Homework #3

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1 R.Paul's methods: Jacobian solution

Joint	θ	d	a	α
1	θ_1	0	0	$-90°$
2	θ_2	0	a_2	$0°$
3	θ_3	d_3	a_3	$90°$
4	θ_4	d_4	0	$-90°$
5	θ_5	0	0	$90°$
6	θ_6	0	0	$0°$

Table 1: PUMA 560 kinematic table

In lecture, $\mathbf{A}_n = \begin{bmatrix} c_n & -s_n c_{\alpha_n} & s_n s_{\alpha_n} & a_n c_n \\ s_n & -c_n c_{\alpha_n} & -c_n s_{\alpha_n} & a_n s_n \\ 0 & s\,\alpha_n & c\,\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$ where $s_n = \sin(n)$ and $c_n = \cos(n)$

Based on Table 1, we have

$$\mathbf{A}_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} c_2 & -s_2 & 0 & c_2 a_2 \\ s_2 & c_2 & 0 & s_2 a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} c_3 & 0 & s_3 & c_3 a_3 \\ s_3 & 0 & -c_3 & s_3 a_3 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Also calculate inverse of each matrix,

$$\mathbf{A}_1^{-1} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_2^{-1} = \begin{bmatrix} c_2 & s_2 & 0 & -a_2 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_3^{-1} = \begin{bmatrix} c_3 & s_3 & 0 & -a_3 \\ 0 & 0 & 1 & -d_3 \\ s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_4^{-1} = \begin{bmatrix} c_4 & s_4 & 0 & 0 \\ 0 & 0 & -1 & d_4 \\ -s_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_5^{-1} = \begin{bmatrix} c_5 & s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_6^{-1} = \begin{bmatrix} c_6 & s_6 & 0 & 0 \\ -s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$\begin{aligned} {}^5\mathbf{T} &= \mathbf{A}_5^{-1}\,\mathbf{A}_4^{-1}\,\mathbf{A}_3^{-1}\,\mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}\,\mathbf{T} \\ &= \mathbf{A}_6 \\ &= \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} {}^4\mathbf{T} &= \mathbf{A}_4^{-1}\,\mathbf{A}_3^{-1}\,\mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}\,\mathbf{T} \\ &= \mathbf{A}_5\,\mathbf{A}_6 \\ &= \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \mathbf{A}_6 \\ &= \begin{bmatrix} c_5c_6 & -c_5s_6 & s_5 & 0 \\ s_5c_6 & -s_5s_6 & -c_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} {}^3\mathbf{T} &= \mathbf{A}_3^{-1}\,\mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}\,\mathbf{T} \\ &= \mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6 \\ &= \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \mathbf{A}_5\,\mathbf{A}_6 \\ &= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & 0 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & 0 \\ -s_5c_6 & s_5s_6 & c_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} {}^2\mathbf{T} &= \mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}\,\mathbf{T} \\ &= \mathbf{A}_3\,\mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6 \\ &= \begin{bmatrix} c_3 & 0 & s_3 & c_3a_3 \\ s_3 & 0 & -c_3 & s_3a_3 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6 \\ &= \begin{bmatrix} c_3(c_4c_5c_6 - s_4s_6) - s_3s_5c_6 & -c_3(c_4c_5s_6 + s_4c_6) + s_3s_5s_6 & c_3c_4s_5 + s_3c_5 & s_3d_4 + c_3a_3 \\ s_3(c_4c_5c_6 - s_4s_6) + c_3s_5c_6 & -s_3(c_4c_5s_6 + s_4c_6) - c_3s_5s_6 & s_3c_4s_5 - c_3c_5 & -c_3d_4 + s_3a_3 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
{}^1\mathbf{T} &= \mathbf{A}_1^{-1}\,\mathbf{T} \\
&= \mathbf{A}_2\,\mathbf{A}_3\,\mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6 \\
&= \begin{bmatrix} c_2 & -s_2 & 0 & c_2a_2 \\ s_2 & c_2 & 0 & s_2a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \mathbf{A}_3\,\mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6 \\
&= \begin{bmatrix} c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6 & -c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6 & c_{23}c_4s_5 + s_{23}c_5 & s_{23}d_4 + c_{23}a_3 + c_2a_2 \\ s_{23}(c_4c_5c_6 - s_4s_6) + c_{23}s_5c_6 & -s_{23}(c_4c_5s_6 + s_4c_6) - c_{23}s_5s_6 & s_{23}c_4s_5 - c_{23}c_5 & -c_{23}d_4 + s_{23}a_3 + s_2a_2 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\mathbf{T} = \mathbf{A}_1 * \mathbf{A}_2 * \mathbf{A}_3 * \mathbf{A}_4 * \mathbf{A}_5 * \mathbf{A}_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{cases}
n_x = c_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) - s_1(s_4c_5c_6 + c_4s_6) \\
n_y = s_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) + c_1(s_4c_5c_6 + c_4s_6) \\
n_z = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6 \\
o_x = -c_1(c_{23}(c_4c_5s_6 + s_4c_6) - s_{23}s_5s_6) + s_1(s_4c_5s_6 - c_4c_6) \\
o_y = -s_1(c_{23}(c_4c_5s_6 + s_4c_6) - s_{23}s_5s_6) - c_1(s_4c_5s_6 - c_4c_6) \\
o_z = s_{23}(c_4c_5s_6 + s_4c_6) + c_{23}s_5s_6 \\
a_x = c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5 \\
a_y = s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5 \\
a_z = -s_{23}c_4s_5 + c_{23}c_5 \\
p_x = c_1(s_{23}d_4 + c_{23}a_3 + c_2a_2) - s_1d_3 \\
p_y = s_1(s_{23}d_4 + c_{23}a_3 + c_2a_2) + c_1d_3 \\
p_z = c_{23}d_4 - s_{23}a_3 - s_2a_2
\end{cases}
$$

