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The method of Polynomial Particular Solutions for Solving Nonlinear Equation of the Poisson Type *

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Abstract: In the present paper, we use the method of polynomial particular solutions to solve nonlinear partial differential equations of Poisson type in the case of one-, two- and three-dimensional. Firstly, we compared the results obtained by using polynomial particular solutions with the results obtained by the monomial method in literature. The method of polynomial particular solutions used in this paper achieved high accuracy in solving two-dimensional and three-dimensional problems. Secondly, it was discovered that the accuracy of results are somewhat influenced by the shape of the computational domain. Finally, we moved the computational domain for equations with singularities in the analytical solution and found that high accuracy is possible.

Key words: nonlinear equation; the Singularity; polynomial particular solutions

1 Introduction

In this paper we use the method of polynomial particular solutions to solve one-, two- and three-dimensional problems with nonlinear partial differential equations of Poisson type.

During the recent decades, predecessors have significantly advanced the study of nonlinear partial differential equations over the last few decades. The methods for solving nonlinear partial differential equations can be categorized into grid method and meshless method depending on whether a grid is present or not. Gridded methods include

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the boundary element method(BEM) proposed by Kasab, Karur and Ramachandram in 1995[4]; the method of fundamental solutions in combination with the analog equation method developed by Li and Zhu[5] in 2009; the combination of the asymptotic method with MFS studied by Tri, Zahrouni and Potier-Ferry in 2011[6] and the homotopy method combined with the MFS proposed by Tsai in 2012[7] are the most important techniques developed in this field recently. At the same time, the meshless methods have been strongly advocated by many domestic and international researchers. The fictitious time integration method(FTIM)[8, 9] combined with the method of fundamental solutions and Chebyshev polynomials to solve Poisson type nonlinear PDEs is proposed by Tsai, Liu and Yeih in 2010[10]. During the recent decades, radial basis functions(RBFs) have played an important role in the development of meshless methods for solving partial differential equations(PDEs)[11, 12]. Among these techniques, the most important one is use the method of particular solutions. In order to solve variable coefficient partial differential equations, the method which combines the MFS and RBFs approximation was proposed. This is the so-called MFS-MPS technique[13]. Recently this technique has been transformed into the method of approximate particular solutions(MAPS)[14, 15]. The method of polynomial particular solutions used in this article is an approximate particular solutions.

The outline of this paper is as follows. In section 2, we give a brief review of polynomial particular solutions. In section 3, 4 and 5, numerical examples are provided for one-dimensional, two-dimensional and three-dimensional cases respectively. In section 6, we draw some conclusions and discuss future work.

2 The method of polynomial particular solutions

In this paper, we use a meshless method of polynomial particular solutions to solve nonlinear differential equation of Poisson type. Let Ω be a bounded and connected regular, or irregular domain in $R^d, d = 1, 2, 3$.

$$Lu(\mathbf{x}) = f(\mathbf{x}) + F(u, u_{x_1}, u_{x_2}, u_{x_3}, \mathbf{x}) \qquad \mathbf{x} = (x_1, x_2, x_3) \equiv (x, y, z) \in \Omega \tag{2.1}$$

$$\mathscr{B}u(\mathbf{x}) = g(\mathbf{x}) \qquad \mathbf{x} \in \partial\Omega \tag{2.2}$$

where L is a linear or nonlinear differential operator, in this dissertation, the differential operator L is equal to Laplace operator $\Delta = \sum_{i=0}^{d} \frac{\partial^2}{\partial x_i^2}$, $\mathscr{B}$ is a boundary operator. We assume the $g(\mathbf{x}), f(\mathbf{x})$ and F are smooth enough functions of each argument. By the method of polynomial particular solutions, we assume $u(\mathbf{x})$ can be approximated by the following linear combination of polynomial particular solutions $\Phi_{nmk}(\mathbf{x})$,

$$u(\mathbf{x}) \approx \hat{u}(\mathbf{x}) = \sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{k=0}^{n-m} a_{nmk}\Phi_{nmk}(\mathbf{x}) \qquad \mathbf{x} \equiv (x, y, z) \in \Omega \tag{2.3}$$

where $\{a_{nmk}\}$ is a set of undetermined coefficient, $\Phi_{nmk}(\mathbf{x})$ is given based on the following analytically solving

$$L\Phi_{nmk}(\mathbf{x}) = x^n y^m z^k \quad 0 \le n \le N, 0 \le m \le n, 0 \le k \le n-m \tag{2.4}$$

Assume $\{\mathbf{x}_i\}_{i=1}^{N_i}$ be interior points in Ω, let $\{\mathbf{x}_i\}_{i=N_i+1}^{N_w}$ be boundary point on Ω and let $N_w = N_i + N_b$. Then coefficient set $\{a_{nmk}\}$ can be obtained by solving

$$f(\mathbf{x}_i) \approx L\hat{u}(\mathbf{x}_i) = \sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{nmk} L\Phi_{nmk}(\mathbf{x}_i) - F(u, u_x, u_y, u_z, \mathbf{x}_i) \quad i = 1, 2, ..., N_i \tag{2.5}$$

$$g(\mathbf{x}_i) \approx \mathscr{B}\hat{u}(\mathbf{x}_i) = \sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{nmk} \mathscr{B}\Phi_{nmk}(\mathbf{x}_i) \quad i = N_i+1, N_i+2, ..., N_w \tag{2.6}$$

