Tsinghua Berkeley Shenzhen Institute (TBSI)
ME233 Advanced Control Systems II
Spring 2024

Homework #2

Assigned:	Saturday, March 16
Due:	Sunday, March 24

1. Consider the stochastic system

$$Y(k) - 0.5Y(k-1) = W(k) - 0.3W(k-1) \tag{1}$$

where $W(k)$ is a wide sense stationary (WSS) zero mean normal white random sequence with unit variance, i.e.

$$m_W = 0 \qquad \Lambda_{WW}(l) = E\{W(k+l)W(k)\} = \delta(l)$$

and $\delta(l)$ is the Kronecker delta function, i.e.

$$\delta(l) = \begin{cases} 1\ , & l = 0 \\ 0\ , & l \neq 0 \end{cases}$$

The system described by Eq. (1) can also be expressed in state space form:

$$\begin{aligned} X(k+1) &= 0.5X(k) + 1W(K) \\ Y(k) &= 0.2X(k) + W(k) \end{aligned} \tag{2}$$

In this problem, we will compare the theoretical value of the relevant covariances with empirical estimates of those quantities computed using MATLAB. [1]

(a) Using the Z-transform, show that the input-output transfer function of the system in (1)

$$Y(z) = G(z)W(z)$$

is given by

$$G(z) = \frac{z - 0.3}{z - 0.5} \tag{3}$$

Here $Y(z) = \mathcal{Z}\{Y(k)\}$ is the Z-transform of $Y(k)$ and $W(z) = \mathcal{Z}\{W(k)\}$ is the Z-transform of $W(k)$. [2]

[1] Since MATLAB requires initial conditions to perform time simulations, the output of the system given by Eq. (1) will not, strictly speaking be WSS. However, if the length of the sample sequence is taken to be sufficiently long, the relevant quantities will be approximately given by time averages.

[2] If you need to review Z-transforms, please read section 3.4 of the textbook "Introduction to Modern Controls with Illustrations in MATLAB and Python", by Chen and Tomizuka, which was distributed as a PDF file.

Hint: To obtain $G(z)$, take the z-transform of Eq. (1),

$$\mathcal{Z}\{Y(k)\} - 0.5\mathcal{Z}\{Y(k-1)\} = \mathcal{Z}\{W(k)\} - 0.3\mathcal{Z}\{W(k-1)\}$$

use the Z-transform property that $\mathcal{Z}\{Y(k-1)\} = z^{-1}Y(z)$ and $\mathcal{Z}\{W(k-1)\} = z^{-1}W(z)$ and solve for $Y(z)$ as a function of $W(z)$.

(b) Using the Z-transform, show that the input-output transfer function of the system in (2) is also given by (3).

Hint: Take the Z-transform of (2) and use the Z-transform property that $\mathcal{Z}\{X(k+1)\} = zX(z) - zX(0)$. Then, set $X(0) = 0$, since we are only concerned with the forced respose of the system.

(c) Do a matlab simulation of the response of this system for one sample sequence $w(k)$:

i. Generate the sample sequence $w(k)$ using `w = randn(N,1);` , where `N` is a large number (e.g. 5000).

ii. Generate the sample output sequence $y(k)$ using

```
[y,k] = lsim(sys1,w,k);
```

Notice that the vector `k` of sample indexes must be defined. [3]

iii. Generate and plot the the estimates of the covariances and cross-covariances $\Lambda_{WW}(j)$, $\Lambda_{WY}(j)$, $\Lambda_{YW}(j)$, $\Lambda_{YY}(j)$, for $j = \{-10, -9, \cdots, 0, \cdots 10\}$ using the matlab command `xcov`, e.g.

```
cov_wy = xcov(w,y,10,'coeff');
```

Read the help on `xcov` to understand what the argument `'coeff'` does. [4]

(d) The cross-covariance (cross-correlation) function $\Lambda_{YW}(l)$ and its Z-transform $\hat{\Lambda}_{YW}(z)$ are respectively defined by

$$\Lambda_{YW}(l) = E\{Y(k+l)W(k)\} \text{ and } \hat{\Lambda}_{YW}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{YW}(l)\,.$$

i. Given that the system is WSS, determine $\hat{\Lambda}_{YW}(z)$ utilizing the result that

$$\hat{\Lambda}_{YW}(z) = G(z)\hat{\Lambda}_{WW}(z) \tag{4}$$

where

$$Y(z) = G(z)W(z)$$

[3] Because we are approximating a stationary system, we only need to compute the zero-initial condition forced response and compute the response for a sufficiently long time so that the statistics of sample the sample sequence $y(k)$ will approximate the stationary response.

