Pointwise Supremum

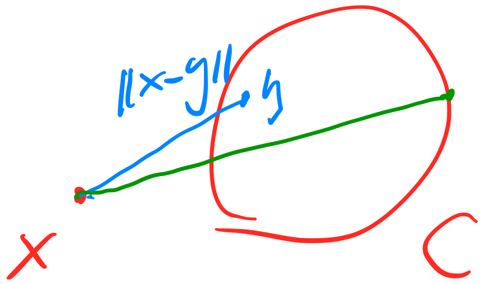
If $f(x,y)$ is convex in x for each $y \in \mathcal{A}$,

then $g(x) \triangleq \sup_{y \in \mathcal{A}} f(x,y)$ is convex.

Ex. distance to a farest point of a set C.

$$f(x) = \sup_{y \in C} \|x - y\|$$

is convex.



Ex. maximum eigenvalue of symmetric matrices. for $X \in S^n$.

$$\lambda_{max}(X) = \sup_{y : \|y\|_2 = 1} (y^T X y) \qquad Tr\, AB = Tr\, BA.$$

$$= \sup_{\|y\|_2 = 1} \underbrace{Tr\,(X \boxed{yy^T})}_{\text{a linear function of } X.}$$

Composition with Scalar functions $(n=1)$

For $g: \mathbb{R} \to \mathbb{R}$, $h: \mathbb{R} \to \mathbb{R}$, with $\operatorname{dom} g = \operatorname{dom} h = \mathbb{R}$.

let $f(x) = h(g(x))$

Assume that g and h are twice differentiable.

Then $f''(x) = h''(g(x))(g'(x))^2 + h'(x)\, g''(x)$.

Second condition f is convex (concave) $\Longleftrightarrow f'' \geq 0$ $(\leq)$.

f		h	$h(x) \le h(y)$ for $x \le y$	g
convex $f'' \geq 0$	if	convex $h'' \geq 0$	, nondecreasing $h' \geq 0$	convex. $g'' \geq 0$.
convex		convex	nonincreasing $h' \leq 0$	concave $g'' \leq 0$
concave		concave	nondecreasing	convex
concave		concave	nonincreasing	concave.

Ex. Suppose that $g(x)$ is convex. $e^{g(x)}$ is convex.

Ex. If g concave & positive, $\begin{cases} \log g(x) & \text{concave} \\ 1/g(x) & \text{convex.} \end{cases}$

For $g: \mathbb{R}^n \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$. let $f(x) = h(g(x))$

(No assumption on differentiability of g & h)

(sufficient conditions)

f		h	$\tilde{h}$	g
convex	if	convex	nondecreasing	convex.
"		"	nonincreasing	concave.
see the previous table				

Proof of the first statement $f = h(g)$

① Assume $x, y \in \mathrm{dom} f. \Rightarrow x, y \in \mathrm{dom}\, g.$

$\Rightarrow g(x), g(y) \in \mathrm{dom}\, h.$

Since g is convex, for $0 \le \theta \le 1$,

$\theta x + (1-\theta) y \in \mathrm{dom}\, g.$

$\Rightarrow g(\theta x + (1-\theta) y) \in \mathrm{dom}\, h.$

g is convex $\quad \le \theta g(x) + (1-\theta) g(y). \in \mathrm{dom}\, h.$

$h(\theta g(x) + (1-\theta) g(y)) < \infty.$

Since $\tilde{h}$ is nondecreasing,

$\Rightarrow \theta x + (1-\theta) y \in \mathrm{dom} f.\ h(g(\theta x + (1-\theta) y)) < \infty.$

$\Rightarrow \mathrm{dom} f$ is convex.

② $h(g(\theta x + (1-\theta) y)) \le$

$\le \theta h(g(x)) + (1-\theta) h(g(y)).$

Examples.

∘ $h(x) = \log x$, $\operatorname{dom} h = \mathbb{R}_{++}$.

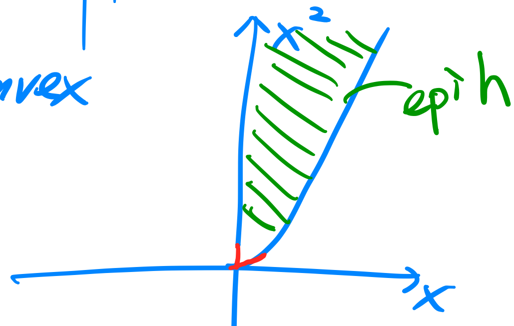
h is concave, $\tilde{h}$ = nondecreasing

$(\tilde{h}(x) = -\infty$ if $x \le 0)$.

△ $h(x) = x^2$, $\boxed{\operatorname{dom} h = \mathbb{R}_+}$, is convex

but $\tilde{h}$ is not nondecreasing

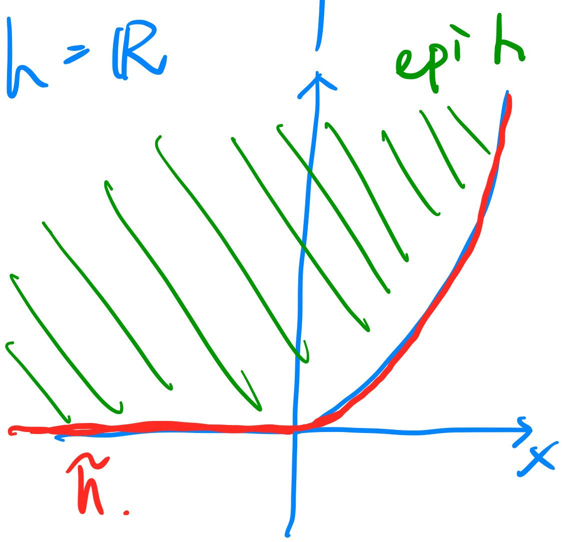
$\tilde{h}(x) = \infty$ for $x < 0$.



