6. If $x_i \preceq_K y_i$, $i=1,2,3,\ldots$ $x_i \to x$, $y_i \to y$. as $i \to \infty$. Then $x \preceq_K y$.

proof. We say $x_i \to x$ if $\forall \epsilon > 0, \exists N_0 > 0$ s.t. for $i \geq N_0$, we have

$$\| x_i - x \|_2 \leq \epsilon.$$

Similarly $y_i \to y$, if $\forall \epsilon > 0, \exists N_1 > 0$ s.t. for $i \geq N_1$, we have

$$\| y_i - y \|_2 \leq \epsilon.$$

For $i \geq \max\{N_0, N_1\}$

$$\| (y_i - x_i) - (y - x) \|_2 \leq \| y_i - y \|_2 + \| x_i - x \|_2 \leq 2\epsilon.$$

$\therefore \lim_{i \to \infty} y_i - x_i = y - x.$

$\because$ K is closed, we have $y - x \in K$.

(A convergent sequence converges to a point in a closed set.).

Similarly.

check these!

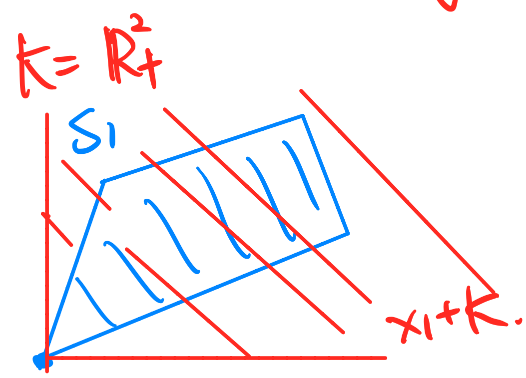
- $x \prec_K y \Rightarrow x \preceq_K y$
- $x \prec_K y$, $\boxed{u \preceq_K v} \Rightarrow x + u \prec_K y + v$
- $x \prec_K y$, $\alpha > 0 \Rightarrow \alpha x \prec_K \alpha y$.
- $x \not\prec_K x$.

We say that $x \in S$ is the <u>minimum</u> element of S if $\forall y \in S$, we have $x \preceq_K y$.

" " " " " " " minimal

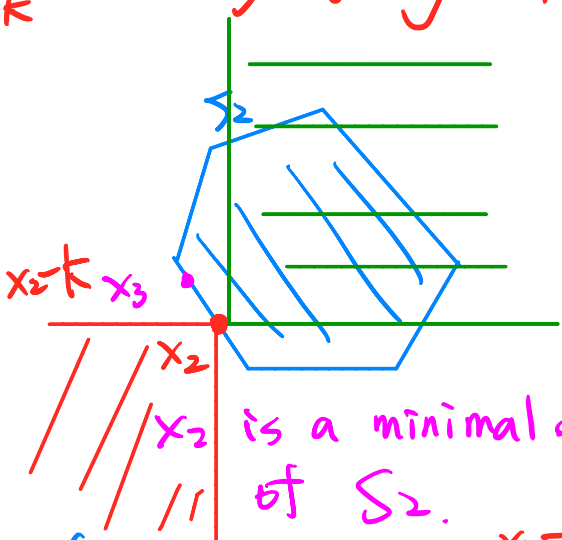
" " " if $y \preceq_K x$ only if $y = x$.

Ex. $\mathbb{R}^2$



x_1 is the minimum.

$S_1 \subseteq x_1+\mathbb{R}^2_+$



x_2 is a minimal element of S_2.

$x_2-K \cap S_2 = \{x_2\}$.

$x \in S$ is minimum if and only if

$$S \subseteq x+K = \{y : y-x \in K\}.$$

$x+z : z \in K$.

This set of points are comparable to x and are greater than or equal to x.

$x \in S$ is minimal if and only if

$$(x-K) \cap S = \{x\}$$

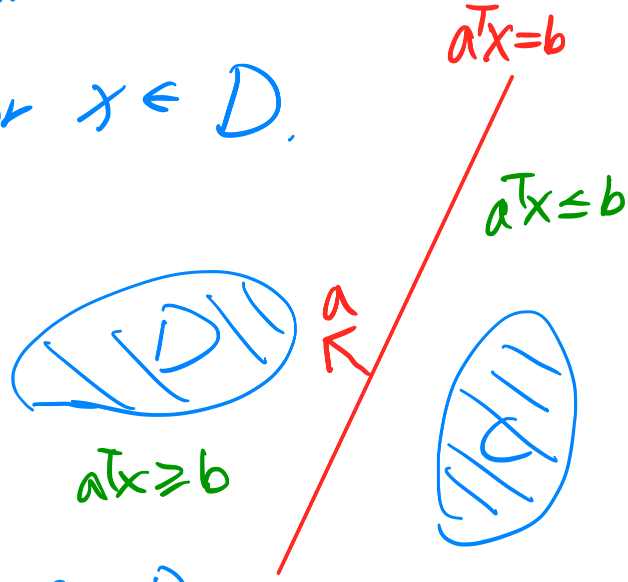
$$\| $$

$$\{y : x-y \in K\}$$

Separating hyperplane theorem.

