Tsinghua Berkeley Shenzhen Institute (TBSI)
ME233 Advanced Control Systems II
Spring 2024

Homework #3

Assigned: March 29 (Friday)
Due: April 7 (Sunday)

1. Let $X \sim N(10, 2)$, $V_1 \sim N(-0.5, 1)$ and $V_2 \sim N(0.5, 2)$ be independent random variables. Assume that you are trying to make a measurement of X with two different instruments. Let $Y = X + V_1$ be the measurement of X using the first instrument and $Z = X + V_2$ be the measurement of X using the second instrument, where V_1 and V_2 are respectively the measurement noises of the first and second instruments.

   (a) We first consider only the measurement of X using only instrument Y. Notice that $\hat{x} = E\{X\} = 10$ and $\Lambda_{XX} = E\{(X - \hat{x})^2\} = 2$.

      i. Determine $\hat{y} = E\{Y\}$ and $\Lambda_{YY} = E\{(Y - \hat{y})^2\}$.
      ii. Determine $\Lambda_{XY} = E\{(X - \hat{x})(Y - \hat{y})\}$.
      iii. Determine $\hat{x}|_{y=9} = E\{X|_{y=9}\}$, i.e. the conditional expectation of X given that the first instrument yielded the measurement $y = 9$.
      iv. Define the conditional random estimator $\hat{X}|_Y = E\{X|Y\}$, and the conditional estimation error $\tilde{X}|_Y = X - \hat{X}|_Y$. Compute $\Lambda_{\tilde{X}|_Y \tilde{X}|_Y} = E\{\tilde{X}|_Y \tilde{X}|_Y^T\}$.

   (b) We now consider only the estimation of X using only instrument Z.

      i. Determine $\hat{z} = E\{Z\}$ and $\Lambda_{ZZ} = E\{(Z - \hat{z})^2\}$.
      ii. Determine $\Lambda_{XZ} = E\{(X - \hat{x})(Z - \hat{z})\}$.
      iii. Determine $\hat{x}|_{z=11} = E\{X|_{z=11}\}$, i.e. the conditional expectation of X given that the second instrument yielded the measurement $z = 11$.
      iv. Define the conditional random estimator $\hat{X}|_Z = E\{X|Z\}$, and the conditional estimation error $\tilde{X}|_Z = X - \hat{X}|_Z$. Compute $\Lambda_{\tilde{X}|_Z \tilde{X}|_Z} = E\{\tilde{X}|_Z \tilde{X}|_Z^T\}$.

   (c) We will now estimate X using both instruments in a non-recursive manner. Let's first define the random vector

$$W = \begin{bmatrix} Y & Z \end{bmatrix}^T$$

      i. Determine $\hat{w} = E\{W\}$ and $\Lambda_{WW} = E\{(W - \hat{w})(W - \hat{w})^T\}$.
      ii. Determine $\Lambda_{XW} = E\{(X - \hat{x})(W - \hat{w})^T\}$.
      iii. Determine $\hat{x}|_{y=9,z=11} = E\{X|_{y=9,z=11}\}$, i.e. the conditional expectation of X given that the first instrument yielded the measurement $y = 9$ and second instrument yielded the measurement $z = 11$, i.e. $\hat{x}\big|_{w=\begin{bmatrix} 9 & 11 \end{bmatrix}^T} = E\{X|w = \begin{bmatrix} 9 & 11 \end{bmatrix}^T\}$
      iv. Define the conditional random estimator $\hat{X}|_W = E\{X|W\}$, and the conditional estimation error $\tilde{X}|_W = X - \hat{X}|_W$. Compute $\Lambda_{\tilde{X}|_W \tilde{X}|_W} = E\{\tilde{X}|_W \tilde{X}|_W^T\}$.

(d) We will now estimate X using both instruments in a recursive manner, using the least squares estimation property 3:

$$\begin{aligned} \hat{x}_{|(y,z)} &= \hat{x}_{|y} + E\left\{\tilde{X}_{|Y}|\tilde{z}_{|y}\right\} \\ \hat{x}_{|(y,z)} &= \hat{x}|_y + \Lambda_{\tilde{X}|_Y\tilde{Z}|_Y}\Lambda^{-1}_{\tilde{Z}|_Y\tilde{Z}|_Y}(z - \hat{z}|_y) \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Lambda_{\tilde{X}|_Y\tilde{Z}|_Y} &= \Lambda_{XZ} - \Lambda_{XY}\Lambda^{-1}_{YY}\Lambda_{XZ} \\ \Lambda_{\tilde{Z}|_Y\tilde{Z}|_Y} &= \Lambda_{ZZ} - \Lambda_{ZY}\Lambda^{-1}_{YY}\Lambda_{XZ} \end{aligned}$$

i. Compute $\hat{z}|_{y=9}$.