[4] The MATLAB function `xcov` is part of the signal processing toolbox. Please make sure to download this toolbox.

ii. Determine $\Lambda_{YW}(l)$ and plot $\Lambda_{YW}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare your results with those empirically obtained using MATLAB. Notice that $\Lambda_{YW}(l)$ is a casual sequence, i.e. $\Lambda_{YW}(l) = 0$ for $l < 0$ and all the poles of $\hat{\Lambda}_{YW}(z)$ will be inside the unit circle.

Hints:

- Taking the inverse Z-transform of (4), we obtain

$$\Lambda_{YW}(l) = \sum_{j=-\infty}^{\infty} g(l-j)\Lambda_{WW}(j) \tag{5}$$

Since

$$\Lambda_{WW}(j) = \delta(j) = \begin{cases} 1\,, & j = 0 \\ 0\,, & j \neq 0 \end{cases}$$

Eq. (5) yields

$$\Lambda_{YW}(l) = g(l) \tag{6}$$

where $g(l) = \mathcal{Z}^{-1}\{G(z)\}$.

- Notice that the inverse Z-transform of $G(z)$ can be calculated from the Z-transform tables [5] as follows

$$\begin{aligned} g(l) &= \mathcal{Z}^{-1}\{G(z)\} = \mathcal{Z}^{-1}\left\{\frac{z-0.3}{z-0.5}\right\} \\ &= \mathcal{Z}^{-1}\left\{\frac{0.4z}{z-0.5} + 0.6\right\} \\ &= \begin{cases} 0.4(0.5)^l + 0.6\,\delta(l) & l \geq 0 \\ 0 & l < 0 \end{cases}. \end{aligned}$$

where again $\delta(l)$ is the Kronecker delta function previously defined.

(e) Determine the cross-covariance (cross-correlation) function

$$\Lambda_{WY}(l) = E\{W(k+l)Y(k)\}$$

and $\hat{\Lambda}_{WY}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{WY}(l)$. Plot $\Lambda_{WY}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare your results with those empirically obtained using MATLAB. Notice that $\Lambda_{WY}(l)$ is an anti-casual sequence, i.e. $\Lambda_{WY}(l) = 0$ for $l > 0$ and all the poles of $\hat{\Lambda}_{WY}(z)$ will be outside the unit circle.

Hints:
Remember from the lecture that $\Lambda_{WY}(l) = \Lambda_{YW}(-l)$ and $\hat{\Lambda}_{WY}(z) = \hat{\Lambda}_{YW}(z^{-1})$.

[5] The Z-transform tables can be found in pages 48 and 49 of the Wu, Tomizuka textbook.

(f) The auto-covariance (cross-correlation) function $\Lambda_{YY}(l)$ and its Z-transform $\hat{\Lambda}_{YY}(z)$ are respectively defined by

$$\Lambda_{YY}(l) = E\{Y(k+l)Y(k)\} \text{ and } \hat{\Lambda}_{YY}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{YY}(l)\,.$$

i. Given that the system is WSS, determine $\hat{\Lambda}_{YY}(z)$ utilizing the result that

$$\hat{\Lambda}_{YY}(z) = G(z)\hat{\Lambda}_{WW}(z)G(z^{-1})$$

Notice that $\hat{\Lambda}_{YY}(z)$ will have poles both outside and inside the unit circle.

(g) Compute $\Lambda_{YW}(0)$ directly from Eq. (1) by multiplying both sides of Eq. (1) by $W(k)$ and taking expectations. Verify that this result agrees with the result found in part 1d.

(h) Compute $\Lambda_{YW}(1)$ directly from Eq. (1) by multiplying both sides of Eq. (1) by $W(k-1)$ and taking expectations. Verify that this result agrees with the result found in part 1e.

(i) From Eq. (1) we have

$$Y(k) = 0.5Y(k-1) + W(k) - 0.3W(k-1)\;. \tag{7}$$

Compute $\Lambda_{YY}(0)$ directly from (7) by squaring both sides of Eq. (7) and taking expectations.

2. Consider a second order discrete time system described by

$$\begin{aligned}\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix} &= \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} W(k) \\ Y(k) &= \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + V(k)\end{aligned} \tag{8}$$

where

- $m_X(0) = E\{X(0)\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\Lambda_{XX}(0,0) = E\{X(0)X^T(0)\} = E\{\tilde{X}(0)\tilde{X}^T(0)\} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$
- $W(k)$ and $V(k)$ are white Gaussian sequences
- $m_W = E\{W(k)\} = 10, \quad m_V = E\{V(k)\} = 0$
- $E\left\{ \begin{bmatrix} W(k+j) - m_W \\ V(k+j) \end{bmatrix} \begin{bmatrix} (W(k) - m_W) & V(k) \end{bmatrix} \right\} = \begin{bmatrix} \Sigma_{WW} & 0 \\ 0 & \Sigma_{VV} \end{bmatrix} \delta(j)$ where $\Sigma_{WW} = 1$ and $\Sigma_{VV} = 0.5$.

