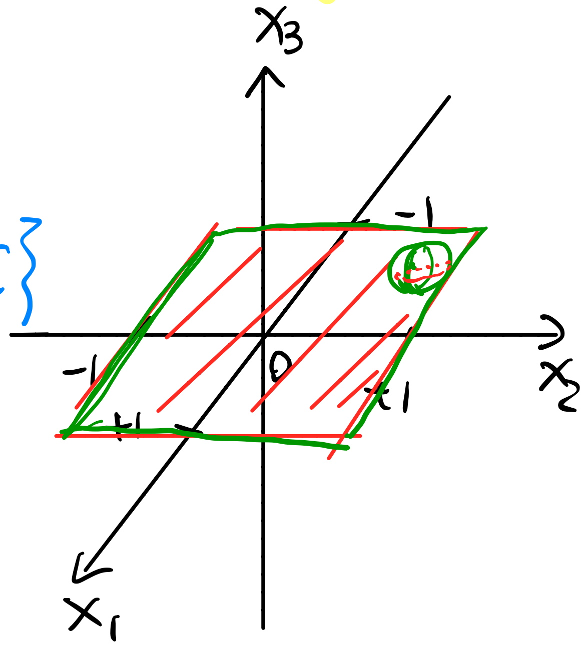
relative boundary of C is defined as

$$\mathrm{cl}\, C \setminus \mathrm{relint}\, C.$$

Ex. $C = \{x \in \mathbb{R}^3 : -1 \le x_1 \le 1, -1 \le x_2 \le 1, x_3 = 0\}$



$\mathrm{aff}(C) = \{x \in \mathbb{R}^3 : x_3 = 0\}$.

$\mathrm{int}(C) = \{y \in \mathbb{R}^3 : \exists\, B(y,r) \subset C\}$
$r > 0$.

$= \emptyset$ empty set.

$\mathrm{relint}(C) = \{x \in \mathbb{R}^3 : -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\}$

$\mathrm{bd}\, C = C$

relative boundary of C is

$$\{x \in \mathbb{R}^3 : \max\{|x_1|, |x_2|\} = 1, x_3 = 0\}.$$

$\mathrm{cl}(C) = C$ since its complement $\mathbb{R}^3 \setminus C$ is open. That is, for $x \in \mathbb{R}^3 \setminus C$, $\exists\, \epsilon > 0$ s.t.

$$B(x, \epsilon) \in \mathbb{R}^3 \setminus C.$$

Convex sets

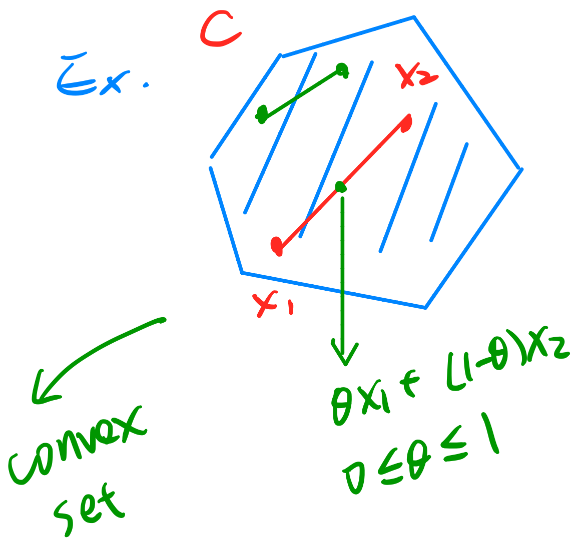
A set C is convex if $x_1, x_2 \in C$, $0 \le \theta \le 1$,

