ECM5901 2019, Assignment #3 due October 31

1. (Kullback-Leibler divergence and the information inequality.) The Kullback-Leibler divergence between $u, v \in \mathbb{R}^n_{++}$ is defined as

$$D_{kl}(u, v) = \sum_{i=1}^{n} (u_i \log(u_i/v_i) - u_i + v_i).$$

   Show that

   (a) $D_{kl}(u, v)$ is convex; (5%)

   (b) $D_{kl}(u, v) \geq 0$ for all $u, v \in \mathbb{R}^n_{++}$. (5%)

   (c) $D_{kl}(u, v) = 0$ if and only if $u = v$. (5%)

   Hint: $D_{kl}(u, v)$ can be expressed as

$$D_{kl}(u, v) = f(u) - f(v) - \nabla f(v)^T (u - v),$$

   where $f(v) = \sum_{i=1}^{n} v_i \log v_i$ is the negative entropy of v for $v \in \mathbb{R}^n_{+}$.

2. Adapt the proof of concavity of the log-determinant function in §3.1.5 of [BV04] to show that

$$f(X) = \mathrm{tr}(X^{-1})$$

   is convex on $\mathrm{dom}\, f = S^n_{++}$. (15%)

3. (Composition rules.) Show that the following functions are convex

   (a) $f(x, u, v) = -\log(uv - x^T x)$ on $\mathrm{dom}\, f = f(x, u, v) : x \in \mathbb{R}^n, u, v \in \mathbb{R}, uv > x^T x, u, v > 0$. (10%)

   (b) Show that

$$f(x) = \frac{\|Ax + b\|_2^2}{c^T x + d}$$

   is convex on $\{x \in \mathbb{R}^n : c^T x + d > 0\}$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. (10%)

   Hint: $x^T x/u$ is convex in (x, u) for $u > 0$.

4. (Conjugate of convex plus affine function) Define $g(x) = f(x) + c^T x + d$ for $c \in \mathbb{R}^n, d \in \mathbb{R}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Express g^* in terms of f^* and c, d. (10%)

5. (Log-concavity of Gaussian cumulative distribution function.) The cumulative distribution function of a Gaussian random variable,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

   is log-concave. (This follows from the general result that the convolution of two log-concave functions is log-concave.) In this problem we guide you through a simple self-contained proof that f is log-concave. Recall that f is log-concave if and only if $f''(x)f(x) \leq f'(x)^2$ for all x.

   (a) Verify that $f''(x)f(x) \leq f'(x)^2$ for $x \geq 0$. That leaves us the hard part, which is to show the inequality for $x < 0$.

   (b) Verify that for any t and x, we have $t^2/2 \geq -x^2/2 + xt$. (Hint: consider the first order condition with the function $g(t) = t^2/2$.)

   (c) Use part (b) to show that $e^{-t^2/2} \leq e^{x^2/2 - xt}$. Conclude that for $x < 0$,

$$\int_{-\infty}^{x} e^{-t^2/2} dt \leq e^{x^2/2} \int_{-\infty}^{x} e^{-xt} dt.$$

   (d) Use part (c) to verify that $f''(x)f(x) \leq f'(x)^2$ for $x \leq 0$.

(20%)

6. (Sublevel sets and epigraph of K-convex functions.) Let $K \subset \mathbb{R}^m$ be a proper convex cone with associated generalized inequality K, and let $f : \mathbb{R}^n \to \mathbb{R}^m$. For $\alpha \in \mathbb{R}^m$, the α-sublevel set of f (with respect to K) is defined as

$$C_\alpha = \{x \in \mathbb{R}^n : f(x) \preceq_K \alpha\}.$$

The epigraph of f, with respect to K, is defined as the set

$$\mathrm{epi}_K f = \{(x, t) \in \mathbb{R}^{n+m} : f(x) \preceq_K t\}.$$

Show the following:

(a) If f is K-convex, then its sublevel sets C_α are convex for all α. (10%)

(b) f is K-convex if and only if $\mathrm{epi}_K f$ is a convex set. (10%)