1.

(a) Proof:

$$Y(k) - 0.5\,Y(k-1) = W(k) - 0.3\,W(k-1)$$

After Z-transform, we have

$$Y(z) - 0.5 z^{-1} Y(z) = W(z) - 0.3 z^{-1} W(z)$$

i.e. $$(1-0.5z^{-1})\,Y(z) = (1-0.3z^{-1})\,W(z)$$

$$Y(z) = \frac{z-0.3}{z-0.5} W(z)$$

So $Y(z) = G(z)W(z)$ is given by $G(z) = \frac{z-0.3}{z-0.5}$

Q.E.D

(b) Proof:

$$X(k+1) = 0.5\,X(k) + W(k)$$
$$Y(k) = 0.2\,X(k) + W(k)$$

After Z-transform, we have

$$zX(z) - zX(0) = 0.5\,X(z) + W(z)$$
$$Y(z) = 0.2\,X(z) + W(z)$$

we set $X(0)=0$, since we are only concerned with the forced response

then $$X(z) = \frac{1}{z-0.5} W(z)$$

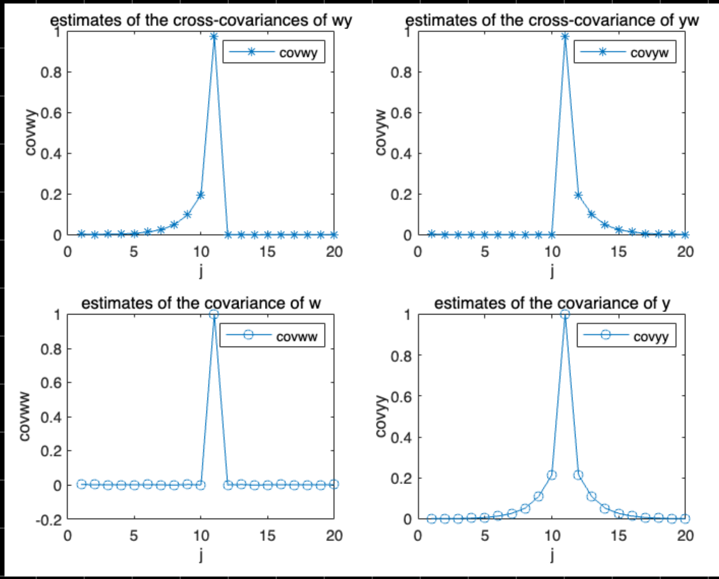
$$Y(z) = \frac{z-0.3}{z-0.5} W(z)$$

so the transfer function $G(z) = \frac{z-0.3}{z-0.5}$ as well

Q.E.D

(c) Solution:

the simulation is shown in next page



(d)

since $\Lambda_{ww}(l) = \delta(l)$, we have $\hat{\Lambda}_{ww}(z) = \sum_{l=-\infty}^{+\infty} z^{-l} \Lambda_{ww}(l) = 1$

