then for any $h \in \mathcal{H}(S_n)$, we have*

$$hC^\lambda_{P,Q} \equiv \sum_{P' \in M(\lambda)} r_h(P', P) C^\lambda_{P',Q} \mod \mathcal{H}(\underset{\mathrm{LR}}{<} \lambda),$$

where $r_h(P', P) \in \mathbb{C}[q^{\pm\frac{1}{2}}]$ only depends on h, P' and P (NOT on Q), and $\mathcal{H}(\underset{\mathrm{LR}}{<} \lambda)$ is the $\mathbb{C}[q^{\pm\frac{1}{2}}]$-submodule spanned by $\{C^\mu_{R,T} \mid \mu \underset{\mathrm{LR}}{\leqslant} \lambda, \mu \neq \lambda, \text{and } R, T \in M(\mu)\}$.

Theorem 4.4 characterizes the cellularity of $\mathcal{H}(S_n)$. It just says that $\mathcal{H}(S_n)$ is a cellular algebra in the sense of [GL96]. We can also restate the theorem in the following form.

Theorem 4.4 $'$.

(1) *The set $\{C_w \mid w \in S_n\}$ is a $\mathbb{C}[q^{\pm\frac{1}{2}}]$-basis of $\mathcal{H}(S_n)$.*

(2) *The $\mathbb{C}[q^{\pm\frac{1}{2}}]$-linear map $(-)^*$ defined by $(C_w)^* = C_{w^{-1}}$ is an anti automorphism of $\mathcal{H}(S_n)$.*

(3) *Let $w \in S_n$ and $\Gamma, \mathcal{C}$ be the left cell and the two-sided cell respectively such that $w \in \Gamma \subseteq \mathcal{C}$. Then for any $h \in \mathcal{H}(S_n)$, we have*

$$hC_w \equiv \sum_{y \in \Gamma} r_h(y, w) C_y \mod \mathcal{H}(\underset{\mathrm{LR}}{<} \mathcal{C}),$$

where $r_h(y, w) \in \mathbb{C}[q^{\pm\frac{1}{2}}]$ only depends on h and the right cells containing y and w (NOT on the left cell Γ).

Corollary 4.5. *Let Γ_1, Γ_2 be two left cells of S_n contained in a two-sided cell $\mathcal{C}$ simultaneously. Then the cell representations $E(\Gamma_1)$ and $E(\Gamma_2)$ are isomorphic.*

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