Fundamentals Of Information Science **2022 Spring**

Homework3

学生：华园（202000120027）） *时间：2022.3.9*

Problem 1.

Coin flips.A fair coin is flipped until the first head occurs. Let X denote the number of flips required. Find the entropy H(X) in bits.

Solution. *Suppose the probability distribution of random variable X is P(X).*

$$\begin{aligned}
P(X=1) &= \frac{1}{2} \\
P(X=2) &= (\frac{1}{2})^2 \\
P(X=3) &= (\frac{1}{2})^3 \\
P(X=4) &= (\frac{1}{2})^4 \\
&\cdots \\
P(X=n) &= (\frac{1}{2})^n
\end{aligned}$$

According to the definition of entropy, we can get the following results:

$$\begin{aligned}
H(X) &= -\sum_{n=1}^{\infty} p(x) log_2 p(x) \\
&= -\sum_{n=1}^{\infty} (\frac{1}{2})^n log_2 (\frac{1}{2})^n \\
&= \sum_{n=1}^{\infty} n(\frac{1}{2})^n \\
&= \frac{1/2}{(1-1/2)^2} \\
&= 2(bits)
\end{aligned}$$

Problem 2.

A die comes up 6 twice as often as it comes up 1.What is the maximum entropy?

Solution. *Suppose the probability distribution of random variable X is P(X).*

$$
\begin{aligned}
P(X=1) &= x \\
P(X=2) &= \frac{1-3x}{4} \\
P(X=3) &= \frac{1-3x}{4} \\
P(X=4) &= \frac{1-3x}{4} \\
P(X=5) &= \frac{1-3x}{4} \\
P(X=6) &= 2x
\end{aligned}
$$

According to the definition of entropy, we can get the following results:

$$
\begin{aligned}
H(X) &= -\sum_{n=1}^{\infty} p(x) lnp(x) \\
&= -(xlnx + 2xlnx + (1-3x)ln(\frac{1-3x}{4}))
\end{aligned}
$$

thus

$$
\begin{aligned}
\frac{\mathrm{d}H(X)}{\mathrm{d}X} &= -(xlnx + 2xlnx + (1-3x)ln(\frac{1-3x}{4})) \\
&= -(lnx + 1 + 2ln2x + 2 - 3ln(\frac{1-3x}{4}) - 3) \\
&= -(lnx + 2lnx + 2ln2 - 3ln(1-3x) + 3ln4) \\
&= -(3ln(\frac{x}{1-3x}) + 4ln4)
\end{aligned}
$$

$$
Let \frac{\mathrm{d}H(X)}{\mathrm{d}X} = 0: \qquad we\ can\ know\ that: \quad x_0 = \frac{1}{3+2^{\frac{8}{3}}}
$$

$$
\begin{aligned}
&when\ x < x_0, \quad \frac{\mathrm{d}H(X)}{\mathrm{d}X} > 0 \\
&when\ x > x_0, \quad \frac{\mathrm{d}H(X)}{\mathrm{d}X} < 0
\end{aligned}
$$

$$
thus\ the\ maximum\ entropy\ is\ H(x_0)
$$

$$
H(X)_{max} = H(x_0) = \frac{1}{3+2^{\frac{8}{3}}} ln(3+2^{\frac{8}{3}}) + \frac{2}{3+2^{\frac{8}{3}}} ln\frac{3+2^{\frac{8}{3}}}{2} - \frac{2^{\frac{8}{3}}}{3+2^{\frac{8}{3}}} ln\frac{2^{\frac{2}{3}}}{3+2^{\frac{8}{3}}}
$$

Problem 3.

AEP and source coding. A discrete memoryless source emits a sequence of statistically independent binary digits with probabilities p(1) = 0.005 and p(0) = 0.995. The digits are taken 100 at a time and a binary codeword is provided for every sequence of 100 digits containing three or fewer 1's.

(a) Assuming that all codewords are the same length, find the minimum length required to provide codewords for all sequences with three or fewer 1's.

Solution. *Assuming that the number of all sequences with three or fewer 1's is N.*

$$\begin{aligned} N &= C_{100}^{1} + C_{100}^{2} + C_{100}^{3} + C_{100}^{0} \\ &= 100 + \frac{(100 \cdot 99)}{2} + \frac{(100 \cdot 99 \cdot 98)}{3 \cdot 2} + 1 \\ &= 166751 \\ &= (10 \quad 1000 \quad 1011 \quad 0101 \quad 1111)_B \end{aligned}$$

In summary,the minimum length is 18.

