ME 233 Spring 2023
Solution to Homework #1

1. (a) First we will define

$$\begin{aligned}
P(A) \quad &- \quad \text{probability that a randomly chosen item comes from factory A} \\
P(B) \quad &- \quad \text{probability that a randomly chosen item comes from factory B} \\
P(C) \quad &- \quad \text{probability that a randomly chosen item comes from factory C} \\
P(D) \quad &- \quad \text{probability that a randomly chosen item is defective} \\
P(N) \quad &- \quad \text{probability that a randomly chosen item is not defective}
\end{aligned}$$

With these definitions, we can state our given information as

$$\begin{aligned}
P(A) &= \frac{1}{4} \qquad P(B) = \frac{1}{2} \qquad P(C) = \frac{1}{4} \\
P(N|A) &= \frac{98}{100} \qquad P(N|B) = \frac{98}{100} \qquad P(N|C) = \frac{99}{100}
\end{aligned}$$

Notice that

$$\begin{aligned}
P(A \cap B) &= P(A \cap C) = P(B \cap C) = 0 \qquad (1) \\
P(N \cap D) &= 0
\end{aligned}$$

Using Bayes' Rule, we can say

$$\begin{aligned}
P(A \cap N) &= P(N|A)P(A) = \frac{98}{100} \times \frac{1}{4} \\
P(B \cap N) &= P(N|B)P(B) = \frac{98}{100} \times \frac{1}{2} \\
P(C \cap N) &= P(N|C)P(C) = \frac{99}{100} \times \frac{1}{4}
\end{aligned}$$

Moreover, since the events that an item is defective and not defective are disjoint,

$$\begin{aligned}
P(A \cap D) &= P(D|A)P(A) = \frac{2}{100} \times \frac{1}{4} \\
P(B \cap D) &= P(D|B)P(B) = \frac{2}{100} \times \frac{1}{2} \\
P(C \cap D) &= P(D|C)P(C) = \frac{1}{100} \times \frac{1}{4}
\end{aligned}$$

With this information and (1), we can determine both $P(N)$ and $P(D)$, the answer to part (a).

$$\begin{aligned}
P(N) &= P(A \cap N) + P(B \cap N) + P(C \cap N) \\
&= \frac{49}{200} + \frac{49}{100} + \frac{99}{400} = \frac{393}{400}
\end{aligned}$$

$$
\begin{aligned}
P(D) &= P(A \cap D) + P(B \cap D) + P(C \cap D) \\
&= \frac{1}{200} + \frac{1}{100} + \frac{1}{400} = \frac{7}{400}
\end{aligned}
$$

(b) Using Bayes' rule, we can determine the probability that a (randomly-chosen) non-defective item comes from factory C.

$$
P(C|N) = \frac{P(C \cap N)}{P(N)} = \frac{99}{393}
$$

We can also construct the array for the joint probabilities shown in Table 1. To construct the last entry in the 'N' column, we add all of the elements above it. To construct the 'D' column, we subtract the 'N' column from the 'Marginal Probabilities' column.

	N	D	Marginal Probabilities
A	$\frac{98}{100} \times \frac{1}{4} = \frac{49}{200}$	$\frac{1}{4} - \frac{49}{200} = \frac{1}{200}$	$\frac{1}{4}$
B	$\frac{98}{100} \times \frac{1}{2} = \frac{49}{100}$	$\frac{1}{2} - \frac{49}{100} = \frac{1}{100}$	$\frac{1}{2}$
C	$\frac{99}{100} \times \frac{1}{4} = \frac{99}{400}$	$\frac{1}{4} - \frac{99}{400} = \frac{1}{400}$	$\frac{1}{4}$
Marginal Probabilities	$\frac{49}{200} + \frac{49}{100} + \frac{99}{400} = \frac{393}{400}$	$\frac{1}{200} + \frac{1}{100} + \frac{1}{400} = \frac{7}{400}$	1

Table 1: Array of joint probability

2. In the "Monty Hall" three-door problem, we will denote the three doors as x, y, and z. Without loss of generality, we will assume that the contestant originally picked door x. We now define C_i to be the event that the car is behind door i and H_j to be the event that the host opens door j.

