1. (a)

$$A = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix},$$

$$B = B_w = [0.34 \quad 0.3]^T,$$

$$C = [0 \quad 3]$$

$$E\{x(k+1)\} = E\{Ax(k) + B(u(k) + w(k))\}$$

Since $u(k)$ is deterministic

$$m_x(k+1) = A\, m_x(k) + Bu(k)$$

Since $x_0 = m_x(0) = [0 \quad 0]^T$,

$$m_x(1) = B u(0)$$

$$m_x(2) = ABu(0) + Bu(1)$$

$$\cdots$$

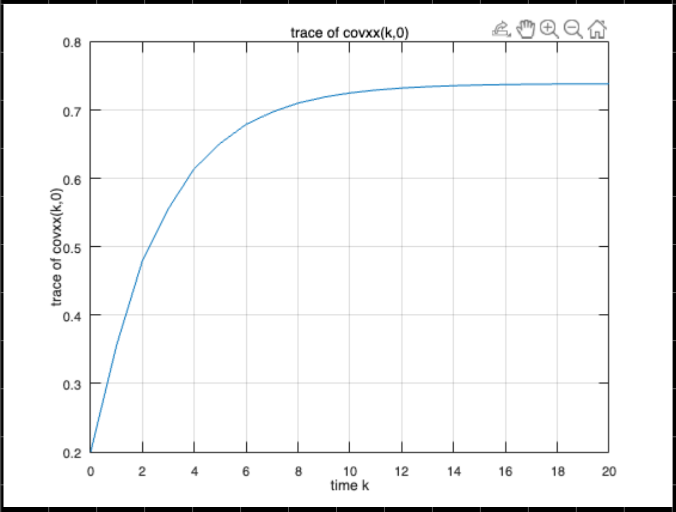
$$m_x(k) = \sum_{i=0}^{k-1} A^{k-i} B u(i)$$

$$m_y(k) = E\{Cx(k) + v(k)\}$$

$$= C\, m_x(k)$$

$$= C \sum_{i=0}^{k-1} A^{k-i} B u(i)$$

(b) the figure of $E\{\tilde{X}^T(k)\tilde{X}(k)\}$, $\Lambda_{\tilde{Y}\tilde{Y}}(k,0)$ are shown below



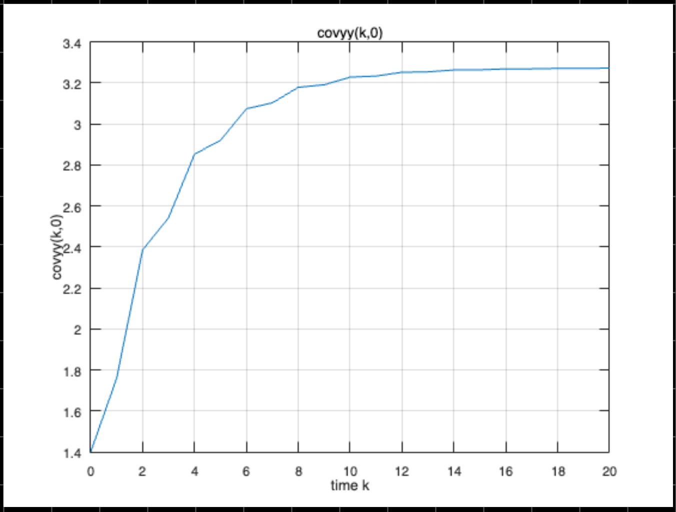


Fig. 1 $E\{\tilde{X}^T(k)\tilde{X}(k)\}$ $(V = 0.5)$

Fig. 2 $\Lambda_{\tilde{Y}\tilde{Y}}(k,0)$ $(V = 0.5)$

Moreover, we have $\operatorname{trace}(\bar{\Lambda}_{\tilde{X}\tilde{X}}(20)) = 0.7385$

$$\Lambda_{\tilde{Y}\tilde{Y}}(20,0) = 3.2712$$

With the Matlab, we have solutions as follows

```
covxx0 = 2x2
    0.4308    0.0276
    0.0276    0.3080

tr_covxx0 = 0.7388
covyy0 = 3.2723
delta_tr_covxx0 = 3.4898e-04
delta_covyy0 = 0.0011
```

i.e. $\text{trace}(\bar{\Lambda}_{\tilde{x}\tilde{x}}(0)) = 0.7388$

$\bar{\Lambda}_{\tilde{y}\tilde{y}}(0) = 3.2723$

$\Delta \text{trace}(\Lambda_{\tilde{x}\tilde{x}}(20,0)) = \text{trace}(\bar{\Lambda}_{xx}(0)) - \text{trace}(\bar{\Lambda}_{\tilde{x}\tilde{x}}(20))$
$= 0.0003$

$\Delta \bar{\Lambda}_{\tilde{y}\tilde{y}}(20,0) = \bar{\Lambda}_{\tilde{y}\tilde{y}}(0) - \Lambda_{\tilde{y}\tilde{y}}(20,0)$
$= 0.0011$

All in all, the system has almost reached stationarity by $k=20$.

