Log-concave and log-convex functions

A positive function f is logarithmically-concave or (log-concave) if $\forall x \in \mathrm{dom} f$, $\log f$ is concave.

$$f(\theta x+(1-\theta)y) \geq f(x)^{\theta} \cdot f(y)^{1-\theta}, \quad 0 \leq \theta \leq 1.$$

$$\Rightarrow \log f(\theta x+(1-\theta)y) \geq \theta \log f(x) + (1-\theta)\log f(y).$$

f is log-convex if $\log f$ is convex.

f is log-convex if and only if $\frac{1}{f}$ is log-concave.

Ex. $f(x) = a^T x + b$. is log-concave on $\{x : a^T x + b > 0\}$

$$f(\theta x + (1-\theta)) = \theta f(x) + (1-\theta) f(y) \underset{\uparrow}{\geq} f(x)^{\theta} f(y)^{(1-\theta)}.$$

$$a^{\theta} b^{1-\theta} \leq \theta a + (1-\theta) b \text{ for } a, b > 0, \ 0 \leq \theta \leq 1.$$

Ex. $f(x) = e^{ax}$

$\log f(x) = ax \Rightarrow f(x)$ is log-convex and log-concave.

Ex. The cumulative distribution function of a Gaussian density

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du.$$

is log-concave. (Exercise.)

- Twice differentiable function f with convex domain

$f: \mathbb{R}^n \to \mathbb{R}$. $\quad f(x) > 0 \ \forall x \in \mathrm{dom} f$.

$$\nabla \log f(x) = \frac{1}{f(x)} \nabla f(x).$$

$$\nabla^2 \log f(x) = \underbrace{\frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^T}_{\preceq 0 \ \text{(Second order condition)}}.$$

f is log-concave if and only if

$$f(x) \nabla^2 f(x) \preceq \nabla f(x) \nabla f(x)^T, \ \forall x \in \mathrm{dom} f.$$

- The product of two log-concave functions is log-concave.

$g(x) f(x) \Rightarrow \log g(x) f(x) = \log g(x) + \log f(x)$.

- The sum of two $\boxed{\text{log-convex}}$ functions is log-convex

Suppose. f & g are log-convex.

$F \triangleq \log f$ & $G \triangleq \log g$ are convex.

$\log(f+g) = \log(e^F + e^G)$ is convex (composition rule).

P.74 of BV04. $\log \sum_i e^{x_i}$ is convex in X.

(Second order condition). verify it

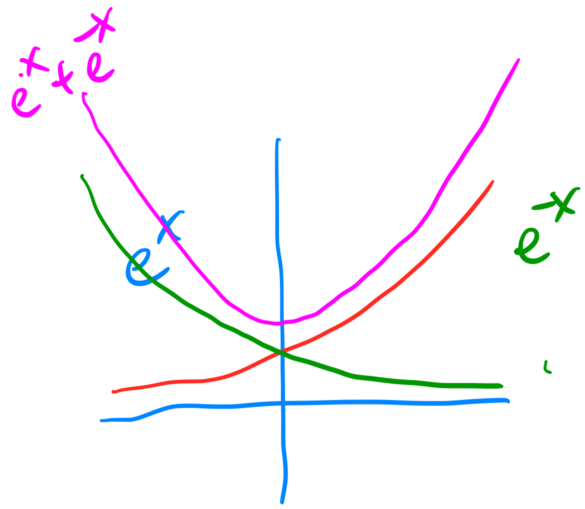
∘ The sum of two <u>log-concave</u> functions is $\boxed{\text{not}}$ always log-concave.

Ex. e^{ax} is log-concave (/log-convex).

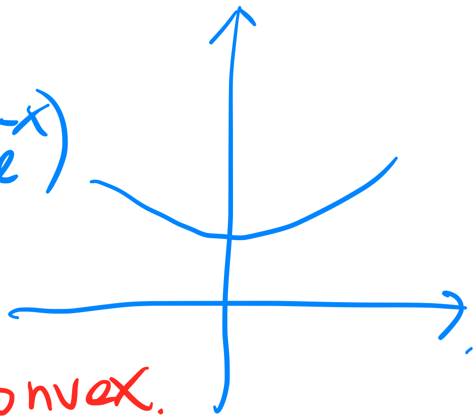
consider $e^{x} + e^{-x}$

$$\frac{\partial^2}{\partial x^2}\left(\log e^{x}+e^{-x}\right) =$$

$$\geq 0$$ (verify this).



$\Rightarrow \log(e^{x}+e^{-x})$



convex.

Suppose that $K \subseteq \mathbb{R}^n$ is a proper cone.

$f: \mathbb{R}^n \to \mathbb{R}$ is K-nondecreasing

if $x \preceq_K y \Longrightarrow f(x) \le f(y)$.

$f: \mathbb{R} \to \mathbb{R}$. f is nondecreasing (increasing).

if $x \le y \Rightarrow f(x) \underset{(<)}{\le} f(y)$.

▷ A differentiable function f with convex domain is K-nondecreasing if and only if

$$\nabla f(x) \succeq_{K^*} 0 \qquad K^* = \{y : \langle y, x \rangle \ge 0 \ \forall x \in K\}.$$

Proof. $\nabla f(x) - 0 \in K^*$.

(Necessity) Suppose not (f is not K-nondecreasing). $\exists x, y \in K$ $x \preceq_K y$, $f(y) < f(x)$

Consider $f(x + t(y-x))$. : $\mathbb{R} \to \mathbb{R}$ which is not nondecreasing

$$\frac{d}{dt} f(x + t(y-x)) = \nabla f(x + t(y-x))^T (y-x).$$

$$\exists t_0 \ni \nabla f(x + t_0(y-x))^T (y-x) < 0.$$

$$y - x \in K \Rightarrow \nabla f(x + t_0(y-x))^T \notin K^*.$$

which is a contradiction that

Ex. $f : S^n \to \mathbb{R}$ defined by $f(X) = tr(WX)$

$W \in S^n$.

If $W \succeq 0$, f is matrix nondecreasing.

proof. for $X \succeq Y$, $(X - Y \succeq 0)$

$\Rightarrow W(X-Y) \succeq 0. \Rightarrow Tr\, W(X-Y) \geq 0.$

$\Rightarrow Tr(WX) \geq Tr(WY)$ #

Ex. $\det X$ is matrix increasing on S^n_{++}.

Suppose $K \subseteq \mathbb{R}^m$ is a proper cone.

$f: \mathbb{R}^n \to \mathbb{R}^m$ is $\underline{K\text{-convex}}$ if $\forall x, y$,
$0 \le \theta \le 1$,

$$f(\theta x + (1-\theta) y) \preceq_K \theta f(x) + (1-\theta) f(y).$$

Ex. $f: \mathbb{R}^{n\times m} \to S_+^n$ defined.

$f(X) = XX^T$ is matrix convex.

consider $X, Y \in \mathbb{R}^{n\times m}$

$$\theta XX^T + (1-\theta) YY^T \succeq (\theta X + (1-\theta) Y)(\theta X + (1-\theta) Y)^T$$

$$\Leftrightarrow \theta(1-\theta) \underbrace{(X-Y)(X-Y)^T \succeq 0}$$

#. .