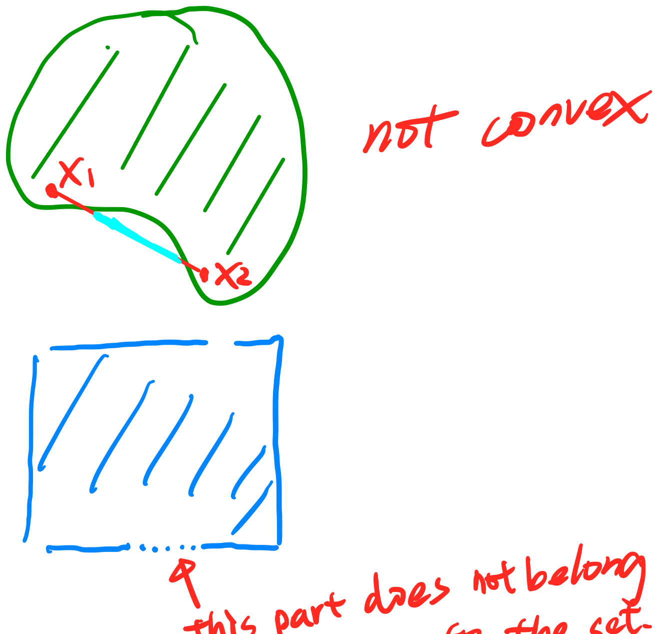
we have $\theta x_1 + (1-\theta) x_2 \in C$.

$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$, $\sum_i \theta_i = 1$, $\theta_i \ge 0$,

is a convex combination of $x_1, \ldots, x_k$

Ex.



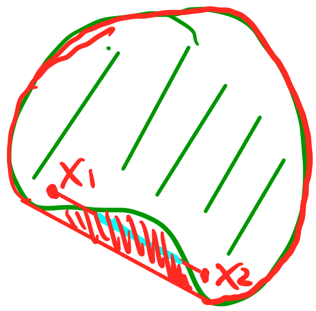


$\Rightarrow$ not convex.

convex hull of C

$$= \{ \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k : x_i \in C, \theta_i \ge 0, \sum_i \theta_i = 1 \}$$

Ex. C

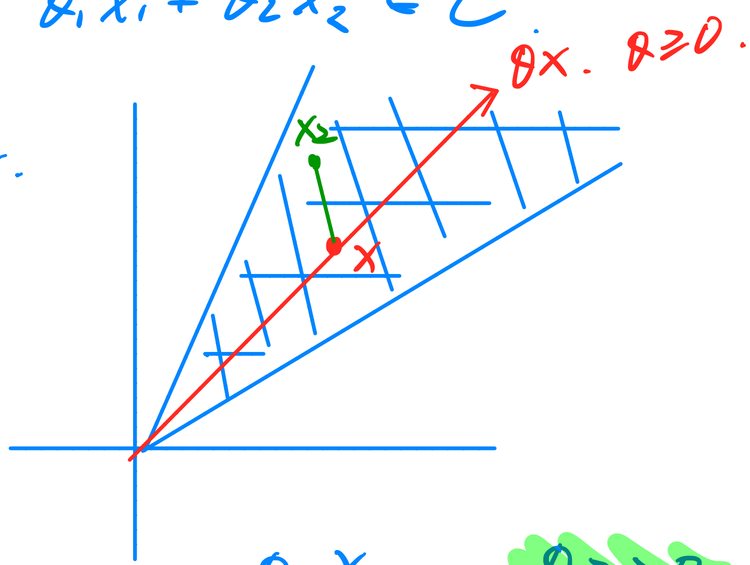


Cones

A set C is a cone if $\forall x \in C$, $\theta \geq 0$ we have $\theta x \in C$.

C is a convex cone if $\forall x_1, x_2 \in C$, and $\theta_1 \geq 0$, $\theta_2 \geq 0$, we have
$$\theta_1 x_1 + \theta_2 x_2 \in C$$

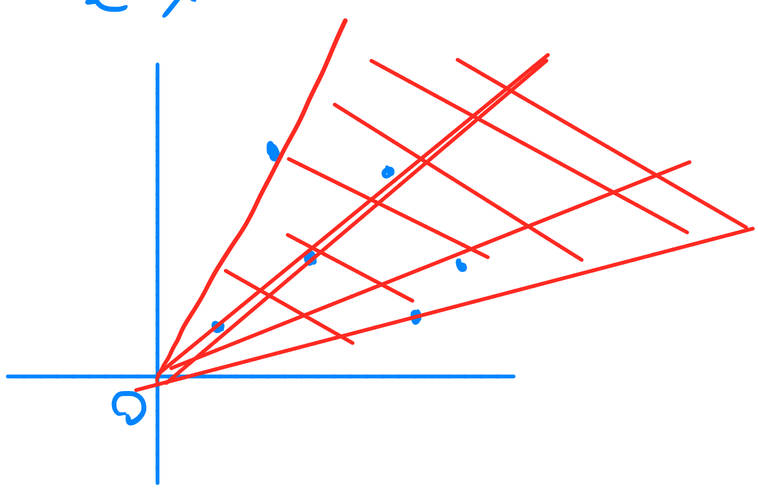
Ex.

