Tsinghua Berkeley Shenzhen Institute (TBSI)
ME233 Advanced Control Systems II
Spring 2024

Homework #6

Assigned: Monday, April 29 2024
Due: Friday, May 10 2024

Infinite Horizon LQR Control of a Geosynchronous Satellite

In this problem we consider the **continuous time (CT)** linear quadratic regulation (LQR) of a geosynchronous satellite. In particular, we want to analyze the effect of using different state penalty costs $Q = C^T C$ on the closed loop eigenvalues.

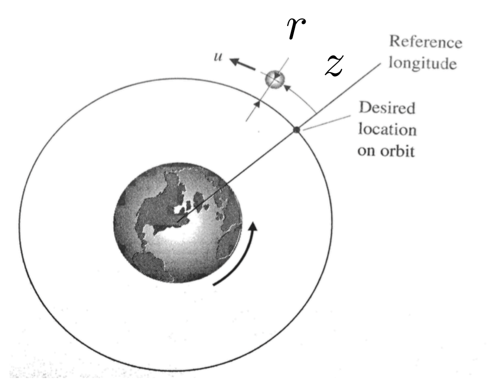

Figure 1: Geosynchronous Satellite (From Franklin and Powell)

The equations of motion for a geosynchronous satellite (such as a weather satellite) are given by

$$\begin{aligned} \ddot{r}(t) - 3\omega^2 r(t) - 2\omega \dot{z}(t) &= 0 \\ \ddot{z}(t) + 2\omega \dot{r}(t) &= u(t) \end{aligned} \tag{1}$$

where r is the satellite's radial perturbation from its desired location while z is the satellite's longitudinal perturbation from its desired location, as shown in Fig. 1. The control action u is the satellite's engine thrust in the z direction. $\omega > 0$ is a constant.

The ultimate objective of the LQR state feedback controller is to maintain the satellite's desired longitudinal and radial locations.

We will consider an infinite horizon LQR state feedback controller $u(t) = -Kx(t)$, which minimizes the cost functional

$$J = \int_0^\infty \left(x^T(t) C^T C x(t) + \rho u^2(t) \right) dt \tag{2}$$

for different matrices C, as explained below, and $\rho > 0$. First, it is necessary to determine if there exists a solution to the LQR problem, which asymptotically stabilizes the closed loop system by analyzing the controllability of the pair (A, B) and detectability of the pairs (A, C).

Defining the state vector as $x = \begin{bmatrix} r & \dot{r} & z & \dot{z} \end{bmatrix}^T \in \mathcal{R}^4$, the state equation for this system is

$$\frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \\ z \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} r \\ \dot{r} \\ z \\ \dot{z} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u \tag{3}$$

1. In order to verify that the system is controllable, determine the controllability matrix

$$P = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

   and compute its rank. In order for the system to be controllable, the rank of P must be 4.

2. Determine if the system is observable when the output is the radial location perturbation,

$$y(t) = r(t) = C_r x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t)$$

   by computing the rank of the observability matrix

$$Q_r = \begin{bmatrix} C_r \\ C_r A \\ C_r A^2 \\ C_r A^3 \end{bmatrix}.$$

   In order for the pair (A, C_r) to be observable, the rank of Q_r must be 4.

3. Determine if the system is observable when the output is the longitudinal location perturbation,

$$y(t) = z(t) = C_z x(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

   by computing the rank of the observability matrix

$$Q_z = \begin{bmatrix} C_z \\ C_z A \\ C_z A^2 \\ C_z A^3 \end{bmatrix}.$$

   In order for the pair (A, C_z) to be observable, the rank of Q_z must be 4.

4. You will now examine the detectability of the system for the two output cases discussed above. When the system is observable, it is also detectable. However, if the system is unobservable, it can still be detectable if the input/output transfer function does not have any unstable (i.e. at the right half complex plane) or limited stable (i.e. at the imaginary axis) pole-zero cancellation.

   Lets first define the Laplace transforms

$$\begin{aligned} R(s) &= \mathcal{L}\{r(t)\}, \qquad Z(s) = \mathcal{L}\{z(t)\} \\ U(s) &= \mathcal{L}\{u(t)\}, \qquad X(s) = \mathcal{L}\{x(t)\} \end{aligned}$$

   You will now compute the input output transfer functions

$$G_r(s) = \frac{R(s)}{U(s)} \qquad \text{and} \qquad G_z(s) = \frac{Z(s)}{U(s)}$$

   directly from Eq. (1) as outlined below.

   (a) Neglecting initial conditions, take the Laplace transform of Eqs. (1) , to obtain the 2×2 matrix $M(s)$ such that

$$\begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \begin{bmatrix} R(s) \\ Z(s) \end{bmatrix} = \begin{bmatrix} 0 \\ U(s) \end{bmatrix}. \tag{4}$$

   (b) Solve Eq. (2) to obtain

$$G_r(s) = \frac{B_r(s)}{s^2(s^2+\omega^2)} \qquad \text{and} \qquad G_z(s) = \frac{B_z(s)}{s^2(s^2+\omega^2)}$$

   and determine the zero polynomials $B_r(s)$ and $B_z(s)$.

   (c) Determine if the system is detectable when the output is the radial location perturbation, $y(t) = r(t)$.

   (d) Determine if the system is detectable when the output is the longitudinal location perturbation, $y(t) = z(t)$.

5. Consider a linear quadratic regulator $u(t) = -Kx(t)$, which minimizes the following infinite horizon cost function

$$J = \int_0^\infty \left(z^2(t) + \rho u^2(t)\right) dt$$

   where ρ is a constant and **assume that** $\omega = 1$.

   Utilize the return difference equality

$$(1 + G_o(-s))(1 + G_o(s)) = 1 + \frac{1}{\rho} G_z(-s) G_z(s)$$

where the return difference is

$$(1 + G_o(s)) = (1 + K(sI - A)^{-1}B) = \frac{A_c(s)}{A(s)},$$

and $A_c(s)$ and $A(s)$ are respectively the closed loop and open loop characteristic polynomials, to sketch the symmetric root locus of the optimal closed loop eigenvalues of the LQR closed loop system (the roots of $A_c(s) = 0$), as well as their unstable counterparts, (the roots of $A_c(-s) = 0$), for $\rho \in (0, \infty)$.