Assume $fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$

So

$$fib(n+1) = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}} = \frac{\varphi \cdot \varphi^n - \psi \cdot \psi^n}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} \cdot \varphi^n - \frac{1-\sqrt{5}}{2} \cdot \psi^n}{\sqrt{5}} = \frac{1}{2}\left(\frac{\varphi^n - \psi^n}{\sqrt{5}} + \varphi^n + \psi^n\right)$$

$$= \frac{1}{2} fib(n) + \frac{\varphi^n + \psi^n}{2}$$

$$fib(n+2) = \frac{1}{2} fib(n+1) + \frac{\varphi^{n+1} + \psi^{n+1}}{2}$$

$$= \frac{1}{4} fib(n) + \frac{\varphi^n + \psi^n}{4} + \frac{\varphi^{n+1} + \psi^{n+1}}{2} = \frac{1}{4} fib(n) + \frac{\varphi^n + \psi^n}{4} + \frac{\varphi^n \cdot (1+\sqrt{5})}{4} + \frac{\psi^n \cdot (1-\sqrt{5})}{4}$$

$$= \frac{1}{4}(fib(n) + 2(\varphi^n + \psi^n) + 5fib(n) = \frac{3}{2} fib(n) + \frac{\varphi^n + \psi^n}{2}$$

$$= fib(n) + fib(n+1)$$

Fib(0) = $\frac{\varphi^0 - \psi^0}{\sqrt{5}}$ = 0

Fib(1) = $\frac{\varphi^1 - \psi^1}{\sqrt{5}}$ = 1

That's prove $fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$ is right.

So:

$$fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} = \frac{\varphi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}}$$

$$\frac{1}{\sqrt{5}} < \frac{1}{2}$$

$$|\psi| = \left|\frac{1-\sqrt{5}}{2}\right| < 1$$

$$\left|\frac{\psi^n}{\sqrt{5}}\right| < \frac{1}{2}$$

$$\left|fib(n) - \frac{\varphi^n}{\sqrt{5}}\right| < \frac{1}{2}$$

So fib(n) is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$.