

# BIOSTAT M280/BIOMATH 280/STAT M230: Statistical Computing

Tue/Thu 1:00pm-2:50pm, CHS 51-279

Instructor: Dr. Hua Zhou, [huazhou@ucla.edu](mailto:huazhou@ucla.edu)

## 1 Lecture 1: Jan 5

### Today

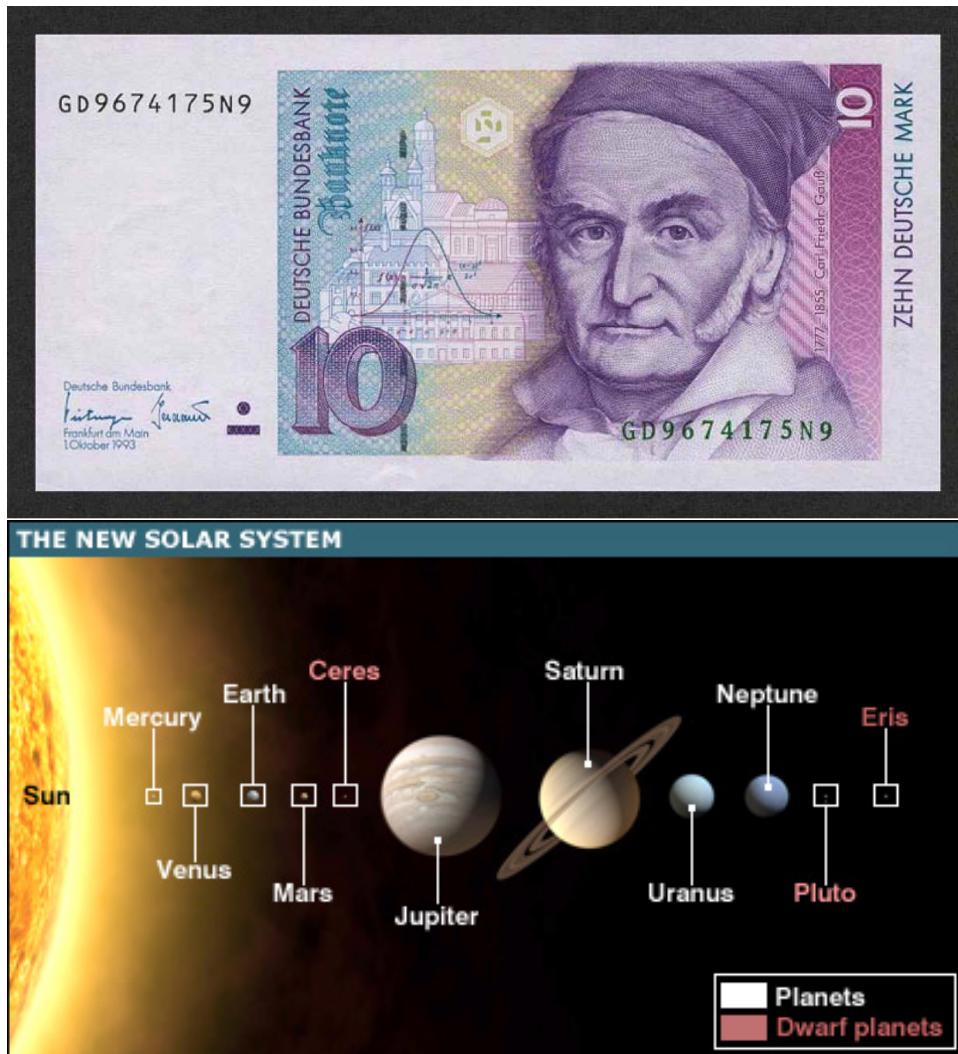
- Introduction and course logistics
- Computer storage and arithmetic
- Homework (not graded): fill out a short survey at <https://www.surveymonkey.com/r/G8BCVVM>
- Homework (not graded): Read the Introduction and Foundations of *Advanced R* by Hadley Wickham <http://adv-r.had.co.nz>. Install R. Try examples while you read the book. Do the quizzes.

### What is statistics?

- People collect data in order to answer certain questions. (Bio)statisticians's job is to help make sense of data.
- Statistics, the science of *data analysis*, is the applied mathematics in the 21st century.
- Read papers *The future of data analysis* by John Tukey (<http://hua-zhou.github.io/teaching/biostatm280-2016winter/readings/Tukey61FutureDataAnalysis.pdf>) and *50 years of data siccence* by David Donoho (<http://hua-zhou.github.io/teaching/biostatm280-2016winter/readings/Donoho50YearsDataScience.pdf>).

[io/teaching/biostatm280-2016winter/readings/Donoho15FiftyYearsDataScience.pdf](http://io/teaching/biostatm280-2016winter/readings/Donoho15FiftyYearsDataScience.pdf)).

## How Gauss became famous?



- 1801, Dr. Carl Friedrich Gauss, 24; proved Fundamental Theorem of Algebra; wrote the book *Disquisitiones Arithmeticæ*, which is still being studied today.
- 1801, Jan 1-Feb 11 (41 days), astronomer Piazzi observed Ceres (a dwarf planet), which was then lost behind sun.
- 1801, Aug–Sep, futile search by top astronomers; Laplace claimed it unsolvable.

- 1801, Oct–Nov, Gauss did calculations by *method of least squares*.
- 1801, Dec 31, astronomer von Zach re-located Ceres according to Gauss' calculation.
- 1802, *Summarische Übersicht der Bestimmung der Bahnen der beiden neuen Hauptplaneten angewandten Methoden*, considered the origin of linear algebra.
- 1807, Professor of Astronomy and (the first) Director of Göttingen Observatory in remainder of his life.
- 1809, *Theoria motus corporum coelestium in sectionibus conicis solum ambientium* (Theory of motion of the celestial bodies moving in conic sections around the Sun); birth of the Gaussian (normal) distribution, as an attempt to rationalize the method of least squares.
- 1810, Laplace consolidated the importance of Gaussian distribution by proving the central limit theorem.
- 1829, Gauss-Markov Theorem. Under Gaussian error assumption (actually only uncorrelated and homoscedastic needed), least square solution is the best linear unbiased estimate (BLUE), i.e., it has the smallest variance and thus MSE among all linear unbiased estimators. Note other estimators such as the James-Stein estimator may have smaller MSE, but they are *nonlinear*.

For more details of the story

- <http://www.keplersdiscovery.com/Asteroid.html>
- Teets and Whitehead (1999)

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# ARTICLES

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## The Discovery of Ceres: How Gauss Became Famous

DONALD TEETS  
KAREN WHITEHEAD  
South Dakota School of Mines and Technology  
Rapid City, SD 57701

*"The Duke of Brunswick has discovered more in his country than a planet: a super-terrestrial spirit in a human body."*

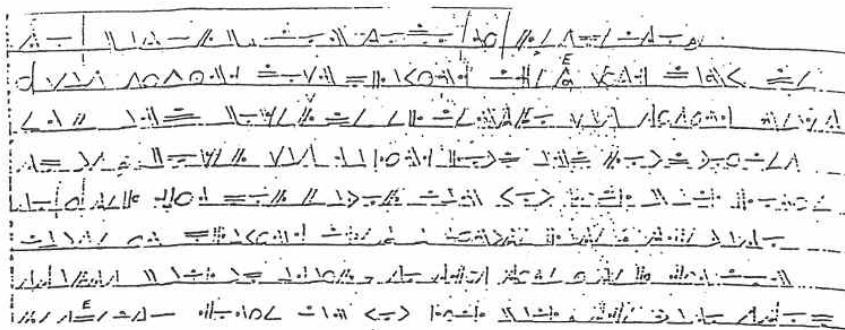
These words, attributed to Laplace in 1801, refer to the accomplishment of Carl Friedrich Gauss in computing the orbit of the newly discovered planetoid *Ceres Ferdinandea* from extremely limited data. Indeed, although Gauss had already achieved some fame among mathematicians, it was his work on the Ceres orbit that "made Gauss a European celebrity—this a consequence of the popular appeal which astronomy has always enjoyed . . ." [2]. The story of Gauss's work on this problem is a good one and is often told in biographical sketches of Gauss (e.g., [2], [3], [6]), but the mathematical details of how he solved the problem are invariably omitted from such historical works. We are left to wonder how did he do it? *Just how did Gauss*

- Stigler (1986) gives a more comprehensive account of the origin of the method of least squares.

Gauss' story

- Motivated by a real problem.
- Heuristic solution: method of least squares.
- Solution readily verifiable: Ceres was re-discovered!
- *Algorithmic development:* linear algebra, Gaussian elimination, FFT (fast Fourier transform).
- Theoretical justification: Gaussian distribution, Gauss-Markov theorem.

## A sampler by Marc Coram



ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS  
NOBLER IN THE MIND TO SUFFER THE SLINGS AND ARROWS OF OUTRAGEOUS  
FORTUNE OR TO TAKE ARMS AGAINST A SEA OF TROUBLES AND BY OPPOSING END

100 ER ENCHDLAE OHIOLO UOZEONORU O UOZED HI CITO HEQSET IUFOPHE HENO ITORUZAEN  
200 ES ELORHNDIE ORRHO UVUEGULOSU O UVEO HR CITO HEQDET JUSOPHE HELO ITOSUVDL  
300 ES ELORHNDIE CHANG UVUEGULOSU O UVEO HA CITO HEQDET JUSOPHE HELO ITOSUVDL  
400 ES ELOHINME OHINO UVUEGULOSU O UVEO HI CATO HEQDET AUSOWHE HELO ATOSUWHL  
500 ES ELOHINME OHINO UDEDEGULOSU O UDEO HI CATO HEQDET AUSOWHE HELO ATOSUWHL  
600 ES ELOHINME OHINO UDEDEGULOSU O UDEO HI CATO HEQDET AUSOWHE HELO ATOSUWHL  
900 ES ELOHINME OHANO UDEDEGULOSU O UDEO HA CITO HEQDET JUSOWHE HELO ITOSUWHL  
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1200 ISTILOHANNITOHANOT ODIO LOS TOT ODIOTHATODERO THIOQUIRTE SOVHITHILOTROS DMIL  
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1600 ESTEL HAMRET HAM TO CE OL SOT TO CE THAT IN THE QUENTIOS WHEHTEL TIM SOCREL  
1700 ESTEL HAMRET HAM TO BE OL SOT TO BE THAT IN THE QUENTIOS WHEHTEL TIM SOBREL  
1800 ESTER HAMLET HAM TO BE OR SOT TO BE THAT IN THE QUENTIOS WHETHER TIM SOBREL  
1900 ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS NOBLER  
2000 ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS NOBLER

to bat-rb. con todo mi respeto. i was sitting down playing chess with  
damny de emf and boxer de el centro was sitting next to us. boxer was  
making loud and loud voices so i tell him por favor can you kick back  
homie cause im playing chess a minute later the vato starts back up again  
so this time i tell him con respeto homie can you kick back. the vato  
stop for a minute and he starts up again so i tell him check this out shut  
the f\*\*k up cause im tired of your voice and if you got a problem with it  
we can go to celda and handle it. i really felt disrespected thaths why i  
told him. anyways after i tell him that the next thing i know that vato  
slashes me and leaves. dy the time i figure im hit i try to get away but  
the c.o. is walking in my direction and he gets me right dy a celda. so i  
go to the hole. when im in the hole my home boys hit doxer so now "b" is  
also in the hole. while im in the hole im getting schoold wrong and

- A consulting project by Marc Coram (then a graduate student in statistics at Stanford); customer is a professor in political science, who wants to understand a cryptic message circulated in a state prison.
- Marc modeled letter sequence by a Markov chain ( $26 \times 26$  transition matrix) and estimated transition probabilities from *War and Peace*.

- Now each mapping  $\sigma$  yields a likelihood  $f(\sigma)$  of the symbol sequence.
- Find the  $\sigma$  that maximizes  $f$ . Sample space is at least  $26! = 4.0329 \times 10^{26}$ . Combinatorial optimization – hard!
- *Metropolis algorithm*: At each iteration, generate a new  $\sigma'$  by random transposition of two letters; accept  $\sigma'$  with probability  $\min\left\{\frac{f(\sigma')}{f(\sigma)}, 1\right\}$ .

Marc Coram's story

- Motivated by a real problem.
- Solution readily verifiable: we can read it!
- *Algorithm development*: Metropolis sampler is one of top 10 algorithms in the 20th century.
- Read Diaconis (2009) for more details.

## What is this course about?

- Not a course on “packages and languages for data analysis”. It does not answer questions such as “How to fit a linear mixed model in SAS, SPSS or R?”
- Not a programming course, although programming is *extremely* important and we do homework in R.
- This course is about “numerical methods in statistics”. Our focus is on *algorithms*.

The form of a mathematical expression and the way the expression should be evaluated in actual practice may be quite different.

James Gentle

For a common numerical task in statistics, say the least squares solution  $\hat{\boldsymbol{\beta}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$ , we need to know which methods/algorithms are out there and what are their advantages and disadvantages. You will *fail* this course if you use

```
solve(t(X) %*% X) %*% t(X) %*% y
```

Using  $\text{lm}(y \sim X)$  is correct (and efficient) but is not the purpose of this course.  
We want to understand what is going on when calling the `lm()` function.

Science , 287 # 5454, Feb 4, 2000, p799

### Algorithms for the Ages

"Great algorithms are the poetry of computation," says Francis Sullivan of the Institute for Defense Analyses' Center for Computing Sciences in Bowie, Maryland. He and Jack Dongarra of the University of Tennessee and Oak Ridge National Laboratory have put together a sampling that might have made Robert Frost beam with pride—if the poet had been a computer jock. Their list of 10 algorithms having "the greatest influence on the development and practice of science and engineering in the 20th century" appears in the January/February issue of *Computing in Science & Engineering*. If you use a computer, some of these algorithms are no doubt crunching your data as you read this. The drum roll, please:

1946: The Metropolis Algorithm for Monte Carlo. Through the use of random processes, this algorithm offers an efficient way to stumble toward answers to problems that are too complicated to solve exactly.

1947: Simplex Method for Linear Programming. An elegant solution to a common problem in planning and decision-making.

1950: Krylov Subspace Iteration Method. A technique for rapidly solving the linear equations that abound in scientific computation.

1951: The Decompositional Approach to Matrix Computations. A suite of techniques for numerical linear algebra.

1957: The Fortran Optimizing Compiler. Turns high-level code into efficient computer-readable code.

1959: QR Algorithm for Computing Eigenvalues. Another crucial matrix operation made swift and practical.

1962: Quicksort Algorithms for Sorting. For the efficient handling of large databases.

1965: Fast Fourier Transform. Perhaps the most ubiquitous algorithm in use today, it breaks down waveforms (like sound) into periodic components.

1977: Integer Relation Detection. A fast method for spotting simple equations satisfied by collections of seemingly unrelated numbers.

1987: Fast Multipole Method. A breakthrough in dealing with the complexity of n-body calculations, applied in problems ranging from celestial mechanics to protein folding.

## Syllabus

Check course website frequently for updates and announcements.

<http://hua-zhou.github.io/teaching/biostatm280-2016winter>

Lecture notes will be updated and posted after each lecture.

# Computer storage and arithmetic

Elementary units of computer storage:

- *bit* = “binary” + “digit” (coined by statistician John Tukey).
- *byte* = 8 bits.
- kB = kilobyte =  $10^3$  bytes.
- MB = megabytes =  $10^6$  bytes.
- GB = gigabytes =  $10^9$  bytes.
- TB = terabytes =  $10^{12}$  bytes.
- PB = petabytes =  $10^{15}$  bytes.

## Storage of characters

ASCII control characters		ASCII printable characters								Extended ASCII characters							
00	NULL (Null character)	32	space	64	@	96	`	128	ç	160	á	192	l	224	ó		
01	SOH (Start of Header)	33	!	65	A	97	a	129	ú	161	í	193	ł	225	ń		
02	STX (Start of Text)	34	"	66	B	98	b	130	é	162	ó	194	ł	226	ő		
03	EOT (End of Text)	35	#	67	C	99	c	131	à	163	ú	195	ł	227	ő		
04	EOT (End of Trans.)	36	\$	68	D	100	d	132	ã	164	ñ	196	-	228	đ		
05	ENQ (Enquiry)	37	%	69	E	101	e	133	à	165	N	197	+	229	ò		
06	ACK (Acknowledgement)	38	&	70	F	102	f	134	á	166	ä	198	å	230	ø		
07	BEL (Bell)	39	,	71	G	103	g	135	ç	167	ö	199	À	231	þ		
08	BS (Backspace)	40	(	72	H	104	h	136	é	168	ż	200	Ł	232	Þ		
09	HT (Horizontal Tab)	41	)	73	I	105	i	137	è	169	®	201	Ł	233	Ӯ		
10	LF (Line feed)	42	*	74	J	106	j	138	ë	170	™	202	Ł	234	ӭ		
11	VT (Vertical Tab)	43	+	75	K	107	k	139	í	171	¼	203	Ł	235	ӻ		
12	FF (Form feed)	44	,	76	L	108	l	140	í	172	½	204	Ł	236	ӻ		
13	CR (Carriage return)	45	-	77	M	109	m	141	í	173	í	205	Ł	237	ӻ		
14	SO (Shift Out)	46	.	78	N	110	n	142	À	174	«	206	Ł	238	—		
15	SI (Shift In)	47	/	79	O	111	o	143	Á	175	»	207	Ł	239	·		
16	DLE (Data link escape)	48	0	80	P	112	p	144	É	176	„	208	Ł	240	≡		
17	DC1 (Device control 1)	49	1	81	Q	113	q	145	æ	177	„	209	Ł	241	±		
18	DC2 (Device control 2)	50	2	82	R	114	r	146	Æ	178	„	210	Ł	242	≈		
19	DC3 (Device control 3)	51	3	83	S	115	s	147	ö	179	„	211	Ł	243	½		
20	DC4 (Device control 4)	52	4	84	T	116	t	148	ö	180	—	212	Ł	244	¶		
21	NAK (Negative acknowl.)	53	5	85	U	117	u	149	ö	181	À	213	Ł	245	§		
22	SYN (Synchronous idle)	54	6	86	V	118	v	150	ú	182	Á	214	Ł	246	+		
23	ETB (End of trans. block)	55	7	87	W	119	w	151	ú	183	Á	215	Ł	247	:		
24	CAN (Cancel)	56	8	88	X	120	x	152	ÿ	184	©	216	Ł	248	°		
25	EM (End of medium)	57	9	89	Y	121	y	153	ó	185	†	217	Ł	249	-		
26	SUB (Substitute)	58	:	90	Z	122	z	154	ú	186	‡	218	Ł	250	.		
27	ESC (Escape)	59	;	91	[	123	{	155	ø	187	„	219	Ł	251	‘		
28	FS (File separator)	60	<	92	\	124		156	£	188	„	220	Ł	252	‘		
29	GS (Group separator)	61	=	93	]	125	}	157	Ø	189	¢	221	Ł	253	‘		
30	RS (Record separator)	62	>	94	^	126	~	158	×	190	¥	222	Ł	254	■		
31	US (Unit separator)	63	?	95	_			159	f	191	„	223	Ł	255	nbsp		
127	DEL (Delete)																

- Plain text files are stored in the form of characters: .r, .c, .cpp, .tex, .html, ...
- ASCII (American Code for Information Interchange): 7 bits, only  $2^7 = 128$  characters, “Hua” corresponds to “48 75 61 (Hex) = 72 117 97 (Dec) = 1001000 1110101 1100001”.

- Extended ASCII: 8 bits,  $2^8 = 256$  characters.
- Unicode: UTF-8, UTF-16 and UTF-32 support many more characters including foreign characters; last 7 digits conform to ASCII. UTF-8 is the current dominant character encoding on internet.

## 2 Lecture 2, Jan 7

### Announcements

- Location change: class meeting is changed to CHS 51-279, effective Jan 7.
- Enrollment cap is raised to 38 (open).
- Use RStudio for R programming.

### Last time

- Introduction. Gauss (least squares to find Ceres)–optimization, Marc Coram (decipher a note circulating in jail)–sampling. Two major modes of statistical computing.
- Course content, logistics.
- Computer representation of characters (ASCII, unicode) and integers (fixed point number system).

### Today

- Computer representation and arithmetic of integers: fixed-point numbers.
- Computer representation and arithmetic of real numbers: floating-point numbers.

### Fixed-point number system

Fixed-point number system  $\mathbb{I}$  is a computer model for integers  $\mathbb{Z}$ . One storage unit may be  $M = 8/16/32/64$  bit.

- The number of bits and method of representing negative numbers vary from system to system. The `integer` type in R has  $M = 32$  bits. MATLAB has `(u)int8`, `(u)int16`, `(u)int32`, `(u)int64`. JULIA has even more choices such as `Int128`, `UInt128`, `BigInt`, ...
- First bit indicates sign: 0 for nonnegative numbers, 1 for negative numbers.

- “two’s complement representation” for negative numbers. (i) Sign bit is set to 1, (ii) remaining bits are set to opposite values, (iii) 1 is added to the result.

+18		
-18	Signed magnitude representation	
	Signed 1's complement representation	
	Signed 2's complement representation	

- Range of representable integers by  $M$ -bit storage is  $[-2^{M-1}, 2^{M-1} - 1]$  (don’t need to represent 0 anymore so *could* have capacity for  $2^{M-1}$  negative numbers).
- For  $M = 8$ ,  $[-128, 127]$ .  
For  $M = 16$ ,  $[-65536, 65535]$ .  
For  $M = 32$ ,  $[-2147483648, 2147483647]$ .
- The smallest representable integer in R is  $-2^{31} + 1 = -2147483647$ .
- For unsigned integers such as in MATLAB, the range is  $[0, 2^M - 1]$ .

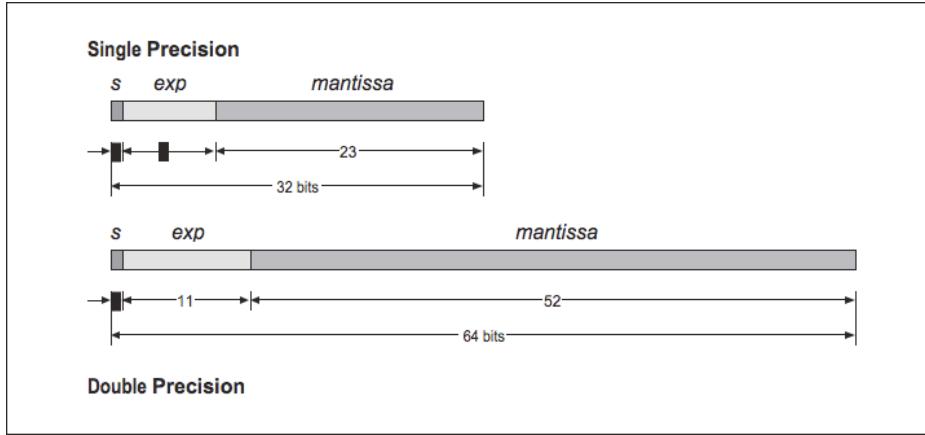
Two's Complement representation using 4 bit binary strings

	$-1$ 1111	$0$ 0000	$1$ 0001
$-2$ 1110			$2$ 0010
$-3$ 1101			$3$ 0011
$-4$ 1100			$4$ 0100
$-5$ 1011			$5$ 0101
$-6$ 1010	$-7$ 1001	$-8$ 1000	$6$ 0110
		$7$ 0111	

- An integer  $-i$  in the interval  $[-2^{M-1}, -1]$  would be represented by the same bit pattern by which the nonnegative integer  $2^M - i$  is represented, treating the sign bit as a regular numeric bit. For example,  $(-18) = 11101110$ ,  $(2^8) - (18) = (238) = 11101110$ . Two's complement is subtracting the nonnegative integer  $i$  from  $1 \dots 1 + 1 = 10 \dots 0 = (2^M)$ .
- Addition and subtraction are much simpler in two's complement representation. Why? It respects modular arithmetic nicely (look at the diagram). E.g.,  $(3) + (4) = 0011 + 0100 = 0111 = (7)$ ,  $(3) + (-5) = 0011 + 1011 = 1110 = (-2)$ ,  $(3) + (7) = 0011 + 0111 = 1010 = (-6)$  (overflow),  $(-3) + (-7) = 1101 + 1001 = 0110 = (6)$  (underflow).

- Keep track of overflow and underflow. If the result of a summation is  $R$ , which must be in the set  $[-2^{M-1}, 2^{M-1} + 1]$ , there are only three possibilities for the true sum:  $R$ ,  $R + 2^M$  (overflow), or  $R - 2^M$  (underflow).
  - When adding two nonnegative integers:  $0XX\dots X + 0YY\dots Y$ , where  $X$  and  $Y$  are arbitrary binary digits, we only need to keep track of overflow in this case. If the resulting binary number has a leading bit of 1, we know overflow occurs since the sum cannot be negative. We should treat that sign bit as a regular numeric bit. In other words, we crossed the upper boundary 100 and should add  $2^M$  to the result. If the resulting binary number has a leading bit of 0, no overflow occurs.
  - When adding two negative integers:  $1XX\dots X + 1YY\dots Y$ , where  $X$  and  $Y$  are arbitrary binary digits, we only need to keep track of underflow in this case. If the resulting binary number has a leading bit of 0, we know underflow occurs since the sum cannot be positive. We crossed the lower boundary 000 and should subtract  $2^M$  from the result. If the resulting binary number has a leading bit of 1, no underflow occurs.
  - When adding a negative integer and a nonnegative integer:  $1XX\dots X + 0YY\dots Y$ , the result is always between the two summands. No overflow or underflow could happen.
- R reports **NA** for integer overflow and underflow. Other languages, e.g., JULIA simply outputs the result according modular arithmetic.

## Floating-point number system



Floating-point number system  $\mathbb{F}$  is a computer model for real numbers  $\mathbb{R}$ .

- A real number is represented by  $\pm d_0.d_1d_2 \cdots d_p \times b^e$  (scientific notation).
- Parameters for a floating-point number system: *base* (or *radix*), range of *fraction* (or *mantissa*, *significand*), range of *exponent*.
- Non-uniqueness of representation. *normalized/denormalized* significant digits.  
E.g.,  $+18 = +1.0010 \times 2^4$  (normalized)  $= +0.10010 \times 2^5$  (denormalized).
- *Bias* (or *excess*): actual exponent is obtained by subtracting bias from the value of exponent evaluated regardless of sign digit.
- IEEE 754.
  - *Single precision* (32 bit): base 2,  $p = 23$  (23 significant bits),  $e_{\max} = 127$ ,  $e_{\min} = -126$  (8 exponent bits), bias=127.  $e_{\min} - 1$  and  $e_{\max} + 1$  are reserved for special numbers. This implies a maximum magnitude of  $\log_{10}(2^{127}) \approx 38$  and precision to  $\log_{10}(2^{23}) \approx 7$  decimal point.  $\pm 10^{\pm 38}$ .
  - *Double precision* (64 bit): base 2,  $p = 52$  (52 significant bits),  $e_{\max} = 1023$ ,  $e_{\min} = -1022$  (11 exponent bits), bias=1023. This implies a maximum magnitude of  $\log_{10}(2^{1023}) \approx 308$  and precision to  $\log_{10}(2^{52}) \approx 16$  decimal point.  $\pm 10^{\pm 308}$ .

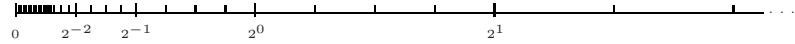
- “ $(+18) = (2^4 + 2^1) = +1.0010 \times 2^4$  in single precision  
 $(0)(10000011)(001000000000000000000000).$

First is sign bit. Next 8 bits are exponent 131 in ordinary base 2 with a bias of 127. Remaining 23 bits represent the fraction beyond the leading bit, known to be 1. In summary it represents  $(+18)$  as  $+1.0010 \times 2^4$  in the binary format.  $(-18)$  is represented by the same bits except changing the sign bit to 1.

- Special floating-point numbers:
  - Exponent  $e_{\max} + 1$  plus a mantissa of 0 means  $\pm\infty$ .
  - Exponent  $e_{\max} + 1$  plus a nonzero mantissa means NaN. NaN could be produced from  $0 / 0$ ,  $0 * \text{Inf}$ , ... In general  $NaN \neq NaN$  bitwise.
  - Exponent  $e_{\min} - 1$  with a mantissa of all 0s represents the real number 0.
  - Exponent  $e_{\min} - 1$  with a nonzero mantissa are for numbers less than  $b^{e_{\min}}$ . Numbers are de-normalized in the range  $(0, b^{e_{\min}})$  – “graceful underflow”.
- $\mathbb{F}$  is not a subset of  $\mathbb{R}$ , although  $\mathbb{I} \subset \mathbb{Z}$ .
- For the history of establishment of IEEE-754 standard, see an interview with William Kahan.  
<http://www.cs.berkeley.edu/~wkahan/ieee754status/754story.html>

To summarize

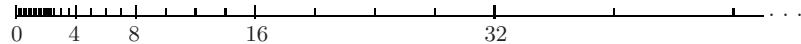
- Single precision: range  $\pm 10^{\pm 38}$  with precision up to 7 decimal digits.
- Double precision: range  $\pm 10^{\pm 308}$  with precision up to 16 decimal digits.
- The floating-point numbers do not occur uniformly over the real number line.



**Fig. 2.4.** The Floating-Point Number Line, Nonnegative Half



**Fig. 2.5.** The Floating-Point Number Line, Nonpositive Half

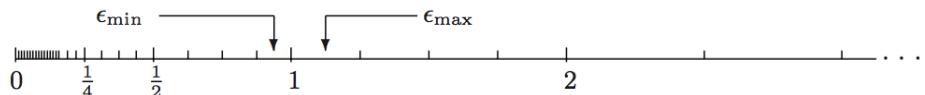


**Fig. 2.6.** The Floating-Point Number Line, Nonnegative Half; Another View



**Fig. 2.7.** The Fixed-Point Number Line, Nonnegative Half

- *Machine epsilon*s are the spacings of numbers around 1.  $\epsilon_{\min} = b^{-p}$  and  $\epsilon_{\max} = b^{1-p}$ .



**Fig. 2.8.** Relative Spacings at 1: “Machine Epsilons”

- The variable `.Machine` in R contains numerical characteristics of the machine.
- How to test `inf` and `nan`? In R, `is.nan()`, `is.finite()`, `is.infinite()`. In MATLAB, `isinf()`, `isnan()`. In JULIA, `isinf()`, `isnan()`.

## Integers and floating-point numbers in Julia

- R only uses double (64-bit) and 32-bit integer. It can be a downside when dealing with big data.

- Julia offers a rich collection of primitive data types. Read (the excellent) documentation. <http://docs.julialang.org/en/release-0.4/manual/integers-and-floating-point-numbers/>  
Notable things include: `Int128`, `UInt128`, half precision numbers `Float16`, arbitrary precision numbers `BigInt` and `BigFloat`, rational numbers, ...

## Consequences of computer storage/arithmetic

- Be memory conscious when dealing with big data. E.g., human genome has about  $3 \times 10^9$  bases, each of which belongs to  $\{A, C, T, G\}$ . How much storage if we store  $10^6$  SNPs (single nucleotide polymorphisms) of 1000 individuals (1000 Genome Project) as characters (1GB), single (4GB), double (8GB), int64(8GB), int32 (4GB), int16 (2GB), int8 (1GB), PLINK binary format 2bit/SNP (250MB)?
- Know the limit. *Overflow* and *Underflow*. For double precision,  $\pm 10^{\pm 308}$ . In most situations, underflow is preferred over overflow. Overflow often causes crashes. Underflow yields zeros. E.g., in logistic regression,  $p_i = \frac{\exp(\mathbf{x}_i^\top \boldsymbol{\beta})}{1+\exp(\mathbf{x}_i^\top \boldsymbol{\beta})} = \frac{1}{1+\exp(-\mathbf{x}_i^\top \boldsymbol{\beta})}$ . The former expression can easily lead to  $\infty/\infty = NaN$ , while the latter expression leads to *graceful underflow*.
- Be aware of non-uniform distribution of floating-point numbers, in contrast to fixed-point numbers. There are the same number of floating-point numbers in  $[b^i, b^{i+1}]$  and  $[b^{i+1}, b^{i+2}]$  for  $e_{\min} \leq i \leq e_{\max} - 2$ . It is more dense when closer to zero.
- “Catastrophic cancellation 1”. Addition or subtraction of two numbers of widely different magnitudes:  $a + b$  or  $a - b$  where  $a \gg b$  or  $a \ll b$ . We loose the precision in the number of smaller magnitude. Consider  $a = x.xxx\dots \times 2^0$  and  $b = y.yyy\dots \times 2^{-53}$ . What happens when computer calculates  $a + b$ ? We get  $a + b = a$ !
- Another example: What happens when compute  $\sum_{x=1}^{\infty} x$  in order? Will the partial sum reach `Inf`? “A divergent series converges.”
- Always try to add numbers of similar magnitude. Rule 1: add small numbers together before adding larger ones. Rule 2: add numbers of like magnitude together (pairing). When all numbers are of same sign and similar magnitude, add in pairs so each stage the summands are of similar magnitude.

$$\begin{array}{c}
 s_1^{(1)} = x_1 + x_2 \quad | \quad s_2^{(1)} = x_3 + x_4 \quad | \quad s_{2m-1}^{(1)} = x_{4m-3} + x_{4m-2} \quad | \quad s_{2m}^{(1)} = \dots \\
 \searrow \qquad \qquad \qquad \searrow \qquad \qquad \qquad \searrow \qquad \qquad \qquad \searrow \\
 s_1^{(2)} = s_1^{(1)} + s_2^{(1)} \quad | \quad \dots \quad | \quad s_m^{(2)} = s_{2m-1}^{(1)} + s_{2m}^{(1)} \quad | \quad \dots \\
 \searrow \qquad \qquad \qquad \searrow \qquad \qquad \qquad \searrow \qquad \qquad \qquad \searrow \\
 s_1^{(3)} = s_1^{(2)} + s_2^{(2)} \quad | \quad \dots \quad | \quad \dots \quad | \quad \dots
 \end{array}$$

- “Catastrophic cancellation 2”. Subtraction of two nearly equal numbers eliminates significant digits.  $a - b$  where  $a \approx b$ . Consider  $a = x.xxxxxxxxxxx1ssss$ ,  $b = x.xxxxxxxxxxx0tttt$ . The result is  $1.vvvvvu...u$  where  $u$  are unassigned digits.
- E.g., evaluating  $e^{-20}$  by Taylor series

$$e^{-x} = 1 - x + x^2/2! - x^3/3! + \dots$$

gets  $6.138\text{e-}09$ , while the true value is about  $2.061\text{e-}09$ . Many cancellations accumulate. Anyway, it's *not* the way to evaluate  $e^x$ !

