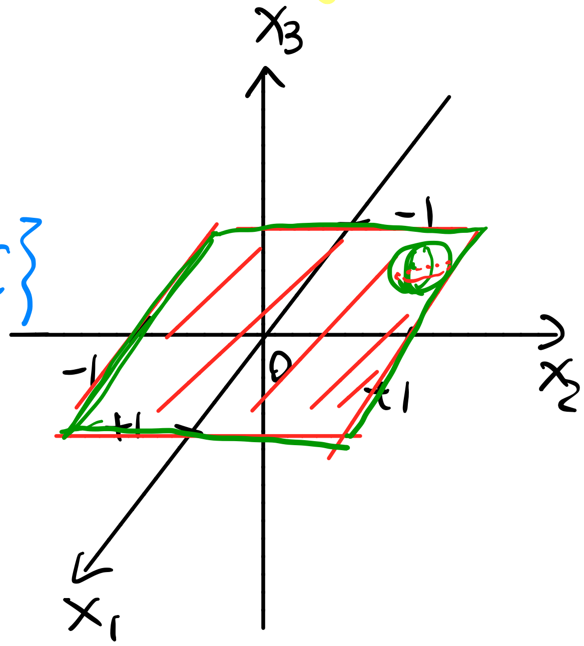
relative boundary of C is defined as
$\mathrm{cl}\, C \setminus \mathrm{relint}\, C$.

Ex. $C = \{x \in \mathbb{R}^3 : -1 \le x_1 \le 1, -1 \le x_2 \le 1, x_3 = 0\}$



$\mathrm{aff}(C) = \{x \in \mathbb{R}^3 : x_3 = 0\}$.

$\mathrm{int}(C) = \{y \in \mathbb{R}^3 : \exists B(y,r) \subset C\}$
$r > 0$.

$= \emptyset$ empty set.

$\mathrm{relint}(C) = \{x \in \mathbb{R}^3 : -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\}$

$\mathrm{bd}\, C = C$

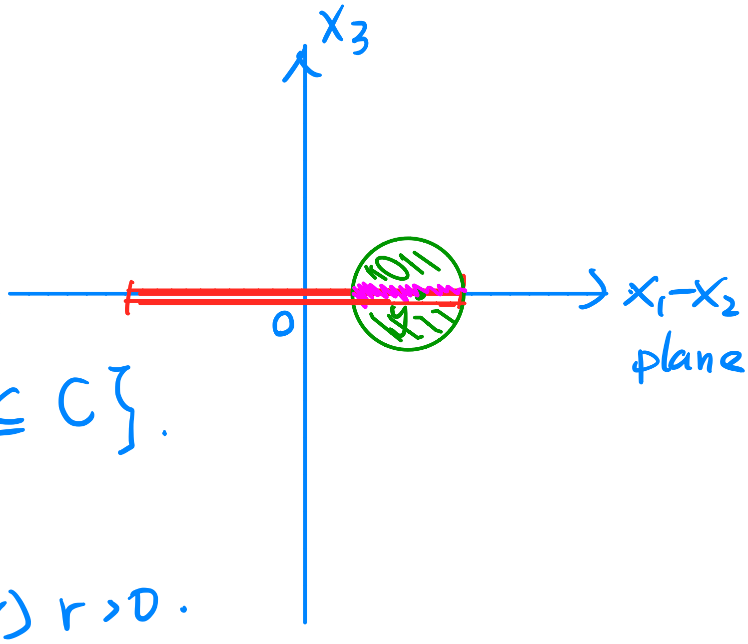
relative boundary of C is

$$\{x \in \mathbb{R}^3 : \max\{|x_1|, |x_2|\} = 1, x_3 = 0\}.$$

$\mathrm{cl}(C) = C$ since its complement $\mathbb{R}^3 \setminus C$ is open. That is, for $x \in \mathbb{R}^3 \setminus C$, $\exists \epsilon > 0$ s.t.

$$B(x, \epsilon) \in \mathbb{R}^3 \setminus C.$$

$$C=\left\{x:\ \begin{array}{l}-1\le x_1\le 1\\ -1\le x_2\le 1\\ x_3=0\end{array}\right\}.$$



$$\operatorname{int} C=\{y:\ \exists\, B(y,r)\ r>0\ \text{s.t. } B(y,r)\subseteq C\}.$$

$$=\emptyset.$$

$$\operatorname{relint} C=\{y:\ \exists\, B(y,r)\ r>0.\ \text{s.t. } B(y,r)\cap \operatorname{aff}(C)\subseteq C\}$$

$\operatorname{aff}(C)=\{x:\ x_3=0\}$ x_1-x_2 plane.