Combine Eqs.(2.5) and Eqs.(2.6), the sequence of $\{a_{nmk}\}$ can be solved. When undetermined coefficient set $\{a_{nmk}\} = \{a_{000}, a_{100}, ..., a_{NNN}\}$ are determined, the approximate solution $\hat{u}(\mathbf{x})$ at any point in the considered domain can be obtained from Eqs.(2.3). To make sure the system of Eqs.(2.5)-(2.6) is solvable, the total number of collocation points N_w has to be larger than $\frac{(N+1)(N+2)(N+3)}{6}$. We will adopt the least squares method to solve Eqs.(2.5)-(2.6). So, the approximate solutions $\hat{u}(\mathbf{x})$ can be given by Eqs.(2.3). In this paper, we use the root mean square error(RMSE) to measure the accuracy of the solutions. It's defined as follows:

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=0}^{N}[\hat{u}(\mathbf{x}_i) - u_{exact}(\mathbf{x}_i)]^2}$$

Lemma 1 Consider a general form second order linear partial differential equation in three variables with constant coefficients:

$$a_1\frac{\partial^2 u_p}{\partial x^2} + a_2\frac{\partial^2 u_p}{\partial y^2} + a_3\frac{\partial^2 u_p}{\partial z^2} + a_4\frac{\partial^2 u_p}{\partial x\partial y} + a_5\frac{\partial^2 u_p}{\partial x\partial z} + a_6\frac{\partial^2 u_p}{\partial y\partial z} + a_7\frac{\partial u_p}{\partial x} + a_8\frac{\partial u_p}{\partial y} + a_9\frac{\partial u_p}{\partial z} + a_{10}u_p = x^n y^m z^k \tag{2.7}$$

where $\{a_i\}_{i=1}^{10}$ are real constants, $a_{10} \ne 0$ and n, m and k are positive integers. Then the polynomial particular solution of Eqs.(2.7) is given by

$$u_p = \frac{1}{a_{10}}\sum_{l=0}^{N}(\frac{-1}{a_{10}})^l L^l(x^n y^m z^k) \tag{2.8}$$

where $N = n + m + k$ and

$$L = a_1\frac{\partial^2}{\partial x^2} + a_2\frac{\partial^2}{\partial y^2} + a_3\frac{\partial^2}{\partial z^2} + a_4\frac{\partial^2}{\partial x\partial y} + a_5\frac{\partial^2}{\partial x\partial z} + a_6\frac{\partial^2}{\partial y\partial z} + a_7\frac{\partial}{\partial x} + a_8\frac{\partial}{\partial y} + a_9\frac{\partial}{\partial z} \tag{2.9}$$

By the method of **Lemma 1**, we can obtain $\Phi_{nmk}(\mathbf{x})$. Similar to **Theorem 1** in [1].So we can easily find the analytical polynomial solutions for any linear ordinary differential operator with constant coefficients.

Proof. Eqs.(2.7) can be written as

$$(L + a_{10}I)u_p = x^n y^m z^k, \tag{2.10}$$

which implies

$$(I + \frac{L}{a_{10}})(a_{10}u_p) = x^n y^m z^k. \tag{2.11}$$

Since L is a differential operator containing various partial derivatives, it is clear that $L^{n+m+k+1}(x^n y^m z^k) = 0$. Hence, the following identity is always true:

$$(I + (\frac{L}{a_{10}})^{N+1})x^n y^m z^k = x^n y^m z^k \tag{2.12}$$

where $N = n + m + k$. By direct algebraic factorization, we have

$$I + (\frac{L}{a_{10}})^{N+1} = (I + \frac{L}{a_{10}}) \sum_{l=0}^{N} (\frac{-1}{a_{10}})^l L^l \tag{2.13}$$

From Eqs.(2.12) and Eqs.(2.13), we have

$$(I + \frac{L}{a_{10}}) \sum_{l=0}^{N} (\frac{1}{a_{10}})^l L^l (x^n y^m z^k) = x^n y^m z^k \tag{2.14}$$

Comparing Eqs.(2.11) and Eqs.(2.14), it follows that

$$a_{10}u_p = \sum_{l=0}^{N} (\frac{1}{a_{10}})^l L^l (x^n y^m z^k). \tag{2.15}$$

Consequently, the particular solution u_p for the above general differential operator is given by

$$u_p = \frac{1}{a_{10}} \sum_{l=0}^{N} (\frac{1}{a_{10}})^l L^l (x^n y^m z^k). \tag{2.16}$$

Using the method of **Lemma 1**, in this dissertation, the differential operator L is equal to Laplace operator Δ. If $a_{10} \neq 0$ rewrite Eqs.(2.1) as follows:

$$(\Delta + 1)u(\mathbf{x}) - u(\mathbf{x}) = f(\mathbf{x}) + F(u, u_{x_1}, u_{x_2}, u_{x_3}, \mathbf{x}) \qquad \mathbf{x} = (x_1, x_2, x_3) \equiv (x, y, z) \in \Omega \tag{2.17}$$

$$\mathscr{B}u(\mathbf{x}) = g(\mathbf{x}) \qquad \mathbf{x} \in \partial\Omega \tag{2.18}$$

with the method of polynomial particular solutions, we approximate the solution to problem Eqs.(2.17)-(2.18) by

$$u(\mathbf{x}) \approx \hat{u}(\mathbf{x}) = \sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{k=0}^{n-m} a_{nmk} \Phi_{nmk}(\mathbf{x}) \quad \mathbf{x} \in \Omega \tag{2.19}$$