Suppose that $C, D \subseteq \mathbb{R}^n$ are two nonempty disjoint convex sets, $C \cap D = \emptyset$.
Then there exists $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$ such that
$$\begin{cases} a^T x \le b & \text{for } x \in C \\ a^T x \ge b & \text{for } x \in D. \end{cases}$$

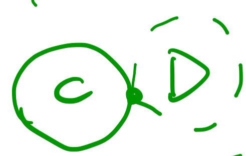
The hyperplane $\{x : a^T x = b\}$ is called a separating hyperplane for the sets C and D.



Proof. distance between two sets C & D.

$$\mathrm{dist}(C, D) \triangleq \inf \{ \| u - v \| : u \in C, v \in D \}$$

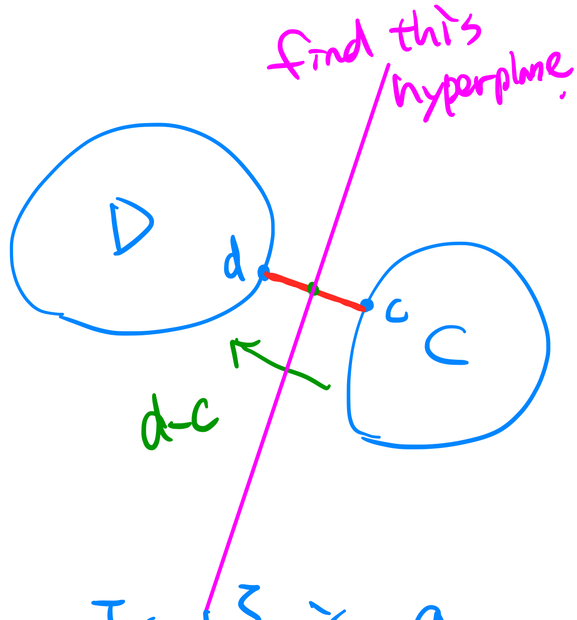
Assume that $\boxed{\mathrm{dist}(C, D) > 0.}$
and there exist $c \in C$ & $d \in D$
such that $\underline{\|c-d\|_2} = \mathrm{dist}(C, D)$.



distance $= 0$
C: closed ball
D: open ball

Define $a = d - c \in \mathbb{R}^n$.

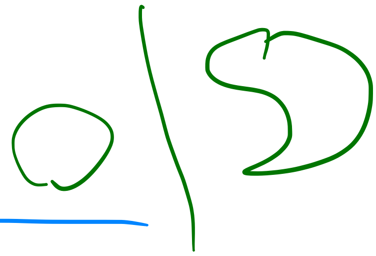
$$b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}.$$



$\boxed{\text{Claim: } f(x) = a^T x - b \text{ is } \begin{cases} \text{nonnegative on } D \\ \text{nonpositive on } C. \end{cases}}$ $\Rightarrow$ $\{x : a^T x = b\}$ is a desired hyperplane.

$$f(x) = (d-c)^T \left(x - \frac{d+c}{2}\right)$$

$$0 = f\Big(\underbrace{\tfrac{d+c}{2}}_{\text{midpoint of } c \,\&\, d.}\Big)$$



~~Assume that f is nonnegative on D.~~

If $\exists u \in D$ s.t. $f(u) < 0$. then

$$f(u) = (d-c)^T \left(u - \frac{d+c}{2}\right).$$

$$= (d-c)^T \left(u - d + \frac{d-c}{2}\right).$$

$$= (d-c)^T (u-d) + \underbrace{(d-c)^T \cdot \left(\frac{d-c}{2}\right)}_{= \frac{1}{2}\|d-c\|_2^2 \geq 0}.$$

