$$a \geq b \iff \begin{cases} a+\epsilon \geq b & \forall \epsilon > 0 \\ a \geq b-\epsilon & \forall \epsilon > 0. \end{cases}$$

$0.999\cdots = 0.\bar{9} = 1.$

$a \geq b(n) \qquad b(n) \to b.$

Claim: $a \geq b$

$b(n) \to b \quad \forall \epsilon > 0, \exists N_0$ s.t. for $n \geq N_0$,

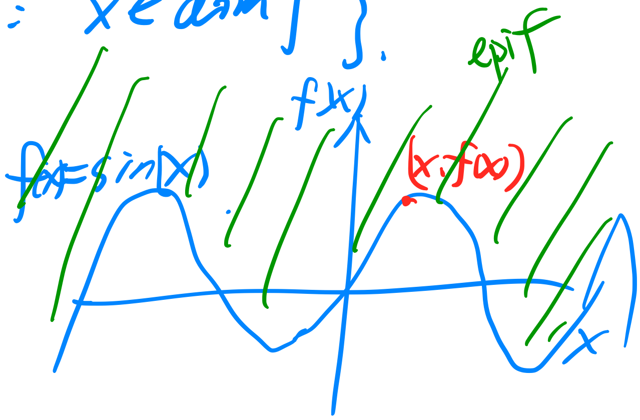
we have $|b(n)-b| \leq \epsilon. \Rightarrow \begin{cases} b(n) \leq b+\epsilon \\ b(n) \geq b-\epsilon \end{cases}$

So $a \geq b(n) \geq b-\epsilon$ for $n \geq N_0$.

$a \geq b-\epsilon \quad \forall \epsilon > 0.$

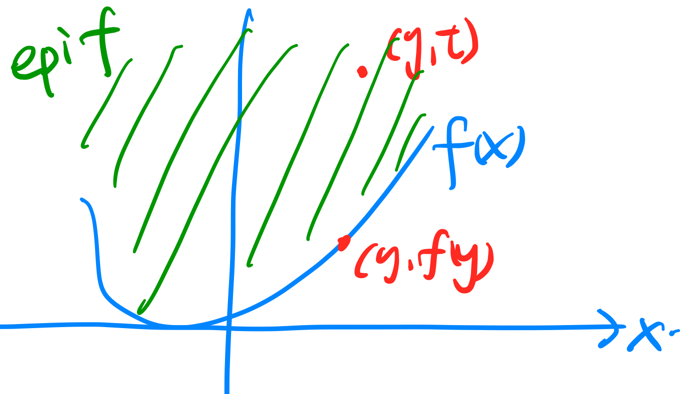
$\Rightarrow \underline{a \geq b}.$

The graph of a function $f: \mathbb{R}^n \to \mathbb{R}$ is defined as $\{(x, f(x)) \in \mathbb{R}^{n+1} : x \in \operatorname{dom} f\}$.



△ The epigraph (upper) of a function $f: \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\operatorname{epi} f = \{(x, t) : x \in \operatorname{dom} f \ f(x) \le t\}$$



△ f is convex if and only if $\operatorname{epi} f$ is a convex set.

$f(x) \le t$

(Necessity) for $(x, t), (y, s) \in \operatorname{epi} f$, $0 \le \theta \le 1$.

$f(y) \le s$

$$f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$$
$$\le \theta t + (1-\theta) s$$

$(\theta x + (1-\theta) y, \theta t + (1-\theta) s) \in \operatorname{epi} f$ by definition

$= \theta (x, t) + (1-\theta)(y, s)$. $\Rightarrow$ $\operatorname{epi} f$ is a convex set.

(Sufficiency)

Assume $\operatorname{epi} f$ is convex for $x, y \in \operatorname{dom} f$

$(x, f(x)), (y, f(y)) \in \operatorname{epi} f$.

$(x, t) \in \operatorname{epi} f \Leftrightarrow f(x) \le t$

For $0 \le \theta \le 1$, $\theta(x, f(x)) + (1-\theta)(y, f(y)) \in \operatorname{epi} f$ by assumption.

