Assignment #5

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ECM5901 - Optimization Theory and Application

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Exercise 1. Let $A_0 = \begin{bmatrix} 10 & 8 & 12 & 15 & 15 \\ 8 & 14 & 8 & 7 & 9 \\ 12 & 8 & 10 & 13 & 9 \\ 15 & 7 & 13 & 4 & 10 \\ 15 & 9 & 9 & 10 & 4 \end{bmatrix}$, $A_1 = \begin{bmatrix} 12 & 11 & 14 & 10 & 3 \\ 11 & 14 & 10 & 14 & 6 \\ 14 & 10 & 16 & 18 & 4 \\ 10 & 14 & 18 & 18 & 8 \\ 3 & 6 & 4 & 8 & 8 \end{bmatrix}$,
$A_2 = \begin{bmatrix} 4 & 13 & 12 & 16 & 6 \\ 13 & 4 & 14 & 9 & 15 \\ 12 & 14 & 6 & 5 & 5 \\ 16 & 9 & 5 & 2 & 6 \\ 6 & 15 & 5 & 6 & 8 \end{bmatrix}$
Suppose $A : \mathbb{R}^2 \to \mathbb{S}^5$ is defined by

$$A(x) = A_0 + x_1 A_1 + x_2 A_2.$$

Let $\lambda_1(x) \geq \lambda_2(x) \geq \lambda_3(x) \geq \lambda_4(x) \geq \lambda_5(x)$ denoted the eigenvalues of $A(x)$.

(a) Formulate the problem of minimizing the spread of the eigenvalues $\lambda_1(x) - \lambda_5(x)$ as an SDP. (15%)

(b) Solve (a) by using MATLAB with the CVX tool. What are the optimal point and optimal value? (25%)

Solution.

(a) Introduce additional variable $t = (t_1, t_2)$. We can use the property that $\lambda_i(x) \leq s$ ($\lambda_i(x) \geq s$) if and only if $A(x) \preceq sI$ ($A(x) \succeq sI$), then we have

$$\begin{cases} \lambda_1(x) \leq t_1 \text{ if and only if } A(x) \preceq t_1 I \\ -\lambda_5(x) \leq -t_2 \text{ if and only if } A(x) \succeq t_2 I \end{cases}$$

Therefore, the problem to minimize $\lambda_1(x) - \lambda_5(x)$ becomes

$$\begin{array}{ll} \text{minimize} & t_1 - t_2 \\ \text{subject to} & t_2 I \preceq A(x) \preceq t_1 I \end{array}$$

It is a semidefinite program (SDP).

(b) MATLAB code

```
% file: hw5_1.m
% assign matrices A0, A1, and A2

cvx_begin sdp quiet
  variables x(2) t(2)
  minimize(t(1)-t(2))
  t(2) * eye(5) <= A0 + x(1) * A1 + x(2) * A2
  A0 + x(1) * A1 + x(2) * A2 <= t(1) * eye(5)
cvx_end

disp(['Optimal value: ', sprintf('%f', cvx_optval)]);
disp('Optimal point:');
disp(['  x = [ ', sprintf('%f ', x ), ']']);
disp(['  t = [ ', sprintf('%f ', t ), ']']);
```

Result

```
>> run hw5_1.m
Optimal value: 28.154385
Optimal point:
  x = [ -0.596605 -0.335843 ]
  t = [ 13.759723 -14.394662 ]
```

Exercise 2. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x+1)(x+4) \leq 0 \end{array}$$

with variable $x \in \mathbb{R}$.

(a) (Analysis of primal problem.) Give the feasible set, the optimal value, and the optimal solution. (5%)

(b) Derive the Lagrange dual function g. (5%)

(c) State the dual problem, and verify that it is a concave maximization problem. (5%)

(d) Find the dual optimal value and dual optimal solution? Does the strong duality hold? (5%)

Solution.

(a) The objective $f_0(x) = x^2 + 1$ is a concave upward parabola with minimization value at point $x = 0$. From the constraint function $f_1(x) = (x+1)(x+4) \leq 0$, we know the feasible set is the interval $[-4, -1]$. Therefore, the optimal point is at $x^* = -1$ with the optimal value $p^* = (x^*)^2 + 1 = 2$.

(b) The Lagrange is

$$L(x,\lambda) = f_0(x) + \lambda f_1(x) = (x^2+1) + \lambda(x+1)(x+4) = (\lambda+1)x^2 + 5\lambda x + (4\lambda+1)$$

The Lagrange dual function is

$$\begin{aligned} g(\lambda) &= \inf_{x\in\mathcal{D}} L(x,\lambda) \\ &= \inf_{x\in\mathcal{D}} \big((\lambda+1)x^2 + 5\lambda x + (4\lambda+1)\big) \\ &= \begin{cases} 4\lambda+1-\dfrac{25\lambda^2}{4\lambda+4}, & \lambda > -1 \\ -\infty, & \lambda \le -1 \end{cases} \end{aligned}$$

where we compute the optimal value of parabola by the formula $(4ac-b^2)/4a$.