- Sometimes catastrophic cancellation can be avoided. Roots of the quadratic function  $ax^2 + bx + c$  are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When one root is close to 0, cancellation can happen. We may evaluate one of the root (away from 0) by the formula and then compute the other by relationship  $x_1x_2 = c/a$ .

- Reading: *What every computer scientist should know about floating-point arithmetic* by David Goldberg.  
<http://hua-zhou.github.io/teaching/biostatm280-2016winter/readings/Goldberg91FloatingPoint.pdf>

### 3 Lecture 3, Jan 12

#### Announcements

- HW1 posted. Due Thu Jan 21 @ 11:59PM. [http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostat\\_m280\\_2016\\_hw1.pdf](http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostat_m280_2016_hw1.pdf)
- Quiz 1 Thu Jan 21 in class.
- Teaching assistant (TA): Max Tolkoff [mtolkoff@ucla.edu](mailto:mtolkoff@ucla.edu), office hours M @ 11A–12P, W @ 2P–3P, at the common area of the 6th floor of Gonda.

#### Last time

- Computer arithmetics: fixed-point numbers and floating-point numbers.
- Consequences of computer arithmetics: range and precision of single/double precision numbers, cancellations, ...

#### Today

- Computer languages.
- Comparison of R, MATLAB, and JULIA.
- R, RStudio, RMarkdown, version control.

#### Computer Languages

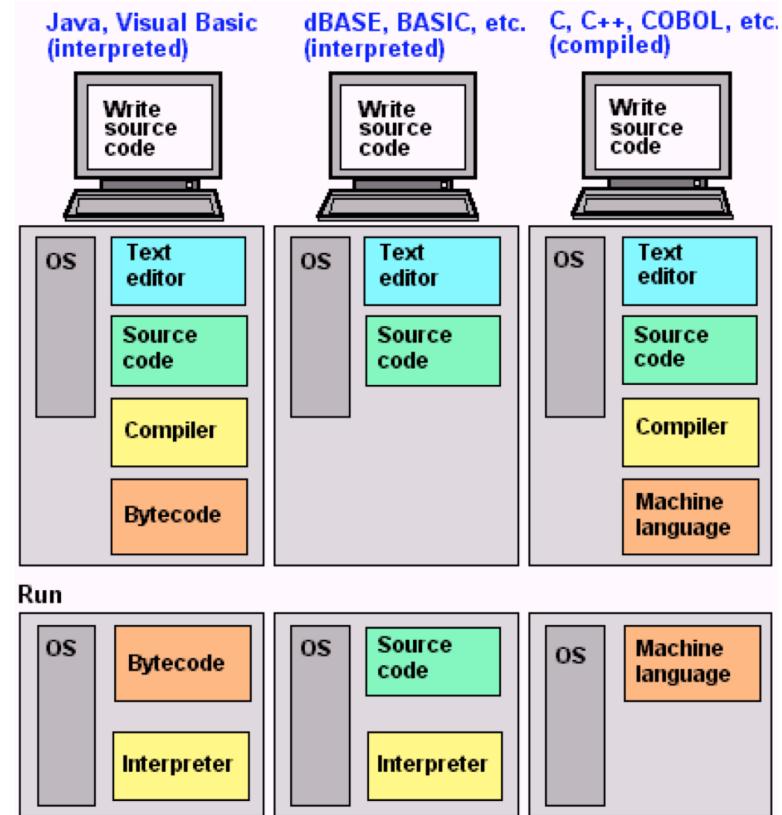
工欲善其事，必先利其器

*To do a good job, an artisan needs the best tools.*

*The Analects by Confucius (about 500 BC)*

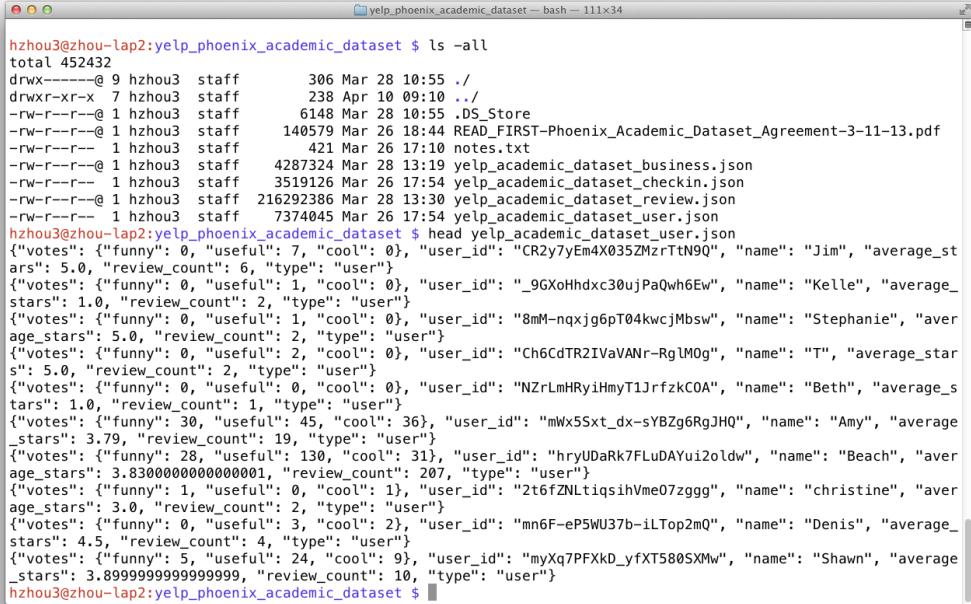
- What features are we looking for in a language?
  - Efficiency (in both run time and memory) for handling big data.
  - IDE support (debugging, profiling).

- Open source.
- Legacy code.
- Tools for generating dynamic report (reproducibility).
- Adaptivity to hardware evolution (parallel and distributed computing).



- Types of languages
  1. Compiled languages: C/C++, FORTRAN, ...
    - Directly compiled to machine code that is executed by CPU
    - Pros: fast, memory efficient
    - Cons: longer development time, hard to debug
  2. Interpreted language: R, MATLAB, SAS IML, JAVASCRIPT, BASIC, ...
    - Interpreted by interpreter
    - Pros: fast prototyping

- Cons: excruciatingly slow for loops
3. Mixed (dynamic) languages: Julia, Python, JAVA, Matlab (JIT), R (JIT), ...
- Compiled to bytecode and then interpreted by virtual machine
  - Pros: relatively short development time, cross-platform, good at data preprocessing and manipulation, rich libraries and modules
  - Cons: not as fast as compiled language
4. Script languages: shell scripts, Perl, ...
- Extremely useful for data preprocessing and manipulation
  - E.g., massage the Yelp ([http://www.yelp.com/dataset\\_challenge](http://www.yelp.com/dataset_challenge)) data before analysis



```

hzhou3@zhou-lap2:yelp_phoenix_academic_dataset $ ls -all
total 452432
drwx-----@ 9 hzhou3 staff      306 Mar 28 10:55 .
drwxr-xr-x  7 hzhou3 staff     238 Apr 10 09:10 ..
-rw-r--r--@ 1 hzhou3 staff    6148 Mar 28 10:55 .DS_Store
-rw-r--r--@ 1 hzhou3 staff   140579 Mar 26 18:44 READ_FIRST-Phoenix_Academic_Dataset_Agreement-3-11-13.pdf
-rw-r--r--  1 hzhou3 staff    421 Mar 26 17:10 notes.txt
-rw-r--r--@ 1 hzhou3 staff   4287324 Mar 28 13:19 yelp_academic_dataset_business.json
-rw-r--r--  1 hzhou3 staff   3519126 Mar 28 17:54 yelp_academic_dataset_checkin.json
-rw-r--r--@ 1 hzhou3 staff  216292386 Mar 28 13:30 yelp_academic_dataset_review.json
-rw-r--r--  1 hzhou3 staff   7374045 Mar 26 17:54 yelp_academic_dataset_user.json
hzhou3@zhou-lap2:yelp_phoenix_academic_dataset $ head yelp_academic_dataset_user.json
{
  "votes": {
    "funny": 0,
    "useful": 7,
    "cool": 0,
    "user_id": "CR2y7yEm4X035ZMzrTtN9Q",
    "name": "Jim",
    "average_stars": 5.0,
    "review_count": 6,
    "type": "user"
  }
}
{
  "votes": {
    "funny": 0,
    "useful": 1,
    "cool": 0,
    "user_id": "_9GXoHhdxc30ujPaQwh6Ew",
    "name": "Kelle",
    "average_stars": 1.0,
    "review_count": 2,
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  }
}
{
  "votes": {
    "funny": 0,
    "useful": 1,
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    "user_id": "8mM-nqxjg6pT04kwcjMbsw",
    "name": "Stephanie",
    "average_stars": 5.0,
    "review_count": 2,
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  }
}
{
  "votes": {
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}
{
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    "user_id": "NZrLmHRyiHmyT1JrfzkCOA",
    "name": "Beth",
    "average_stars": 1.0,
    "review_count": 1,
    "type": "user"
  }
}
{
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    "funny": 30,
    "useful": 45,
    "cool": 36,
    "user_id": "mWx5Sxt_dx-sYBZg6RgJHQ",
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    "average_stars": 3.79,
    "review_count": 19,
    "type": "user"
  }
}
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    "useful": 130,
    "cool": 31,
    "user_id": "hryUdaRk7FLuDAYui2oldw",
    "name": "Beach",
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}
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    "cool": 1,
    "user_id": "3.8300000000000001",
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    "name": "Denis",
    "average_stars": 4.5,
    "review_count": 4,
    "type": "user"
  }
}
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    "useful": 24,
    "cool": 9,
    "user_id": "myXq7PFXkD_yfXT580SXm",
    "name": "Shawn",
    "average_stars": 3.899999999999999,
    "review_count": 10,
    "type": "user"
  }
}
hzhou3@zhou-lap2:yelp_phoenix_academic_dataset $

```

5. Database languages: SQL, Hadoop. Data analysis never happens if we do not know how to retrieve data from databases.

- Messages

- To be versatile in the big data era, be proficient in at least one language in each category.

- To improve efficiency of interpreted languages such as R or MATLAB, avoid loops as much as possible. Aka, vectorize code  
“The only loop you are allowed to have is that for an iterative algorithm.”  
For some tasks where looping is necessary, consider coding in C or FORTRAN. It is “convenient” to incorporate compiled code into R or MATLAB.  
But do this **only after profiling!**  
Success stories: `glmnet` and `lars` packages in R are based on FORTRAN.
- Modern languages such as Julia are trying to bridge the gap between interpreted and compiled languages. That is to achieve efficiency without vectorizing code.

- Language features of R, MATLAB, and JULIA:

---

Features	R	MATLAB	JULIA
Open source	☺	☺	☺
IDE	RStudio ☺☺	☺☺☺	☺
Dynamic document	☺☺☺(RMarkdown)	☺☺☺	☺☺(IJulia)
Multi-threading	parallel pkg	☺	☺
JIT	compiler pkg	☺	☺
Call C/Fortran	wrapper, Rcpp	wrapper	no glue code
Call shared library	wrapper	wrapper	no glue code
Typing	☺	☺☺	☺☺☺
Pass by reference	☺	☺	☺☺☺
Linear algebra	☺	MKL, Arpack	OpenBLAS, Eigpack
Distributed computing	☺	☺	☺☺☺
Sparse linear algebra	☺ (Matrix package)	☺☺☺	☺☺☺
Documentation	☺	☺☺☺	☺☺
Profiler	☺	☺☺☺	☺☺☺

---

- Benchmark code `R-benchmark-25.R` from <http://r.research.att.com/benchmarks/R-benchmark-25.R> covers many commonly used numerical operations used in statistics. We ported (literally) to MATLAB and Julia and report the run times (averaged over 5 runs) here.

Matlab benchmark code:

[http://hua-zhou.github.io/teaching/biostatm280-2016winter/benchmark\\_matlab.m](http://hua-zhou.github.io/teaching/biostatm280-2016winter/benchmark_matlab.m)

Julia benchmark code:

[http://hua-zhou.github.io/teaching/biostatm280-2016winter/benchmark\\_julia.jl](http://hua-zhou.github.io/teaching/biostatm280-2016winter/benchmark_julia.jl)

Machine specs: Intel i7 @ 2.6GHz (4 physical cores, 8 threads), 16G RAM, Mac OS 10.11.2.

Test	R 3.2.2	MATLAB R2014a	JULIA 0.4.2
Matrix creation, trans, deformation ( $2500 \times 2500$ )	0.77	0.17	<b>0.14</b>
Power of matrix ( $2500 \times 2500$ , $A^{1000}$ )	0.26	<b>0.11</b>	0.21
Quick sort ( $n = 7 \times 10^6$ )	0.76	<b>0.24</b>	0.69
Cross product ( $2800 \times 2800$ , $A^T A$ )	10.72	0.35	<b>0.31</b>
LS solution ( $n = p = 2000$ )	1.28	<b>0.07</b>	0.09
FFT ( $n = 2,400,000$ )	0.40	<b>0.04</b>	0.64
Eigen-values ( $600 \times 600$ )	0.82	<b>0.31</b>	0.57
Determinant ( $2500 \times 2500$ )	3.76	0.18	<b>0.17</b>
Cholesky ( $3000 \times 3000$ )	4.23	<b>0.15</b>	0.20
Matrix inverse ( $1600 \times 1600$ )	3.37	<b>0.16</b>	0.21
Fibonacci (vector calc)	0.34	<b>0.17</b>	0.68
Hilbert (matrix calc)	0.21	0.07	<b>0.01</b>
GCD (recursion)	0.31	0.14	<b>0.12</b>
Toeplitz matrix (loops)	0.36	<b>0.0014</b>	0.02
Escoufiers (mixed)	0.45	0.40	<b>0.21</b>

- A slightly more complicated (or realistic) example taken from Doug Bates's slides <http://www.stat.wisc.edu/~bates/JuliaForRProgrammers.pdf>. The task is to use Gibbs sampler to sample from bivariate density

$$f(x, y) = kx^2 \exp(-xy^2 - y^2 + 2y - 4x), x > 0,$$

using the conditional distributions

$$\begin{aligned} X|Y &\sim \Gamma\left(3, \frac{1}{y^2 + 4}\right) \\ Y|X &\sim \mathcal{N}\left(\frac{1}{1+x}, \frac{1}{2(1+x)}\right). \end{aligned}$$

Let's sample 10,000 points from this distribution with a thinning of 500.

- How long does R take? 42 seconds.  
[http://hua-zhou.github.io/teaching/biostatm280-2016winter/gibbs\\_r.html](http://hua-zhou.github.io/teaching/biostatm280-2016winter/gibbs_r.html)
- How long does Julia take? 0.43 seconds.  
[http://hua-zhou.github.io/teaching/biostatm280-2016winter/gibbs\\_julia.html](http://hua-zhou.github.io/teaching/biostatm280-2016winter/gibbs_julia.html)
- With similar coding efforts, Julia offers  $\sim 100$  fold speed-up! JIT in R didn't kick in. Neither does Matlab, which took about 20 seconds.
- Julia offers the capability of strong typing of variables. This facilitates the optimization by compiler.
- With little extra efforts, we can do parallel and distributed computing using Julia.

Benchmark of the same example in other languages including Rcpp is available in the blogs by Darren Wilkinson (<http://bit.ly/IWhJ52>) and Dirk Eddelbuettel's (<http://dirk.eddelbuettel.com/blog/2011/07/14/>).

*“As some of you may know, I have had a (rather late) mid-life crisis and run off with another language called Julia. <http://julialang.org>”*

Doug Bates (on the knitr Google Group)

## Reproducible research (in computational science)

*An article about computational result is advertising, not scholarship. The actual scholarship is the full software environment, code and data, that produced the result.*

Buckheit and Donoho (1995)

also see Claerbout and Karrenbach (1992)

- 3 stories of *not* being reproducible.
  - Duke Potti Scandal.

How a New Hope in Cancer Fell Apart – NYTimes.com

www.nytimes.com/2011/07/08/health/research/08genes.html?\_r=0

Stat 790-003 Stat 758 Hua Zhou | 胡华 NCSU Statistics Statistics... phics and Fun Google Lost in the Moment... InterfaceLIFT Google Maps Yahoo! News RStudio The Duke Saga Starter Set | Simply Statistics bioinformatics.mdanderson.org/Supplements/Re... How a New Hope in Cancer Fell Apart – NYTimes...

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## How Bright Promise in Cancer Testing Fell Apart

Michael Stravato for The New York Times

Keith Baggerly, left, and Kevin Coombes, statisticians at M. D. Anderson Cancer Center, found flaws in research on tumors.

By GINA KOLATA  
Published: July 7, 2011

When Juliet Jacobs found out she had lung [cancer](#), she was terrified, but realized that her hope lay in getting the best treatment medicine could offer. So she got a second opinion, then a third. In February of 2010, she ended up at [Duke University](#), where she entered a research study whose promise seemed stunning.

**Well** Tara Parker-Pope on Health

Blueberries May Lower Blood Pressure January 14, 2015, 11:10 AM

Can Compression Clothing Enhance Your Workout? January 14, 2015, 8:00 AM

Many Who Take a Daily Aspirin Don't Need It January 13, 2015

Varied Routes to Safer Streets January 12, 2015

Naps May Be Good for a Baby's Learning January 12, 2015

Health & Fitness Tools

BMI Calculator What's your score? »

MOST EMAILED MOST VIEWED

1. WOMEN AT WORK Speaking While Female
2. DAVID BROOKS The Child in the Basement
3. MODERN LOVE To Fall in Love With Anyone, Do This

Potti et al. (2006) Genomic signatures to guide the use of chemotherapeutics, *Nature Medicine*, 12(11):1294–1300.

Baggerly and Coombes (2009) Deriving chemosensitivity from cell lines: Forensic bioinformatics and reproducible research in high-throughput biology, *Ann. Appl. Stat.*, 3(4):1309–1334. <http://projecteuclid.org/euclid.aoas/1267453942>

More information is available at

[http://en.wikipedia.org/wiki/Anil\\_Potti](http://en.wikipedia.org/wiki/Anil_Potti)

<http://simplystatistics.org/2012/02/27/the-duke-saga-starter-set/>

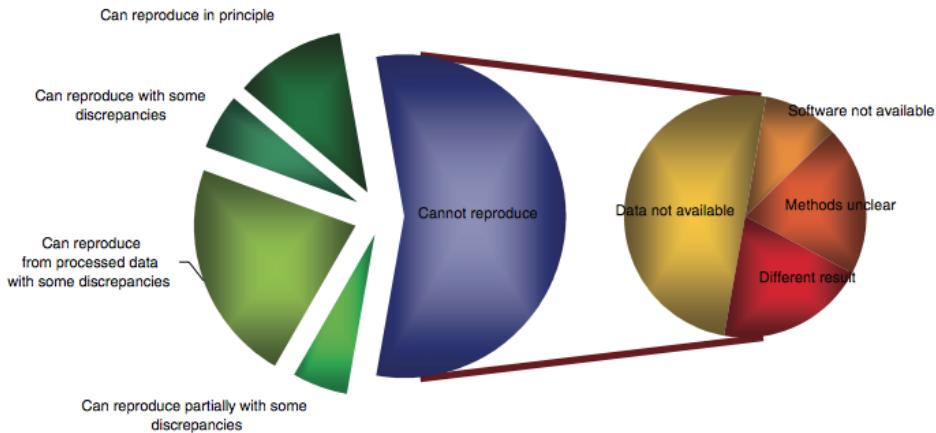
- Nature Genetics (2013 Impact Factor: 29.648). 20 articles about microarray profiling published in *Nature Genetics* between Jan 2005 and Dec 2006.

## Repeatability of published microarray gene expression analyses

John P A Ioannidis<sup>1–3</sup>, David B Allison<sup>4</sup>, Catherine A Ball<sup>5</sup>, Issa Coulibaly<sup>4</sup>, Xiangqin Cui<sup>4</sup>, Aedín C Culhane<sup>6,7</sup>, Mario Falchi<sup>8,9</sup>, Cesare Furlanello<sup>10</sup>, Laurence Game<sup>11</sup>, Giuseppe Jurman<sup>10</sup>, Jon Mangion<sup>11</sup>, Tapan Mehta<sup>4</sup>, Michael Nitzberg<sup>5</sup>, Grier P Page<sup>4,12</sup>, Enrico Petretto<sup>11,13</sup> & Vera van Noort<sup>14</sup>

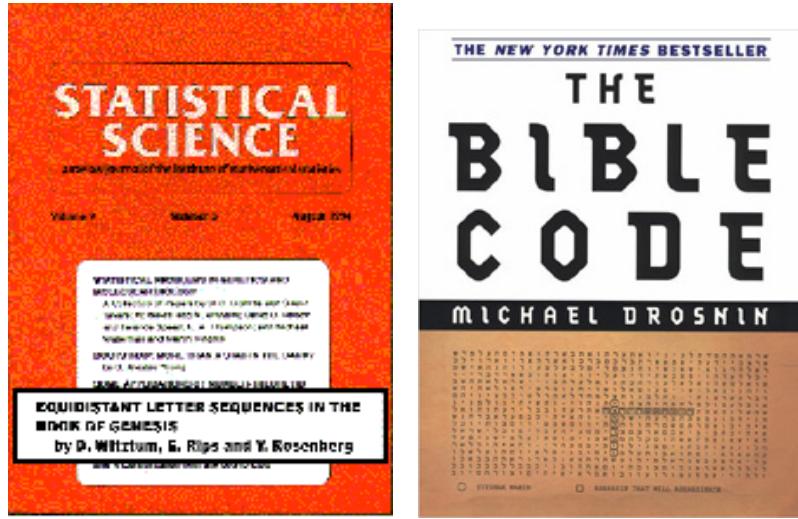
**All rights reserved.**  
 Given the complexity of microarray-based gene expression studies, guidelines encourage transparent design and public data availability. Several journals require public data deposition and several public databases exist. However, not all data are publicly available, and even when available, it is unknown whether the published results are reproducible by independent scientists. Here we evaluated the replication of data analyses in 18 articles on microarray-based gene expression profiling published in *Nature Genetics* in 2005–2006. One table or figure from each article was independently evaluated by two teams of analysts. We reproduced two analyses in principle

research, the Uniform Guidelines of the International Committee of Medical Journal Editors state that authors should “identify the methods, apparatus and procedures in sufficient detail to allow other workers to reproduce the results”<sup>12</sup>. Making primary data publicly available has many challenges but also many benefits<sup>13</sup>. Public data availability allows other investigators to confirm the results of the original authors, exactly replicate these results in other studies and try alternative analyses to see whether results are robust and to learn new things. Journals such as *Nature Genetics* require public data deposition as a prerequisite for publication for microarray-based research. Yet, the extent to which data are indeed made fully and accurately publicly available and permit con-



**Figure 1** Summary of the efforts to replicate the published analyses.

- Bible code.



Witztum et al. (1994) Equidistant letter sequences in the book of genesis. *Statist. Sci.*, 9(3):429438. <http://projecteuclid.org/euclid.ss/1177010393>

McKay et al. (1999) Solving the Bible code puzzle, *Statist. Sci.*, 14(2):150–173.

<https://www.math.washington.edu/~greenber/BibleCode.html>

- Why reproducible research?
  - Replicability has been a foundation of science. It helps accumulate scientific knowledge.
  - Better work habit boosts quality of research.
  - Greater research impact.
  - Better teamwork. For you, it means better communication with your advisor (Buckheit and Donoho, 1995).
  - ...
- Readings.
  - Buckheit and Donoho (1995) Wavelab and reproducible research, in *Wavelets and Statistics*, volume 103 of *Lecture Notes in Statistics*, page 55–81.

Springer Newt York. <http://statweb.stanford.edu/~donoho/Reports/1995/wavelab.pdf>

Donoho (2010) An invitation to reproducible computational research, *Bio-statistics*, 11(3):385-388.

- Peng (2009) Reproducible research and biostatistics, *Biostatistics*, 10(3):405–408.

Peng (2011) Reproducible research in computational science, *Science*, 334(6060):1226–1227.

Roger Peng's blogs *Treading a New Path for Reproducible Research*.

<http://simplystatistics.org/2013/08/21/treading-a-new-path-for-reproducible->

<http://simplystatistics.org/2013/08/28/evidence-based-data-analysis-treading>

<http://simplystatistics.org/2013/09/05/implementing-evidence-based-data-anal>

- *Reproducible research with R and RStudio* by Christopher Gandrud. It covers many useful tools: R, RStudio, L<sup>A</sup>T<sub>E</sub>X, Markdown, *knitr*, Github, Linux shell, ...

This book is nicely reproducible. Git clone the source from <https://github.com/christophergandrud/Rep-Res-Book> and you should be able to compile into a pdf.

- *Reproducibility in Science* at  
<http://ropensci.github.io/reproducibility-guide/>

- How to be reproducible in statistics?

*When we publish articles containing figures which were generated by computer, we also publish the complete software environment which generates the figures.*

Buckheit and Donoho (1995)

- For theoretical results, include all detailed proofs.
- For data analysis or simulation study
  - \* Describe your computational results with painstaking details.

- \* Put your code on your website or in an online supplement (required by many journals, e.g., *Biostatistics*, *JCGS*, ...) that allow replication of entire analysis or simulation study. A good example:  
[http://stanford.edu/~boyd/papers/admm\\_distr\\_stats.html](http://stanford.edu/~boyd/papers/admm_distr_stats.html)
  - \* Create a dynamic version of your simulation study/data analysis.
- What can we do *now*? At least make your homework reproducible!
    - Document everything!
    - Everything is a text file (.csv, .tex, .bib, .Rmd, .R, ...) They aid future proof and are subject to version control.  
 Word/Excel are not text files.
    - All files should be human readable. Abundant comments and adopt a good style.
    - Tie your files together.
    - Use a dynamic document generation tool (weaving/knitting text, code, and output together) for documentation.
    - Use a version control system proactively.
    - Print `sessionInfo()` in R.

For your homework, submit (put in the `master` branch) a final pdf or html report and all files and instructions necessary to reproduce all results.

- Tools for dynamic document/report generation.
  - R: RMarkdown, knitr, Sweave.
  - Matlab: automatic report generator.
  - Python: IPython, Pweave.
  - Julia: IJulia.

For this course, please write homework report in RMarkdown (or IJulia if you use Julia).

## 4 Lecture 4, Jan 14

### Last time

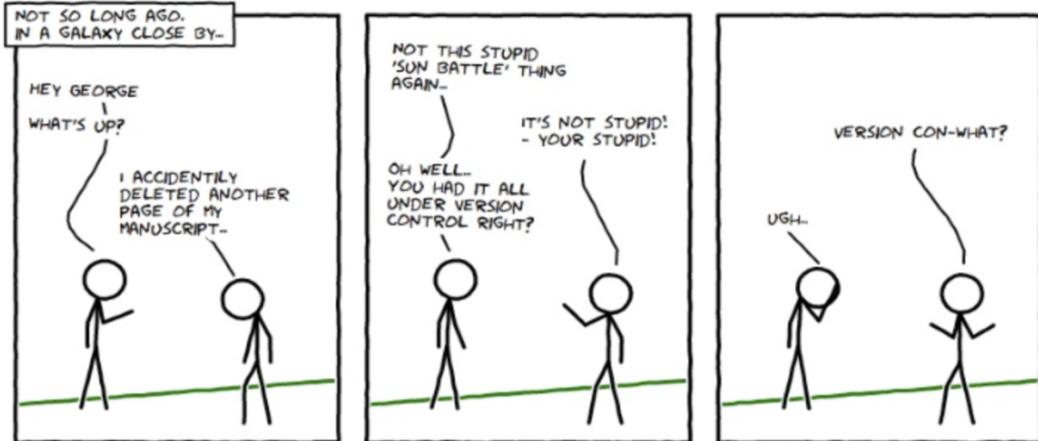
- Computer languages.
- Comparison of R, MATLAB, and JULIA.
- R, RStudio, RMarkdown.

### Today

- Version control.

### Version control by Git

If it's not in source control, it doesn't exist.



- Collaborative research. Statisticians, as opposed to “closet mathematicians”, rarely do things in vacuum.
  - We talk to scientists/clients about their data and questions.
  - We write code (a lot!) together with team members or coauthors.
  - We run code/program on different platforms.
  - We write manuscripts/reports with co-authors.

- ...
- 4 things distinguish professional programmers from amateurs:
  - *Use a version control system.*
  - Automate repetitive tasks.
  - Systematic testing.
  - Use debugging aids rather than print statements.
- Why version control?
  - A centralized repository helps coordinate multi-person projects.
  - Time machine. Keep track of all the changes and revert back easily (reproducible).
  - Storage efficiency.
  - Synchronize files across multiple computers and platforms.
  - `github.com` is becoming a *de facto* central repository for open source development. E.g., all packages in Julia are distributed through `github.com`.
  - Advertise yourself thru `github.com`.
- Available version control tools.
  - Open source: cvs, subversion (aka svn), Git, ...
  - Proprietary: Visual SourceSafe (VSS), ...
  - Dropbox? Mostly for file back and sharing, limited version control (1 month?), ...

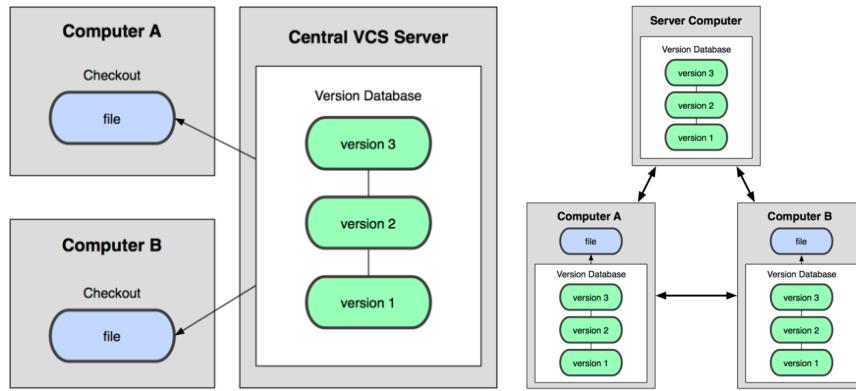
We use Git in this course.

- Why Git?
  - The Eclipse Community Survey in 2014 shows Git is the most widely used source code management tool *now*. Git (33.3%) vs svn (30.7%).
  - History: Initially designed and developed by Linus Torvalds in 2005 for Linux kernel development. “git” is the British English slang for “unpleasant person”.

I'm an egotistical bastard, and I name all my projects after myself. First 'Linux', now 'git'.

Linus Torvalds

- A fundamental difference between svn (centralized version control system, left plot) and Git (distributed version control system, right plot):



- Advantages of Git.
  - \* Free and open source.
  - \* Speed and simple (?) design.
  - \* Strong support for non-linear development (1000s of parallel branches).  
Scalable to large projects like the Linux kernel project.
  - \* Fully distributed. Fast, no internet required, disaster recovery,
- Be aware that svn is still widely used in IT industry (Apache, GCC, SourceForge, Google Code, ...) and R development. E.g., type  
`svn log -v -l 5 https://svn.r-project.org/R`  
on command line to get a glimpse of what R development core team is doing. Good to master some basic svn commands.
- What do I need to use Git?
  - A Git server enabling multi-person collaboration through a centralized repository.
    - \* [github.com](https://github.com): unlimited public repositories, private repositories costs \$, academic user can get 5 private repositories for free.

- \* `bitbucket.org`: unlimited private repositories for academic account (register for free using your UCLA email).
- \* Some institutes and departments have their own git server.

We use `bitbucket.org` in this course.

- Git client.
  - \* Linux: installed on many servers. If not, install on CentOS by `yum install git`.
  - \* Mac: install by `port install git`.
  - \* Windows: GitHub for Windows (GUI), TortoiseGIT (is this good?)

Don't rely on GUI. Learn to use Git on command line.

- Life cycle of a (research) project.

Stage 1:

- A project (idea) is started on `bitbucket.org`, with directories say `codebase`, `datasets`, `manuscripts`, `talks`, ...
- Advantage of `bitbucket.org`: privacy of research ideas (free private repositories).
- Downside of `bitbucket.org`: not as widely known as `github.com`.

Stage 2:

- Hopefully, research idea pans out and we want to distribute a standalone software development, e.g., an R package, repository at `github.com`. Read the (free) book *R Packages* (<http://r-pkgs.had.co.nz>) by Hadley Wickham for development of R packages and distribution via `github.com`.
- This usually inherits from the `codebase` folder and happens when we submit a paper.
- Challenges: keep all version history. It's doable.

Stage 3:

- Active maintenance of the public software repository.