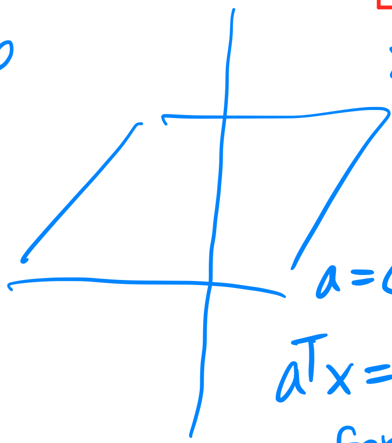
Hyperplane

$H = \{ x \in \mathbb{R}^n : \underline{\underline{a}}^T x = \underline{b} \}$ for $\boxed{a \in \mathbb{R}^n,\ \cancel{a \neq 0}.\ b \in \mathbb{R}}$

Suppose $x_0 \in H$. $\overset{\text{s.t.}}{\exists}$ $a^T x_0 = b$

Then $H = \{ x : a^T(x - x_0) = 0 \}$

$= \{ x_0 + v : a^T v = 0,\ v \in \mathbb{R}^n \}$



x-y plane.

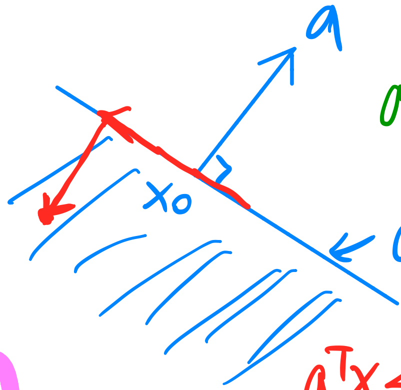
$(x, y, z=0)$

$a = (0, 0, 1)^T$

$a^T x = 0$

for $x \in$ x-y plane

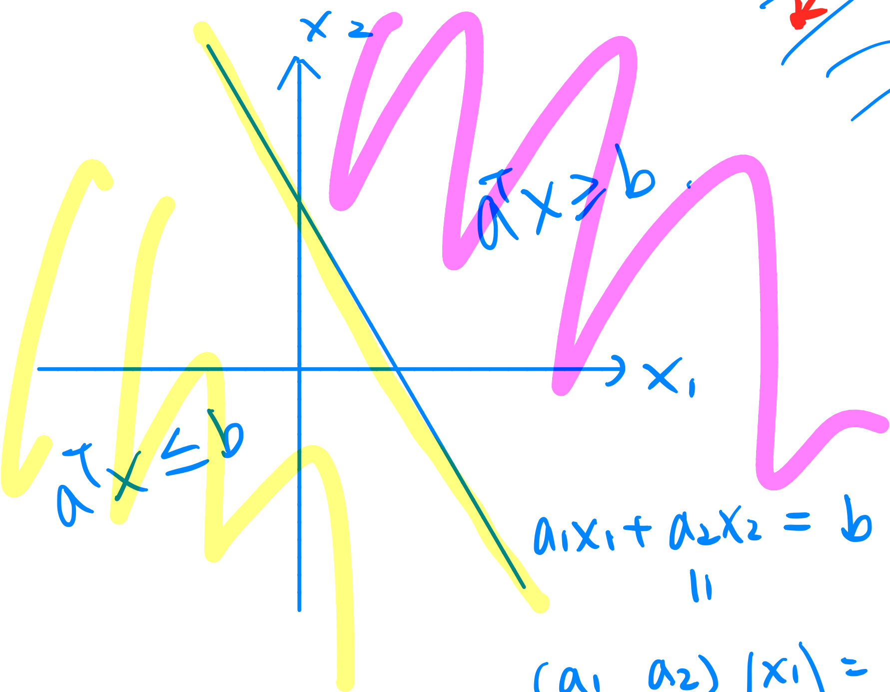
$$\boxed{\begin{array}{l} \text{(closed) Half space } \langle a, x \rangle \leq b \\ \{ x \in \mathbb{R}^n : a^T x \leq b \} \\ \text{(open) } a^T x < b. \end{array}}$$



$a^T x \geq b$

$a^T x = b$

$a^T x \leq b$



$a_1 x_1 + a_2 x_2 = b$

$\shortparallel$

$(a_1\ a_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b.$

$a^T x = b$

Ellipsoid $\mathcal{E} = \{x : (x-x_c)^T P^{-1} (x-x_c) \le 1\}$

where $P > 0$ .i.e. P is symmetric & positive definite.