Jacobian Matrix

$$
\begin{bmatrix} {}^{T_6}d_x \\ {}^{T_6}d_y \\ {}^{T_6}d_z \\ {}^{T_6}\delta_x \\ {}^{T_6}\delta_y \\ {}^{T_6}\delta_z \end{bmatrix} = \begin{bmatrix} {}^{T_6}d_{1x} & {}^{T_6}d_{2x} & {}^{T_6}d_{3x} & {}^{T_6}d_{4x} & {}^{T_6}d_{5x} & {}^{T_6}d_{6x} \\ {}^{T_6}d_{1y} & {}^{T_6}d_{2y} & {}^{T_6}d_{3y} & {}^{T_6}d_{4y} & {}^{T_6}d_{5y} & {}^{T_6}d_{6y} \\ {}^{T_6}d_{1z} & {}^{T_6}d_{2z} & {}^{T_6}d_{3z} & {}^{T_6}d_{4z} & {}^{T_6}d_{5z} & {}^{T_6}d_{6z} \\ {}^{T_6}\delta_{1x} & {}^{T_6}\delta_{2x} & {}^{T_6}\delta_{3x} & {}^{T_6}\delta_{4x} & {}^{T_6}\delta_{5x} & {}^{T_6}\delta_{6x} \\ {}^{T_6}\delta_{1y} & {}^{T_6}\delta_{2y} & {}^{T_6}\delta_{3y} & {}^{T_6}\delta_{4y} & {}^{T_6}\delta_{5y} & {}^{T_6}\delta_{6y} \\ {}^{T_6}\delta_{1z} & {}^{T_6}\delta_{2z} & {}^{T_6}\delta_{3z} & {}^{T_6}\delta_{4z} & {}^{T_6}\delta_{5z} & {}^{T_6}\delta_{6z} \end{bmatrix} * \begin{bmatrix} dq_1 \\ dq_2 \\ dq_3 \\ dq_4 \\ dq_5 \\ dq_6 \end{bmatrix}
$$

$$
{}^{T_N}d_i = \begin{bmatrix} p_xn_y - n_xp_y \\ p_xo_y - o_xp_y \\ p_xa_y - a_xp_y \end{bmatrix} \texttt{ and } {}^{T_N}\delta_i = \begin{bmatrix} n_z \\ o_z \\ a_z \end{bmatrix}
$$

