Assignment #2

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Exercise 1. Suppose $C \subset \mathbb{R}^m$ is convex and $f : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function. Show that the inverse image of the convex set C

$$f^{-1}(C) = \{x \mid f(x) \in C\}$$

is convex. (10%)

Proof. f is an affine function $\Rightarrow f(x) = Ax + b$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$
Let $x_1, x_2 \in f^{-1}(C)$, we need to prove $\theta x_1 + (1-\theta)x_2 \in f^{-1}(C)$ where $0 \le \theta \le 1$.

$$\begin{aligned}
x_1, x_2 \in f^{-1}(C) &\Rightarrow f(x_1), f(x_2) \in C \\
&\Rightarrow \theta f(x_1) + (1-\theta) f(x_2) \in C \\
&\Rightarrow \theta(Ax_1 + b) + (1-\theta)(Ax_2 + b) \in C \\
&\Rightarrow A(\theta x_1 + (1-\theta)x_2) + b \in C \\
&\Rightarrow f(\theta x_1 + (1-\theta)x_2) \in C \\
&\Rightarrow \theta x_1 + (1-\theta)x_2 \in f^{-1}(C)
\end{aligned}$$

Therefore, $f^{-1}(C)$ is convex. $\square$

Exercise 2. The second-order cone is the norm cone for the Euclidean norm, *i.e.*,

$$\begin{aligned}
C &= \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \le t\} \\
&= \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le 0, t \ge 0 \right\}
\end{aligned}$$

Show that C is convex. (10%)

Proof. Let $(x_1, t_1), (x_2, t_2) \in C$, we need to prove for $0 \le \theta \le 1$, the following holds

$$\theta(x_1, t_1) + (1-\theta)(x_2, t_2) = (\theta x_1 + (1-\theta)x_2, \theta t_1 + (1-\theta)t_2) \in C$$

We have $\|x_1\|_2 \le t_1$, $\|x_2\|_2 \le t_2$, check whether the above vector is in norm cone

$$\begin{aligned}
\|\theta x_1 + (1-\theta)x_2\|_2 &\le \theta\|x_1\|_2 + (1-\theta)\|x_2\|_2 \quad \text{(triangle inequality of norm)} \\
&\le \theta t_1 + (1-\theta)t_2
\end{aligned}$$

Therefore, C is convex. $\square$

Exercise 3. The distance between two sets C and D is defined as

$$\inf\{\|u-v\|_2 \mid u \in C, v \in D\}.$$

What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbb{R}^n \mid a^T x = b_2\}$ for $a \in \mathbb{R}^n, b_1, b_2 \in \mathbb{R}$? (10%)

Solution. The required distance is the same as the distance of the two points x_1, x_2 where the hyperplanes intersects the line through origin and parallel to vector a. We have

$$x_1 = \frac{b_1}{\|a\|_2^2}a \quad x_2 = \frac{b_2}{\|a\|_2^2}a$$

Therefore, $\textbf{dist}(C,D) = \|x_1 - x_2\|_2 = \dfrac{|b_1 - b_2|}{\|a\|_2}$

Exercise 4. (Hyperbolic sets.)

(a) Show that the hyperbolic set $\{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$ is convex. (5%)

(b) As a generalization, show that $\{x \in \mathbb{R}_+^n \mid \prod_{i=1}^n x_i \geq 1\}$ is convex. (5%)

(Hint: If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b$; see §3.1.9.)

Proof. Employ general arithmetic-geometric mean inequality in §3.1.9.

(a) Let $(a_1, a_2), (b_1, b_2)$ in the set, *i.e.*, $a_1 a_2 \geq 1, b_1 b_2 \geq 1,\ a_1, a_2, b_1, b_2 \in \mathbb{R}_+$
Check whether $(\theta a_1 + (1-\theta)b_1, \theta a_2 + (1-\theta)b_2)$ in the set, for $0 \leq \theta \leq 1$

$$[\theta a_1 + (1-\theta)b_1][\theta a_2 + (1-\theta)b_2] \geq (a_1^\theta b_1^{1-\theta})(a_2^\theta b_2^{1-\theta}) = (a_1 a_2)^\theta (b_1 b_2)^{1-\theta} \geq 1$$

Therefore, the hyperbolic set is convex.