3 One dimensional problems

In this part, we consider the following problem of two-point boundary value problem

$$\Delta u(x) = F(u, u_x, x) + f(x) \quad x \in (a, b) \tag{3.1}$$

$$\mathscr{B}u(a) = g(a), \quad \mathscr{B}u(b) = g(b) \tag{3.2}$$

where Δ is a Laplace operator, $F(u, u_x, x)$ is a nonlinear term

3.1 Main algorithm

The application methods of polynomial particular solutions in one-dimensional problems are as follows. Let $\varphi_n(x)$ be some of system of basis functions on $[a, b]$, in this part, we consider the following form:

$$\varphi_n(x) = x^n \qquad n = 1, 2, ...N \tag{3.3}$$

We assume that the approximate solution of Eqs.(3.1)-(3.2) is

$$u(x) \approx \hat{u}(x) = \sum_{n=0}^{N} a_n \Phi_n(x) \qquad x \in (a, b) \tag{3.4}$$

where $\Phi_n(x)$ is obtained by following analytic expression

$$(\Delta + 1)\Phi_n(x) = \varphi_n(x) \qquad 0 \leq n \leq N \tag{3.5}$$

according the method of **Lemma 1**, here $a_{10} = 1$. Then

$$f(x) \approx \sum_{n=0}^{N} a_n \varphi_n(x) - \hat{u}(x) - F(u, u_x, x) \tag{3.6}$$

$$g(x) \approx \mathscr{B}\hat{u}(x) = \sum_{n=0}^{N} a_n \mathscr{B}\Phi_n(x) \tag{3.7}$$

The coefficient sequence $\{a_n\}$ can be obtained by bringing the interior points in (a, b) and boundary point a, b into the Eqs.(3.6) and Eqs.(3.7) respectively, and then the approximate solution can be obtained.

Remark 1: The result matrix obtained using the polynomial particular solutions is extremely ill-conditioned as the polynomial basis function's order increases. Therefore, multi-scale technique is essential to reduce the condition number of the result matrix and improve the numerical accuracy. In this paper, the multi-scale technique is applied to all the numerical examples. However, this article does not provide the process of multi-scale technique.

As for the nonlinear term of $F(u, u_x, x)$, we use the Picard method to carry out the iteration[16]. The process of iteration is as follows:

Algorithm 1

Step 1: Construct a sequence $\{\mathbf{u}^{(i)} : i \in \mathbf{N} \cup \{0\}\}$ such that $\mathbf{u}^{(0)} = 0$ and

$$\Delta \mathbf{u}^{(i+1)} = \mathbf{f}(\mathbf{u}^{(i)}) \quad \text{in } \Omega$$
$$\mathbf{u}^{(i+1)} = \mathbf{g} \quad \text{on } \partial\Omega$$

Step 2: At each iteration, compute $\mathbf{u}^{(i+1)}$ using modified MPS.

Step 3: If $\mid \mathbf{u}^{(i+1)} - \mathbf{u}^{(i)} \mid < \varepsilon$

stop,

end.

Step 4: $\mathbf{u}^{(i+1)}$ at the final iteration is the required approximate solution.

3.2 Numerical implementation and examples

In this part, we give two examples of one-dimensional nonlinear equation, and use the method of polynomial particular solutions to solve them. Let $0 \leq x_1 < x_2 < ... < x_N \leq 1$ be collocation points. In particular we use the following two methods to determine the collocation points:

1. the uniform distribution

$$x_n = \frac{n-1}{N-1} \tag{3.8}$$

2.the Chebishev collocation points

$$x_n = \frac{1}{2}[1 + cos(\frac{\pi(n-1)}{N-1})] \tag{3.9}$$

where N is the number of collocation points.

Example1: In this example, we consider the following nonlinear differential equation with constant coefficients:

$$\Delta u(x) = u^2 + 2\pi^2 cos(2\pi x) - sin^4(\pi x) \qquad x \in (0, 1)$$

$$u(0) = 0, \quad u(1) = 0$$

The analytical solution considered for this problem is

$$u_{exact}(x) = sin^2(\pi x) \quad x \in [0, 1]$$

The number of interior points and test points are 100 and 102, the boundary points are 0 and 1. We compared our results with those obtained by the method proposed in [2]. The data in Table 1 are obtained from [2] and the data in Table 2 are the results obtained by using the method of polynomial particular solutions. The data in the second and third rows correspond to the uniform distribution and Chebyshev's distribution of the collocation points. From Table 1 and Table 2, we can see that the accuracy of the RMSE obtained by using the method of polynomial particular solutions is the same as that obtained using the method in [2]. However, as the order of the polynomial basis function increases, the accuracy of RMSE obtained by the method used in [2] is much higher than the RMSE obtained by using the method of polynomial particular solutions, which is evident in the Table 1 and Table 2. But one important feature of the polynomial particular solutions is the high numerical stability, as shown in Figure 1a is the result with uniform distribution and Figure 1b is the result with Chebishev collocation points, as the order of the polynomial basis function increases, the numerical accuracy remains stable in both collocation point methods.

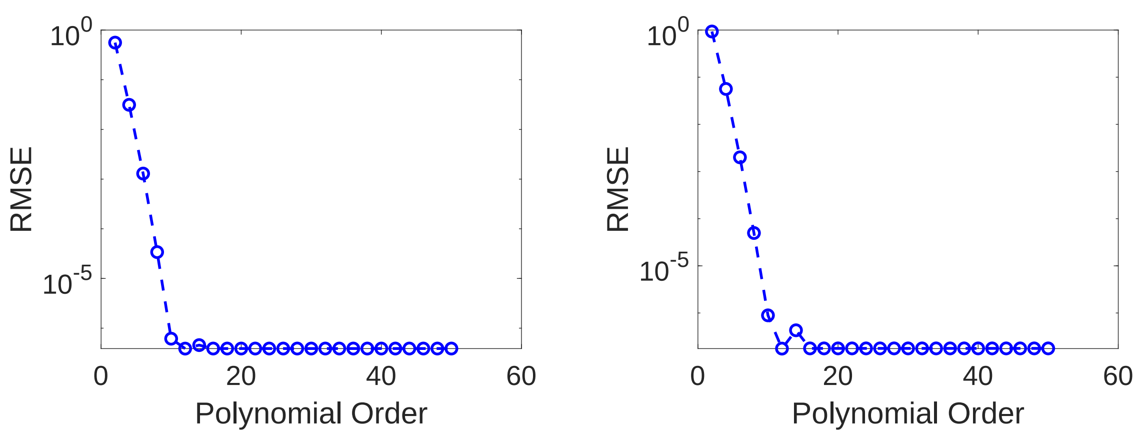
It can be seen that for this example of one-dimensional nonlinear equation, the method of polynomial particular solutions is not ideal, but there are more methods that give better results than the method of polynomial particular solutions.