let $\mathbf{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$\begin{cases} {}^{T_6}d_{1x} = p_x n_y - n_x p_y = (s_{23}d_4 + c_{23}a_3 + c_2 a_2)(s_4 c_5 c_6 + c_4 s_6) - d_3(c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_5 c_6) \\ {}^{T_6}d_{1y} = p_x o_y - o_x p_y = -(s_{23}d_4 + c_{23}a_3 + c_2 a_2)(s_4 c_5 s_6 - c_4 c_6) + d_3(c_{23}(c_4 c_5 s_6 + s_4 c_6) - s_{23} s_5 s_6) \\ {}^{T_6}d_{1z} = p_x a_y - a_x p_y = (s_{23}d_4 + c_{23}a_3 + c_2 a_2) s_4 s_5 - d_3(c_{23} c_4 s_5 + s_{23} c_5) \\ {}^{T_6}\delta_{1x} = n_z = -s_{23}(c_4 c_5 c_6 - s_4 s_6) - c_{23} s_5 c_6 \\ {}^{T_6}\delta_{1y} = o_z = s_{23}(c_4 c_5 s_6 + s_4 c_6) + c_{23} s_5 s_6 \\ {}^{T_6}\delta_{1z} = a_z = -s_{23} c_4 s_5 + c_{23} c_5 \end{cases}$$

let ${}^1\mathbf{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$\begin{cases} {}^{T_6}d_{2x} = p_x n_y - n_x p_y = (c_4 c_5 c_6 - s_4 s_6)(d_4 + a_2 s_3) + s_5 c_6 (a_3 + c_3 a_2) \\ {}^{T_6}d_{2y} = p_x o_y - o_x p_y = -(c_4 c_5 s_6 + s_4 c_6)(d_4 + a_2 s_3) - s_5 s_6 (a_3 + c_3 a_2) \\ {}^{T_6}d_{2z} = p_x a_y - a_x p_y = c_4 s_5 (d_4 + a_2 s_3) - c_5 (a_3 + c_3 a_2) \\ {}^{T_6}\delta_{2x} = n_z = s_4 c_5 c_6 + c_4 s_6 \\ {}^{T_6}\delta_{2y} = o_z = -s_4 c_5 s_6 + c_4 c_6 \\ {}^{T_6}\delta_{2z} = a_z = s_4 s_5 \end{cases}$$

let ${}^2\mathbf{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$\begin{cases} {}^{T_6}d_{3x} = p_x n_y - n_x p_y = d_4(c_4 c_5 c_6 - s_4 s_6) + a_3 s_5 c_6 \\ {}^{T_6}d_{3y} = p_x o_y - o_x p_y = -d_4(c_4 c_5 s_6 + s_4 c_6) - a_3 s_5 s_6 \\ {}^{T_6}d_{3z} = p_x a_y - a_x p_y = d_4 c_4 s_5 - a_3 c_5 \\ {}^{T_6}\delta_{3x} = n_z = s_4 c_5 c_6 + c_4 s_6 \\ {}^{T_6}\delta_{3y} = o_z = -s_4 c_5 s_6 + c_4 c_6 \\ {}^{T_6}\delta_{3z} = a_z = s_4 s_5 \end{cases}$$

let ${}^3\mathbf{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$\begin{cases} {}^{T_6}d_{4x} = p_x n_y - n_x p_y = 0 \\ {}^{T_6}d_{4y} = p_x o_y - o_x p_y = 0 \\ {}^{T_6}d_{4z} = p_x a_y - a_x p_y = 0 \\ {}^{T_6}\delta_{4x} = n_z = -s_5 c_6 \\ {}^{T_6}\delta_{4y} = o_z = s_5 s_6 \\ {}^{T_6}\delta_{4z} = a_z = c_5 \end{cases}$$

let ${}^4\mathbf{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$\begin{cases} {}^{T_6}d_{5x} = p_x n_y - n_x p_y = 0 \\ {}^{T_6}d_{5y} = p_x o_y - o_x p_y = 0 \\ {}^{T_6}d_{5z} = p_x a_y - a_x p_y = 0 \\ {}^{T_6}\delta_{5x} = n_z = s_6 \\ {}^{T_6}\delta_{5y} = o_z = c_6 \\ {}^{T_6}\delta_{5z} = a_z = 0 \end{cases}$$

let ${}^5\mathbf{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$\begin{cases} {}^{T_6}d_{6x} = p_x n_y - n_x p_y = 0 \\ {}^{T_6}d_{6y} = p_x o_y - o_x p_y = 0 \\ {}^{T_6}d_{6z} = p_x a_y - a_x p_y = 0 \\ {}^{T_6}\delta_{6x} = n_z = 0 \\ {}^{T_6}\delta_{6y} = o_z = 0 \\ {}^{T_6}\delta_{6z} = a_z = 1 \end{cases}$$