- At least three branches: `develop`, `master`, `gh-pages`.
    - `develop`: main development area.
    - `master`: software release.
    - `gh-pages`: software webpage.
  - Basic workflow of Git.
- The diagram illustrates the basic Git workflow. It shows two main areas: Local (left) and Remote (right). In the Local area, there are three components: working directory (green), staging area (yellow), and local repo (blue). In the Remote area, there is a single component: remote repo (orange). Red arrows indicate the flow of operations:

  - From the working directory to the staging area: `git add`
  - From the staging area to the local repo: `git commit`
  - From the local repo to the remote repo: `git push`
  - From the remote repo back to the local repo: `git fetch`
  - From the local repo to the working directory: `git checkout`
  - From the local repo to the working directory: `git merge`
- Synchronize local Git directory with remote repository (`git pull`).
  - Modify files in local working directory.
  - Add snapshots of them to staging area (`git add`).
  - Commit: store snapshots permanently to (local) Git repository (`git commit`).
  - Push commits to remote repository (`git push`).
  - Basic Git usage.
  - Register for an account on a Git server, e.g., [bitbucket.org](https://bitbucket.org). Fill out your profile, upload your public key to the server, ...
  - Identify yourself at local machine:
 

```
git config --global user.name "Hua Zhou"
git config --global user.email "huazhou@ucla.edu"
```

 Name and email appear in each commit you make.

- Initialize a project:
  - \* Create a repository, e.g., `biostat-m280-2016-winter`, on the server `bitbucket.org`. Then clone to local machine  
`git clone git@bitbucket.org:username/biostat-m280-2016-winter.git`
  - \* Alternatively use following commands to initialize a Git directory from a local folder and then push to the Git server  
`git init`  
`git remote add origin git@bitbucket.org:username/biostat-m280-2016-winter`  
`git push -u origin master`
- Edit working directory.
  - `git pull` update local Git repository with remote repository (fetch + merge).
  - `git status` displays the current status of working directory.
  - `git log filename` displays commit logs of a file.
  - `git diff` shows differences (by default difference from the most recent commit).
  - `git add ...` adds file(s) to the staging area.
  - `git commit` commits changes in staging area to Git directory.
  - `git push` publishes commits in local Git directory to remote repository.

Following demo session is on my local Mac machine.

```

hzhou3@Hua-Zhou$ pwd
/Users/hzhou3/github.ncsu/mglm
hzhou3@Hua-Zhou$ ls
.DS_Store .gitignore datasets/ manuscripts/
.git/ codebase/ literature/ talks/
hzhou3@Hua-Zhou$ git pull
remote: Counting objects: 5, done.
remote: Compressing objects: 100% (2/2), done.
remote: Total 5 (delta 3), reused 5 (delta 3)
Unpacking objects: 100% (5/5), done.
From github.ncsu.edu:hzhou3/vctest
 80be212..b22d29f master    -> origin/master
Updating 80be212..b22d29f
Fast-forward
  manuscripts/letter-skat-famskat/Letter_to_the_editor.tex | 4 +---
  1 file changed, 2 insertions(+), 2 deletions(-)
hzhou3@Hua-Zhou$ echo "hello st790 class" > gitdemo.txt
hzhou3@Hua-Zhou$ ls
.DS_Store .gitignore datasets/ literature/ talks/
.git/ codebase/ gitdemo.txt manuscripts/

```

```

hzhou3@Hua-Zhous-MacBook-Pro:mglm $ git add gitdemo.txt
hzhou3@Hua-Zhous-MacBook-Pro:mglm $ git status .
On branch master
Your branch is up-to-date with 'origin/master'.
Changes to be committed:
  (use "git reset HEAD <file>..." to unstage)

    new file:   gitdemo.txt

Untracked files:
  (use "git add <file>..." to include in what will be committed)

    codebase/Example_RNAseq_top100/
    codebase/MGLM/R.Rhistory
hzhou3@Hua-Zhous-MacBook-Pro:mglm $ git commit -m "git demo for st790 class"
[master ea636ff] git demo for st790 class
 1 file changed, 1 insertion(+)
 create mode 100644 gitdemo.txt
hzhou3@Hua-Zhous-MacBook-Pro:mglm $ git log gitdemo.txt
commit ea636ff5665bc26bf8a79751b75d0e9d67bdb7d1
Author: Hua Zhou <hua_zhou@ncsu.edu>
Date:   Sun Jan 11 17:06:23 2015 -0500

  git demo for st790 class
hzhou3@Hua-Zhous-MacBook-Pro:mglm $ git push
Counting objects: 3, done.
Delta compression using up to 8 threads.
Compressing objects: 100% (2/2), done.
Writing objects: 100% (3/3), 301 bytes | 0 bytes/s, done.
Total 3 (delta 1), reused 0 (delta 0)
To git@github.ncsu.edu:hzhou3/mglm.git
  77145d2..ea636ff  master -> master

```

git reset --soft HEAD 1 undo the last commit.

git checkout filename go back to the last commit.

git rm different from rm.

Although git rm deletes files from working directory. They are still in Git history and can be retrieved whenever needed. So always be cautious to put large data files or binary files into version control.

```

hzhou3@Hua-Zhou-MacBook-Pro:mglm $ echo "bye st790 class" >> gitdemo.txt
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ cat gitdemo.txt
hello st790 class
bye st790 class
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ git diff gitdemo.txt
diff --git a/gitdemo.txt b/gitdemo.txt
index ece6d4e..2bb77f8 100644
--- a/gitdemo.txt
+++ b/gitdemo.txt
@@ -1 +1,2 @@
    hello st790 class
+bye st790 class
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ git checkout gitdemo.txt
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ cat gitdemo.txt
hello st790 class
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ git rm gitdemo.txt
rm 'gitdemo.txt'
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ git status .
On branch master
Your branch is up-to-date with 'origin/master'.
Changes to be committed:
  (use "git reset HEAD <file>..." to unstage)

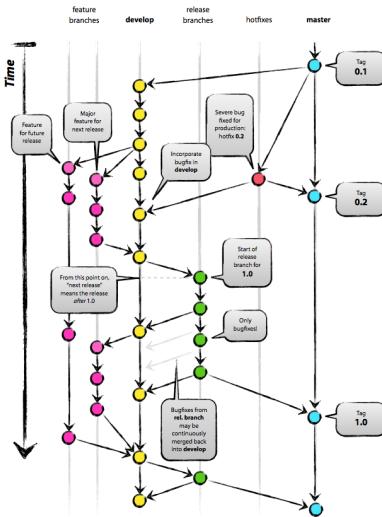
    deleted:   gitdemo.txt

Untracked files:
  (use "git add <file>..." to include in what will be committed)

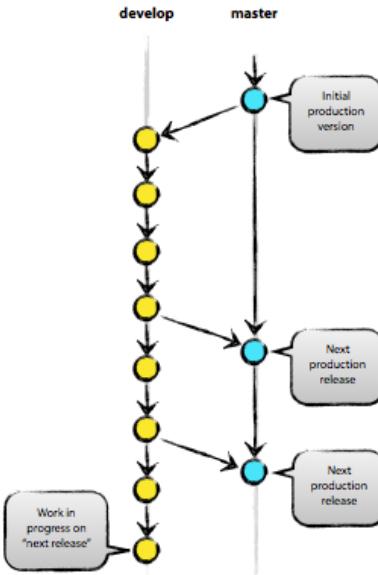
    codebase/Example_RNAseq_top100/
    codebase/MGLM/R/.Rhistory
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ git commit -m "delete the git demo file for st790"
[master 4b4f9c5] delete the git demo file for st790
 1 file changed, 1 deletion(-)
 delete mode 100644 gitdemo.txt
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ git push
Counting objects: 2, done.
Delta compression using up to 8 threads.
Compressing objects: 100% (2/2), done.
Writing objects: 100% (2/2), 238 bytes | 0 bytes/s, done.
Total 2 (delta 1), reused 0 (delta 0)
lTo git@github.ncsu.edu:hzhou3/mglm.git
 ea636ff..4b4f9c5 master -> master
hzhou3@Hua-Zhou-MacBook-Pro:mglm $ ls
.DS_Store  .gitignore  datasets/  manuscripts/
.git/      codebase/  literature/  talks/

```

- Branching in Git.
  - Branches in a project:



- For this course, you need to have two branches: `develop` for your own development and `master` for releases (homework submission). Note `master` is the default branch when you initialize the project; create and switch to `develop` branch immediately after project initialization.



- Commonly used commands:
  - `git branch branchname` creates a branch.
  - `git branch` shows all project branches.
  - `git checkout branchname` switches to a branch.
  - `git tag` shows tags (major landmarks).

`git tag tagname` creates a tag.

- Let's look at a typical branching and merging workflow.

- \* Now there is a bug in v0.0.3 ...



```
gitdemo — bash — 80x11
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git branch
  develop
* master
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git tag
v0.0.1
v0.0.2
v0.0.3
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ ls
.git/    bug.txt  code.txt
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $
```

How to organize version number of your software? Read blog “R Package Versioning” by Yihui Xie

<http://yihui.name/en/2013/06/r-package-versioning/>

```
gitdemo — bash — 80x31
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git checkout develop
Switched to branch 'develop'
Your branch is up-to-date with 'origin/develop'.
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git branch
* develop
  master
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ ls
.git/    code.txt
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git pull origin master
From github.ncsu.edu:hzhou3/gitdemo
 * branch           master      -> FETCH_HEAD
Updating 44dd1d1..da047cf
Fast-forward
 bug.txt | 1 +
 1 file changed, 1 insertion(+)
 create mode 100644 bug.txt
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ ls
.git/    bug.txt  code.txt
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git rm bug.txt
rm 'bug.txt'
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git commit -m "debug"
[develop 1085b4c] debug
 1 file changed, 1 deletion(-)
 delete mode 100644 bug.txt
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git push
Counting objects: 1, done.
Writing objects: 100% (1/1), 180 bytes | 0 bytes/s, done.
Total 1 (delta 0), reused 0 (delta 0)
To git@github.ncsu.edu:hzhou3/gitdemo.git
 44dd1d1..1085b4c  develop -> develop
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $
```

Now ‘debug’ in develop branch is ahead of master branch.

- \* Merge bug fix to the `master` branch.

```
gitdemo — bash — 80x26
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git checkout master
Switched to branch 'master'
Your branch is up-to-date with 'origin/master'.
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git branch
  develop
* master
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git pull origin develop
From github.ncsu.edu:hzhou3/gitdemo
 * branch            develop    -> FETCH_HEAD
Updating da047cf..1085b4c
Fast-forward
 bug.txt | 1 -
 1 file changed, 1 deletion(-)
 delete mode 100644 bug.txt
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ ls
.git/   code.txt
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git status .
On branch master
Your branch is ahead of 'origin/master' by 1 commit.
  (use "git push" to publish your local commits)
nothing to commit, working directory clean
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git push origin master
Total 0 (delta 0), reused 0 (delta 0)
To git@github.ncsu.edu:hzhou3/gitdemo.git
  da047cf..1085b4c  master -> master
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $
```

\* Tag a new release v0.0.4.

```

hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git tag v0.0.4
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git tag
v0.0.1
v0.0.2
v0.0.3
v0.0.4
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git show v0.0.4
commit 1085b4c97ed29fc847442bd0640db2b6fed4d0af
Author: Hua Zhou <hua_zhou@ncsu.edu>
Date:   Tue Jan 13 11:24:19 2015 -0500

        debug

diff --git a/bug.txt b/bug.txt
deleted file mode 100644
index 0a1d6ac..0000000
--- a/bug.txt
+++ /dev/null
@@ -1 +0,0 @@
-There is a bug
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $ git push origin v0.0.4
Total 0 (delta 0), reused 0 (delta 0)
To git@github.ncsu.edu:hzhou3/gitdemo.git
 * [new tag]      v0.0.4 -> v0.0.4
hzhou3@Hua-Zhous-MacBook-Pro:gitdemo $

```

- Further resources:
  - Book *Pro Git*, <http://git-scm.com/book/en/v2>
  - Google
- Some etiquettes of using Git and version control systems in general.
  - Be judicious what to put in repository
    - \* Not too less: Make sure collaborators or yourself can reproduce everything on other machines.
    - \* Not too much: No need to put all intermediate files in repository.
  - Strictly version control system is for source files only. E.g. only `xxx.tex`, `xxx.bib`, and figure files are necessary to produce a pdf file. Pdf file doesn't need to be version controlled or frequently committed.
  - “Commit early, commit often and don't spare the horses”

- Adding an informative message when you commit is *not* optional. Spending one minute now saves hours later for your collaborators and yourself. Read the following sentence to yourself 3 times:  
“Write every commit message like the next person who reads it is an axe-wielding maniac who knows where you live.”

## 5 Lecture 5, Jan 19

### Announcements

- HW1 due this Thu @ 11:59PM.
- Quiz 1 this Thu, in class, closed-book.
- Demo of JIT in R. <http://hua-zhou.github.io/teaching/biostatm280-2016winter/jit.html>

### Last time

- Version control.

### Today

- Version control (cont'd).
- Algorithm: general concepts.
- Numerical linear algebra: introduction, BLAS.

### Algorithms

- *Algorithm* is loosely defined as a set of instructions for doing something. Input → Output.
- Knuth (2005): (1) finiteness, (2) definiteness, (3) input, (4) output, (5) effectiveness
- Basic unit for measuring efficiency is flop. A *flop* (floating point operation) consists of a floating point addition, subtraction, multiplication, division, or comparison, and the usually accompanying fetch and store. Some books count multiplication followed by an addition (fused multiply-add, FMA) as one flop. This results a factor of 2 difference in flop counts.
- How to measure efficiency of an algorithm? Big O notation. If  $n$  is the size of a problem, an algorithm has order  $O(f(n))$ , where the leading term in the number of flops is  $c \cdot f(n)$ .

- E.g., matrix-vector multiplication  $\mathbf{A} \times \mathbf{b}$ , where  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{b} \in \mathbb{R}^n$ , takes  $2mn$  or  $O(mn)$  flops. Matrix-matrix multiplication  $\mathbf{A} \times \mathbf{B}$ , where  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{B} \in \mathbb{R}^{n \times p}$ , takes  $2mnp$  or  $O(mnp)$  flops.
- A hierarchy of computational complexity:  
*Exponential* order  $O(b^n)$  (NP-hard=“horrible”)  
*Polynomial* order  $O(n^q)$  (doable)  
 $O(n \log n)$  (fast)  
Linear order  $O(n)$  (fast)  
Log order  $O(\log n)$  (super fast).
- One goal of this course is to get familiar with the flop counts for some common numerical tasks in statistics.

The form of a mathematical expression and the way the expression should be evaluated in actual practice may be quite different.

- Compare flops of the following two R expressions:

```
G %*% Xt %*% y
G %*% (Xt %*% y)
```

where  $\mathbf{G} \in \mathbb{R}^{p \times p}$ ,  $\mathbf{Xt} \in \mathbb{R}^{p \times n}$ , and  $\mathbf{y} \in \mathbb{R}^n$ . “Matrix multiplication is expensive.”

- Hardware advancement, e.g., CPU clock rate, only affects constant  $c$ . Unfortunately, data size  $n$  is increasing too and often at a faster rate.
- Classification of data sets by Huber (1994, 1996).

Data Size	Bytes	Storage Mode
Tiny	$10^2$	Piece of paper
Small	$10^4$	A few pieces of paper
Medium	$10^6$ (megabyte)	A floppy disk
Large	$10^8$	Hard disk
Huge	$10^9$ (gigabytes)	Hard disk(s)
Massive	$10^{12}$ (terabytes)	RAID storage

- Difference of  $O(n^2)$  and  $O(n \log n)$  on massive data. Suppose we have a teraflop supercomputer capable of doing  $10^{12}$  flops per second. For a problem of size  $n = 10^{12}$ ,  $O(n \log n)$  algorithm takes about  $10^{12} \log(10^{12})/10^{12} \approx 27$  seconds.  $O(n^2)$  algorithm takes about  $10^{12}$  seconds, which is approximately 31710 years!
- QuickSort and FFT are celebrated algorithms that turn  $O(n^2)$  operations into  $O(n \log n)$ . *Divide-and-conquer* is a powerful technique. Another example is the Strassen's method, which turns  $O(n^3)$  matrix multiplication into  $O(n^{\log_2 7})$ .

## Numerical linear algebra

- The first big chunk of this course is numerical linear algebra.
- Topics in numerical algebra: BLAS, solve linear equations  $\mathbf{Ax} = \mathbf{b}$ , regression computations  $\mathbf{X}^T \mathbf{X} \boldsymbol{\beta} = \mathbf{X}^T \mathbf{y}$ , eigen-problems  $\mathbf{Ax} = \lambda \mathbf{x}$  and generalized eigen-problems  $\mathbf{Ax} = \lambda \mathbf{Bx}$ , singular value decompositions  $\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^T$ , iterative methods, ...
- Most of these numerical linear algebra tasks are implemented in the BLAS and LAPACK libraries. Our major goal is to (1) know the computational cost of each task, (2) be familiar with the BLAS and LAPACK functions (what they do), and (3) do *not* re-invent wheels by implementing these subroutines by yourself.

All high-level languages (R, MATLAB, JULIA) call BLAS and LAPACK for numerical linear algebra. JULIA offers more flexibility by exposing interfaces to many BLAS/LAPACK subroutines directly. <http://docs.julialang.org/en/release-0.4/stdlib/linalg/?highlight=blas#module-Base.LinAlg.BLAS>

- Please review the following linear algebra facts by yourself.

## Linear algebra review

### Vector and matrix norms

KL chapter 6. Norm measures the “size”.

- Vector norm  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ . (a)  $\|\mathbf{x}\| \geq 0$ , (b)  $\|\mathbf{x}\| = 0$  if and only if  $\mathbf{x} = \mathbf{0}$ , (c)  $\|c\mathbf{x}\| = c\|\mathbf{x}\|$  (homogeneity), (d)  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$  (triangle inequality).

- $\ell_p$  norm.  $\|\mathbf{x}\|_p = (\sum_i |x_i|^p)^{1/p}$ ,  $p \in [1, \infty]$ .  $\ell_1$  is the Manhattan norm.  $\ell_2$  is the Euclidean norm.  $\ell_\infty$  is the sup norm.
- For matrix norm  $\|\cdot\| : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ , we further require (e)  $\|\mathbf{AB}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|$ .
- Frobenius norm  $\|\mathbf{A}\|_{\text{F}} = (\sum_i \sum_j a_{ij}^2)^{1/2}$ . Properties of (e) is checked by Cauchy-Schwartz inequality.
- Induced matrix norm (or operator norm):  $\|\mathbf{A}\| = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|} = \sup_{\|\mathbf{x}\|=1} \|\mathbf{Ax}\|$  for any fixed vector norm. To check property (e), let  $\mathbf{y} = \mathbf{Bx}$ , then  $\|\mathbf{AB}\| = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Ay}\|}{\|\mathbf{y}\|} \frac{\|\mathbf{Bx}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\| \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Bx}\|}{\|\mathbf{x}\|} = \|\mathbf{A}\| \|\mathbf{B}\|$ .
- Matrix-1 norm,  $\|\mathbf{A}\|_1 = \max_j \sum_i |a_{ij}|$ .  
Matrix-2 norm,  $\|\mathbf{A}\|_2 = \sqrt{\rho(\mathbf{A}^\top \mathbf{A})} = \max_{\|\mathbf{u}\|_2=1, \|\mathbf{v}\|_2=1} \mathbf{u}^\top \mathbf{A} \mathbf{v}$ , which reduces to  $\rho(\mathbf{A})$  if  $\mathbf{A}$  is symmetric.  $\rho$  is the spectral radius of a matrix, the absolute value of the dominant eigenvalue.  
Matrix- $\infty$  norm,  $\|\mathbf{A}\|_\infty = \max_i \sum_j |a_{ij}|$ .
- When  $\mathbf{A}$  is a column vector, these matrix induced norms reduce to the original vector norm.
- $\rho(\mathbf{A}) \leq \|\mathbf{A}\|$  for any induced matrix norm. For any  $\mathbf{A}$  and  $\epsilon > 0$ , there exists an induced matrix norm such that  $\|\mathbf{A}\| \leq \rho(\mathbf{A}) + \epsilon$ .

## Rank

Assume  $\mathbf{A} \in \mathbb{R}^{m \times n}$ .

- $\text{rank}(\mathbf{A})$  is the maximum number of linearly independent rows (or columns) of a matrix.
- $\text{rank}(\mathbf{A}) \leq \min\{m, n\}$ .
- A matrix is *full rank* if  $\text{rank}(\mathbf{A}) = \min\{m, n\}$ . It is *full row rank* if  $\text{rank}(\mathbf{A}) = m$ . It is *full column rank* if  $\text{rank}(\mathbf{A}) = n$ .
- A square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is *singular* if  $\text{rank}(\mathbf{A}) < n$  and *non-singular* if  $\text{rank}(\mathbf{A}) = n$ .

- $\text{rank}(\mathbf{AB}) \leq \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$ . “Matrix multiplication cannot increase the rank.”
- $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) = \text{rank}(\mathbf{A}^T \mathbf{A}) = \text{rank}(\mathbf{AA}^T)$ .
- $\text{rank}(\mathbf{AB}) = \text{rank}(\mathbf{A})$  if  $\mathbf{B}$  has full row rank.
- $\text{rank}(\mathbf{AB}) = \text{rank}(\mathbf{B})$  if  $\mathbf{A}$  has full column rank.
- $\text{rank}(\mathbf{A} + \mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$ .

## Trace

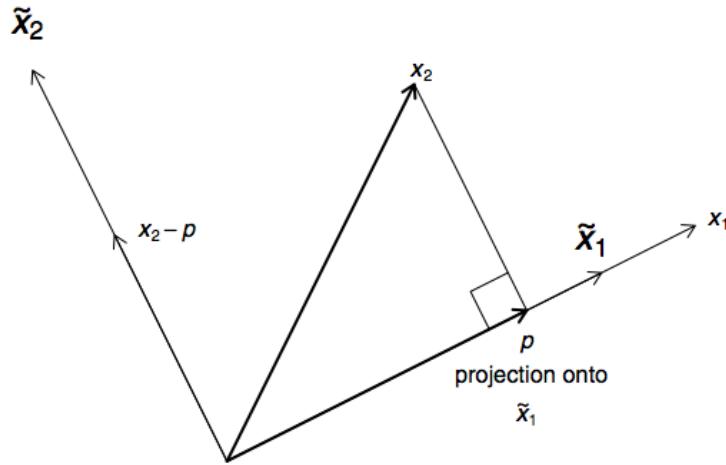
$\mathbf{A} \in \mathbb{R}^{n \times n}$  a square matrix.

- $\text{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$
- $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$
- $\text{tr}(\lambda \mathbf{A}) = \lambda \text{tr}(\mathbf{A})$  where  $\lambda$  is a scalar
- $\text{tr}(\mathbf{A}^\top) = \text{tr}(\mathbf{A})$
- Invariance under cycle permutation:  $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$ . In general,  $\text{tr}(\mathbf{A}_1 \cdots \mathbf{A}_k) = \text{tr}(\mathbf{A}_{j+1} \cdots \mathbf{A}_k \mathbf{A}_1 \cdots \mathbf{A}_j)$ .

## Orthogonality and orthogonalization

- $\mathbf{v}_1$  is *orthogonal* to  $\mathbf{v}_2$ , written  $\mathbf{v}_1 \perp \mathbf{v}_2$ , if  $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1^\top \mathbf{v}_2 = 0$ . They are *orthonormal* if  $\mathbf{v}_1 \perp \mathbf{v}_2$  and  $\|\mathbf{v}_i\|_2 = 1$ ,  $i = 1, 2$ .
- Gram-Schmidt transformation orthonormalizes two non-zero vectors  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .

$$\begin{aligned}\tilde{\mathbf{x}}_1 &= \frac{1}{\|\mathbf{x}_1\|_2} \mathbf{x}_1 \\ \tilde{\mathbf{x}}_2 &= \frac{1}{\|\mathbf{x}_2 - \langle \tilde{\mathbf{x}}_1, \mathbf{x}_2 \rangle \tilde{\mathbf{x}}_1\|_2} (\mathbf{x}_2 - \langle \tilde{\mathbf{x}}_1, \mathbf{x}_2 \rangle \tilde{\mathbf{x}}_1)\end{aligned}$$



**Fig. 2.2.** Orthogonalization of  $x_1$  and  $x_2$

- A set of nonzero, mutually orthogonal vectors are linearly independent.
- A real square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is orthogonal if  $\mathbf{A}^\top \mathbf{A} = \mathbf{I}_n$ , i.e., its rows/columns are orthonormal. Orthogonal matrix is of full rank, thus  $\mathbf{A}^\top = \mathbf{A}^{-1}$  and  $\mathbf{A}\mathbf{A}^\top = \mathbf{I}_n$ .

### Positive (semi)definite matrix

Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is *symmetric*.

- A real symmetric matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is *positive semidefinite* (or *nonnegative definite*) if  $\mathbf{x}^\top \mathbf{A} \mathbf{x} \geq 0$  for all  $\mathbf{x}$ . Notation:  $\mathbf{A} \succeq \mathbf{0}_{n \times n}$ .
- E.g., the *Gramian matrix*  $\mathbf{X}^\top \mathbf{X}$  or  $\mathbf{X} \mathbf{X}^\top$ .
- If inequality is strict for all  $\mathbf{x} \neq \mathbf{0}$ , then  $\mathbf{A}$  is *positive definite*. Notation:  $\mathbf{A} \succ \mathbf{0}_{n \times n}$ .
- $\mathbf{A} \succeq \mathbf{B}$  means  $\mathbf{A} - \mathbf{B} \succeq \mathbf{0}_{n \times n}$ .
- If  $\mathbf{A} \succeq \mathbf{B}$ , then  $\det(\mathbf{A}) \geq \det(\mathbf{B})$  with equality if and only if  $\mathbf{A} = \mathbf{B}$ .
- $\mathbf{A} \in \mathbb{R}^{n \times n}$  is positive semidefinite if and only if  $\mathbf{A}$  is a covariance matrix of a random vector in  $\mathbb{R}^n$ .

## Matrix inverses

Assume  $\mathbf{A} \in \mathbb{R}^{m \times n}$ .

- The *Moore-Penrose inverse* of  $\mathbf{A}$  is a matrix  $\mathbf{A}^+ \in \mathbb{R}^{n \times m}$  with following properties
  - (a)  $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$ . (*Generalized inverse*,  $g_1$  *inverse*, or *inner pseudo-inverse*)
  - (b)  $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$ . (*Outer pseudo-inverse*. Any  $g_1$  inverse that satisfies this condition is called a  $g_2$  *inverse*, or *reflexive generalized inverse* and is denoted by  $\mathbf{A}^*$ .)
  - (c)  $\mathbf{A}^+\mathbf{A}$  is symmetric.
  - (d)  $\mathbf{A}\mathbf{A}^+$  is symmetric.
- $\mathbf{A}^+$  exists and is unique for any matrix  $\mathbf{A}$ .
- *Generalized inverse* (or  $g_1$  *inverse*, denoted by  $\mathbf{A}^-$  or  $\mathbf{A}^g$ ): property (a).
- $g_2$  *inverse* (denoted by  $\mathbf{A}^*$ ): properties (a)+(b).
- *Moore-Penrose inverse* (denoted by  $\mathbf{A}^+$ ): properties (a)+(b)+(c)+(d).
- If  $\mathbf{A}$  is square and full rank, then the generalized inverse is unique and denoted by  $\mathbf{A}^{-1}$  (*inverse*).
- In practice, the Moore-Penrose inverse  $\mathbf{A}^+$  is easily computed from the singular value decomposition (SVD) of  $\mathbf{A}$ .
- $(\mathbf{A}^-)^T$  is a generalized inverse of  $\mathbf{A}^T$ .
- $\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{A}\mathbf{A}^-)$  and  $\mathcal{C}(\mathbf{A}^T) = \mathcal{C}((\mathbf{A}^-\mathbf{A})^T)$ .  
 $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}^-) = \text{rank}(\mathbf{A}^-\mathbf{A})$ .  
“Multiplication by generalized inverse does not change rank.”
- $\text{rank}(\mathbf{A}^-) \geq \text{rank}(\mathbf{A})$ . “Generalized inverse has equal or a larger rank than the original matrix.”

## System of linear equations

$\mathbf{A}\mathbf{x} = \mathbf{b}$  where  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{b} \in \mathbb{R}^m$ .

- When is there a solution? The following statements are equivalent.
  1. The linear system  $\mathbf{A}\mathbf{x} = \mathbf{b}$  has a solution (*consistent*)
  2.  $\mathbf{b} \in \mathcal{C}(\mathbf{A})$ .
  3.  $\text{rank}((\mathbf{A}, \mathbf{b})) = \text{rank}(\mathbf{A})$ .
  4.  $\mathbf{A}\mathbf{A}^{-1}\mathbf{b} = \mathbf{b}$ .

The last equivalence gives intuition why  $\mathbf{A}^{-1}$  is called an inverse.

- What are the solutions to a homogeneous system  $\mathbf{A}\mathbf{x} = \mathbf{0}$ ?  
 $\mathcal{N}(\mathbf{A}) = \mathcal{C}(\mathbf{I}_n - \mathbf{A}^{-1}\mathbf{A})$ .
- If  $\mathbf{A}\mathbf{x} = \mathbf{b}$  is consistent, then  $\tilde{\mathbf{x}}$  is a solution to  $\mathbf{A}\mathbf{x} = \mathbf{b}$  if and only if

$$\tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{b} + (\mathbf{I}_n - \mathbf{A}^{-1}\mathbf{A})\mathbf{q}$$

for some  $\mathbf{q} \in \mathbb{R}^n$ .

Interpretation: “a specific solution” + “a vector in the null space of  $\mathbf{A}$ ”.

- $\mathbf{A}\mathbf{x} = \mathbf{b}$  is consistent for *all*  $\mathbf{b}$  if and only if  $\mathbf{A}$  has full row rank.
- If a system is consistent, its solution is unique if and only if  $\mathbf{A}$  has full column rank.
- If  $\mathbf{A}$  has full row and column rank, then  $\mathbf{A}$  is non-singular and the unique solution is  $\mathbf{A}^{-1}\mathbf{b}$ .

## Gramian matrix $\mathbf{A}^\top \mathbf{A}$

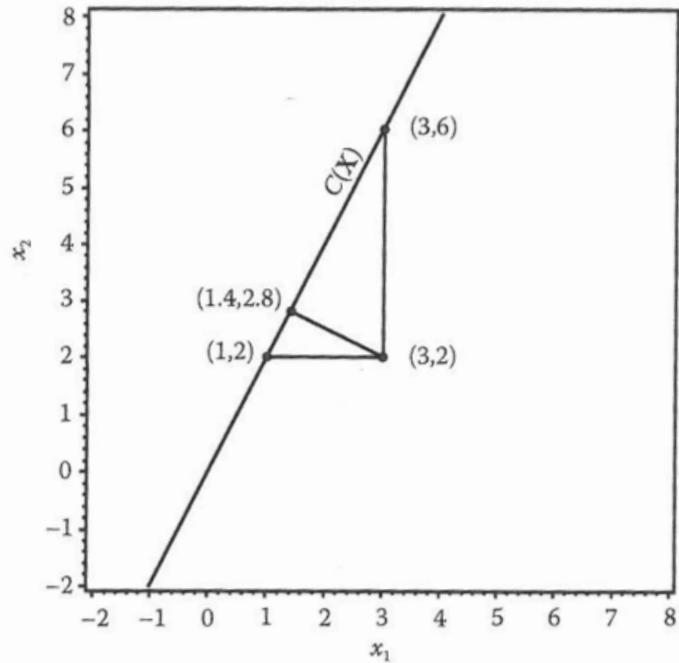
- $\mathbf{A}^\top \mathbf{A}$  is symmetric and positive semidefinite.
- $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^\top) = \text{rank}(\mathbf{A}^\top \mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}^\top)$ .
- $\mathbf{A}^\top \mathbf{A} = \mathbf{0}$  if and only if  $\mathbf{A} = \mathbf{0}$ .
- $\mathbf{B}\mathbf{A}^\top \mathbf{A} = \mathbf{C}\mathbf{A}^\top \mathbf{A}$  if and only if  $\mathbf{B}\mathbf{A}^\top = \mathbf{C}\mathbf{A}^\top$ .

- $\mathbf{A}^\top \mathbf{A}\mathbf{B} = \mathbf{A}^\top \mathbf{A}\mathbf{C}$  if and only if  $\mathbf{AB} = \mathbf{AC}$ .
- For any generalized inverse  $(\mathbf{A}^\top \mathbf{A})^-$ ,  $[(\mathbf{A}^\top \mathbf{A})^-]^\top$  is also a generalized inverse of  $\mathbf{A}^\top \mathbf{A}$ . Note  $(\mathbf{A}^\top \mathbf{A})^-$  is not necessarily symmetric.
- $(\mathbf{A}^\top \mathbf{A})^- \mathbf{A}^\top$  is a generalized inverse of  $\mathbf{A}$ .
- $\mathbf{AA}^+ = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^- \mathbf{A}^\top$ , where  $\mathbf{A}^+$  is the Moore-Penrose inverse of  $\mathbf{A}$ .
- $\mathbf{P}_\mathbf{A} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^- \mathbf{A}^\top$  is symmetric, idempotent, invariant to the choice of generalized inverse  $(\mathbf{A}^\top \mathbf{A})^-$ , and projects onto  $\mathcal{C}(\mathbf{A})$ .

### Idempotent matrix and projection

Assume  $\mathbf{P} \in \mathbb{R}^{n \times n}$ .

