Assignment #4

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Exercise 1. For a subspace V of $\mathbb{R}^n$, its orthogonal complement, denoted by $V^\perp$, is defined as $V^\perp = \{x \mid \langle x, z\rangle = 0 \text{ for all } z \in V\}$. Suppose that $A \in \mathbb{R}^{m\times n}$. Show that

$$\mathcal{N}(A) = (\mathcal{R}(A^T))^\perp.$$

where $\mathcal{N}(A)$ is the null space of A and $\mathcal{R}(A^T)$ is the range of A^T. (15%)

Proof. The definition of range is $\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$, and the definition of nullspace is $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$. Let $A = [a_1\ a_2\ \cdots\ a_m]^T$ where a_i are row vectors of A, and $A^T = [a_1\ a_2\ \cdots\ a_m]$ where a_i are column vectors of A^T.

Separate the proof into two parts.

(i) $\mathcal{N}(A) \subseteq (\mathcal{R}(A^T))^\perp$, *i.e.*, $v \in \mathcal{N}(A) \Rightarrow v \in (\mathcal{R}(A^T))^\perp$

$$\begin{aligned}
v \in \mathcal{N}(A) &\Rightarrow Av = 0 \\
&\Rightarrow a_i^T v = 0,\ i = 1, \ldots, m \\
&\Rightarrow c_1 a_1^T v + c_2 a_2^T v + \cdots + c_m a_m^T v = 0 \\
&\Rightarrow w^T v = 0 \quad \forall\, w = c_1 a_1 + c_2 a_2 + \cdots + c_m a_m = A^T c \\
&\Rightarrow w^T v = 0 \quad \forall\, w \in \mathcal{R}(A^T) \\
&\Rightarrow v \in (\mathcal{R}(A^T))^\perp
\end{aligned}$$

where $c = (c_1, c_2, \ldots, c_m) \in \mathbb{R}^m$

(ii) $\mathcal{N}(A) \supseteq (\mathcal{R}(A^T))^\perp$, *i.e.*, $v \in (\mathcal{R}(A^T))^\perp \Rightarrow v \in \mathcal{N}(A)$

$$\begin{aligned}
v \in (\mathcal{R}(A^T))^\perp &\Rightarrow w^T v = 0 \quad \forall\, w \in \mathcal{R}(A^T) \\
&\Rightarrow w^T v = 0 \quad \forall\, w = A^T c = c_1 a_1 + c_2 a_2 + \cdots + c_m a_m \\
&\Rightarrow a_i^T v = 0,\ i = 1, \ldots, m \\
&\Rightarrow Av = 0
\end{aligned}$$

where choose $c = e_1, e_2, \ldots, e_m$

(iii) From (i) and (ii), we know that $\mathcal{N}(A) = (\mathcal{R}(A^T))^\perp$ □

Exercise 2. Prove that $x^* = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = (1/2)x^T Px + q^T x + r \\ \text{subject to} & -1 \leq x_i \leq 1, \; i = 1, 2, 3 \end{array}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \; q = \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}, \; r = 1.$$

(20%)

Proof. First check whether $f_0(x)$ is convex. Since f_0 is twice differentiable and $P \in \mathbb{S}^3$, *i.e.*, check $\nabla^2 f_0(x) = P \succeq 0$. After some calculations, eigenvalues of P are $(\lambda_1, \lambda_2, \lambda_3) \approx (27.898, 13.843, 0.259) \succeq 0$. Therefore, P is positive semidefinite.

Since f_0 is convex and inequality constraint functions are convex (intersection of halfspaces), the original problem becomes a convex optimization problem. Thus we can apply equation (4.21) in §4.2.3 to check whether $x^* = (1, 1/2, -1)$ is optimal.

$$\begin{aligned} \nabla f_0(x^*)^T (y - x^*) &= \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 - 1 \\ y_2 - 1/2 \\ y_3 + 1 \end{bmatrix} \\ &= -1(y_1 - 1) + 2(y_3 + 1) \geq 0 \end{aligned} \tag{1}$$

Since inequality (1) holds for all y in the feasible set, x^* is proved to be an optimal. □

Exercise 3. Consider a problem of the form

$$\begin{array}{ll} \text{minimize} & f_0(x)/(c^T x + d) \\ \text{subject to} & f_i(x) \leq 0, \; i = 1, \ldots, m \\ & Ax = b \end{array} \tag{2}$$

where $f_0, f_1, \ldots, f_m$ are convex, and the domain of the objective function is defined as $\{x \in \mathbf{dom}\, f_0 \mid c^T x + d > 0\}$.