(b) Calculate the probability of observing a source sequence for which no codeword has been assigned.

Solution. *Assuming that the probability of observing a source sequence for which no codeword has been assigned is P.*

$$\begin{aligned} P &= 1 - C_{100}^{0} * (0.995)^{100} - C_{100}^{1} * (0.995)^{99} * 0.005 - C_{100}^{2} * (0.995)^{98} * (0.005)^2 - C_{100}^{3} * (0.995)^{97} * (0.005)^3 \\ &= 1 - 0.60577 - 0.304407 - 0.075719 - 0.012429 \\ &\approx 0.0017 \end{aligned}$$

(c) Use Chebyshev's inequality (search online if you don't know Chebyshev's inequality) to bound the probability of observing a source sequence for which no codeword has been assigned. Compare this bound with the actual probability computed in part (b).

Solution. *Suppose that the random variable X represents the number of 1 in the sequence*

$$X \sim B(100, 0.005), E(X) = 0.5, D(X) = 0.4975$$

$$P(|X - E(X)| \geq 3.5) \leq \frac{D(X)}{(3.5)^2}$$

$$P(|X - E(X)| \geq 3.5) \leq 0.04061$$

$$0.0017 < 0.04061$$

From this, we can get that the estimation of Chebyshev inequality is relatively rough, and there is a certain gap with the actual probability value.

Problem 4.

Entropy and pairwise independence. Let X, Y, Z be three binary Bernoulli random variables that are pairwise independent; that is, I(X; Y) = I(X;Z) = I(Y ;Z) = 0.
(a) Under this constraint, what is the minimum value for H(X, Y, Z)?

Solution.

$$X = Y = Z = \begin{cases} 1 & P = \frac{1}{2} \\ 0 & P = \frac{1}{2} \end{cases}$$

$$H(X) = H(Y) = H(Z) = \frac{1}{2}log_2(2) + \frac{1}{2}log_2(2) = 1$$

$$\begin{aligned} H(X,Y,Z) &= H(X) + H(Y|X) + H(Z|Y,X) \\ &= H(X) + H(Y) + H(Z|Y,X) \end{aligned}$$

$$when\ H(Z|Y,X) = 0,\ \ H(X)\ is\ the\ minimum.$$
$$H(X)_{min} = 2$$

(b) Give an example achieving this minimum.

Solution.

Suppose a letter is randomly selected from A, B, C and D, and the probability of each letter is 0.25 ,Now set up the following events:

Event X: The extracted letter is A or B, P(X)=0.5.
Event Y: The extracted letter is A or C, P(Y)=0.5.
Event Z: The extracted letter is B or C, P(Z)=0.5.

We assume that random variables X, Y and Z correspond to event X, event Y and event Z respectively. X = 1 means that event X occurs, X= 0 means that event X does not occur, and so on.

(1).X,Y,Z are three binary Bernoulli(0.5) random variables.
(2).P(X,Y)=P(X)P(Y),P(X,Z)=P(X)P(Z),P(Z,Y)=P(Z)P(Y)
so X,Y,Z are pairwise independent.

$$when\ (X=0,Y=0):\ P(Z=1)=0,\ P(Z=0)=1. \qquad [P(X=0,Y=0)=\frac{1}{4}]$$
$$when\ (X=0,Y=1):\ P(Z=1)=1,\ P(Z=0)=0. \qquad [P(X=0,Y=1)=\frac{1}{4}]$$
$$when\ (X=1,Y=0):\ P(Z=1)=1,\ P(Z=0)=0. \qquad [P(X=1,Y=0)=\frac{1}{4}]$$
$$when\ (X=1,Y=1):\ P(Z=1)=0,\ P(Z=0)=1. \qquad [P(X=1,Y=1)=\frac{1}{4}]$$

$$H(Z|Y,X) = \frac{1}{4}log_2(1) + \frac{1}{4}log_2(1) + \frac{1}{4}log_2(1) + \frac{1}{4}log_2(1) = 0$$

$$H(X,Y,Z) = H(X) + H(Y) = 2.(achieving\ the\ minimum)$$