- A matrix  $\mathbf{P} \in \mathbb{R}^{n \times n}$  is *idempotent* if and only if  $\mathbf{P}^2 = \mathbf{P}$ .
- A matrix  $\mathbf{P}$  is a *projection* on a vector space  $\mathcal{V}$  if (a)  $\mathbf{P}$  is idempotent, (b)  $\mathbf{Px} \in \mathcal{V}$  for all  $\mathbf{x}$ , and (c)  $\mathbf{Pz} = \mathbf{z}$  for all  $\mathbf{z} \in \mathcal{V}$ .
- An idempotent matrix  $\mathbf{P}$  is a projection onto  $\mathcal{C}(\mathbf{P})$ .
- For a *general* matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , the matrices  $\mathbf{AA}^-$  are projections onto  $\mathcal{C}(\mathbf{A})$  and  $\mathbf{I}_n - \mathbf{A}^- \mathbf{A}$  are projections onto  $\mathcal{N}(\mathbf{A})$ .



**Figure A.1:** Three projections onto a column space.

### Symmetric idempotent matrix and orthogonal projection

Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$ .

- A symmetric, idempotent matrix is called an *orthogonal projection*.
- An orthogonal projection  $\mathbf{P}$  satisfies  $\mathbf{y} - \mathbf{P}\mathbf{y} \perp \mathbf{v}$  for all  $\mathbf{v} \in \mathcal{C}(\mathbf{P})$ .
- The orthogonal projection onto a vector space is unique.
- If a symmetric, idempotent matrix  $\mathbf{P}$  projects onto  $\mathcal{V}$ , then  $\mathbf{I} - \mathbf{P}$  projects onto the orthogonal complement  $\mathcal{V}^\perp$ .
- Pythagorean theorem: For  $\mathbf{P}$  an orthogonal projection,

$$\|\mathbf{y}\|_2^2 = \|\mathbf{P}\mathbf{y}\|_2^2 + \|(\mathbf{I} - \mathbf{P})\mathbf{y}\|_2^2.$$

- Many books use the term “projection” in the sense of of orthogonal projection.

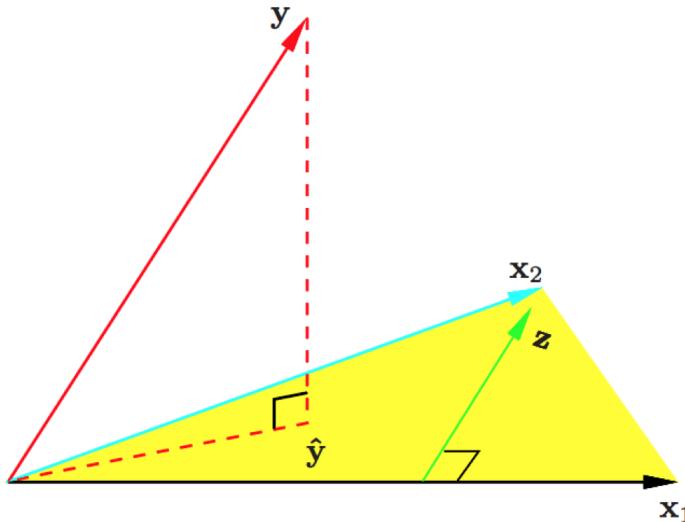
## Method of least squares

- Goal: Approximate  $\mathbf{y} \in \mathbb{R}^n$  by a linear combination of columns of  $\mathbf{X} \in \mathbb{R}^{n \times p}$ .
- Least squares criterion:  $\min Q(\mathbf{b}) = \|\mathbf{y} - \mathbf{X}\mathbf{b}\|_2^2$ .
- Any solution to the normal equation  $\mathbf{X}^T \mathbf{X}\mathbf{b} = \mathbf{X}^T \mathbf{y}$  (always consistent) is a minimizer of the least squares criterion  $Q(\mathbf{b})$ .
- Solutions to the normal equation:

$$\hat{\mathbf{b}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} + (\mathbf{I}_p - (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X}) \mathbf{q},$$

where  $\mathbf{q} \in \mathbb{R}^q$  is arbitrary.

- $(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$  is a generalized inverse of  $\mathbf{X}$ . Therefore the least squares solution applies even when the system  $\mathbf{X}\mathbf{b} = \mathbf{y}$  is consistent.
- Least squares solution is unique if and only if  $\mathbf{X}$  has full column rank.
- $\mathbf{P}_{\mathbf{X}} = \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$  is *the* orthogonal projection onto  $\mathcal{C}(\mathbf{X})$ .
- Geometry: The fitted value from the least squares solution  $\hat{\mathbf{y}} = \mathbf{P}_{\mathbf{X}} \mathbf{y}$  is the orthogonal projection of the response vector  $\mathbf{y}$  onto the column space  $\mathcal{C}(\mathbf{X})$ .



- $\mathbf{I}_n - \mathbf{P}_{\mathbf{X}}$  is the orthogonal projection onto  $\mathcal{N}(\mathbf{X}^T)$ .

- Decomposition of  $\mathbf{y}$ :

$$\mathbf{y} = \mathbf{P}_X \mathbf{y} + (\mathbf{I}_n - \mathbf{P}_X) \mathbf{y} = \hat{\mathbf{y}} + \hat{\mathbf{e}},$$

where  $\hat{\mathbf{y}} \perp \hat{\mathbf{e}}$  and

$$\|\mathbf{y}\|_2^2 = \|\hat{\mathbf{y}}\|_2^2 + \|\hat{\mathbf{e}}\|_2^2.$$

## Eigenvalues and eigenvectors

KL chapter 8. Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$  a square matrix.

- *Eigenvalues* are defined as roots of the characteristic equation  $\det(\lambda \mathbf{I}_n - \mathbf{A}) = 0$ .
- If  $\lambda$  is an eigenvalue of  $\mathbf{A}$ , then there exist non-zero  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{Ax} = \lambda \mathbf{x}$  and  $\mathbf{y}^T \mathbf{A} = \lambda \mathbf{y}^T$ .  $\mathbf{x}$  and  $\mathbf{y}$  are called the (column) *eigenvector* and *row eigenvector* of  $\mathbf{A}$  associated with the eigenvalue  $\lambda$ .
- $\mathbf{A}$  is singular if and only if it has at least one 0 eigenvalue.
- Eigenvectors associated with distinct eigenvalues are linearly independent.
- Eigenvalues of an upper or lower triangular matrix are its diagonal entries:  $\lambda_i = a_{ii}$ .
- Eigenvalues of an idempotent matrix are either 0 or 1.
- Eigenvalues of an orthogonal matrix have complex modulus 1.
- In most statistical applications, we deal with eigenvalues/eigenvectors of symmetric matrices.  
The eigenvalues and eigenvectors of a real *symmetric* matrix are real.
- Eigenvectors associated with distinct eigenvalues of a symmetry matrix are orthogonal.
- Eigen-decomposition of a symmetric matrix:  $\mathbf{A} = \mathbf{U} \Lambda \mathbf{U}^T$ , where
  - $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$
  - columns of  $\mathbf{U}$  are the eigenvectors which are (or can be chosen to be) mutually orthonormal

- A real symmetric matrix is positive semidefinite (positive definite) if and only if all eigenvalues are nonnegative (positive).
- Spectral radius  $\rho(\mathbf{A}) = \max_i |\lambda_i|$ .
- $\mathbf{A} \in \mathbb{R}^{n \times n}$  a square matrix (not required to be symmetric), then  $\text{tr}(\mathbf{A}) = \sum_i \lambda_i$  and  $|\mathbf{A}| = \prod_i \lambda_i$ .

## Singular value decomposition

KL chapter 9. Assume  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $p = \min\{m, n\}$ .

- Singular value decomposition (SVD):  $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ , where
  - $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_m) \in \mathbb{R}^{m \times m}$  is orthogonal
  - $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathbb{R}^{n \times n}$  is orthogonal
  - $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_p) \in \mathbb{R}^{m \times n}$ ,  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0$ .
- $\sigma_i$  are called the *singular values*,  $\mathbf{u}_i$  are the *left singular vectors*, and  $\mathbf{v}_i$  are the *right singular vectors*.
- $\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i$  and  $\mathbf{A}^T \mathbf{u}_i = \sigma_i \mathbf{v}_i$  for  $i = 1, \dots, p$ .
- Thin SVD. Assume  $m \geq n$ .  $\mathbf{A}$  can be factored as  $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ , where
  - $\mathbf{U} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{U}^\top \mathbf{U} = \mathbf{I}_n$
  - $\mathbf{V} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{V}^\top \mathbf{V} = \mathbf{I}_n$
  - $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$
- Relation to eigen-decomposition. Using thin SVD,

$$\begin{aligned}\mathbf{A}^\top \mathbf{A} &= \mathbf{V} \Sigma \mathbf{U}^\top \mathbf{U} \Sigma \mathbf{V}^\top = \mathbf{V} \Sigma^2 \mathbf{V}^\top \\ \mathbf{A} \mathbf{A}^\top &= \mathbf{U} \Sigma \mathbf{V}^\top \mathbf{V} \Sigma \mathbf{U}^\top = \mathbf{U} \Sigma^2 \mathbf{U}^\top.\end{aligned}$$

- Another relation to eigen-decomposition. Using thin SVD,

$$\begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{A}^\top \\ \mathbf{A} & \mathbf{0}_{m \times m} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{V} & \mathbf{V} \\ \mathbf{U} & -\mathbf{U} \end{pmatrix} \begin{pmatrix} \Sigma & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & -\Sigma \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{V}^\top & \mathbf{U}^\top \\ \mathbf{V}^\top & -\mathbf{U}^\top \end{pmatrix}.$$

Hence any symmetric eigen-solver can produce the SVD of a matrix  $\mathbf{A}$  without forming  $\mathbf{A}\mathbf{A}^\top$  or  $\mathbf{A}^\top \mathbf{A}$ .

- Yet another relation to eigen-decomposition: If the eigendecomposition of a real symmetric matrix is  $\mathbf{A} = \mathbf{W}\Lambda\mathbf{W}^T = \mathbf{W}\text{diag}(\lambda_1, \dots, \lambda_n)\mathbf{W}^T$ , then

$$\mathbf{A} = \mathbf{W}\Lambda\mathbf{W}^T = \mathbf{W} \begin{pmatrix} |\lambda_1| & & \\ & \ddots & \\ & & |\lambda_n| \end{pmatrix} \begin{pmatrix} \text{sgn}(\lambda_1) & & \\ & \ddots & \\ & & \text{sgn}(\lambda_n) \end{pmatrix} \mathbf{W}^T$$

is the SVD of  $\mathbf{A}$ .

- Relation to the Moore-Penrose (MP) inverse: Using thin SVD,

$$\mathbf{A}^+ = \mathbf{V}\Sigma^+\mathbf{U}^\top,$$

where  $\Sigma^+ = \text{diag}(\sigma_1^{-1}, \dots, \sigma_r^{-1}, 0, \dots, 0)$ ,  $r = \text{rank}(\mathbf{A})$ .

- Denote  $\sigma(\mathbf{A}) = (\sigma_1, \dots, \sigma_p)$ . Then

- $\text{rank}(\mathbf{A}) = \# \text{ nonzero singular values} = \|\sigma(\mathbf{A})\|_0$
- $\mathbf{A} = \mathbf{U}_r\Sigma_r\mathbf{V}_r^T = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T$
- $\|\mathbf{A}\|_\text{F} = (\sum_{i=1}^p \sigma_i^2)^{1/2} = \|\sigma(\mathbf{A})\|_2$
- $\|\mathbf{A}\|_2 = \sigma_1 = \|\sigma(\mathbf{A})\|_\infty$

- Assume  $\text{rank}(\mathbf{A}) = r$  and partition  $\mathbf{U} = (\mathbf{U}_r, \tilde{\mathbf{U}}_r) \in \mathbb{R}^{m \times m}$  and  $\mathbf{V} = (\mathbf{V}_r, \tilde{\mathbf{V}}_r) \in \mathbb{R}^{n \times n}$ , then

- $\mathcal{C}(\mathbf{A}) = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ ,  $\mathcal{N}(\mathbf{A}^T) = \text{span}\{\mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$
- $\mathcal{N}(\mathbf{A}) = \text{span}\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ ,  $\mathcal{C}(\mathbf{A}^T) = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$
- $\mathbf{U}_r\mathbf{U}_r^T$  is the orthogonal projection onto  $\mathcal{C}(\mathbf{A})$
- $\tilde{\mathbf{U}}_r\tilde{\mathbf{U}}_r^T$  is the orthogonal projection onto  $\mathcal{N}(\mathbf{A}^T)$
- $\mathbf{V}_r\mathbf{V}_r^T$  is the orthogonal projection onto  $\mathcal{C}(\mathbf{A}^T)$
- $\tilde{\mathbf{V}}_r\tilde{\mathbf{V}}_r^T$  is the orthogonal projection onto  $\mathcal{N}(\mathbf{A})$

## Preliminaries of numerical linear algebra

Numerical linear algebra concerns how matrix/vector computations are done in computer. We first look at some basic linear algebra operations.

## Flop counts of some basic linear algebra subroutines (BLAS)

See <http://www.netlib.org/blas/> for a complete listing of BLAS functions.

Level	Example Operation	Name	Dimension	Flops
1	$\alpha \leftarrow \mathbf{x}^T \mathbf{y}$	dot product	$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$	$2n$
	$\mathbf{y} \leftarrow \mathbf{y} + a\mathbf{x}$	saxpy	$a \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$	$2n$
2	$\mathbf{y} \leftarrow \mathbf{y} + \mathbf{A}\mathbf{x}$	gaxpy	$\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$	$2mn$
	$\mathbf{A} \leftarrow \mathbf{A} + \mathbf{y}\mathbf{x}^T$	rank one update	$\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$	$2mn$
3	$\mathbf{C} \leftarrow \mathbf{C} + \mathbf{A}\mathbf{B}$	matrix multiplication	$\mathbf{A} \in \mathbb{R}^{m \times p}, \mathbf{B} \in \mathbb{R}^{p \times n}, \mathbf{C} \in \mathbb{R}^{m \times n}$	$2mnp$
	$\mathbf{A} \leftarrow \mathbf{A}\mathbf{D}$	column scaling	$\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{D} = \text{diag}(d_1, \dots, d_n)$	$mn$
	$\mathbf{A} \leftarrow \mathbf{D}\mathbf{A}$	row scaling	$\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{D} = \text{diag}(d_1, \dots, d_m)$	$mn$

- Go over the “mxmult” session in R.

<http://hua-zhou.github.io/teaching/biostatm280-2016winter/matmult.html>

Different ways to compute  $\mathbf{X}^T \mathbf{W}^{-1} \mathbf{X}$ , such as in the weighted least squares.

$\mathbf{X} \in \mathbb{R}^{n \times p}$ ,  $\mathbf{W} = \text{diag}(w_1, \dots, w_n) \in \mathbb{R}^{n \times n}$ .

1. `t(X) %*% solve(W) %*% X`:  $O(n^3 + n^2p + np^2)$  flops, why need to do the expensive matrix inversion?
2. `t(X) %*% diag(1 / w) %*% X`:  $O(n^2p + np^2)$  flops, why need to save a diagonal matrix and do an extra matrix multiplication?
3. `(t(X) / w) %*% X`: wrong!  $w$  recycled incorrectly
4. `t(X) %*% (X / w)`:  $O(np^2 + np) = O(np^2)$  flops
5. `crossprod(X, X / w)`: same as 4, skip the transpose operation

- Another example: Fisher information matrix of a generalized linear model (GLM):  $\mathbf{X}^T \mathbf{W} \mathbf{X}$ , where  $\mathbf{X} \in \mathbb{R}^{n \times p}$  and  $\mathbf{W} = \text{diag}(w_1, \dots, w_n)$  are the observation weights.
- Bottom line: Always be flop-aware when writing code.

The form of a mathematical expression and the way the expression should be evaluated in actual practice may be quite different.

- But, for high-performance matrix computations, it is *not* enough to minimize flops. Pipelining, effective use of memory hierarchy, data layout in memory, ... play important role too.

## Vector computer

- Most modern computers are vector machines, which perform vector calculations (saxpy, inner product) fast.
- Vector processing by *pipelining*. E.g., vector addition  $z = x + y$

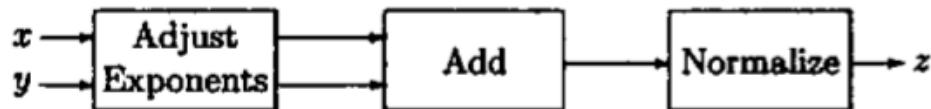


FIG. 1.4.1 A 3-Cycle Adder

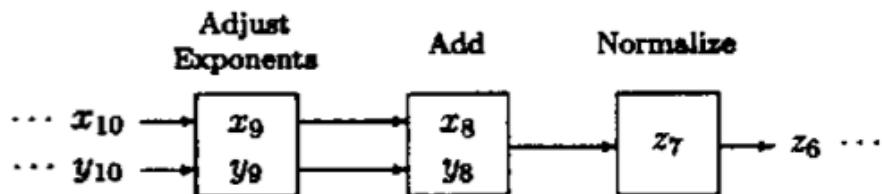


FIG. 1.4.2 Pipelined Addition

- For example, for Intel Sandy Bridge/Ivy Bridge:
  - 8 DP FLOPs/cycle: 4-wide AVX addition + 4-wide AVX multiplication
  - 16 SP FLOPs/cycle: 8-wide AVX addition + 8-wide AVX multiplication
- and for Intel Haswell/Broadwell/Skylake:
  - 16 DP FLOPs/cycle: two 4-wide FMA (fused multiply-add) instructions
  - 32 SP FLOPs/cycle: two 8-wide FMA (fused multiply-add) instructions

- One implication of the pipelining technology is that we need to ship vectors to the pipeline fast enough to keep the arithmetic units (ALU) busy and maintain high throughput.

# 6 Lecture 6, Jan 21

## Announcements

- HW1 due today @ 11:59PM. Tag and push to the `master` branch.
- HW2 posted. Due Tue Feb 2 @ 11:59PM.
- Quiz 1 at end of today's class (2:35PM–2:50PM).

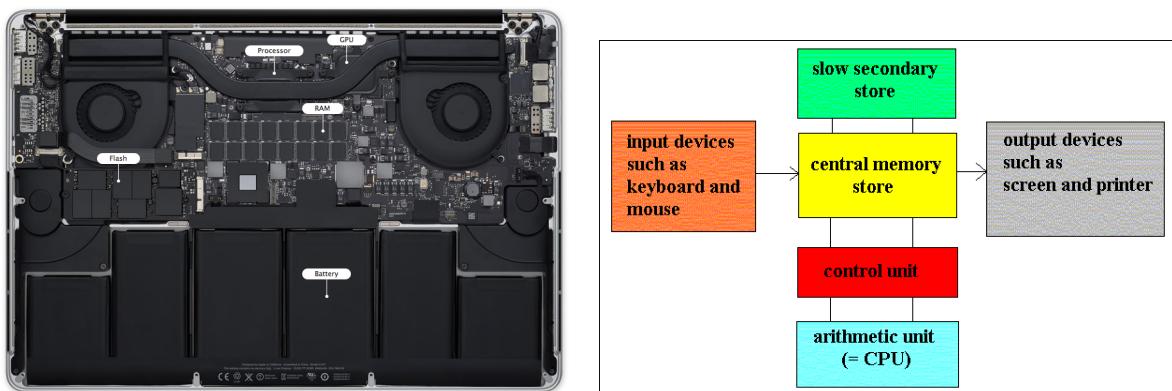
## Last time

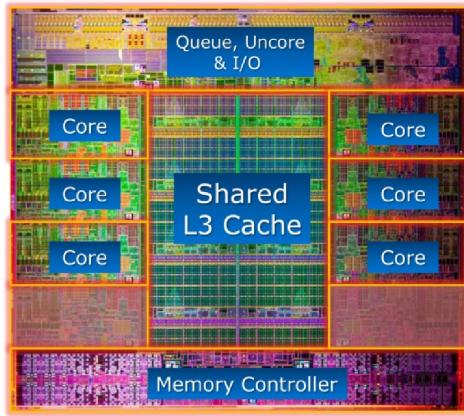
- Version control with Git.
- Algorithm, flop, computational complexity.
- Numerical linear algebra: introduction, BLAS.

## Today

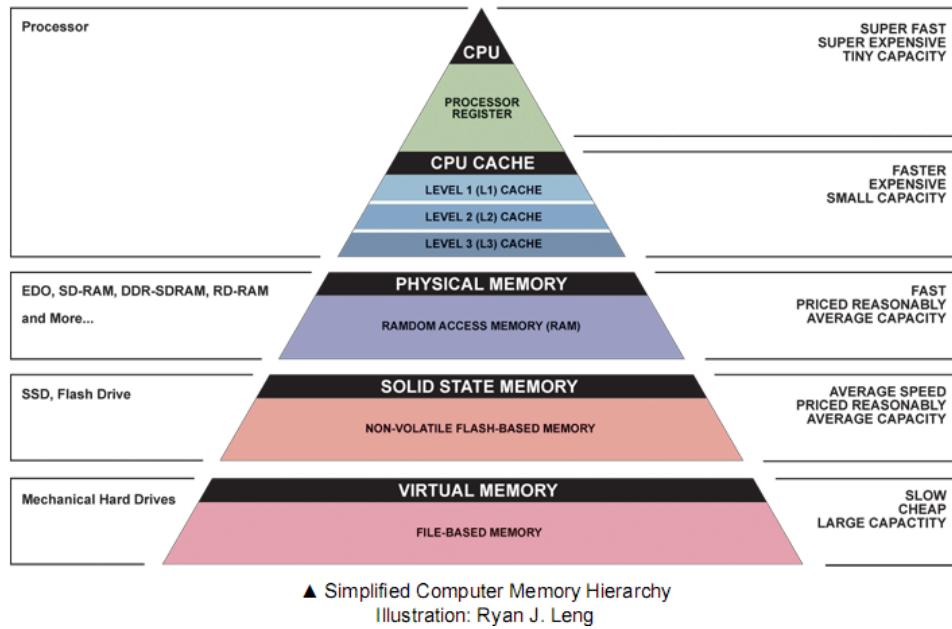
- BLAS (con'td).
- Triangular systems.
- Gaussian elimination and LU decomposition.

## Computer architecture and high-level BLAS





- Memory hierarchy:



Upper the hierarchy, faster the memory accessing speed, and more expensive the memory units.

Key to high performance is effective use of memory hierarchy. True on all architectures.

- Can we keep the super fast arithmetic units busy with enough deliveries of matrix data and ship the results to memory fast enough to avoid backlog?  
Answer: use high-level BLAS as much as possible.

- Why high-level BLAS?

BLAS	Dimension	Mem Refs	Flops	Ratio
Level 1: $\mathbf{y} \leftarrow \mathbf{y} + a\mathbf{x}$	$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$	$3n$	$2n$	3:2
Level 2: $\mathbf{y} \leftarrow \mathbf{y} + \mathbf{A}\mathbf{x}$	$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{n \times n}$	$n^2$	$2n^2$	1:2
Level 3: $\mathbf{C} \leftarrow \mathbf{C} + \mathbf{A}\mathbf{B}$	$\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times n}$	$4n^2$	$2n^3$	2:n

- BLAS 1 tend to be *memory bandwidth-limited*. E.g., Xeon X5650 CPU has a theoretical throughput of 128 DP GFLOPS but a max memory bandwidth of 32GB/s.
- Higher level BLAS (3 or 2) make more effective use of arithmetic logic units (ALU) by keeping them busy.
- Message: Although we state many algorithms (solving linear equations, least squares, eigen-decomposition, SVD, ...) in terms of inner product and saxpy, the actual implementation in BLAS and LAPACK may be quite different.
- A distinction between LAPACK and LINPACK (older version of R uses LINPACK) is that LAPACK makes use of higher level BLAS as much as possible (usually by smart partitioning) to increase the so-called *level-3 fraction*.

## Effect of data layout

- Data layout in memory effects execution speed too. It is much faster to move chunks of data in memory than retrieving/writing scattered data.
- Storage mode: column-major (FORTRAN, MATLAB, R, Julia) vs row-major (C/C++).

When you call BLAS from C/C++, use the CBLAS library instead of the traditional BLAS library implemented in Fortran.

- Take matrix multiplication as an example. Assume the storage is column-major, such as in FORTRAN.  $\mathbf{C} \leftarrow \mathbf{C} + \mathbf{A}\mathbf{B}$ , where  $\mathbf{A} \in \mathbb{R}^{m \times p}$ ,  $\mathbf{B} \in \mathbb{R}^{p \times n}$ ,  $\mathbf{C} \in \mathbb{R}^{m \times n}$ . There are 6 variants of the algorithms according to the order in the triple loops. We pay attention to the innermost loop, where the vector calculation occurs,

<i>JKI</i> or <i>KJI</i> :	<b>for</b> $i = 1:m$ $C(i,j) = C(i,j) + A(i,k)B(k,j)$ <b>end</b>
<i>IKJ</i> or <i>KIJ</i> :	<b>for</b> $j = 1:n$ $C(i,j) = C(i,j) + A(i,k)B(k,j)$ <b>end</b>
<i>ijk</i> or <i>jik</i> :	<b>for</b> $k = 1:p$ $C(i,j) = C(i,j) + A(i,k)B(k,j)$ <b>end</b>

and the associated *stride* when accessing the three matrices in memory (assuming column-major storage)

Variant	A Stride	B Stride	C Stride
<i>JKI</i> or <i>KJI</i>	Unit	0	Unit
<i>IKJ</i> or <i>KIJ</i>	0	Non-Unit	Non-Unit
<i>ijk</i> or <i>jik</i>	Non-Unit	Unit	0

Apparently the variants *JKI* or *KJI* are preferred.

- Message: data storage mode effects algorithm implementation too.
- A numerical experiment in JULIA: [http://hua-zhou.github.io/teaching/biostatm280-2016winter/matmul\\_loop.html](http://hua-zhou.github.io/teaching/biostatm280-2016winter/matmul_loop.html).

## Solving linear equations

We consider algorithms for solving linear equations  $\mathbf{Ax} = \mathbf{b}$ , a ubiquitous task in statistics. Idea: turning original problem into an “easy” one, e.g., triangular system.

## Triangular system

- *Forward substitution* to solve  $\mathbf{Lx} = \mathbf{b}$ , where  $\mathbf{L} \in \mathbb{R}^{n \times n}$  is lower triangular

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 = b_1/a_{11}$$

$$x_2 = (b_2 - a_{21}x_1)/a_{22}$$

$$x_3 = (b_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$$

$$\vdots$$

$$x_m = b_m - a_{m1}x_1 - a_{m2}x_2 - \dots - a_{m,m-1}x_{m-1})/a_{mm}$$

$n^2$  flops ( $n^2/2$  multiplications/divisions and  $n^2/2$  additions/subtractions) and  $\mathbf{L}$  is accessed by row.

- Back substitution to solve  $\mathbf{U}\mathbf{x} = \mathbf{b}$  where  $\mathbf{U} \in \mathbb{R}^{n \times n}$  is upper triangular

$$\begin{bmatrix} a_{11} & \dots & a_{1,m-1} & a_{1m} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & a_{m-1,m-1} & a_{m-1,m} \\ 0 & \dots & 0 & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_m = b_m/a_{mm}$$

$$x_{m-1} = (b_{m-1} - a_{m-1,m}x_m)/a_{m-1,m-1}$$

$$x_{m-2} = (b_{m-2} - a_{m-2,m-1}x_{m-1} - a_{m-2,m}x_m)/a_{m-2,m-2}$$

$$\vdots$$

$$x_1 = b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1m}x_m)/a_{11}$$

$n^2$  flops ( $n^2/2$  multiplications/divisions and  $n^2/2$  additions/subtractions) and  $\mathbf{U}$  is accessed by row.

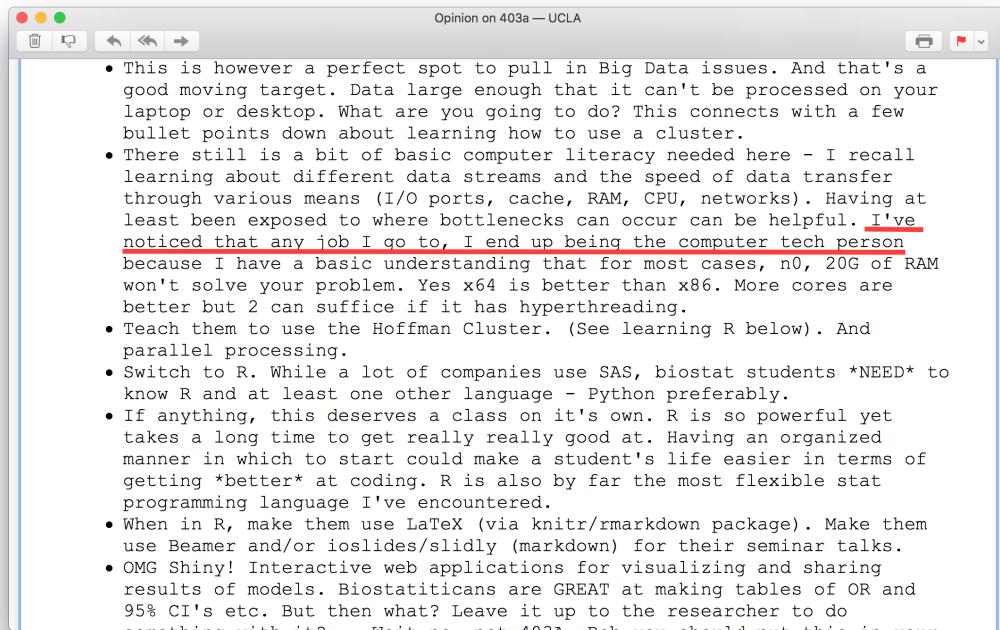
- Column version: reverse the order of looping.
- BLAS level 2 function: `?trsv` (triangular solve with one right hand side)
- BLAS level 3 function: `?trsm` (matrix triangular solve, i.e., multiple right hand sides)

- In R, `forwardsolve()` and `backsolve()` (wrappers of `dtrsm`).
- In JULIA,  $\mathbf{A} \setminus \mathbf{b}$  uses forward or backward substitution when  $\mathbf{A}$  is a triangular matrix. Or we can simply call BLAS functions directly.
- Eigenvalues of  $\mathbf{L}$  are diagonal entries  $\lambda_i = \ell_{ii}$ .  $\det(\mathbf{L}) = \prod_i \ell_{ii}$ .
- A *unit triangular matrix* is a triangular matrix with all diagonal entries being 1.
- The algebra of triangular matrices (HW2)
  - The product of two upper (lower) triangular matrices is upper (lower) triangular.
  - The inverse of an upper (lower) triangular matrix is upper (lower) triangular.
  - The product of two unit upper (lower) triangular matrices is unit upper (lower) triangular.
  - The inverse of a unit upper (lower) triangular matrix is unit upper (lower) triangular.

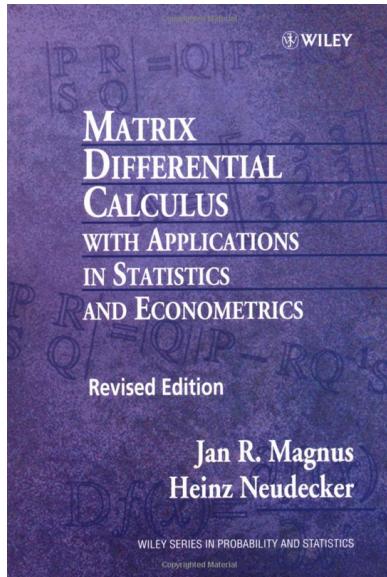
## 7 Lecture 7, Jan 26

### Announcements

- Quiz 1 returned:  $7.73 \pm 1.39$ . Statistical culture, future of statistics, computer literacy, ... An email from Jeffrey Robbins (Biostatistics alumna)



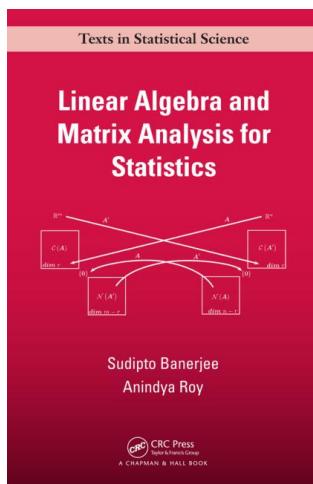
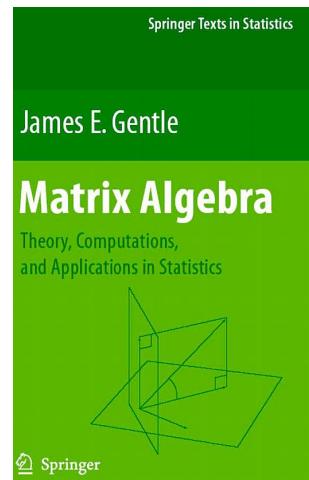
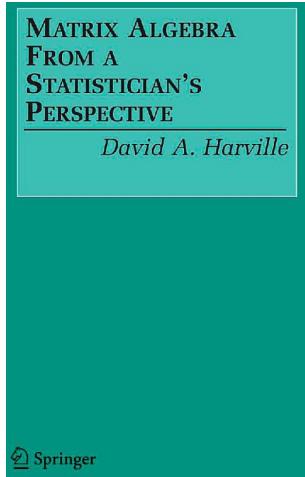
- HW2 due next Tuesday. FAQs at <http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostatm280winter2016/2016/01/26/hw2-hints.html>
- Any good reference books for linear algebra? Magnus and Neudecker (1999), Horn and Johnson (1985), Harville (1997), Gentle (2007), Banerjee and Roy (2014).