Table 1: The RMSE obtained by the method proposed in [2].

M	5	10	14	20	30
$RMSE(uni)$	1.8e-03	2.6e-05	2.5e-07	2.0e-10	4.0e-13
$RMSE(Cheb)$	5.8e-03	8.8e-06	2.0e-08	2.2e-11	1.7e-15

Table 2: The RMSE obtained by the method of polynomial particular solutions.

M	5	10	14	20	30
$RMSE(uni)$	1.7e-02	2.6e-05	1.3e-07	1.4e-07	1.4e-07
$RMSE(Cheb)$	7.5e-03	5.4e-06	1.5e-07	1.5e-07	1.5e-07



(a) (b)

Figure 1: (a) Uniform distribution; (b) Chebishev collocation points.

Example2: Consider the following problem:

$$\Delta u(x) + \pi^2 e^{-u} = 0 \qquad x \in (0, 1)$$

$$u(0) = 0, \quad u(1) = 0$$

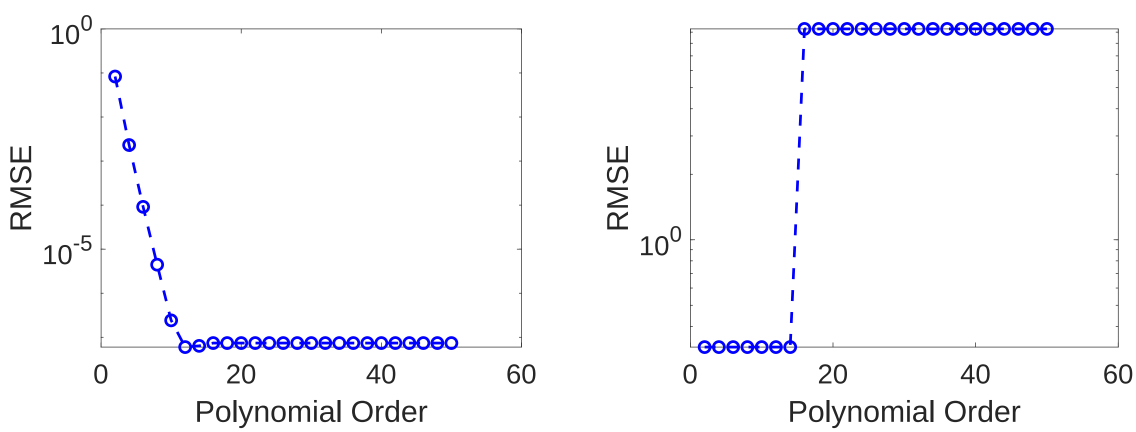
The analytical solution considered for this problem is

$$u_{exact}(x) = ln[1 + sin(\pi x)] \quad x \in [0, 1]$$

The number of interior points and test points are 100 and 102, the number of boundary points are 0 and 1. In this example, we compare the results of two type distributions of collocation point(the uniform distribution and the Chebyshev distribution) at different polynomial basis function's order, Table 3 shows the type of distribution of the collocation points has little impact on the accuracy of the RMSE. In addition, we compare the effects of using multi-scale technique versus not using it on the process of the polynomial particular solutions. As seen in Figure 2a represents the results of using multi-technique, and Figure 2b represents the results of not using multi-scale technique. The accuracy of the RMSE is improved by using this technique, which significantly lowers the condition number of the result matrix. It can be concluded that multi-scale technique plays an important role in the polynomial particular solutions.

Table 3: The RMSE obtained by the uniform distribution and the Chebishev collocation points.

$order$	2	6	10	14	18	22
$RMSE(uni)$	4.08e-02	5.42e-05	1.53e-07	9.78e-08	1.19e-07	1.19e-07
$RMSE(Cheb)$	8.30e-02	9.13e-05	2.42e-07	6.39e-08	7.39e-08	7.39e-08



(a) (b)

Figure 2: (a) with Multi-scale Technique; (b) without Multi-scale Technique.

