ME 233 Spring 2023
Solution to Homework #4

1. (a) We have $m_w \neq 0$, so:

$$
\begin{aligned}
M(k) &= E\{\tilde{x}^o(k)\tilde{x}^{oT}(k)\} \\
\hat{x}^o(k) &= A\hat{x}^o(k-1) + Bu(0) + B_w m_w \\
x(k) &= Ax(k-1) + Bu(k-1) + B_w w(k-1) \\
\tilde{x}^o(k) &= A\tilde{x}(k-1) + B_w(w(k-1) - m_w)
\end{aligned}
$$

Then we can deduce:

$$
\begin{aligned}
M(k) &= E\{[A\tilde{x}^o(k-1) + B_w(w(k-1) - m_w)][A\tilde{x}^o(k-1) + B_w(w(k-1) - m_w)]^T\} \\
&= AE\{A\tilde{x}^o(k-1)\tilde{x}^{oT}(k-1)\}A^T + B_w E\{(w(k-1) - m_w)(w(k-1) - m_w)^T\}B_w \\
&= AZ(k-1)A + B_w W B_w
\end{aligned}
$$

The following equation is still valid for $Z(k)$:

$$
Z(k) = M(k) - M(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)
$$

And we have finally:

$$
\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0) = CM(k)C^T + V
$$

The initial condition is:

$$
M(0) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}
$$

to iteratively find the estimation error covariances. To quantify when the estimation error covariance matrices are approaching their steady state value, use matrix norms. For example, one could say that they are approaching their steady state values when

$$
\max\left\{\left\|M(k) - M(k-1)\right\|_{i2}, \left\|Z(k) - Z(k-1)\right\|_{i2}\right\} < 10^{-6}
$$

Using these equations and this termination condition, the computed steady state estimation error covariances are

$$
\begin{aligned}
M_{ss} &= \begin{bmatrix} 0.1608 & 0.0764 \\ 0.0764 & 0.1586 \end{bmatrix} \\
Z_{ss} &= \begin{bmatrix} 0.1335 & 0.0198 \\ 0.0198 & 0.0411 \end{bmatrix} \\
\left(\Lambda_{\tilde{y}^o\tilde{y}^o}\right)_{ss} &= 1.9275
\end{aligned}
$$

It is easy to verify that

$$
M_{ss} - Z_{ss} > 0
$$

Thus, our a-posteriori estimates always do "better" than the a-priori estimates in the sense that they have a smaller covariance.
After we can deduce $F(k)$ and $L(k)$ thanks to the two following equations:

$$\begin{aligned} F(k) &= M(k)C^T[CM(k)C^T + V]^{-1} \\ L(k) &= AM(k)C^T[CM(k)C^T + V]^{-1} \end{aligned}$$

We obtain:

$$\begin{aligned} L &= \begin{bmatrix} -0.2564 \\ 0.1079 \end{bmatrix} \\ F &= \begin{bmatrix} 0.1189 \\ 0.2469 \end{bmatrix} \end{aligned}$$

(b) Figure 1 shows the a-priori output estimation error covariance as a function of time.

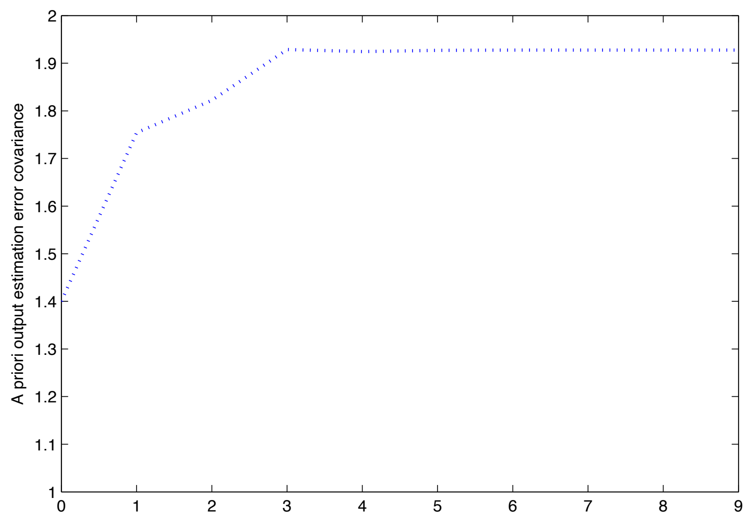

Figure 1: A priori output estimation error covariance vs. time

2. In this problem, we will derive a Kalman filter for a system in which the input noise $w(k)$ and $v(k)$ are correlated. To do so, we will transform the plant equations