# MATRIX ANALYSIS

ROGER A. HORN  
AND  
CHARLES R. JOHNSON

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## Last time

- Memory hierarchy, high-level BLAS, effect of data layout. A few messages from the matrix multiplication example (in Julia): [http://hua-zhou.github.io/teaching/biostatm280-2016winter/matmul\\_loop.html](http://hua-zhou.github.io/teaching/biostatm280-2016winter/matmul_loop.html)
  1. Order of looping matters. Access a contiguous chunk of data in memory is much more efficient than accessing scattered data.
  2. Don't re-invent wheels and use BLAS/LAPACK whenever possible.
  3. Be alert to unnecessary allocation of intermediate variables. Memory transactions and garbage collection are *expensive*.
- Solving triangular systems.  $n^2$  flops for the backward or forward substitution algorithms.

## Today

- Gaussian elimination and LU decomposition.
- Cholesky decomposition.
- Reading: *The decompositional approach to matrix computation*, by G. W. Stewart.  
<http://hua-zhou.github.io/teaching/biostatm280-2016winter/readings/decomp.pdf>

## Gaussian elimination and LU decomposition

Given a system of linear algebraic equations

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Step 1: Each row times  $a_{11}/a_{k1}$ ,

then use row one to subtract other rows.

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{a}_{n2} & \dots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$$

Step 2: The second row and down multiply by  $\tilde{a}_{22}/\tilde{a}_{k2}$ ,

then use row two to subtract every row below.

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & \dots & \tilde{a}_{2n} \\ 0 & 0 & \tilde{a}_{33} & \dots & \tilde{a}_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \tilde{a}_{n3} & \dots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$$

Step 3: Similar to the previous two steps, repeat until all elements in the lower triangle of the matrix  $A$  become zeros.

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{a}_{n2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$$

- History: It is one by-product of Gauss's efforts to re-discover the dwarf planet Ceres using method of least squares. No linear algebra in 1801!
- Solve  $\mathbf{Ax} = \mathbf{b}$  for a general matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ .
- A toy example: [https://en.wikipedia.org/wiki/Gaussian\\_elimination#Example\\_of\\_the\\_algorithm](https://en.wikipedia.org/wiki/Gaussian_elimination#Example_of_the_algorithm).

- Idea: a series of *elementary operations* that turn  $\mathbf{A}$  into a triangular system.
- It turns out Gaussian elimination not only transforms  $\mathbf{A}$  into a triangular system,
- We consider the square  $\mathbf{A}$  case first.
- *Elementary operator matrix*  $\mathbf{E}_{jk}(c)$  is the identity with the 0 in position  $(j, k)$  replaced by  $c$ . For any vector  $\mathbf{x}$ ,

$$\mathbf{E}_{jk}(c)\mathbf{x} = (x_1, \dots, x_{j-1}, x_j + cx_k, x_{j+1}, \dots, x_n)^\top.$$

Applying  $\mathbf{E}_{jk}(c)$  to both sides of the system  $\mathbf{Ax} = \mathbf{b}$  replaces the  $j$ -th equation  $\mathbf{a}_{j*}^\top \mathbf{x} = b_j$  by  $\mathbf{a}_{j*}^\top \mathbf{x} + c\mathbf{a}_{k*}^\top \mathbf{x} = b_j + cb_k$ . For  $j > k$ ,  $\mathbf{E}_{jk}(c) = \mathbf{I} + ce_j e_k^\top$  is unit lower triangular and full rank.  $\mathbf{E}_{jk}^{-1}(c) = \mathbf{E}_{jk}(-c)$ .

- Zeroing the first column

$$\begin{aligned}\mathbf{E}_{21}(c_2^{(1)})\mathbf{Ax} &= \mathbf{E}_{21}(c_2^{(1)})\mathbf{b} \\ \mathbf{E}_{31}(c_3^{(1)})\mathbf{E}_{21}(c_2^{(1)})\mathbf{Ax} &= \mathbf{E}_{31}(c_3^{(1)})\mathbf{E}_{21}(c_2^{(1)})\mathbf{b} \\ &\vdots \\ \mathbf{E}_{n1}(c_n^{(1)}) \cdots \mathbf{E}_{31}(c_3^{(1)})\mathbf{E}_{21}(c_2^{(1)})\mathbf{Ax} &= \mathbf{E}_{n1}(c_n^{(1)}) \cdots \mathbf{E}_{31}(c_3^{(1)})\mathbf{E}_{21}(c_2^{(1)})\mathbf{b}\end{aligned}$$

where  $c_i^{(1)} = -a_{i1}/a_{11}$ . Denote  $\mathbf{M}_1 = \mathbf{E}_{n1}(c_n^{(1)}) \cdots \mathbf{E}_{31}(c_3^{(1)})\mathbf{E}_{21}(c_2^{(1)})$ .

- Then zero the  $k$ -th column for  $k = 2, \dots, n-1$  sequentially. This results in a transformed linear system  $\mathbf{Ux} = \tilde{\mathbf{b}}$ , where  $\mathbf{U} = \mathbf{M}_{n-1} \cdots \mathbf{M}_1 \mathbf{A}$  is upper triangular and  $\tilde{\mathbf{b}} = \mathbf{M}_{n-1} \cdots \mathbf{M}_1 \mathbf{b}$ .  $\mathbf{M}_k$  has the shape

$$\mathbf{M}_k = \mathbf{E}_{n,k}^{(k)} \cdots \mathbf{E}_{k+1,k}^{(k)} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & c_{k+1}^{(k)} & 1 & \\ & & \vdots & & \ddots \\ & & c_n^{(k)} & & 1 \end{pmatrix},$$

where  $c_i^{(k)} = -\tilde{a}_{ik}^{(k-1)}/\tilde{a}_{kk}^{(k-1)}$ .  $\mathbf{M}_k$  is unit lower triangular and full rank.  $\mathbf{M}_k$  are called the *Gauss transformations*.

- Let  $\mathbf{L} = \mathbf{M}_1^{-1} \cdots \mathbf{M}_{n-1}^{-1}$ . We have the decomposition

$$\mathbf{A} = \mathbf{L}\mathbf{U}.$$

$\mathbf{M}_k$  is unit upper triangular, so  $\mathbf{M}_k^{-1}$  and thus  $\mathbf{L}$  is unit lower triangular.

- Where is  $\mathbf{L}$ ? Note  $\mathbf{M}_k = \mathbf{I} + (0, \dots, 0, c_{k+1}^{(k)}, \dots, c_n^{(k)})^\top \mathbf{e}_k^\top$ . By Sherman-Morrison,  $\mathbf{M}_k^{-1} = \mathbf{I} - (0, \dots, 0, c_{k+1}^{(k)}, \dots, c_n^{(k)})^\top \mathbf{e}_k^\top$ . So the entries of  $\mathbf{L}$  are simply  $\ell_{ik} = -c_i^{(k)}$ ,  $i > k$ , the negative of multipliers in GE.
- The whole LU procedure is done in place, i.e.,  $\mathbf{A}$  is overwritten by  $\mathbf{L}$  and  $\mathbf{U}$ .
- Implementation: outer product LU ( $kij$  loop), block outer product LU (higher level-3 fraction), Crout's algorithm ( $jki$  loop), ...

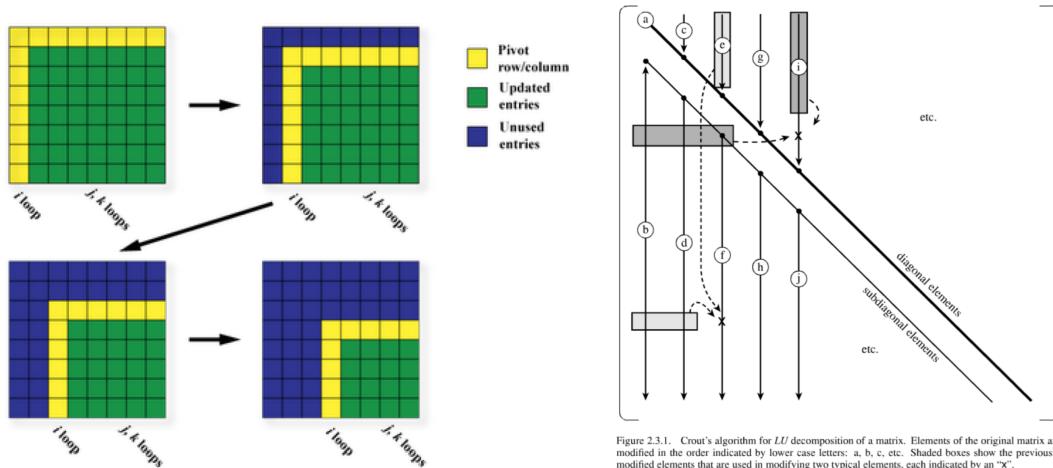


Figure 2.3.1. Crout's algorithm for LU decomposition of a matrix. Elements of the original matrix are modified in the order indicated by lower case letters: a, b, c, etc. Shaded boxes show the previously modified elements that are used in modifying two typical elements, each indicated by an "X".

- Exercise: work out the LU decomposition of a 3-by-3 example on paper. [https://en.wikipedia.org/wiki/Gaussian\\_elimination#Example\\_of\\_the\\_algorithm](https://en.wikipedia.org/wiki/Gaussian_elimination#Example_of_the_algorithm)
- LU decomposition exists if the principal sub-matrix  $A(1 : k, 1 : k)$  is non-singular for  $k = 1, \dots, n-1$ . If the LU decomposition exists and  $\mathbf{A}$  is non-singular, then the LU decomposition is unique and  $\det(\mathbf{A}) = \prod_{i=1}^n u_{ii}$ .
- This LU decomposition costs  $2(n-1)^2 + 2(n-2)^2 + \dots + 2 \cdot 1^2 \approx \frac{2}{3}n^3$  flops ( $n^3/3$  multiplications and  $n^3/3$  additions).

- Given LU, one right hand side costs  $2n^2$  flops: one forward substitution to solve  $\mathbf{Ly} = \mathbf{b}$  and then one backward substitution to solve  $\mathbf{Ux} = \mathbf{y}$ .
- For *matrix inversion*, there are  $n$  right hand sides  $\mathbf{e}_i$ . However, taking advantage of zeros reduces  $2n^3$  flops to  $\frac{4}{3}n^3$ . So matrix inversion costs  $\frac{2}{3}n^3 + \frac{4}{3}n^3 = 2n^3$  flops in total.
- No inversion mentality*: Whenever we see matrix inverse, we should think in terms of solving linear equations. We do *not* compute matrix inverse unless (i) it is necessary to compute standard errors, (2) number of right hand sides is much larger than  $n$ , (3)  $n$  is small.

## Pivoting for LU

- What if we encounter a *pivot*  $\tilde{a}_{kk}^{(k-1)}$  being 0 or close to 0 due to underflow?
- Think about  $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Does it have a solution for arbitrary  $\mathbf{b}$ ? Does GE work?
- Work on the example

$$\begin{aligned} 0.0001x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2, \end{aligned}$$

which has solution  $x_1 = 1.0001$  and  $x_2 = 0.9999$ . Suppose we have 3 digits of precision. After first step of elimination, we have (catastrophic cancellation happens)

$$\begin{aligned} 0.0001x_1 + x_2 &= 1 \\ -10,000x_2 &= -10,000 \end{aligned}$$

and the solution by back substitution is  $x_2 = 1.000$  and  $x_1 = 0.000$ .

- Message: zero or very small pivots cause trouble.  
Solution: *pivoting*.
- Partial pivoting*: at the  $k$ -th stage the equation with  $\max_{i=k}^n |\tilde{a}_{ik}^{(k-1)}|$  is moved into the  $k$ -th row. Thus we have  $\mathbf{M}_{n-1}\mathbf{P}_{n-1} \cdots \mathbf{M}_1\mathbf{P}_1\mathbf{A} = \mathbf{U}$ .

- With partial pivoting, it can be shown that

$$\mathbf{P}\mathbf{A} = \mathbf{L}\mathbf{U},$$

where  $\mathbf{P} = \mathbf{P}_{n-1} \cdots \mathbf{P}_1$ ,  $\mathbf{L}$  is unit lower triangular with  $|\ell_{ij}| \leq 1$ , and  $\mathbf{U}$  is upper triangular.

- $\det(\mathbf{P}) \det(\mathbf{A}) = \det(\mathbf{U}) = \prod_{i=1}^n u_{ii}$ .
- To solve  $\mathbf{Ax} = \mathbf{b}$ , we solve two triangular systems

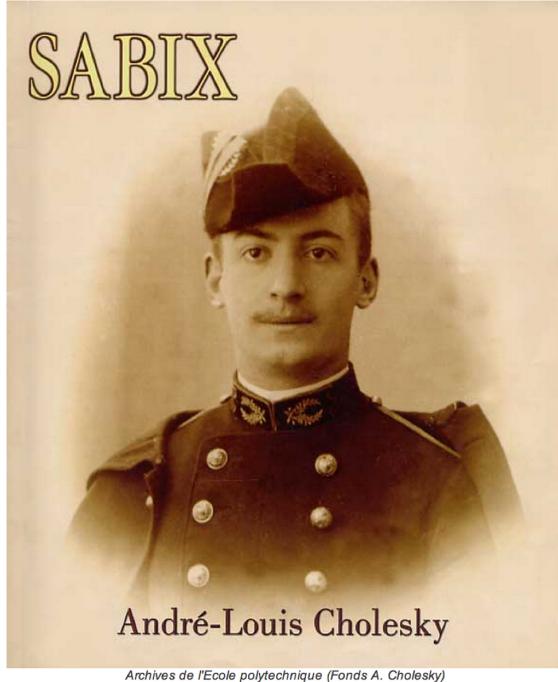
$$\mathbf{Ly} = \mathbf{Pb} \quad \text{and} \quad \mathbf{Ux} = \mathbf{y},$$

costing  $2n^2$  flops.

- Complete pivoting:* Do both row and column interchanges so that the largest entry in the sub matrix  $\mathbf{A}(k : n, k : n)$  is permuted to the  $(k, k)$ -th entry. This yields the decomposition  $\mathbf{PAQ} = \mathbf{LU}$ , where  $|\ell_{ij}| \leq 1$ .
- Warning: In the actual LAPACK implementation, we do not really need to interchange rows/columns for pivoting. Just keep track of the indices we have interchanged.
- Gaussian elimination with partial pivoting is one of the most commonly used methods for solving *general* linear systems. Complete pivoting is stable but costs more computation. Partial pivoting is stable most of times.
- LAPACK: ?GETRF does  $\mathbf{PA} = \mathbf{LU}$  (LU decomposition with partial pivoting) in place.
- In R, `solve()` implicitly performs LU decomposition (wrapper of LAPACK routine DGESV). `solve()` allows specifying a single or multiple right hand sides. If none, it computes the matrix inverse. The `matrix` package contains `lu()` function that outputs L, U, and pivot.
- In JULIA, `lu()` and `lufact()` perform LU decomposition with partial pivoting, or call LAPACK function directly using `getrf!()`.

## Cholesky decomposition (symmetric LU)

Reference: KL 7.7.



- A basic tenet in numerical analysis:

The structure should be exploited whenever solving a problem.

Common structures include: symmetry, definiteness, sparsity, Kronecker product, low rank, ...

- LU decomposition (GE) is *not* used in statistics so often because most of time statisticians deal with positive (semi)definite matrix.
- Consider solving the normal equation  $\mathbf{X}^T \mathbf{X} \boldsymbol{\beta} = \mathbf{X}^T \mathbf{y}$  for linear regression. The coefficient matrix  $\mathbf{X}^T \mathbf{X}$  is symmetric and positive semidefinite. How to exploit this structure?
- Theorem: Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be symmetric and positive definite. Then  $\mathbf{A} = \mathbf{L} \mathbf{L}^\top$ , where  $\mathbf{L}$  is lower triangular with positive diagonal entries and is unique.

*Proof by induction.* If  $n = 1$ , then  $\ell = \sqrt{a}$ . For  $n > 1$ , the block equation

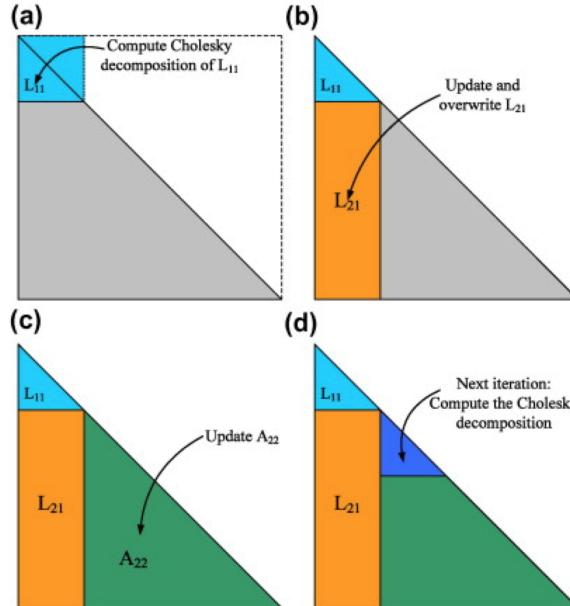
$$\begin{pmatrix} a_{11} & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{A}_{22} \end{pmatrix} = \begin{pmatrix} \ell_{11} & \mathbf{0}_{n-1}^\top \\ \mathbf{l} & \mathbf{L}_{22} \end{pmatrix} \begin{pmatrix} \ell_{11} & \mathbf{l}^\top \\ \mathbf{0}_{n-1} & \mathbf{L}_{22}^\top \end{pmatrix}.$$

has solution

$$\begin{aligned} \ell_{11} &= \sqrt{a_{11}} \\ \mathbf{l} &= \ell_{11}^{-1} \mathbf{a} \\ \mathbf{L}_{22} \mathbf{L}_{22}^\top &= \mathbf{A}_{22} - \mathbf{l} \mathbf{l}^\top = \mathbf{A}_{22} - a_{11}^{-1} \mathbf{a} \mathbf{a}^\top. \end{aligned}$$

Now  $a_{11} > 0$  (why?), so  $\ell_{11}$  and  $\mathbf{l}$  are uniquely determined.  $\mathbf{A}_{22} - a_{11}^{-1} \mathbf{a} \mathbf{a}^\top$  is positive definite because  $\mathbf{A}$  is positive definite (why?). By induction hypothesis,  $\mathbf{L}_{22}$  exists and is unique.  $\square$

- The constructive proof completely specifies the algorithm.



- Exercise: work out the Cholesky decomposition of a 3-by-3 example on paper.  
[https://en.wikipedia.org/wiki/Cholesky\\_decomposition#Example](https://en.wikipedia.org/wiki/Cholesky_decomposition#Example)
- Computational cost:  $\frac{1}{2}[2(n-1)^2 + 2(n-2)^2 + \dots + 2 \cdot 1^2] \approx \frac{1}{3}n^3$  (half the cost of LU decomposition due to symmetry) plus  $n$  square roots.

- Avoid square roots:  $\mathbf{LDL}^\top$  decomposition.
- In general Cholesky decomposition is very stable. Failure of the decomposition simply means  $\mathbf{A}$  is not positive definite. It is an efficient way to test positive definiteness.

If zero pivots  $\tilde{a}_{ii} = 0$  are encountered, we can still continue the algorithm by setting  $\ell_{ii} = 0$  and  $\mathbf{l} = \mathbf{0}$ . A better alternative is to use Cholesky decomposition with symmetric pivoting.

## Cholesky with symmetric pivoting

Assume  $\mathbf{A} \succeq \mathbf{0}_{n \times n}$  (p.s.d.)

- When  $\mathbf{A}$  does not have full rank, e.g.,  $\mathbf{X}^T \mathbf{X}$  with a non-full column rank  $\mathbf{X}$ , we encounter  $\tilde{a}_{kk} = 0$  during the procedure.
- *Symmetric pivoting.* At each stage  $k$ , we permute both row and column such that  $\max_{i=k}^n \tilde{a}_{kk}$  becomes the pivot. If we encounter  $\max_{i=k}^n \tilde{a}_{kk} = 0$ , then  $A(k : n, k : n) = \mathbf{0}$  (why?) and the algorithm terminates.
- With symmetric pivoting:  $\mathbf{P} \mathbf{A} \mathbf{P}^T = \mathbf{L} \mathbf{L}^T$ , where  $\mathbf{P}$  is a permutation matrix and  $\mathbf{L} \in \mathbb{R}^{n \times r}$ ,  $r = \text{rank}(\mathbf{A})$ .
- In R, `chol()` is a wrapper function for LAPACK routines DPOTRF (p.d. no pivoting) and DPSTRF (p.s.d. with pivoting).
  - Option `pivot = FALSE` calls DPOTRF. It does  $\mathbf{A} = \mathbf{R}^T \mathbf{R}$  and gives error message if  $\mathbf{A}$  is not full rank.
  - Option `pivot = TRUE` calls DPSTRF. It does symmetric pivoting  $\mathbf{P} \mathbf{A} \mathbf{P}^T = \mathbf{R}^T \mathbf{R}$  and yields `rank` and `pivot`.
  - Option `tol` passes the tolerance to LAPACK for deciding zero pivots. Default is  $n \cdot \text{machine epsilon} \cdot \max(\text{diag}(\mathbf{A}))$ .
- In JULIA, `chol()`, `cholfact()`, `ldlfact()`, or call LAPACK functions directly.

## Applications of Cholesky decomposition

There are numerous applications of Cholesky decomposition.

- *No inversion mentality:* Whenever we see matrix inverse, we should think in terms of solving linear equations. If the matrix is positive (semi)definite, Cholesky decomposition applies.
- Example: multivariate normal density  $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ ,  $\boldsymbol{\Sigma}$  is p.d.

$$-\frac{n}{2} \ln(2\pi) - \frac{1}{2} \ln \det \boldsymbol{\Sigma} - \frac{1}{2} (\mathbf{y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}).$$

- Method 1: (a) compute explicit inverse  $\boldsymbol{\Sigma}^{-1}$  ( $2n^3$  flops), (b) compute quadratic form ( $2n^2 + 2n$  flops), (c) compute determinant ( $2n^3/3$  flops).
- Method 2: (a) Cholesky decomposition  $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^T$  ( $n^3/3$  flops), (b) Solve  $\mathbf{L}\mathbf{x} = \mathbf{y} - \boldsymbol{\mu}$  by forward substitutions ( $n^2$  flops), (c) compute quadratic form  $\mathbf{x}^T \mathbf{x}$  ( $2n$  flops), and (d) compute determinant from Cholesky factor ( $n$  flops).

Which method is better?

- Compute Moore-Penrose inverse  $\mathbf{A}^+$ .
- Cholesky decomposition is *one* approach to solve linear regression.

## 8 Lecture 8, Jan 28

### Announcements

- HW2 due next Tuesday. FAQs at <http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostatm280winter2016/2016/01/26/hw2-hints.html>

### Last time

- LU decomposition  $\mathbf{A} = \mathbf{LU}$ .  $2n^3/3$  flops.
- LU with partial pivoting  $\mathbf{PA} = \mathbf{LU}$ .  $2n^3/3$  flops.
- Cholesky decomposition  $\mathbf{A} = \mathbf{LL}^T$ .  $n^3/3$  flops.
- Cholesky decomposition with symmetric pivoting:  $\mathbf{PAP}^T = \mathbf{LL}^T$ .  $n^3/3$  flops.  
Yields the rank of the psd matrix.
- Note the actual implementation in LAPACK may be quite different from our description.
- *No inversion mentality.* Matrix inversion takes an extra of  $4/3n^3$  flops, which is unnecessary in most applications.
- Multivariate normal density evaluation.

### Today

- Linear regression by Cholesky.
- QR and linear regression.
- QR by (modified) Gram-Schmidt.
- QR by Householder.
- QR by Givens.
- Sweep operator.

## Linear regression by Cholesky (method of normal equations)

Assume  $\mathbf{X} \in \mathbb{R}^{n \times p}$  has full column rank. (For rank deficient  $\mathbf{X}$ , use Cholesky with symmetric pivoting.)

- Cholesky on the augmented matrix  $(\mathbf{X} \quad \mathbf{y})^T (\mathbf{X} \quad \mathbf{y})$  yields

$$\begin{pmatrix} \mathbf{X}^\top \mathbf{X} & \mathbf{X}^\top \mathbf{y} \\ \mathbf{y}^\top \mathbf{X} & \mathbf{y}^\top \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{l}^\top & d \end{pmatrix} \begin{pmatrix} \mathbf{L}^\top & \mathbf{l} \\ \mathbf{0}^\top & d \end{pmatrix} = \begin{pmatrix} \mathbf{L} \mathbf{L}^\top & \mathbf{L} \mathbf{l} \\ \mathbf{l}^\top \mathbf{L}^\top & \|\mathbf{l}\|_2^2 + d^2 \end{pmatrix}.$$

Normal equation  $\mathbf{X}^\top \mathbf{X} \boldsymbol{\beta} = \mathbf{X}^\top \mathbf{y}$  implies the equation

$$\mathbf{L} \mathbf{L}^\top \boldsymbol{\beta} = \mathbf{L} \mathbf{l} \text{ or } \mathbf{L}^\top \boldsymbol{\beta} = \mathbf{l},$$

which we can solve for  $\boldsymbol{\beta}$  in  $p^2$  flops. Since  $\mathbf{l} = \mathbf{L}^{-1} \mathbf{X}^\top \mathbf{y}$ , we have

$$\mathbf{l}^\top \mathbf{l} = \mathbf{y}^\top \mathbf{X} (\mathbf{L} \mathbf{L}^\top)^{-1} \mathbf{X}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{X} (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{P}_X \mathbf{y} = \|\hat{\mathbf{y}}\|_2^2$$

and

$$d^2 = \mathbf{y}^\top \mathbf{y} - \mathbf{l}^\top \mathbf{l} = \mathbf{y}^\top (\mathbf{I} - \mathbf{P}_X) \mathbf{y} = \|\mathbf{y} - \hat{\mathbf{y}}\|_2^2 = \text{SSE}.$$

If standard errors are needed, we need inversion  $(\mathbf{X}^\top \mathbf{X})^{-1} = (\mathbf{L} \mathbf{L}^\top)^{-1} = \mathbf{L}^{-T} \mathbf{L}^{-1}$ . Use `chol2inv()` in R function for this purpose.

- In summary, linear regression by Cholesky, aka the method of normal equations:
  - Form the lower triangular part of  $(\mathbf{X}, \mathbf{y})^T (\mathbf{X}, \mathbf{y})$ .  
 $n(p+1)^2 \approx np^2$  flops.
  - Cholesky decomposition of the augmented system  $\begin{pmatrix} \mathbf{X}^\top \mathbf{X} & \mathbf{X}^\top \mathbf{y} \\ \mathbf{y}^\top \mathbf{X} & \mathbf{y}^\top \mathbf{y} \end{pmatrix}$ .  
 $(p+1)^3/3 \approx p^3/3$  flops.
  - Solve  $\mathbf{L}^\top \boldsymbol{\beta} = \mathbf{l}$  for regression coefficients  $\hat{\boldsymbol{\beta}}$ .  
 $p^2$  flops.
  - If want standard errors, estimate  $\sigma^2$  by  $\hat{\sigma}^2 = d^2/(n-p)$  and compute  
 $\hat{\sigma}^2 (\mathbf{X}^\top \mathbf{X})^{-1} = \hat{\sigma}^2 (\mathbf{L} \mathbf{L}^\top)^{-1}$ .  
 $4p^3/3$  flops.

Total cost is  $p^3/3 + np^2$  flops (without s.e.) or  $5p^3/3 + np^2$  flops (with s.e.).

## QR decomposition and linear regression

A second approach for linear regression uses QR decomposition. This is how the `lm()` function in R does linear regression. Assume  $\mathbf{X} \in \mathbb{R}^{n \times p}$  has full column rank.

- QR decomposition:  $\mathbf{X} = \mathbf{Q}\mathbf{R}$ , where  $\mathbf{Q} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}_n$ , and  $\mathbf{R} \in \mathbb{R}^{n \times p}$ .
  - first  $p$  columns of  $\mathbf{Q}$  form an orthonormal basis of  $\mathcal{C}(\mathbf{X})$
  - last  $n - p$  columns of  $\mathbf{Q}$  form an orthonormal basis of  $\mathcal{N}(\mathbf{X}^T)$
- (Thin/Skinny QR) Then  $\mathbf{X} = \mathbf{Q}_1 \mathbf{R}_1$ , where  $\mathbf{Q}_1 \in \mathbb{R}^{n \times p}$  has orthogonal columns,  $\mathbf{Q}_1^\top \mathbf{Q}_1 = \mathbf{I}_p$ , and  $\mathbf{R}_1 \in \mathbb{R}^{p \times p}$  is an invertible upper triangular matrix with positive diagonal entries.
- For linear regression, we only need skinny QR.  
Note  $\mathbf{X}^\top \mathbf{X} = \mathbf{R}_1^\top \mathbf{R}_1$  yields the Cholesky factor of  $\mathbf{X}^\top \mathbf{X}$ .
- (Skinny) QR on the augmented matrix yields

$$(\mathbf{X} \quad \mathbf{y}) = (\mathbf{Q} \quad \mathbf{q}) \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}_p^\top & d \end{pmatrix} = (\mathbf{Q}\mathbf{R} \quad \mathbf{Q}\mathbf{r} + d\mathbf{q}).$$

Normal equation  $\mathbf{X}^\top \mathbf{X} \boldsymbol{\beta} = \mathbf{X}^\top \mathbf{y}$  implies

$$\mathbf{R}\boldsymbol{\beta} = \mathbf{R}^{-T} \mathbf{X}^\top \mathbf{y} = \mathbf{R}^{-T} \mathbf{R}^\top \mathbf{Q}^\top \mathbf{y} = \mathbf{Q}^\top \mathbf{y} = \mathbf{r},$$

which is easy to solve for  $\boldsymbol{\beta}$ . The fitted value is  $\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{Q}\mathbf{R}\mathbf{R}^{-1}\mathbf{r} = \mathbf{Q}\mathbf{r}$ .  
The residual is

$$\hat{\mathbf{e}} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{y} - \mathbf{Q}\mathbf{r} = d\mathbf{q}$$

and SSE =  $\|\hat{\mathbf{e}}\|_2^2 = d^2$ . The projection matrix is

$$\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X} = \mathbf{Q}\mathbf{R}(\mathbf{R}^\top \mathbf{R})^{-1} \mathbf{R}^\top \mathbf{Q}^\top = \mathbf{Q}\mathbf{Q}^\top.$$

- Three numerical methods to compute QR: (modified) Gram-Schmidt, Householder transform, (fast) Givens transform.

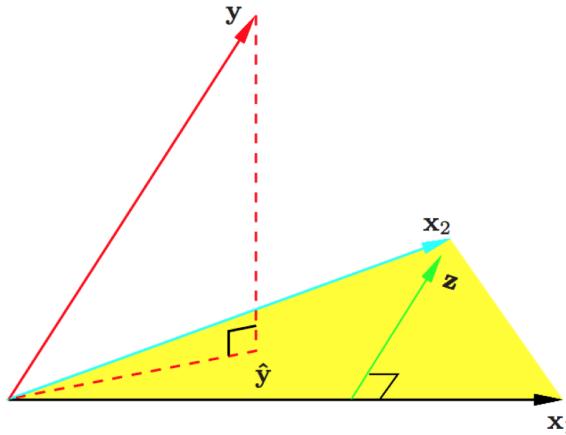
## QR by (modified) Gram-Schmidt



Jørgen Pedersen Gram in an undated photo



Erhard Schmidt (courtesy MFO)



Assume  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_p) \in \mathbb{R}^{n \times p}$  has full column rank.

- Gram-Schmidt (GS) algorithm produces the skinny QR:  $\mathbf{X} = \mathbf{Q}\mathbf{R}$ , where  $\mathbf{Q} \in \mathbb{R}^{n \times p}$  has orthogonal columns and  $\mathbf{R} \in \mathbb{R}^{p \times p}$  is an upper triangular matrix.
- Gram-Schmidt algorithm orthonormalizes a set of non-zero, *linearly independent* vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$ . Initialize  $\mathbf{q}_1 = \mathbf{x}_1 / \|\mathbf{x}_1\|_2$ ; then for  $k = 2, \dots, p$ ,

$$\begin{aligned} \mathbf{v}_k &= \mathbf{x}_k - \mathbf{P}_{\mathcal{C}\{\mathbf{q}_1, \dots, \mathbf{q}_{k-1}\}} \mathbf{x}_k = \mathbf{x}_k - \sum_{j=1}^{k-1} \langle \mathbf{x}_k, \mathbf{q}_j \rangle \mathbf{q}_j \\ \mathbf{q}_k &= \mathbf{v}_k / \|\mathbf{v}_k\|_2 \end{aligned}$$

- For  $j = 1, \dots, p$ ,  $\mathcal{C}\{\mathbf{x}_1, \dots, \mathbf{x}_j\} = \mathcal{C}\{\mathbf{q}_1, \dots, \mathbf{q}_j\}$ , and  $\mathbf{q}_j \perp \mathcal{C}\{\mathbf{x}_1, \dots, \mathbf{x}_{j-1}\}$ .
- Collectively, we have  $\mathbf{X} = \mathbf{Q}\mathbf{R}$  (skinny QR), where
  - $\mathbf{Q} \in \mathbb{R}^{n \times p}$  has orthonormal columns  $\mathbf{q}_k$  and thus  $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}_p$ .
  - Where  $\mathbf{R}?$   $\mathbf{R} = \mathbf{Q}^T \mathbf{X} \in \mathbb{R}^{p \times p}$  has entries  $r_{jk} = \langle \mathbf{q}_j, \mathbf{x}_k \rangle$ , which are available from the algorithm. Note  $r_{jk} = 0$  for  $j > k$ . Thus  $\mathbf{R}$  is upper triangular.
- In GS algorithm,  $\mathbf{X}$  is over-written by  $\mathbf{Q}$  and  $\mathbf{R}$  is stored in a separate array.
- The regular Gram-Schmidt is *unstable* (we loose orthogonality due to roundoff errors) when columns of  $\mathbf{X}$  are collinear.
- *Modified Gram-Schmidt* (MGS): after each normalization step of  $\mathbf{v}_k$ , we replace  $\tilde{\mathbf{x}}_j$ ,  $j > k$ , by its residual.
- Why MGS is better than GS? Read <http://cavern.uark.edu/~arnold/4353/CGSMGS.pdf>
- Computational cost of GS and MGS is  $\sum_{k=1}^p 4n(k-1) \approx 2np^2$ .

