HW5 Solutions

December 17, 2019

1. (a) The problem with SDP form is

$$\begin{array}{ll} \text{minimize} & \lambda_1(x) - \lambda_5(x) \\ \text{subject to} & A(x) \preceq \lambda_1(x)\mathbb{I} \\ & A(x) \succeq \lambda_5(x)\mathbb{I}. \end{array}$$

   (b) The optimal value is 28.1544 with optimal point [-0.5966, -0.3358], which is solved by cvx in MATLAB.

2. (a) The feasible set is the interval [-4,-1]. The (unique) optimal point is $x^\star = -1$, and the optimal value is $p^\star = 2$.

   (b) The Lagrangian is

$$L(x, \lambda) = (1 + \lambda)x^2 + 5\lambda x + (4\lambda + 1)$$

   which is an convex function of x. It follows that the dual function are given by

$$\frac{\partial L(x, \lambda)}{\partial x} = 2x(1 + \lambda) + 5\lambda = 0$$

$$\Rightarrow x = \frac{-5\lambda}{2(1 + \lambda)}.$$

   If $\lambda > -1$

$$g(\lambda) = \frac{-9\lambda^2 + 20\lambda + 4}{4(1 + \lambda)}.$$

   If $\lambda \leq -1$, $L(x, \lambda)$ is unbounded

$$g(\lambda) = -\infty.$$

   (c) The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & g(\lambda) = \frac{-9\lambda^2+20\lambda+4}{4(1+\lambda)} \\ \text{subject to} & \lambda \geq 0. \end{array}$$

   Furthermore, $g''(\lambda) = \frac{-25}{2(1+\lambda)^3} < 0$ with $\lambda \geq 0$, hence $g(\lambda)$ is a concave function.

(d) We solve the problem in (c) that

$$\begin{aligned} g'(\lambda) &= 0 \\ \Rightarrow -9\lambda^2 - 18\lambda + 16 &= 0 \\ \Rightarrow \lambda^* &= \frac{2}{3} \quad (\lambda \geq 0). \end{aligned}$$

Finally, $g(\lambda^*) = 2$ is equal to the optimal value p^* in (a), and the strong duality is holds.

3. The Lagrangian is

$$\begin{aligned} L(x, \lambda, \nu) &= c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b) \\ &= (c^T + \lambda^T G + \nu^T A)x - h\lambda^T - \nu^T b, \end{aligned}$$

which is an affine function of x. It follows that the dual function is given by

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -\lambda^T h - \nu^T b & c + G^T\lambda + A^T\nu = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem is

$$\begin{aligned} &\text{maximize} && g(\lambda, \nu) \\ &\text{subject to} && \lambda \succeq 0. \end{aligned}$$

After making the implicit constraints explicit, we obtain

$$\begin{aligned} &\text{maximize} && -\lambda^T h - \nu^T b \\ &\text{subject to} && c + G^T\lambda + A^T\nu = 0 \\ &&& \lambda \succeq 0. \end{aligned}$$

4. We derive the dual of the problem

$$\begin{aligned} &\text{minimize} && -\sum_{i=1}^m \log y_i \\ &\text{subject to} && y = b - Ax, \end{aligned}$$

where $A \in \mathbb{R}^{m\times m}$ has a_i^T as its its row. The Lagrangian is

$$L(x, y, \nu) = -\sum_{i=1}^m \log y_i + \nu^T (y - b + Ax)$$

and the dual function is

$$g(\nu) = \inf_{x,y} (-\sum_{i=1}^m \log y_i + \nu^T (y - b + Ax)).$$

The term $\nu^T Ax$ is unbounded below as a function of x unless $A^T\nu = 0$. The terms in y are unbounded below if $\nu \not\succ 0$, and achieve their minimum for $y_i = 1/\nu_i$ otherwise. We therefore find the dual function

$$g(\nu) = \begin{cases} \sum_{i=1}^m \log \nu_i + m - b^T\nu & A^T\nu = 0,\ \nu \succ 0 \\ -\infty & \text{otherwise} \end{cases}$$

and the dual problem

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} \log \nu_i - b^T \nu + m \\ \text{subject to} & A^T \nu = 0. \end{array}$$