$$L = -b^T v + \langle y, x \rangle.$$

$$\inf_x L(v, x) = -b^T v + \inf_x \langle y, x \rangle.$$

$$\inf_x \langle y, x \rangle = 0 \quad \text{if} \quad y = 0.$$

If $y \neq 0$. $\inf_x \langle y, x \rangle = -\infty$.

$$x = y \qquad \langle y, x \rangle = \|y\|^2$$

$$x = -1000\, y \quad \langle y, x \rangle = -1000\, \|y\|^2.$$

$$x = a \cdot y \quad \langle y, x \rangle = a\, \|y\|^2.$$

$$a \to -\infty, \quad \langle y, x \rangle \to -\infty.$$

For $\lambda \geq 0$, any ν. $g(\lambda, \nu) \leq p^*$.

The Lagrange dual problem associated with (1) is defined as

$$\begin{aligned} &\text{maximize} && g(\lambda, \nu) \\ &\text{subject to} && \lambda \geq 0. \end{aligned}$$

- Find the best lower bound on p^* that can be obtained from the dual function.
- Let d^* to denote the optimal value of the dual problem.
- λ, ν are dual feasible if $\lambda \geq 0$ and $\lambda, \nu \in \operatorname{dom} g = \{(\lambda, \nu) : g(\lambda, \nu) > -\infty\}$.
- (λ^*, ν^*) are dual optimal if they are optimal for the problem

$$g(\lambda^*, \nu^*) = d^*.$$

Lagrange dual of LP.

Standard form

minimize $c^T x$

s.t. $Ax = b$

$x \geq 0.$

Inequality form

minimize $c^T x$

s.t. $Ax \leq b.$

– dual function

$$g(\lambda, \nu) = \begin{cases} -b^T \nu, & \text{if } c + A^T \nu - \lambda = 0. \\ -\infty. & \text{otherwise.} \end{cases}$$

$$g(\lambda, \nu) = \begin{cases} -b^T \lambda, & \text{if } c + A^T \lambda = 0. \\ -\infty. & \text{otherwise.} \end{cases}$$

– dual problem

maximize $-b^T \nu$

s.t. $\begin{cases} c + A^T \nu - \lambda = 0 \\ \lambda \geq 0 \end{cases}$

$c + A^T \nu = \lambda \geq 0.$

$\Leftrightarrow$

$$\boxed{\begin{array}{ll} \text{maximize} & -b^T \nu \\ \text{s.t.} & c + A^T \nu \geq 0. \end{array}}$$

also a LP

maximize $-b^T \lambda$

s.t. $c + A^T \lambda = 0.$

$\lambda \geq 0.$

also a LP.

compare

Weak duality : $d^* \le p^*$

always holds (for convex and nonconvex problem).

- The difference $p^* - d^*$ (≥ 0) is called the optimal duality gap of the original problem.

- The dual problem is always convex

$\Rightarrow$ a nontrivial lower bound for a difficult problem.

Example. minimize $x^T W x$. $W \in S^n$

s.t. $x_i^2 = 1$, $i = 1, \dots, n$. $x \in \mathbb{R}^n$.

$x_i \in \{+1, -1\}$.

The feasible set is $x \in \{+1, -1\}^n$. ← discrete. of 2^n points

ex. $x = (+1), (+1), (+1), \dots, (-1), (-1)$.

- This problem is not convex.

— The Lagrangian is

$$L(x,\nu) = x^T W x + \boxed{\sum_i \nu_i (x_i^2 - 1)}$$

$$= x^T W x - \sum_i \nu_i + \underbrace{(x_1\ x_2 \cdots x_n) \begin{pmatrix} \nu_1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_n \end{pmatrix}}_{\mathrm{diag}(\nu)} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x^T (W + \mathrm{diag}(\nu)) x - 1^T \nu$$

this is the all-1 vector.

— $g(\nu) = \inf_x L(x,\nu)$

$$= -1^T\nu + \inf_x x^T (W + \underbrace{\mathrm{diag}(\nu)}_{= \sum \lambda_i y_i y_i^T}) x$$

$\sum_i \lambda_i \|x_i^T y_i\|_2^2$.

if $\lambda_j < \infty$. $x = -y_j \cdot a$

$$= \begin{cases} -1^T\nu, & \text{if } W + \mathrm{diag}(\nu) \succeq 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

choose $x = 0$. in the inf ( )

This is not necessarily optimal
simply a lower bound for p^*

— The dual problem is (SDP).

maximize $-1^T\nu = -\sum_i \nu_i$

s.t. $W + \mathrm{diag}(\nu) \succeq 0$.

minimize $\sum_i \nu_i$

$W = U \begin{pmatrix} \pi_1 & & & \\ & \pi_2 & & \\ & & \ddots & \\ & & & \pi_n \end{pmatrix} U^T$

$W + \mathrm{diag}(\nu) = U \begin{pmatrix} \pi_1+\nu_1 & & & \\ & \pi_2+\nu_2 & & \\ & & \ddots & \\ & & & \pi_n+\nu_n \end{pmatrix} U^T \succeq 0$

$\nu_i \geq -\pi_{min}$

Choose $\mathrm{diag}\,\nu = -\pi_{min} \cdot \mathbb{1}$

$\Rightarrow p^* \geq n\pi_{min}$

Strong duality $d^* = p^*$.

- does not hold in general.
- It $\boxed{\text{usually}}$ holds for convex problems.

- Slater's constraint qualification.

The strong duality holds for the convex problem (1). if there exists an x^*
$x^* \in \text{relint } D$ such that.

$$f_i(x) < 0 \quad i = 1, \cdots, m.$$

$$Ax = b$$

(The problem is strictly feasible.).

Then $\exists$ a dual feasible (λ^*, ν^*) with
$g(\lambda^*, \nu^*) = d^* = p^*$ if $d^* \neq -\infty$.

(See P. 234).