$= (\theta x + (1-\theta) y, \theta f(x) + (1-\theta) f(y)) \in \operatorname{epi} f$.

By definition $f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$. $\Rightarrow$ f is convex.

Jensen's inequality

(probability theory or information theory)

A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if $\operatorname{dom} f$ convex

$$\forall x, y \in \operatorname{dom} f \quad f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y).$$

$$0 \le \theta \le 1$$

If f is convex and x is a random variable such that $x \in \operatorname{dom} f$ with probability 1, then

$$f(\mathbb{E}\{x\}) \le \mathbb{E}\{f(x)\}.$$

provided the expection exists.

If x is a discrete random variable $\{x_i, p(x_i), i \in I\}$

$$\mathbb{E}\{x\} \triangleq \sum_{i \in I} p(x_i) \cdot x_i$$

$x_1 \quad p(x_1) = \theta$

$x_2 \quad p(x_2) = 1-\theta.$

continuous

$$\mathbb{E}\{x\} \triangleq \int_{t \in A} p(t)\, t \, dt$$

Ex. generalized arithmetic-geometric mean inequality

$$a^{\theta} b^{1-\theta} \le \theta a + (1-\theta) b.$$

when $\theta = 1/2$, $\sqrt{ab} \le \frac{a+b}{2}$.

proof.

$-\log x$ is convex. for $a, b > 0$

$$\Rightarrow -\log(\theta a + (1-\theta) b) \le -\theta \log a - (1-\theta) \log b.$$

$e \qquad e$

(take exponential on both sides).

Hölder's inequality

for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $x, y \in \mathbb{R}^n$

we have $$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}.$$

proof: Apply the generalized arithmetic-geometric mean inequality with $a^{\theta} b^{(1-\theta)} \le \theta a + (1-\theta) b$.

$$a = \frac{|x_i|^p}{\sum_{j=1}^{n} |x_j|^p}, \quad b = \frac{|y_i|^q}{\sum_{j=1}^{n} |y_j|^q}, \quad \theta = \frac{1}{p}$$

Then summarizing over $\underline{\underline{i}}$ (check it).

Practical method for establishing convexity of a function

①. verify the definition (often simplified by restricting to a line).

② For twice-differentiable functions, show $\nabla^2 f(x) \succeq 0$.

③ Show that f is obtained from simple convex functions by operations that preserve convexity.

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1. Nonnegative weighted sums.

Suppose f convex. $\alpha \geq 0$

$\Rightarrow \alpha f$ is convex.

$\{f_i\}$ convex, $w_i \geq 0$

$\Rightarrow \sum_i w_i f_i$ is convex.

2. composition with an affine mapping

$f(Ax+b)$ is convex (concave) if f is convex (concave).

Ex. $f(x) = \| Ax+b \|$ is convex since $\| \cdot \|$ is convex.

3. Pointwise maximum

If $f_1, f_2, \dots, f_m$ convex, $f(x) = \max\{f_1(x), f_2(x), \dots, f_m(x)\}$ with $\mathrm{dom} f = \bigcap_{i=1}^{n} \mathrm{dom} f_i$ is convex.

Proof. for $x, y \in \mathrm{dom} f \Rightarrow x, y \in \mathrm{dom} f_i \ \forall i$.

$$
\begin{aligned}
f(\theta x + (1-\theta) y) &= \max_i \{ f_i(\theta x + (1-\theta) y) \}. \\
&\le \max_i \{ \theta f_i(x) + (1-\theta) f_i(y) \}. \quad (f_i \text{ convex}). \\
&\le \theta \max_i \{ f_i(x) \} + (1-\theta) \max_i \{ f_i(y) \}, \\
&= \theta f(x) + (1-\theta) f(y). \ \#.
\end{aligned}
$$

Ex. $f(x) = \max_{i=1,\dots,m} \{ a_i^T x + b \}$ is convex.

Pointwise Supremum

If $f(x,y)$ is convex in x for each $y \in \mathcal{A}$,

then $g(x) \triangleq \sup_{y \in \mathcal{A}} f(x,y)$ is convex.

Ex. distance to a farest point of a set C.

$$f(x) = \sup_{y \in C} \| x - y \|$$

is convex.