4 Two dimensional problems

In this part, we consider the two-dimensional nonlinear differential equation of the following form:

$$\Delta u(x,y) = F(u,u_x,u_y,x,y) + f(x,y) \qquad (x,y) \in \Omega \tag{4.1}$$

$$\mathscr{B}u(x,y) = g(x,y) \qquad (x,y) \in \partial\Omega \tag{4.2}$$

where Δ is a Laplace operator, $F(u,u_x,u_y,x,y)$ is a nonlinear term

4.1 Main algorithm

In the two-dimensional case, we consider the basis functions of the following form

$$\varphi_{nm}(x,y) = x^{n-m}y^m \qquad n = 1,2,...N \quad m = 1,2...n \tag{4.3}$$

we assume that the approximate solution of Eqs.(4.1)-(4.2) is

$$u(x,y) \approx \hat{u}(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{n} a_{nm}\Phi_{nm}(x,y) \qquad (x,y) \in \Omega \tag{4.4}$$

where $\Phi_{nm}(x,y)$ is obtained by following analytic expression

$$(\Delta + 1)\Phi_{nm}(x,y) = \varphi_{nm}(x,y) \tag{4.5}$$

that is, according the method of **Lemma 1**, here $a_{10} = 1$. Then

$$f(x,y) \approx \sum_{n=0}^{N}\sum_{m=0}^{n} a_{nm}\varphi_{nm}(x,y) - \hat{u}(x,y) - F(u,u_x,u_y,x,y) \tag{4.6}$$

$$g(x,y) \approx \mathscr{B}\hat{u}(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{n} a_{nm}\mathscr{B}\Phi_{nm}(x,y) \quad (x,y) \in \partial\Omega \tag{4.7}$$

Then, obtain the interior points in the domain of Ω and the boundary points on the $\partial\Omega$, and bring them into Eqs.(4.6) and (4.7) respectively to obtain the value of the undetermined coefficient $\{a_{nm}\}$.

For the nonlinear terms of $F(u,u_x,u_y,x,y)$, we also used the Picard method to carry out the iteration. The iteration is as **Algorithm 1**.

4.2 Numerical implementation and examples

In this section, we will solve three numerical examples, for these numerical experiments, we consider several domains including regular and irregular geometries. The boundary of the domain is defined as follows:

$$\rho(\theta) = 0.3\sqrt{cos(2\theta) + \sqrt{1.1 - sin^2(2\theta)}} \qquad 0 \leq \theta \leq 2\pi \tag{4.8}$$

is the peanut-shaped domain

$$\rho(\theta) = 1 + \frac{1}{10}tan(10sin(12\theta)) \qquad 0 \leq \theta \leq 2\pi \tag{4.9}$$

is the gear-shaped domain

$$\rho(\theta) = 1 + cos^2(4\theta) \qquad 0 \leq \theta \leq 2\pi \tag{4.10}$$

is the star-shaped domain

$$x_1 = 1.5cos(\theta) \qquad x_2 = sin(\theta) \qquad 0 \leq \theta \leq 2\pi \tag{4.11}$$

is the ellipse domain

Example3: Consider the following nonlinear differential equation:

$$\Delta u(x, y) = 4u^4(x, y) \qquad (x, y) \in \Omega$$

$$u(x, y) = u_{exact}(x, y) \qquad (x, y) \in \partial\Omega$$

The analytical solution considered for this problem is

$$u_{exact}(x, y) = \frac{1}{4 + x + y} \qquad (x, y) \in \overline{\Omega}$$

In this example, the computational domain is the peanut-shaped from Eqs.(4.9) The number of interior points, boundary points and test points are 217, 150 and 367 respectively. We use the polynomial particular solutions to solve this problem. Figure 3a shows the change of the RMSE as the polynomial basis function's order gradually increases. We can observe as the order of polynomial basis function increases, the numerical accuracy of RMSE gradually stabilizing, and the accuracy of RMSE reaches a stable state when the order reaches about 10. Therefore, we decide to move the entire computational domain by d units when order=10.

In Table 4, the M in the first row represents the number of nonlinear equations, the data in the second row is obtained from the monomial method in [2] and the data in the third row is obtained by the method of polynomial particular solutions. It can be seen that the results obtained by the method of polynomial particular solutions have higher accuracy than the results obtained from [2]. The data in the third row are the RMSEs for d=0,6,31,50 respectively. Obviously, the RMSEs obtained for different d are not same, and the accuracy of RMSE obtained when d=0 is the lowest, that is, the accuracy of the RMSE can be improved by moving the computational domain appropriately. However, as the moving unit d rises, there is no discernible pattern in the change of RMSE. Figure 3b shows the variation of RMSE with increasing d for polynomial basis function of order=10. When a suitable value of d is obtained, the RMSE is more sensitive and can reach a high accuracy.

Table 4: The RMSE obtained by the monomials and polynomial method.

M	10	21	36
$RMSE(mon)$	3.7e-09	9.1e-10	7.4e-11
$RMSE(poly \quad d=0)$	2.21e-11	2.23e-11	2.23e-11
$RMSE(poly \quad d=6)$	3.01e-13	4.03e-13	5.48e-11
$RMSE(poly \quad d=31)$	7.78e-13	1.96e-16	4.02e-17
$RMSE(poly \quad d=50)$	8.99e-14	1.74e-13	2.99e-13

