Formalizing Category Theory in Agda

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- study category theory;
- study the proof assistant;
- study other fields using the formalized category theory.

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libraries	proof assistants	foundation	LoC
[Peebles et al., 2018]	Agda 2.5.2	MLTT + K + irrelevance	11770
[Timany and Jacobs, 2016]	Coq 8.11.1	CIC	14711
[Wiegley, 2019]	Coq 8.10.2	CIC	23003
[Huet and Saïbi, 2000]	Coq 8.12.0	CIC	7879
[Voevodsky et al., , Ahrens et al., 2015]	Coq 8.12.0	HoTT	96366
[Gross et al., 2014]	Hoq 8.12	HoTT with HIT	10604
[mathlib Community, 2020]	Lean	CIC	14975
[Stark, 2016, Stark, 2017, Stark, 2020]	Isabelle	HOL	82782

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 - Prove more properties

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 - Most of basic definitions
 - Many lemmas and theorems
- A decent amount of enriched category theory and higher category theory

It Implements

Just to name a few (non-exhaustively):

- Concepts:
 - category, functor, natural transformation, adjoint functors;
 - various monoidal categories, cartesian closed category, comma category,
 - initial / terminal, (co)product, (co)end, etc.

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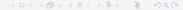
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- Constructions:
 - the category of categories, of set(oid)s, of presheaves, of monoidal categories, etc.
- Properties:
 - the Yoneda lemma,
 - Freyd's adjoint functor theorem,
 - Lambek's lemma,
 - Right adjoints preserve limits,
 - (local) cartesian closure of Setoids,
 - etc.



Basic Design Principles

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- Hom-setoids
- universe polymorphism
- definitional duality
- records for encapsulation
- predicate versus structure

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```
record Category (o ℓ e : Level) : Set (suc (o □ ℓ □ e)) where
   field
      Obi : Set o
      \Rightarrow_ : (A B : Obj) \rightarrow Set \ell
      \circ : \forall {A B C} \rightarrow B \Rightarrow C \rightarrow A \Rightarrow B \rightarrow A \Rightarrow C
      \approx : \forall {A B} \rightarrow (f g : A \Rightarrow B) \rightarrow Set e
      equiv : \forall \{A B\} \rightarrow IsEquivalence (= \approx \{A\} \{B\})
      -- ignore other laws
```

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Many concepts, e.g. Monad and NaturalTransformation, require similar additional laws.



Duality: Redundant Definitions

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- Dual concepts are defined individually makes their end-use considerably clearer.
 - products and coproducts
 - monads and comonads
- Conversions between duals are defined in *.Duality.
 - Help us to ensure we got the definition right.

```
Comonad⇔coMonad : ∀ (M : Comonad C) →
  coMonad⇒Comonad (Comonad⇒coMonad M) ≡ M
Comonad⇔coMonad _ = ≡.ref1
The proof body must be ≡.ref1.
```

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- record Monad {o ℓ e} (C : Category o ℓ e) : Set (o \sqcup ℓ \sqcup e) where field
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 - η : NaturalTransformation idF F

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Given a Monad M,
```

we can have

```
 \begin{array}{ll} Functor.F_0 & (\texttt{Monad}.F \ \texttt{M}) \\ Functor.F_1 & (\texttt{Monad}.F \ \texttt{M}) \end{array}
```

 ${\tt NaturalTransformation.commute}~({\tt Monad}.\eta~{\tt M})~{\tt f}$

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■ record Monad {o \ell e} (C : Category o \ell e) : Set (o \sqcup \ell \sqcup e) where
    field
      F : Endofunctor C
      η : NaturalTransformation idF F
    module F = Functor F
    module \eta = NaturalTransformation \eta
  Given a Monad M.
                                                       we can have
  Functor.Fo (Monad.F M)
  Functor.F<sub>1</sub> (Monad.F M)
  NaturalTransformation.commute (Monad.\eta M) f
```

- We use records to encode concepts; plays well with the record modules feature of Agda.
- record Monad $\{o \ \ell \ e\}$ (C : Category $o \ \ell \ e$) : Set $(o \sqcup \ell \sqcup e)$ where field F : Endofunctor C η : NaturalTransformation idF F module F = Functor Fmodule η = NaturalTransformation η Given a Monad M. after declaring module M = Monad M. we can have M.F.Fo M.F.F₁ $M.\eta.$ commute f

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 - Predicate: "is-a" relation;
 - Structure: "has-a" relation.
- This choice is fundamental in systems with automated search machinery.
- In Agda, more about usability and namespace management.
- In fact, we see benefits in combining both choices!

