1. Let $A \in \mathbb{R}^{p\times n}$ and $b \in \mathbb{R}^p$. Consider the following primal problem with variable $x \in \mathbb{R}^n$:

$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax = b. \end{array}$$

   (a) In the case of $n = 2$, show that $f(x) = x_1^2 + x_2^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$ is **strictly** convex. (10%)
   (b) What is the corresponding dual problem with dual variable $\nu \in \mathbb{R}^p$? (10%)
   (c) What are the KKT conditions for $x^\star$ and $\nu^\star$ being optimal? What is the KKT system? (10%)

2. Show that $f(x, y) = -\log y - \log(\log y - x)$ on $\{(x, y) : e^x < y\}$ is **self-concordant**. (Recall that for a convex function $g : \mathbb{R} \to \mathbb{R}$ with $\operatorname{dom} g = \mathbb{R}_{++}$ and $|g'''(x)| \le 3\frac{g''(x)}{x}$ for all $x > 0$, $f(x) = -\log(-g(x)) - \log x$ is self-concordant on $\{x|x > 0, g(x) < 0\}$.) (20%)

3. Consider the function $f(x) = x_1^2 + x_2^2$ with $\operatorname{dom} f = \mathbb{R}^2$. Suppose we are using the following method.

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given an initial point $x \in \operatorname{dom} f$.
repeat 1. $\Delta x = -\nabla f(x)$.
  2. Line search. Choose step size t via exact line search.
  3. Update. $x := x + t\Delta x$.
until $||\nabla f(x)||_2 \le 0.005$.

   (a) Suppose the initial point is $x^{(0)} = (2, 3)$. What is the initial descent direction Δx? (5%)
   (b) In exact line search, the step size t is determined by

$$\arg\min_{s\ge 0} f(x + s\Delta x).$$

   What is the step size t at the first iteration? (10%)
   (c) What is $p^\star = \inf_x f(x)$? How many iterations are requried? (10%)

4. Consider the following equality-constrained optimization problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b. \end{array}$$

The Newton step Δx_{nt} of f at feasible x is given by the **solution** v of $\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$.
Suppose $Q \succeq 0$. The following modified problem

$$\begin{array}{ll} \text{minimize} & f(x) + (Ax - b)^T (Ax - b) \\ \text{subject to} & Ax = b. \end{array}$$

is equivalent to the original problem. Is the Newton step for this problem the same as the Newton step for the original problem? Prove or disprove it. (15%)

5. Consider the following inequality-constrained optimization problem with variable $x \in \mathbb{R}^2_+$:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \succeq \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{array}$$

   (a) Consider the generalized logarithm $\psi(x) = \log x_1 + \log x_2$ for $\mathbb{R}^2_+$. Show that there exists $\theta > 0$ such that for $s > 0$, $y >_{\mathbb{R}^2_+} 0$, $\psi(sy) = \psi(y) + \theta \log s$. (5%)
   (b) What is the corresponding centering problem if the generalized logarithm ψ is used? (5%)
   (c) What is the centrality condition? (5%)
   (d) Suppose in the centering problem $t = 1$ and $x^{(0)} = (1, 2)$. What is the initial Newton step? (15%)