1. A product is produced by three different factories: A, B, and C. Factories A, B, and C respectively produce 25%, 50%, and 25% of the total production. In factories A and B, 98% of the items produced are not defective, whereas in factory C, 99% are not defective. Calculate: (a) The probability that a randomly-chosen item is defective and (b) the probability that a (randomly-chosen) non-defective item comes from factory C.

   Hint: Let events A, B, and C respectively correspond to a randomly-chosen item coming from factory A, factory B, and factory C. Then, for example, $P(A) = 0.25$. Let events D and N respectively correspond to the event that the item is defective and not defective. The statement that "in factory C, 99% of the items are not defective" means that the conditional probability of an item being not defective given that it is produced by factory C is 0.98, i.e. $P(N/C) = 0.98$. In part (a) you are asked to compute $P(D)$ whereas in part (b) you are asked to compute $P(C/N)$.

Solution:

Let events A, B, C respectively correspond to a randomly-chosen item coming from factory A, B, C

then we have $P(A) = P(C) = 0.25 \quad P(B) = 0.5$

Let events D, N respectively correspond to the event that the item is defective and not defective

then we have $P(N|A) = P(N|B) = 0.98 \quad P(N|C) = 0.99$

(a) $P(N) = P(A)P(N|A) + P(B)P(N|B) + P(C)P(N|C)$

$= 0.25 \times 0.98 + 0.5 \times 0.98 + 0.25 \times 0.99$

$= 0.9825$

(b) $P(C|N) = \dfrac{P(C)P(N|C)}{P(N)} = \dfrac{0.25 \times 0.99}{0.9825} = 0.2519$

So the probability that a randomly-chosen item is defective is 0.9825

the probability that a non-defective item comes from C is 0.2519

2. In the "Monty Hall" three-door problem, a contestant is asked to choose one of three doors. One of the three doors conceals a prize while the other two do not. After the contestant chooses, Monty Hall (the master of ceremonies of the Let's Make a Deal television show) opens one of the two doors the player did not choose to reveal one door that does not conceal the prize. The contestant is then permitted to either stay with their original choice or switch to the other unopened door. Determine the contestant's probability of getting the prize if she switches. Assume that before Monty Hall opens one of the doors, the prize is equally likely to be hidden behind each of the three doors.

Hint: Let the doors be called x, y, and z. Let C_x be the event that the prize is behind door x and so on. Let H_x be the event that the host opens door x and so on. Assuming that you choose door x, the probability that you win a car if you then switch your choice is given by

$$P((H_z \cap C_y) \cup (H_y \cap C_z)) .$$

Notice that, by Baye's rule, $P(H_z \cap C_y) = P(H_z|C_y)P(C_y)$.

Solution:

the notation is as same as it shown in the hint

then we have

$$P(C_x) = P(C_y) = P(C_z) = \frac{1}{3}$$

$$P(C_x \cap C_y) = P(C_x \cap C_z) = P(C_y \cap C_z) = 0$$

w.l.o.g. we assume that door x is chosen

the probability we win can be given by $P((H_z \cap C_y) \cup (H_y \cap C_z))$

$$P((H_z \cap C_y) \cup (H_y \cap C_z)) = P(H_z \cap C_y) + P(H_y \cap C_z) - P((H_z \cap C_y) \cap (H_y \cap C_z))$$

since $P(C_y \cap C_z) = 0$,

$$P((H_z \cap C_y) \cap (H_y \cap C_z)) = 0$$

then

$$\begin{aligned} P((H_z \cap C_y) \cup (H_y \cap C_z)) &= P(H_z \cap C_y) + P(H_y \cap C_z) \\ &= P(H_z | C_y) P(C_y) + P(H_y | C_z) P(C_z) \\ &= 1 \times \frac{1}{3} + 1 \times \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

All in all, the probability we win after switching is $\frac{2}{3}$

3. Consider three independent random variables, X_1, X_2, and X_3, each of which is uniformly distributed between 0 and 1. Obtain the probability density function (PDF) for:

- $Y = X_1 + X_2$

Hint: Recall from lecture 3 that, if X_1 and X_2 are two independent random variables and $Y = X_1 + X_2$, then the PDF of Y, $p_Y(y)$, is the convolution of the PDF of X_1, $p_{X_1}(x)$, and the PDF of X_2, $p_{X_2}(x)$, i.e.

$$p_Y(y) = p_{X_1}(\cdot) * p_{X_2}(\cdot) = \int_{-\infty}^{\infty} p_{X_1}(x_1) p_{X_2}(y - x_1) dx_1 \ .$$

Notice that the PDF of X_1 only takes on the values of 1 and 0. Thus, we only need to integrate $p_{X_2}(y - x_1)$ over the regions where $p_{X_1}(x_1)$ has a value of 1. Thus,

$$p_Y(y) = \int_0^1 p_{X_2}(y - x_1) dx_1 \ .$$

Solution:

$$P_{X_1}(x) = P_{X_2}(x) = P_{X_3}(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{others} \end{cases}$$

$$Y = X_1 + X_2$$

$$\begin{aligned} P_Y(y) &= P_{X_1} * P_{X_2} \\ &= \int_{-\infty}^{+\infty} P_{X_1}(x_1) P_{X_2}(y - x_1) \, dx_1 \\ &= \int_0^1 P_{X_2}(y - x_1) \, dx_1 \\ &= \begin{cases} y & y \in [0,1) \\ 2-y & y \in [1,2] \\ 0 & \text{others} \end{cases} \end{aligned}$$

the PDF of Y is $P_Y(y) = \begin{cases} y & y \in [0,1) \\ 2-y & y \in [1,2] \\ 0 & \text{others.} \end{cases}$

4. Let $X \sim N(m_X, \sigma_X^2)$, i.e. let X be a Gaussian random variable with mean m_X and variance σ_X^2. Then the moment generating function of X is

$$P_X(j\omega) = \mathcal{F}\{p_X(\cdot)\} = E\left\{e^{-j\omega X}\right\} = \exp\left(-j\omega m_X - \frac{\sigma_X^2 \omega^2}{2}\right).$$

Now let X and Y be independent, where $Y \sim N(m_Y, \sigma_Y^2)$. Show that if $Z = X + Y$, then $Z \sim N(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$.

Proof:

$$X \sim N(m_X, \sigma_X^2) \qquad Y \sim N(m_Y, \sigma_Y^2)$$

$$\mathcal{F}\{P_X(\cdot)\} = P_X(jw) = e^{-jwm_X - \frac{\sigma_X^2 w^2}{2}}$$

$$\mathcal{F}\{P_Y(\cdot)\} = P_Y(jw) = e^{-jwm_Y - \frac{\sigma_Y^2 w^2}{2}}$$

since X, Y are independent

$$Z = X+Y \qquad P_Z = P_X * P_Y$$

then we have

$$\mathcal{F}\{P_Z(\cdot)\} = \mathcal{F}\{P_X(\cdot)\} \cdot \mathcal{F}\{P_Y(\cdot)\}$$

$$= e^{-jwm_X - \frac{\sigma_X^2 w^2}{2}} \cdot e^{-jwm_Y - \frac{\sigma_Y^2 w^2}{2}}$$

$$= e^{-jw(m_X + m_Y) - \frac{(\sigma_X^2 + \sigma_Y^2) w^2}{2}}$$

After inverse Flourie transform.

we can have $P_Z(z)$. and it can show that

$$Z \sim N(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$$

Q.E.D