(c) when $V=10$, the figure of $E\{\tilde{x}^T(k)\tilde{x}(k)\}$, $\Lambda_{\tilde{y}\tilde{y}}(k,0)$ are shown below

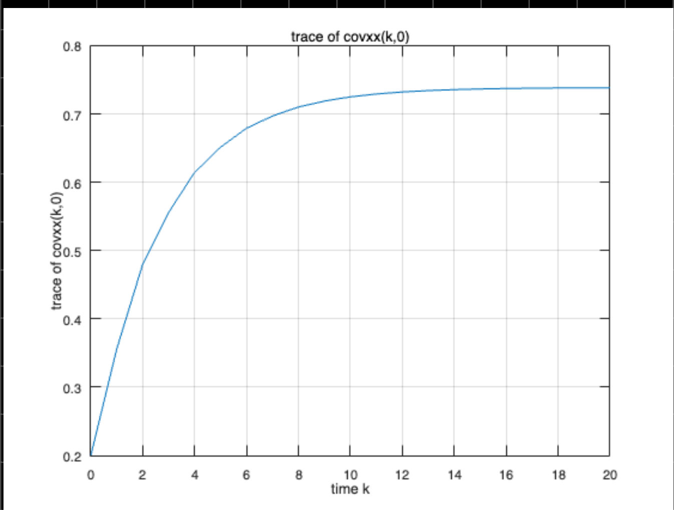


Fig. 3 $E\{\tilde{x}^T(k)\tilde{x}(k)\}$
$(V=10)$

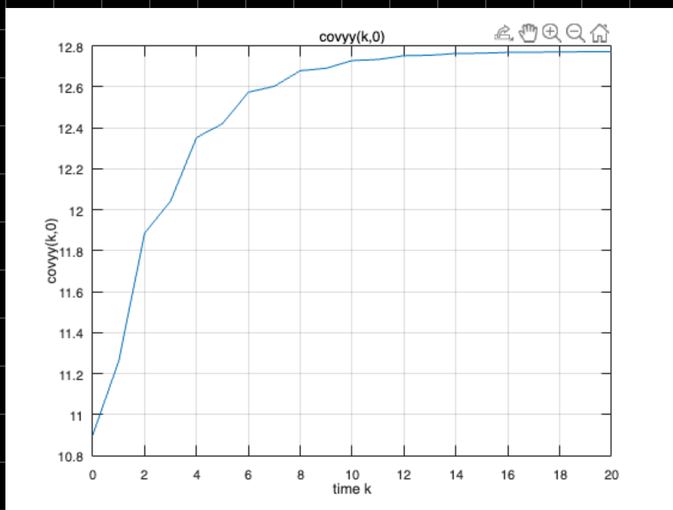


Fig. 4 $\Lambda_{\tilde{y}\tilde{y}}(k,0)$
$(V=10)$

Moreover, we have $\text{trace}(\bar{\Lambda}_{\tilde{x}\tilde{x}}(20)) = 0.7385$

$\Lambda_{\tilde{y}\tilde{y}}(20,0) = 12.7712$

With the Matlab, we have solutions as follows

```
covxx0 = 2x2
    0.4308    0.0276
    0.0276    0.3080

tr_covxx0 = 0.7388
covyy0 = 12.7723

delta_tr_covxx0 = 3.4898e-04
delta_covyy0 = 0.0011
```

i.e. $\text{trace}(\bar{\Lambda}_{\tilde{x}\tilde{x}}(0)) = 0.7388$

$\bar{\Lambda}_{\tilde{y}\tilde{y}}(0) = 12.7723$

$\Delta \text{trace}(\Lambda_{\tilde{x}\tilde{x}}(20,0)) = \text{trace}(\bar{\Lambda}_{xx}(0)) - \text{trace}(\bar{\Lambda}_{\tilde{x}\tilde{x}}(20))$
$= 0.0003$

$\Delta \bar{\Lambda}_{\tilde{y}\tilde{y}}(20,0) = \bar{\Lambda}_{\tilde{y}\tilde{y}}(0) - \Lambda_{\tilde{y}\tilde{y}}(20,0)$

$\Delta \Lambda_{\tilde{y}\tilde{y}}(20,0) = \bar{\Lambda}_{\tilde{y}\tilde{y}}(0) - \Lambda_{\tilde{y}\tilde{y}}(20,0)$

$= 0.0011$

All in all, the system has almost reached stationarity by $k=20$.