```
record Monoidal {o ℓ e}
(C : Category o ℓ e)
: Set (o ⊔ ℓ ⊔ e) where
```

```
record MonoidalCategory o ℓ e
: Set (suc (o ⊔ ℓ ⊔ e)) where
field
U : Category o ℓ e
monoidal : Monoidal U
```

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- But MonoidalCategory is more convenient for:

record MonoidalFunctor

```
(C : MonoidalCategory o ( e)
```

```
(D : MonoidalCategory o' \ell' e')
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 - predicate: properties of one instance of the concept.
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```
record MonoidalFunctor record MonoidalFunctor'  (\texttt{C} : \texttt{MonoidalCategory} \ o \ \ell \ e) \qquad \{\texttt{C} : \texttt{Category} \ o \ \ell \ e\} \ \{\texttt{D} : \texttt{Category} \ o' \ \ell' \ e'\}   (\texttt{MC} : \texttt{Monoidal} \ \texttt{C}) \qquad (\texttt{MD} : \texttt{Monoidal} \ \texttt{D})   : \texttt{Set} \ \_ \ \texttt{where}   : \texttt{Set} \ \_ \ \texttt{where}
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More type theoretic constructs:

- Picking type theoretic definitions
- natural isos *between Hom-sets* as adjunctions + mates
- set theoretic quantification as adjoint equivalence
 - finite categories



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record Adjoint {C : Category o \ell e} {D : Category o' \ell' e'}
      (L : Functor C D) (R : Functor D C) : Set _ where
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```
record Adjoint {C : Category o \ell e} {D : Category o' \ell' e'} (L : Functor C D) (R : Functor D C) : Set _ where L \dashv R iff Hom_D(L-,-) \simeq Hom_C(-,R-).
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- $lue{}$ Universe levels of C and D are unrelated \Rightarrow Hom functors cannot be directly related.
 - (Ugly) use lifting functors:

$$Lift \circ Hom_{D}(L-,-) \simeq Lift \circ Hom_{C}(-,R-)$$

Consider adjoint functors:

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■ This form of lifting is required in many definitions / statements in general.

Unit-Counit Definition of Adjoint Functors

• We instead use the unit-counit definition of Adjoint functors:

Definition

Functors $L: \mathcal{C} \Rightarrow \mathcal{D}$ and $R: \mathcal{D} \Rightarrow \mathcal{C}$ are adjoint, $L \dashv R$, if there exist two natural transformations, unit $\eta: 1_C \Rightarrow RL$ and counit $\epsilon: LR \Rightarrow 1_D$, so that the triangle identities below hold:

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- Lesson: unlearn set-theoretic constructs when formalizing categories in type theory!



Mate: Relating Two Adjunctions

■ We sometimes need to relate two adjunctions $F \dashv G$ and $F' \dashv G'$:

$$Hom(F'X, Y) \xrightarrow{\simeq} Hom(X, G'Y)$$
 $Hom(\alpha_X, Y) \downarrow \qquad \qquad \downarrow Hom(X, \beta_Y)$
 $Hom(FX, Y) \xrightarrow{\simeq} Hom(X, GY)$

where $\alpha: F \Rightarrow F'$ and $\beta: G' \Rightarrow G$.

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■ It also has a definition purely in terms of morphism equality:

Definition

Two natural transformation $\alpha: F \Rightarrow F'$ and $\beta: G' \Rightarrow G$ form a mate for two pairs of adjunctions $(\eta, \epsilon): F \dashv G$ and $(\eta', \epsilon'): F' \dashv G'$, if the following two diagrams commute:

$$\begin{array}{cccc}
\mathbf{1}_{C} & \xrightarrow{\eta} & GF & FG' & \xrightarrow{\alpha G'} F'G' \\
\eta' \downarrow & & \downarrow G\alpha & F\beta \downarrow & \downarrow \epsilon' \\
G'F' & \xrightarrow{\beta F'} & GF' & FG & \xrightarrow{\epsilon} & \mathbf{1}_{\mathcal{D}}
\end{array}$$

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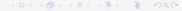


Example: Closed Monoidal Categories

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- This definition of closed monoidal categories has no universe level issues

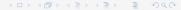


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 - for $f_i: Y_i \Rightarrow Y_i', i \in [1, n]$, $\alpha: L(-, f_1, \dots, f_n)$ and $\beta: R(f_1, \dots, f_n, -)$ form a mate \Rightarrow naturality in all Y_i .