$$\hat{\Lambda}_{yw}(z) = G(z)\hat{\Lambda}_{ww}(z) = \frac{z-0.3}{z-0.5}$$

$$\Lambda_{yw}(l) = Z^{-1}\{\hat{\Lambda}_{yw}(z)\}$$

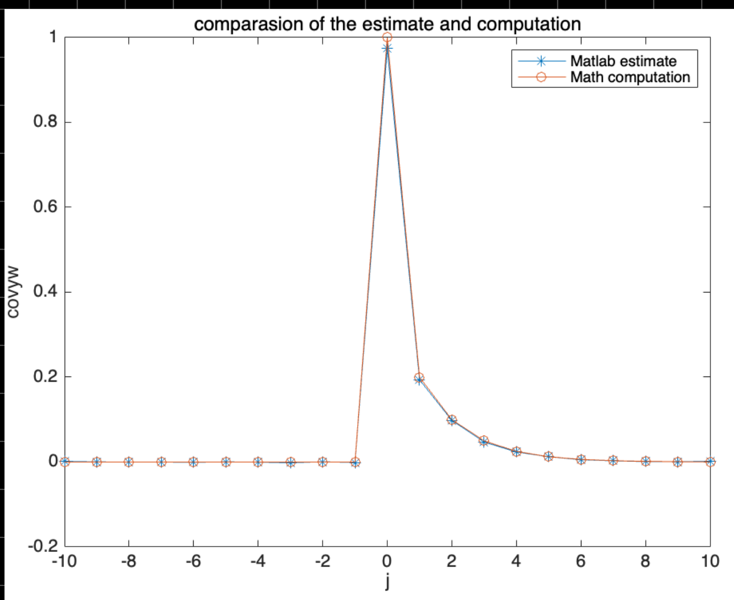
$$= Z^{-1}\left\{\frac{z-0.3}{z-0.5}\right\}$$

$$= Z^{-1}\left\{\frac{0.4z}{z-0.5} + 0.6\right\}$$

$$= \begin{cases} 0.4 \times (0.5)^{l} + 0.6\,\delta(l) & l \geq 0 \\ 0 & l < 0 \end{cases}$$

where $\delta(l)$ is the Kronecker delta function

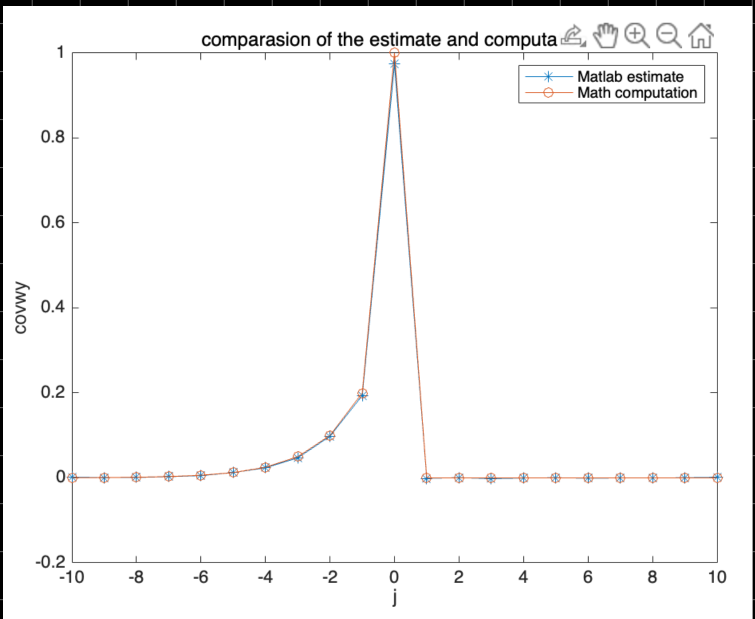
Matlab outcome is shown here



(e) $\hat{\Lambda}_{wy}(z) = \hat{\Lambda}_{yw}(z^{-1}) = \frac{z^{-1}-0.3}{z^{-1}-0.5} = \frac{1-0.3z}{1-0.5z}$

$$\Lambda_{wy}(l) = \Lambda_{wy}(-l) = \begin{cases} 0.4 \times (0.5)^{-l} + 0.6\,\delta(l) & l \leq 0 \\ 0 & l > 0 \end{cases}$$

Matlab outcome is shown in next page



(f) $\hat{\Lambda}_{YY}(z) = G(z)\hat{\Lambda}_{WW}(z)G(z^{-1})$

$$= \frac{z-0.3}{z-0.5} \times \frac{z^{-1}-0.3}{z^{-1}-0.5}$$

$$= \frac{-0.3(z+z^{-1})+1.09}{(z-0.5)(z^{-1}-0.5)}$$

(g) $Y(k) - 0.5Y(k-1) = W(k) - 0.3W(k-1)$

$\Lambda_{YW}(0) = E\{\tilde{Y}(k)\tilde{W}(k)\}$

$= E\{[0.5\tilde{Y}(k-1) + \tilde{W}(k) - 0.3\tilde{W}(k-1)]\tilde{W}(k)\}$

$= E\{\tilde{W}^2(k)\} + 0.5E\{\tilde{Y}(k-1)\tilde{W}(k)\} - 0.3E\{\tilde{W}(k-1)\tilde{W}(k)\}$

