HW2 Solutions

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1. Let $x_1, x_2 \in f^{-1}(C)$, and $f(x_1) = Ax_1 + b \in C, f(x_2) = Ax_2 + b \in C$, where $0 \leq \theta \leq 1$.

$$\begin{aligned} f(\theta x_1 + (1-\theta)x_2) &= A(\theta x_1 + (1-\theta)x_2) + b \\ &= \theta A x_1 + (1-\theta)Ax_2 + (1-\theta)b \\ &= \theta(Ax_1 + b) + (1-\theta)(Ax_2 + b) \\ &= \theta f(x_1) + (1-\theta)f(x_2). \end{aligned}$$

   Because C is convex, $\theta f(x_1) + (1-\theta)f(x_2) \in C$, and hence $\theta x_1 + (1-\theta)x_2 \in f^{-1}(C)$.
   $\Rightarrow f^{-1}(C)$ is convex.

2. Let $(x_1, t), (x_2, s) \in C$, and then $\|x_1\|_2 \leq t, \|x_2\|_2 \leq s$, where t, s ≥ 0.
   We show that $(\theta x_1 + (1-\theta)x_2, \theta t + (1-\theta)s) \in C$, where $0 \leq \theta \leq 1$.

$$\begin{aligned} \|\theta x_1 + (1-\theta)x_2\|_2 &\leq \|\theta x_1\|_2 + \|(1-\theta)x_2\|_2 \;\; \text{(triangle inequlity)} \\ &= \theta \|x_1\|_2 + (1-\theta)\|x_2\|_2 \\ &\leq \theta t + (1-\theta)s. \end{aligned}$$

   Hence $\theta(x_1, t) + (1-\theta)(x_2, s) = (\theta x_1 + (1-\theta)x_2, \theta t + (1-\theta)s) \in C$.
   $\Rightarrow$ C is convex.

3. The distance between the two hyperplanes is also the distance between the two points x_1 and x_2 where the hyperplane intersects the line through the origin and parallel to the normal vector a. These points are given by

$$x_1 = (b_1/\|a\|_2^2)a, \qquad x_2 = (b_2/\|a\|_2^2)a,$$

   and the distance is

$$\|x_1 - x_2\|_2 = |b_1 - b_2| / \|a\|_2 .$$

4. (a) Consider a convex combination z of two points (x_1, x_2) and (y_1, y_2) in the set. If $x \succeq y$, then $z = \theta x + (1-\theta) y \succeq y$ and obviously $z_1 z_2 \geq y_1 y_2 \geq 1$. Similar proof if $y \succeq x$. Suppose $y \not\succeq x$ and $x \not\succeq y$, *i.e.*, $(y_1 - x_1)(y_2 - x_2) < 0$. Then

$$\begin{aligned}
&(\theta x_1 + (1-\theta) y_1)(\theta x_2 + (1-\theta) y_2) \\
&= \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2 + \theta(1-\theta) x_1 y_2 + \theta(1-\theta) x_2 y_1 \\
&= \theta x_1 x_2 + (1-\theta) y_1 y_2 - \theta(1-\theta)(y_1 - x_1)(y_2 - x_2) \\
&\geq 1.
\end{aligned}$$

(We can also use the hint for(a).)

(b) Assume that $\prod_i x_i \geq 1$ and $\prod_i y_i \geq 1$. Using the inequality in the hint, we have

$$\prod_i (\theta x_i + (1-\theta) y_i) \geq \prod x_i^\theta y_i^{1-\theta} = (\prod_i x_i)^\theta (\prod_i y_i)^{1-\theta} \geq 1.$$

5. We consider two points $(\bar{x}, \bar{y}_1 + \bar{y}_2), (\tilde{x}, \tilde{y}_1 + \tilde{y}_2) \in S$, *i.e.*, with

$$(\bar{x}, \bar{y}_1) \in S_1, \qquad (\bar{x}, \bar{y}_2) \in S_2, \qquad (\tilde{x}, \tilde{y}_1) \in S_1, \qquad (\tilde{x}, \tilde{y}_2) \in S_2.$$

