Ch.4 Convex Optimization Problem

Optimization problem

standard form

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \le 0, \quad i=1,\dots,m \\ \text{(s.t.)} \quad & h_j(x) = 0, \quad j=1,\dots,r. \end{aligned} \qquad \underline{\underline{(1)}}$$

- $x \in \mathbb{R}^n$ optimization variable.
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ objective function
- $f_i : \mathbb{R}^n \to \mathbb{R}$ inequality constraint functions.
- $h_j : \mathbb{R}^n \to \mathbb{R}$ equality constraint functions.
- domain of (1) is

$$D = \left(\bigcap_i \operatorname{dom} f_i\right) \cap \left(\bigcap_j \operatorname{dom} h_j\right).$$

$$\begin{cases} \max f_0(x) \Rightarrow \min -f_0(x). \\ f_i(x) \ge 0 \Rightarrow -f_i(x) \le 0. \\ b \le g(x) \le a \Rightarrow \begin{cases} g(x) - a \le 0 \\ b - g(x) \le 0. \end{cases} \end{cases}$$

$\triangle$ If $m=0$, $r=0$. the problem is unconstrained.

- $x \in D$ is **feasible** if it satisfies the constraints $f_i(x) \le 0$, $h_j(x) = 0$.
- If $\exists$ feasible $x \in D$, (1) is called **feasible**; otherwise, **infeasible**.

- Optimal value $p^* = \inf \left\{ f_0(x) : f_i(x) \le 0,\ i=1,\dots,m,\ h_j(x)=0,\ j=1,\dots,r \right\}$.

  $p^* = \infty$ if the problem is infeasible.

  $p^* = -\infty$ if the problem is unbounded below.

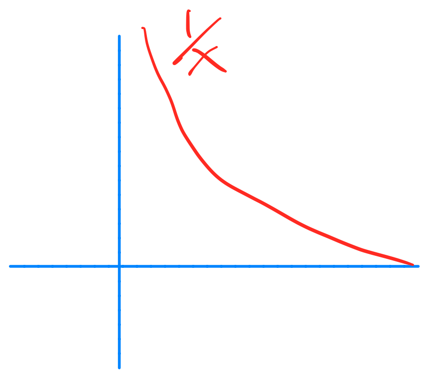
- A feasible x is optimal if $f_0(x) = p^*$.

- If there exists an optimal point for the problem, we say that the optimal value is attained or achieved. (the problem is solvable.)

Ex. $f_0(x) = \frac{1}{x}$, $x \in \mathbb{R}_{++}$

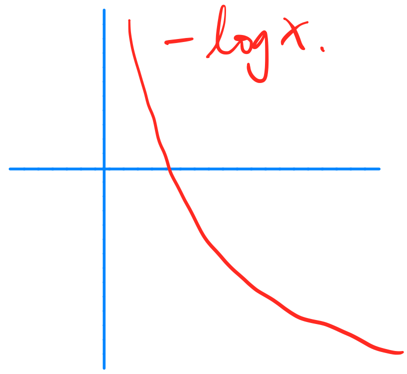
$\inf\left(\frac{1}{x}\right) = 0$, $p^* = 0$.

(no optimal point).



Ex. $f_0(x) = -\log x, \quad x \in \mathbb{R}_{++}$

$p^* = -\infty$

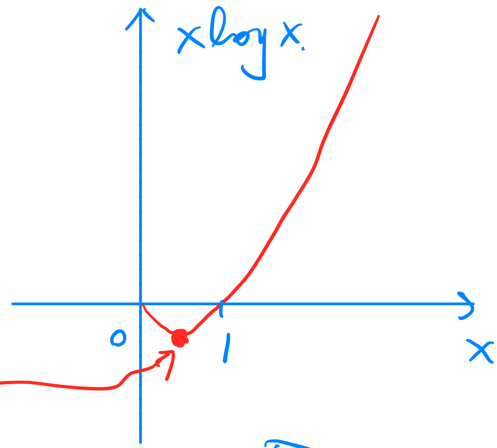


Ex. $f_0(x) = x \log x, \quad x \in \mathbb{R}_{++}$

$0 = f_0'(x) = \log x + 1$

$\Rightarrow x^* = e^{-1}$

$\Rightarrow p^* = -e^{-1}$



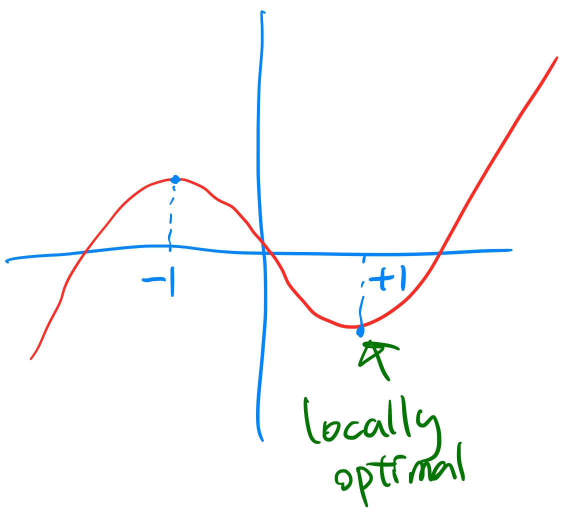
▷ x_0 is locally optimal if there exists $R > 0$ s.t. x_0 is optimal for