$$x(k+1) = Ax(k) + Bu(k) + w(k) \tag{1}$$
$$y(k) = Cx(k) + v(k) \tag{2}$$

into an equivalent system

$$x(k+1) = \bar{A}x(k) + \bar{B}\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} + w'(k) \tag{3}$$
$$y(k) = Cx(k) + v(k) \tag{4}$$

where $w'(k)$ and $v(k)$ are uncorrelated. Notice that $\begin{bmatrix} u(k) & y(k) \end{bmatrix}^T$ is a deterministic quantity, since both $u(k)$ and $y(k)$ are outcomes at step k.

Let n be the dimension of the state and p be the dimension of the output. As the first step, we first need to define a new input noise $w'(k)$, which is a linear combination of $w(k)$ and $v(k)$, but is uncorrelated with $v(k)$, For simplicity, lets choose $w'(k)$ to be

$$w'(k) := w(k) + Tv(k)$$

where the matrix $T \in \mathcal{R}^{n \times p}$ has to be determined.

With this choice of transformation, we get

$$\begin{aligned} \Lambda_{W'V}(0) &= E\left\{[w(k) + Tv(k)]\, v^T(k)\right\} \\ &= S + TV \end{aligned}$$

To make $w'(k)$ uncorrelated with $v(k)$, we will choose $T = -SV^{-1}$. The covariance of $w'(k)$ is given by

$$\begin{aligned} \Lambda_{W'W'}(0) &= E\left\{[w(k) + Tv(k)]\,[w(k) + Tv(k)]^T\right\} \\ &= W + ST^T + TS^T + TVT^T \\ &= W - SV^{-1}S^T \end{aligned}$$

Noting that

$$\begin{aligned} w(k) &= w'(k) + SV^{-1}v(k) \\ &= w'(k) + SV^{-1}\big[y(k) - Cx(k)\big] \end{aligned}$$

our governing equations become

$$\begin{aligned} x(k+1) &= \big[A - SV^{-1}C\big]x(k) + \begin{bmatrix} B & SV^{-1} \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} + w'(k) \\ y(k) &= Cx(k) + v(k) \end{aligned}$$

where $w'(k)$ and $v(k)$ are uncorrelated. Thus,

$$\begin{aligned} \bar{A} &= \big[A - SV^{-1}C\big] \\ \bar{B} &= \begin{bmatrix} B & SV^{-1} \end{bmatrix} \end{aligned}$$

Again, notice that that $\begin{bmatrix} u(k) & y(k) \end{bmatrix}^T$ is a deterministic quantity.

Using the Kalman filter results derived in class, we get

$$\begin{aligned} \hat{x}(k) &= \hat{x}^o(k) + M(k)C^T\left[CM(k)C^T + V\right]^{-1}\tilde{y}^o(k) \\ \hat{x}^o(k+1) &= \big[A - SV^{-1}C\big]\hat{x}(k) + \begin{bmatrix} B & SV^{-1} \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \\ Z(k) &= M(k) - M(k)C^T\left[CM(k)C^T + V\right]^{-1}CM(k) \\ M(k+1) &= \big[A - SV^{-1}C\big]Z(k)\big[A - SV^{-1}C\big]^T + \big[W - SV^{-1}S^T\big] \end{aligned}$$