As stated in the hint, the probability that the contestant wins a car if she switches her choice is

$$
P(\text{win}|\text{she switches}) = P((H_z \cap C_y) \cup (H_y \cap C_z)) \,.
$$

Since there is equal probability that the car is behind any of the doors, it should be clear that, if the contestant does not switch her choice, the probability of winning the prize is

$$
P(\text{win}|\text{she does not switch}) = P(C_x) = 1/3 \,.
$$

Solution 1:

Notice that the mutually exclusive events C_x, $C_y \cap H_z$, and $C_z \cap H_y$ cover the sample space, i.e.

$$
1 = P(C_x) + P(C_y \cap H_z) + P(C_z \cap H_y).
$$

If the contestant switches her guess, the probability that she will win is given by $P((C_y \cap H_z) \cup (C_z \cap H_y))$. Since the event $C_y \cap H_z$ is disjoint from the event $C_z \cap H_y$, we can say that

$$
\begin{aligned}
P(\text{win}|\text{she switches}) &= P((C_y \cap H_z) \cup (C_z \cap H_y)) \\
&= P(C_y \cap H_z) + P(C_z \cap H_y) \\
&= 1 - P(C_x) = \frac{2}{3}.
\end{aligned}
$$

Solution 2:

We can compute $P(C_y \cap H_z)$ and $P(C_z \cap H_y)$ using Bayes' rule. Remember that the contestant has initially chosen door x.

$$P(C_y \cap H_z) = P(H_z|C_y)\,P(C_y)$$

Notice that $P(H_z|C_y)$ is the probability that the host (Monti) will open door z given that the prize is in door y. Since the contestant has initially chosen door x, and Monti cannot reveal the prize, his only choice is to open door z. Thus,

$$\begin{aligned} P(H_z|C_y) &= 1 \\ P(C_y \cap H_z) &= P(H_z|C_y)\,P(C_y) = 1/3\,. \end{aligned}$$

Similarly,

$$\begin{aligned} P(H_y|C_z) &= 1 \\ P(C_z \cap H_y) &= P(H_y|C_z)\,P(C_z) = 1/3\,. \end{aligned}$$

Thus,

$$P(\text{win}|\text{she switches}) = P((H_z \cap C_y) \cup (H_y \cap C_z)) = 2/3\,.$$

3. (a) First we will define $Y = X_1 + X_2$. Now, since X_1 and X_2 are independent, we can apply the property that the PDF of Y is the convolution of the PDF of X_1 and the PDF of X_2:

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X_1}(x_1) p_{X_2}(y - x_1) dx_1$$

Note that the PDF of X_1 only takes on the values of 1 and 0. Thus, we only need to integrate $p_{X_2}(y - x_1)$ over the regions where $p_{X_1}(x_1)$ has a value of 1. Thus,

$$p_Y(y) = \int_0^1 p_{X_2}(y - x_1) dx_1$$

Now note that the following conditions hold

$$\begin{aligned} & p_{X_2}(y - x_1) = 1 \\ \Leftrightarrow\ & 0 \le y - x_1 \le 1 \\ \Leftrightarrow\ & y - 1 \le x_1 \le y \end{aligned}$$

Thus, we get

$$\begin{aligned} p_Y(y) &= \begin{cases} \int_0^y dx_1 & \text{for } 0 \le y \le 1 \\ \int_{y-1}^1 dx_1 & \text{for } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} y & \text{for } 0 \le y \le 1 \\ 2 - y & \text{for } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Part b) is an additional part that was not assigned in the homework. Here we want to calculate the probability density function (PDF) of the randome variable

$$Z = X_1 + X_2 + X_3$$

(b) First we define $Z = X_1 + X_2 + X_3 = Y + X_3$. Following a similar procedure as before,

$$\begin{aligned} p_Z(z) &= \int_{-\infty}^{\infty} p_{X_3}(x_3) p_Y(z - x_3) dx_3 \\ &= \int_0^1 p_Y(z - x_3) dx_3 \\ p_Y(z - x_3) &= \begin{cases} z - x_3 & \text{for } z - 1 \le x_3 \le z \\ 2 - z + x_3 & \text{for } z - 2 \le x_3 \le z - 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus, we get that

