1.

(a) (i) $\hat{y} = E\{Y\} = E\{X + V_1\} = E\{X\} + E\{V_1\} = 10 - 0.5 = 9.5$

$$\Lambda_{YY} = E\{(Y - \hat{y})^2\}$$
$$= E\{Y^2\} - E^2\{Y\}$$
$$= E\{(X + V_1)^2\} - (\hat{x} + \hat{v}_1)^2$$
$$= E\{X^2\} + E\{V_1^2\} + 2E\{XV_1\} - \hat{x}^2 - \hat{v}_1^2 - 2\hat{x}\hat{v}_1$$

since $\Lambda_{XX} = E\{(X - \hat{x})^2\} = E\{X^2\} - \hat{x}^2$

$\Lambda_{V_1V_1} = E\{V_1^2\} - \hat{v}_1^2$

$\Lambda_{XV_1} = E\{(X - \hat{x})(V_1 - \hat{v}_1)\} = E\{XV_1\} - \hat{x}\hat{v}_1$

then we have $\Lambda_{YY} = \Lambda_{XX} + \Lambda_{V_1V_1} + 2\Lambda_{XV_1}$

since X, V_1 are independent, $\Lambda_{XV_1} = 0$

then $\Lambda_{YY} = \Lambda_{XX} + \Lambda_{V_1V_1} = 2 + 1 = 3$

(ii) $\Lambda_{XY} = E\{(X - \hat{x})(Y - \hat{y})\}$
$$= E\{(X - \hat{x})(X + V_1 - \hat{x} - \hat{v}_1)\}$$
$$= E\{X^2 + XV_1 - X\hat{x} - X\hat{v}_1 - \hat{x}X - \hat{x}V_1 + \hat{x}^2 + \hat{x}\hat{v}_1\}$$
$$= E\{X^2\} + E\{XV_1\} - \hat{x}^2 - \hat{x}\hat{v}_1 - \hat{x}^2 - \hat{x}\hat{v}_1 + \hat{x}^2 + \hat{x}\hat{v}_1$$
$$= E\{X^2\} + E\{XV_1\} - \hat{x}^2 - \hat{x}\hat{v}_1$$
$$= \Lambda_{XX} + \Lambda_{XV_1}$$
$$= 2 + 0$$
$$= 2$$

(iii) $\hat{x}_{|y=9} = \hat{x} + \Lambda_{XY}\Lambda_{YY}^{-1}(y - \hat{y})$
$$= 10 + 2 \times \frac{1}{3} \times (9 - 9.5)$$
$$= 9\frac{2}{3}$$

(iv) $\hat{X}_{|Y} = \hat{x} + \Lambda_{XY}\Lambda_{YY}^{-1}(Y - \hat{y}) = 10 + \frac{2}{3}(Y - 9.5)$

$\check{X}_{|Y} = X - \hat{X}_{|Y} = X - \hat{x} - \Lambda_{XY}\Lambda_{YY}^{-1}(Y - \hat{y}) = X - \frac{2}{3}Y - \frac{11}{3}$

$$\Lambda_{\tilde{x}|y\tilde{x}|y} = E\{\tilde{X}|_Y \tilde{X}|_Y^T\}$$

$$= E\{(X - \tfrac{2}{3}Y - \tfrac{11}{3})^2\}$$

$$= E\{X^2 + \tfrac{4}{9}Y^2 + \tfrac{121}{9} - \tfrac{4}{3}XY - \tfrac{22}{3}X + \tfrac{44}{9}Y\}$$

$$= E\{X^2\} + \tfrac{4}{9}E\{Y^2\} - \tfrac{4}{3}E\{XY\} + \tfrac{121}{9} - \tfrac{22}{3}\hat{x} + \tfrac{44}{9}\hat{y}$$

$$= E^2\{X\} + \Lambda_{xx} + \tfrac{4}{9}[E^2\{Y\} + \Lambda_{YY}] - \tfrac{4}{3}[E\{X\}E\{Y\} + \Lambda_{XY}] + \tfrac{121}{9} - \tfrac{22}{3}\hat{x} + \tfrac{44}{9}\hat{y}$$

$$= 10^2 + 2 + \tfrac{4}{9}(9.5^2 + 3) - \tfrac{4}{3}(9.5\times 10 + 2) + \tfrac{121}{9} - \tfrac{22}{3}\times 10 + \tfrac{44}{9}\times 9.5$$