Since the system is casual, $E\{\tilde{Y}(k-1)\tilde{W}(k)\} = 0$

Since W is white, $E\{\tilde{W}(k-1)\tilde{W}(k)\} = 0$

then we have $\Lambda_{YW}(0) = E\{\tilde{W}^2(k)\} = \Lambda_{WW}(0) = 1$

from 1(d) we know

$$\Lambda_{YW}(l) = \begin{cases} 0.4 \times (0.5)^l + 0.6\,\delta(l) & l \geq 0 \\ 0 & l < 0 \end{cases}$$

$\Lambda_{YW}(0) = 1$

So, the results of $\Lambda_{YW}(0)$ are same from different equations.

(h) Since it is wide-sense stationary, and with Eq(1), we have

$$\Lambda_{YW}(1) = E\{\tilde{Y}(k+1)\tilde{W}(k)\}$$
$$= E\{\tilde{Y}(k)\tilde{W}(k-1)\}$$
$$= E\{[0.5\tilde{Y}(k-1) + \tilde{W}(k) - 0.3\tilde{W}(k-1)]\tilde{W}(k-1)\}$$
$$= 0.5E\{\tilde{Y}(k-1)\tilde{W}(k-1)\} + E\{\tilde{W}(k)\tilde{W}(k-1)\} - 0.3E\{\tilde{W}^2(k-1)\}$$

from the result of 1(g)

$$\Lambda_{YW}(1) = 0.5 \times 1 + 0 - 0.3 = 0.2$$

from 1(d) we know

$$\Lambda_{YW}(l) = \begin{cases} 0.4 \times (0.5)^l + 0.6\,\delta(l) & l \geq 0 \\ 0 & l < 0 \end{cases}$$

$$\Lambda_{YW}(1) = 0.2$$

So, the results of $\Lambda_{YW}(1)$ are same from different equations.

(i) from Eq(7), $Y^2(k) = [0.5Y(k-1) + W(k) - 0.3W(k-1)]^2$

$$\Lambda_{YY}(0) = E\{\tilde{Y}^2(k)\}$$
$$= 0.25E\{\tilde{Y}^2(k-1)\} + E\{\tilde{W}^2(k)\} + 0.09E\{\tilde{W}^2(k-1)\}$$
$$+ E\{\tilde{Y}(k-1)\tilde{W}(k)\} - 0.3E\{\tilde{Y}(k-1)\tilde{W}(k-1)\} - 0.6\{\tilde{W}(k)\tilde{W}(k-1)\}$$
$$= 0.25\Lambda_{YY}(0) + \Lambda_{WW}(0) + 0.09\Lambda_{WW}(0) + 0 - 0.3\Lambda_{YW}(0) - 0$$

then we can find

$$0.75\Lambda_{YY}(0) = 0.79$$
$$\Lambda_{YY}(0) = 1.0533$$

2. (a) (b) (c) (d) Matlab outcome is shown in Fig. 1. 2. 3. 4

(a) from Fig. 1. $\quad \bar{m}_y = \lim\limits_{k\to\infty} m_y(k) = 10.08$

(b) from Fig. 2 $\quad \bar{\Lambda}_{YY}(0) = \lim\limits_{k\to\infty} \Lambda_{YY}(k,0) = 3.27$

(c) from Fig. 3 $\quad \bar{\Lambda}_{YY}(5) = \lim\limits_{k\to\infty} \Lambda_{YY}(k,5) = 0.27$