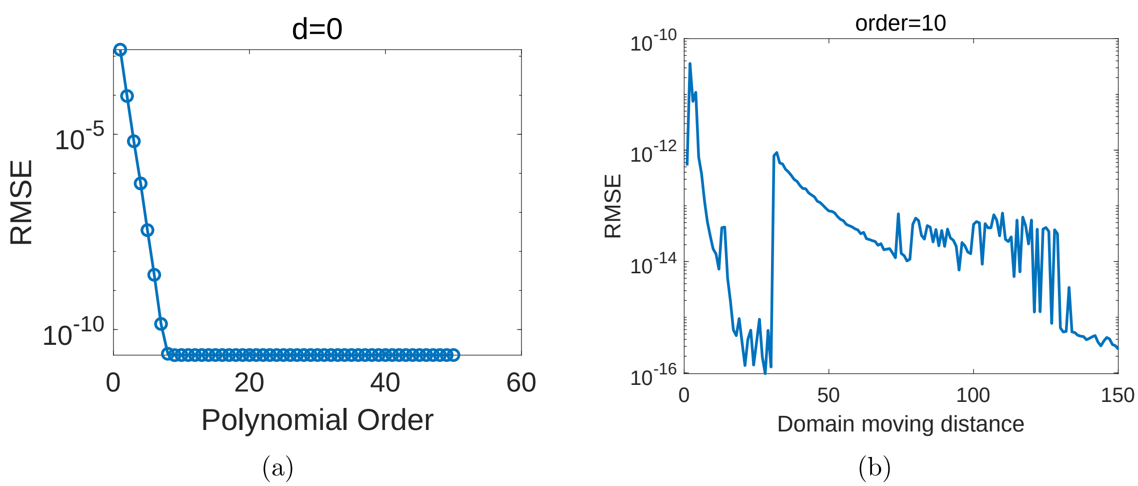


Figure 3: (a) RMSE versus the polynomial order using polynomial basis; (b) RMSE versus the domain moving distance(order=10).

Example4: Consider the following problem:

$$\Delta u(x,y) = u^2(x,y) + 6x - x^6 - 4x^4y - 4x^2y^2 \qquad (x,y) \in \Omega$$

$$u(x,y) = u_{exact}(x,y) \qquad (x,y) \in \partial\Omega$$

The analytical solution considered for this problem is

$$u_{exact}(x,y) = x^3 + 2xy \qquad (x,y) \in \overline{\Omega}$$

In this example, we select both regular domains(ellipse, square) and irregular domains(peanut-shaped, star-shaped, gear-shaped) as computational domain. We also use the method of polynomial particular solutions to solve this problem. Figure 4 shows the variation of the RMSE with increasing polynomial basis function's order in five different computational domains. It can be seen that the RMSE gradually tends to be stable and has high stability as the order of polynomial basis function increases. However, the accuracy of the RMSE achieved varies for different computational domains, as shown in Table 5. It can be observed that among the five different domains we selected, the accuracy of the RMSE obtained in peanut-shaped domain is the highest. That is, the selection of computational domain has a certain impact on the accuracy of the results.

For this example, the analytical solution does not contain singularities, we also moved its computational domain, but discovered that performing calculations using the relocated computational domain produced no results.

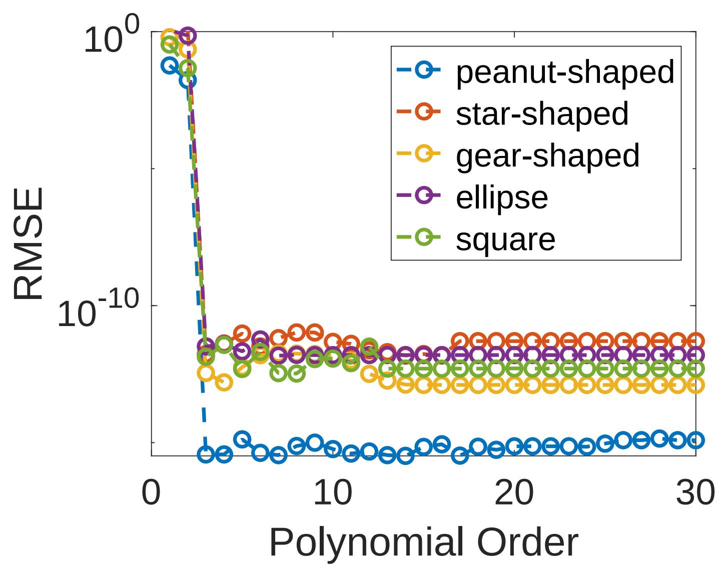


Figure 4: RMSE versus the polynomial order using polynomial basis function in five different computational domains.

Table 4: The RMSE obtained in five different domains.

domain/order	5	15	25
$peanut-shaped$	5.39e-16	8.77e-16	6.81e-16
$gear-shaped$	6.10e-13	1.29e-13	1.30e-13
$star-shaped$	9.45e-13	1.49e-12	4.59e-12
ellipse	2.29e-12	1.69e-12	1.69e-12
square	4.78e-13	5.23e-13	5.24e-13

Example5: Consider the following problem:

$$\Delta u(x,y) = 4u^3(x,y) - (\frac{\partial u}{\partial x})^2 - (\frac{\partial u}{\partial y})^2 + \frac{2}{(4+x+y)^4} \qquad (x,y) \in \Omega$$

$$u(x,y) = u_{exact}(x,y) \qquad (x,y) \in \partial\Omega$$

The analytical solution considered for this problem is

$$u_{exact}(x,y) = \frac{1}{4+x+y} \qquad (x,y) \in \overline{\Omega}$$

For this example, singularities included in the analytical solution. Using the method of polynomial particular solutions to solve this problem. If the computational domain is not moved, no result will be obtained as the order increases. Therefore, in order to obtain high precision results, we move the computational domain by d units. Figure 5 shows the variation of the RMSE with increasing of polynomial basis function's order in five different computational domains(same as Example 4) when d=100. As can be seen from Figure 5, the accuracy of the results varies in five different computational domains, that is, for equations containing singularities in the analytical solution, the accuracy of the RMSE is also influenced by the shape of computational domain. Additionally, we compared the RMSEs obtained using polynomial and monomial method[2] in elliptic domain, as shown in Table 5, where the first two rows are the RMSEs obtained using the monomial method and the last two rows are the RMSEs obtained using the polynomial method after shifting the computational domain by an appropriate distance. As can be seen from Table 5, after appropriately moving the computational domain, the accuracy of the RMSE obtained by using the method of polynomial particular solutions is higher than that obtained by using the monomial method.

Table 5: The RMSE obtained by the monomials and polynomial method in ellipse domain.

