1.

$$\min_{U_0}\{J\} = \min_{U_0} [y_d(N) - y(N)]^T \bar{Q}_f [y_d(N) - y(N)] + \sum_{k=0}^{N-1}\{[y_d(k) - y(k)]^T \bar{Q} [y_d(k) - y(k)] + u^T(k) R u(k)\}$$

where $U_k = \{u(k), u(k+1), \cdots u(N-1)\}$

when $k \in [0, N-1]$ we can define

$$J_k^\circ[x(k)] = \min_{U_k} [y_d(N) - y(N)]^T \bar{Q}_f [y_d(N) - y(N)] + \sum_{i=k}^{N-1}\{[y_d(i) - y(i)]^T \bar{Q} [y_d(i) - y(i)] + u^T(i) R u(i)\}$$

especially, for $k=N$

$$J_N^\circ[x(N)] = [y_d(N) - y(N)]^T \bar{Q}_f [y_d(N) - y(N)]$$

since $y_d(k)$ is specified for all k, and $y(k) = C x(k)$

we have

$$J_N^\circ[x(N)] = [y_d(N) - Cx(N)]^T \bar{Q}_f [y_d(N) - Cx(N)]$$

$$= x^T(N) C^T \bar{Q}_f C x(N) + y_d^T(N) \bar{Q}_f y_d(N) - y_d^T(N) \bar{Q}_f C x(N) - x^T(N) C^T \bar{Q}_f y_d(N)$$

since $y_d^T(N) \bar{Q}_f C x(N) = x^T(N) C^T \bar{Q}_f y_d(N)$

then $J_N^\circ[x(N)] = x^T(N) C^T \bar{Q}_f C x(N) + y_d^T(N) \bar{Q}_f y_d(N) - 2 x^T(N) C^T \bar{Q}_f y_d(N)$

we can define $P(N) = C^T \bar{Q}_f C$, $b(N) = -2 C^T \bar{Q}_f y_d(N)$, $c(N) = y_d^T(N) \bar{Q}_f y_d(N)$

so that $J_N^\circ[x(N)]$ can be simplified as below

$$J_N^\circ[x(N)] = x^T(N) P(N) x(N) + x^T(N) b(N) + c(N)$$

With Bellman's principle, we can obtain $J_{k-1}^\circ[x(k-1)]$ from $J_k^\circ[x(k)]$

Assume $J_k^\circ[x(k)]$ has the form that

$$J^\circ[x(k)] = x^T(k) P(k) x(k) + x^T(k) b(k) + c(k)$$

since $x(k) = Ax(k-1) + Bu(k-1)$

$$J^\circ[x(k)] = [Ax(k-1) + Bu(k-1)]^T P(k) [Ax(k-1) + Bu(k-1)] + [Ax(k-1) + Bu(k-1)]^T b(k) + c(k)$$

$$= x^T(k-1) A^T P(k) A x(k-1) + u^T(k-1) B^T P(k) B u(k-1) + 2u^T(k-1) B^T P(k) A x(k-1) + x^T(k-1) A^T b(k) + u^T(k-1) B^T b(k) + c(k)$$

then

$$J^\circ_{k-1}[x(k-1)] = \min_{u(k-1)} \{ [y_d(k-1) - y(k-1)]^T \bar{Q} [y_d(k-1) - y(k-1)] + u^T(k-1) R u(k-1) + J^\circ_k[x(k)] \}$$

$$= \min_{u(k-1)} \{ x^T(k-1) C^T \bar{Q} C x(k-1) + y_d^T(k-1) \bar{Q} y_d(k-1) - y_d^T(k-1) \bar{Q} C x(k-1) - x^T(k-1) C^T \bar{Q} y_d(k-1) + u^T(k-1) R u(k-1) + x^T(k-1) A^T P(k) A x(k-1) + u^T(k-1) B^T P(k) B u(k-1) + 2u^T(k-1) B^T P(k) A x(k-1) + x^T(k-1) A^T b(k) + u^T(k-1) B^T b(k) + c(k) \}$$

$$= \min_{u(k-1)} \{ x^T(k-1) [C^T \bar{Q} C + A^T P(k) A] x(k-1) + x^T(k-1) [A^T b(k) - 2C^T \bar{Q} y_d(k-1)] + u^T(k-1) [R + B^T P(k) B] u(k-1) + u^T(k-1) B^T [2P(k) A x(k-1) + b(k)] + y_d^T(k-1) \bar{Q} y_d(k-1) + c(k) \}$$

Then, we can take the partial derivative of the term, and set it to 0.