$\boxed{\text{Ball: } P = r^2 I}$

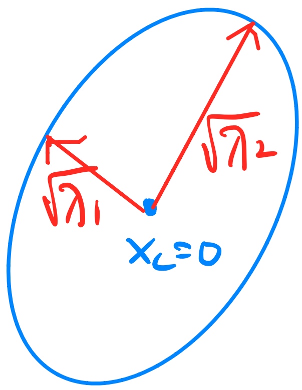
$B : \{x : (x-x_c)^T (x-x_c) \le r^2\}$

$P = Q \Lambda Q^T$, where $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $\lambda_i > 0$.

$P^{-1} = Q \Lambda^{-1} Q^T$.

Ex $n=2$ $\left\{x : (x_1 \ \ x_2) \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le 1 \right\}$

$\Leftrightarrow \dfrac{x_1^2}{\lambda_1} + \dfrac{x_2^2}{\lambda_2} \le 1$.



Taking $A = P^{1/2} \triangleq Q \Lambda^{1/2} Q^T$.

$\mathcal{E}' = \{x_c + Au : \|u\|_2 \le 1\}$ $\{u : \|u\|_2 \le 1\}$ $B(0,1)$

$\boxed{\mathcal{E}' \subseteq \mathcal{E}}$ & $\boxed{\mathcal{E}' \supseteq \mathcal{E}}$ $\Rightarrow \mathcal{E}' = \mathcal{E}$.

verify it

proof. suppose that $x : (x-x_c)^T P^{-1} (x-x_c) \le 1$

$x \in \mathcal{E}$, we want to show $x \in \mathcal{E}'$.

$\| P^{-1/2} (x-x_c) \|_2^2 \le 1$ $Au + x_c$ (affine)

let $u = P^{-1/2}(x-x_c) \Rightarrow x = \underbrace{P^{1/2} u + x_c}_{\in \mathcal{E}'}$ #.

Polyhedra.

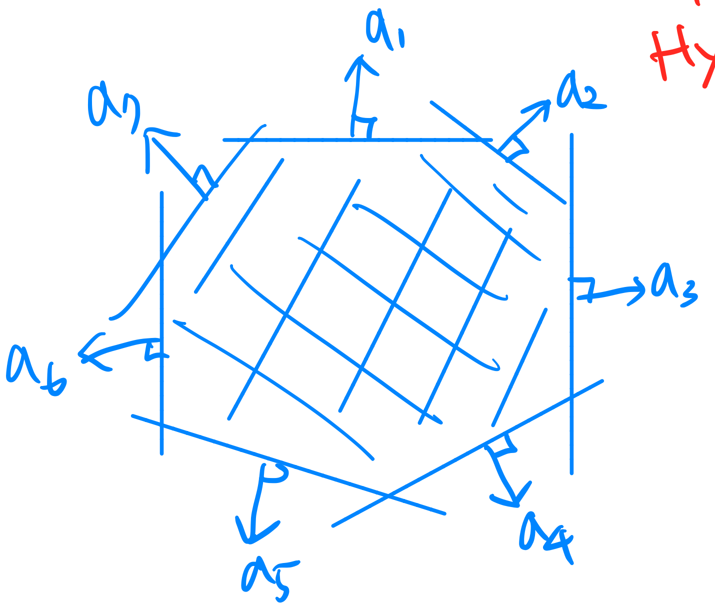
$$P = \Big\{ x \in \mathbb{R}^n : \underbrace{\boxed{a_j^T x \le b_j}}_{}, \; j = 1, \dots, m, \quad \underline{c_j^T x = d_j}, \; j = 1, \dots, p \Big\}$$

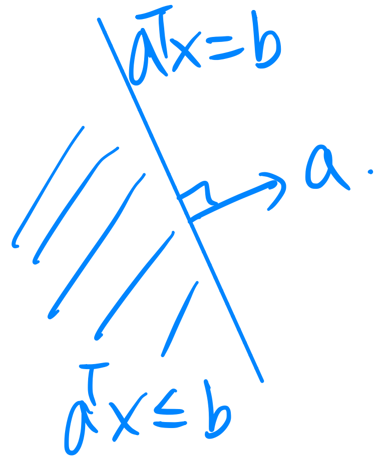
halfspaces

Hyperplances.