2 R.Paul's methods: inverse Jacobian solution

From $\mathbf{A}_1^{-1}\,\mathbf{T} = \mathbf{A}_2\,\mathbf{A}_3\,\mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6$,

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{T} = \begin{bmatrix} \dots & \dots & \dots & s_{23}d_4 + c_{23}a_3 + c_2 a_2 \\ \dots & \dots & s_4 s_5 s & -c_{23}d_4 + s_{23}a_3 + s_2 a_2 \\ \dots & \dots & \dots & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have $-s_1 p_x + c_1 p_y = d_3$,
differentiate with respect to θ_1

$$d\,\theta_1 = \frac{-s_1 dp_x + c_1 dp_y}{c_1 p_x + s_1 p_y}$$

Singular point: when $c_1 p_x + s_1 p_y = 0$
From $\mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}\,\mathbf{T} = \mathbf{A}_3\,\mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6$,

$$\begin{bmatrix} c_1 c_2 & s_1 c_2 & -s_2 & -a_2 \\ -c_1 s_2 & -s_1 s_2 & -c_2 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{T} = \begin{bmatrix} \dots & \dots & \dots & s_3 d_4 + c_3 a_3 \\ \dots & \dots & \dots & -c_3 d_4 + s_3 a_3 \\ \dots & \dots & \dots & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have

$$s_3 d_4 + c_3 a_3 = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2}$$

differentiate with respect to θ_3,

$$d\,\theta_3 = \frac{p_x dp_x + p_y dp_y + p_z dp_z}{-a_2(s_3 a_3 - c_3 d_4)}$$

Singular point: when $s_3 a_3 - c_3 d_4 = 0$
We also have $c_2(c_1 p_x + s_1 p_y) - s_2 p_z = s_3 d_4 + c_3 a_3 + a_2$, differentiate with respect to θ_2,

$$\theta_2 = \frac{c_2 dc_1 p_x + c_1 c_2 dp_x + c_2 ds_1 p_y + c_2 s_1 dp_y - s_2 dp_z - d_4 ds_3 - a_3 dc_3}{s_2 c_1 p_x + s_2 s_1 p_y + c_2 p_z}$$

Singular point: when $s_2 c_1 p_x + s_2 s_1 p_y + c_2 p_z = 0$
From $\mathbf{A}_3^{-1}\,\mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}\,\mathbf{T} = \mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6$,

$$\begin{bmatrix} c_1 c_{23} & s_1 c_{23} & -s_{23} & -c_3 a_2 - a_3 \\ -s_1 & c_1 & 0 & -d_3 \\ c_1 s_{23} & s_1 s_{23} & c_{23} & -s_3 a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{T} = \begin{bmatrix} \dots & \dots & c_4 s_5 & \dots \\ \dots & \dots & s_4 s_5 & \dots \\ -s_5 c_6 & s_5 s_6 & \dots & \dots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have

$$\tan\theta_4 = \frac{-s_1 a_x + c_1 a_y}{c_1 c_{23} a_x + s_1 c_{23} a_y - s_{23} a_z}$$

differentiate with respect to θ_4,

$$d\,\theta_4 = \frac{[(-s_1 da_x + c_1 da_y)(c_1 c_{23} a_x + s_1 c_{23} a_y - s_{23} a_z) - (-s_1 a_x + c_1 a_y)(c_1 c_{23} da_x + s_1 c_{23} da_y - s_{23} da_z)]c_4^2}{(c_1 c_{23} a_x + s_1 c_{23} a_y - s_{23} a_z)^2}$$

