ECM5901 2019, Assignment #3 due November 12

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1. For a subspace V of $\mathbb{R}^n$, its orthogonal complement, denoted by $V^\perp$, is defined as $V^\perp = \{x : \langle x, z\rangle = 0 \text{ for all } z \in V\}$. Suppose that $A \in \mathbb{R}^{m\times n}$. Show that

$$\mathcal{N}(A) = (\mathcal{R}(A^T))^\perp.$$

where $\mathcal{N}(A)$ is the null space of A and $\mathcal{R}(A^T)$ is the range of A^T.

(15%)

2. Prove that $x^\star = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = (1/2)x^T P x + q^T x + r \\ \text{subject to} & -1 \le x_i \le 1, \quad i = 1, 2, 3 \end{array}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \; q = \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}, \; r = 1.$$

(20%)

3. Consider a problem of the form

$$\begin{array}{ll} \text{minimize} & f_0(x)/(c^T x + d) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

where $f_0, f_1, \ldots, f_m$ are convex, and the domain of the objective function is defined as $\{x \in \operatorname{dom} f_0 : c^T x + d > 0\}$.

(a) Show that the problem is equivalent to

$$\begin{array}{ll} \text{minimize} & g_0(y, t) \\ \text{subject to} & g_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ay = bt \\ & c^T y + dt = 1 \end{array}$$

where g_i is the perspective of f_i (see §3.2.6). The variables are $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. (15%)

(b) Show that this problem is convex. (10%)

4. (Network flow problem.) Consider a network of n nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: x_{ij} will denote the flow from node i to node j. The cost of the flow along the link from node i to node j is given by $c_{ij}x_{ij}$, where c_{ij} are given constants. The total cost across the network is

$$C = \sum_{i,j=1}^{n} c_{ij} x_{ij}.$$

Each link flow x_{ij} is also subject to a given lower bound l_{ij} (usually assumed to be nonnegative) and an upper bound u_{ij}. The external supply at node i is given by b_i, where $b_i > 0$ means an external flow enters the network at node i, and $b_i < 0$ means that at node i, an amount $|b_{ij}|$ flows out of the network. We assume that $\sum_i b_i = 0$, i.e., the total external supply equals total external demand. At each node we have conservation of flow: the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero. The problem is to minimize the total cost of flow through the network, subject to the constraints described above. Formulate this problem as a linear program. (20%)

5. Give an explicit solution of the following QCQPs. (Minimizing a linear function over an ellipsoid)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (x - x_c)^T A (x - x_c) \leq 1 \end{array}$$

where $A \in S^n_{++}$ and $c \neq 0$. (20%)