$$= \tfrac{2}{3}$$

(b) (i) Similar to $\hat{y}$, Λ_{YY} determined in (a)

$$\hat{z} = E\{Z\} = E\{X + V_2\} = E\{X\} + E\{V_2\} = 10 + 0.5 = 10.5$$

$$\Lambda_{zz} = \Lambda_{xx} + \Lambda_{V_2V_2} = 2 + 2 = 4$$

(ii) Similar to Λ_{XY}

$$\Lambda_{XZ} = \Lambda_{XX} + \Lambda_{XV_2} = 2 + 0 = 2$$

(iii)
$$\hat{x}|_{z=11} = \hat{x} + \Lambda_{XZ}\Lambda_{ZZ}^{-1}(z - \hat{z})$$

$$= 10 + 2 \times \tfrac{1}{4} \times (11 - 10.5)$$

$$= 10.25$$

(iv)
$$\hat{X}|_Z = \hat{x} + \Lambda_{XZ}\Lambda_{ZZ}^{-1}(Z - \hat{z}) = 10 + \tfrac{1}{2}(Z - 10.5)$$

$$\tilde{X}|_Z = X - \hat{X}|_Z = X - \tfrac{1}{2}Z - \tfrac{19}{4}$$

$$\Lambda_{\tilde{X}|Z\tilde{X}|Z} = E\{\tilde{X}|_Z \tilde{X}_Z^T\}$$

$$= E\{(X - \tfrac{1}{2}Z - \tfrac{19}{4})^2\}$$

$$= E\{X^2\} + \tfrac{1}{4}E\{Z^2\} + (\tfrac{19}{4})^2 - E\{XZ\} - \tfrac{19}{2}E\{X\} + \tfrac{19}{4}E\{Z\}$$

$$= 102 + \tfrac{1}{4}\times(\tfrac{21^2}{4} + 4) + \tfrac{19^2}{16} - (2 + 105) - \tfrac{19}{2}\times 10 + \tfrac{19}{4}\times\tfrac{21}{2}$$

$$= 1$$

(c) (i) $\hat{w} = E\{W\} = [\hat{y} \quad \hat{z}]^T = [9.5 \quad 10.5]^T$

$$\Lambda_{ww} = E\{(W-\hat{w})(W-\hat{w})^T\}$$

$$= E\left\{\begin{bmatrix} \tilde{Y}\tilde{Y}^T & \tilde{Y}\tilde{Z}^T \\ \tilde{Z}\tilde{Y}^T & \tilde{Z}\tilde{Z}^T \end{bmatrix}\right\}$$

$$= E\left\{\begin{bmatrix} \Lambda_{YY} & \Lambda_{YZ} \\ \Lambda_{ZY} & \Lambda_{ZZ} \end{bmatrix}\right\}$$

$$\Lambda_{YZ} = E\{(\tilde{X}+\tilde{V}_1)(\tilde{X}+\tilde{V}_2)^T\}$$

$$= E\{\tilde{X}\tilde{X}^T + \tilde{V}_1\tilde{X} + \tilde{X}\tilde{V}_2 + \tilde{V}_1\tilde{V}_2\}$$

$$= \Lambda_{XX} + \Lambda_{V_1X} + \Lambda_{XV_2} + \Lambda_{V_1V_2}$$

$$= 2$$

$$\Lambda_{ZY} = 2$$

then $\Lambda_{ww} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$

(ii) $\Lambda_{XW} = E\{\tilde{X}\tilde{W}^T\}$

$$= E\{[\tilde{X}\tilde{X}^T + \tilde{X}\tilde{V}_1^T \quad \tilde{X}\tilde{X}^T + \tilde{X}\tilde{V}_2^T]\}$$

$$= [\Lambda_{XX} + \Lambda_{XV_1} \quad \Lambda_{XX} + \Lambda_{XV_2}]$$

$$= [2 \quad 2]$$

(iii) $\hat{x}|_{w=[9 \quad 11]^T} = \hat{x} + \Lambda_{XW}\Lambda_{WW}^{-1}(w-\hat{w})$

$$= 10 + [2 \quad 2]\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 9-9.5 \\ 11-10.5 \end{bmatrix}$$

$$= 10 + [2 \quad 2]\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{8} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= 9.875$$