Singular point: when $c_1 c_{23} a_x + s_1 c_{23} a_y - s_{23} a_z = 0$
From $\mathbf{A}_1^{-1}\,\mathbf{T} = \mathbf{A}_2\,\mathbf{A}_3\,\mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6$,
We have $-s_1 a_x + c_1 a_y = s_4 s_5$
differentiate with respect to θ_5,

$$d\,\theta_5 = \frac{-s_1 da_x + c_1 da_y - s_5 ds_4}{s_4 c_5}$$

Singular point: when $s_4 c_5 = 0$
From $\mathbf{A}_3^{-1}\,\mathbf{A}_2^{-1}\,\mathbf{A}_1^{-1}\,\mathbf{T} = \mathbf{A}_4\,\mathbf{A}_5\,\mathbf{A}_6$,
We have

$$\tan\theta_6 = \frac{c_1 s_{23} o_x + s_1 s_{23} o_y + c_{23} o_z}{-(c_1 s_{23} n_x + s_1 s_{23} n_y + c_{23} n_z)} = \frac{P}{Q}$$

differentiate with respect to θ_6,

$$d\,\theta_6 = \frac{[(c_1 s_{23} do_x + s_1 s_{23} do_y + c_{23} do_z)Q + (c_1 s_{23} dn_x + s_1 s_{23} dn_y + c_{23} dn_z)P]c_6^2}{Q^2}$$

3 Jacobian derived by wrist coordinate frame

From paper "An Efficient Solution of Differential Inverse Kinematics Problem for Wrist-Partitioned Robots", we know

$$\mathbf{J}_w = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & 0 & 0 & 0 \\ \mathbf{J}_{21} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & 0 & 0 & 0 \\ \mathbf{J}_{41} & 0 & 0 & 0 & \mathbf{J}_{45} & \mathbf{J}_{46} \\ 0 & 1 & 1 & 0 & \mathbf{J}_{55} & \mathbf{J}_{56} \\ \mathbf{J}_{61} & 0 & 0 & 1 & 0 & \mathbf{J}_{66} \end{bmatrix} \texttt{where}$$

$$\begin{cases} \mathbf{J}_{11} = -\,\mathbf{J}_{61}\,d_3 \\ \mathbf{J}_{21} = a_2\cos\theta_2 + a_3\,\mathbf{J}_{61} - d_4\,\mathbf{J}_{41} \\ \mathbf{J}_{31} = \mathbf{J}_{41}\,d_3 \\ \mathbf{J}_{41} = -\sin(\theta_2+\theta_3) \\ \mathbf{J}_{61} = \cos(\theta_2+\theta_3) \\ \mathbf{J}_{12} = a_2\sin\theta_3 + d_4 \\ \mathbf{J}_{32} = -a_2 cos\,\theta_3 - a_3 \\ \mathbf{J}_{13} = d_4 \\ \mathbf{J}_{33} = -a_3 \\ \mathbf{J}_{45} = -\sin\theta_4 \\ \mathbf{J}_{55} = \cos\theta_4 \\ \mathbf{J}_{46} = \cos\theta_4\sin\theta_5 \\ \mathbf{J}_{56} = \sin\theta_4\sin\theta_5 \\ \mathbf{J}_{66} = \cos\theta_5 \end{cases}$$

$$d\bar{x} = \mathbf{J}_w\,d\bar{q} \quad d\bar{x} = \begin{bmatrix} dx & dy & dz & \delta_x & \delta_y & \delta_z \end{bmatrix}^T$$

$$\mathbf{W}_i = \bar{\delta}_i + \epsilon\bar{d}_i \quad \bar{\delta}_i = \begin{bmatrix} \delta x_i & \delta y_i & \delta z_i \end{bmatrix}^T \quad \bar{d}_i = \begin{bmatrix} dx_i & dy_i & dz_i \end{bmatrix}^T$$

$$\mathbf{W}_{\delta_T} = \bar{\delta} + \epsilon\bar{d} = \mathbf{W}_1\,dq_1 + \mathbf{W}_2\,dq_2 + \ldots + \mathbf{W}_N\,dq_N$$