We must have

$$(d-c)^T (u-d) < 0.$$

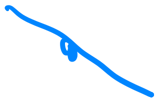
Observe that

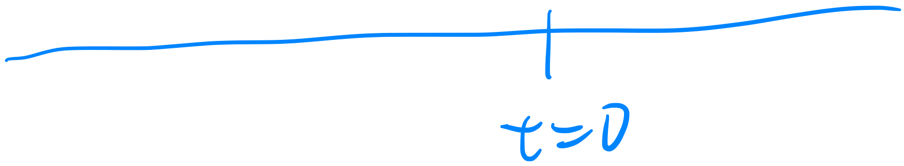
$$\frac{d}{dt} \| d + t(u-d) - c \|_2^2 \Big|_{t=0}.$$

$$= (u-d)^T (d + t(u-d-c)) + (d + t(u-d) - c)^T (u-d) \Big|_{t=0}.$$

$$= 2(d-c)^T (u-d) < 0$$

$$\|d + t(u-d) - c\|_2^2$$





For some small value of $0 < t_1 \le 1$

we have $\|d + t_1(u-d) - c\|_2 < \|\underline{\underline{d}} - c\|_2$

$$\| \left((1-t_1)d + t_1 u \right) - c \|_2$$

$\Rightarrow$ a contradiction that d is the closest point to C.

$u, d \in D$, which is convex.

$\Rightarrow \underline{(1-t_1)d + t_1 u \in D}$

Strict separation:

$$\begin{cases} a^T x < b & \text{for } x \in C \\ a^T x > b & \text{for } x \in D. \end{cases}$$

(does not hold in general. See Exercise 2.23).

Example 2.20 of BV.

Let C be a closed convex set and $x_0 \notin C$.

Then there exists a hyperplane that strictly separates $\{x_0\}$ and C.

$\downarrow$

Exercise 2.22 $\Rightarrow$ general proof for the separating hyperplane theorem

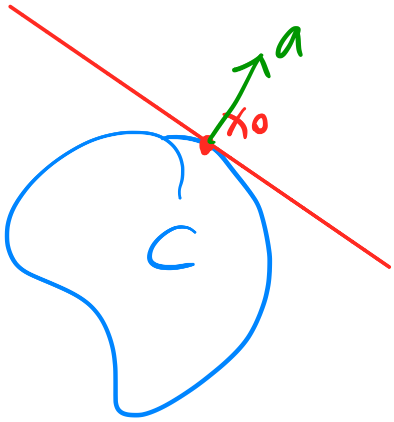
(If you are interested).

Supporting hyperplanes.

Suppose that $C \subseteq \mathbb{R}^n$, $x_0 \in \text{bd}\, C$

If $a \neq 0 \in \mathbb{R}^n$ satisfies that $a^T x \le a^T x_0 \ \forall x \in C$,

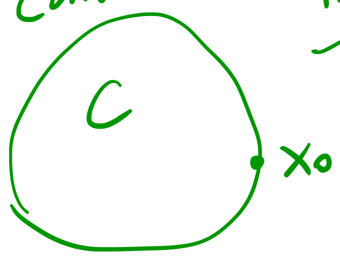
then $\{x : a^T x = a^T x_0\}$ is called a supporting hyperplane to C at x_0.



Supporting hyperplane theorem

For any nonempty convex set C, and any $x_0 \in \text{bd}\, C$, there exists a supporting hyperplane to C at x_0.

① if $\text{int}\, C \neq \emptyset$.



$\text{Int}\, C$, $\{x_0\}$

use separating hyperplane theorem

② if $\text{int}\, C = \emptyset$.

$C \subseteq \text{aff}\, C \neq \mathbb{R}^n$

Then any hyperplane containing aff C contains C & x_0.

Dual Cone

Let K be a cone. The set

$$K^* = \{ y : \underline{x^T y \geq 0} \ \forall x \in K \}$$

is called the dual cone of K.

> K^* is always a convex cone.
verify it.

Ex. Suppose that $V \subseteq \mathbb{R}^n$ is subspace.
V is a cone. $\theta x \in V \ \forall x \in V. \ \theta \geq 0.$

The dual cone of V is

$$V^\perp = \{ y : y^T x = 0 \ \forall x \in V \}.$$

proof. ① $V^\perp \subseteq V^*$ trivially.

② Suppose that $V = \text{span}(\{ v_1, v_2, \ldots, v_m \})$ $m < n$
We can extend $\{v_1, \ldots, v_m\}$ to a basis of $\mathbb{R}^n$
by $\{ v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n \}$.