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{s.t.} & f_i(x) \le 0, \quad h_j(x) = 0, \\ & \|x - x_0\|_2 \le R. \end{array}$$

Ex. $f_0(x) = x^3 - 3x$

$f_0'(x) = 3(x^2 - 1)$

$= 3(x-1)(x+1)$



Feasibility Problem

$$\begin{aligned} \text{minimize} \quad & 0 \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i=1,\dots,m \\ & h_j(x) = 0, \quad j=1,\dots,r \end{aligned}$$

- $p^* = 0$ if the problem is feasible.
- $p^* = \infty$ if infeasible.

$$\begin{aligned} \text{Find} \quad & x \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i=1,\dots,m \\ & h_j(x) = 0, \quad j=1,\dots,p. \end{aligned}$$

Equivalent Problems

Informally two problems are equivalent if the solution of one is readily obtained from the solution of the other, and vice versa.

Ex. minimize $|ax+b|$

minimize $\alpha(ax+b)^2$, $\alpha > 0$.

Slack variables

$f_i(x) \le 0 \iff$ There exists $s_i \ge 0$ s.t.

$$f_i(x) + s_i = 0.$$

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{s.t.} & s_i \ge 0 & i=1,\dots,m \\ & f_i(x) + s_i = 0. & i=1,\dots,m \\ & h_j(x) = 0 & j=1,\dots,r. \end{array} \qquad (2)$$

- this problem has $n+m$ variables $x \in \mathbb{R}^n, s \in \mathbb{R}^m$
- If (x, s) is feasible for (2), then x is feasible for (1).
- If x is feasible for (1), choose $s_i = -f_i(x)$ and then (x, s) is feasible for (2).

Eliminating equality constraints

In the case that the equality constraints are linear

$$Ax = b \quad (Ax - b = 0)$$

- If $b \notin R(A) = \{Ax : x \in \mathbb{R}^n\}$, the problem is infeasible.
- If $b \in R(A)$. let x_0 be $Ax_0 = b$.

Let F be an $n \times k$ matrix, where $k = n - \text{rank}(A)$, such that

$$\underbrace{R(F)}_{\text{range of } F} = \underbrace{N(A)}_{\text{null space of } A} = \{x : Ax = 0\}.$$

$\{Fz : z \in \mathbb{R}^k\}$. $\quad (AF = 0)$,

Then a general solution to $Ax = b$ is

$$Fz + x_0, \quad z \in \mathbb{R}^k.$$

$$(A(Fz + x_0) = Ax_0 = b.)$$

$$\begin{aligned} \text{minimize} \quad & f_0(Fz + x_0) \\ \text{s.t.} \quad & f_i(Fz + x_0) \le 0 \\ & i = 1, \ldots, m. \end{aligned}$$

- variable : $z \in \mathbb{R}^k$
- no equality constraints
- rank(A) fewer variables.

Optimizing over some variables.

$$\inf_{x,y} f(x,y) = \inf_x \left(\inf_y f(x,y) \right)$$

(iterated infimum).

Proof.

① $\inf_{x',y'} f(x',y') \le f(x,y), \ \forall x,y,$

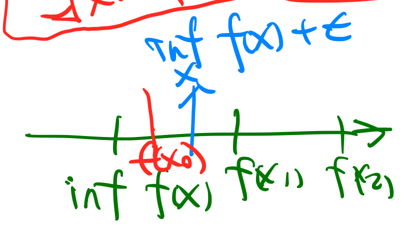
$\Rightarrow \inf_{x',y'} f(x',y') \le \inf_y f(x,y), \ \forall x,$

$\Rightarrow \inf_{x',y'} f(x',y') \le \inf_x \left(\inf_y f(x,y) \right).$

② Let $\epsilon > 0$, $\exists x_0, y_0$, s.t. $f(x_0,y_0) \le \inf_{x,y} f(x,y) + \epsilon$.

$$\inf_x \left(\inf_y f(x,y) \right) \le \inf_y f(x_0,y)$$
$$\le f(x_0,y_0)$$
$$\le \inf_{x,y} f(x,y) + \epsilon.$$

$\boxed{\exists x_0 \ f(x_0) < \inf f + \epsilon}$



We have $\inf_x \left(\inf_y f(x,y) \right) \le \inf_{x,y} f(x,y)$.

#.