(c) The Lagrange dual problem is

$$\begin{aligned} &\text{maximize} \quad g(\lambda) = \begin{cases} 4\lambda+1-\dfrac{25\lambda^2}{4\lambda+4}, & \lambda > -1 \\ -\infty, & \lambda \le -1 \end{cases} \\ &\text{subject to} \quad \lambda \ge 0 \end{aligned}$$

With $\lambda \ge 0$ and abuse of the terminology, the Lagrange dual problem is

$$\begin{aligned} &\text{maximize} \quad g(\lambda) = 4\lambda+1-\frac{25\lambda^2}{4\lambda+4} \\ &\text{subject to} \quad \lambda \ge 0 \end{aligned}$$

$f(\lambda)$ is a concave function since

(i) $4\lambda+1$ is an affine function

(ii) $\dfrac{25\lambda^2}{4\lambda+4}$ is affine mapping of the quadratic-over-linear function, thus convex

(iii) $g(\lambda)$ is an affine function minus a convex function, thus concave

The Lagrange dual problem is a concave maximization problem.

(d) Find optimal point by the derivative of g

$$g'(\lambda)|_{\lambda=\lambda^*} = 4 - \frac{25\lambda^*(\lambda^*+2)}{4(\lambda^*+1)^2} = 0 \Rightarrow \lambda^* = \frac{2}{3}, -\frac{8}{3} \text{ (invalid)}$$

Then the optimal value is

$$d^* = g(\lambda^*) = g(\frac{2}{3}) = \frac{8}{3} + 1 - \frac{\frac{100}{9}}{\frac{8}{3}+4} = 2$$

Therefore the strong duality $d^* = p^*$ holds.

Exercise 3. (Dual of general LP). Find the dual function of the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Give the dual problem, and make the implicit equality constraints explicit. (20%)

Solution.

(a) Lagrange dual function
The Lagrange is

$$\begin{aligned} L(x, \lambda, \nu) &= f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \\ &= c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b) \\ &= -\lambda^T h - \nu^T b + (c + G^T \lambda + A^T \nu)^T x \end{aligned}$$

where h_i is equality constraint function and unrelated to h in the origin problem.
The Lagrange dual function is

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= -\lambda^T h - \nu^T b + \inf_{x \in \mathcal{D}} (c + G^T \lambda + A^T \nu)^T x \\ &= \begin{cases} -\lambda^T h - \nu^T b, & c + G^T \lambda + A^T \nu = 0 \\ -\infty, & \text{otherwise} \end{cases} \end{aligned}$$

(b) Lagrange dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) = \begin{cases} -\lambda^T h - \nu^T b, & c + G^T \lambda + A^T \nu = 0 \\ -\infty, & \text{otherwise} \end{cases} \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

(c) Make dual constraints explicit

$$\begin{array}{ll} \text{maximize} & -\lambda^T h - \nu^T b \\ \text{subject to} & c + G^T \lambda + A^T \nu = 0 \\ & \lambda \succeq 0 \end{array}$$

Exercise 4. Derive a dual problem for

$$\text{minimize} \quad -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

with domain $\{x \mid a_i^T x < b_i,\ i = 1, \ldots, m\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^T x$. (20%)

Solution. The original problem is equivalent to

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^{m} \log y_i \\ \text{subject to} & Ax + y - b = 0 \end{array}$$

where x and y are variables, and $A = [a_1\ a_2\ \cdots\ a_m]^T \in \mathbb{R}^{m \times n}$
The Lagrange is

$$\begin{aligned} L(x, y, \nu) &= f_0(x) + \sum_{i=1}^{p} \nu_i h_i(x) \\ &= -\sum_{i=1}^{m} \log y_i + \nu^T (Ax + y - b) \end{aligned}$$

The Lagrange dual function is

$$\begin{aligned} g(\nu) &= \inf_{x,y \in \mathcal{D}} L(x, y, \nu) \\ &= \inf_{x,y \in \mathcal{D}} \left(-\sum_{i=1}^{m} \log y_i + \nu^T (Ax + y - b) \right) \\ &= \begin{cases} \sum_{i=1}^{m} \log \nu_i + m - b^T \nu, & A^T \nu = 0,\ \nu \succ 0 \\ -\infty, & \text{otherwise} \end{cases} \end{aligned}$$

and we use these properties to compute the result

(i) $\nu^T Ax$ is unbounded below, or is zero when $A^T \nu = 0$

(ii) since $y \succ 0$ is the domain of y, $\nu^T y$ is unbounded below if $\nu \not\succ 0$

(iii) by analysis the derivative of terms in y, we know that it achieves the minimum at $y_i = 1/\nu_i$, *i.e.*, $-\sum_{i=1}^{m} \log y_i + \nu^T y = \sum_{i=1}^{m} \log \nu_i + m$

The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} \log \nu_i + m - b^T \nu \\ \text{subject to} & A^T \nu = 0 \\ & \nu \succ 0 \end{array}$$