(b) Let a, b in the set, *i.e.*, $\prod_{i=1}^n a_i \geq 1, \prod_{i=1}^n b_i \geq 1,\ a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}_+$

$$\prod_{i=1}^n (\theta a_i + (1-\theta)b_i) \geq \prod_{i=1}^n a_i^\theta b_i^{1-\theta} = (\prod_{i=1}^n a_i)^\theta (\prod_{i=1}^n b_i)^{1-\theta} \geq 1 \text{ where } 0 \leq \theta \leq 1$$

Therefore, the set is convex. □

Exercise 5. Show that if S_1 and S_2 are convex sets in $\mathbb{R}^{m+n}$, then so is their partial sum $S = \{(x, y_1 + y_2) \mid x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}$. (10%)

Proof. Let $(a_0, a_1 + a_2), (b_0, b_1 + b_2) \in S$ where $a_0, b_0 \in \mathbb{R}^m, a_1, a_2, b_1, b_2 \in \mathbb{R}^n$, we need to prove $(\theta a_0 + (1-\theta)b_0, \theta(a_1 + a_2) + (1-\theta)(b_1 + b_2)) \in S$.

$$\begin{cases} (a_0, a_1 + a_2) \in S & \Rightarrow (a_0, a_1) \in S_1, (a_0, a_2) \in S_2 \\ (b_0, b_1 + b_2) \in S & \Rightarrow (b_0, b_1) \in S_1, (b_0, b_2) \in S_2 \end{cases}$$

Since S_1, S_2 are convex, for $0 \leq \theta \leq 1$

$$\begin{cases} (\theta a_0 + (1-\theta)b_0,\ \theta a_1 + (1-\theta)b_1) \in S_1 \\ (\theta a_0 + (1-\theta)b_0,\ \theta a_2 + (1-\theta)b_2) \in S_2 \end{cases}$$
$$\Rightarrow (\theta a_0 + (1-\theta)b_0,\ \theta(a_1+a_2) + (1-\theta)(b_1+b_2)) \in S$$

Therefore, S is convex. $\square$

Exercise 6. Linear-fractional functions and convex sets. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be the linear-fractional function $f(x) = (Ax+b)/(c^T x + d)$, $\mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$. In this problem we study the inverse image of a convex set C under f, *i.e.*,

$$f^{-1}(C) = \{x \in \mathbf{dom}\, f \mid f(x) \in C\}.$$

For each of the following sets $C \subset \mathbb{R}^n$, give a simple description of $f^{-1}(C)$.

(a) The halfspace $C = \{y \mid g^T y \leq h\}$ (with $g \neq 0 \in \mathbb{R}^n$ and $h \in \mathbb{R}$). (5%)

(b) The ellipsoid $C = \{y \mid y^T P^{-1} y \leq 1\}$ (where $P \succ 0$). (5%)

Solution.

(a)
$$\begin{aligned} f^{-1}(C) &= \{x \in \mathbf{dom}\, f \mid f(x) \in C\} \\ &= \{x \mid g^T f(x) \leq h, c^T x + d > 0, g \neq 0\} \\ &= \{x \mid g^T(\frac{Ax+b}{c^T x + d}) \leq h, c^T x + d > 0, g \neq 0\} \\ &= \{x \mid (A^T g - hc)^T x \leq hd - g^T b, c^T x + d > 0, g \neq 0\} \end{aligned}$$
It is a new halfspace intersected with $\mathbf{dom}\, f$.

(b)
$$\begin{aligned} f^{-1}(C) &= \{x \in \mathbf{dom}\, f \mid f(x) \in C\} \\ &= \{x \mid f(x)^T P^{-1} f(x) \leq 1, c^T x + d > 0, P \succ 0\} \\ &= \{x \mid (Ax+b)^T P^{-1}(Ax+b) \leq (c^T x + d)^2, c^T x + d > 0, P \succ 0\} \\ &= \{x \mid x^T(A^T P^{-1} A - cc^T)x + 2(b^T P^{-1} A + dc^T)x \leq d^2 - b^T P^{-1} b, c^T x + d > 0, P \succ 0\} \end{aligned}$$
If $A^T P^{-1} A \succ cc^T$, it is a new ellipsoid intersected with $\mathbf{dom}\, f$.