(d) when $V=0.01$, the figure of $E\{\tilde{x}^T(k)\tilde{x}(k)\}$, $\Lambda_{\tilde{y}\tilde{y}}(k,0)$ are shown below

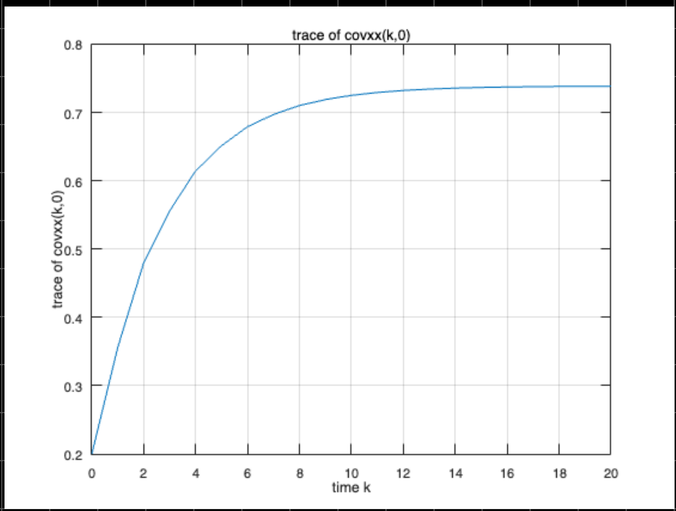


Fig. 5 $E\{\tilde{x}^T(k)\tilde{x}(k)\}$ $(V=0.01)$

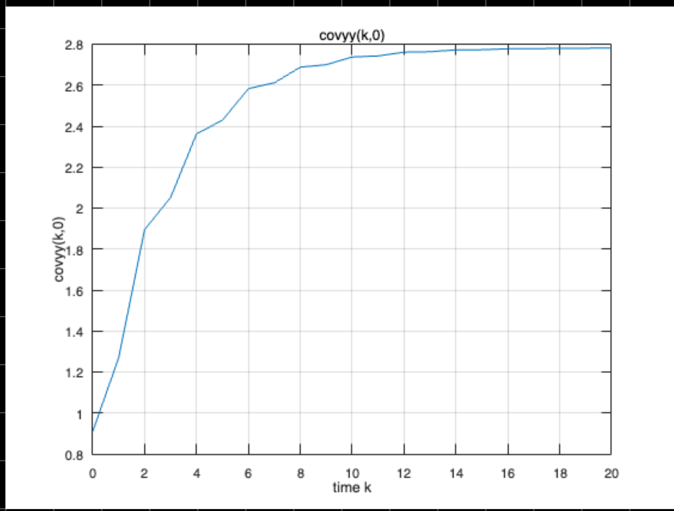


Fig. 6 $\Lambda_{\tilde{y}\tilde{y}}(k,0)$ $(V=0.01)$

Moreover, we have $\mathrm{trace}(\bar{\Lambda}_{\tilde{x}\tilde{x}}(20)) = 0.7385$

$\Lambda_{\tilde{y}\tilde{y}}(20,0) = 2.7812$

With the Matlab, we have solutions as follows

```
covxx0 = 2x2
      0.4308    0.0276
      0.0276    0.3080

tr_covxx0 = 0.7388
covyy0 = 2.7823
delta_tr_covxx0 = 3.4898e-04
delta_covyy0 = 0.0011
```

i.e. $\mathrm{trace}(\bar{\Lambda}_{\tilde{x}\tilde{x}}(0)) = 0.7388$

$\bar{\Lambda}_{\tilde{y}\tilde{y}}(0) = 2.7823$

$\Delta \mathrm{trace}(\Lambda_{\tilde{x}\tilde{x}}(20,0)) = \mathrm{trace}(\bar{\Lambda}_{xx}(0)) - \mathrm{trace}(\bar{\Lambda}_{\tilde{x}\tilde{x}}(20))$

$= 0.0003$

$\Delta \bar{\Lambda}_{\tilde{y}\tilde{y}}(20,0) = \bar{\Lambda}_{\tilde{y}\tilde{y}}(0) - \Lambda_{\tilde{y}\tilde{y}}(20,0)$

$= 0.0011$

All in all, the system has almost reached stationarity by $k=20$.