M	15	36	66
$RMSE(mon)$	2.3e-04	1.0e-05	5.3e-08
$order$	14	35	50
$RMSE(poly)$	1.34e-09	1.57e-09	4.39e-09
	d=127	d=194	d=179

5 Three dimensional problems

Applications in 3D:

$$\Delta u(x,y,z) = F(u, u_x, u_y, u_z, x, y, z) + f(x,y,z) \qquad (x,y,z) \in \Omega \tag{5.1}$$

$$\mathscr{B}u(x,y,z) = g(x,y,z) \qquad (x,y,z) \in \partial\Omega \tag{5.2}$$

where Δ is a Laplace operator, $F(u, u_x, u_y, u_z, x, y, z)$ is a nonlinear term

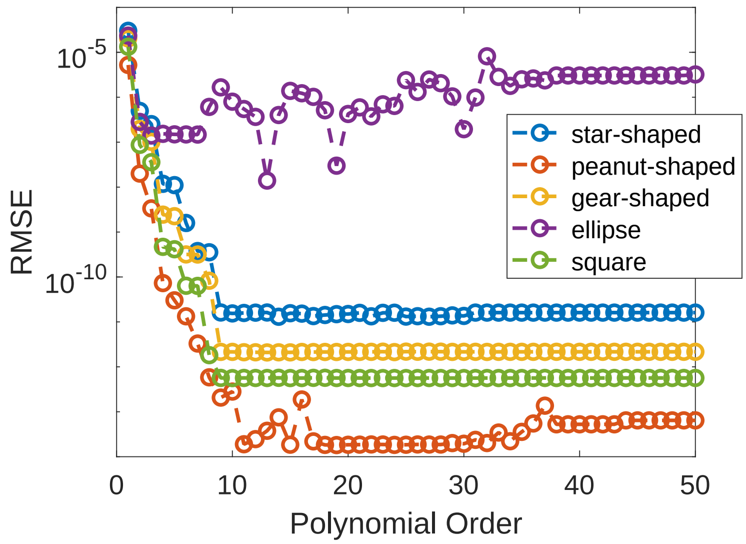


Figure 5: RMSE versus the polynomial order using polynomial basis in five different solution domains, d=100

5.1 Main algorithm

For the calculation of three-dimensional problems, the main process is as follows. Let φ_{nmk} be the basis function on Ω, in this section, we consider the following form:

$$\varphi_{nmk}(x,y,z) = x^{n-m-k}y^m z^k \quad n = 0,1,2...N \quad m = 0,1,2...n \quad k = 0,1,2...n-m \tag{5.3}$$

we assume that the approximate solution of Eqs.(5.1)-(5.2) is

$$u(x,y,z) \approx \hat{u}(x,y,z) = \sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{nmk}\Phi_{nmk}(x,y,z) \quad (x,y,z) \in \Omega \tag{5.4}$$

where $\Phi_{nmk}(x,y,z)$ is obtained by following analytic expression

$$(\Delta + 1)\Phi_{nmk}(x,y,z) = \varphi_{nmk}(x,y,z) \tag{5.5}$$

according to the method of **Lemma 1**, here $a_{10} = 1$. Then

$$f(x,y,z) \approx \sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{nmk}\varphi_{nmk}(x,y,z) - \hat{u}(x,y,z) - F(u,u_x,u_y,u_z,x,y,z) \tag{5.6}$$

$$g(x,y,z) \approx \mathscr{B}\hat{u}(x,y,z) = \sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{nmk}\mathscr{B}\Phi_{nmk}(x,y,z) \tag{5.7}$$

Then, by bringing the interior points in Ω and the boundary points on $\partial\Omega$ into Eqs.(5.6) and Eqs.(5.7), we obtain the undetermined coefficient sequence $\{a_{nmk}\}$.

For the nonlinear term of $F(u, u_x, u_y, u_z, x, y, z)$, we still using the iteration of the Picard method.

5.2 Numerical implementation and examples

In this part, we use the method of polynomial particular solutions to solve the following two numerical examples

Example6: Consider the following nonlinear differential equation:

$$\Delta u(x,y,z) = \frac{2}{u(x,y,z)} + \frac{3}{u^3(x,y,z)} \qquad (x,y,z) \in \Omega$$

$$u(x,y,z) = u_{exact}(x,y,z) \qquad (x,y,z) \in \partial\Omega$$

The analytical solution considered for this problem is

$$u_{exact}(x,y,z) = \sqrt{3 + x^2 + y^2 + z^2} \qquad (x,y,z) \in \overline{\Omega}$$

In this example, we select sphere as the computational domain and the number of interior points and boundary points are 2103 and 470. Figure 6 shows the variation of the RMSE with polynomial basis function's order. From Figure 6, it can be seen that the accuracy of the RMSE increases with the order of the polynomial basis function and the RMSE reaches stability when the polynomial basis function's order reaches a certain value. Moving the computational domain of a three-dimensional problem without singularity in the analytical solution is ineffective, just like in the two-dimensional case.