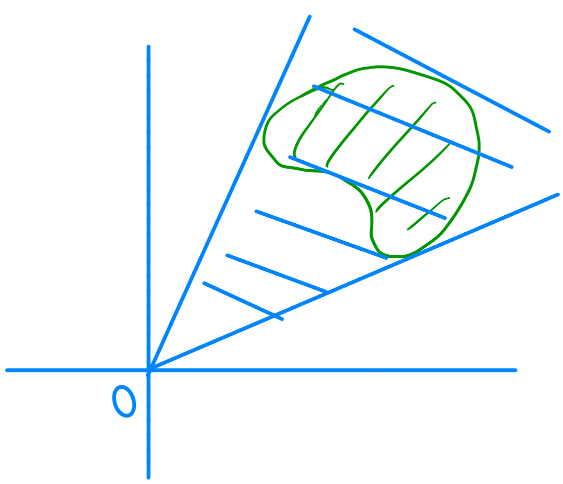


- $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$, $\theta_i \geq 0$, is a conic combination of $x_1, x_2, \ldots, x_k$..

- conic hull of $= \{ \sum_i \theta_i x_i : x_i \in C, \theta_i \geq 0 \}$

Ex





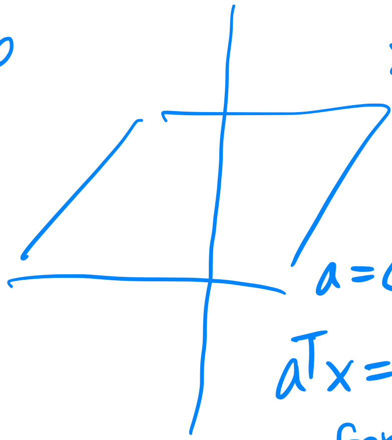
Hyperplane

$H = \{ x \in \mathbb{R}^n : a^T x = b \}$ for $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$.

Suppose $x_0 \in H$. s.t. $\ni a^T x_0 = b$

Then $H = \{ x : a^T (x - x_0) = 0 \}$

$= \{ x_0 + v : a^T v = 0,\ v \in \mathbb{R}^n \}$



x-y plane.

$(x, y, z=0)$

$a = (0, 0, 1)^T$

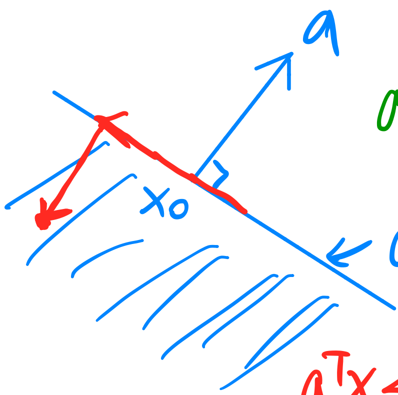
$a^T x = 0$

for $x \in$ x-y plane

(closed) Half space

$\{ x \in \mathbb{R}^n : a^T x \leq b \}$

(open) $a^T x < b$.



$a^T x \geq b$

$a^T x = b$

$a^T x \leq b$

Euclidean ball

$$B(x_c, r) = \{x : \|x - x_c\|_2 \le r\}$$
$$= \{x : (x - x_c)^T (x - x_c) \le r^2\}$$
$$= \{x_c + ru : \|u\|_2 \le 1\}$$

This norm ball is convex.

proof. If $\|x_1 - x_c\|_2 \le r$, $\|x_2 - x_c\|_2 \le r$

For $0 \le \theta \le 1$,

$$\|(\theta x_1 + (1-\theta)x_2) - x_c\|_2$$
$$= \|\theta(x_1 - x_c) + (1-\theta)(x_2 - x_c)\|$$
$\le$ triangle inequality $\|\theta(x_1 - x_c)\| + \|(1-\theta)(x_2 - x_c)\|$

$\le$ homogenity $\theta\|(x_1 - x_c)\| + (1-\theta)\|x_2 - x_c\|$

$\le r$.