ECM5901 Assignment #6, 2019 due December 24

1. Derive a dual problem for

$$\text{minimize} \quad \sum_{j=1}^{N} \|A_i x - b_i\|_2 + \frac{1}{2}\|x - x_0\|_2^2$$

The problem data are $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$, and $x_0 \in \mathbb{R}^n$. First introduce new variables $y_i \in \mathbb{R}^{m_i}$ and equality constraints $y_i = A_i x - b_i$. (20%)

2. (A convex problem in which strong duality fails.) Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \le 0 \end{array}$$

with variables x and y, and domain $\mathcal{D} = \{(x, y) : y > 0\}$.

   (a) Verify that this is a convex optimization problem. Find the optimal value. (5%)

   (b) Give the Lagrange dual problem, and find the optimal solution $\lambda^\star$ and optimal value $d^\star$ of the dual problem. (10%)

   (c) What is the optimal duality gap? Does Slater's condition hold for this problem? (5%)

3. Prove (without using any linear programming code) that the optimal solution of the LP

$$\begin{array}{ll} \text{minimize} & 47x_1 + 93x_2 + 17x_3 - 93x_4 \\ \text{subject to} & \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \le \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} \end{array}$$

is unique, and given by $x^\star = (1, 1, 1, 1)$. (20%)

4. (SDP relaxations of two-way partitioning problem). We consider the two-way partitioning problem (5.7), described on page 219,

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \; i = 1, \dots, n \end{array} \tag{1}$$

with variable $x \in \mathbb{R}^n$. The Lagrange dual of this (nonconvex) problem is given by the SDP

$$\begin{array}{ll} \text{maximize} & -\sum_{j=1}^{n} \nu_j \\ \text{subject to} & W + \operatorname{diag}(\nu) \ge 0 \end{array} \tag{2}$$

with variable $\nu \in \mathbb{R}^n$. The optimal value of this SDP gives a lower bound on the optimal value of the partitioning problem Eq. (1). In this exercise we derive another SDP that gives a lower bound on the optimal value of the two-way partitioning problem, and explore the connection between the two SDPs.

   (a) Two-way partitioning problem in matrix form. Show that the two-way partitioning problem can be cast as

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(WX) \\ \text{subject to} & X \ge 0, \; \operatorname{rank}(X) = 1 \\ & X_{ii} = 1, \; i = 1, \dots, n. \end{array}$$

   with variable $X \in S^n$. Hint. Show that if X is feasible, then it has the form $X = xx^T$, where $x \in \mathbb{R}^n$ satisfies $x_i \in \{+1, -1\}$ (and vice versa). (5%)

(b) (SDP relaxation of two-way partitioning problem.) Using the formulation in part (a), we can form the relaxation

$$\begin{array}{ll} \text{minimize} & \mathrm{tr}(WX) \\ \text{subject to} & X \succeq 0, \\ & X_{ii} = 1, \ i = 1, \ldots, n. \end{array} \tag{3}$$

with variable $X \in S^n$. This problem is an SDP, and therefore can be solved efficiently. Explain why its optimal value gives a lower bound on the optimal value of the two-way partitioning problem (1). What can you say if an optimal point $X^\star$ for this SDP has rank one? (5%)

(c) We now have two SDPs that give a lower bound on the optimal value of the two-way partitioning problem (1): the SDP relaxation (3) found in part (b), and the Lagrange dual of the two-way partitioning problem, given in (2). What is the relation between the two SDPs? What can you say about the lower bounds found by them? Hint: Relate the two SDPs via duality. (10%)

5. The pure Newton method. Newton's method with fixed step size $t = 1$ can diverge if the initial point is not close to $x^\star$. Consider

$$\text{minimize} \qquad f(x) = \log(e^x + e^{-x})$$

$f(x)$ has a unique minimizer $x^\star = 0$. Run Newton's method with fixed step size $t = 1$, starting at $x^{(0)} = 1$ and at $x^{(0)} = 1.2$. Show the errors of the first four iterates. (You can do it by hand or using MATLAB.)

(20%)