$$
\begin{aligned}
p_Z(z) &= \begin{cases} \int_0^z (z - x_3)dx_3 & \text{for } 0 \le z \le 1 \\ \int_0^{z-1}(2 - z + x_3)dx_3 + \int_{z-1}^1 (z - x_3)dx_3 & \text{for } 1 \le z \le 2 \\ \int_{z-2}^1 (2 - z + x_3)dx_3 & \text{for } 2 \le z \le 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2}z^2 & \text{for } 0 \le z \le 1 \\ -z^2 + 3z - \frac{3}{2} & \text{for } 1 \le z \le 2 \\ \frac{1}{2}z^2 - 3z + \frac{9}{2} & \text{for } 2 \le z \le 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Figure 1 shows the PDFs of X_1, Y, and Z. Notice that each time an extra variable is added onto the random variable being looked at, the mean moves to the right, the maximum value moves to the right, and, to compensate, the maximum value of the PDF starts to drop. Also, the plot of the PDF starts to look more and more like a Gaussian distribution.

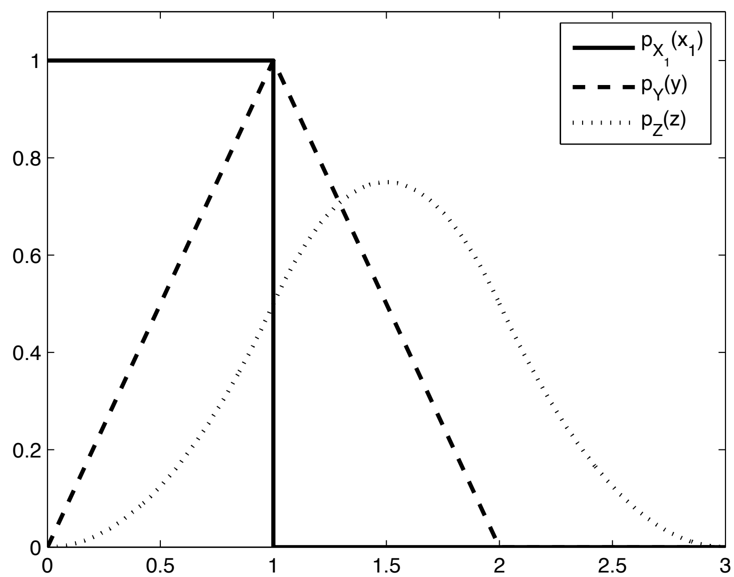

Figure 1: PDFs for X_1, Y, and Z

4. Since X and Y are Gaussian random variables, their moment generating functions are given by

$$
P_X(j\omega) = \mathcal{F}\{p_X(\cdot)\} = \exp\left(-j\omega m_X - \frac{\sigma_X^2\omega^2}{2}\right)
$$

$$
P_Y(j\omega) = \mathcal{F}\{p_Y(\cdot)\} = \exp\left(-j\omega m_Y - \frac{\sigma_Y^2\omega^2}{2}\right)
$$

Since $Z = X + Y$ is the sum of two independent random variables, we can say that

$$
P_Z(j\omega) = \mathcal{F}\{p_Z(\cdot)\} = \mathcal{F}\{p_X(\cdot)\}\,\mathcal{F}\{p_Y(\cdot)\}
$$

Substituting our expressions for the moment generating functions of X and Y then gives

$$
\begin{aligned}
P_Z(j\omega) &= \exp\left\{\left(-j\omega m_X - \frac{\sigma_X^2\omega^2}{2}\right) + \left(-j\omega m_Y - \frac{\sigma_Y^2\omega^2}{2}\right)\right\} \\
&= \exp\left\{-j\omega\left(m_X + m_Y\right) - \frac{\left(\sigma_X^2 + \sigma_Y^2\right)\omega^2}{2}\right\}
\end{aligned}
$$

Note that this is the moment generating function of a Gaussian random variable with mean $m_X + m_Y$ and variance $\sigma_X^2 + \sigma_Y^2$. Therefore, $Z \sim N(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$.