$a_j \in \mathbb{R}^n$, $b_j \in \mathbb{R}$, $c_j \in \mathbb{R}^n$, $d_j \in \mathbb{R}$

Ex.



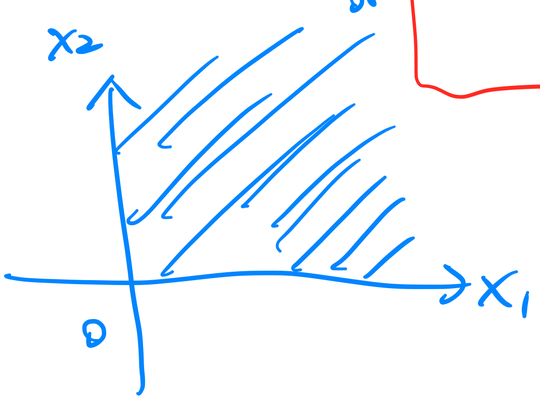


Ex. $\mathbb{R}^2_+ = \{ x \in \mathbb{R}^2 : x_1 \ge 0, \; x_2 \ge 0 \}$

or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ge 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

vector

entry-wise inequality



$k+1$ points $v_0, v_1, \ldots, v_k \in \mathbb{R}^n$ are affinely independent if $v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0$ are linearly independent.

The <u>simplex</u> defined by $v_0, v_1, \ldots, v_k$ is

$$C = \operatorname{conv}\{v_0, v_1, v_2, \ldots, v_k\}$$

$$= \{\theta_0 v_0 + \theta_1 v_1 + \cdots + \theta_k v_k : \theta \geq 0, \ \textstyle\sum_i \theta_i = 1\}$$

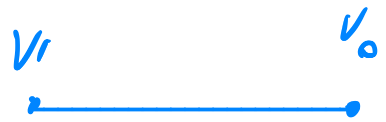
$$\theta = \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{pmatrix} \quad \theta_i \geq 0$$

$\dim \operatorname{aff}(C) = k$

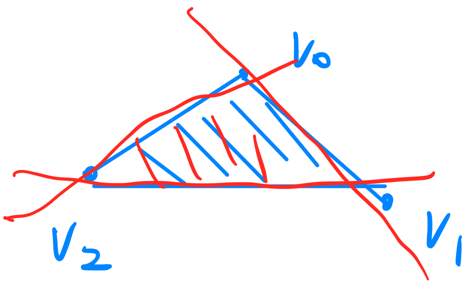
k-dimensional simplex in $\mathbb{R}^n$.

Ex.

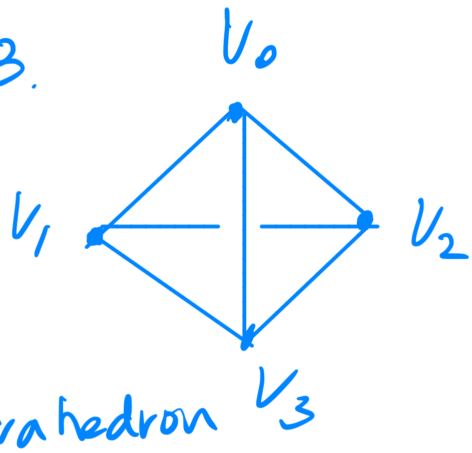
$k=1$



$k=2$



$k=3$.



<u>tetrahedron</u>

▷ A <u>simplex</u> is a <u>polyhedron</u>.

Convex hull description of polyhedra

The convex combination of a finite set $\{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$

, $\mathrm{conv}\{v_1, \dots, v_k\}$,

is a polyhedron & bounded, but

it is ~~not~~ easy to express it in the form

$$\{x \in \mathbb{R}^n : \begin{matrix} Ax \le b \\ Cx \le d \end{matrix}\}. \quad A = \begin{bmatrix} -a_1^T- \\ -a_2^T- \\ \vdots \\ -a_m^T- \end{bmatrix}$$

$$C = \begin{bmatrix} -c_1^T- \\ \vdots \\ -c_p^T- \end{bmatrix}$$

Ex. unit ball in the l_∞ norm in $\mathbb{R}^n$.

