TBSI ME 233 Spring 2024
Solution to Homework #2

1. (a) Figure 1 shows the MATLAB estimates of the auto-covariances and cross-covariances of W and Y. As we would expect, $\Lambda_{WW}(j)$ is approximately a unit pulse and $\Lambda_{YY}(j)$ is approximately symmetric. Also, $\Lambda_{YW}(-j) \approx \Lambda_{WY}(j)$ is approximately 0 for positive j, as causality dictates.

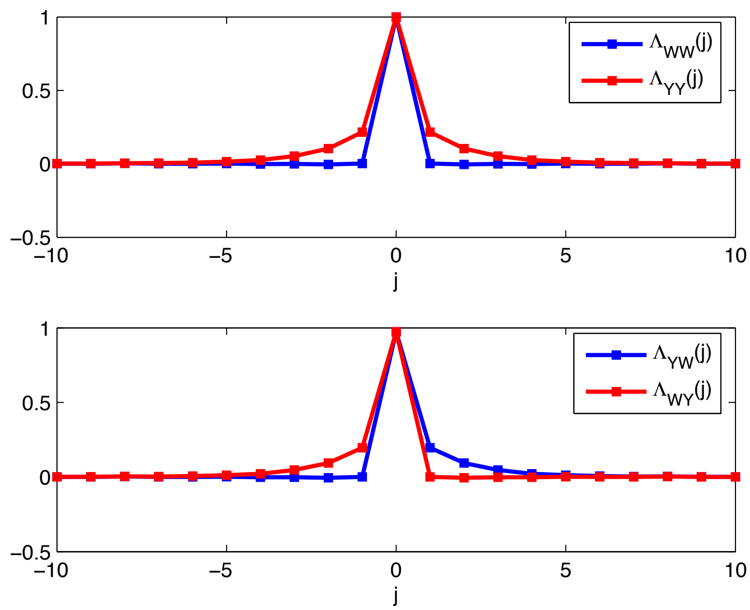

Figure 1: MATLAB estimates of auto-covariances and cross-covariances

(b) To find $\Lambda_{YW}(l)$, it is easiest to first find $\hat{\Lambda}_{YW}(z)$. Thus, we first note that

$$
\begin{aligned}
\hat{\Lambda}_{YW}(z) &= G(z)\hat{\Lambda}_{WW}(z) \\
G(z) &= \frac{z-0.3}{z-0.5} \\
\hat{\Lambda}_{WW}(z) &= \mathcal{Z}\{\delta(l)\} = 1 \\
\Rightarrow \hat{\Lambda}_{YW}(z) &= \frac{z-0.3}{z-0.5}.
\end{aligned}
$$

Now, with the aid of inverse Z-transform tables, we get that

$$
\begin{aligned}
\Lambda_{YW}(l) &= \mathcal{Z}^{-1}\left\{\frac{0.4z}{z-0.5} + 0.6\right\} \\
&= \begin{cases} 0.4(0.5)^l + 0.6\delta(l) & l \geq 0 \\ 0 & l < 0 \end{cases}.
\end{aligned}
$$

Figure 2 shows that the values of $\Lambda_{YW}(l)$ determined through MATLAB simulation match up well with the values determined above.

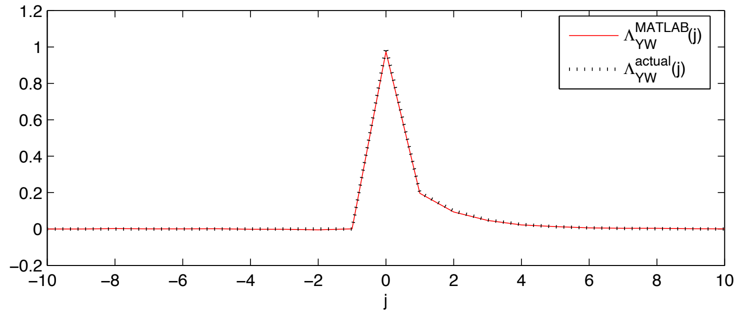

Figure 2: Comparison of MATLAB-determined cross-covariance to actual values

(c) Now that we have $\Lambda_{YW}(l)$, finding $\Lambda_{WY}(l)$ is a trivial matter. Using the property that $\Lambda_{YW}(l) = \Lambda_{WY}(-l)$, we see that

$$\Lambda_{WY}(l) = \begin{cases} 0.4(0.5)^{-l} + 0.6\delta(l) & l \leq 0 \\ 0 & l > 0 \end{cases}.$$

Figure 3 shows that the values of $\Lambda_{WY}(l)$ determined through MATLAB simulation match up well with the values determined above.