Exercise 7. Give an example of two closed convex sets that are disjoint but cannot be *strictly* separated. (You have to verify that the two sets you provide are closed and convex.) (10%)

Solution. Take set $C = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$ and $D = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$.

- set C is convex: $a, b \in C \Rightarrow \theta a_2 + (1-\theta)b_2 \leq 0 \Rightarrow \theta a + (1-\theta)b \in C$
- set C is closed: $\mathbb{R}^2 \setminus C = \mathbf{int}\, \mathbb{R}^2 \setminus C$, boundary is $x_2 = 0$
- set D is convex: refer to **Exercise 4.(a)**
- set D is closed: $\mathbb{R}^2 \setminus D = \mathbf{int}\, \mathbb{R}^2 \setminus D$, boundary is $x_1 x_2 = 1$

The separating hyperplane is $x_2 = 0$, and no strict separating hyperplane exists.

Exercise 8. Copositive matrices. A matrix $X \in \mathbb{S}^n$ is called copositive if $z^T X z \geq 0$ for all $z \succeq 0 \in \mathbb{R}^n$. Verify that the set of copositive matrices is a proper cone. Find its dual cone. (10%)

Solution. X is a cone trivially, now check four requirements of a proper cone.

- convex and closed: the set of all copositive matrices can be expressed by intersection of infinite closed halfspaces, *i.e.*, $\bigcap_{\forall z \succ 0} \{X \in \mathbb{S}^n \mid z^T X z = \sum_{ij} z_i z_j X_{ij} \geq 0\}$. Note that intersection of closed halfspaces must be convex and closed.
- solid: K includes positive semidefinite cone; hence has interior
- pointed: $X \in K, -X \in K \Rightarrow z^T X z = 0$ for all $z \succeq 0 \Rightarrow X = 0$

The dual cone of the origin cone is the set of normal vectors of all homogeneous halfspaces containing the origin cone (plus the origin). Therefore,

$$K^* = \mathbf{conv}\{zz^T \mid z \succeq 0\}$$

Exercise 9. Properties of dual cones. Let K^* be the dual cone of a convex cone K, as defined in (2.19) of the textbook by Boyd and Vandenberghe. Prove the following. (25%)

(a) K^* is indeed a convex cone.

(b) Two sets $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$

(c) K^* is closed

(d) If K has nonempty interior, then K^* is pointed

(e) K^{**} is the closure of K. (Hence if K is closed, $K^{**} = K$)

Proof. Dual cone definition: $K^* = \{y \mid x^T y \geq 0 \text{ for all } x \in K\}$ where K is a cone.

(a) K^* is the intersection of homogeneous halfspaces $\Rightarrow$ K^* is a convex cone

(b) $y \in K_2^* \Rightarrow x^T y \geq 0$ for all $x \in K_2 \Rightarrow x^T y \geq 0$ for all $x \in K_1 \Rightarrow y \in K_1^*$
Hence, $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$.

(c) K^* is the intersection of closed halfspaces $\Rightarrow$ K^* is a closed

(d) $$\begin{aligned} K^* \text{ is not pointed } &\Rightarrow y \in K^*, -y \in K^*, y \neq 0 \\ &\Rightarrow x^T y \geq 0, -x^T y \geq 0, y \neq 0 \text{ for all } x \in K \\ &\Rightarrow x^T y = 0, y \neq 0 \text{ for all } x \in K \\ &\Rightarrow K \text{ has empty interior} \end{aligned}$$
Hence, K has nonempty interior $\Rightarrow$ K^* is pointed.

(e) The intersection of all homogeneous halfspaces containing convex cone K is the closure of K. Hence,
$\mathbf{cl}\, K = \bigcap_{y \in K^*} \{x \mid y^T x \geq 0\} = \{x \mid y^T x \geq 0 \text{ for all } y \in K^*\} = K^{**}$ ☐