2. (a)

With the matlab, we have the value of $M(k)$, $Z(k)$, $\mathrm{Trace}(Z(k))$, and $\Lambda_{\bar{y}^o\bar{y}^o}(k,0)$, in which k from 0 to 10

$M(k)$

	1
1	[0.1000,0;0,0.1000]
2	[0.1520,0.0928;0.0928,0.1394]
3	[0.1602,0.0733;0.0733,0.1469]
4	[0.1600,0.0762;0.0762,0.1587]
5	[0.1608,0.0765;0.0765,0.1583]
6	[0.1608,0.0764;0.0764,0.1586]
7	[0.1608,0.0764;0.0764,0.1586]
8	[0.1608,0.0764;0.0764,0.1586]
9	[0.1608,0.0764;0.0764,0.1586]
10	[0.1608,0.0764;0.0764,0.1586]
11	[0.1608,0.0764;0.0764,0.1586]

$Z(k)$

	1
1	[0.1000,0;0,0.0357]
2	[0.1077,0.0265;0.0265,0.0397]
3	[0.1337,0.0201;0.0201,0.0403]
4	[0.1329,0.0198;0.0198,0.0412]
5	[0.1334,0.0199;0.0199,0.0411]
6	[0.1335,0.0198;0.0198,0.0411]
7	[0.1335,0.0198;0.0198,0.0411]
8	[0.1335,0.0198;0.0198,0.0411]
9	[0.1335,0.0198;0.0198,0.0411]
10	[0.1335,0.0198;0.0198,0.0411]
11	[0.1335,0.0198;0.0198,0.0411]

$\mathrm{Trace}(Z(k))$

	1
1	0.1357
2	0.1475
3	0.1740
4	0.1740
5	0.1746
6	0.1747
7	0.1747
8	0.1747
9	0.1747
10	0.1747
11	0.1747

$\Lambda_{\bar{y}^o\bar{y}^o}(k,0)$

	1
1	1.4000
2	1.7542
3	1.8221
4	1.9287
5	1.9245
6	1.9272
7	1.9275
8	1.9275
9	1.9275
10	1.9275
11	1.9275

(b) the figure of $\mathrm{Trace}(Z(k))$, $\Lambda_{\bar{y}^o\bar{y}^o}(k,0)$ are shown below

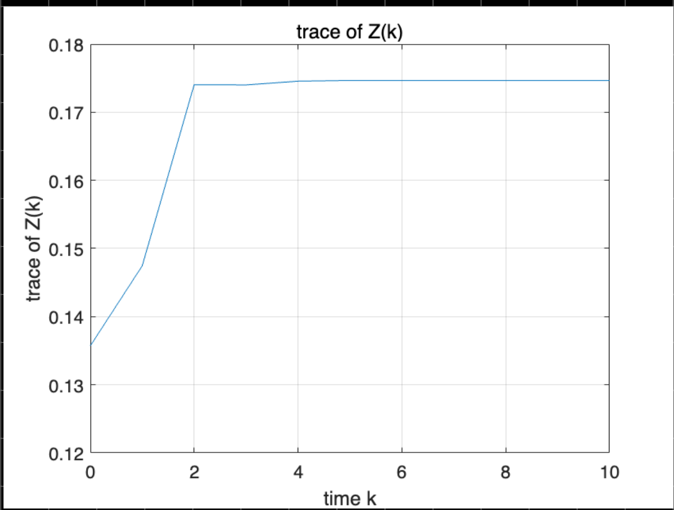


Fig. 1 $\mathrm{Trace}(Z(k))$
($V = 0.5$)

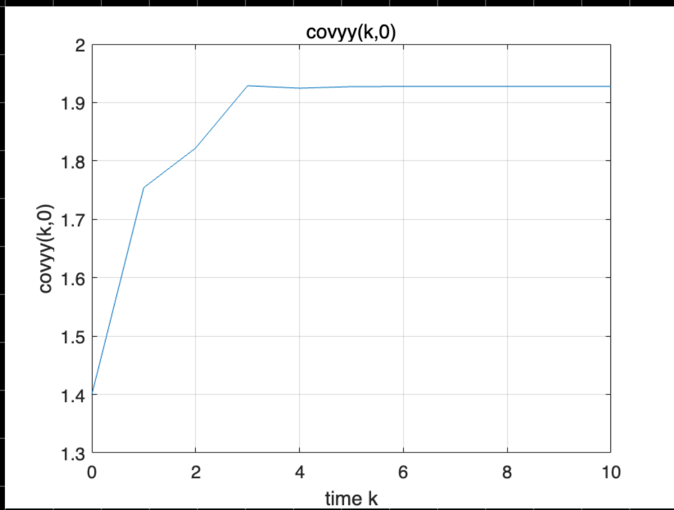


Fig. 2 $\Lambda_{\bar{y}^o\bar{y}^o}(k,0)$
($V = 0.5$)