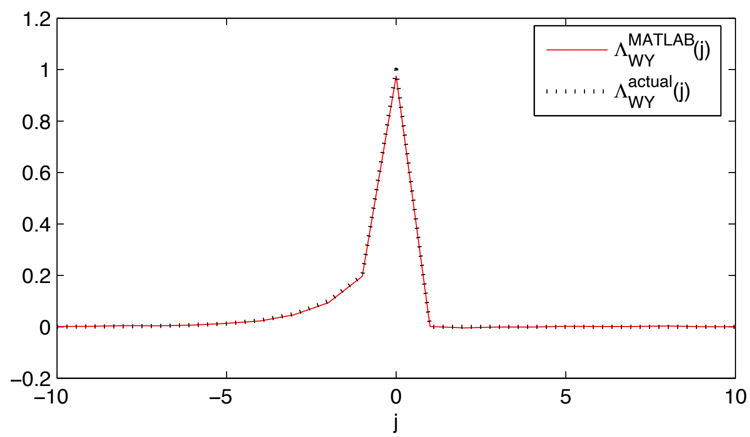

Figure 3: Comparison of MATLAB-determined cross-covariance to actual values

To find $\hat{\Lambda}_{WY}(z)$, it is easiest to recognize that the following general property applies to any random variables X and U:

$$\begin{aligned} \hat{\Lambda}_{XU}(z) &= \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{XU}(l) \\ &= \sum_{l=-\infty}^{\infty} (z^{-1})^{l}\Lambda_{UX}(-l) \\ &= \sum_{l=-\infty}^{\infty} (z^{-1})^{-l}\Lambda_{UX}(l) \\ &= \hat{\Lambda}_{UX}(z^{-1}). \end{aligned}$$

Applying this property to our system here gives

$$\hat{\Lambda}_{WY}(z) = \hat{\Lambda}_{YW}(z^{-1}) = \frac{z^{-1} - 0.3}{z^{-1} - 0.5} = \frac{0.3z - 1}{0.5z - 1}.$$

(d) We have the following:

$$
\begin{aligned}
\hat{\Lambda}_{YY}(z) &= \left(\frac{z-0.3}{z-0.5}\right)\left(\frac{z^{-1}-0.3}{z^{-1}-0.5}\right) \\
&= \frac{-0.3(z+z^{-1})+1.09}{(z-0.5)(z^{-1}-0.5)}.
\end{aligned}
$$

Figure 4 shows that the values of $\Lambda_{YY}(l)$ determined through MATLAB simulation and its exact value. You were not asked to plot this.

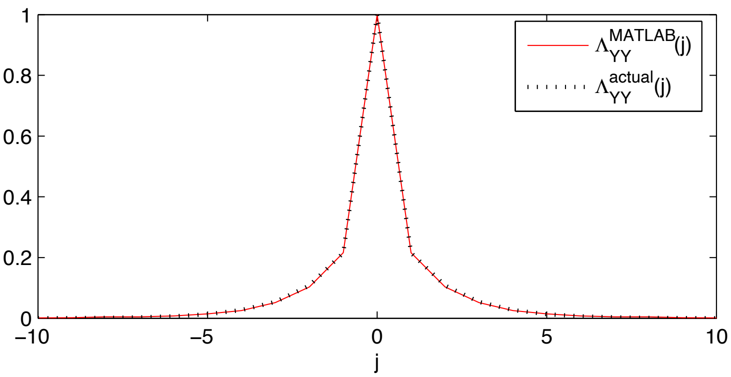

Figure 4: Comparison of MATLAB-determined auto-covariance to actual values

(e) Here, we want to compute covariances using the orginal series equation and compare our results to those obtained using transforms. To start, note that

$$
\begin{aligned}
\Lambda_{YW}(0) &= E\{Y(k)W(k)\} \\
&= E\Big\{\big[0.5Y(k-1)+W(k)-0.3W(k-1)\big]W(k)\Big\} \\
&= E\left\{W^2(k)\right\}+0.5E\{Y(k-1)W(k)\}-0.3E\{W(k-1)W(k)\}.
\end{aligned}
$$

