1.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 0 \end{bmatrix} \qquad B = [0\ 0\ 0\ 1]^T$$

$$P = [B\ \ AB\ \ A^2B\ \ A^3B] = \begin{bmatrix} 0 & 0 & 2w & 0 \\ 0 & 2w & 0 & -2w^3 \\ 0 & 1 & 0 & -4w^2 \\ 1 & 0 & -4w^2 & 0 \end{bmatrix}$$

since $w > 0$, $\det(P) = -12w^4 \neq 0$

then we have $\operatorname{rank}(P) = 4$. i.e P has full rank

the system is controllable

2. $C_r = [1\ 0\ 0\ 0]$

$$Q_r = \begin{bmatrix} C_r \\ C_rA \\ C_rA^2 \\ C_rA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & -w^2 & 0 & 0 \end{bmatrix}$$

$w > 0$

$\operatorname{rank}(Q_r) = 3 < 4$, so when the output is the radial location perturbation, the system is not observable

3. $C_z = [0\ 0\ 1\ 0]^T$

$$Q_z = \begin{bmatrix} C_z \\ C_zA \\ C_zA^2 \\ C_zA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 0 \\ -6w^3 & 0 & 0 & -4w^2 \end{bmatrix}$$

since $w > 0$, $\det(Q_z) = -12w^4 \neq 0$

then we have $\operatorname{rank}(Q_z) = 4$. i.e Q_z has full rank

the system is observable

4. (a) from Eqs. (1), we have

$$s^2R(s) - 3w^2R(s) - 2wsZ(s) = 0$$
$$s^2Z(s) + 2wsR(s) = U(s)$$
$$\Rightarrow \begin{bmatrix} s^2-3w^2 & -2ws \\ 2ws & s^2 \end{bmatrix}\begin{bmatrix} R(s) \\ Z(s) \end{bmatrix} = \begin{bmatrix} 0 \\ U(s) \end{bmatrix}$$

i.e. $M(s) = \begin{bmatrix} s^2-3w^2 & -2ws \\ 2ws & s^2 \end{bmatrix}$ such that $M(s)\begin{bmatrix} R(s) \\ Z(s) \end{bmatrix} = \begin{bmatrix} 0 \\ U(s) \end{bmatrix}$

(b)

from Eqs. (1)

we also have $G_r(s) = \dfrac{R(s)}{U(s)} = \dfrac{2ws}{s^4 + w^2s^2}$

$$G_z(s) = \frac{Z(s)}{U(s)} = \frac{s^2 - 3w^2}{s^4 + w^2s^2}$$

i.e. $B_r(s) = 2ws$, $B_z(s) = s^2 - 3w^2$

such that $G_r(s) = \dfrac{B_r(s)}{s^2(s^2+w^2)}$

$$G_z(s) = \frac{B_z(s)}{s^2(s^2+w^2)}$$

(c) the poles of $G_r(s)$ are $0, 0, +wj, -wj$

i.e. the eigenvalue λ of A are $0, 0, +wj, -wj$

for $\lambda = 0$

$$\text{rank}\begin{bmatrix} A - \lambda I \\ C_r \end{bmatrix} = 3 < 4$$

so it's not detectable

(d) when $y(t) = z(t)$

the system is observable, so it is detectable.

5.

$$sX(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} X(s) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} U(s)$$

$$Y(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} X(s)$$

$$G(s) = C(sI-A)^{-1}B = \frac{s^2 - 4.4\times10^{-16}s - 3}{s^2(s^2+1)}$$

4 open loop poles: $0, 0, -j, j$

2 open loop zeros

since $G_z(s) = \dfrac{Z(s)}{U(s)} = \dfrac{s^2 - 3w^2}{s^2(s^2+w^2)} = \dfrac{s^2-3}{s^2(s^2+1)}$

$$(1+G_0(-s))(1+G_0(s)) = 1 + \frac{1}{\rho} G_z(s)\, G_z(-s)$$

$$= 1 + \frac{1}{\rho}\cdot\frac{s^2-3}{s^2(s^2+1)}\cdot\frac{s^2-3}{s^2(s^2+1)}$$

$$= 1 + \frac{1}{\rho}\,\frac{(s^2-3)^2}{s^4(s^2+1)^2}$$

8 poles: $0, 0, 0, 0, +j, -j, +j, -j$

4 zeros: $\sqrt{3}, -\sqrt{3}, \sqrt{3}, -\sqrt{3}$

$$\varphi = \frac{(2k+1)\pi}{8-4} = 45^\circ$$