i.e. $[R + B^T P(k) B] u^\circ(k-1) + \frac{1}{2} B^T [2P(k) A x(k-1) + b(k)] = 0$

then $u^\circ(k-1) = -[R + B^T P(k) B]^{-1} B^T [P(k) A x(k-1) + \frac{1}{2} b(k)]$

bring the $u^\circ(k-1)$ back to the $J^\circ_{k-1}[x(k-1)]$, we have

$$J^\circ_{k-1}[x(k-1)] = x^T(k-1) \{ C^T \bar{Q} C + A^T P(k) A - A^T P(k) B [R + B^T P(k) B]^{-1} B^T P(k) A \} x(k-1) + x^T(k-1) \{ A^T b(k) - 2C^T \bar{Q} y_d(k-1) - A^T P(k) B [R + B^T P(k) B]^{-1} B^T b(k) \} + \{ y_d^T(k-1) \bar{Q} y_d(k-1) + c(k) - \frac{1}{4} b^T(k) B [R + B^T P(k) B]^{-1} B^T b(k) \}$$

so, we have $P(k-1) = C^T Q C + A^T P(k) A - A^T P(k) B [R + B^T P(k) B]^{-1} B^T P(k) A$

$b(k-1) = A^T b(k) - 2C^T \bar{Q} y_d(k-1) - A^T P(k) B [R + B^T P(k) B]^{-1} B^T b(k)$

$c(k-1) = y_d^T(k-1) \bar{Q} y_d(k-1) + c(k) - \frac{1}{4} b^T(k) B [R + B^T P(k) B]^{-1} B^T b(k)$

and $J_{k-1}^{\circ}[x(k-1)] = x^T(k-1) P(k-1) x(k-1) + x^T(k-1) b(k-1) + c(k-1)$

2.

define $\bar{x}(k) = [\, x(k) \quad x_w(k) \,]^T$ $\quad \bar{A} = \begin{bmatrix} A & B_w C_w \\ 0 & A_w \end{bmatrix}$

$\bar{B} = [\, B \quad 0 \,]^T \qquad \bar{B}_\eta = [\, 0 \quad B_n \,]^T$

then the augmented state equation can be written as

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{B}_\eta \eta(k)$$

then we have

$$E\{\bar{x}(0)\} = \bar{x}_0 = [\, x_0 \quad x_{w0} \,]^T$$

$$E\{(\bar{x}(0) - \bar{x}_0)(\bar{x}(0) - \bar{x}_0)^T\} = \bar{X}_0 = \begin{bmatrix} X_0 & 0 \\ 0 & X_{w0} \end{bmatrix}$$

define $\bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{Q}_f = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}$

then $J = E\left\{ \bar{x}^T(N) \bar{Q}_f \bar{x}(N) + \sum_{k=0}^{N-1} [\bar{x}^T(k) \bar{Q} \bar{x}(k) + u^T(k) R u(k)] \right\}$

(a) if $u(k)$ is allowed to be a function of both $x(0), \cdots, x(k)$, $x_w(0), \cdots, x_w(k)$

then $u(k)$ can be a function of $\bar{x}(k)$

so from previous study, we have $u(k) = -K(k+1)\bar{x}(k)$

$K(k) = (\bar{B}^T P(k) \bar{B} + R)^{-1} \bar{B}^T P(k) \bar{A}$

$P(N) = \bar{Q}_f$

$P(k-1) = \bar{A}^T P(k) \bar{A} + \bar{Q} - \bar{A}^T P(k) \bar{B} (\bar{B}^T P(k) \bar{B} + R)^{-1} \bar{B}^T P(k) \bar{A}$

(b) if we can not measure the state, and only have access to the output

since $y(k) = Cx(k) + v(k)$

we can let $\bar{C} = [C \quad 0]$

then $y(k) = \bar{C}\bar{x}(k) + v(k)$

then we can have the kalman filter

$$\hat{\bar{x}}^{o}(k+1) = \bar{A}\hat{\bar{x}}(k) + \bar{B}u(k)$$

$$\hat{\bar{x}}(k) = \hat{\bar{x}}^{o}(k) + F(k)[y(k) - \bar{C}\hat{\bar{x}}^{o}(k)]$$

where, $F(k) = M(k)\bar{C}^T[\bar{C}M(k)\bar{C}^T + V]^{-1}$

$$M(k+1) = \bar{A}Z(k)\bar{A}^T + \bar{B}_\eta \Gamma \bar{B}_\eta^T$$

$$Z(k) = M(k) - M(k)\bar{C}^T[\bar{C}M(k)\bar{C}^T + V]^{-1}\bar{C}M(k)$$

$$M(0) = \bar{X}_0$$

then we use the state estimation to have $u(k)$

$u(k) = -K(k+1)\hat{\bar{x}}(k)$

$K(k) = (\bar{B}^T P(k)\bar{B} + R)^{-1}\bar{B}^T P(k)\bar{A}$

$P(N) = \bar{Q}_f$

$P(k-1) = \bar{A}^T P(k)\bar{A} + \bar{Q} - \bar{A}^T P(k)\bar{B}(\bar{B}^T P(k)\bar{B} + R)^{-1}\bar{B}^T P(k)\bar{A}$