ME 233 Spring 2023
Solution to Homework #3

1. (a) To begin, we find the conditional expectation of X given y:

   i.
   $$\hat{y} = E\{Y\} = E\{X\} + E\{V_1\} = 10 - 0.5 = 9.5$$
   Since X and V_1 are two independent normal distributed random variables,
   $$\Lambda_{YY} = \Lambda_{XX} + \Lambda_{V_1V_1} = 2 + 1 = 3$$

   ii. Noting that $X - \hat{x}$ is independent of V_1, we calculate the cross-covariance of X and Y as
   $$\begin{aligned}
   \Lambda_{XY} &= E\left[(X-\hat{x})(Y-\hat{y})\right] \\
   &= E\left[(X-\hat{x})(X+V_1-\hat{x}-\hat{v}_1)\right] \\
   &= E\left[(X-\hat{x})^2\right] + E\left[(X-\hat{x})(V_1-\hat{v}_1)\right] \\
   &= E\left[(X-\hat{x})^2\right] + E\left[X-\hat{x}\right]E\left[V_1-\hat{v}_1\right] \\
   &= E\left[(X-\hat{x})^2\right] \\
   &= \Lambda_{XX} = 2
   \end{aligned}$$

   iii. Substituting the relevant values gives
   $$\hat{x}|_{y=9} = \hat{x} + \Lambda_{XY}\Lambda_{YY}^{-1}(y-\hat{y})$$
   $$10 + \frac{2}{3}(9-9.5) = 9\frac{2}{3}$$

   iv. The error covariance of the estimate given y is given by
   $$\Lambda_{\tilde{X}_{|Y}\tilde{X}_{|Y}} = \Lambda_{XX} - \Lambda_{XY}\Lambda_{YY}^{-1}\Lambda_{YX} = 2 - 2 \times \frac{1}{3} \times 2 = \frac{2}{3}$$

   (b) To begin, we find the conditional expectation of X given z:

   i.
   $$\hat{z} = E\{Z\} = E\{X\} + E\{V_2\} = 10 + 0.5 = 10.5$$
   Since X and V_2 are two independent normal distributed random variables,
   $$\Lambda_{ZZ} = \Lambda_{XX} + \Lambda_{V_2V_2} = 2 + 2 = 4$$

   ii. Noting that $X - \hat{x}$ is independent of V_2, we calculate the cross-covariance of X and Z as
   $$\begin{aligned}
   \Lambda_{XZ} &= E\left[(X-\hat{x})(Z-\hat{z})\right] \\
   &= E\left[(X-\hat{x})(X+V_2-\hat{x}-\hat{v}_2)\right] \\
   &= E\left[(X-\hat{x})^2\right] + E\left[(X-\hat{x})(V_2-\hat{v}_2)\right] \\
   &= E\left[(X-\hat{x})^2\right] + E\left[X-\hat{x}\right]E\left[V_2-\hat{v}_2\right] \\
   &= E\left[(X-\hat{x})^2\right] \\
   &= \Lambda_{XX} = 2
   \end{aligned}$$

iii. Substituting the relevant values gives

$$\hat{x}|_{z=11} = \hat{x} + \Lambda_{XZ}\Lambda_{ZZ}^{-1}(z - \hat{z})$$
$$10 + \frac{2}{4}(11 - 10.5) = 10\frac{1}{4}$$

iv. The error covariance of the estimate given y is given by

$$\Lambda_{\tilde{X}_{|Z}\tilde{X}_{|Z}} = \Lambda_{XX} - \Lambda_{XZ}\Lambda_{ZZ}^{-1}\Lambda_{ZX} = 2 - 2 \times \frac{1}{4} \times 2 = 1$$

(c) First, we define the random vector W as

$$W = \begin{bmatrix} Y \\ Z \end{bmatrix}$$

The mean and covariance of this vector are given by

$$\begin{aligned} \hat{w} &= \begin{bmatrix} \hat{y} \\ \hat{z} \end{bmatrix} \\ \Lambda_{WW} &= \begin{bmatrix} \Lambda_{YY} & \Lambda_{YZ} \\ \Lambda_{ZY} & \Lambda_{ZZ} \end{bmatrix} \end{aligned}$$

As before,

$$\begin{aligned} \Lambda_{YY} &= \Lambda_{XX} + \Lambda_{V_1V_1} \\ \Lambda_{ZZ} &= \Lambda_{XX} + \Lambda_{V_2V_2} \end{aligned}$$