$$\begin{cases} \mathbf{W}_1^w = \mathbf{J}_{41}\,\bar{i}^w + \mathbf{J}_{61}\,\bar{k}^w + \epsilon(\mathbf{J}_{11}\,\bar{i}^w + \mathbf{J}_{21}\,\bar{j}^w + \mathbf{J}_{31}\,\bar{k}^w) \\ \mathbf{W}_2^w = \bar{j}^w + \epsilon(\mathbf{J}_{12}\,\bar{i}^w + \mathbf{J}_{32}\,\bar{k}^w) \\ \mathbf{W}_3^w = \bar{j}^w + \epsilon(\mathbf{J}_{13}\,\bar{i}^w + \mathbf{J}_{33}\,\bar{k}^w) \\ \mathbf{W}_4^w = \bar{k}^w \\ \mathbf{W}_5^w = \mathbf{J}_{45}\,\bar{i}^w + \mathbf{J}_{55}\,\bar{j}^w \\ \mathbf{W}_6^w = \mathbf{J}_{46}\,\bar{i}^w + \mathbf{J}_{56}\,\bar{j}^w + \mathbf{J}_{66}\,\bar{k}^w \end{cases}$$

$$\begin{cases} \mathbf{N}_1^w = \epsilon\bar{j}^w \\ \mathbf{N}_2^w = \epsilon(\mathbf{J}_{33}\,\bar{i}^w - \mathbf{J}_{13}\,\bar{k}^w) \\ \mathbf{N}_3^w = \epsilon(\mathbf{J}_{13}\,\bar{i}^w + \mathbf{J}_{33}\,\bar{k}^w) \\ \mathbf{N}_4^w = \bar{k}^w \\ \mathbf{N}_5^w = \mathbf{J}_{45}\,\bar{i}^w + \mathbf{J}_{55}\,\bar{j}^w \\ \mathbf{N}_6^w = \mathbf{J}_{55}\,\bar{i}^w - \mathbf{J}_{45}\,\bar{j}^w \end{cases}$$

$$\begin{cases} dq_1 = \frac{dy_w}{\mathbf{J}_{21}} \\ dq_2 = \frac{A\,\mathbf{J}_{33} - B\,\mathbf{J}_{13}}{\mathbf{J}_{12}\,\mathbf{J}_{33} - \mathbf{J}_{13}\,\mathbf{J}_{32}} \\ dq_3 = \frac{(A - \mathbf{J}_{12}\,dq_2)\,\mathbf{J}_{13} + (B - \mathbf{J}_{32}\,dq_2)\,\mathbf{J}_{33}}{\mathbf{J}_{13}^2 + \mathbf{J}_{33}^2} \\ dq_4 = \delta z_w - \mathbf{J}_{61}\,dq_1 - \mathbf{J}_{66}\,dq_6 \\ dq_5 = (C - \mathbf{J}_{46}\,dq_6)J_{45} + (D - \mathbf{J}_{56}\,dq_6)\,\mathbf{J}_{55} \\ dq_6 = \frac{C\,\mathbf{J}_{55} - D\,\mathbf{J}_{45}}{\mathbf{J}_{46}\,\mathbf{J}_{55} - \mathbf{J}_{45}\,\mathbf{J}_{56}} \end{cases}$$

$$\mathbf{J}_w^{-1} = \begin{bmatrix} 0 & \mathbf{G}_{12} & 0 & 0 & 0 & 0 \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} & 0 & 0 & 0 \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{G}_{33} & 0 & 0 & 0 \\ \mathbf{G}_{41} & \mathbf{G}_{42} & \mathbf{G}_{43} & \mathbf{G}_{44} & \mathbf{G}_{45} & 1 \\ \mathbf{G}_{51} & \mathbf{G}_{52} & \mathbf{G}_{53} & \mathbf{G}_{54} & \mathbf{G}_{55} & 0 \\ \mathbf{G}_{61} & \mathbf{G}_{62} & \mathbf{G}_{63} & \mathbf{G}_{64} & \mathbf{G}_{65} & 0 \end{bmatrix} \texttt{where}$$

$$\mathbf{G}_{12} = 1/\,\mathbf{J}_{21} \quad \mathbf{G}_{21} = \mathbf{J}_{33}\,/\,\mathbf{L} \quad \mathbf{G}_{22} = \mathbf{G}_{12}\,\mathbf{M}\,/\,\mathbf{L} \quad \mathbf{G}_{23} = -\,\mathbf{J}_{13}\,/\,\mathbf{L}$$

