Let S be a set in $\mathbb{R}^n$.

$$f(S) = \{f(x) : x \in S\}.$$

S_1, S_2. two sets.

$$S_1 + S_2 = \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\}.$$

$$S_1 \times S_2 = \{\underline{(x_1, x_2)} : x_1 \in S_1, x_2 \in S_2\}.$$

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}.$$

$$\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}\}.$$

The perspective function

$P: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ with $\text{dom} P = \mathbb{R}^n \times \mathbb{R}_{++}$.

$(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. $t > 0$.

$\mathbb{R}_{++} = \{x \in \mathbb{R}; x > 0\}$

$$P(x,t) = \frac{x}{t}.$$

Ex. $P(\underbrace{1,2}_{x},\underbrace{3}_{t}) = \underbrace{\left(\frac{1}{3},\frac{2}{3}\right)}_{\frac{x}{t}}$.

If $C \subseteq \text{dom} P$ is convex, then $P(C) = \{P(x) : x \in C\}$ is convex.

proof. for $x = (\hat{x}, x_{n+1})$, $y = (\hat{y}, y_{n+1}) \in \mathbb{R}^{n+1}$ where $x_{n+1}, y_{n+1} > 0$.

$$P(x) = \frac{\hat{x}}{x_{n+1}}, \quad P(y) = \frac{\hat{y}}{y_{n+1}}.$$

For $0 \le \theta \le 1$, $\theta x + (1-\theta) y = (\theta \hat{x} + (1-\theta)\hat{y}, \theta x_{n+1} + (1-\theta) y_{n+1})$

$\in C$.

Since C is convex

$$P(\theta x + (1-\theta)y) = \frac{\theta \hat{x} + (1-\theta)\hat{y}}{\theta x_{n+1} + (1-\theta) y_{n+1}} \in P(C).$$

Let $\mu = \dfrac{\theta x_{n+1}}{\theta x_{n+1} + (1-\theta) y_{n+1}} \in [0,1]$.

μ is monotonic in θ for $\theta \in [0,1]$.

$$= \mu P(x) + (1-\mu) P(y).$$

$\gamma P(x) + (1-\gamma) P(y) \in [P(x), P(y)] \subseteq P(C)$.

$\therefore P(\underbrace{[x,y]}_{\text{line segment in } \mathbb{R}^{n+1}}) = \underbrace{[P(x), P(y)]}_{\text{line segment in } \mathbb{R}^n} \subseteq P(C)$.

$\Rightarrow P(C)$ is convex.

If $C \subseteq \mathbb{R}^n$ is convex, then

$$P^{-1}(C) = \left\{ (x,t) \in \mathbb{R}^{n+1} : t > 0, \frac{x}{t} \in C \right\}$$

is convex. (Exercise).

$$P^{-1}\left(\frac{x}{t}\right) = (ax, at) \quad a \neq 0$$

$$P(ax, at) = \frac{x}{t} \quad \forall a \neq 0.$$

$P : \mathbb{R}^{n+1} \to \mathbb{R}^n$. is <u>not</u> one-to-one

Linear fractional functions.

Suppose that $g : \mathbb{R}^n \to \mathbb{R}^{m+1}$ is affine.

$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} Ax+b \\ c^Tx+d \end{bmatrix}, \text{ where } A \in \mathbb{R}^{m\times n}, \; b \in \mathbb{R}^m, \; c \in \mathbb{R}^n, \; d \in \mathbb{R}.$$

The function $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$f(x) = P \circ g(x) = \frac{Ax+b}{c^Tx+d}$$

, with $\operatorname{dom} f = \{x : c^Tx+d > 0\}$,

is called a linear fractional function.

If $C \subseteq \operatorname{dom} f$ is convex, then

$$f(C) = \underbrace{P}_{\text{perspective function}} \circ \underbrace{g}_{\text{affine function.}}(C) \text{ is convex.}$$

(or you can prove it by definition.)

