ECM5901 2019, Assignment #1 due October 1

1. Verify that the ℓ_1-norm on $\mathbb{R}^n$ defined by $\|x\|_1 = \sum_{i=1}^n |x|$ for $x \in \mathbb{R}^n$ is a norm on $\mathbb{R}^n$. (10%)

2. The operator norm on $\mathbb{R}^{m\times n}$ induced by two norms $\|\cdot\|_a$ on $\mathbb{R}^m$ and $\|\cdot\|_b$ on $\mathbb{R}^n$ is defined by

$$\|X\|_{a,b} = \sup\{\|Xu\|_a : \|u\|_b \le 1\}$$

for $X \in \mathbb{R}^{m\times n}$. Verify that

$$\|X\|_{1,1} = \max_{j=1,\ldots,n} \sum_{i=1}^n |X_{ij}|.$$

(10%)

3. Show that the dual norm of ℓ_1-norm is the ℓ_∞-norm. (10%)

4. The trace of a square matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $\mathrm{tr}A = \sum_{i=1}^n a_{ii}$. Show that:
   (a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for $A, B \in \mathbb{R}^{n\times n}$. (5%)
   (b) $\mathrm{tr}(tA + B) = t\mathrm{tr}(A) + \mathrm{tr}(B)$ for $A, B \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$. (5%)

5. Let $\langle\ ,\ \rangle$ be the inner product on $\mathbb{R}^n$. Prove the Cauchy-Schwarz inequality that for $x, y \in \mathcal{R}^n$,

$$|\langle x, y\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle.$$

(10%)

6. Verify that that Frobenious inner product on $\mathbb{R}^{m\times n}$ defined by

$$\langle\ X, Y\ \rangle_F = \mathrm{tr}(X^TY)$$

for $X, Y \in \mathbb{R}^{m\times n}$ is an inner product. (10%)

7. Read Appendix A.4. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Consider $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = ||Ax - b||_2^2$ for $x \in \mathbb{R}^n$. Show that
   (a) $\nabla f(x) = 2A^T(Ax - b)$. (10%)
   (b) $\nabla^2 f(x) = 2A^TA$. (10%)

8. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, it has an eigenvalue (spectral) decomposition

$$A = Q\Lambda Q^T,$$

where $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix such that $Q^TQ = QQ^T = I$ and $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix with entries that are eigenvalues of A.
   (a) Show that

$$\mathrm{tr}A = \sum_{i=1}^n \lambda_i.$$

   (5%)
   (b) Show that $x^TAx \ge 0$ for any $x \in \mathbb{R}^n$ if and only if $\lambda_i \ge 0$ for $i = 1, \ldots, n$. (A is called positive semidefinite if all the eigenvalues λ_i are nonnegative.) (10%)
   (c) Let $\lambda_{\max} = \max\{\lambda_1, \ldots, \lambda_n\}$ and $\lambda_{\min} = \min\{\lambda_1, \ldots, \lambda_n\}$. Show that

$$\lambda_{\min} x^Tx \le x^TAx \le \lambda_{\max} x^Tx.$$

   (10%)