(a) Show that the problem is equivalent to

$$\begin{array}{ll} \text{minimize} & g_0(y, t) \\ \text{subject to} & g_i(y, t) \leq 0, \; i = 1, \ldots, m \\ & Ay = bt \\ & c^T y + dt = 1 \end{array} \tag{3}$$

where g_i is the perspective of f_i (see §3.2.6). The variables are $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. (15%)

(b) Show that this problem is convex. (10%)

Proof. We have $g_i(y,t) = tf_i(y/t)$ by definition of perspective function.

(a) Separate the proof into two parts.

(i) If x is feasible in (2), then (y,t) is feasible in (3).
Define $y = \dfrac{x}{c^Tx+d}$ and $t = \dfrac{1}{c^Tx+d} > 0$ (since $x \in \mathbf{dom}\, f_0$), then

$$\begin{cases} (y,t) \in \mathbf{dom}\, g_i = \{(y,t) \mid y/t \in \mathbf{dom}\, f_i,\ t > 0\},\ i = 1,\dots,m \\ g_i(y,t) = tf_i(y/t) = tf_i(x) \le 0,\ i = 1,\dots,m \\ Ay = \dfrac{Ax}{c^Tx+d} = \dfrac{b}{c^Tx+d} = bt \\ c^Ty + dt = \dfrac{c^Tx+d}{c^Tx+d} = 1 \end{cases}$$

Thus (y,t) is feasible with $g_0(y,t) = f_0(x)/(c^Tx+d)$.

(ii) If (y,t) is feasible in (3), then x is feasible in (2).
Define $x = y/t$ ($t > 0$ by definition of the perspective function), then

$$\begin{cases} (y,t) \in \mathbf{dom}\, g_i = \{(y,t) \mid y/t \in \mathbf{dom}\, f_i,\ t > 0\} \Rightarrow x \in \mathbf{dom}\, f_i,\ i = 1,\dots,m \\ f_i(x) = f_i(y/t) = g_i(y,t)/t \le 0,\ i = 1,\dots,m \\ Ax = \dfrac{Ay}{t} = b \end{cases}$$

Thus x is feasible with $f_0(x)/(c^Tx+d) = g_0(y,t)/(c^Txt+dt) = g_0(y,t)$.

(iii) From (i) and (ii), we know that problem (2) is equivalent to problem (3).

(b) As discussed in §3.2.6, the perspective operation preserves convexity, *i.e.*, $g_i(y,t)$ is convex since $f_i(x)$ is convex where $i = 0,\dots,m$. In conclusion, the problem (3) have a convex objective function, m convex inequality constraint functions, and the row mumber of A plus 1 affine equality constraint functions. Therefore, (3) is a convex optimization problem. □

Exercise 4. (Network flow problem.) Consider a network of n nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: x_{ij} will denote the flow from node i to node j. The cost of the flow along the link from node i to node j is given by $c_{ij}x_{ij}$, where c_ij are given constants. The total cost across the network is

$$C = \sum_{i,j=1}^{n} c_{ij}x_{ij}$$

Each link flow x_{ij} is also subject to a given lower bound l_{ij} (usually assumed to be nonnegative) and an upper bound u_{ij}. The external supply at node i is given by b_i, where $b_i > 0$ means an external flow enters the network at node i, and $b_i < 0$ means that at node i, an amount $|b_i|$ flows out of the network. We assume that $\sum_i b_i = 0$, *i.e.*, the total external supply equals total external demand. At each node we have conservation of flow: the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero. The problem is to minimize the total cost of flow through the network, subject to the constraints described above. Formulate this problem as a linear program. (20%)

Solution. The problem can be formulated by linear programming

$$\begin{array}{ll} \text{minimize} & C = \sum_{i,j=1}^{n} c_{ij} x_{ij} \\ \text{subject to} & b_i + \sum_{j=1}^{n} x_{ji} - \sum_{j=1}^{n} x_{ij} = 0,\ i = 1, \dots, n \\ & l_{ij} \leq x_{ij} \leq u_{ij} \end{array}$$

Exercise 5. Give an explicit solution of the following **QCQP**s. (Minimizing a linear function over an ellipsoid)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (x - x_c)^T A (x - x_c) \leq 1 \end{array}$$

where $A \in \mathbb{S}_{++}^n$ and $c \neq 0$. (20%)

Solution. Let $y = A^{1/2}(x - x_c)$, *i.e.*, $x = A^{-1/2} y + x_c$, then the problem becomes

$$\begin{array}{ll} \text{minimize} & f_0(x) = c^T A^{-1/2} y + c^T x_c \\ \text{subject to} & y^T y \leq 1 \end{array}$$

We have $\nabla f_0(x) = A^{1/2} c$, and the solution is $y^* = -\dfrac{A^{1/2} c}{\|A^{1/2} c\|_2}$, *i.e.*,

$$x^* = x_c - \frac{A^{-1}}{\|A^{1/2} c\|_2} c$$