Moreover, we have $\mathrm{trace}(Z(10)) = 0.1747$ $\Lambda_{\bar{y}^o\bar{y}^o}(k,0) = 1.9275$
the system has almost reached stationarity by $k=10$.

(c) the figure of $\mathrm{Trace}(Z(k))$, $\Lambda_{\bar{y}^o\bar{y}^o}(k,0)$ are shown below

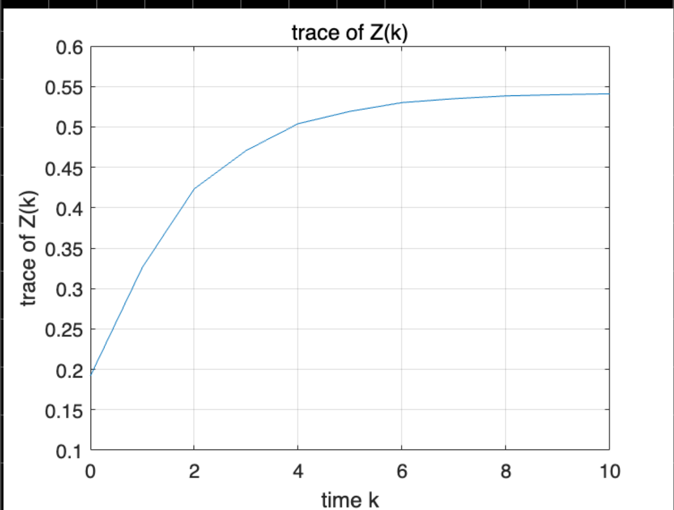


Fig. 3 $\mathrm{Trace}(Z(k))$ ($V = 10$)

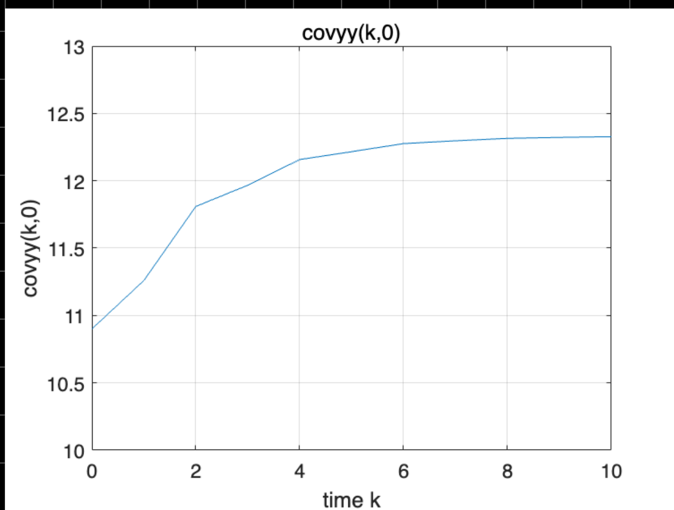


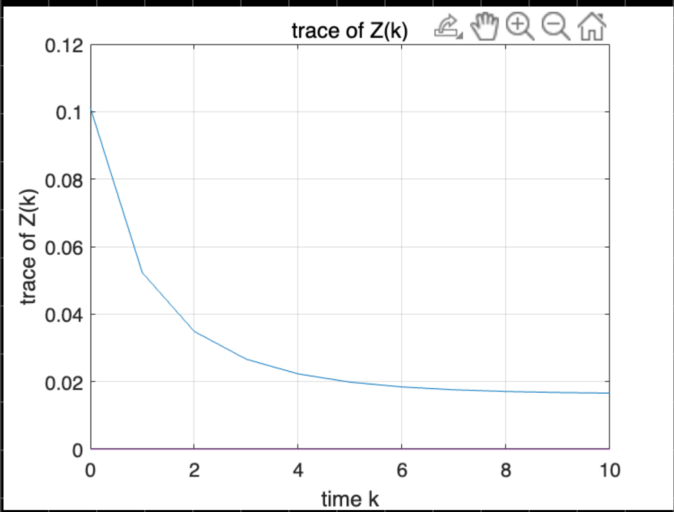
Fig. 4 $\Lambda_{\bar{y}^o\bar{y}^o}(k,0)$ ($V = 10$)

Moreover, we have $trace(Z(10)) = 0.5409$

$$\Lambda_{\bar{y}^o\bar{y}^o}(k,0) = 12.3272$$

the system has almost reached stationarity by $k=10$.

