$$\begin{aligned} \text{maximize} \quad & -\mathrm{tr}\, GZ \rightarrow \text{minimize} \ \mathrm{tr}\, GZ. \\ \text{s.t.} \quad & Z \succeq 0 \Rightarrow -Z \preceq 0 \\ & \mathrm{tr}\, F_i Z + c_i = 0, \quad i = 1, \dots, n. \end{aligned}$$

$Y \in S^k$. $x \in \mathbb{R}^n$ $\quad (Y \succeq 0)$.

$$L(\underline{\underline{Z}}, Y, \underset{(\nu)}{x}) = \mathrm{tr}\, GZ + \langle Y, -Z \rangle - \sum_{i=1}^{n} \underset{(\nu_i)}{x_i} (\mathrm{tr}\, F_i Z + c_i)$$

$$g(Y, x) = \inf L(Z, Y, x)$$

$$= -\sum_{i=1}^{n} x_i c_i + \inf_{Z \succeq 0} \Big\{ \underbrace{-\mathrm{tr}\Big(\Big(\sum_i x_i F_i - G + Y\Big) Z\Big)}_{\langle -\sum_i x_i F_i - G - Y,\ Z \rangle = 0} \Big\}$$

$$= \begin{cases} -c^T x, & \text{if } \underbrace{-\sum_i x_i F_i + G - Y = 0.\ Y \succeq 0.}_{-\sum_i x_i F_i - G = \underline{\underline{Y}} \succeq 0} \\ -\infty, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} -c^T x, & \text{if } \sum_i x_i F_i - G \preceq 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \text{maximize} \quad & -c^T x \longrightarrow \text{minimize} \ c^T x \\ \text{s.t.} \quad & \sum_i x_i F_i - G \preceq 0. \end{aligned}$$

(the primal problem)

Optimality conditions

Assume that the primal and dual optimal values are equal $d^* = p^*$ and x^*, λ^*, ν^* are optimal points.

Complementary Slackness.

$$\mathcal{L}(x^*, \lambda^*, \nu^*).$$

$$f_0(x^*) = g(\lambda^*, \nu^*).$$

$$\le \underbrace{f_0(x^*) + \underbrace{\sum_{i=1}^{m} \lambda_i^T f_i(x^*)}_{\le 0} + \underbrace{\sum_{j=1}^{q} \nu_j^* h_j(x^*)}_{= 0}}_{= \mathcal{L}(x^*, \lambda^*, \nu^*)}$$

$$\le f_0(x^*).$$

Two inequalities will hold with equality.

$$\Rightarrow \begin{cases} \lambda_i^{*T} f_i(x^*) = 0, & i = 1, \dots, m. \\ x^* \text{ minimizes } \mathcal{L}(x, \lambda^*, \nu^*). \end{cases}$$

$$\langle \lambda_i^*, f_i(x^*) \rangle = 0. \quad i = 1, \dots, m.$$

$$\begin{cases} \text{if } \lambda_i^* > 0, \Rightarrow f_i(x^*) = 0. \\ \text{if } f_i(x^*) < 0, \Rightarrow \lambda_i^* = 0. \end{cases}$$

If. $a, b \in \mathbb{R}$.
$ab = 0$
$\Leftrightarrow a = 0$
or $b = 0$.

It is possible that $a.b \neq 0$. but $\langle a, b \rangle = 0$.
$a.b \in \mathbb{R}^n$.

KKT conditions

Assume that the functions f_i, h_j are differentiable.

x^* minimizes $L(x, \lambda^*, \nu^*)$,

$$\left(0 = \nabla_x L(x^*, \lambda^*, \nu^*)\right)$$

$$\nabla f_0(x^*) + \sum_{i=1}^{m} Df_i(x^*)^T_i \lambda_i^* + \sum_{j=1}^{q} \nu_j^* \nabla h_j(x^*) = 0.$$

where $Df_i(x^*) \in \mathbb{R}^{k_i \times n}$ is the derivative of f_i evaluated at x^*.

- If strong duality holds, any primal optimal x^* and dual optimal λ^*, ν^*, must satisfy that

① $f_i(x^*) \preceq_{K_i} 0$, $i = 1, \dots, m$

$h_j(x^*) = 0$. $j = 1, \dots, q$.

② $\lambda_i^* \succeq_{K_i^*} 0$. $i = 1, \dots m$.

③ $\lambda_i^{*T} f_i(x^*) = 0$. $i = 1, \dots m$.

④

- If the primal problem is convex, the KKT conditions are sufficient for the optimality of (x^*, λ^*, ν^*).

Hw #5 deadline Dec. 17. 2019.

Final Exam : Jan. 7. 2020.

(Tuesday).

Hw # 6.

Hw # 7. on the way.