## QR by Householder



Photograph by [Paul Halmos](#)

Assume  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_p) \in \mathbb{R}^{n \times p}$  has full column rank.

- Idea:  $\mathbf{H}_p \cdots \mathbf{H}_2 \mathbf{H}_1 \mathbf{X} = \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix}$ , where  $\mathbf{H}_j \in \mathbb{R}^{n \times n}$  are the Householder transformation matrix. It yields the full QR where  $\mathbf{Q} = \mathbf{H}_1 \cdots \mathbf{H}_p \in \mathbb{R}^{n \times n}$ . Recall that GS/MGS only produces the thin QR decomposition.
- For arbitrary  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  with  $\|\mathbf{v}\|_2 = \|\mathbf{w}\|_2$ , we can construct a *Householder matrix*  $\mathbf{H} = \mathbf{I}_n - 2\mathbf{u}\mathbf{u}^\top$ ,  $\mathbf{u} = -\frac{1}{\|\mathbf{v}-\mathbf{w}\|_2}(\mathbf{v} - \mathbf{w})$ , that carries  $\mathbf{v}$  to  $\mathbf{w}$ :

$$\mathbf{H}\mathbf{v} = \mathbf{w}.$$

$\mathbf{H}$  is symmetric and orthogonal. Calculation of Householder vector  $\mathbf{u}$  costs  $4n$  flops.

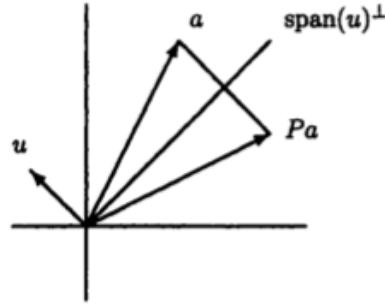


FIG. 2.3.1. *Reflection of a vector  $a$  in a hyperplane with normal  $u$ .*

- Now choose  $\mathbf{H}_1$  to zero the first column of  $\mathbf{X}$  below diagonal

$$\mathbf{H}_1 \mathbf{x}_1 = \begin{pmatrix} \|\mathbf{x}_1\|_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Take  $\mathbf{H}_2$  to zero the second column below diagonal; ...

$$\mathbf{H}_2 \mathbf{H}_1 \mathbf{A} = \left[ \begin{array}{ccccc} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \blacksquare & \times & \times \\ 0 & 0 & \blacksquare & \times & \times \\ 0 & 0 & \blacksquare & \times & \times \\ 0 & 0 & \blacksquare & \times & \times \end{array} \right]$$

In general, choose the  $j$ -th Householder transform  $\mathbf{H}_j = \mathbf{I}_n - 2\mathbf{u}_j\mathbf{u}_j^\top$ , where  $\mathbf{u}_j = \begin{pmatrix} \mathbf{0}_{j-1} \\ \tilde{\mathbf{u}}_j \end{pmatrix}$ ,  $\tilde{\mathbf{u}}_j \in \mathbb{R}^{n-j+1}$ , to zero the  $j$ -th column below diagonal.  $\mathbf{H}_j$  takes the form

$$\mathbf{H}_j = \begin{pmatrix} \mathbf{I}_{j-1} & \\ & \mathbf{I}_{n-j+1} - 2\tilde{\mathbf{u}}_j\tilde{\mathbf{u}}_j^T \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{j-1} & \\ & \tilde{\mathbf{H}}_j \end{pmatrix}.$$

- Applying a Householder transform  $\mathbf{H} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^\top$  to a matrix  $\mathbf{X} \in \mathbb{R}^{n \times p}$

$$\mathbf{H}\mathbf{X} = \mathbf{X} - 2\mathbf{u}(\mathbf{u}^\top \mathbf{X})$$

costs  $4np$  flops. *We never explicitly form the Householder matrices.*

Note applying  $\mathbf{H}_j$  to  $\mathbf{X}$  only needs  $4(n - j + 1)(p - j + 1)$  flops.

- QR by Householder:  $\mathbf{H}_p \cdots \mathbf{H}_1 \mathbf{X} = \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix}$ .
- The process is done in place. Upper triangular part of  $\mathbf{X}$  is overwritten by  $\mathbf{R}_1$  and the essential Householder vectors ( $\tilde{\mathbf{u}}_{j1}$  is normalized to 1) are stored in  $\mathbf{X}(j : n, j)$ .
- At  $j$ -th stage
  1. computing the Householder vector  $\tilde{\mathbf{u}}_j$  costs  $4(n - j + 1)$  flops
  2. applying the Householder transform  $\tilde{\mathbf{H}}_j$  to the  $\mathbf{X}(j : n, j : p)$  block costs  $4(n - j + 1)(p - j + 1)$  flops

In total we need  $\sum_{j=1}^p [3(n - j + 1) + 4(n - j + 1)(p - j + 1)] \approx 2np^2 - \frac{2}{3}p^3$  flops.

- Where is  $\mathbf{Q}$ ?  $\mathbf{Q} = \mathbf{H}_1 \cdots \mathbf{H}_p$ .

In some applications, it's necessary to form the orthogonal matrix  $\mathbf{Q}$ .

Accumulating  $\mathbf{Q}$  costs another  $2np^2 - \frac{2}{3}p^3$  flops.

- When computing  $\mathbf{Q}^\top \mathbf{v}$  or  $\mathbf{Q}\mathbf{v}$  as in some applications (e.g., solve linear equation using QR), no need to form  $\mathbf{Q}$ . Simply apply Householder transforms successively.  
`qr.qy()` and `qr.qty()` in R do this.
- Computational cost of Householder QR for linear regression:  $2np^2 - \frac{2}{3}p^3$  (regression coefficients and  $\hat{\sigma}^2$ ) or more (fitted values, s.e., ...).

## Rank deficient $\mathbf{X}$ : Householder QR with column pivoting

$\mathbf{X} \in \mathbb{R}^{n \times p}$  may not have full column rank.

- Idea (due to Businger and Golub 1965): at the  $j$ -th stage, swap the column in  $\mathbf{X}(j : n, j : p)$  with maximum  $\ell_2$  norm to be the pivot column. If the maximum  $\ell_2$  norm is 0, it stops, ending with

$$\mathbf{X}\boldsymbol{\Pi} = \mathbf{Q} \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{0}_{(n-r) \times r} & \mathbf{0}_{(n-r) \times (p-r)} \end{pmatrix},$$

where  $\boldsymbol{\Pi} \in \mathbb{R}^{p \times p}$  is a permutation matrix and  $r$  is the rank of  $\mathbf{X}$ . QR with column pivoting is rank revealing.

- The overhead of re-computing the column norms can be reduced by the property

$$\mathbf{Q}\mathbf{z} = \begin{pmatrix} \alpha \\ \boldsymbol{\omega} \end{pmatrix} \Rightarrow \|\boldsymbol{\omega}\|_2^2 = \|\mathbf{z}\|_2^2 - \alpha^2$$

for any orthogonal matrix  $\mathbf{Q}$ .

- In R, the `qr()` function is a wrapper for various LINPACK (default) and LAPACK routines. It performs Householder QR with column pivoting and returns
  - `$qr`: a matrix of same size as input matrix
  - `$rank`: rank of the input matrix
  - `$pivot`: pivot vector
  - `$aux`: normalizing constants of Householder vectors

Auxiliary functions `qr.coef()`, `qr.resid()`, `qr.qy()`, `qr.qty()`, `qr.solve()`, ... are very helpful.

## QR by Givens rotation



- Householder transform  $\mathbf{H}_j$  introduces batch of zeros into a vector.
- Givens transform (aka Givens rotation, Jacobi rotation, plane rotation) selectively zeros one element of a vector.
- Overall QR by Givens rotation is less efficient than the Householder method, but is better suited for matrices with structured patterns of nonzero elements.
- Givens/Jacobi rotations:

$$\mathbf{G}(i, k, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & c & s & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -s & c & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $c = \cos(\theta)$  and  $s = \sin(\theta)$ .  $\mathbf{G}(i, k, \theta)$  is orthogonal.

- Pre-multiplication by  $\mathbf{G}(i, k, \theta)^T$  rotates counterclockwise  $\theta$  radians in the  $(i, k)$  coordinate plane. If  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{y} = \mathbf{G}(i, k, \theta)^T \mathbf{x}$ , then

$$y_j = \begin{cases} cx_i - sx_k & j = i \\ sx_i + cx_k & j = k \\ x_j & j \neq i, k \end{cases}.$$

Apparently if we choose  $\tan(\theta) = -x_k/x_i$ , or equivalently,

$$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}}, \quad s = \frac{-x_k}{\sqrt{x_i^2 + x_k^2}},$$

then  $y_k = 0$ .

- Pre-applying Givens transform  $\mathbf{G}(i, k, \theta)^T \in \mathbb{R}^{n \times n}$  to a matrix  $\mathbf{A} \in \mathbb{R}^{n \times m}$  only effects two rows of  $\mathbf{A}$ :

$$\mathbf{A}([i, k], :) \leftarrow \begin{pmatrix} c & s \\ -s & c \end{pmatrix}^T \mathbf{A}([i, k], :),$$

costing  $6m$  flops.

- Post-applying Givens transform  $\mathbf{G}(i, k, \theta) \in \mathbb{R}^{m \times m}$  to a matrix  $\mathbf{A} \in \mathbb{R}^{n \times m}$  only effects two columns of  $\mathbf{A}$ :

$$\mathbf{A}(:, [i, k]) \leftarrow \mathbf{A}(:, [i, k]) \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

costing  $6n$  flops.

- QR by Givens:  $\mathbf{G}_t^T \cdots \mathbf{G}_1^T \mathbf{X} = \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix}$

$$\begin{array}{c} \left[ \begin{array}{ccc} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{array} \right] \xrightarrow{(3,4)} \left[ \begin{array}{ccc} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \end{array} \right] \xrightarrow{(2,3)} \left[ \begin{array}{ccc} \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{array} \right] \xrightarrow{(1,2)} \\ \left[ \begin{array}{ccc} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{array} \right] \xrightarrow{(3,4)} \left[ \begin{array}{ccc} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{array} \right] \xrightarrow{(2,3)} \left[ \begin{array}{ccc} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{array} \right] \xrightarrow{(3,4)} R \end{array}$$

- Zeros in  $\mathbf{X}$  can also be introduced row-by-row.
- If  $\mathbf{X} \in \mathbb{R}^{n \times p}$ , the total cost is  $3np^2 - p^3$  flops and  $O(np)$  square roots.
- Note each Givens transform can be summarized by a single number, which is stored in the zeroed entry of  $\mathbf{X}$ .
- Fast Givens transform avoids taking square roots.

# 9 Lecture 9, Feb 2

## Announcements

- HW1 graded. Feedback:
  - Sketch of solution: <http://hua-zhou.github.io/teaching/biostatm280-2016winter/hw01sol.html>. Please compare to your code carefully and understand why.
  - Q2: *Rounding to even rule* in R.
  - Q5: be familiar with R functions: `crossprod`, `tcrossprod`, `outer`, `row`, `col`, `rowSums`, `colSums`, ...
  - Q7: vectorize code (no looping necessary).
  - Code style. Be professional!
  - Frequent commits in git. I want to see how you developed the solutions.
- HW2 due today @ 11:59PM.
- HW3 posted. Due ~~Feb 11~~ Feb 16 @ 11:59PM. [http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostat\\_m280\\_2016\\_hw3.pdf](http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostat_m280_2016_hw3.pdf)
- Quiz 2 ~~this Thu Feb 4~~ next Tue Feb 9. In class, closed book.

## Last time

- Linear regression by Cholesky.  $np^2 + p^3/3$  flops for  $\hat{\beta}$  or more for s.e.
- Linear regression by QR  $\mathbf{X} = \mathbf{QR}$ .
- QR by MGS.  $2np^2$  flops to obtain  $\mathbf{Q}_1$  and  $\mathbf{R}$ .
- QR by Householder.  $2np^2 - 2p^3/3$  flops to overwrite  $\mathbf{X}$  by  $\mathbf{R}$  and Householder vectors. It is the algorithm R uses for linear regression.
- QR by Givens.  $3np^2 - p^3$  to overwrite  $\mathbf{X}$  by  $\mathbf{R}$  and Givens rotations.

## Today

- Sweep operator.
- Summary of numerical methods for linear regression.
- Summary of solving linear equations.
- Condition number.

## Sweep operator

Assume  $\mathbf{A} \succeq \mathbf{0}_{n \times n}$ .

- The popular statistical software SAS uses sweep operator for linear regression and matrix inversion.
- Read KL 7.4-7.6 for introduction.

Also see “*A tutorial on the SWEEP operator*” by James H. Goodnight. <http://www.jstor.org/stable/2683825>

- *Sweep* on the  $k$ -th diagonal entry  $a_{kk} \neq 0$  yields  $\hat{\mathbf{A}}$  with entries

$$\begin{aligned}\hat{a}_{kk} &= -\frac{1}{a_{kk}} \\ \hat{a}_{ik} &= \frac{a_{ik}}{a_{kk}} \\ \hat{a}_{kj} &= \frac{a_{kj}}{a_{kk}} \\ \hat{a}_{ij} &= a_{ij} - \frac{a_{ik}a_{kj}}{a_{kk}}, \quad i \neq k, j \neq k.\end{aligned}$$

$n^2$  flops (taking into account of symmetry).

- *Inverse sweep* sends  $\mathbf{A}$  to  $\check{\mathbf{A}}$  with entries

$$\begin{aligned}\check{a}_{kk} &= -\frac{1}{a_{kk}} \\ \check{a}_{ik} &= -\frac{a_{ik}}{a_{kk}} \\ \check{a}_{kj} &= -\frac{a_{kj}}{a_{kk}} \\ \check{a}_{ij} &= a_{ij} - \frac{a_{ik}a_{kj}}{a_{kk}}, \quad i \neq k, j \neq k.\end{aligned}$$

$n^2$  flops (taking into account of symmetry).

- $\check{\mathbf{A}} = \mathbf{A}$ .
- Successively sweeping all diagonal entries of  $\mathbf{A}$  yields  $-\mathbf{A}^{-1}$ .
- Exercise: invert a  $2 \times 2$  matrix, say  $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ , on paper using sweep operator.
- Block form of sweep: Let the symmetric matrix  $\mathbf{A}$  be partitioned as  $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$ . If possible, sweep on the diagonal entries of  $\mathbf{A}_{11}$  yields

$$\hat{\mathbf{A}} = \begin{pmatrix} -\mathbf{A}_{11}^{-1} & \mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{pmatrix}.$$

Order does *not* matter. The block  $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$  is recognized as the *Schur complement* of  $\mathbf{A}_{11}$ .

- Pd and determinant:  $\mathbf{A}$  is pd if and only if each diagonal entry can be swept in succession and is positive until it is swept. When a diagonal entry of a pd matrix  $\mathbf{A}$  is swept, it becomes negative and remains negative thereafter. Taking the product of diagonal entries just before each is swept yields the determinant of  $\mathbf{A}$ .

- Linear regression by sweep. Sweep on the Gram matrix  $\begin{pmatrix} \mathbf{X}^\top \mathbf{X} & \mathbf{X}^\top \mathbf{y} \\ \mathbf{y}^\top \mathbf{X} & \mathbf{y}^\top \mathbf{y} \end{pmatrix}$  yields
- $$\begin{pmatrix} -(\mathbf{X}^\top \mathbf{X})^{-1} & (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y} \\ \mathbf{y}^\top \mathbf{X} (\mathbf{X} \mathbf{X})^{-1} & \mathbf{y}^\top \mathbf{y} - \mathbf{y}^\top \mathbf{X} (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sigma^2} \text{Cov}(\hat{\boldsymbol{\beta}}) & \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\beta}}^\top & \|\mathbf{y} - \hat{\mathbf{y}}\|_2^2 \end{pmatrix}.$$

In total  $np^2 + p^3$  flops.

- Sweep is useful for stepwise regression, (conditional) multivariate normal density calculation, MANOVA, ...
- LAPACK does *not* implement sweep operator.
- Warning: the `sweep()` function in R has nothing to do with the sweep operator here.
- Demo code: <http://hua-zhou.github.io/teaching/biostatm280-2016winter/sweep.html>

## Summary of linear regression: Table on KL p105

Method	Flops	Remarks	Software	Stability
Sweep	$np^2 + p^3$	$(\mathbf{X}^\top \mathbf{X})^{-1}$ available	SAS	less stable
Cholesky	$np^2 + p^3/3$			less stable
QR by Householder	$2np^2 - 2p^3/3$		R	stable
QR by MGS	$2np^2$	$\mathbf{Q}_1$ available		most stable

Table 1: Numerical methods for linear regression. In order of stability.

Remarks:

- When  $n \gg p$ , sweep and Cholesky are twice faster than QR and need less space.
- Sweep and Cholesky is based on the Gram matrix  $\mathbf{X}^T \mathbf{X}$ , which can be dynamically updated with incoming data. They can handle huge  $n$ , moderate  $p$  data sets that cannot fit into memory.
- QR methods are more stable and produce numerically more accurate solution.
- Although sweep is slower than Cholesky, it yields standard errors and so on.
- Sweep is useful for stepwise regression, multivariate normal calculation, and numerous other statistical applications.
- MGS appears slower than Householder, but it yields  $\mathbf{Q}_1$ .

There is simply no such thing as a universal ‘gold standard’ when it comes to algorithms.

anonymous reviewer

## Summary of solving linear equations

Consider linear system  $\mathbf{A}\mathbf{x} = \mathbf{b}$ .

- We now know some good numerical methods for the least squares problem, which is essentially “solving” an overdetermined system (a tall  $\mathbf{A}$ ).

- Table 2 compares the flops of some methods (in order of stability) for solving a square (unstructured)  $\mathbf{A} \in \mathbb{R}^{n \times n}$ .

Method	Flops	Stability
Gaussian elimination	$2n^3/3$	less stable
QR by Householder	$4n^3/3$	
QR by MGS	$2n^3$	
SVD	$12n^3$	most stable

Table 2: Flops of different numerical methods for  $n \times n$  square linear systems, assuming availability of the right hand side at time of decomposition.

- Solve an underdetermined system (a flat  $\mathbf{A} \in \mathbb{R}^{m \times n}$  of full row rank) by QR – version 1. First compute QR on  $\mathbf{A}^T$

$$\mathbf{A}^T = \mathbf{Q}\mathbf{R} = \mathbf{Q} \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0}_{(n-m) \times m} \end{pmatrix}.$$

Then  $\mathbf{A}\mathbf{x} = \mathbf{b}$  becomes

$$(\mathbf{Q}\mathbf{R})^T \mathbf{x} = \left( \mathbf{R}_1^T \quad \mathbf{0}_{m \times (n-m)} \right) \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \mathbf{b},$$

where

$$\mathbf{Q}^T \mathbf{x} = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}.$$

$\mathbf{z}_1$  is determined from  $\mathbf{R}_1^T \mathbf{z}_1 = \mathbf{b}$ . If we take  $\mathbf{z}_2 = \mathbf{0}_{n-m}$ , then we obtain the minimum norm solution (why?).

- Solve an underdetermined system (a flat  $\mathbf{A} \in \mathbb{R}^{m \times n}$  of full row rank) by QR – version 2. First compute QR with column pivoting on  $\mathbf{A}$

$$\mathbf{A}\Pi = \mathbf{Q} (\mathbf{R}_1 \quad \mathbf{R}_2),$$

where  $\mathbf{R}_1 \in \mathbb{R}^{m \times m}$  is upper triangular and  $\mathbf{R}_2 \in \mathbb{R}^{m \times (n-m)}$ . Thus  $\mathbf{A}\mathbf{x} = \mathbf{b}$  transforms to

$$(\mathbf{R}_1 \quad \mathbf{R}_2) \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \mathbf{Q}^T \mathbf{b},$$

where

$$\Pi^T \mathbf{x} = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}.$$

One solution is obtained by  $\mathbf{z}_1 = \mathbf{R}_1^{-1} \mathbf{Q}^T \mathbf{b}$  and  $\mathbf{z}_2 = \mathbf{0}_{n-m}$ . It is not guaranteed to be of minimum norm.

## Condition number for linear equations (matrix inversion)

- Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is nonsingular and consider the system of linear equation  $\mathbf{Ax} = \mathbf{b}$ . The solution is  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ . We want to know how the solution changes with a small perturbation of the input  $\mathbf{b}$  (or  $\mathbf{A}$ ).
- Let  $\tilde{\mathbf{b}} = \mathbf{b} + \Delta\mathbf{b}$ . Then  $\tilde{\mathbf{x}} = \mathbf{A}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = \mathbf{x} + \Delta\mathbf{x}$ . Thus

$$\|\Delta\mathbf{x}\| = \|\mathbf{A}^{-1}\Delta\mathbf{b}\| \leq \|\mathbf{A}^{-1}\| \|\Delta\mathbf{b}\|.$$

Because  $\mathbf{b} = \mathbf{Ax}$ ,  $\frac{1}{\|\mathbf{x}\|} \leq \|\mathbf{A}\| \frac{1}{\|\mathbf{b}\|}$ . This results

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\| \|\mathbf{A}^{-1}\| \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}.$$

- $\kappa(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|$  is called the *condition number for inversion*. It depends on the matrix norm being used.  $\kappa_p$  means condition number defined from matrix- $p$  norm.
- Large condition number means “bad”.
- Some useful facts

$$\begin{aligned} \kappa(\mathbf{A}) &= \kappa(\mathbf{A}^{-1}) \\ \kappa(c\mathbf{A}) &= \kappa(\mathbf{A}) \\ \kappa(\mathbf{A}) &\geq 1 \\ \kappa_2(\mathbf{A}) &= \kappa_2(\mathbf{A}^\top) = \frac{\sigma_1(\mathbf{A})}{\sigma_n(\mathbf{A})} \\ \kappa_2(\mathbf{A}^\top \mathbf{A}) &= \frac{\lambda_1(\mathbf{A}^\top \mathbf{A})}{\lambda_n(\mathbf{A}^\top \mathbf{A})} = \kappa_2^2(\mathbf{A}) \geq \kappa_2(\mathbf{A}). \end{aligned}$$

The last fact says that the condition number of  $\mathbf{A}^\top \mathbf{A}$  can be much larger than that of  $\mathbf{A}$ .

- The smallest singular value  $\sigma_n$  indicates the “distance to the trouble”.
- Condition number for the least squares problem is more complicated. Roughly speaking, the method based on normal equation (Cholesky, sweep) has a condition depending on  $\kappa_2(\mathbf{X})^2$ . QR for a “small residuals” least squares problem has a condition depending on  $\kappa_2(\mathbf{X})$ .
- Numerically, consider the simple case

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 10^{-3} & 0 \\ 0 & 10^{-3} \end{pmatrix}.$$

Forming normal equation yields a singular Gramian matrix

$$\mathbf{X}^\top \mathbf{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

if executed with a precision of 6 digits.

- In R, the `kappa()` function (wrapper of `DTRCON` in LAPACK and `DTRCO` in LINPACK) computes or approximates (default) the condition number of a matrix.
- In regression problems, standardizing the predictors could improve the condition. See demo on the Longley data <http://hua-zhou.github.io/teaching/biostatm280-2016winter/longleycond.html>.
- In design of experiments (DoE), people favor orthogonal design. Why?

# 10 Lecture 10, Feb 4

## Announcements

- HW3 posted. Due ~~Feb 11~~ Feb 16 @ 11:59PM. [http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostat\\_m280\\_2016\\_hw3.pdf](http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostat_m280_2016_hw3.pdf)
- Quiz 2 ~~this Thu Feb 4~~ next Tue Feb 9. In class, closed book.
- TA office hour location: common area of Gonda 6th floor or Gonda 6545 (Max's office).

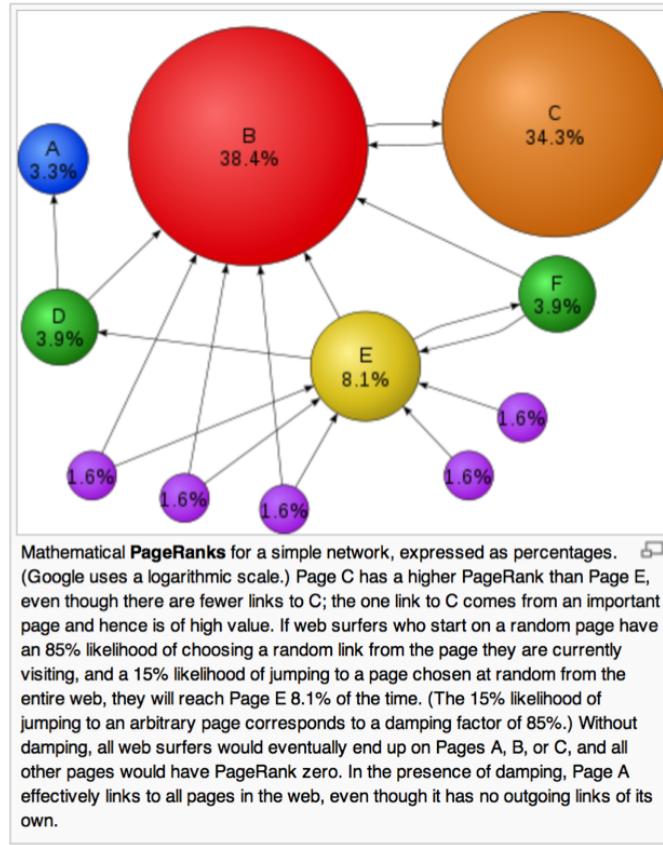
## Last time

- Sweep operator.
- Summary of numerical methods for linear regression.
- Summary of solving linear equations.
- Condition number.

## Today

- Iterative methods for solving linear equations.
- A catalog of easy linear systems.

## Iterative method for solving linear equations: introduction



- Direct method (flops fixed *a priori*) vs iterative methods:
  - Direct method (GE/LU, Cholesky, QR, SVD): good for dense, small or moderate sized, unstructured  $\mathbf{A}$
  - Iterative methods (Jacobi, Gauss-Seidal, SOR, conjugate-gradient, GMRES): good for large, sparse, or structured linear system, parallel computing, warm start
- PageRank (HW3):
  - $\mathbf{A} \in \{0, 1\}^{n \times n}$  the connectivity matrix with entries

$$a_{ij} = \begin{cases} 1 & \text{if page } i \text{ links to page } j \\ 0 & \text{otherwise} \end{cases}.$$

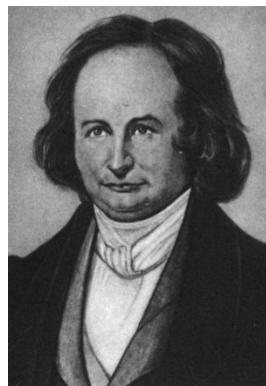
$n \approx 10^9$  in Feb 2016.

- $r_i = \sum_j a_{ij}$  is the out-degree of page  $i$ .
- Larry Page: Imagine a random surfer wandering on internet according to following rules:
  - \* From a page  $i$  with  $r_i > 0$ 
    - with probability  $p$ , (s)he randomly chooses a link on page  $i$  (uniformly) and follows that link to the next page
    - with probability  $1 - p$ , (s)he randomly chooses one page from the set of all  $n$  pages (uniformly) and proceeds to that page
  - \* From a page  $i$  with  $r_i = 0$  (a dangling page), (s)he randomly chooses one page from the set of all  $n$  pages (uniformly) and proceeds to that page

The process defines a Markov chain on the space of  $n$  pages. Stationary distribution of this Markov chain gives the ranks (probabilities) of each page.

- Stationary distribution is the top left eigenvector of the transition matrix  $\mathbf{P}$  corresponding to eigenvalue 1. Equivalently it can be cast as a linear equation.
- Largest matrix computation in world (?).
- GE/LU will take  $2 \times (10^9)^3 / 3 / 10^{12} \approx 6.66 \times 10^{14}$  seconds  $\approx 20$  million years on a tera-flop supercomputer!
- Iterative methods come to the rescue.

## Jacobi method



Solve linear system  $\mathbf{A}\mathbf{x} = \mathbf{b}$ .

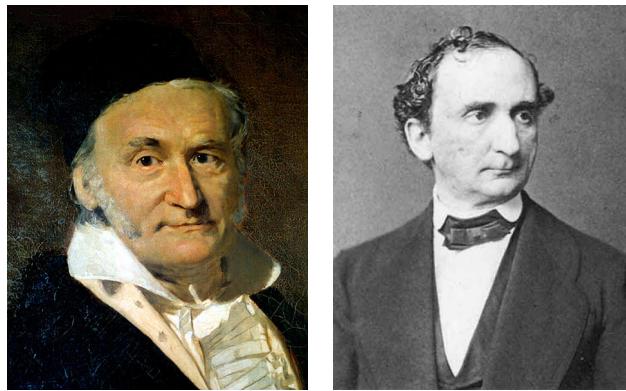
- Jacobi iteration:

$$x_i^{(t+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(t)} - \sum_{j=i+1}^n a_{ij}x_j^{(t)}}{a_{ii}}.$$

- Requires non-zero diagonal element!
- One round costs  $2n^2$  flops with an unstructured  $\mathbf{A}$ . Gain over GE/LU if converges in  $o(n)$  iterations. Saving is huge for *sparse* or *structured*  $\mathbf{A}$ . By structured, we mean the matrix-vector multiplication  $\mathbf{Av}$  is fast.
- Splitting:  $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$ .
- Jacobi:  $\mathbf{D}\mathbf{x}^{(t+1)} = -(\mathbf{L} + \mathbf{U})\mathbf{x}^{(t)} + \mathbf{b}$ , i.e.,

$$\mathbf{x}^{(t+1)} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\mathbf{x}^{(t)} + \mathbf{D}^{-1}\mathbf{b}.$$

## Gauss-Seidel



- Gauss-Seidel iteration:

$$x_i^{(t+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(t+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(t)}}{a_{ii}}.$$

- With splitting,  $(\mathbf{D} + \mathbf{L})\mathbf{x}^{(t+1)} = -\mathbf{U}\mathbf{x}^{(t)} + \mathbf{b}$ , i.e.,

$$\mathbf{x}^{(t+1)} = -(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}\mathbf{x}^{(t)} + (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b}.$$

- GS converges for any  $\mathbf{x}^{(0)}$  for symmetric and pd  $\mathbf{A}$ .
- Convergence rate of Gauss-Seidel is the spectral radius of the  $(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$ .
- Comparing Jacobi and GS, Jacobi is particularly attractive for parallel computing.

## Successive over-relaxation (SOR)

- SOR:  $x_i^{(t+1)} = \omega(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(t+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(t)})/a_{ii} + (1-\omega)x_i^{(t)}$ , i.e.,  

$$\mathbf{x}^{(t+1)} = (\mathbf{D} + \omega\mathbf{L})^{-1}[(1-\omega)\mathbf{D} - \omega\mathbf{U}]\mathbf{x}^{(t)} + (\mathbf{D} + \omega\mathbf{L})^{-1}(\mathbf{D} + \mathbf{L})^{-1}\omega\mathbf{b}.$$
- Need to pick  $\omega \in [0, 1]$  beforehand, with the goal of improving convergence rate.

## Conjugate gradient method

Solving  $\mathbf{Ax} = \mathbf{b}$  is equivalent to minimizing the quadratic function  $\frac{1}{2}\mathbf{x}^T \mathbf{Ax} - \mathbf{b}^T \mathbf{x}$ . To do later, when we study optimization. Conjugate gradient and its variants are the top-notch iterative methods for solving huge, structured linear systems.

**Table 1. Kershaw's results for a fusion problem.**

Method	Number of iterations
Gauss Seidel	208,000
Block successive overrelaxation methods	765
Incomplete Cholesky conjugate gradients	25

## A list of “easy” linear systems

Consider  $\mathbf{Ax} = \mathbf{b}$ ,  $\mathbf{A} \in \mathbb{R}^{n \times n}$ . Or, consider matrix inverse (if you want).  $\mathbf{A}$  can be huge. Keep massive data in mind: 1000 Genome Project, NetFlix, Google PageRank, finance, spatial statistics, ... We should be alert to many easy linear systems. *Don't waste computing resources by bad choices of algorithms!*

- Diagonal:  $n$  flops.