For $y \in V^*$, $y = \sum_{i=1}^{n} \alpha_i v_i$. for $\alpha_i \in \mathbb{R}$.

$$0 \leq \langle y , \beta_j v_j \rangle = \alpha_j \beta_j \ \forall \beta_j$$

by def of dual cone
& $\beta_j v_j \in V$. for $\beta_j \in \mathbb{R}$

$\Rightarrow \alpha_j = 0$.
for $j = 1, \ldots, m$.

$\therefore y = \sum_{i=m+1}^{n} \alpha_i v_i \in V^\perp$

$\Rightarrow V^* \subseteq V^\perp$.

Ex. $(S_n^+)^* = S_n^+$. That is, (self-dual).

$\operatorname{tr} XY \geq 0 \ \forall X \geq 0 \iff Y \geq 0$

① "$\Leftarrow$" For $X \geq 0$, $X = \sum_i \lambda_i q_i q_i^T$, $\lambda_i \geq 0$.

$$q_i^T q_j = \delta_{ij}$$

$$\operatorname{Tr} XY = \operatorname{Tr}\left(\sum_i \lambda_i q_i q_i^T Y\right)$$

$\operatorname{Tr} AB = \operatorname{Tr} BA$

$$= \sum_i \lambda_i \operatorname{Tr} q_i q_i^T Y$$

$$= \sum_i \underbrace{\lambda_i}_{\geq 0} \underbrace{q_i^T Y q_i}_{\geq 0} \geq 0.$$

② "$\Rightarrow$" Suppose that $\operatorname{tr} XY \geq 0 \ \forall X \geq 0$

For $z \in \mathbb{R}^n$ $z^T Y z = \operatorname{tr}(z z^T Y) \geq 0$.

$\therefore Y \geq 0$.

K^* is closed & convex

⋮

K^{**} is the closure of the convex hull of K.
(Exercise 2.3)

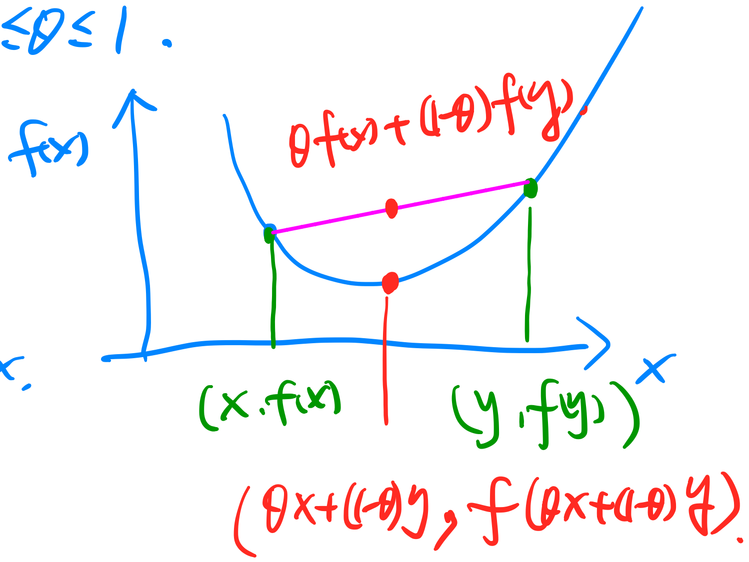
(If K is convex & closed, $K^{**} = K$.)

Chapter 3. Convex functions

A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if $\text{dom} f$ is convex, and

$$f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \text{dom} f$, $0 \le \theta \le 1$.



- f is strictly convex if $\text{dom} f$ is convex and

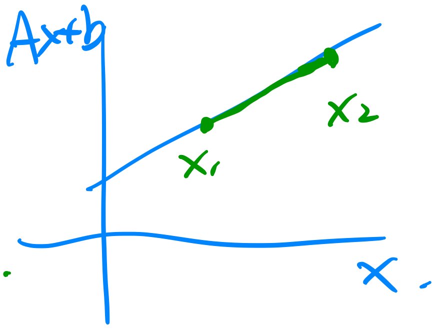
$$f(\theta x + (1-\theta) y) < \theta f(x) + (1-\theta) f(y)$$

for $x \ne y \in \text{dom} f$. $0 \le \theta \le 1$.

- f is (strictly) concave if $-f$ is (strictly) convex.

Examples.