$\Rightarrow$ We can minimize a function by first minimizing over some of the variables and then minimizing over the remaining variables.

$$\begin{aligned} \text{minimize} \quad & f(x_1, x_2) \\ \text{s.t.} \quad & f_i(x_1) \le 0 \quad i = 1, \dots, m_1 \\ & h_j(x_2) \le 0 \quad j = 1, \dots, m_2 \end{aligned}$$

Define $\tilde{f}_0(x_1) = \inf_{z} \{ f_0(x_1, z) : h_j(z) \le 0 \quad j = 1, \dots, m_2 \}$.

$$\begin{aligned} \text{minimize} \quad & \tilde{f}_0(x_1) \\ \text{s.t.} \quad & f_i(x_1) \le 0, \quad i = 1, \dots, m_1. \end{aligned}$$

Ex. $P_{11}, P_{22} \in S^n_{++}$.

$$\begin{aligned} \text{minimize} \quad & x_1^T P_{11} x_1 + x_2^T P_{22} x_2 + 2 x_1^T P_{12} x_2 \\ \text{s.t.} \quad & f_i(x_1) \le 0. \quad i = 1, \dots, m \end{aligned}$$

$\inf_{x_2} \left(x_1^T P_{11} x_1 + x_2^T P_{22} x_2 + 2 x_1^T P_{12} x_2 \right)$.

no constraints on x_2

$$0 = \frac{\partial}{\partial x_2}\left(\quad \right) = 2 x_2^T P_{22} + 2 x_1^T P_{12}.$$

$$\Rightarrow \quad x_2 = - P_{22}^{-1} P_{12}^T x_1.$$

$$\begin{aligned} \text{minimize} \quad & x_1^T \left(P_{11} - P_{12} P_{22}^{-1} P_{12}^T \right) x_1 \\ \text{s.t.} \quad & f_i(x_1) \le 0, \quad i = 1, \dots, m. \end{aligned}$$

Implicit & explicit constraints

- $f_i(x) \le 0$, $h_j(x) = 0$. are explicit constraints.
- The standard form optimization problem has an implicit constraint

$$x \in \mathcal{D} = \left(\bigcap_i \operatorname{dom} f_i\right) \cap \left(\bigcap_j \operatorname{dom} h_j\right).$$

$\triangle$ The unconstrained problem

$$\text{minimize} \quad F(x)$$

where $F(x) = f_0(x)$ for $x \in \operatorname{dom} F$

$$\operatorname{dom} F = \{x \in \operatorname{dom} f_0 : f_i(x) \le 0,\ h_j(x) = 0\}.$$

Ex. $f(x) = \begin{cases} x^T x, & \text{if } Ax = b; \\ \infty, & \text{otherwise.} \end{cases}$

minimize $f(x)$. $\iff$ $\begin{array}{ll} \text{minimize} & x^T x \\ \text{s.t} & Ax = b. \end{array}$

Convex optimization Problems

standard form

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i=1,\dots,m \\ & a_j^T x = b_j, \quad j=1,\dots,r. \\ & (Ax=b). \end{aligned}$$

— $f_0, f_1, \dots, f_m$ are convex.

Note: The feasible set of a convex optimization problem is convex.

If x, y feasible, then $\theta x + (1-\theta) y$ is feasible for $0 \le \theta \le 1$. $f_i(\theta x + (1-\theta) y) \le \theta f_i(x) + (1-\theta) f_i(y)$

- If the objective function is strictly convex, then the set of optimal points contains at most one point.

If not, x, y optimal. $f_0(x) = f_0(y) = p^*$.

$\theta x + (1-\theta) y$ is feasible.

$f_0(\theta x + (1-\theta) y) < \theta f_0(x) + (1-\theta) f_0(y) = p^*$.

a contradiction that p^* is optimal.

Example:

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(x) = \frac{x_1}{1+x_2^2} \le 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0. \end{aligned}$$

This is not a convex optimization problem.

h_1 is not affine

f_1 is not convex.

An equivalent problem:

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & \tilde{f}_1(x) = x_1 \le 0. \\ & \tilde{h}_1(x) = x_1 + x_2 = 0. \end{aligned}$$

(convex).

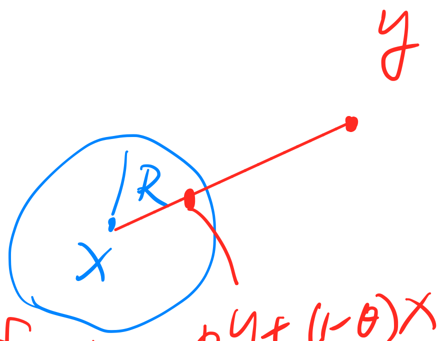
▷ Any locally optimal point of a convex problem is also (globally) optimal.

proof. Suppose that x is locally optimal. $\exists R>0$ s.t. for feasible z and $\|z-x\|_2<R$, we have $f_0(z)\geq f_0(x)$.

Assume that $\exists$ feasible y with $f_0(y)<f_0(x)$.

Consider $z_0=\theta y+(1-\theta)x$

$$\theta=\frac{R}{2\|y-x\|_2}<\frac{1}{2}$$



① z_0 is feasible.

$f_0(z_0)\leq\theta f_0(y)+(1-\theta)f_0(x)<f_0(x)$

$(f_0(y)<f_0(x))$.

② Also $\|z_0-x\|_2=\|\theta y+(1-\theta)x-x\|_2=\theta\|y-x\|_2$

$=\frac{R}{2}<R$.

$\begin{cases} z_0\in B(x,R) \\ f_0(z_0)<f_0(x) \end{cases}$

a contradiction that x is locally optimal.