Assignment #6

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ECM5901 - Optimization Theory and Application

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Exercise 1. Derive a dual problem for

$$\text{minimize} \quad \sum_{i=1}^{N} \|A_i x - b_i\|_2 + \frac{1}{2}\|x - x_0\|_2^2$$

The problem data are $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$, and $x_0 \in \mathbb{R}^n$. First introduce new variables $y_i \in \mathbb{R}^{m_i}$ and equality constraints $y_i = A_i x - b_i$. (20%)

Solution. The original problem is equivalent to

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{N} \|y_i\|_2 + \frac{1}{2}\|x - x_0\|_2^2 \\ \text{subject to} \quad & y_i = A_i x - b_i, \ i = 1, \ldots, N \end{aligned}$$

The Lagrange is

$$L(x, z_1, \ldots, z_N) = \sum_{i=1}^{N} \|y_i\|_2 + \frac{1}{2}\|x - x_0\|_2^2 + \sum_{i=1}^{N} z_i^T (y_i - A_i x + b_i)$$

Exercise 2. (A convex problem in which strong duality fails.) Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & e^{-x} \\ \text{subject to} \quad & x^2/y \le 0 \end{aligned}$$

with variables x and y, and domain $\mathcal{D} = \{(x, y) \mid y > 0\}$.

(a) Verify that this is a convex optimization problem. Find the optimal value. (5%)

(b) Give the Lagrange dual problem, and find the optimal solution λ^* and optimal value d^* of the dual problem. (10%)

(c) What is the optimal duality gap? Does Slater's condition hold for this problem? (5%)

Solution.

(a) a

(b) b

(c) c

Exercise 3. Prove (without using any linear programming code) that the optimal solution of the LP

$$\text{minimize} \quad 47x_1 + 93x_2 + 17x_3 - 93x_4$$

$$\text{subject to} \quad \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \preceq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix}$$

is unique, and given by $x^* = (1, 1, 1, 1)$. (20%)

Proof. □

Exercise 4. (SDP relaxations of two-way partitioning problem). We consider the two-way partitioning problem (5.7), described on page 219,

$$\begin{aligned} \text{minimize} \quad & x^T W x \\ \text{subject to} \quad & x_i^2 = 1, \; i = 1, \ldots, n \end{aligned} \tag{1}$$

with variable $x \in \mathbb{R}^n$. The Lagrange dual of this (nonconvex) problem is given by the SDP

$$\begin{aligned} \text{maximize} \quad & -\mathbf{1}^T \nu \\ \text{subject to} \quad & W + \mathbf{diag}(\nu) \succeq 0 \end{aligned} \tag{2}$$

with variable $\nu \in \mathbb{R}^n$. The optimal value of this SDP gives a lower bound on the optimal value of the partitioning problem (1). In this exercise we derive another SDP that gives a lower bound on the optimal value of the two-way partitioning problem, and explore the connection between the two SDPs.

(a) Two-way partitioning problem in matrix form. Show that the two-way partitioning problem can be cast as

$$\begin{aligned} \text{maximize} \quad & \mathbf{tr}(WX) \\ \text{subject to} \quad & X \succeq 0, \; \mathbf{rank}(X) = 1 \\ & X_{ii} = 1, \; i = 1, \ldots, n \end{aligned}$$

with variable $X \in \mathbb{S}^n$. Hint. Show that if X is feasible, then it has the form $X = xx^T$, where $x \in \mathbb{R}^n$ satisfies $x_i \in \{+1, -1\}$ (and vice versa). (5%)

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(b) (SDP relaxation of two-way partitioning problem.) Using the formulation in part (a), we can form the relaxation

$$\begin{array}{ll} \text{maximize} & \mathbf{tr}(WX) \\ \text{subject to} & X \succeq 0 \\ & X_{ii} = 1, \ i = 1, \ldots, n \end{array} \qquad (3)$$

with variable $X \in \mathbb{S}^n$. This problem is an SDP, and therefore can be solved efficiently. Explain why its optimal value gives a lower bound on the optimal value of the two-way partitioning problem (1). What can you say if an optimal point X^* for this SDP has rank one? (5%)

(c) We now have two SDPs that give a lower bound on the optimal value of the two-way partitioning problem (1): the SDP relaxation (3) found in part (b), and the Lagrange dual of the two-way partitioning problem, given in (2). What is the relation between the two SDPs? What can you say about the lower bounds found by them? Hint: Relate the two SDPs via duality. (10%)

Proof.

(a) a

(b) b

(c) c

$\square$

Exercise 5. The pure Newton method. Newton's method with fixed step size $t = 1$ can diverge if the initial point is not close to x^*. Consider

$$\text{minimize} \quad f(x) = \log(e^x + e^{-x})$$

$f(x)$ has a unique minimizer $x^* = 0$. Run Newton's method with fixed step size $t = 1$, starting at $x^{(0)} = 1$ and at $x^{(0)} = 1.2$. Show the errors of the first four iterates. (You can do it by hand or using MATLAB.)

Solution.

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1    for ok
2    while ok
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