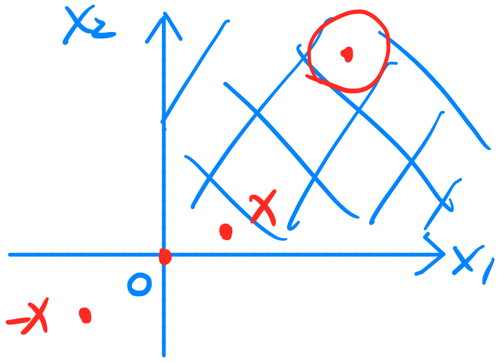
If $C \subseteq \mathbb{R}^n$ is convex, then $f^{-1}(C)$ is convex.

Generalized inequalities

A cone $K \subseteq \mathbb{R}^n$ is called a proper cone if

- K is convex. if $x \in \text{int } K$ if $\exists\, \epsilon > 0$ s.t. $\{y : \|y-x\|_2 \le \varepsilon\} \subseteq K$.
- K is closed
- K is solid, ($\text{int } K \ne \phi$)
- K is pointed. (If $x, -x \in K$, we have $x=0$).

Ex. $\mathbb{R}^2_+$:



$\text{bd}(\mathbb{R}^2_+) = \{(x, y) : x_1 = 0, x_2 \ge 0$ or $x_1 \ge 0, x_2 = 0\}$

A proper cone defines a partial ordering on $\mathbb{R}^n$.

For $x, y \in \mathbb{R}^n$.

$$x \preceq_K y \iff y - x \in K. \quad y \succeq_K x$$

$$(\preceq_K.\ \le_K)$$

Similary, $x \prec_K y \iff y - x \in \text{int } K$.

or $(y \succ_K x)$.

Ex

$K = \mathbb{R}^2_+$. If $(x_1, x_2) \preceq_K (y_1, y_2)$

$(y_1 - x_1, y_2 - x_2) \in K = \mathbb{R}^2_+$.

$\Leftrightarrow \quad y_1 - x_1 \geq 0, \quad y_2 - x_2 \geq 0.$

Ex $\quad (1, 2) \prec_K (3, 4)$. $\quad (3,4) - (1,2) = (2,2) \in \operatorname{int} K$.

$(1, 3) \qquad (0, 4)$

$(1, 2) \preceq_K (1, 3) \quad \underbrace{(1,3) - (1,2) = (0,1) \in K}_{\in \operatorname{bd}(K)}$.

Ex. $K = S^n_+$ is a $\boxed{\text{proper cone}}$ in S^n? verify it yourself

$X, Y \in S^n_+$. $\quad X \preceq_K Y \Leftrightarrow Y - X \in K = S^n_+$

$Y - X \succeq 0$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$.

$1 > 0.5$

$0 < 0.5$.

Properties of $\preceq_K$. K: proper cone.

1. If $x \preceq_K y$, $u \preceq_K v$, then $x+u \preceq_K y+v$.

Proof: $y-x \in K$. $v-u \in K$.

$\because$ K is convex, $(y-x)+(v-u) \in K$

$$\begin{gathered}(y-x)+(v-u) \\ \| \\ (y+v)-(x+u) \in K\end{gathered}$$

$\therefore$ $x+u \preceq_K y+v$ by definition.

2. If $x \preceq_K y$, $y \preceq_K z$, then $x \preceq_K z$.

Proof. $y-x \in K$. $z-y \in K$.

$$\Rightarrow \underbrace{y-x+z-y}_{= z-x} \in K.$$

3. nonnegative scaling.

If $x \preceq_K y$, $\alpha \geq 0$, then $\alpha x \preceq_K \alpha y$

(Check it).

4. reflexive: $x \preceq_K x$

$0 \in K$

$\boxed{x, -x \in K \Rightarrow x = 0.}$

5. If $x \preceq_K y$, $y \preceq_K x$, then $y = x$.

proof. $x-y \in K$. $y-x \in K$.

$$y-x = -(x-y).$$

$\because$ K is pointed. $x-y=0 \Rightarrow y=x$. #.