The cross-covariance between Y and Z can be calculated as

$$\begin{aligned} \Lambda_{ZY} = \Lambda_{YZ} &= E\left[(X + V_1 - \hat{y})\,(X + V_2 - \hat{z})\right] \\ &= E\left[(X + V_1 - \hat{x} - \hat{v}_1)\,(X + V_2 - \hat{x} - \hat{v}_2)\right] \\ &= E\left[(X - \hat{x})^2\right] + E\left[(X - \hat{x})\left((V_1 - \hat{v}_1) + (V_2 - \hat{v}_2)\right)\right] + E\left[(V_1 - \hat{v}_1)(V_2 - \hat{v}_2)\right] \\ &= E\left[(X - m_X)^2\right] \\ &= \Lambda_{XX} \end{aligned}$$

The cross-covariance between X and W can be expressed as

$$\Lambda_{XW} = \begin{bmatrix} \Lambda_{XY} & \Lambda_{XZ} \end{bmatrix} = \begin{bmatrix} \Lambda_{XX} & \Lambda_{XX} \end{bmatrix}$$

Thus,

$$\begin{aligned} \hat{x}_{|y=9,z=11} &= \hat{x}_{|w=[9\ \ 11]^T} \\ &= \hat{x} + \Lambda_{XW}\Lambda_{WW}^{-1}(w - \hat{w}) \\ &= 10 + \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 9 \\ 11 \end{bmatrix} - \begin{bmatrix} 9.5 \\ 10.5 \end{bmatrix} \right) \\ &= 9\frac{7}{8} \end{aligned}$$

The error covariance can now be calculated as

$$\begin{aligned} \Lambda_{\tilde{X}_{|W}\tilde{X}_{|W}} &= \Lambda_{XX} - \Lambda_{XW}\Lambda_{WW}^{-1}\Lambda_{WX} \\ &= 2 - \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{2} \end{aligned}$$

Note that the Y measurement has a greater impact on the conditional mean for X than the Z measurement. This means that our estimation is making use of the fact that Y is a more "reliable" measurement than Z, i.e. $\Lambda_{YY} < \Lambda_{ZZ}$.

(d) i. The conditional expectation of z given y can be calculated as

$$\hat{z}_{|y=9} = \hat{z} + \Lambda_{ZY}\Lambda_{YY}^{-1}(y - \hat{y})$$

$$\begin{aligned}
\Lambda_{ZY} &= E\{(Z-\hat{z})(Y-\hat{y})\} \\
&= E\{(X+V_2-\hat{x}-\hat{v}_2)(Y--\hat{x}-\hat{v}_1)\} \\
&= E\{(X-\hat{x})^2\} + E\{(X-\hat{x})(V_1-\hat{v}_1)\} + E\{(X-\hat{x})(V_2-\hat{v}_2)\} + E\{(V_1-\hat{v}_1)(V_2-\hat{v}_2)\} \\
&= E\{(X-\hat{x})^2\} = \Lambda_{XX} = 2
\end{aligned}$$

$$\hat{z}_{|y=9} = 10.5 + 2\frac{1}{3}(9 - 9.5) = 10\frac{1}{6}$$

ii.

$$\begin{aligned}
\Lambda_{\tilde{X}_{|Y}\tilde{z}_{|Y}} &= \Lambda_{XZ} - \Lambda_{XY}\Lambda_{YY}^{-1}\Lambda_{YZ} \\
&= \Lambda_{XX} - \Lambda_{XX}\Lambda_{YY}^{-1}\Lambda_{XX} \\
&= 2 - 2 \times \frac{1}{3} \times 2 = \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\Lambda_{\tilde{Z}_{|Y}\tilde{z}_{|Y}} &= \Lambda_{ZZ} - \Lambda_{ZY}\Lambda_{YY}^{-1}\Lambda_{YZ} \\
&= \Lambda_{ZZ} - \Lambda_{XX}\Lambda_{YY}^{-1}\Lambda_{XX} \\
&= 4 - 2 \times \frac{1}{3} \times 2 = \frac{8}{3}
\end{aligned}$$

iii. The conditional expectation of x given y and z can now be calculated in terms of the conditional expectation of x given y as

$$\begin{aligned}
\hat{x}_{|(y,z)} &= \hat{x}_{|y} + \Lambda_{\tilde{X}_{|Y}\tilde{z}_{|Y}}\Lambda_{\tilde{Z}_{|Y}\tilde{z}_{|Y}}^{-1}(z - \hat{z}_{|y}) \\
\hat{x}_{|(y=9,z=11)} &= 9\frac{2}{3} + \frac{2}{3} \times \frac{3}{8} \times (11 - 10\frac{1}{6}) = 9\frac{7}{8}
\end{aligned}$$

iv. The error covariance can now be calculated as

$$\begin{aligned}
\Lambda_{\tilde{X}_{|(Y,Z)}\tilde{X}_{|(Y,Z)}} &= \Lambda_{\tilde{X}_{|Y}\tilde{X}_{|Y}} - \Lambda_{\tilde{X}_{|Y}\tilde{Z}_{|Y}}\Lambda_{\tilde{Z}_{|Y}\tilde{Z}_{|Y}}^{-1}\Lambda_{\tilde{Z}_{|Y}\tilde{Z}_{|Y}} \\
&= \frac{2}{3} - \frac{2}{3} \times \frac{8}{3} \times \frac{2}{3} = \frac{1}{2}
\end{aligned}$$