(iv) $\hat{X}_{|W} = \hat{x} + \Lambda_x \Lambda_{xw}^{-1} \Lambda_{ww}^{-1} (w - \hat{w})$

$= 10 + \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} y - 9.5 \\ z - 10.5 \end{bmatrix}$

$\tilde{X}_{|W} = X - \hat{X}_{|W} = X - 10 - \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} (w - \hat{w})$

$\Lambda_{\tilde{X}|W \tilde{X}|W} = E\{ \tilde{X}_{|W} \tilde{X}_{|W}^T \}$

$= E\left\{ \left[X - 10 - \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} (w - \hat{w})\right] \left[X^T - 10 - (w - \hat{w})^T \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \right] \right\}$

$= E\left\{ X^2 - 10X - \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} (w - \hat{w}) X \right.$

$- 10X + 100 + 10 \times \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} (w - \hat{w})$

$- X (w - \hat{w})^T \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} + 10 (w - \hat{w})^T \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$

$\left. + \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} (w - \hat{w})(w - \hat{w})^T \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \right\}$

$= 102 - 100 - 1.5 - 100 + 100 + 0 - 1.5 + 0 + 1.5$

$= 0.5$

so, we have $\Lambda_{\tilde{X}|W \tilde{X}|W} = 0.5$

(d) (i) $\hat{z}_{|y=9} = \hat{z} + \Lambda_{zy} \Lambda_{yy}^{-1} (y - \hat{y})$

$= 10.5 + 2 \times \frac{1}{3} \times (9 - 9.5)$

$= \frac{61}{6}$

(ii)

$\Lambda_{\tilde{X}|Y \tilde{Z}|Y} = \Lambda_{xz} - \Lambda_{xy} \Lambda_{yy}^{-1} \Lambda_{xz}$

$= 2 - 2 \times \frac{1}{3} \times 2$

$= \frac{2}{3}$

$\Lambda_{\tilde{Z}|Y \tilde{Z}|Y} = \Lambda_{zz} - \Lambda_{zy} \Lambda_{yy}^{-1} \Lambda_{xz}$

$= 4 - 2 \cdot \frac{1}{3} \times 2$

$= \frac{8}{3}$

(iii) $\hat{x}_{|(y=9,\, z=11)} = \hat{x}|_{y=9} + \Lambda_{\tilde{x}|y\,\tilde{z}|y} \Lambda^{-1}_{\tilde{z}|y\,\tilde{z}|y} (z - \hat{z}|_y)$

$= 9\frac{2}{3} + \frac{2}{3} \times \frac{3}{8} \times (11 - \frac{61}{6})$

$= 9.875$

the result is as same as it in (c)(iii)

(iv) $\Lambda_{\tilde{x}|w\,\tilde{x}|w} = \Lambda_{\tilde{x}|(y,z)\,\tilde{x}|(y,z)}$

$= \Lambda_{\tilde{x}|y\,\tilde{x}|y} - \Lambda_{\tilde{x}|y\,\tilde{z}|y} \Lambda^{-1}_{\tilde{z}|y\,\tilde{z}|y} \Lambda_{\tilde{z}|y\,\tilde{x}|y}$

$= \frac{2}{3} - \frac{2}{3} \times \frac{3}{8} \times \frac{2}{3}$

$= 0.5$ the result is as same as it in (c)(iv)

2.

$\Lambda_{X\bar{Y}(k)} = E\{X\, \bar{Y}^T(k)\}$

$= E\{[X\,Y(0) \quad X\,Y(1) \quad \cdots \quad X\,Y(k)]\}$

$= E\{[X^2 + XV(0) \quad X^2 + XV(1) \quad \cdots \quad X^2 + XV(k)]\}$

$= [X_0 \quad X_0 \quad \cdots \quad X_0]$

$\Lambda_{\bar{Y}(k)\bar{Y}(k)} = E\{\bar{Y}(k)\, \bar{Y}^T(k)\}$

$$= E\left\{ \begin{bmatrix} Y^2(0) & Y(0)Y(1) & \cdots & Y(0)Y(k) \\ Y(1)Y(0) & Y^2(1) & \cdots & Y(1)Y(k) \\ \vdots & & & \vdots \\ Y(k)Y(0) & Y(k)Y(1) & \cdots & Y^2(k) \end{bmatrix} \right\}$$