Since the system is causal we know that the system's output should not depend on future inputs. Thus, the system's output should be independent of future inputs. Also, since W is white, its value should be independent of its value at any other timestep. Using these two facts gives

$$
\begin{aligned}
\Lambda_{YW}(0) &= E\left\{W^2(k)\right\}+E\{W(k)\}\left[0.5E\{Y(k-1)\}-0.3E\{W(k-1)\}\right] \\
&= E\left\{W^2(k)\right\}=1
\end{aligned}
$$

where we have used the fact that W is zero-mean. Note that this result agrees with the result found in part (b).

(f) Using the wide-sense stationarity of the signals and the results from the previous part,

$$
\begin{aligned}
\lambda_{YW}(1) &= E\{Y(k+1)W(k)\} \\
&= E\{Y(k)W(k-1)\} \\
&= -0.3E\left\{W^2(k-1)\right\}+0.5E\{Y(k-1)W(k-1)\}+E\{W(k)W(k-1)\} \\
&= -0.3E\left\{W^2(k-1)\right\}+0.5E\{Y(k-1)W(k-1)\} \\
&= -0.3E\left\{W^2(k)\right\}+0.5E\{Y(k)W(k)\} \\
&= -0.3+0.5\Lambda_{YW}(0)=0.2.
\end{aligned}
$$

Note that this result agrees with the result found in part (b).

(g) To solve this problem, we will first note that

$$
Y^2(k) = \left[0.5Y(k-1)+W(k)-0.3W(k-1)\right]^2.
$$

Taking the expected value of both sides gives

$$\begin{aligned}
\Lambda_{YY}(0) &= 0.25E\left\{Y^2(K-1)\right\} + E\left\{W^2(k)\right\} + 0.09E\left\{W^2(k-1)\right\} \\
&\quad + E\left\{Y(k-1)W(k)\right\} - 0.3E\left\{Y(k-1)W(k-1)\right\} - 0.6E\left\{W(k)W(k-1)\right\} \\
&= 0.25\Lambda_{YY}(0) + 1 + 0.09 + 0 - 0.3\Lambda_{YW}(0) + 0 \\
&= \frac{0.79}{0.75} = 1.0533.
\end{aligned}$$

Note that this result agrees with the result found in part (e).

2. (a) First, we express our system as

$$\begin{aligned}
X(k+1) &= AX(k) + BW(k) \\
Y(k) &= CX(k) + V(k).
\end{aligned}$$

Taking expectation of our system equations gives

$$\begin{aligned}
m_x(k+1) &= Am_x(k) + Bm_w(k) \\
m_y(k) &= Cm_x(k).
\end{aligned}$$

Thus, finding $m_y(k)$ is equivalent to finding a step response of this system with magnitude 10. Figure 5 shows a plot of $m_y(k)$ versus the time step. Note that because $W(k)$ is not a zero-mean sequence, $Y(k)$ does not settle out to 0; the steady state value of m_y is given by