2. (a) First, we define

$$Z := \begin{bmatrix} Y(0) & Y(1) & \cdots & Y(k) \end{bmatrix}^T.$$

And Z takes the outcome of $\bar{y}(k) = \begin{bmatrix} y(0) & \cdots & y(k) \end{bmatrix}^T$.
With this notation in mind, we are interested in finding $\hat{x}_{|z}$. Recall that

$$\begin{aligned}
\hat{x}_{|\bar{y}(k)} &= E\{X\} + \Lambda_{XZ}\Lambda_{ZZ}^{-1}(\bar{y}(k) - E\{Z\}) \\
&= \Lambda_{XZ}\Lambda_{ZZ}^{-1}\bar{y}(k).
\end{aligned}$$

Note that we used that X and Z are zero mean. In order to find this quantity, we need to find expressions for Λ_{XZ} and Λ_{ZZ}^{-1}. First, we will start by finding Λ_{XZ}. Note that

$$\begin{aligned}E\{XY(j)\} &= E\{X^2\} + E\{XV(j)\} \\ &= X_0.\end{aligned}$$

Thus, if we define

$$w = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}^T \in \mathbb{R}^{k+1}$$

we can express

$$\Lambda_{XZ} = X_0 w^T$$

Now we turn our attention to finding Λ_{ZZ}^{-1}. Note that

$$\begin{aligned}E\{Y(k+j)Y(k)\} &= E\{(X+V(k+j))(X+V(k))\} \\ &= E\{X^2\} + E\{XV(k)\} + E\{XV(k+j)\} + E\{V(k+j)V(k)\} \\ &= X_0 + \Sigma_V \delta(j).\end{aligned}$$

Thus, we can express

$$\begin{aligned}\Lambda_{ZZ} &= \Sigma_V I + X_0 ww^T \\ &= \Sigma_V \left(I + \frac{X_0}{\Sigma_V} ww^T\right).\end{aligned}$$

In order to invert this matrix, we must use the matrix inversion lemma, which states that

$$(I+AB)^{-1} = I - A(I+BA)^{-1}B.$$

Using this, we can say that

$$\begin{aligned}\Lambda_{ZZ}^{-1} &= \frac{1}{\Sigma_V}\left(I + \frac{X_0}{\Sigma_V} ww^T\right)^{-1} \\ &= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V} w \left(1 + \frac{X_0}{\Sigma_V} w^T w\right)^{-1} w^T\right] \\ &= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V} \cdot \frac{\Sigma_V}{\Sigma_V + (k+1)X_0} ww^T\right] \\ &= \frac{1}{\Sigma_V}\left[I - \frac{X_0}{\Sigma_V + (k+1)X_0} ww^T\right].\end{aligned}$$

Thus the estimate of X is given by

$$\begin{aligned}\hat{x}(k) = \hat{x}_{|\bar{y}(k)} &= \frac{X_0}{\Sigma_V} w^T \left[I - \frac{X_0}{\Sigma_V + (k+1)X_0} ww^T\right] \bar{y}(k) \\ &= \frac{X_0}{\Sigma_V}\left[1 - \frac{X_0}{\Sigma_V + (k+1)X_0} w^T w\right] w^T \bar{y}(k) \\ &= \frac{X_0}{\Sigma_V + (k+1)X_0} w^T \bar{y}(k) \\ &= \frac{X_0}{\Sigma_V + (k+1)X_0} \sum_{i=0}^{k} y(i).\end{aligned}$$

The covariance of the estimate is given by

$$\begin{aligned}\Lambda_{\tilde{X}\tilde{X}}(k,0) &= \Lambda_{XX} - \Lambda_{XZ}\Lambda_{ZZ}^{-1}\Lambda_{ZX} \\ &= X_0 - \left(\frac{X_0}{\Sigma_V + (k+1)X_0} w^T\right)\left(X_0 w\right) \\ &= \frac{X_0 \Sigma_V}{\Sigma_V + (k+1)X_0}.\end{aligned}$$

(b) Using the results of the previous part, it is trivial to see that

$$\lim_{X_0 \to \infty} \hat{x}(k) = \frac{1}{k+1} \sum_{i=0}^{k} y(k)$$

$$\lim_{X_0 \to \infty} \Lambda_{\tilde{X}\tilde{X}}(k, 0) = \frac{\Sigma_V}{k+1}.$$