Example7: Consider the following nonlinear differential equation:

$$\Delta u(x,y,z) = 6u^3 - (\frac{\partial u}{\partial x})^2 - (\frac{\partial u}{\partial y})^2 - (\frac{\partial u}{\partial z})^2 + \frac{3}{(4+x+y+z)^4} \qquad (x,y,z) \in \Omega$$

$$u(x,y,z) = u_{exact}(x,y,z) \qquad (x,y,z) \in \partial\Omega$$

The analytical solution considered for this problem is

$$u_{exact}(x,y,z) = \frac{1}{4+x+y+z} \qquad (x,y,z) \in \overline{\Omega}$$

In this numerical example, we select two regular space domain: Cubic domain and Sphere domain as shown in Figure 7(a)-(b), two irregular space domain: the Bumpy-shaped domain and magnify the size of the Stanford Bunny domain, as shown in Figure 7(c)-(d). For this equation with singularity in the analytical solution, as in the two-dimensional case, the results is not obtained when the space domain is not moved. When

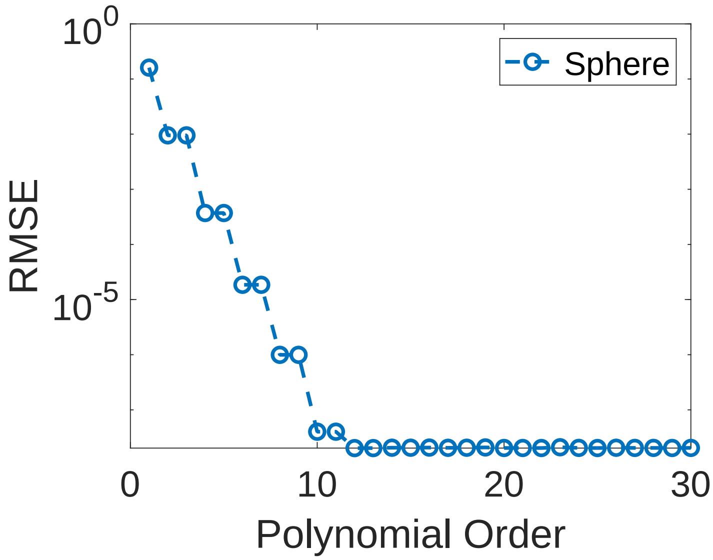
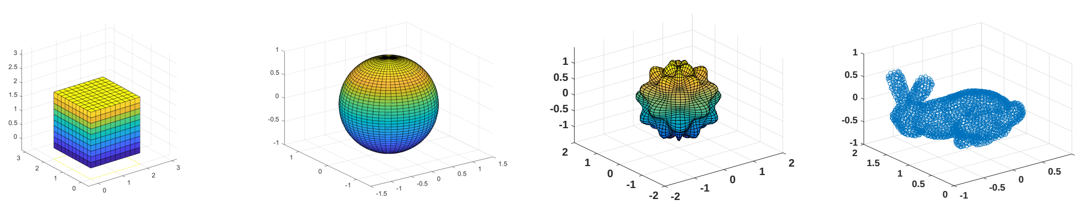


Figure 6: RMSE versus the polynomial order using polynomial basis function in Sphere domain

the space domain is appropriately moved, high accuracy results can be obtained. This result proves that for the three-dimensional case with singularities in the analytic solution, moving the space domain by an appropriate distance can improve the accuracy of the results. After moving the space domain by d=50 units in four different space domains, Figure 8 shows the variation of RMSE with the order of the polynomial basis function. The figure illustrates how the RMSE obtained varies for different space domain, that is the three-dimensional case is similar to the two-dimensional case, and the selection of space domain affects the accuracy of the RMSE.



(a) (b) (c) (d)

Figure 7: (a) Cubic domain; (b) Sphere domain; (c) Bumpy-Shaped domain; (d) Stanford Bunny domain

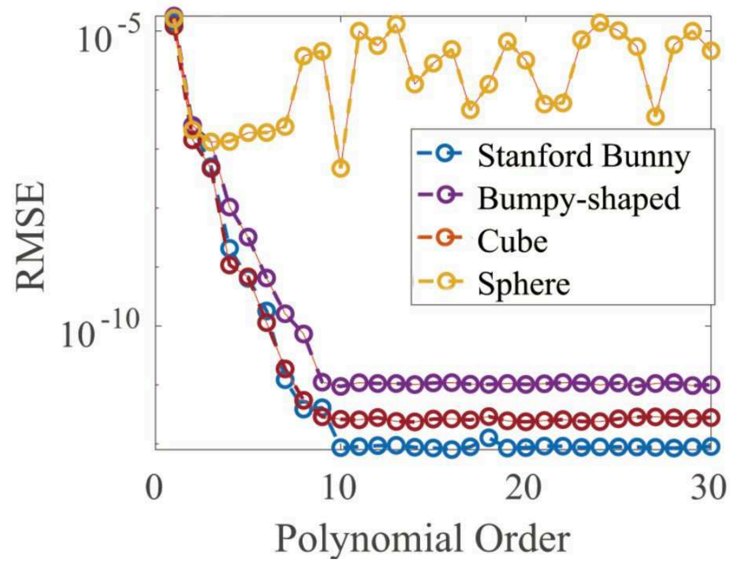


Figure 8: RMSE versus the polynomial order using polynomial basis in four space domain, d=50

6 Conclusion

In this paper, we used the method of polynomial particular solutions to solve one-, two- and three-dimensional nonlinear partial differential equation of Poisson type, and the RMSE is used to measure the accuracy of the results. From the results, it can be seen that the application of polynomial particular solutions is superior to the method in [2] in both two-dimensional and three-dimensional. However, the accuracy of the results is not as high as the method used in [2] for one-dimensional case when use the method of polynomial particular solutions to solve nonlinear equations of Poisson type.

In addition, we move the computational domain by d units and use polynomial particular solutions to solve two-dimensional and three-dimensional problems with singularities in the analytical solution. When d is appropriate, the calculation results can achieve high accuracy. That is the accuracy of the results for equations with singularities in the analytical solution can be greatly increased by appropriately moving the computational domain and using polynomial particular solutions to the problem. This method can be extended to high-dimensional nonlinear partial differential equation with singularities in the analytical solution, this will be the subject of further studies.

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