(d) the figure of $trace(Z(k))$, $\Lambda_{\bar{y}^o\bar{y}^o}(k,0)$ are shown below



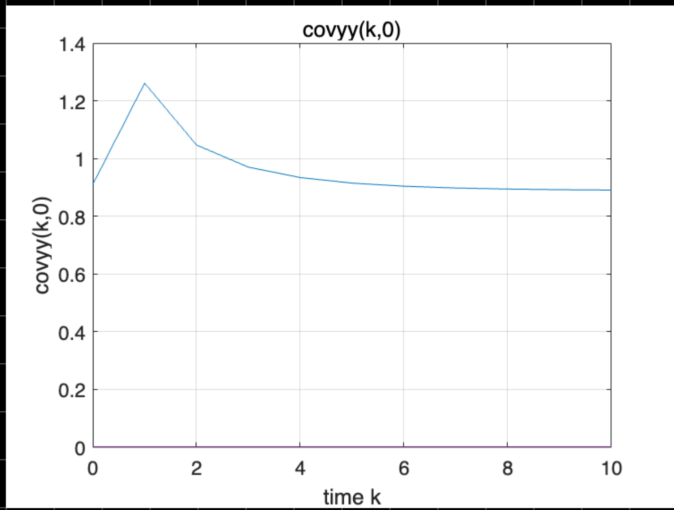


Fig. 5 $trace(Z(k))$ $(V=0.01)$

Fig. 6 $\Lambda_{\bar{y}^o\bar{y}^o}(k,0)$ $(V=0.01)$

Moreover, we have $trace(Z(10)) = 0.0166$ $\Lambda_{\bar{y}^o\bar{y}^o}(k,0) = 0.8906$

the system has almost reached stationarity by $k=10$.

3.

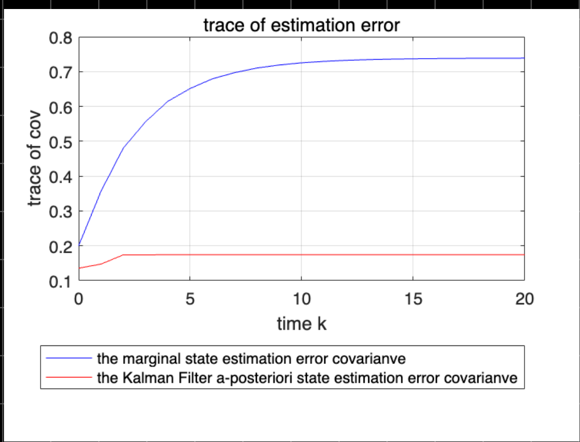


Fig. 1 $V=0.5$, $W=1$

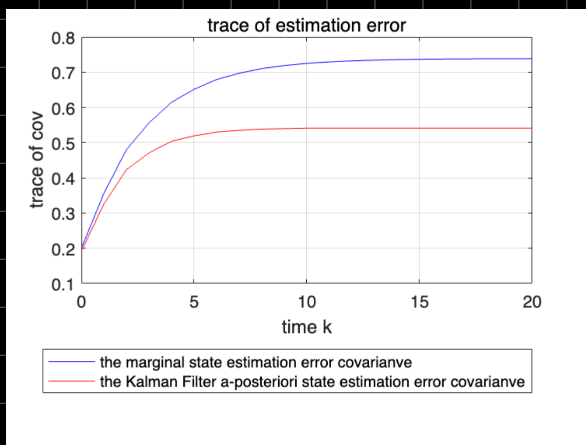


Fig. 2 $V=10$, $W=1$

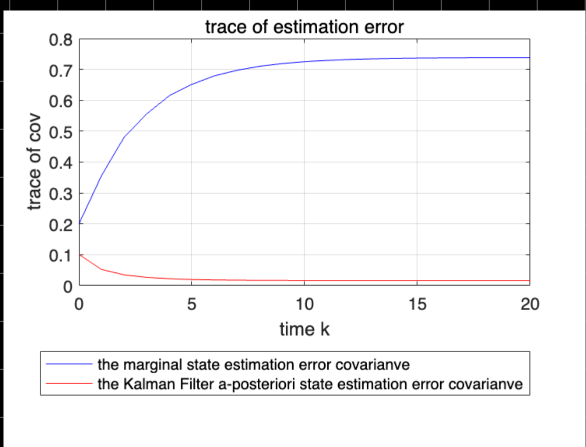


Fig. 3 $V=0.01$, $W=1$

from Fig. 1 – Fig. 3, we can find that

① the estimation error after using Kalman filter is quite smaller

② Kalman filter have much better performance than the previous one.