- Tridiagonal/banded: Band LU, band Cholesky, ... roughly  $O(n)$  flops
- Triangular:  $n^2$  flops
- Block diagonal: Suppose  $n = \sum_i n_i$ .  $(\sum_i n_i)^3$  vs  $\sum_i n_i^3$ .
- Kronecker product:  $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$ ,  $(\mathbf{C}^\top \otimes \mathbf{A})\text{vec}\mathbf{B} = \text{vec}(\mathbf{ABC})$  fits iterative method.
- Sparsity: iterative method, or sparse matrix decomposition.  
Remark: Probably the easiest recognizable structure. Familiarize yourself with the sparse matrix computation tools in JULIA, MATLAB, R (**Matrix** package), MKL (sparse BLAS), ... as much as possible.
- Easy plus low rank:  $\mathbf{U} \in \mathbb{R}^{n \times m}$ ,  $\mathbf{V} \in \mathbb{R}^{n \times m}$ ,  $m \ll n$ . Woodbury formula

$$(\mathbf{A} + \mathbf{U}\mathbf{V}^\top)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{I}_m + \mathbf{V}^\top\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^\top\mathbf{A}^{-1}.$$

Keep HW2 Q5 in mind.

- Easy plus border: For  $\mathbf{A}$  pd and  $\mathbf{V}$  full row rank,

$$\begin{pmatrix} \mathbf{A} & \mathbf{V}^\top \\ \mathbf{V} & \mathbf{0} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{V}^\top(\mathbf{V}\mathbf{A}^{-1}\mathbf{V}^\top)^{-1}\mathbf{V}\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{V}^\top(\mathbf{V}\mathbf{A}^{-1}\mathbf{V}^\top)^{-1} \\ (\mathbf{V}\mathbf{A}^{-1}\mathbf{V}^\top)^{-1}\mathbf{V}\mathbf{A}^{-1} & -(\mathbf{V}\mathbf{A}^{-1}\mathbf{V}^\top)^{-1} \end{pmatrix}.$$

- Orthogonal:  $n^2$  flops *at most*. Permutation matrix, Householder matrix, Jacobi matrix, ... take less.
- Toeplitz systems:

$$\mathbf{T} = \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \\ r_{-1} & r_0 & r_1 & r_2 \\ r_{-2} & r_{-1} & r_0 & r_1 \\ r_{-3} & r_{-2} & r_{-1} & r_0 \end{pmatrix}.$$

$\mathbf{T}\mathbf{x} = \mathbf{b}$ , where  $\mathbf{T}$  is pd and Toeplitz, can be solved in  $O(n^2)$  flops. Durbin algorithm (Yule-Walker equation), Levinson algorithm (general  $\mathbf{b}$ ), Trench algorithm (inverse). These matrices occur in auto-regressive models and econometrics.

- Circulant systems: Toeplitz matrix with wraparound

$$C(\mathbf{z}) = \begin{pmatrix} z_0 & z_4 & z_3 & z_2 & z_1 \\ z_1 & z_0 & z_4 & z_3 & z_2 \\ z_2 & z_1 & z_0 & z_4 & z_3 \\ z_3 & z_2 & z_1 & z_0 & z_4 \\ z_4 & z_3 & z_2 & z_1 & z_0 \end{pmatrix},$$

FFT type algorithms: DCT (discrete cosine transform) and DST (discrete sine transform).

- Vandermonde matrix: such as in interpolation and approximation problems

$$\mathbf{V}(x_0, \dots, x_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{pmatrix}.$$

$\mathbf{V}\mathbf{x} = \mathbf{b}$  or  $\mathbf{V}^T\mathbf{x} = \mathbf{b}$  can be solved in  $O(n^2)$  flops.

- Cauchy-like matrices:

$$\Omega \mathbf{A} - \mathbf{A} \Lambda = \mathbf{R} \mathbf{S}^T,$$

where  $\Omega = \text{diag}(\omega_1, \dots, \omega_n)$  and  $\Lambda = (\lambda_1, \dots, \lambda_n)$ .  $O(n)$  flops for LU and QR.

- Structured-rank problems: semiseparable matrices (LU and QR takes  $O(n)$  flops), quasiseparable matrices, ...
- Fast multiple method (FMM) for kernel matrix.
- ...

Other computations such as matrix-vector multiplication with these “easy” matrices are typically fast too.

Bottom line: Don’t blindly use `solve()`.

## Linear algebra review: eigen-decomposition

Our last topic on numerical linear algebra is eigen-decomposition and singular value decomposition (SVD). We already saw the wide applications of QR decomposition in least squares problem and solving square and under-determined linear equations. Eigen-decomposition and SVD can be deemed as more thorough orthogonalization of a matrix. We start with a brief review of the related linear algebra.

- *Eigenvalues* are defined as roots of the characteristic equation  $\det(\lambda \mathbf{I}_n - \mathbf{A}) = 0$ .
- If  $\lambda$  is an eigenvalue of  $\mathbf{A}$ , then there exist non-zero  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{Ax} = \lambda\mathbf{x}$  and  $\mathbf{y}^T \mathbf{A} = \lambda \mathbf{y}^T$ .  $\mathbf{x}$  and  $\mathbf{y}$  are called the (column) *eigenvector* and *row eigenvector* of  $\mathbf{A}$  associated with the eigenvalue  $\lambda$ .
- $\mathbf{A}$  is singular if and only if it has at least one 0 eigenvalue.
- Eigenvectors associated with distinct eigenvalues are linearly independent.
- Eigenvalues of an upper or lower triangular matrix are its diagonal entries:  $\lambda_i = a_{ii}$ .
- Eigenvalues of an idempotent matrix are either 0 or 1.
- Eigenvalues of an orthogonal matrix have complex modulus 1.
- In most statistical applications, we deal with eigenvalues/eigenvectors of symmetric matrices. The eigenvalues and eigenvectors of a real *symmetric* matrix are real.
- Eigenvectors associated with distinct eigenvalues of a symmetry matrix are orthogonal.
- Eigen-decomposition of a symmetric matrix:  $\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^\top$ , where
  - $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$
  - columns of  $\mathbf{U}$  are the eigenvectors, which are (or can be chosen to be) mutually orthonormal. Thus  $\mathbf{U}$  is an orthogonal matrix.
- A real symmetric matrix is positive semidefinite (positive definite) if and only if all eigenvalues are nonnegative (positive).

- Spectral radius  $\rho(\mathbf{A}) = \max_i |\lambda_i|$ .
- $\mathbf{A} \in \mathbb{R}^{n \times n}$  a square matrix (not required to be symmetric), then  $\text{tr}(\mathbf{A}) = \sum_i \lambda_i$  and  $\det(\mathbf{A}) = \prod_i \lambda_i$ .

## Linear algebra review: SVD

- Singular value decomposition (SVD): For a rectangular matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , let  $p = \min\{m, n\}$ , then we have the SVD

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top,$$

where

- $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_m) \in \mathbb{R}^{m \times m}$  is orthogonal
- $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathbb{R}^{n \times n}$  is orthogonal
- $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_p) \in \mathbb{R}^{m \times n}$ ,  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0$ .

$\sigma_i$  are called the *singular values*,  $\mathbf{u}_i$  are the *left singular vectors*, and  $\mathbf{v}_i$  are the *right singular vectors*.

- Thin SVD. Assume  $m \geq n$ .  $\mathbf{A}$  can be factored as

$$\mathbf{A} = \mathbf{U}_n \Sigma_n \mathbf{V}^\top = \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^T,$$

where

- $\mathbf{U}_n \in \mathbb{R}^{m \times n}$ ,  $\mathbf{U}_n^\top \mathbf{U}_n = \mathbf{I}_n$
- $\mathbf{V} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{V}^\top \mathbf{V} = \mathbf{I}_n$
- $\Sigma_n = \text{diag}(\sigma_1, \dots, \sigma_n) \in \mathbb{R}^{n \times n}$ ,  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$

- Denote  $\sigma(\mathbf{A}) = (\sigma_1, \dots, \sigma_p)^T$ . Then

- $r = \text{rank}(\mathbf{A}) = \# \text{ nonzero singular values} = \|\sigma(\mathbf{A})\|_0$
- $\mathbf{A} = \mathbf{U}_r \Sigma_r \mathbf{V}_r^\top = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T$
- $\|\mathbf{A}\|_{\text{F}} = (\sum_{i=1}^p \sigma_i^2)^{1/2} = \|\sigma(\mathbf{A})\|_2$
- $\|\mathbf{A}\|_2 = \sigma_1 = \|\sigma(\mathbf{A})\|_\infty$

- Assume  $\text{rank}(\mathbf{A}) = r$  and partition  $\mathbf{U} = (\mathbf{U}_r, \tilde{\mathbf{U}}_r) \in \mathbb{R}^{m \times m}$  and  $\mathbf{V} = (\mathbf{V}_r, \tilde{\mathbf{V}}_r) \in \mathbb{R}^{n \times n}$ , then

- $\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{U}_r)$ ,  $\mathcal{N}(\mathbf{A}^T) = \mathcal{C}(\tilde{\mathbf{U}}_r)$
- $\mathcal{N}(\mathbf{A}) = \mathcal{C}(\tilde{\mathbf{V}}_r)$ ,  $\mathcal{C}(\mathbf{A}^T) = \mathcal{C}(\mathbf{V}_r)$
- $\mathbf{U}_r \mathbf{U}_r^T$  is the orthogonal projection onto  $\mathcal{C}(\mathbf{A})$
- $\tilde{\mathbf{U}}_r \tilde{\mathbf{U}}_r^T$  is the orthogonal projection onto  $\mathcal{N}(\mathbf{A}^T)$
- $\mathbf{V}_r \mathbf{V}_r^T$  is the orthogonal projection onto  $\mathcal{C}(\mathbf{A}^T)$
- $\tilde{\mathbf{V}}_r \tilde{\mathbf{V}}_r^T$  is the orthogonal projection onto  $\mathcal{N}(\mathbf{A})$

- Relation to eigen-decomposition. Using thin SVD,

$$\begin{aligned}\mathbf{A}^T \mathbf{A} &= \mathbf{V} \Sigma \mathbf{U}^T \mathbf{U} \Sigma \mathbf{V}^T = \mathbf{V} \Sigma^2 \mathbf{V}^T \\ \mathbf{A} \mathbf{A}^T &= \mathbf{U} \Sigma \mathbf{V}^T \mathbf{V} \Sigma \mathbf{U}^T = \mathbf{U} \Sigma^2 \mathbf{U}^T.\end{aligned}$$

In principle we can obtain singular triplets of  $\mathbf{A}$  by doing two eigen-decompositions.

- Another relation to eigen-decomposition. Using thin SVD,

$$\begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0}_{m \times m} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{V} & \mathbf{V} \\ \mathbf{U} & -\mathbf{U} \end{pmatrix} \begin{pmatrix} \Sigma & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & -\Sigma \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{V}^T & \mathbf{U}^T \\ \mathbf{V}^T & -\mathbf{U}^T \end{pmatrix}.$$

Hence any symmetric eigen-solver can produce the SVD of a matrix  $\mathbf{A}$  without forming  $\mathbf{A} \mathbf{A}^T$  or  $\mathbf{A}^T \mathbf{A}$ .

- Yet another relation to eigen-decomposition: If the eigendecomposition of a real symmetric matrix is  $\mathbf{A} = \mathbf{W} \Lambda \mathbf{W}^T = \mathbf{W} \text{diag}(\lambda_1, \dots, \lambda_n) \mathbf{W}^T$ , then

$$\mathbf{A} = \mathbf{W} \Lambda \mathbf{W}^T = \mathbf{W} \begin{pmatrix} |\lambda_1| & & \\ & \ddots & \\ & & |\lambda_n| \end{pmatrix} \begin{pmatrix} \text{sgn}(\lambda_1) & & \\ & \ddots & \\ & & \text{sgn}(\lambda_n) \end{pmatrix} \mathbf{W}^T$$

is the SVD of  $\mathbf{A}$ .

# 11 Lecture 11, Feb 9

## Announcements

- HW3 due next Tue Feb 16 @ 11:59PM.
- Quiz 2 today. In class, closed book.

## Last time

- Iterative methods for solving linear equations: Jacobi, Gauss-Seidel, successive over-relaxation, conjugate gradient (to do later).
- A catalog of easy linear systems.
- Linear algebra review: eigen-decomposition and SVD.

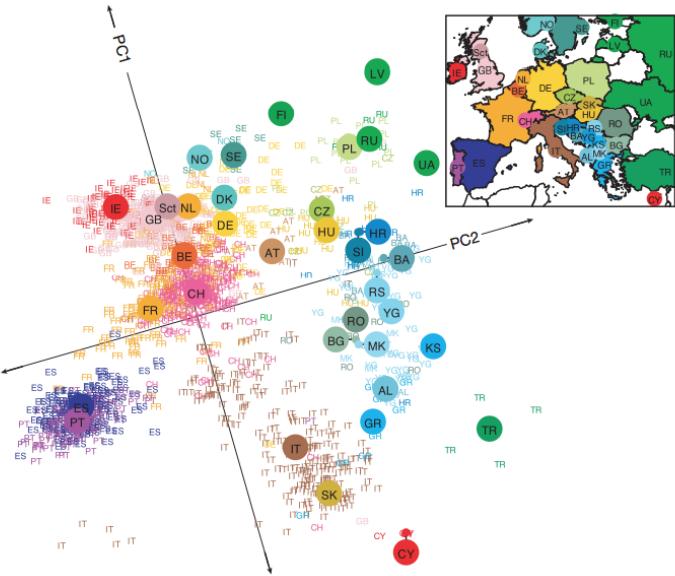
## Today

- Applications of eigen-decomposition and SVD.
- Algorithms for eigen-decomposition and SVD.
- Concluding remarks of numerical linear algebra.

## Applications of eigen-decomposition and SVD

- Principal components analysis (PCA).  $\mathbf{X} \in \mathbb{R}^{n \times p}$  is a centered data matrix. Perform SVD  $\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^\top$  or equivalently  $\mathbf{X}^\top\mathbf{X} = \mathbf{V}\Sigma^2\mathbf{V}^\top$ . The linear combinations  $\tilde{\mathbf{x}}_i = \mathbf{X}\mathbf{v}_i$  are the principal components (PC) and have variance  $\sigma_i^2$ . Usages:

1. Dimension reduction: reduce dimensionality  $p$  to  $q \ll p$ . Use top PCs  $\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_q$  in downstream analysis.
2. Use PCs to adjust for confounding – a serious issue in association studies in large data sets.



Above picture is from the article *Genes mirror geography within Europe* by Novembre et al. (2008) published in *Nature* <http://www.nature.com/nature/journal/v456/n7218/full/nature07331.html>. Use of PCA to adjust for confounding in modern genetic studies is proposed in the paper *Principal components analysis corrects for stratification in genome-wide association studies* by Price et al. (2006) published in *Nature Genetics* <http://www.nature.com/ng/journal/v38/n8/full/ng1847.html>. It has been cited 4,286 times as of Feb 9, 2016.

- Low rank approximation, e.g, image/data compression. Find a low rank approximation of data matrix  $\mathbf{X}$ .

Eckart-Young theorem:

$$\min_{\text{rank}(\mathbf{Y})=r} \|\mathbf{X} - \mathbf{Y}\|_F^2$$

is achieved by  $\mathbf{Y} = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$  with optimal value  $\sum_{i=1}^{r-1} \sigma_i^2$ , where  $(\sigma_i, \mathbf{u}_i, \mathbf{v}_i)$  are singular values and vectors of  $\mathbf{X}$ .

Gene Golub's  $2691 \times 598$  picture requires  $2691 \times 598 \times 6 = 9,655,308$  bytes (RGB 16 bit per channel). Rank 120 approximation requires  $120 \times (2691 + 598) \times 6 = 2,368,080$  bytes. Rank 50 approximation requires  $50 \times (2691 + 598) \times 6 = 986,700$  bytes. Rank 12 approximation requires  $12 \times (2691 + 598) \times 8 = 236,808$  bytes.

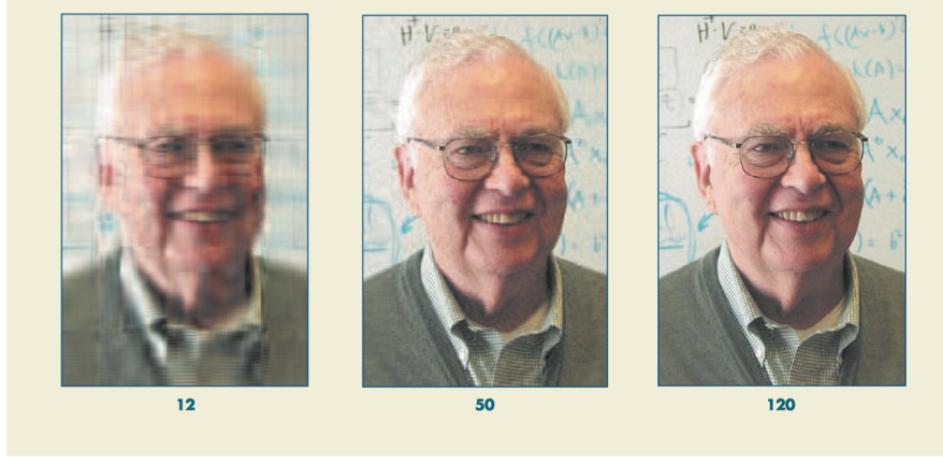


Figure 2. Rank 12, 50, and 120 approximations to a rank 598 color photo of Gene Golub.

- Moore-Penrose (MP) inverse: Using thin SVD,

$$\mathbf{A}^+ = \mathbf{V}\Sigma^+\mathbf{U}^\top,$$

where  $\Sigma^+ = \text{diag}(\sigma_1^{-1}, \dots, \sigma_r^{-1}, 0, \dots, 0)$ ,  $r = \text{rank}(\mathbf{A})$ . This is how the `ginv()` function is implemented in **MASS** package.

- Least squares. Given thin SVD  $X = \mathbf{U}\Sigma\mathbf{V}^T$ ,

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= (\mathbf{X}^T \mathbf{X})^- \mathbf{X}^T \mathbf{y} \\ &= (\mathbf{V} \Sigma^2 \mathbf{V}^T)^+ \mathbf{V} \Sigma \mathbf{U}^T \mathbf{y} \\ &= \mathbf{V} (\Sigma^2)^+ \mathbf{V}^T \mathbf{V} \Sigma \mathbf{U}^T \mathbf{y} \\ &= \mathbf{V}_r \Sigma_r^{-1} \mathbf{U}_r^T \mathbf{y} \\ &= \sum_{i=1}^r (\mathbf{u}_i^T \mathbf{y} / \sigma_i) \mathbf{v}_i\end{aligned}$$

and

$$\hat{\mathbf{y}} = \mathbf{X} \hat{\boldsymbol{\beta}} = \mathbf{U}_r \mathbf{U}_r^T \mathbf{y}.$$

In general, SVD is more expensive than the approaches (Cholesky, Sweep, QR) we learnt. In some applications, SVD is computed for other purposes then we get least squares solution for free.

- ridge regression (HW3,4), least squares over a sphere, ...
- See KL Chapters 8 and 9 and do HW4 for some more applications of SVD.

## Algorithm: One eigen-pair - power method

Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$  symmetric.

- *Power method* iterates according to

$$\begin{aligned}\mathbf{x}^{(t)} &\leftarrow \frac{1}{\|\mathbf{A}\mathbf{x}^{(t-1)}\|_2} \mathbf{A}\mathbf{x}^{(t-1)} \\ \lambda_1^{(t)} &\leftarrow \mathbf{x}^{(t)\top} \mathbf{A}\mathbf{x}^{(t)}\end{aligned}$$

from an initial guess  $\mathbf{x}^{(0)}$  of unit norm.

- Suppose we arrange  $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$  (the first inequality strict) with corresponding eigenvectors  $\mathbf{u}_i$ , and expand  $\mathbf{x}^{(0)} = c_1\mathbf{u}_1 + \dots + c_n\mathbf{u}_n$ , then

$$\begin{aligned}\mathbf{x}^{(t)} &= \frac{(\sum_i \lambda_i^t \mathbf{u}_i \mathbf{u}_i^\top) (\sum_i c_i \mathbf{u}_i)}{\|(\sum_i \lambda_i^t \mathbf{u}_i \mathbf{u}_i^\top) (\sum_i c_i \mathbf{u}_i)\|_2} \\ &= \frac{\sum_i c_i \lambda_i^t \mathbf{u}_i}{\|\sum_i c_i \lambda_i^t \mathbf{u}_i\|_2} \\ &= \frac{c_1 \mathbf{u}_1 + c_2 (\lambda_2/\lambda_1)^t \mathbf{u}_2 + \dots + c_n (\lambda_n/\lambda_1)^t \mathbf{u}_n}{\|c_1 \mathbf{u}_1 + c_2 (\lambda_2/\lambda_1)^t \mathbf{u}_2 + \dots + c_n (\lambda_n/\lambda_1)^t \mathbf{u}_n\|_2} \left(\frac{\lambda_1}{|\lambda_1|}\right)^t.\end{aligned}$$

Thus  $\mathbf{x}^{(t)} - \frac{c_1 \mathbf{u}_1}{\|c_1 \mathbf{u}_1\|_2} \left(\frac{\lambda_1}{|\lambda_1|}\right)^t \rightarrow 0$  as  $t \rightarrow \infty$ . The convergence rate is  $|\lambda_2|/|\lambda_1|$ .

- $\mathbf{x}^{(t)\top} \mathbf{A}\mathbf{x}^{(t)}$  converges to  $\lambda_1$ .

- *Inverse power method* for finding the eigenvalue of smallest absolute value: Substitute  $\mathbf{A}$  by  $\mathbf{A}^{-1}$  in the power method. (E.g., pre-compute LU or Cholesky of  $\mathbf{A}$ ).
- *Shifted inverse power*: Substitute  $(\mathbf{A} - \mu \mathbf{I})^{-1}$  in the power method. It converges to an eigenvalue close to the given  $\mu$ .
- Power method also applies to asymmetric  $\mathbf{A}$ , e.g., PageRank problem costs  $O(n)$  per iteration.

## Algorithm: Top $r$ eigen-pairs - orthogonal iteration

Generalization of power method to higher dimensional invariant subspace.

- *Orthogonal iteration:* Initialize  $\mathbf{Q}^{(0)} \in \mathbb{R}^{n \times r}$  with orthonormal columns. For  $t = 1, 2, \dots$ ,

$$\begin{aligned}\mathbf{Z}^{(t)} &\leftarrow \mathbf{A}\mathbf{Q}^{(t-1)} \quad (2n^2r \text{ flops}) \\ \mathbf{Q}^{(t)}\mathbf{R}^{(t)} &\leftarrow \mathbf{Z}^{(t)} \quad (\text{QR factorization})\end{aligned}$$

- $\mathbf{Z}^{(t)}$  converges to the eigenspace of the largest  $r$  eigenvalues if they are real and separated from remaining spectrum. The convergence rate is  $|\lambda_{r+1}|/|\lambda_r|$ .

### Algorithm: (Impractical) full eigen-decomposition - QR iteration

Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$  symmetric.

- take  $r = n$  in the orthogonal iteration. Then  $\mathbf{Q}^{(t)}$  converges to the eigenspace  $\mathbf{U}$  of  $\mathbf{A}$ . This implies that

$$\mathbf{T}^{(t)} := \mathbf{Q}^{(t) T} \mathbf{A} \mathbf{Q}^{(t)}$$

converges to a diagonal form  $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n)$ .

- Note how to compute  $\mathbf{T}^{(t)}$  from  $\mathbf{T}^{(t-1)}$

$$\begin{aligned}\mathbf{T}^{(t-1)} &= \mathbf{Q}^{(t-1) T} \mathbf{A} \mathbf{Q}^{(t-1)} = \mathbf{Q}^{(t-1) T} (\mathbf{A} \mathbf{Q}^{(t-1)}) = (\mathbf{Q}^{(t-1) T} \mathbf{Q}^{(t)}) \mathbf{R}^{(t)} \\ \mathbf{T}^{(t)} &= \mathbf{Q}^{(t) T} \mathbf{A} \mathbf{Q}^{(t)} = \mathbf{Q}^{(t) T} \mathbf{A} \mathbf{Q}^{(t-1)} \mathbf{Q}^{(t-1) T} \mathbf{Q}^{(t)} = \mathbf{R}^{(t)} (\mathbf{Q}^{(t-1) T} \mathbf{Q}^{(t)}).\end{aligned}$$

- *QR iteration:* Initialize  $\mathbf{U}^{(0)} \in \mathbb{R}^{n \times n}$  orthogonal and set  $\mathbf{T}^{(0)} = \mathbf{U}^{(0) T} \mathbf{A} \mathbf{U}^{(0)}$ . For  $t = 1, 2, \dots$

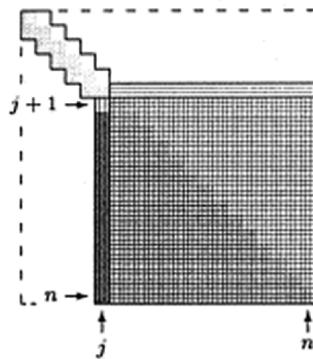
$$\begin{aligned}\mathbf{U}^{(t)} \mathbf{R}^{(t)} &\leftarrow \mathbf{T}^{(t-1)} \quad (\text{QR factorization}) \\ \mathbf{T}^{(t)} &\leftarrow \mathbf{R}^{(t)} \mathbf{U}^{(t)}\end{aligned}$$

- QR iteration is expensive:  $O(n^3)$  per iteration and linear convergence rate.

## QR algorithm for symmetric eigen-decomposition

Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$  symmetric.

- Reading: *The QR algorithm* by Beresford N. Parlett. <http://hua-zhou.github.io/teaching/biostatm280-2016winter/readings/qr.pdf>
- This is the algorithm implemented in LAPACK: `eigen()` in R, `eig()` in Matlab and Julia.
- Idea: Tri-diagonalization (by Householder) + QR iteration on the tri-diagonal system with implicit shift
  - Step 1: Householder tri-diagonalization:  $4n^3/3$  for eigenvalues only,  $8n^3/3$  for both eigenvalues and eigenvectors. (Why can't we apply Householder to make it diagonal directly?)



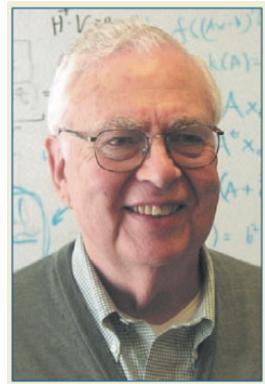
- Step 2: QR iteration on the tridiagonal matrix. Implicit shift accelerates convergence rate. On average 1.3-1.6 QR iteration per eigenvalue,  $\sim 20n$  flops per QR iteration. So total operation count is about  $30n^2$ . Eigenvectors need an extra of about  $6n^3$  flops.

	Eigenvalue	Eigenvector
Householder reduction	$4n^3/3$	$4n^3/3$
QR with implicit shift	$\sim 30n^2$	$\sim 6n^3$

- Message: Don't request eigenvectors unless necessary: set `only.values = TRUE` when calling `eigen()` in R.

- The *unsymmetric QR algorithm* obtains the real Schur decomposition of an asymmetric matrix  $\mathbf{A}$ .

## Algorithm for singular value decomposition (SVD)



Gene Golub's license plate, photographed by Professor P. M. Kroonenberg of Leiden University.

Assume  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and we seek the SVD  $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ .

- “Golub-Kahan-Reinsch” algorithm:
  - Stage 1: Transform  $\mathbf{A}$  to an upper bidiagonal form  $\mathbf{B}$  (by Householder).

$$\mathbf{U}_B^T \mathbf{A} \mathbf{V}_B = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} d_1 & f_1 & & \cdots & 0 \\ 0 & d_2 & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & f_{n-1} \\ 0 & \cdots & 0 & & d_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{U_1} \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{V_1}$$

$$\begin{bmatrix} \times & \times & 0 & 0 \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{U_2} \begin{bmatrix} \times & \times & 0 & 0 \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{V_2}$$

$$\begin{bmatrix} \times & \times & 0 & 0 \\ 0 & \times & \times & 0 \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{U_3} \begin{bmatrix} \times & \times & 0 & 0 \\ 0 & \times & \times & 0 \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ 0 & 0 & 0 & \times \end{bmatrix} \xrightarrow{U_4} \begin{bmatrix} \times & \times & 0 & 0 \\ 0 & \times & \times & 0 \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

– Stage 2: Apply implicit-shift QR step to the tridiagonal matrix  $\mathbf{B}^\top \mathbf{B}$  implicitly.

- See Golub and Van Loan (1996, Section 8.6) for more details.
- $4m^2n + 8mn^2 + 9n^3$  flops for a tall ( $m \geq n$ ) matrix.
- `svd()` in R and Matlab: wrapper of the `_GESVD` subroutine in LAPACK.

## 12 Lecture 12, Feb 11

### Announcements

- HW3 due Tue Feb 16 @ 11:59PM.
- HW4 posted. Due Tue Feb 23 @ 11:59PM. [http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostat\\_m280\\_2016\\_hw4.pdf](http://hua-zhou.github.io/teaching/biostatm280-2016winter/biostat_m280_2016_hw4.pdf)
- Quiz 2 returned ( $8.0 \pm 2.1$ ). Q1 1pt, Q2 1pt, Q3 1pt, Q4 1pt, Q5 2pt, Q6 4pt.
- HW2 graded. Feedback:
  - Cheating is not tolerated. From syllabus, “*... giving or receiving answers or code to or from another student is cheating ...*” First time, homework score is set to zero. Second time, disciplinary action.
  - Check the solution sketch to make sure you learn something. <http://hua-zhou.github.io/teaching/biostatm280-2016winter/hw02sol.html>

### Last time

- Applications of eigen-decomposition and SVD.
- Power algorithm and *QR iteration* for top eigen-pairs.
- *QR algorithm* for symmetric eigen-decomposition.
- Golub-Kahan-Reinsch algorithm SVD.

### Today

- Iterative methods for eigen-problem and SVD.
- Jacobi algorithm for eigen-decomposition (parallel computing).
- Misc. topics: generalized eigen-problem, variants of least squares.
- Concluding remarks of numerical linear algebra.
- Optimization: introduction.

## Lanczos/Arnoldi iterative method for top eigen-pairs

- Motivation
  - Consider the Google PageRank problem. We want to find the top left eigenvector of the transition matrix  $\mathbf{P}$ . Direct methods such as (unsymmetric) QR or SVD takes forever. Iterative methods such as power method is feasible. However power method may take a huge number of iterations.
  - Consider adjusting for confounding by PCA in modern GWAS (genome-wide association studies). We want to find the top singular values/vectors of a genotype matrix  $\mathbf{X} \in \mathbb{R}^{n \times p}$ , where  $n \sim 10^3$  and  $p \sim 10^6$ .
- *Krylov subspace methods* are the state-of-art iterative method for obtaining the top eigen-values/vectors or singular values/vectors of large *sparse* or *structured* matrices.
- Lanczos method: top eigen-pairs of a large *symmetric* matrix.
- Arnoldi method: top eigen-pairs of a large *asymmetric* matrix.
- Both methods are also adapted to obtain top singular values/vectors of large sparse or structured matrices.
- We will give an overview of these methods together with the conjugate gradient method for solving large linear system.
- `eigs()` and `svds()` in Matlab and Julia are wrappers of the ARPACK package, which implements Lanczos and Arnoldi methods. In R, try to construct sparse matrix using the `Matrix` package (by Doug Bates) and try the `irlba` package.  
<http://cran.r-project.org/web/packages/irlba/index.html>

## Jacobi method for symmetric eigen-decomposition (KL 8.2)

Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is symmetric and we seek the eigen-decomposition  $\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^T$ .

- Idea: Systematically reduce off-diagonal entries

$$\text{off}(\mathbf{A}) = \sum_i \sum_{j \neq i} a_{ij}^2$$

by Jacobi rotations.

- Jacobi/Givens rotations:

$$\mathbf{J}(p, q, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$\mathbf{J}(p, q, \theta)$  is orthogonal.

- Consider  $\mathbf{B} = \mathbf{J}^\top \mathbf{A} \mathbf{J}$ .  $\mathbf{B}$  preserves the symmetry and eigenvalues of  $\mathbf{A}$ .

Taking

$$\begin{cases} \tan(2\theta) = 2a_{pq}/(a_{qq} - a_{pp}) & \text{if } a_{pp} \neq a_{qq} \\ \theta = \pi/4 & \text{if } a_{pp} = a_{qq} \end{cases}$$

forces  $b_{pq} = 0$ .

- Since orthogonal transform preserves Frobenius norm, we have

$$b_{pp}^2 + b_{qq}^2 = a_{pp}^2 + a_{qq}^2 + 2a_{pq}^2.$$

(Just check the 2-by-2 block)

- Since  $\|\mathbf{A}\|_F = \|\mathbf{B}\|_F$ , this implies that the off-diagonal part

$$\text{off}(\mathbf{B}) = \text{off}(\mathbf{A}) - 2a_{pq}^2$$

is decreased whenever  $a_{pq} \neq 0$ .

- One Jacobi rotation costs  $O(n)$  flops.
- *Classical Jacobi*: search for the largest  $|a_{ij}|$  at each iteration.
- $\text{off}(\mathbf{A}) \leq n(n-1)a_{ij}^2$  and  $\text{off}(\mathbf{B}) = \text{off}(\mathbf{A}) - 2a_{ij}^2$  together implies

$$\text{off}(\mathbf{B}) \leq \left(1 - \frac{2}{n(n-1)}\right) \text{off}(\mathbf{A}).$$

Thus Jacobi method converges in  $O(n^2)$  iterations.

- In practice, cyclic-by-row implementation, to avoid the costly  $O(n^2)$  search in the classical Jacobi.
- Jacobi method attracts a lot recent attention because of its rich inherent parallelism.
- *Parallel Jacobi*: “merry-go-round” to generate parallel ordering.