$$\overline{m}_y = 10.084.$$

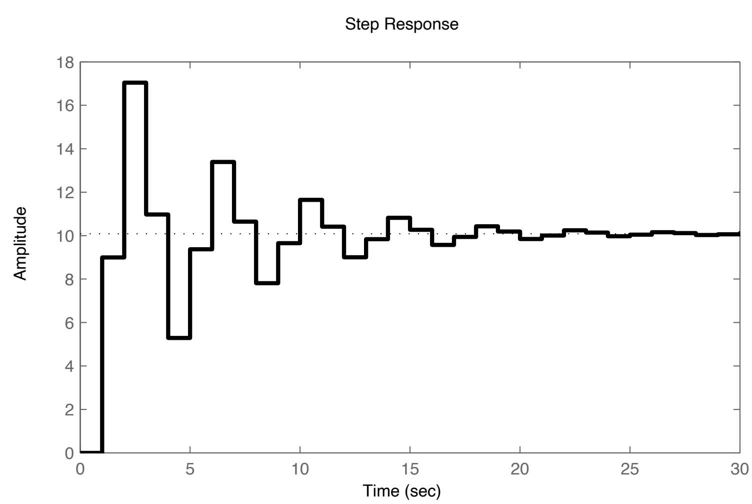

Figure 5: Evolution of $m_y(k)$ with time

(b) As discussed in lecture, the covariance of X propagates in the following way:

$$\Lambda_{XX}(k+1,0) = A\Lambda_{XX}(k,0)A^T + B\Sigma_{WW}(k)B^T.$$

Since we know the initial condition $\Lambda_{XX}(0,0)$, we can find $\Lambda_{XX}(k,0)$ iteratively using this Lyapunov equation. To find the covariance of Y, note that

$$\begin{aligned}
\Lambda_{YY}(k,0) &= E\left\{\tilde{Y}^2(k)\right\} \\
&= E\left\{\left[C\tilde{X}(k) + V(k)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\
&= C\Lambda_{XX}(k,0)C^T + \Sigma_{vv}
\end{aligned}$$

where we made use of the fact that $X(k)$ and $V(k)$ are uncorrelated. Thus, we can use our simulation results for $\Lambda_{XX}(k,0)$ to find $\Lambda_{YY}(k,0)$. Figure 6 shows the results of simulating the evolution of $\Lambda_{XX}(k,0)$ and then using it to find $\Lambda_{YY}(k,0)$. This set of simulations terminated when

$$\left\|\Lambda_{XX}(k,0) - \Lambda_{XX}(k-1,0)\right\|_{i2} \leq 10^{-5}.$$

Note that we could have used any matrix norm in this termination condition (Frobenius norm, i1 norm, i2 norm, i∞ norm, etc). The steady state covariance of y was found to be

$$\Lambda_{YY}(0) = 3.27.$$

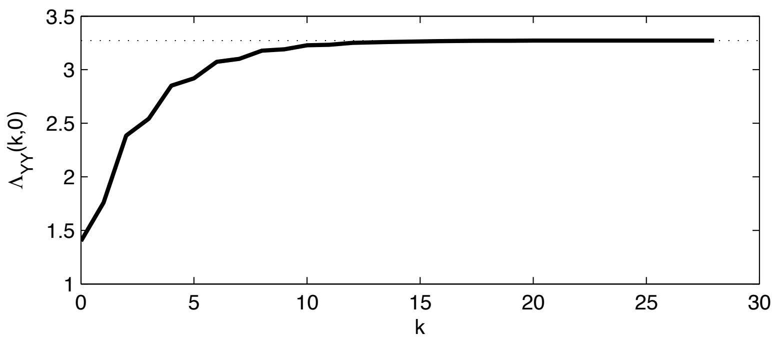

Figure 6: Evolution of $\Lambda_{YY}(k,0)$ with time

(c) To find $\Lambda_{XX}(5)$, recall that

$$\begin{aligned} \Lambda_{XX}(k,l) &= A^l \Lambda_{XX}(k,0) \\ \Rightarrow \Lambda_{XX}(k,5) &= A^5 \Lambda_{XX}(k,0). \end{aligned}$$