4. since in this problem, $w(k)$ and $v(k)$

we can set $w'(k)$. which are not correlated with $v(k)$

i.e. $\Lambda_{w'v} = \Lambda_{vw'} = 0$

Assume $w'(k) = w(k) + T v(k)$

$$\Lambda_{w'v}(0) = E\{[w(k) + Tv(k)]\, v^T(k)\}$$
$$= S + TV$$

to make $\Lambda_{w'v}(0) = 0$, let $T = -SV^{-1}$

then, we have $\Lambda_{w'w'}(0) = E\{[w(k) + Tv(k)][w(k) + Tv(k)]^T\}$

$$= W + ST^T + TS^T + TVT^T$$
$$= W - SV^{-1}S^T$$

Since $w(k) = w'(k) + SV^{-1}v(k)$

$$= w'(k) + SV^{-1}[y(k) - Cx(k)]$$

state equations can be rewritten as below

$$x(k+1) = [A - SV^{-1}C]\, x(k) + [B \quad SV^{-1}] \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} + w'(k)$$

$$y(k) = Cx(k) + v(k)$$

so that, we can let $A' = A - SV^{-1}C$, $B' = [B \quad SV^{-1}]$, $u' = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$

u', w' can be considered as new input and input noise

moreover, in a-priori Kalman Filter, $u(k)$ is deterministic as well.

then we can use the results in PPT

$\hat{x}^o(0) = x_0$, $M(0) = X_0$

$\tilde{y}^o(k) = y(k) - C\hat{x}^o(k)$

$\hat{x}(k) = \hat{x}^o(k) + M(k)C^T[CM(k)C^T + V]^{-1}\tilde{y}^o(k)$

$Z(k) = M(k) - M(k)C^T[CM(k)C^T + V]^{-1}CM(k)$

① $\hat{x}^o(k+1) = A'\hat{x}(k) + B'u'(k)$

$$= (A - SV^{-1}C)\{\hat{x}^o(k) + M(k)C^T[CM(k)C^T + V]^{-1}[y(k) - C\hat{x}^o(k)]\}$$

$$+ Bu(k) + SV^{-1}y(k)$$

$$= A\hat{x}^o(k) - SV^{-1}C\hat{x}^o(k) + Bu(k) + SV^{-1}y(k)$$

$$+ (A - SV^{-1}C)M(k)C^T[CM(k)C^T + V]^{-1}[y(k) - C\hat{x}^o(k)]$$

Compute the marked item seperately, we have

$$-SV^{-1}C\hat{x}^o(k) + SV^{-1}y(k) = SV^{-1}[y(k) - C\hat{x}^o(k)]$$

then, $\hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)[y(k) - C\hat{x}^o(k)]$