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lelism of the latter algorithm. To illustrate this, suppose  $n = 4$  and group the six subproblems into three *rotation sets* as follows:

$$\begin{aligned} \text{rot.set}(1) &= \{(1,2), (3,4)\} \\ \text{rot.set}(2) &= \{(1,3), (2,4)\} \\ \text{rot.set}(3) &= \{(1,4), (2,3)\} \end{aligned}$$

Note that all the rotations within each of the three rotation sets are “non-conflicting.” That is, subproblems (1,2) and (3,4) can be carried out in parallel. Likewise the (1,3) and (2,4) subproblems can be executed in parallel as can subproblems (1,4) and (2,3). In general, we say that

$$(i_1, j_1), (i_2, j_2), \dots, (i_N, j_N) \quad N = (n-1)n/2$$

is a *parallel ordering* of the set  $\{(i,j) \mid 1 \leq i < j \leq n\}$  if for  $s = 1:n-1$  the rotation set  $\text{rot.set}(s) = \{(i_r, j_r) : r = 1 + n(s-1)/2:ns/2\}$  consists of nonconflicting rotations. This requires  $n$  to be even, which we assume of throughout this section. (The odd  $n$  case can be handled by bordering  $A$  with a row and column of zeros and being careful when solving the subproblems that involve these augmented zeros.)

A good way to generate a parallel ordering is to visualize a chess tournament with  $n$  players in which everybody must play everybody else exactly once. In the  $n = 8$  case this entails 7 “rounds.” During round one we have the following four games:

1	3	5	7
2	4	6	8

$$\text{rot.set}(1) = \{(1,2), (3,4), (5,6), (7,8)\}$$

i.e., 1 plays 2, 3 plays 4, etc. To set up rounds 2 through 7, player 1 stays put and players 2 through 8 embark on a merry-go-round:

1	2	3	5
4	6	8	7

$$\text{rot.set}(2) = \{(1,4), (2,6), (3,8), (5,7)\}$$

1	4	2	3
6	8	7	5

$$\text{rot.set}(3) = \{(1,6), (4,8), (2,7), (3,5)\}$$

1	6	4	2
8	7	5	3

$$\text{rot.set}(4) = \{(1,8), (6,7), (4,5), (2,3)\}$$

1	8	6	4
7	5	3	2

$$\text{rot.set}(5) = \{(1,7), (5,8), (3,6), (2,4)\}$$

1	7	8	6
5	3	2	4

$$\text{rot.set}(6) = \{(1,5), (3,7), (2,8), (4,6)\}$$

#### 8.4.4 JACOBI METHODS

$$\begin{array}{|c|c|c|c|} \hline 1 & 5 & 7 & 8 \\ \hline 3 & 2 & 4 & 6 \\ \hline \end{array} \quad \text{rot.set}(7) = \{(1,3), (2,5), (4,7), (6,8)\}$$

We can encode these operations in a pair of integer vectors  $\text{top}(1:n/2)$  and  $\text{bot}(1:n/2)$ . During a given round  $\text{top}(k)$  plays  $\text{bot}(k)$ ,  $k = 1:n/2$ . The pairings for the next round is obtained by updating  $\text{top}$  and  $\text{bot}$  as follows:

```
function: [new.top,new.bot] = music(top,bot,n)
m = n/2
for k = 1:m
    if k = 1
        new.top(1) = 1
    else if k = 2
        new.top(k) = bot(1)
    elseif k > 2
        new.top(k) = top(k-1)
    end
    if k = m
        new.bot(k) = top(k)
    else
        new.bot(k) = bot(k+1)
    end
end
```

Using `music` we obtain the following parallel order Jacobi procedure.

**Algorithm 8.4.4 (Parallel Order Jacobi )** Given a symmetric  $A \in \mathbb{R}^{n \times n}$  and a tolerance  $\text{tol} > 0$ , this algorithm overwrites  $A$  with  $V^T A V$  where  $V$  is orthogonal and  $\text{off}(V^T A V) \leq \text{tol} \|A\|_F$ . It is assumed that  $n$  is even.

```
V = I_n
eps = tol \|A\|_F
top = 1:2:n; bot = 2:2:n
while off(A) > eps
    for set = 1:n-1
        for k = 1:n/2
            p = min(top(k),bot(k))
            q = max(top(k),bot(k))
            (c, s) = sym.schur2(A,p,q)
            A = J(p,q,theta)^T A J(p,q,theta)
            V = V J(p,q,theta)
        end
        [top,bot] = music(top,bot,n)
    end
end
```

## Generalized eigen-problem

- Generalized eigen-problem:  $\mathbf{Ax} = \lambda \mathbf{Bx}$ , where  $\mathbf{A}$  psd and  $\mathbf{B}$  pd.
- Applications: canonical correlation analysis (CCA), partial least squares (PLS), sliced inverse regression (SIR).

- Method 1:  $\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ . Non-symmetric eigen-problem  $\odot$ .
- Method 2: Cholesky  $\mathbf{B} = \mathbf{L}\mathbf{L}^\top$ . Then  $\mathbf{L}^{-1}\mathbf{A}\mathbf{L}^{-T}\mathbf{y} = \lambda\mathbf{y}$  where  $\mathbf{y} = \mathbf{L}^\top\mathbf{x}$ .
- Method 3 (most numerically stable,  $\mathbf{B}$  can be rank deficient): QZ algorithm.
- `eig()` and `qz()` in Matlab and Julia implement QZ. No native function in R? Check the `geneig` package. <https://cran.r-project.org/web/packages/geigen/geigen.pdf>

## Generalized singular value decomposition

- $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{B} \in \mathbb{R}^{p \times n}$ . Then there exists orthogonal  $\mathbf{U} \in \mathbb{R}^{m \times m}$  and  $\mathbf{V} \in \mathbb{R}^{p \times p}$  and an invertible  $\mathbf{X} \in \mathbb{R}^{n \times n}$  such that

$$\begin{aligned}\mathbf{U}^T \mathbf{A} \mathbf{X} &= \mathbf{C} = \text{diag}(c_1, \dots, c_n), \quad c_i \geq 0 \\ \mathbf{V}^T \mathbf{B} \mathbf{X} &= \mathbf{S} = \text{diag}(s_1, \dots, s_q), \quad s_i \geq 0,\end{aligned}$$

where  $q = \min\{p, n\}$ .

- Applications: quadratically inequality-constrained least squares problem (LSQI).
- `gsvd()` in Matlab implements generalized SVD. No native function in R?

## In the zoo of least squares (self-study)

### Weighted least squares

- In weighted least squares, we minimize  $\sum_{i=1}^n w_i(y_i - \mathbf{x}_i^T \boldsymbol{\beta})^2$ , where  $w_i > 0$  are observation weights.
- Let  $\mathbf{W} = \text{diag}(w_1, \dots, w_n)$ . Then the criterion is  $\|\mathbf{W}^{1/2}\mathbf{y} - \mathbf{W}^{1/2}\mathbf{X}\boldsymbol{\beta}\|_2^2$ , which can be solved by standard methods for least squares with  $\tilde{\mathbf{y}} = \mathbf{W}^{1/2}\mathbf{y}$  and  $\tilde{\mathbf{X}} = \mathbf{W}^{1/2}\mathbf{X}$ .

### General least squares

- In Aitken model:  $E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$ ,  $\text{Cov}(\mathbf{y}) = \sigma^2\mathbf{V}$ , where  $\mathbf{V}$  is a positive semidefinite matrix. We minimize the generalized least squares criterion

$$(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T \mathbf{M}^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}),$$

where  $\mathbf{M} \in \mathbb{R}^{n \times n}$  is some positive semidefinite matrix, e.g.,  $\mathbf{M} = \mathbf{V}$  for non-singular  $\mathbf{V}$  or  $\mathbf{M} = \mathbf{V} + \mathbf{X}\mathbf{X}^T$  for singular  $\mathbf{V}$ .

- Let  $\mathbf{M} = \mathbf{B}\mathbf{B}^T$  for some  $\mathbf{B} \in \mathbb{R}^{n \times n}$  (e.g., the Cholesky factor). One approach is to minimize

$$\|\mathbf{B}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\|_2^2.$$

*Unfortunately*, when  $\mathbf{B}$  is poorly conditioned (or even not invertible), the procedure produces a poor solution.

- Paige's method. The generalized least squares problem is equivalent to

$$\begin{aligned} & \text{minimize} && \mathbf{v}^T \mathbf{v} \\ & \text{subject to} && \mathbf{X}\boldsymbol{\beta} + \mathbf{B}\mathbf{v} = \mathbf{y}. \end{aligned}$$

To solve this problem, first compute the QR of  $\mathbf{X}$

$$\mathbf{X} = (\mathbf{Q}_1, \mathbf{Q}_2) \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix}.$$

Compute another QR for the (flat) matrix  $\mathbf{Q}_2^T \mathbf{B}$  such that

$$\mathbf{Q}_2^T \mathbf{B} = (\mathbf{0}, \mathbf{S}) \begin{pmatrix} \mathbf{Z}_1^T \\ \mathbf{Z}_2^T \end{pmatrix},$$

where  $\mathbf{S}$  is upper triangular and  $(\mathbf{Z}_1, \mathbf{Z}_2) \in \mathbb{R}^{n \times n}$  is orthogonal. Then the constraint becomes

$$\begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \mathbf{Q}_1^T \mathbf{B} \mathbf{Z}_1 & \mathbf{Q}_1^T \mathbf{B} \mathbf{Z}_2 \\ \mathbf{0} & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{Z}_1^T \mathbf{v} \\ \mathbf{Z}_2^T \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_1^T \mathbf{y} \\ \mathbf{Q}_2^T \mathbf{y} \end{pmatrix}.$$

From the bottom half we can solve for  $\mathbf{v}$  from the equation (how?)

$$\mathbf{S} \mathbf{Z}_2^T \mathbf{v} = \mathbf{Q}_2^T \mathbf{y}.$$

Then we solve for  $\boldsymbol{\beta}$  from the equation

$$\mathbf{R}_1 \boldsymbol{\beta} = \mathbf{Q}_1^T \mathbf{y} - (\mathbf{Q}_1^T \mathbf{B} \mathbf{Z}_1 \mathbf{Z}_1^T + \mathbf{Q}_1^T \mathbf{B} \mathbf{Z}_2 \mathbf{Z}_2^T) \mathbf{v} = \mathbf{Q}_1^T \mathbf{y} - \mathbf{Q}_1^T \mathbf{B} \mathbf{Z}_2 (\mathbf{Z}_2^T \mathbf{v}).$$

- Paige's method also works for singular  $\mathbf{X}$  and  $\mathbf{B}$  (using QR with column pivoting).
- MATLAB's `lscov()` function implements Paige's method for singular covariance  $\mathbf{V}$ . No R implementation (?)

## Ridge regression

- In ridge regression, we minimize

$$\|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda\|\beta\|_2^2,$$

where  $\lambda$  is a tuning parameter.

- Ridge regression by augmented linear regression. Ridge regression problem is equivalent to

$$\left\| \begin{pmatrix} \mathbf{y} \\ \mathbf{0}_p \end{pmatrix} - \begin{pmatrix} \mathbf{X} \\ \sqrt{\lambda} \mathbf{I}_p \end{pmatrix} \beta \right\|_2^2.$$

Therefore any methods for linear regression can be applied.

- Ridge regression by method of normal equation. The normal equation for the ridge problem is

$$(\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I}_p) \beta = \mathbf{X}^T \mathbf{y}.$$

Therefore Cholesky or sweep can be used.

- Ridge regression by SVD. If we obtain the (thin) SVD of  $\mathbf{X}$

$$\mathbf{X} = \mathbf{U} \Sigma_{p \times p} \mathbf{V}^T.$$

Then the normal equation reads

$$(\Sigma^2 + \lambda \mathbf{I}_p) \mathbf{V}^T \beta = \Sigma \mathbf{U}^T \mathbf{y}$$

and we get

$$\hat{\beta}(\lambda) = \sum_{i=1}^p \frac{\sigma_i \mathbf{u}_i^T \mathbf{y}}{\sigma_i^2 + \lambda} \mathbf{v}_i = \sum_{i=1}^r \frac{\sigma_i \mathbf{u}_i^T \mathbf{y}}{\sigma_i^2 + \lambda} \mathbf{v}_i, \quad r = \text{rank}(\mathbf{X}).$$

It is clear that

$$\lim_{\lambda \rightarrow 0} \hat{\beta}(\lambda) = \hat{\beta}_{\text{OLS}}$$

and  $\|\hat{\beta}(\lambda)\|_2$  is monotone decreasing as  $\lambda$  increases.

- Only one SVD is needed for all  $\lambda$  (!), in contrast to the method of augmented linear regression, Cholesky, or sweep.

## Least squares over a sphere

- Ridge regression “shrinks” the solution via penalty. Alternatively we can simply fit a least squares problem subject to the constraint that the solution lives in a sphere

$$\begin{aligned} & \text{minimize} && \|\mathbf{y} - \mathbf{X}\beta\|_2^2 \\ & \text{subject to} && \|\beta\|_2 \leq \alpha. \end{aligned}$$

- Suppose we obtain the (thin) SVD  $\mathbf{X} = \mathbf{U}\Sigma_{p \times p}\mathbf{V}^T$ . If the ordinary least squares solution

$$\hat{\beta}_{\text{OLS}} = \sum_{i=1}^r \frac{\mathbf{u}_i^T \mathbf{y}}{\sigma_i} \mathbf{v}_i$$

has  $\ell_2$  norm less than  $\alpha$ , then we are done. If not, we use the method of Lagrangian multipliers

$$\psi(\beta, \lambda) = \frac{1}{2}\|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \frac{\lambda}{2}(\|\beta\|_2^2 - \alpha^2).$$

Setting the gradient to 0, we have the shifted normal equation

$$(\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I}_p) \beta = \mathbf{X}^T \mathbf{y},$$

which has solution

$$\hat{\beta}(\lambda) = \sum_{i=1}^r \frac{\sigma_i \mathbf{u}_i^T \mathbf{y}}{\sigma_i^2 + \lambda} \mathbf{v}_i.$$

We need to choose the  $\lambda$  such that  $\|\hat{\beta}(\lambda)\|_2 = \alpha$ . That is we need to find the (unique) zero of the function

$$f(\lambda) = \|\hat{\beta}(\lambda)\|_2^2 - \alpha^2 = \sum_{i=1}^r \left( \frac{\sigma_i \mathbf{u}_i^T \mathbf{y}}{\sigma_i^2 + \lambda} \right)^2 - \alpha^2.$$

This is easily achieved by Newton's or other methods.

## Least squares with equality constraints

- In many applications, there are *a priori* constraints on the regression parameters. Let's consider how to solve linear regression with equality constraints (LSE)

$$\begin{aligned} & \text{minimize} && \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 \\ & \text{subject to} && \mathbf{B}\boldsymbol{\beta} = \mathbf{d}. \end{aligned}$$

- LSE by QR. First compute QR of  $\mathbf{B}^T \in \mathbb{R}^{p \times m}$

$$\mathbf{B}^T = \mathbf{Q} \begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix}$$

and set

$$\mathbf{X}\mathbf{Q} = (\mathbf{X}_1, \mathbf{X}_2) \quad \text{and} \quad \mathbf{Q}^T \boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}.$$

Then the original minimization problem becomes

$$\begin{aligned} & \text{minimize} && \|\mathbf{y} - \mathbf{X}_1\boldsymbol{\beta}_1 - \mathbf{X}_2\boldsymbol{\beta}_2\|_2^2 \\ & \text{subject to} && \mathbf{R}^T \boldsymbol{\beta}_1 = \mathbf{d}. \end{aligned}$$

Now  $\boldsymbol{\beta}_1$  is determined from the constraint  $\mathbf{R}^T \boldsymbol{\beta}_1 = \mathbf{d}$  and  $\boldsymbol{\beta}_2$  is solved from the unconstrained least squares problem

$$\text{minimize } \|(\mathbf{y} - \mathbf{X}_1\boldsymbol{\beta}_1) - \mathbf{X}_2\boldsymbol{\beta}_2\|_2^2.$$

Finally we recover the solution from

$$\boldsymbol{\beta} = \mathbf{Q} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}.$$

- LSE by augmented system. Define the Lagrangian function

$$\phi(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \frac{1}{2} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 - \boldsymbol{\lambda}^T (\mathbf{B}\boldsymbol{\beta} - \mathbf{d}).$$

Setting gradient to zero yields

$$\begin{aligned} \mathbf{X}^T \mathbf{X} \boldsymbol{\beta} - \mathbf{B}^T \boldsymbol{\lambda} &= \mathbf{X}^T \mathbf{y} \\ \mathbf{B} \boldsymbol{\beta} &= \mathbf{d}, \end{aligned}$$

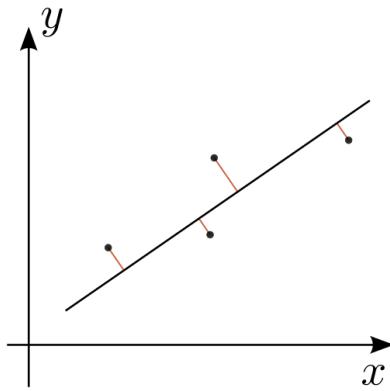
suggesting the augmented system

$$\begin{pmatrix} \mathbf{X}^T \mathbf{X} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ -\boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{X}^T \mathbf{y} \\ \mathbf{d} \end{pmatrix}.$$

This linear system is non-singular when  $\mathbf{X}$  and  $\mathbf{B}$  have full rank and can be solved by Cholesky, sweep, and so on.

- LSE by generalized SVD.

### Total least squares (TLS)



TLS considers the case both predictors and observations are subject to errors. It is solved by SVD. Read KL 9.3.6 if interested.

### Tikhonov regularization

Tikhonov regularization is an extension of the ridge regression

$$\|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda \|\mathbf{B}\boldsymbol{\beta}\|_2^2,$$

where  $\mathbf{B} \in \mathbb{R}^{m \times p}$  is a fixed regularization matrix and  $\lambda$  is a tuning parameter. It is solved by the generalized singular value decomposition (GSVD).

### Least squares with quadratic inequality constraint (LSQI)

Least squares with quadratic inequality constraint (LSQI) minimizes the least squares criterion over a hyper-ellipsoid:

$$\begin{aligned} &\text{minimize} && \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 \\ &\text{subject to} && \|\mathbf{B}\boldsymbol{\beta}\|_2 \leq \alpha, \end{aligned}$$

where  $\mathbf{B} \in \mathbb{R}^{m \times p}$  is a fixed regularization matrix. It is solved by the generalized singular value decomposition (GSVD). See Golub and Van Loan (1996, Section 2.1.1).

## Concluding remarks on numerical linear algebra

- Numerical linear algebra forms the building blocks of most computation we do.
- Be flop and memory aware.

The form of a mathematical expression and the way the expression should be evaluated in actual practice may be quite different.

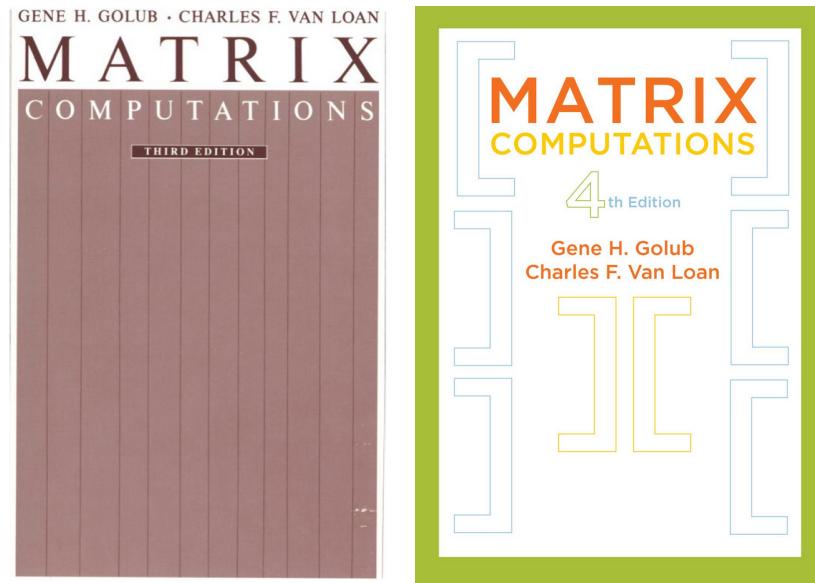
- Be alert to problem structure and make educated choice of software/algorithm.

The structure should be exploited whenever solving a problem.

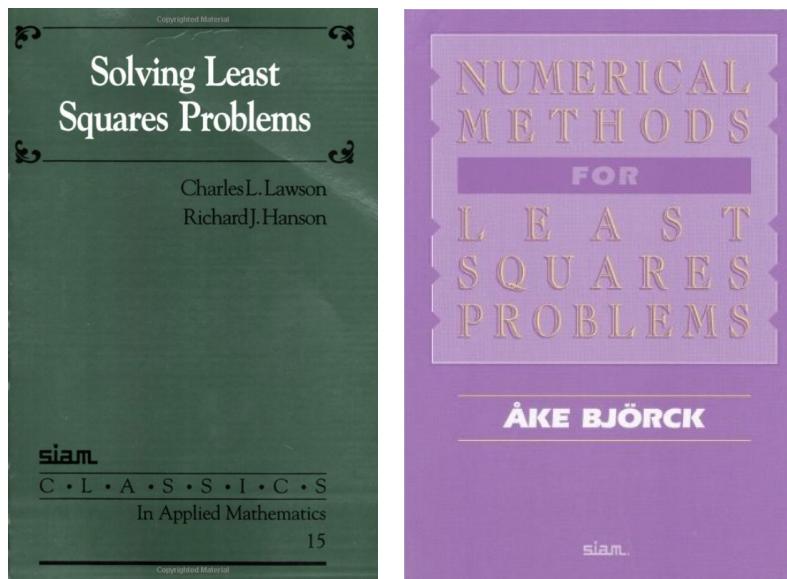
- Do not write your own matrix computation routines unless for good reason.  
Utilize BLAS and LAPACK as much as possible!
- In contrast, for optimization, often we need to devise problem specific optimization routines, or even “mix and match” them.

## Reference books on numerical linear algebra

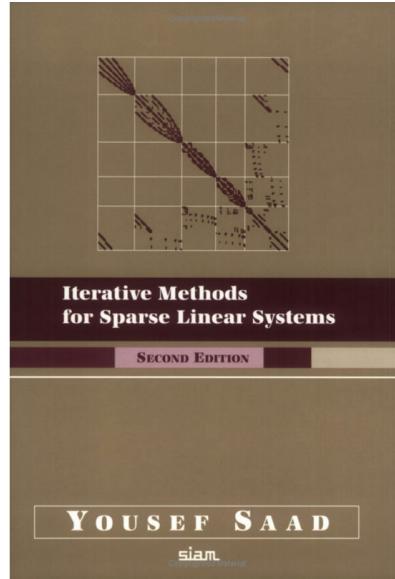
- Golub and Van Loan (1996): “Bible” in numerical linear algebra. Good for reference.



- Lawson and Hanson (1987) and Björck (1996): classical monographs on solving least squares problems.



- Saad (2003): standard reference for iterative methods



## MLE (as a motivation for optimization)

A great idea due to Fisher in 20s, and made rigorous by Cramer and others in 40s.

- Notations:

- Density:  $f(\mathbf{x}|\boldsymbol{\theta})$ , where  $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^p$
- Log-likelihood function:  $L(\boldsymbol{\theta}) = \ln f(\mathbf{x}|\boldsymbol{\theta})$
- (Column) Gradient/score vector:  $\nabla L(\boldsymbol{\theta}) \in \mathbb{R}^{p \times 1}$
- Differential:  $dL(\boldsymbol{\theta}) = [\nabla L(\boldsymbol{\theta})]^\top \in \mathbb{R}^{1 \times p}$
- Hessian:  $d^2L(\boldsymbol{\theta}) = \nabla^2 L(\boldsymbol{\theta})$
- Observed information matrix:  $-d^2L(\boldsymbol{\theta})$
- Expected (Fisher) information matrix:  $\mathbf{I}(\boldsymbol{\theta}) = \mathbf{E}_{\boldsymbol{\theta}}[-d^2L(\boldsymbol{\theta})]$
- Given iid observations  $\mathbf{x}_1, \dots, \mathbf{x}_n$  from  $f(\cdot|\boldsymbol{\theta})$ ,

$$L_n(\boldsymbol{\theta}) = \sum_{i=1}^n \ln f(\mathbf{x}_i|\boldsymbol{\theta})$$

- Maximum likelihood estimator (MLE):

$$\hat{\boldsymbol{\theta}}_{\text{MLE}} = \operatorname{argmax}_{\boldsymbol{\theta}} L_n(\boldsymbol{\theta})$$

- Consistency of MLE

- Under the true parameter value  $\boldsymbol{\theta}_0$ ,

$$M_n(\boldsymbol{\theta}) = \frac{1}{n} [L_n(\boldsymbol{\theta}) - L_n(\boldsymbol{\theta}_0)] \\ \rightarrow M(\boldsymbol{\theta}) = \mathbf{E}_{\boldsymbol{\theta}_0} [\ln f(\mathbf{X}|\boldsymbol{\theta}) - \ln f(\mathbf{X}|\boldsymbol{\theta}_0)]$$

for all  $\boldsymbol{\theta}$  almost surely.

- Note that  $M(\boldsymbol{\theta})$  is the negative Kullback-Leibler divergence between distribution at  $\boldsymbol{\theta}$  and distribution at  $\boldsymbol{\theta}_0$ .

Assuming *identifiability*, by the *information inequality*,  $M(\boldsymbol{\theta})$  achieves maximum uniquely at  $\boldsymbol{\theta}_0$ . We hope the MLE

$$\hat{\boldsymbol{\theta}}_n = \operatorname{argmax}_{\boldsymbol{\theta}} M_n(\boldsymbol{\theta})$$

converges to

$$\boldsymbol{\theta}_0 = \operatorname{argmax}_{\boldsymbol{\theta}} M(\boldsymbol{\theta}).$$

- Need *uniform convergence* of  $M_n(\boldsymbol{\theta})$  to  $M(\boldsymbol{\theta})$ , i.e.,

$$\sup_{\Theta} |M_n(\boldsymbol{\theta}) - M(\boldsymbol{\theta})|$$

converges to 0 in probability. A set of sufficient conditions for uniform convergence:

- \* compactness of the parameter space  $\Theta$
- \* continuity of  $M(\boldsymbol{\theta})$  in  $\boldsymbol{\theta}$  for any  $\mathbf{x}$
- \*  $M(\boldsymbol{\theta})$  dominated by an integrable function
- Example of non-uniform convergence:  $f_n(x) = 1_{\{n,n+1\}}$  (or a triangle on  $[n, n+1]$  if we want  $f_n$  to be continuous).  $f_n \rightarrow f \equiv 0$  pointwise but not uniformly.

- Asymptotic normality of MLE

- Assume  $\hat{\boldsymbol{\theta}}_n$  is consistent for  $\boldsymbol{\theta}_0$ .

- Taylor expansion on  $\mathbf{0}_p = \frac{1}{n} \nabla L_n(\hat{\boldsymbol{\theta}}_n)$  gives

$$\begin{aligned}\mathbf{0}_p &= \frac{1}{n} \nabla L_n(\boldsymbol{\theta}_0) + \left[ \frac{1}{n} d^2 L_n(\boldsymbol{\theta}_0) \right] (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \\ &\quad + \frac{1}{2} [\mathbf{I}_p \otimes (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0)^\top] \left[ \frac{1}{n} D d^2 L_n(\tilde{\boldsymbol{\theta}}_n) \right] (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0),\end{aligned}$$

where  $\tilde{\boldsymbol{\theta}}_n$  is somewhere between  $\boldsymbol{\theta}_0$  and  $\hat{\boldsymbol{\theta}}_n$ . If  $\frac{1}{n} D d^2 L_n(\tilde{\boldsymbol{\theta}}_n) = O_p(1)$  (bounded in probability), then the third term is  $o_p(1)(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0)$  and

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) = \left[ -\frac{1}{n} d^2 L_n(\boldsymbol{\theta}_0) + o_p(1) \right]^{-1} \frac{\sqrt{n}}{n} \nabla L_n(\boldsymbol{\theta}_0).$$

Now

- \*  $-\frac{1}{n} d^2 L_n(\boldsymbol{\theta}_0) + o_p(1) \rightarrow \mathbf{E}_{\boldsymbol{\theta}_0}[-d^2 L(\boldsymbol{\theta}_0)] = \mathbf{I}(\boldsymbol{\theta}_0)$  almost surely by the law of large number.
- \*  $n^{-1/2} \nabla L_n(\boldsymbol{\theta}_0)$  converges to a multivariate normal with mean  $\mathbf{0}_p$  and variance

$$\mathbf{E}_{\boldsymbol{\theta}_0}[\nabla L(\boldsymbol{\theta}_0) dL(\boldsymbol{\theta}_0)],$$

which equals  $\mathbf{I}(\boldsymbol{\theta}_0)$  under exchangeability of integral and differentiation.

Then by the Slutsky theorem,

$$\begin{aligned}&\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \\ &\rightarrow N_p \left( \mathbf{0}_p, \mathbf{I}^{-1}(\boldsymbol{\theta}_0) \cdot \mathbf{E}_{\boldsymbol{\theta}_0}[\nabla \ln f(\boldsymbol{\theta}_0) d \ln f(\boldsymbol{\theta}_0)] \cdot \mathbf{I}^{-1}(\boldsymbol{\theta}_0) \right) \\ &= N_p(\mathbf{0}_p, \mathbf{I}^{-1}(\boldsymbol{\theta}_0))\end{aligned}$$

in distribution.

- In practice, we can estimate the variance by
  - \* Fisher information matrix  $\mathbf{I}^{-1}(\hat{\boldsymbol{\theta}})$ ,
  - \* observed information matrix  $[-(1/n) d^2 L_n(\hat{\boldsymbol{\theta}})]^{-1}$ , or
  - \* the sandwich estimator
- Asymptotic efficiency of MLE.

“Cramer-Rao theorem” says the variance of any unbiased estimator is “at least”  $(n \mathbf{I}(\boldsymbol{\theta}_0))^{-1}$  (the difference is psd). So MLE has the smallest asymptotic variance within the class of unbiased estimators.

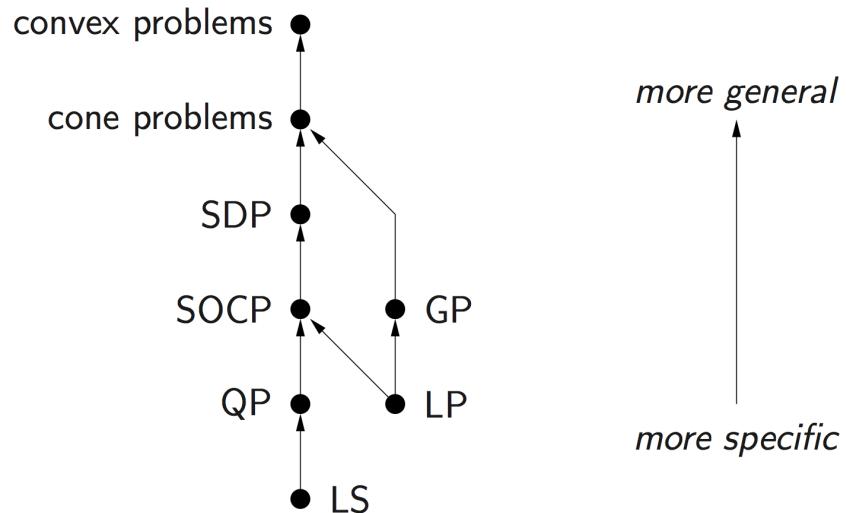
## Hierarchy of optimization problems

Difficulty of optimization problems *in general*

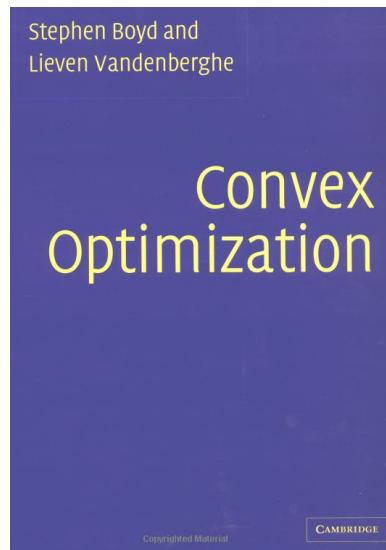
Harder	Easier
discrete (combinatorial) optimization	continuous optimization
non-smooth	smooth
non-convex	convex
constrained	un-constrained
inequality constraint	equality constrained

## Convex optimization

- *Extremely important* skill to recognize or transform to convex problems



- Examples:  $\ell_\infty$  regression,  $\ell_1$  regression, quantile regression, and many more.
- *Convex Optimization* by Boyd and Vandenberghe and accompanying slides  
<http://www.stanford.edu/~boyd/cvxbook/>



- Lecture videos:  
<http://www.stanford.edu/class/ee364a/videos.html>  
<http://www.stanford.edu/class/ee364b/videos.html>
- UCLA courses by Lieven Vandenberghe: EE236A (Linear Programming), EE236B (Convex Optimization), EE236C (Optimization Methods for Large-scale Systems).
- Convex programming (LS, LP, QP, GP, SOCP, SDP) is almost becoming a technology (**Cplex**, **Gurobi**, **Mosek**, **cvx**, **Matlab**, **JuliaOpt**, ...), somewhat like numerical linear algebra libraries BLAS and LAPACK. Current technology can solve convex problems with up to thousands of variables and constraints.
- Non-convex optimization still occurs in many natural statistical applications. Statisticians have specialized tools to deal with them (Fisher scoring method, EM algorithm, simulated annealing, ...)