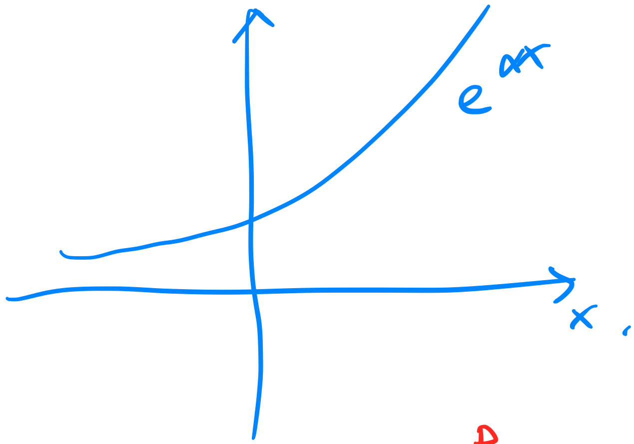
1. affine. $f(x) = Ax + b$. for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$.



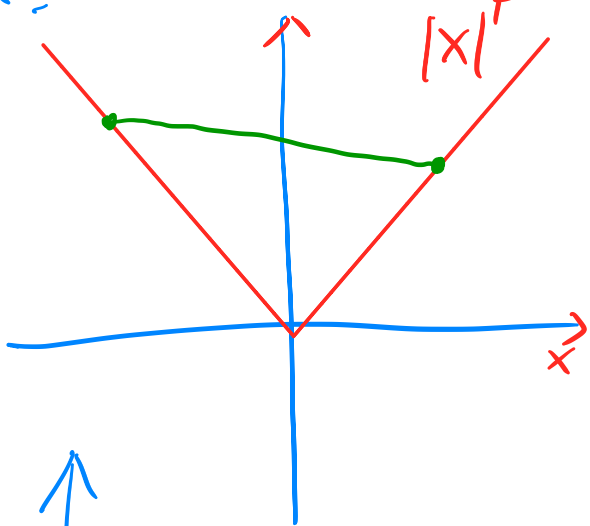
$f(x)$ is both convex and concave.

$f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta) f(y)$.

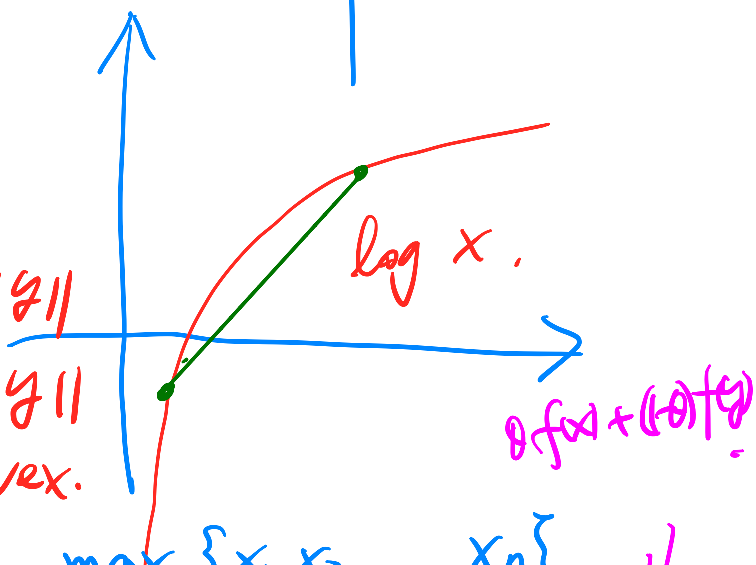
2. $f(x) = e^{\alpha x}$ for $\alpha \in \mathbb{R}$ is convex.



3. $|x|^p$, for $p \geq 1$, $x \in \mathbb{R}$.



4. $\log x$ is concave on $\mathbb{R}$



5. norm $\|x\|$: $x \in \mathbb{R}^n$.

$\|\theta x + (1-\theta) y\| \leq \|\theta x\| + (1-\theta)\|y\|$

$= \theta\|x\| + (1-\theta)\|y\|$

triangle inequality

$\therefore$ norm is convex.

6. max function : $f(x) = \max\{x_1, x_2, \ldots, x_n\}$.

convex. $f(\theta x + (1-\theta) y) = \max_i \{\theta x_i + (1-\theta) y_i\}$

$\leq \max_i \{\theta x_i\} + \max_i \{(1-\theta) y_i\}$

$f: \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

$g: \mathbb{R} \to \mathbb{R}$, defined by

$g(t) = f(x + tv)$,

$\mathrm{dom}\, g = \{t : x + tv \in \mathrm{dom} f\}$,

is convex in t for $x \in \mathrm{dom} f$; $v \in \mathbb{R}^n$.