② $L(k) = SV^{-1} + (A - SV^{-1}C)M(k)C^T[CM(k)C^T + V]^{-1}$

$$= SV^{-1} + AM(k)C^T[CM(k)C^T + V]^{-1}$$

$$-SV^{-1}CM(k)C^T[CM(k)C^T + V]^{-1}$$

Compute the marked item seperately, we have

$$SV^{-1} - SV^{-1}CM(k)C^T[CM(k)C^T + V]^{-1}$$

$$= SV^{-1}[CM(k)C^T + V][CM(k)C^T + V]^{-1}$$

$$- SV^{-1}CM(k)C^T[CM(k)C^T + V]^{-1}$$

$$= SV^{-1}V[CM(k)C^T + V]^{-1}$$

$$= S[CM(k)C^T + V]^{-1}$$

then, $L(k) = [S + AM(k)C^T][CM(k)C^T + V]^{-1}$

③ $Z(k) = M(k) - M(k)C^T[CM(k)C^T + V]^{-1}CM(k)$

$W' = \Lambda_{w'w'}(0) = W - SV^{-1}S^T$ which is computed at first

$M(k+1) = A'Z(k)A'^T + W'$

$$= (A - SV^{-1}C)\{M(k) - M(k)C^T[CM(k)C^T + V]^{-1}CM(k)\}(A - SV^{-1}C)^T + (W - SV^{-1}S^T)$$

$$= (A - SV^{-1}C)M(k)(A - SV^{-1}C)^T + (W - SV^{-1}S^T)$$

$$- (A - SV^{-1}C)M(k)C^T[CM(k)C^T + V]^{-1}CM(k)(A - SV^{-1}C)^T$$

$$= A M(k) A^T + W$$
$$- S V^{-1} S^T$$
$$- S V^{-1} C M(k) A^T$$
$$- A M(k) C^T V^{-1} S^T$$
$$+ S V^{-1} C M(k) C^T V^{-1} S^T$$
$$- A M(k) C^T [C M(k) C^T + V]^{-1} C M(k) A^T$$
$$+ S V^{-1} C M(k) C^T [C M(k) C^T + V]^{-1} C M(k) A^T$$
$$+ A M(k) C^T [C M(k) C^T + V]^{-1} C M(k) C^T V^{-1} S^T$$
$$- S V^{-1} C M(k) C^T [C M(k) C^T + V]^{-1} C M(k) C^T V^{-1} S^T$$

the blue part $= -(A - SV^{-1}C) M(k) C^T V^{-1} S^T$
$$+ (A - SV^{-1}C) M(k) C^T [C M(k) C^T + V]^{-1} C M(k) C^T V^{-1} S^T$$

$$= -(A - SV^{-1}C) M(k) C^T [C M(k) C^T + V]^{-1} [C M(k) C^T + V] V^{-1} S^T$$
$$+ (A - SV^{-1}C) M(k) C^T [C M(k) C^T + V]^{-1} C M(k) C^T V^{-1} S^T$$

$$= -(A - SV^{-1}C) M(k) C^T [C M(k) C^T + V]^{-1} S^T$$

the pink part $= - S V^{-1} C M(k) A^T$
$$-(A - SV^{-1}C) M(k) C^T [C M(k) C^T + V]^{-1} C M(k) A^T$$

$$= - S V^{-1} [C M(k) C^T + V] [C M(k) C^T + V]^{-1} C M(k) A^T$$
$$-(A - SV^{-1}C) M(k) C^T [C M(k) C^T + V]^{-1} C M(k) A^T$$

$$= -(S + A M(k) C^T) [C M(k) C^T + V]^{-1} C M(k) A^T$$

$$M(k+1) = A M(k) A^T + W - S V^{-1} S^T$$
$$- (A - SV^{-1}C) M(k) C^T [C M(k) C^T + V]^{-1} S^T$$
$$-(S + A M(k) C^T) [C M(k) C^T + V]^{-1} C M(k) A^T$$

$$= A M(k) A^T + W - S V^{-1} S^T$$
$$- A M(k) C^T [C M(k) C^T + V]^{-1} S^T$$
$$- A M(k) C^T [C M(k) C^T + V]^{-1} C M(k) A^T$$

$$+ SV^{-1}C\,M(k)C^T[CM(k)C^T+V]^{-1}S^T$$
$$- S[CM(k)C^T+V]^{-1}CM(k)A^T$$

since $-SV^{-1}S^T + SV^{-1}C\,M(k)C^T[CM(k)C^T+V]^{-1}S^T$

$$= -SV^{-1}[CM(k)C^T+V][CM(k)C^T+V]^{-1}S^T$$
$$+ SV^{-1}C\,M(k)C^T[CM(k)C^T+V]^{-1}S^T$$
$$= -S[CM(k)C^T+V]^{-1}S^T$$

Finally, we have

$$
\begin{aligned}
M(k+1) = {} & AM(k)A^T + W \\
& - AM(k)C^T[CM(k)C^T+V]^{-1}S^T \\
& - AM(k)C^T[CM(k)C^T+V]^{-1}CM(k)A^T \\
& - S[CM(k)C^T+V]^{-1}S^T \\
& - S[CM(k)C^T+V]^{-1}CM(k)A^T \\
= {} & AM(k)A^T + W - [AM(k)C^T+S][CM(k)C^T+V]^{-1}[CM(k)A^T+S^T]
\end{aligned}
$$

in conclusion, when w and v are correlated

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)[y(k) - C\hat{x}^o(k)]$$

$$L(k) = [S + AM(k)C^T][CM(k)C^T + V]^{-1}$$

$$M(k+1) = AM(k)A^T + W - [AM(k)C^T+S][CM(k)C^T+V]^{-1}[CM(k)A^T+S^T]$$

with the initial condition $\hat{x}^o(0) = x_0$, and $M(0) = X_0$

Q. E. D.