For $0 \leq \theta \leq 1$,
$\theta(\bar{x}, \bar{y}_1 + \bar{y}_2) + (1-\theta)(\tilde{x}, \tilde{y}_1 + \tilde{y}_2) = (\theta \bar{x} + (1-\theta)\tilde{x}, (\theta \bar{y}_1 + (1-\theta)\tilde{y}_1) + (\theta \bar{y}_2 + (1-\theta)\tilde{y}_2))$
is in S because by convexity of S_1 and S_2,

$$(\theta \bar{x} + (1-\theta)\tilde{x}, (\theta \bar{y}_1 + (1-\theta)\tilde{y}_1) \in S_1, \qquad (\theta \bar{x} + (1-\theta)\tilde{x}, (\theta \bar{y}_2 + (1-\theta)\tilde{y}_2) \in S_2.$$

6. (a)
$$\begin{aligned}
f^{-1}(C) &= \{x \in \mathbf{dom}\, f \mid g^T f(x) \leq h\} \\
&= \{x \mid g^T (Ax + b)/(c^T x + d) \leq h,\ c^T x + d > 0\} \\
&= \{x \mid (A^T g - hc)^T x \leq hd - g^T b,\ c^T x + d > 0\},
\end{aligned}$$
which is another halfspace, intersected with $\mathbf{dom}\, f$.

(b)
$$\begin{aligned}
f^{-1}(C) &= \{x \in \mathbf{dom}\, f \mid f(x)^T P^{-1} f(x) \leq 1\} \\
&= \{x \mid (Ax + b)^T P^{-1} (Ax + b) \leq (c^T x + d)^2\} \\
&= \{x \mid x^T Q x + 2q^T x \leq r,\ c^T x + d > 0\},
\end{aligned}$$
where $Q = A^T P^{-1} A - cc^T$, $q = A^T P^{-1} b - dc$, $r = d^2 - b^T P^{-1} b$. If $A^T P^{-1} A \succ cc^T$, this is an ellipsoid intersected with $\mathbf{dom}\, f$.

7. Take $C = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$ and $D = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$. (D is convex from Problem 4.)

8. We denote by K the set of copositive matrices in S^n. K is a closed convex cone because it is the intersection of (infinitely many) halfspaces defined by homogeneous inequalities

$$z^T X z = \sum_{i,j} z_i z_j X_{ij} \geq 0.$$

K has nonempty interior, because it includes the cone of positive semidefinite matrices, which has nonempty interior. K is pointed because $X \in K, -X \in K$ means $z^T X z = 0$ for all $z \succeq 0$, hence $X = 0$.

By definition, the dual cone of a cone K is the set of normal vectors of all homogeneous halfspaces containing K (plus the origin). Therefore,

$$K^* = \mathbf{conv}\{zz^T \mid z \succeq 0\}.$$

9. (a) K^* is the intersection of a set of homogeneous halfspaces (meaning, halfspaces that include the origin as a boundary point). Hence it is a closed convex cone.

   (b) $y \in K_2^*$ means $x^T y \geq 0$ for all $x \in K_2$, which is includes K_1, therefore $x^T y \geq 0$ for all $x \in K_1$.

   (c) See part (a).

   (d) Suppose K^* is not pointed, *i.e.*, there exists a nonzero $y \in K^*$ such that $-y \in K^*$. This means $y^T x \geq 0$ and $-y^T x \geq 0$ for all $x \in K$, *i.e.*, $y^T x = 0$ for all $x \in K$, hence K has empty interior.

   (e) By definition of K^*, $y \neq 0$ is the normal vector of a (homogeneous) halfspace containing K if and only if $y \in K^*$. The intersection of all homogeneous halfspaces containing a convex cone K is the closure of K. Therefore the closure of K is

$$\mathbf{cl}\, K = \bigcap_{y \in K^*} \{x \mid y^T x \geq 0\} = \{x \mid y^T x \geq 0 \text{ for all } y \in K^*\} = K^{**}.$$