Example: Entropy maximization

$$\text{minimize} \quad f_0(x) = \sum_{i=1}^{n} x_i \log x_i$$

$$\text{s.t.} \quad Ax \leq b$$

$$\sum_i x_i = 1.$$

Suppose (λ^*, ν^*) is dual optimal.

$$L(x, \lambda^*, \nu^*) = \sum_{i=1}^{n} x_i \log x_i + (\lambda^*)^T (Ax - b) + \nu^* \left(\sum_i x_i - 1\right).$$

To find x^*,

$$0 = \frac{\partial L(x, \lambda^*, \nu^*)}{\partial x_i}$$

$$\Rightarrow x_i^* = \frac{1}{e^{\left(\sum_j \lambda_j^* A_{ji} + \nu^* + 1\right)}}$$

$i = 1, \ldots, n.$

Proof of Strong duality under constraint qualification

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \; i=1,\dots,m. \\ & Ax = b. \qquad A = q \times n. \end{aligned}$$

Suppose $f_0, \dots, f_m$ are convex.

Assume that there exists feasible $\tilde{x} \in \operatorname{relint} D$ $\;\; \operatorname{relint} D = \operatorname{int} D$

Assume that D has nonempty interior.

$\operatorname{rank}(A) = q$.

Suppose that p^* is finite.

1°. $\mathcal{A} = \{ (u, v, t) \in \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R} : \exists x \in D,$

$f_i(x) \le u_i, \; h_j(x) = v_j, \; f_0(x) \le t \}$ is convex.

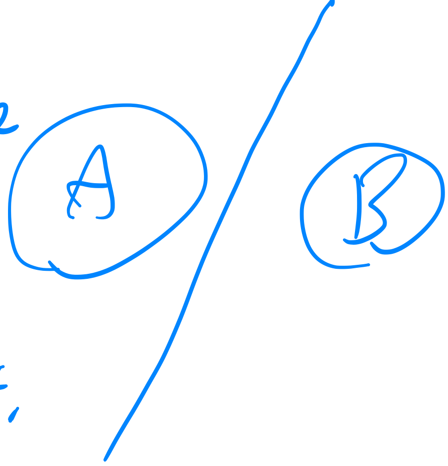
2°. Define a convex set B as

$$B = \{ (0, 0, s) \in \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R} : s < p^* \}$$

Claim: $A \cap B = \emptyset$. Let $(0,0,t)$ with $t < p^*$ belong to $\mathcal{A}$.

$\exists x \in D$ $f_i(x) \le 0, \; h_j(x) = 0, \; f_0(x) \le t < p^*$.

3°. By the separating hyperplane theorem, there exists



$(\tilde{\lambda}, \tilde{\nu}, \mu) \neq 0$, $\alpha \in \mathbb{R}$ s.t.

$$\begin{cases} (u, v, t) \in \mathcal{A} \Rightarrow \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \geq \alpha & \text{— (1)} \\ (u, v, t) \in \mathcal{B} \Rightarrow \tilde{\lambda}^T u + \tilde{\nu}^T v + \mu t \leq \alpha. & \text{— (2)} \end{cases}$$

(1) $\Rightarrow$ $\underline{\tilde{\lambda}^T u + \mu t}$ is unbounded below over $\mathcal{A}$

unless $\tilde{\lambda} \geq 0$ and $\mu \geq 0$.

since $\begin{cases} u \\ t \end{cases}$ can go to $\begin{cases} +\infty \\ +\infty \end{cases}$.

(2) $(u, v, t) \in \mathcal{B} \Rightarrow u = 0,\ v = 0. \Rightarrow \mu t \leq \alpha.$

This is true for all $t < p^*$.

$\Rightarrow \mu(p^* - \epsilon) \leq \alpha.\ \forall \epsilon > 0$

$\Rightarrow \underline{\mu p^* \leq \alpha.}$

$a - \epsilon \leq b\ \forall \epsilon > 0.$

$\Rightarrow a \leq b$

Thus for $x \in D$, $\underline{(f_i(x), Ax - b, f_0(x)) \in \mathcal{A}}$.

(1)

$$\sum_{i=1}^{m} \tilde{\lambda}_i f_i(x) + \tilde{\nu}^T (Ax - b) + \mu f_0(x) \geq \alpha \text{ — (3)}$$

$$\geq \mu p^*.$$

4°. Assume $\mu > 0$

divide (3) by μ on both sides

$$\Rightarrow \quad L(x, \hat{\lambda}/\mu, \hat{\nu}/\mu) \geq p^* \quad \forall x \in D.$$

Thus $g(\lambda, \nu) \geq p^*$, where $\lambda = \hat{\lambda}/\mu$, $\nu = \hat{\nu}/\mu$.

Also, the weak duality says that

$$g(\lambda, \nu) \leq p^*.$$

$$\Rightarrow \quad g(\lambda, \nu) = p^*.$$

5°. Assume $\mu = 0$.