- $E\left\{\begin{bmatrix} W(k)-m_W \\ V(k) \end{bmatrix} X^T(0)\right\} = 0$

The state space system in (8) can be expressed more compactly as follows

$$\begin{aligned} X(k+1) &= AX(k) + B_w W(k) \\ Y(k) &= CX(k) + V(k) \end{aligned} \tag{9}$$

where

$$\begin{aligned} X(k) &= \begin{bmatrix} X_1(k+1) & X_2(k+1) \end{bmatrix}^T \\ A &= \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix}, \quad B_w = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 3 \end{bmatrix} \end{aligned}$$

(a) Taking expectations in Eq. (9), we obtain

$$\begin{aligned} m_X(k+1) &= Am_X(k) + B_w m_W(k) \\ m_Y(k) &= Cm_X(k) \end{aligned} \tag{10}$$

Use MATLAB to plot $m_y(k) = E\{Y(k)\}$ for $k = 0, 1, \ldots$ until $m_y(k)$ reaches a value close to its steady state value $\bar{m}_Y = \lim_{k\to\infty} m_Y(k)$.

(b) Using MATLAB, compute

$$\Lambda_{XX}(k,0) = E\{(X(k) - m_X(k))(X(k) - m_X(k))^T\}$$

utilizing the covariance propagation Lyapunov equation

$$\Lambda_{XX}(k+1,0) = A\Lambda_{XX}(k,0)A^T + B_w \Sigma_{WW} B_w^T \tag{11}$$

and plot

$$\Lambda_{YY}(k,0) = E\{(Y(k) - m_Y(k))^2\} = C\Lambda_{XX}(k,0)C^T + \Sigma_{VV}$$

for $k = 0, 1, \ldots$ until $\Lambda_{YY}(k,0)$ reaches a value that is very close to its steady state value $\bar{\Lambda}_{YY}(0) = \lim_{k\to\infty} \Lambda_{YY}(k,0)$.

(c) Using MATLAB, compute

$$\Lambda_{XX}(k,5) = E\{(X(k+5) - m_x(k+5))(X(k) - m_x(k))^T\}$$

using the result

$$\Lambda_{XX}(k,j) = A^j \Lambda_{XX}(k,0)$$

and plot

$$\Lambda_{YY}(k,5) = E\{(Y(k+5) - m_y(k+5))(Y(k) - m_y(k))\}$$

for $k = 0, 1, \ldots$, until it reaches a value that is very close to its steady state value $\bar{\Lambda}_{YY}(5) = \lim_{k\to\infty} \Lambda_{YY}(k,5)$.

(d) The steady state auto-covariance

$$\bar{\Lambda}_{XX}(0) = \lim_{k\to\infty} \Lambda_{XX}(k,0)$$

can be obtained by setting

$$\Lambda_{XX}(k+1,0) = \Lambda_{XX}(k,0) = \bar{\Lambda}_{XX}(0)$$

in the covariance propagation Lyapunov equation (11) to obtain the covariance algebraic Lyapunov equation

$$A\bar{\Lambda}_{XX}(0)A^T - \bar{\Lambda}_{XX}(0) = -B_w \Sigma_{WW} B_w^T \tag{12}$$

Use the MATLAB function `dlyap` to compute $\bar{\Lambda}_{XX}(0)$. Then compute and plot the steady state covariances $\bar{\Lambda}_{YY}(j)$ for $j = \{-10,\ -9,\ \cdots,\ 0,\ 1,\ \cdots 10\}$.

(e) Let $G(z) = C(zI - A)^{-1}B$ be the transfer function from $W(z) = \mathcal{Z}\{W(k)\}$ to $Y(z) = \mathcal{Z}\{Y(k)\}$ so that

$$Y(z) = G(z)W(z) + V(z)\ .$$

Assume that $G(\omega)$, Σ_{ww} and Σ_{vv} are known and obtain an expression for the (steady state) output spectral density, $\Phi_{YY}(\omega)$ in terms of $G(\omega)$, Σ_{ww} and Σ_{vv}. Where

$$G(\omega) = G(z)|_{z=e^{jw}} = G(e^{j\omega}).$$

You do not have to explicitly determine $G(z)$, $G(\omega)$, or substitute in the numerical values for Σ_{ww} and Σ_{vv} in this problem.

Hint:
Remember that $W(k)$ and $V(k)$ are independent white random sequences, and $\Lambda_{WV}(l) = E\{(W(k+l) - m_w)V(k)\} = 0$ for all k, l.