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

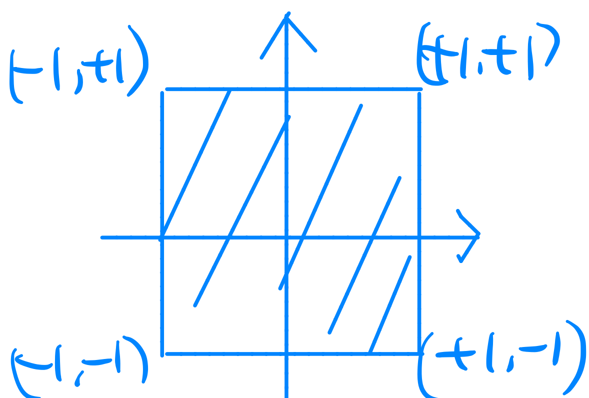
$$C = \{x \in \mathbb{R}^n : \|x\|_\infty \le 1\}$$

$$= \{x \in \mathbb{R}^n : |x_i| \le 1, \ i = 1, \dots n\}$$

$$= \{x \in \mathbb{R}^n : \underbrace{\pm e_i^T x \le 1, \ i = 1, \dots, n}_{2n \text{ inequalities}}\}$$

where $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith entry. unit vector.

$n=2$.



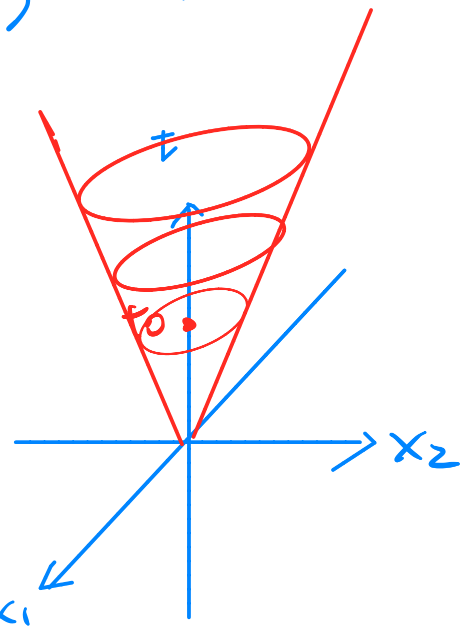
$= \mathrm{conv}(v_1, v_2, \dots, v_{2^n}) \leftarrow$ 2^n vectors

where $v_i \in \{+1, -1\}^n$.

Norm cones.

$$C = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| \le t\} \subseteq \mathbb{R}^{n+1}$$

Ex. n=2.



$\{(x,t) \in \mathbb{R}^3 : \|x\|_2 \le t\}$

for a fixed $t_0 \ge 0$

$\{x : \|x\|_2 \le t_0\}$ is a norm ball

Positive semidefinite cone

$$S^n = \{A \in \mathbb{R}^{n\times n} : A^T = A\}$$

$$S_+^n = \{A \in S^n : A \succeq 0\} \Leftrightarrow x^TAx \ge 0 \ \forall x \in \mathbb{R}^n.$$

S_+^n is a convex cone:

proof. For $A, B \in S_+^n$, $\theta_1, \theta_2 \ge 0$,

Want to show that $\underline{x^T(\theta_1 A + \theta_2 B)x \ge 0 \ \forall x \in \mathbb{R}^n}$

$$\underset{\ge 0}{\theta_1}\underset{\ge 0}{(x^TAx)} + \underset{\ge 0}{\theta_2}\underset{\ge 0}{(x^TBx)}$$

Operations that preserve convexity.

If C is convex. function f $\to$ $f(C)$ convex

1. Intersection.

If S_1, S_2 are convex sets, then

$S_1 \cap S_2$ is convex. (trivial to prove)

If $S_\alpha : \alpha \in \mathcal{A}$ are convex,

$\bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex. $|\mathcal{A}|$: can be infinite.