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 - Using both, regain the naturality of all X, Y_i and Z without using anything set theoretic.

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where Obj is the type representing objects and Bijection is a predicate showing a bijection between Obj and Fin n.

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- There are problems:
 - We are forced to talk about equality between objects.
 - We are implicitly assuming objects are a set.



Adjoint Equivalence

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Definition

Two categories $\mathcal C$ and $\mathcal D$ are adjoint equivalent if there are two functors $L:\mathcal C\to\mathcal D$ and $R:\mathcal D\to\mathcal C$ forming an adjunction $L\dashv R$ where the unit and counit are natural isomorphisms.

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- We do not want to talk about the finiteness of *any* type as it is not categorical.
- However, we can do so if a concrete type is given.
- A finite diagram is a special category in which objects and morphisms are finite:

Definition

Given $n : \mathbb{N}$ as the number of objects and a function $|a, b| : \mathbb{N}$ for a, b : Fin n, a finite diagram is a category with

- Fin n as objects, and
- 2 Fin |a, b| as morphisms for a, b : Fin n.

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 - if a finite diagram is the index category of some (co)limit, then adjoint equivalence demonstrates an isomorphic (co)limit with a general finite category as the index.
- One could consider other notions of equivalence depending on the purpose.

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- Check our paper for more discussions!

Bibliography I



Ahrens, B., Kapulkin, K., and Shulman, M. (2015).

Univalent categories and the rezk completion.

Mathematical Structures in Computer Science, 25(5):1010-1039.



Gross, J., Chlipala, A., and Spivak, D. I. (2014).

Experience implementing a performant category-theory library in cog.

In Klein, G. and Gamboa, R., editors, Interactive Theorem Proving - 5th International Conference, ITP 2014, Held as Part of the Vienna Summer of Logic, VSL 2014, Vienna, Austria, July 14-17, 2014. Proceedings, volume 8558 of Lecture Notes in Computer Science, pages 275–291. Springer.



Huet, G. P. and Saïbi, A. (2000).

Constructive category theory.

In Plotkin, G. D., Stirling, C., and Tofte, M., editors, *Proof, Language, and Interaction, Essays in Honour of Robin Milner*, pages 239–276. The MIT Press



mathlib Community, T. (2020).

The lean mathematical library.

In Blanchette, J. and Hritcu, C., editors, Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs, CPP 2020, New Orleans, LA, USA, January 20-21, 2020, pages 367–381. ACM.



Peebles, D., Deikun, J., Norell, U., Doel, D., Jahandarie, D., and Cook, J. (2018).

categories: Categories parametrized by morphism equality in agda.



Bibliography II



Stark, E. W. (2016).

Category theory with adjunctions and limits.

Archive of Formal Proofs.

http://isa-afp.org/entries/Category3.html, Formal proof development.



Stark, E. W. (2017).

Monoidal categories.

Archive of Formal Proofs

http://isa-afp.org/entries/MonoidalCategory.html, Formal proof development.



Stark, E. W. (2020).

Bicategories.

Archive of Formal Proofs.

http://isa-afp.org/entries/Bicategory.html, Formal proof development.



Timany, A. and Jacobs, B. (2016).

Category theory in cog 8.5.

In Kesner, D. and Pientka, B., editors, 1st International Conference on Formal Structures for Computation and Deduction, FSCD 2016, June 22-26, 2016, Porto. Portugal, volume 52 of LIPIcs, pages 30:1-30:18, Schloss Dagstuhl - Leibniz-Zentrum für Informatik.



Voevodsky, V., Ahrens, B., Grayson, D., et al.

UniMath — a computer-checked library of univalent mathematics. available at https://github.com/UniMath/UniMath.



Wiegley, J. (2019).

category-theory: Category theory in coq.