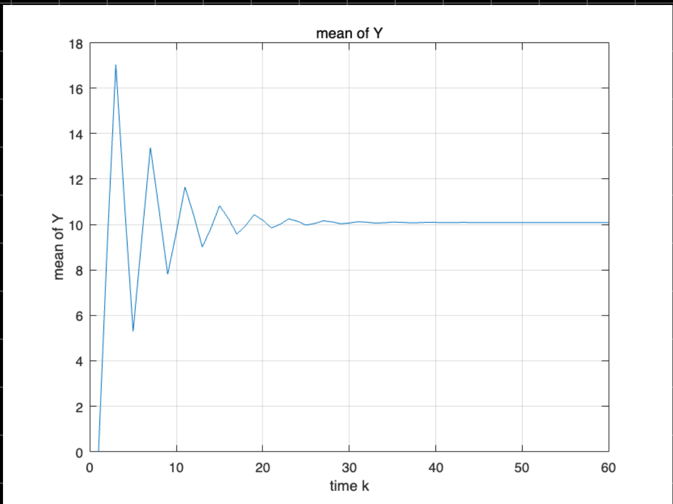


Fig. 1 $\quad m_y(k) = E\{Y(k)\}$

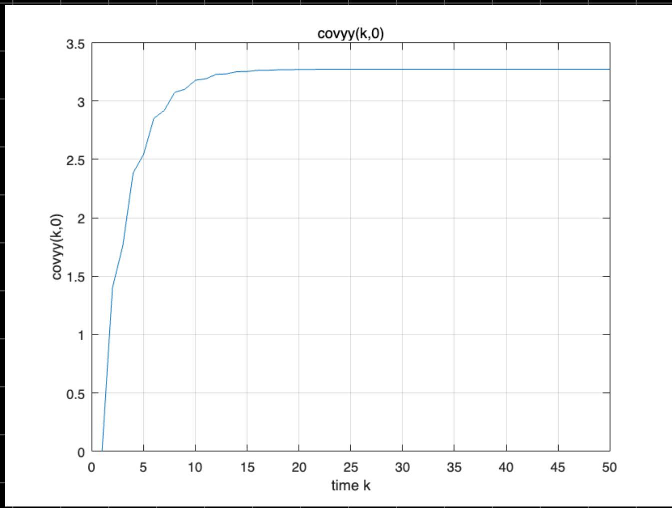


Fig. 2. $\Lambda_{YY}(k,0)$

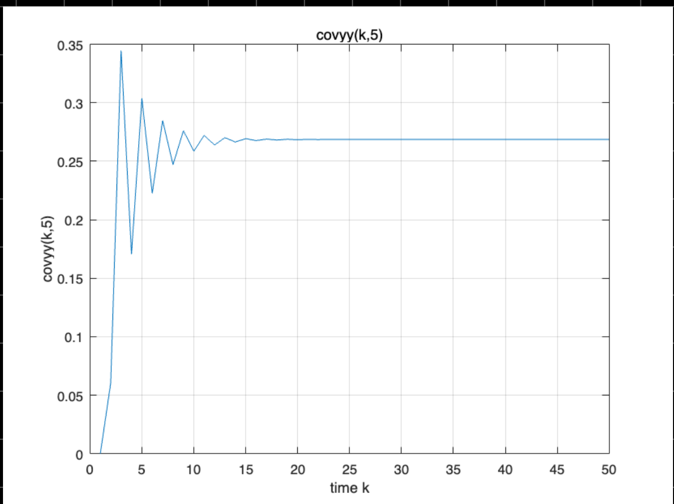


Fig. 3. $\Lambda_{YY}(k,5)$

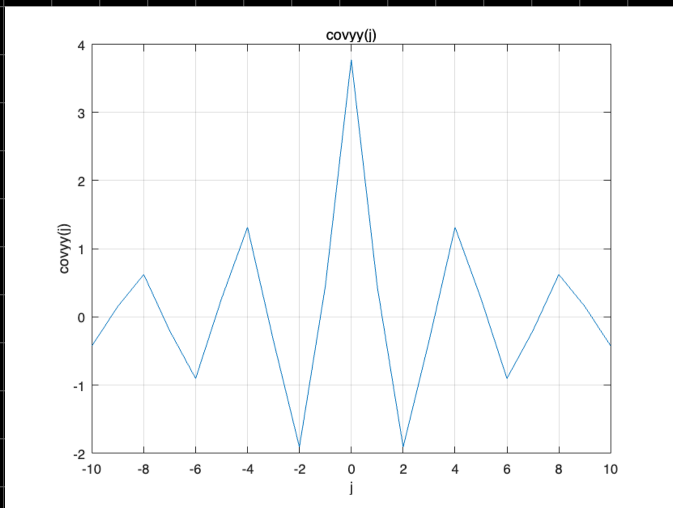


Fig. 4. $\bar{\Lambda}_{YY}(j)$

(e)

$$\Lambda_{YY}(l) = E\{\tilde{Y}(k+l)\,\tilde{Y}^T(k)\}$$

$$= E\left\{\left[\sum_{i=-\infty}^{\infty} g(i)\,W(k-i+l) + V(k+l) - m_y(k+l)\right]\left[\sum_{j=-\infty}^{\infty} g(j)\,W(k-j) + V(k) - m_y(k)\right]^T\right\}$$

$$= E\left\{\left[\sum_{i=-\infty}^{\infty} g_i\,\tilde{W}(k-i+l) + \tilde{V}(k+l)\right]\left[\sum_{j=-\infty}^{+\infty} g(j)\,\tilde{W}(k-j) + \tilde{V}(k)\right]^T\right\}$$

Since $\Lambda_{WV}(l) = 0$ for all l.

$$\Lambda_{YY}(l) = E\left\{\sum_{i=-\infty}^{\infty} g_i\,\tilde{W}(k-i+l) \sum_{j=-\infty}^{+\infty} \tilde{W}^T(k+j)\,g^T(j) + \tilde{V}(k+l)\,\tilde{V}(k)^T\right\}$$

