Tsinghua Berkeley Shenzhen Institute (TBSI)
ME233 Advanced Control Systems II
Spring 2024

Homework #5

Assigned: April 13 (Saturday)
Due: April 22 (Monday)

1. **Finite-Horizon Optimal Tracking Problem:**

   Consider the discrete time system

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k) \\
y(k) &= C\,x(k)
\end{aligned}
$$

   where $x \in \mathbb{R}^n$ and $y,\ u \in \mathbb{R}^m$. Assume the existence of a *known* reference output sequence

$$
\big(y_d(0),\, y_d(1),\, \ldots,\, y_d(N)\big)
$$

   The optimal control is sought to minimize the finite-horizon quadratic performance index:

$$
\begin{aligned}
J = {} & [y_d(N) - y(N)]^T\, \bar{Q}_f\, [y_d(N) - y(N)] \\
& + \sum_{k=0}^{N-1} \left\{ [y_d(k) - y(k)]^T\, \bar{Q}\, [y_d(k) - y(k)] + u^T(k)\, R u(k) \right\}
\end{aligned}
$$

   where $\bar{Q}_f$, $\bar{Q}$ and R are symmetric and positive definite matrices of the appropriate dimensions. Find the optimal control law by applying dynamic programming and utilizing the Bellman equation

$$
J_k^o[x(k)] = \min_{u(k)} \big(L[x(k), u(k)] + J_{k+1}^o[x(k+1)] \big)
$$

   where

$$
\begin{aligned}
L[x(k), u(k)] &= [y_d(k) - y(k)]^T\, \bar{Q}\, [y_d(k) - y(k)] + u^T(k) R u(k) \\
J_N^o[x(N)] &= [y_d(N) - y(N)]^T\, \bar{Q}_f\, [y_d(N) - y(N)]\,. \qquad (1)
\end{aligned}
$$

   Hint: Show that the optimal cost $J_N^o[X(N)]$ in Eq. (1) can be written as

$$
J_k^o[x(N)] = x^T(N) P(N) x(N) + x^T(N) b(N) + c(N)\,.
$$

   Then, using induction and the Bellman equation, show that the optimal cost to go from state $x(k)$ to the final state can be expressed as

$$
J_k^o[x(k)] = x^T(k) P(k) x(k) + x^T(k) b(k) + c(k)\,.
$$

   Obtain recursive expressions for $P(k)$, $b(k)$, and $c(k)$ (starting at $k = N$ and going backwards to $k = 0$).

2. **LQG Problem with input colored noise**

A discrete-time system is described by

$$x(k+1) = A\,x(k) + B\,u(k) + B_w\,w(k) \tag{2}$$

where $x(k) \in \mathcal{R}^{p_1}$ is the state, $u(k) \in \mathcal{R}^{p_2}$ is the control input. $w(k) \in \mathcal{R}^{p_3}$ is stationary Gaussian colored noise given by the output of the state space model

$$x_w(k+1) = A_w\,x_w(k) + B_n\,\eta(k) \tag{3}$$

$$w(k) = C_w\,x_w(k) \tag{4}$$

where $n(k) \in \mathcal{R}^{p_5}$ is stationary and white and $x_w(k) \in \mathcal{R}^{p_4}$. Assume that $x(0)$, $x_w(0)$ and $\eta(k)$ are independent and normally distributed with

$$E\{x(0)\} = x_o, \quad E\{(x(0)-x_o)(x(0)-x_o)^T\} = X_o$$

$$E\{x_w(0)\} = x_{wo}, \quad E\{(x_w(0)-x_{wo})(x_w(0)-x_{wo})^T\} = X_{wo}$$

$$E\{\eta(k)\} = 0, \quad E\{\eta(k)\eta(k+l)^T\} = \Gamma\delta(l)$$

The performance index to be minimized is

$$J = E\left\{ x^T(N)\,Q_f\,x(N) + \sum_{k=0}^{N-1}\left[x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right]\right\}$$

where the expectation must be taken over all underlying random quantities in each of the following situations that will be described below. For each situation obtain the equations that must be solved to find the optimal control:

(a) $u(k)$ is allowed to be a function of both $x(0), \ldots, x(k)$ **and** $x_w(0), \ldots, x_w(k)$

(b) $u(k)$ is allowed to be a function of $y(0), \ldots, y(k)$ where $y(k) = C\,x(k) + v(k)$ and $v(k)$ is a zero-mean, white and Gaussian noise with covariance $E\{v(k)v(k+l)\} = V\,\delta(l)$ and is independent from $x(0)$, $x_w(0)$ and $\eta(k)$.

Hint: Notice that the overall system dynamics in state equation (2) and in state and output equations (3) and (4) can be described by a single state equation by the defining the augmented state

$$\bar{x}^T(k) = \begin{bmatrix} x(k) & x_w(k) \end{bmatrix}^T$$

and the augmented state equation

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{B}_\eta\eta(k)$$

Obtain the experssions for $\bar{A}$, $\bar{B}$ and $\bar{B}_\eta$ and write down the LQR and Kalman filter equations that are needed for each case.