Ex. $S_+^+ = \{A \in S^n : x^T A x \geq 0 \ \forall x \in \mathbb{R}^n\}$

$= \bigcap_{x \in \mathbb{R}^n} \{A \in S^n : x^T A x \geq 0\}$.

$x^T A x = Tr(xx^T A) = Tr(B^T A)$, $B = xx^T$. $\quad Tr\,AB = Tr\,BA$

$\bigcap_{B = B^T,\ rank(B) = 1} \{A \in S^n : \underbrace{\langle B, A \rangle \geq 0}_{\text{half space in } S^n}\}$.

2. Affine functions.

$f: \mathbb{R}^n \to \mathbb{R}^m$ is affine if $f(x) = Ax + b$

for $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$. (if $b=0$, f is linear.)

Suppose that $S \subseteq \mathbb{R}^n$ is convex. Then $f(S)$ is convex.

$f(S)$ is the image of f.

proof. for $y_1, y_2 \in f(S)$,

we have $y_1 = f(x_1)$, $y_2 = f(x_2)$, for $x_1, x_2 \in S$.

For $0 \le \theta \le 1$,

$$\Rightarrow \theta y_1 + (1-\theta) y_2 = \theta f(x_1) + (1-\theta) f(x_2)$$

$$= \theta (Ax_1 + b) + (1-\theta)(Ax_2 + b)$$

$$= A(\theta x_1 + (1-\theta) x_2) + b \in f(S).$$

$\because$ S is convex, $\theta x_1 + (1-\theta) x_2 \in S$.

#

▷ Inverse Image.

$f^{-1}(S) = \{x : f(x) \in S\}$ is convex.

(Exercise)

▷ Scaling

$\alpha \in \mathbb{R}$. $\alpha S = \{\alpha x : x \in S\}$ is convex.

▷ Translation.

$x \in \mathbb{R}^n$: $x + S = \{x + y : y \in S\}$ is convex.

▷ The sum of two sets

Suppose S_1, S_2 convex.

$$\begin{cases} S_1+S_2 = \{x_1+x_2 : x_1\in S_1, x_2\in S_2\} \subseteq \mathbb{R}^n \\ S_1\times S_2 = \{(x_1,x_2) : x_1\in S_1, x_2\in S_2\} \subseteq \mathbb{R}^n\times\mathbb{R}^n \end{cases}$$

It is trivial to see that $S_1\times S_2$ is convex. verify this

Consequently, $S_1+S_2 = f(S_1\times S_2)$,

where $f : S_1\times S_2 \to \mathbb{R}$ is defined by

$$f(x_1,x_2) = x_1+x_2 = (1\ \ 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which is a linear function. (affine).

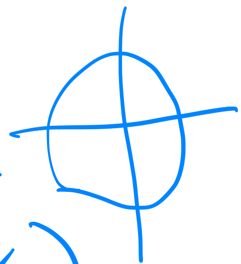
$S_1\times S_2$ convex, f preserves convexity

$\Rightarrow f(S_1\times S_2) = S_1+S_2$ convex.

Ex. Ellipsoid $\mathcal{E} : \{x : (x-x_c)^T P^{-1}(x-x_c)\le 1\}$.

$P > 0$

$B = \{x : \|x\|_2 \le 1\}$ is convex



① $\mathcal{E} = f(B)$, where $f(u) = P^{1/2}(u+x_c)$

for $u\in\mathbb{R}^n$. affine mapping.

Thus $\mathcal{E}$ is convex.

② $\mathcal{E} = g^{-1}(B)$, where $g(x) = P^{-1/2}(x-x_c)$.

g is affine. $\mathcal{E}$ is the inverse image of g on a convex set B.

The perspective function

$P: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. with $\operatorname{dom} P = \mathbb{R}^n \times \mathbb{R}_{++}$.

$$P(x,t) = \frac{x}{t}. \qquad t > 0.$$

$\mathbb{R}_{++} = \{x \in \mathbb{R} ; x > 0\}$

If $C \subseteq \operatorname{dom} P$ is convex, then $P(C) = \{P(x) : x \in C\}$ is convex.

proof. for $x = (\hat{x}, x_{n+1})$. $y = (\hat{y}, y_{n+1}) \in \mathbb{R}^{n+1}$ where $x_{n+1}, y_{n+1} > 0$.

$$f(x) = \frac{Ax + b}{c^T x + d}$$ linear functional