since $E\{Y(k+j)\,Y(k)\} = E\{X^2 + V(k+j)X + V(k)X + V(k+j)V(k)\}$

$= X_0 + \Sigma_V\, \delta(j)$

$$\Lambda_{\bar{Y}(k)\bar{Y}(k)} = \begin{bmatrix} X_0 + \Sigma_V & X_0 & \cdots & X_0 \\ X_0 & X_0 + \Sigma_V & \cdots & X_0 \\ \vdots & & & \vdots \\ X_0 & X_0 & \cdots & X_0 + \Sigma_V \end{bmatrix}$$

let $w(k) = [1 \quad 1 \quad \cdots \quad 1]^T \in \mathbb{R}^{k+1}$

$$\Lambda_{x\bar{y}(k)} = X_0 w^T(k)$$

$$\Lambda_{\bar{y}(k)\bar{y}(k)} = \Sigma_v I + X_0 w(k) w^T(k)$$

(b) $$\Lambda^{-1}_{\bar{y}(k)\bar{y}(k)} = \frac{1}{\Sigma_v} I - \frac{1}{1 + w^T(k)\Sigma_v^{-1} I X_0 w(k)} \times \Sigma_v^{-1} I \times X_0 w(k) w^T(k) \Sigma_v^{-1}$$

$$= \Sigma_v^{-1} I - \frac{\Sigma_v^{-2} X_0}{1 + \Sigma_v^{-1} X_0 (k+1)} w(k) w^T(k)$$

then $$\hat{x}_{|\bar{y}(k)} = \hat{x} + \Lambda_{x\bar{y}(k)} \Lambda^{-1}_{\bar{y}(k)\bar{y}(k)} \bar{y}(k)$$

$$= 0 + X_0 w^T(k) \left(\Sigma_v^{-1} I - \frac{\Sigma_v^{-2} X_0}{1 + \Sigma_v^{-1} X_0 (k+1)} w(k) w^T(k) \right) \bar{y}(k)$$

$$= X_0 \Sigma_v^{-1} w^T(k) \bar{y}(k) - \frac{\Sigma_v^{-2} X_0}{1 + \Sigma_v^{-1} X_0 (k+1)} w^T(k) \bar{y}(k)$$

$$= \left(X_0 \Sigma_v^{-1} - \frac{\Sigma_v^{-2} X_0^2 (k+1)}{1 + \Sigma_v^{-1} X_0 (k+1)} \right) w^T(k) \bar{y}(k)$$

$$= \frac{X_0 \Sigma_v^{-1}}{1 + \Sigma_v^{-1} X_0 (k+1)} w^T(k) \bar{y}(k)$$

$$\Lambda_{\bar{x}|\bar{y}(k)\,\bar{x}|\bar{y}(k)} = \Lambda_{xx} - \Lambda_{x\bar{y}(k)} \Lambda^{-1}_{\bar{y}(k)\bar{y}(k)} \Lambda_{\bar{y}(k)x}$$

$$= X_0 - X_0 w^T(k) \left(\Sigma_v^{-1} I - \frac{\Sigma_v^{-2} X_0}{1 + \Sigma_v^{-1} X_0 (k+1)} w(k) w^T(k) \right) w(k) X_0$$

$$= X_0 - X_0^2 \Sigma_v^{-1} (k+1) + \frac{\Sigma_v^{-2} X_0^3 (k+1)^2}{1 + \Sigma_v^{-1} X_0 (k+1)}$$

$$= \frac{X_0}{1 + \Sigma_v^{-1} X_0 (k+1)}$$

(c)

$$\lim_{X_0 \to \infty} \hat{x}_{|\bar{y}(k)} = \lim_{X_0 \to \infty} \frac{X_0 \Sigma_v^{-1}}{1 + \Sigma_v^{-1} X_0 (k+1)} w^T(k) \bar{y}(k)$$

$$= \frac{1}{1+k} w^T(k) \bar{y}(k)$$

$$= \frac{1}{1+k} \sum_{i=0}^{k} y(i)$$

$$\lim_{X_0 \to \infty} \Lambda_{\bar{X}|\bar{Y}(k)\,\bar{X}|\bar{Y}(k)} = \lim_{X_0 \to \infty} \frac{X_0}{1 + \Sigma_V^{-1} X_0 (k+1)}$$

$$= \frac{1}{\Sigma_V^{-1}(k+1)}$$

$$= \frac{\Sigma_V}{k+1}$$