$$\mathbf{G}_{31} = (\mathbf{J}_{13} - \mathbf{G}_{21}\,\mathbf{P})/\,\mathbf{N} \quad \mathbf{G}_{32} = -(\mathbf{G}_{12}\,\mathbf{Q} + \mathbf{G}_{22}\,\mathbf{P})/\,\mathbf{N} \quad \mathbf{G}_{33} = (\mathbf{J}_{33} - \mathbf{G}_{23}\,P)/\,\mathbf{N}$$

$$\mathbf{G}_{41} = -\,\mathbf{J}_{66}\,\mathbf{G}_{61} \quad \mathbf{G}_{42} = -\,\mathbf{J}_{61}\,\mathbf{G}_{12} - \mathbf{J}_{66}\,\mathbf{G}_{62} \quad \mathbf{G}_{43} = -\,\mathbf{J}_{66}\,\mathbf{G}_{63}$$

$$\mathbf{G}_{44} = -\,\mathbf{J}_{66}\,\mathbf{G}_{64} \quad \mathbf{G}_{45} = -\,\mathbf{J}_{66}\,\mathbf{G}_{65}$$

$$\mathbf{G}_{51} = -\,\mathbf{J}_{55}(\mathbf{G}_{21} + \mathbf{G}_{31}) - \mathbf{G}_{61}\,\mathbf{S} \quad \mathbf{G}_{52} = -\,\mathbf{J}_{41}\,\mathbf{J}_{45}\,\mathbf{G}_{12} - \mathbf{J}_{55}(\mathbf{G}_{22} + \mathbf{G}_{32}) - \mathbf{G}_{62}\,\mathbf{S}$$

$$\mathbf{G}_{53} = -\,\mathbf{J}_{55}(\mathbf{G}_{23} + \mathbf{G}_{33}) - \mathbf{G}_{63}\,\mathbf{S} \quad \mathbf{G}_{54} = \mathbf{J}_{45} - \mathbf{G}_{64}\,\mathbf{S} \quad \mathbf{G}_{55} = \mathbf{J}_{55} - \mathbf{G}_{65}\,\mathbf{S}$$

$$\mathbf{G}_{61} = \mathbf{J}_{45}(\mathbf{G}_{21} + \mathbf{G}_{31})/\,\mathbf{R} \quad \mathbf{G}_{62} = [\mathbf{J}_{45}(\mathbf{G}_{22} + \mathbf{G}_{32}) - \mathbf{J}_{41}\,\mathbf{J}_{55}\,\mathbf{G}_{12}]/\,\mathbf{R}$$

$$\mathbf{G}_{63} = (\mathbf{G}_{23} + \mathbf{G}_{33})\,\mathbf{J}_{45}\,/\,\mathbf{R} \quad \mathbf{G}_{64} = \mathbf{J}_{55}\,/\,\mathbf{R} \quad \mathbf{G}_{65} = -\,\mathbf{J}_{45}\,/\,\mathbf{R}$$

where

$$\mathbf{L} = \mathbf{J}_{12}\,\mathbf{J}_{33} - \mathbf{J}_{13}\,\mathbf{J}_{32} \quad \mathbf{M} = \mathbf{J}_{13}\,\mathbf{J}_{31} - \mathbf{J}_{11}\,\mathbf{J}_{33} \quad \mathbf{N} = \mathbf{J}_{13}^2 + \mathbf{J}_{33}^2 \quad \mathbf{P} = \mathbf{J}_{12}\,\mathbf{J}_{13} + \mathbf{J}_{32}\,\mathbf{J}_{33}$$

$$\mathbf{Q} = \mathbf{J}_{11}\,\mathbf{J}_{13} + \mathbf{J}_{31}\,\mathbf{J}_{33} \quad \mathbf{R} = \mathbf{J}_{46}\,\mathbf{J}_{55} - \mathbf{J}_{45}\,\mathbf{J}_{56} \quad \mathbf{S} = \mathbf{J}_{45}\,\mathbf{J}_{46} + \mathbf{J}_{55}\,\mathbf{J}_{56}$$