Tsinghua Berkeley Shenzhen Institute (TBSI)
ME233 Advanced Control Systems II
Spring 2024

Homework #4

Assigned: April 8 (Monday)
Due: April 16 (Tuesday)

1. Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} (u(k) + w(k)) \tag{1}$$
$$y(k) = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + v(k)$$

where $u(k)$ is a deterministic *known* input and $x(0)$ is Gaussian and $w(k)$ and $w(k)$ are zero-mean stationary Gaussian sequences (i.e. **white**), with the following statistics:

- $x_o = E\{x(0)\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\tilde{x}(0) = x(0) - x_0$ and $X_o = E\{\tilde{x}(0)\tilde{x}^T(0)\} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$
- $E\{w(k)\} = 0$, $\tilde{w}(k) = w(k) - m_w$, $E\{v(k)\} = 0$, and $\tilde{v}(k) = v(k)$.
- $E\left\{ \begin{bmatrix} \tilde{w}(k+j) \\ \tilde{v}(k+j) \end{bmatrix} \begin{bmatrix} \tilde{w}(k) & \tilde{v}(k) \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(j) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \delta(j)$
- $E\left\{ \begin{bmatrix} \tilde{w}(k) \\ \tilde{v}(k) \end{bmatrix} \tilde{x}^T(0) \right\} = 0$

Notice that in this system, the control input $u(k)$ and the input noise $w(k)$ are injected at the same location, i.e. $B = B_w$. This is not necessarily the case in many systems.

(a) Assume that the control input $u(k)$ is a given deterministic sequence. Obtain the state equations that define the **marginal** expected value of the state and the output equation for the **marginal** expected value of the output

$$\begin{aligned} m_x(k) &= \begin{bmatrix} m_{x_1}(k) \\ m_{x_1}(k) \end{bmatrix} = \begin{bmatrix} E\{x_1(k)\} \\ E\{x_1(k)\} \end{bmatrix} \\ m_y(k) &= E\{y(k)\} \end{aligned}$$

(b) Define the **marginal** state and output estimation errors by

$$\begin{aligned} \tilde{X}(k) &= x(k) - m_x(k) \\ \tilde{Y}(k) &= y(k) - m_y(k) \end{aligned}$$

(Notice that we are using capital letters to distinguish the marginal estimation errors from the Kalman filter a-posteriori estimation errors, which will be written using lower case letters.)

The state and output equations for the **marginal** state and output estimation errors can be written as

$$\begin{aligned} \tilde{X}(k+1) &= A\tilde{X}(k) + Bw(k) \\ \tilde{Y}(k) &= C\tilde{X}(k) + v(k) \end{aligned} \tag{2}$$

i. The marginal state and output estimation covariances are

$$\begin{aligned} \Lambda_{\tilde{X}\tilde{X}}(k,j) &= E\{\tilde{X}(k+j)\tilde{X}^T(k)\} \\ \Lambda_{\tilde{Y}\tilde{Y}}(k,j) &= E\{\tilde{Y}(k+j)\tilde{Y}^T(k)\} \end{aligned}$$

As discuss in class $\Lambda_{\tilde{X}\tilde{X}}(k,0)$ evolves according to the following Lyapunov equation (remember that $B = B_w$)

$$\Lambda_{\tilde{X}\tilde{X}}(k+1,0) = A\Lambda_{\tilde{X}\tilde{X}}(k,0)A^T + B_w W B_w^T \qquad \Lambda_{\tilde{X}\tilde{X}}(0,0) = X_o\,. \tag{3}$$

Write a matlab code that will compute $\Lambda_{\tilde{X}\tilde{X}}(k,0)$, $\Lambda_{\tilde{Y}\tilde{Y}}(k,0)$ and

$$E\{\tilde{X}^T(k)\tilde{X}(k)\} = \text{trace}(\Lambda_{\tilde{X}\tilde{X}}(k,0))$$

recursively forward in time, starting from $k=0$ until $k=20$.