To find $\Lambda_{YY}(5)$, note that

$$\begin{aligned} \Lambda_{YY}(k,5) &= E\left\{\left[C\tilde{X}(k+5) + V(k+5)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\ &= C\Lambda_{XX}(k,5)C^T \end{aligned}$$

where we have used that the measurement noise is white and uncorrelated with the state. Figure 7 shows the simulation results. At steady state,

$$\Lambda_{YY}(5) = 0.27.$$

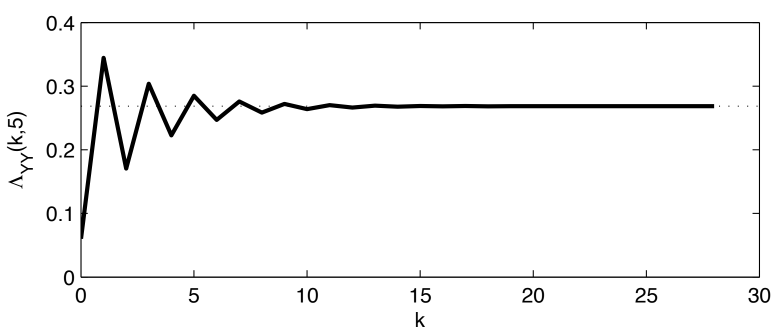

Figure 7: Evolution of $\Lambda_{YY}(k,5)$ with time

(d) At steady state,

$$A\Lambda_{XX}(0)A^T - \Lambda_{XX}(0) = -B\Sigma_{ww}B^T.$$

A call to `dlyap(A,B*Sigma_ww*B')` gives

$$\Lambda_{XX}(0) = \begin{bmatrix} 0.4308 & 0.0276 \\ 0.0276 & 0.3080 \end{bmatrix}.$$

At steady state, the stationary covariances of x and y are given by

$$\begin{aligned} \Lambda_{XX}(l) &= \begin{cases} \Lambda_{XX}(0)\left(A^{-l}\right)^T & l < 0 \\ \Lambda_{XX}(0) & l = 0 \\ A^l \Lambda_{XX}(0) & l > 0 \end{cases} \\ \Lambda_{YY}(l) &= E\left\{\left[C\tilde{X}(k+l) + V(k+l)\right]\left[C\tilde{X}(k) + V(k)\right]^T\right\} \\ &= C\Lambda_{XX}(l)C^T + \Sigma_{vv}\delta(l). \end{aligned}$$

Figure 8 shows the computed stationary covariance of Y. As expected, the plot is symmetric and the largest value occurs at $j = 0$. Note that the values of $\Lambda_{YY}(0)$ and $\Lambda_{YY}(5)$ are the same as the steady state covariances found in the two previous parts.

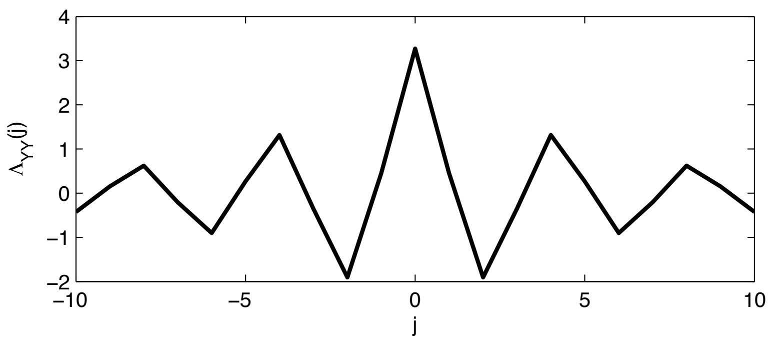

Figure 8: Stationary covariance of Y

(e) First, we define

$$\begin{aligned} \overline{W}(k) &= \begin{bmatrix} W(k) \\ V(k) \end{bmatrix} \\ \overline{G}(z) &= \begin{bmatrix} G(z) & 1 \end{bmatrix} \end{aligned}$$

so that our governing equations in the Z domain become

$$Y(z) = \overline{G}(z)\overline{W}(z).$$

Thus, the output spectral density is given by

$$\begin{aligned} \Phi_{YY}(\omega) &= \overline{G}(\omega)\Phi_{\overline{WW}}(\omega)\overline{G}^T(-\omega) \\ &= \begin{bmatrix} G(\omega) & 1 \end{bmatrix} \begin{bmatrix} \Sigma_{ww} & 0 \\ 0 & \Sigma_{vv} \end{bmatrix} \begin{bmatrix} G(-\omega) \\ 1 \end{bmatrix} \\ &= |G(\omega)|^2\Sigma_{ww} + \Sigma_{vv}. \end{aligned}$$