ii. Compute $\Lambda_{\tilde{X}|_Y\tilde{Z}|_Y}$ and $\Lambda_{\tilde{Z}|_Y\tilde{Z}|_Y}$.

iii. Compute $\hat{x}_{|(y=9,z=11)}$ using Eq. (1) and compare it with the result obtained in part 1(c)iii.

iv. Compute $\Lambda_{\tilde{X}|_W\tilde{X}|_W} = \Lambda_{\tilde{X}|_{(Y,Z)}\tilde{X}|_{(Y,Z)}}$ using the recursive relation

$$\Lambda_{\tilde{X}|_{(Y,Z)}\tilde{X}|_{(Y,Z)}} = \Lambda_{\tilde{X}|_Y\tilde{X}|_Y} - \Lambda_{\tilde{X}|_Y\tilde{Z}|_Y}\Lambda^{-1}_{\tilde{Z}|_Y\tilde{Z}|_Y}\Lambda_{\tilde{Z}|_Y\tilde{X}|_Y}$$

and compare it with the result obtained in part 1(c)iv.

2. A random variable X is repeatedly measured, but the measurements are noisy. Assume that the measurement process can be described by

$$Y(k) = X + V(k)$$

where $X, V(0), V(1), V(2), \ldots$ are jointly Gaussian random variables with

$$\begin{aligned} E\{X\} &= 0 & E\{X^2\} &= X_0 \\ E\{V(k)\} &= 0 & E\{V(k+j)V(k)\} &= \Sigma_V\,\delta(j) \\ E\{XV(k)\} &= 0\,. \end{aligned}$$

where $\delta(k)$ is the Kroneker delta function. Let $y(k)$ be the k-th measurement (i.e. the k-th outcome of $Y(k)$). We now the define the random vector

$$\bar{Y}(k) = \begin{bmatrix} Y(0) & \cdots & Y(k) \end{bmatrix}^T \in \mathcal{R}^{k+1}.$$

and the outcome vector

$$\bar{y}(k) = \begin{bmatrix} y(0) & \cdots & y(k) \end{bmatrix}^T \in \mathcal{R}^{k+1}.$$

(a) Obtain the deterministic vector $w(k) \in \mathcal{R}^{k+1}$ so that covariance matrices $\Lambda_{X\bar{Y}(k)}$ and $\Lambda_{\bar{Y}(k)\bar{Y}(k)}$ can be expressed as follows

$$\begin{aligned} \Lambda_{X\bar{Y}(k)} &= E\{X\bar{Y}^T(k)\} = X_0 w(k)^T \\ \Lambda_{\bar{Y}(k)\bar{Y}(k)} &= E\{\bar{Y}(k)\bar{Y}^T(k)\} = \Sigma_V I + X_0 w(k)w(k)^T \end{aligned}$$

(3)

(b) Utilizing the non-recursive conditional estimation equation for normal variables,

$$\hat{x}_{|\bar{y}(k)} = \hat{x} + \Lambda_{X\bar{Y}(k)} \Lambda^{-1}_{\bar{Y}(k)\bar{Y}(k)} \bar{y}(k)$$

obtain an expression for the least squares estimate of X given the $k+1$ measurements $y(0), \ldots, y(k)$. Also obtain an expression for the corresponding estimation error covariance,

$$\Lambda_{\tilde{X}_{|\bar{Y}(k)} \tilde{X}_{|\bar{Y}(k)}} = \Lambda_{XX} - \Lambda_{X\bar{Y}(k)} \Lambda^{-1}_{\bar{Y}(k)\bar{Y}(k)} \Lambda_{\bar{Y}(k)X}$$

Hint: You do not need to invert a general $(k+1) \times (k+1)$ matrix to find these quantities. Instead, utilize the matrix inversion lemma to invert the matrix

$$\Lambda_{\bar{Y}(k)\bar{Y}(k)} = \Sigma_V I + X_0 w(k) w(k)^T .$$

(Remember that Σ_V and X_0 are scalars).

The matrix inversion lemma says that if

$$M = A + uv^T ,$$

then

$$\begin{aligned} M^{-1} &= A^{-1} - A^{-1}u(I + v^T A^{-1} u)^{-1} v^T A^{-1} \\ &= A^{-1} - \frac{1}{1 + v^T A^{-1} u} A^{-1} u v^T A^{-1} . \end{aligned}$$

(c) We now examine the case when $X_0 \to \infty$, i.e. when no prior information is available on X. Show the following:

$$\begin{aligned} \lim_{X_0 \to \infty} \left(\hat{x}_{|\bar{y}(k)} \right) &= \frac{1}{k+1} \left[y(0) + y(1) + \cdots + y(k) \right] \\ \lim_{X_0 \to \infty} \left(\Lambda_{\tilde{X}_{|\bar{y}(k)} \tilde{X}_{|\bar{y}(k)}} \right) &= \frac{\Sigma_V}{k+1} . \end{aligned}$$