ii. **Plot $E\{\tilde{X}^T(k)\tilde{X}(k)\}$ and $\Lambda_{\tilde{Y}\tilde{Y}}(k,0)$ as a function of k.**

iii. Since that the input noise $w(k)$ and measurement noise $v(k)$ are white, the marginal estimation error dynamics in Eq. (2) will converge to a stationary system. As a consequence,

$$\lim_{k\to\infty} \Lambda_{\tilde{X}\tilde{X}}(k,0) = \bar{\Lambda}_{\tilde{X}\tilde{X}}(0)\,.$$

where the steady state marginal state estimation covariance, $\bar{\Lambda}_{\tilde{X}\tilde{X}}(0)$, is the solution of the following algebraic Lyapunov equation

$$A\bar{\Lambda}_{\tilde{X}\tilde{X}}(0)A^T - \bar{\Lambda}_{\tilde{X}\tilde{X}}(0) = -B_w W B_w^T\,. \tag{4}$$

Use the matlab function `dlyap`, to solve the Lyapunov equation (4) and compute the steady state marginal state estimation covariance $\bar{\Lambda}_{\tilde{X}\tilde{X}}(0)$ and its trace. Then compute the the steady state marginal output estimation covariance:

$$\begin{aligned} \bar{\Lambda}_{\tilde{X}\tilde{X}}(0) &= \lim_{k\to\infty} \Lambda_{\tilde{X}\tilde{X}}(k,0) = \lim_{k\to\infty} E\{\tilde{X}(k)\tilde{X}^T(k)\} \\ \text{trace}(\bar{\Lambda}_{\tilde{X}\tilde{X}}(0)) &= \lim_{k\to\infty} E\{\tilde{X}^T(k)\tilde{X}(k)\} \\ \bar{\Lambda}_{\tilde{Y}\tilde{Y}}(0) &= \lim_{k\to\infty} = \Lambda_{\tilde{Y}\tilde{Y}}(k,0) = \lim_{k\to\infty} E\{\tilde{Y}^2(k)\} \end{aligned}$$

Also use matlab to compute the differences

$$\Delta\text{trace}(\Lambda_{\tilde{X}\tilde{X}}(20,0)) = \text{trace}(\bar{\Lambda}_{\tilde{X}\tilde{X}}(0)) - \text{trace}(\bar{\Lambda}_{\tilde{X}\tilde{X}}(20))$$

and

$$\Delta\bar{\Lambda}_{\tilde{Y}\tilde{Y}}(20,0) = \bar{\Lambda}_{\tilde{Y}\tilde{Y}}(0) - \Lambda_{\tilde{Y}\tilde{Y}}(20,0)$$

to check if the system has reached stationarity by $k=20$.

(c) Run the matlab code that executes steps 1(b)i - 1(b)iii, only that in this case use $V = 10$.

(d) Run the matlab code that executes steps 1(b)i - 1(b)iii, only that in this case use $V = 0.01$.

2. Consider again the system in Eq. (1). We will now analyze the response of a Kalman filter to estimate the state of the system using a sequence of output measurements $\{y(k)\}$ for $k = 0, 1, 2, \cdots$.

   The a-posteriori estimation structure of the Kalman filter can be written as follows

$$\begin{aligned}
\tilde{y}^o(k) &= y(k) - C\hat{x}^o(k) && (5)\\
\hat{x}(k) &= \hat{x}^o(k) + F(k)\tilde{y}^o(k) \\
F(k) &= M(k)C^T[AM(k)C^T + V]^{-1} \\
Z(k) &= M(k) - M(k)C^T[AM(k)C^T + V]^{-1}CM(k) && (6)\\
\hat{x}^o(k+1) &= A\hat{x}(k) + Bu(k) \\
M(k+1) &= AZ(k)A^T + B_w W(k) B_w^T && (7)
\end{aligned}$$

   where the a-posteriori and a-priori state estimates and estimation errors are define as follows. Given $Y_k = \begin{bmatrix} y(0), y(1), \cdots y(k) \end{bmatrix}$,

$$\begin{aligned}
\hat{x}(k) &= E\{x(k)|Y_k\} & \tilde{x}(k) &= x(k) - \hat{x}(k) \\
\hat{x}^o(k) &= E\{x(k)|Y_{k-1}\} & \tilde{x}^o(k) &= x(k) - \hat{x}^o(k)
\end{aligned}$$

   Their respective estimation error covariances are

$$\begin{aligned}
Z(k) &= E\{\tilde{x}(k)\tilde{x}^T(k)\} \\
M(k) &= E\{\tilde{x}^o(k)\tilde{x}^{oT}(k)\}
\end{aligned}$$