To make the algebra easier, we will define

$$\begin{aligned} L(k) &:= \big[A - SV^{-1}C\big]\left[M(k)C^T\right]\left[CM(k)C^T + V(k)\right]^{-1} + SV^{-1} \\ &= \left[AM(k)C^T - SV^{-1}CM(k)C^T\right]\left[CM(k)C^T + V(k)\right]^{-1} \\ &\quad + \left[SV^{-1}CM(k)C^T + SV^{-1}V\right]\left[CM(k)C^T + V(k)\right]^{-1} \\ &= \left[AM(k)C^T + S\right]\left[CM(k)C^T + V(k)\right]^{-1} \end{aligned}$$

Simplifying the state estimation equations gives

$$
\begin{aligned}
\hat{x}^o(k+1) &= \left[A - SV^{-1}C\right]\Big(\hat{x}^o(k) + M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}\tilde{y}^o(k)\Big) \\
&\quad +Bu(k) + SV^{-1}y(k) \\
&= \left[A - SV^{-1}C\right]\hat{x}^o(k) + \left[L(k) - SV^{-1}\right]\tilde{y}^o(k) + Bu(k) + SV^{-1}y(k) \\
&= A\hat{x}^o(k) + \left[L(k) - SV^{-1}\right]\tilde{y}^o(k) + Bu(k) + SV^{-1}\tilde{y}^o(k) \\
&= A\hat{x}^o(k) + Bu(k) + L(k)\tilde{y}^o(k) \\
&= A\hat{x}^o(k) + Bu(k) + L(k)\left[y(k) - C\hat{x}^o(k)\right]
\end{aligned}
$$

Simplifying the state estimation covariance equations gives

$$
\begin{aligned}
M(k+1) &= \left[A - SV^{-1}C\right]M(k)\left[A - SV^{-1}C\right]^T + \left[W - SV^{-1}S^T\right] \\
&\quad - \left[A - SV^{-1}C\right]M(k)C^T\left[CM(k)C^T + V\right]^{-1}CM(k)\left[A - SV^{-1}C\right] \\
&= \left[A - SV^{-1}C\right]M(k)\left[A - SV^{-1}C\right]^T + \left[W - SV^{-1}S^T\right] \\
&\quad - \left[L(k) - SV^{-1}\right]\Big(\left[L(k) - SV^{-1}\right]\left[CM(k)C^T + V\right]\Big)^T \\
&= AM(k)A^T - SV^{-1}CM(k)A^T - \left[A - SV^{-1}C\right]M(k)C^TV^{-1}S^T \\
&\quad +W - SV^{-1}S^T - L(k)\left[CM(k)C^T + V\right]L^T(k) \\
&\quad +SV^{-1}\left[CM(k)C^T + V\right]L^T(k) + \left[L(k) - SV^{-1}\right]\left[CM(k)C^T + V\right]V^{-1}S^T
\end{aligned}
$$

Note that the third term in the last expression cancels with the last term in that expressions. Thus,

$$
\begin{aligned}
M(k+1) &= AM(k)A^T - SV^{-1}CM(k)A^T + W - SV^{-1}S^T \\
&\quad -L(k)\left[CM(k)C^T + V\right]L^T(k) + SV^{-1}\left[CM(k)C^T + V\right]L^T(k) \\
&= AM(k)A^T + W - L(k)\left[CM(k)C^T + V\right]L^T(k) \\
&\quad -SV^{-1}\left[AM(k)C^T + S\right]^T + SV^{-1}\left[CM(k)C^T + V\right]L^T(k) \\
&= AM(k)A^T + W - \left[AM(k)C^T + S\right]\left[CM(k)C^T + V\right]^{-1}\left[CM(k)A^T + S^T\right]
\end{aligned}
$$

Thus, in summary

$$
\begin{aligned}
\hat{x}^o(k+1) &= A\hat{x}^o(k) + Bu(k) + L(k)\left[y(k) - C\hat{x}^o(k)\right] \\
L(k) &= \left[AM(k)C^T + S\right]\left[CM(k)C^T + V\right]^{-1} \\
M(k+1) &= AM(k)A^T + W - \left[AM(k)C^T + S\right]\left[CM(k)C^T + V(k)\right]^{-1}\left[CM(k)A^T + S^T\right]
\end{aligned}
$$