since when $j \neq i-l$, $\hat{W}(k-i+l)\hat{W}^T(k+j) = 0$

$$\Lambda_{YY}(l) = E\left\{\sum_{i=-\infty}^{+\infty} g(i)\hat{W}(k-i+l)\hat{W}^T(k-i+l)g^T(i-l) + \tilde{V}(k+l)\tilde{V}^T(k)\right\}.$$

$$= \sum_{i=-\infty}^{+\infty} g(i)\Sigma_{ww} g^T(i-l) + \Sigma_{vv}\delta(l)$$

$$\Phi_{YY}(z) = Z(\Lambda_{YY}(l))$$

$$= \sum_{l=-\infty}^{+\infty}\sum_{i=-\infty}^{+\infty} g(i)\Sigma_{ww} g^T(i-l) z^{-l} + \Sigma_{vv}$$

$$= \sum_{i=-\infty}^{+\infty} g(i)\Sigma_{ww}\sum_{l=-\infty}^{+\infty} g^T(i-l)z^{-l} + \Sigma_{vv}$$

$$= \sum_{i=-\infty}^{+\infty} g(i)\Sigma_{ww} G^T(z^{-1})\cdot z^{-i} + \Sigma_{vv}$$

$$= \sum_{i=-\infty}^{+\infty} g(i)z^{-i}\cdot\Sigma_{ww} G^T(z^{-1}) + \Sigma_{vv}$$

$$= G(z)\Sigma_{ww} G^T(z^{-1}) + \Sigma_{vv}$$

let $z = e^{jw}$

$$\Phi_{YY}(w) = G(w)\Sigma_{ww} G^*(w) + \Sigma_{vv}$$

So, we have $\Phi_{YY}(w) = G(w)\Sigma_{ww} G^*(w) + \Sigma_{vv}$