   As discussed in class, the update equations for $Z(k)$ and $M(k)$, Eqs. (6) and (7), can be combined into a single **Riccati** equation for updating $M(k)$ (remember that $B = B_w$):

$$M(k+1) = AM(k)A^T + B_w W B_w^T - AM(k)C^T[CM(k)C^T + V]^{-1}CM(k)A^T, \quad M(0) = X_o \quad (8)$$

   Finally, the innovation covariance, also known as the a-posteriori output error covariance, is given by

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k, 0) = E\{\tilde{y}^o(k)^2\} = CM(k)C^T + V \ .$$

   (a) Write a matlab program that will compute:
      - $M(k)$
      - $Z(k)$
      - $E\{\tilde{x}^T(k)\tilde{x}(k)\} = \text{trace}(Z(k))$
      - $\Lambda_{\tilde{y}^o\tilde{y}^o}(k, 0)$

recursively forward in time, starting from $k = 0$ and until $k = 10$. (Remember to reset $V = 0.5$.)

(b) **Plot trace$(Z(k))$ and $\Lambda_{\tilde{y}^o\tilde{y}^o}(k, 0)$ as a function of k.** Check if these quantities appear to be converging to steady state values. If they don't you may need to extend the simulation beyond $k = 10$.

(c) Run the matlab code that executes steps 2a and 2b, only that in this case use $V = 10$.

(d) Run the matlab code that executes steps 2a and 2b, only that in this case use $V = 0.01$.

3. Discuss the performance of the Kalman filter in Problem 2 as a function the magnitude of the measurement noise intensities $V = 10$, $V = 0.5$ and $V = 0.01$ relative to the constant magnitude of the input noise intensity $W = 1$. Do so by comparing

   - The trace of the Kalman filter a-posteriori state estimation error covariance,

   $$E\{\tilde{x}^T(k)\tilde{x}(k)\} = \text{trace}(Z(k))$$

   with the trace of the marginal state estimation error covariance

   $$E\{\tilde{X}^T(k)\tilde{X}(k)\} = \text{trace}(\Lambda_{\tilde{X}\tilde{X}}(k, 0))$$

   in Problem 1

   for the same value of output noise intensity V.

4. **Kalman filter with correlated input and measurement noise:**

   Consider the discrete-time system given by

$$x(k+1) = Ax(k) + Bu(k) + w(k) \tag{9}$$
$$y(k) = Cx(k) + v(k) \tag{10}$$

where $E\{x(0)\} = x_o$, $E\{w(k)\} = 0$, $E\{v(k)\} = 0$, $E\{(x(0) - x_o)(x(0) - x_o)^T\} = X_o$, $E\{(x(0) - x_o)w^T(k)\} = 0$, $E\{(x(0) - x_o)v^T(k)\} = 0$, and

$$E\left\{\begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w(j)^T & v(j)^T \end{bmatrix}\right\} = \begin{bmatrix} W & S \\ S^T & V \end{bmatrix} \delta(k - j)$$

where $V \in \mathbb{R}^{m \times m}$ is positive definite matrix. The a-priori Kalman filter for this system can be written as

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)[y(k) - C\hat{x}^o(k)] \tag{11}$$
$$L(k) = [AM(k)C^T + S][CM(k)C^T + V]^{-1} \tag{12}$$
$$M(k+1) = AM(k)A^T + W - [AM(k)C^T + S][CM(k)C^T + V]^{-1}[CM(k)A^T + S^T] \tag{13}$$

with initial conditions $\hat{x}^o(0) = x_o$ and $M(0) = X_o$.

Derive Eqs. (11)–(13) using previously-derived results in Kalman filtering and noticing that Eqs. (9)–(10) can be written as

$$x(k+1) = A^{'}x(k) + Bu(k) + w^{'}(k) + SV^{-1}y(k)\,,$$

where $A^{'} = A - SV^{-1}C$ and

$$E\left\{\begin{bmatrix} w^{'}(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w^{'}(j)^T & v(j)^T \end{bmatrix}\right\} = \begin{bmatrix} W^{'} & 0 \\ 0 & V \end{bmatrix} \delta(k-j), \qquad W^{'} = W - SV^{-1}S^T\ .$$