∘ $h(x) = (\max\{x, 0\})^2$. $\operatorname{dom} h = \mathbb{R}$

$\tilde{h}$ is nondecreasing

$(\tilde{h}(x) = 0$ for $x < 0)$



Vector Composition

$g: \mathbb{R}^n \to \mathbb{R}^k$, $h: \mathbb{R}^k \to \mathbb{R}$.

$f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$.

where $g_i(x): \mathbb{R}^n \to \mathbb{R}$.

Ex.

f		h	$\tilde{h}$	$\boxed{g_i}$
convex	if	convex	nondecreasing	convex

For general results, the monotonicity of $\tilde{h}$ must hold.

Ex. $\sum_{i=1}^{m} \log g_i(x)$ is concave if g_i are concave and positive,

Ex. $h(z) = \log \sum_{i=1}^{k} e^{z_i}$ is convex and nondecreasing in each argument so $\log \sum_{i=1}^{k} e^{g_i(x)}$ is convex if g_i are convex.

Minimization

If $f(x,y)$ is convex in (x,y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x,y)$$

with $\operatorname{dom} g = \{x : (x,y) \in \operatorname{dom} f \text{ for some } y \in C\}$.

is convex in x, provided that

$g(x) > -\infty$ for some x.

$\Rightarrow g(x) > -\infty \ \forall x$

proof. If $g(x_0) > -\infty$, $f(x_0, y) > -\infty \ \forall y \in C$.

Consider $x \neq x_0$, let $x' = 2x_0 - x \Rightarrow x_0 = \frac{x+x'}{2}$

$$f\left(\frac{x+x'}{2}, y\right) \leq \underbrace{\frac{1}{2} f(x,y)}_{-\infty <} + \underbrace{\frac{1}{2} f(x', y)}_{-\infty <}$$

$$\| $$

$$-\infty < f(x_0, y)$$

$\Rightarrow g(x) > -\infty$.

Proof. (we are trying to show that

$g(\theta x_1 + (1-\theta)x_2) \leq \theta g(x_1) + (1-\theta) g(x_2)$.).

$$\boxed{a \geq b - \epsilon \;\; \forall \epsilon > 0 \iff a \geq b}$$

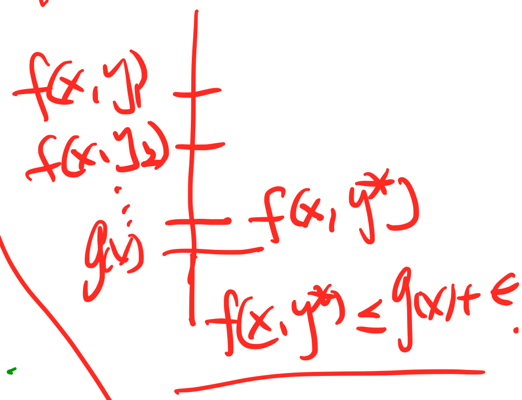
Let $\varepsilon > 0$.

For $x_1, x_2 \in \operatorname{dom} g$

$\exists\, y_1, y_2 \in C$ such that

$$\left(g(x) = \inf_{y \in C} f(x, y) \right)$$

$$\begin{cases} f(x_1, y_1) \leq g(x_1) + \epsilon \\ f(x_2, y_2) \leq g(x_2) + \epsilon . \end{cases}$$



$$\Rightarrow \underline{g(\theta x_1 + (1-\theta)x_2)} \overset{\text{def.}}{=} \inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y)$$

$$\leq f(\theta x_1 + (1-\theta)x_2, \underbrace{\theta y_1 + (1-\theta) y_2}_{\text{a fixed } y}).$$

$$\overset{f \text{ convex}}{\leq} \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2)$$

$$\leq \theta (g(x_1) + \epsilon) + (1-\theta)(g(x_2) + \epsilon)$$

$$= \theta g(x_1) + (1-\theta) g(x_2) + \epsilon .$$

$$g(\theta x_1 + (1-\theta)x_2) \leq \theta g(x_1) + (1-\theta) g(x_2) + \epsilon .$$

This holds $\forall \epsilon > 0$.

$$\Rightarrow g(\theta x_1 + (1-\theta)x_2) \leq \theta g(x_1) + (1-\theta) g(x_2)$$

$\Rightarrow g$ is convex.

Ex. $\operatorname{dist}(x, S) = \inf_{y \in S} \|x - y\|$

is convex iff S is convex.

Perspective of a function

The perspective of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$g(x, t) = t f(x/t)$$

$$\operatorname{dom} g = \{ (x,t) : x/t \in \operatorname{dom} f,\ t > 0 \}.$$

$\triangle$ g is convex (concave) if f is convex (concave).

(verify it).

Ex. $f(x) = x^T x$ is convex.

hence $g(x,t) = t \cdot \frac{x^T}{t} \cdot \frac{x}{t} = x^T x / t$ is convex for $t > 0$.

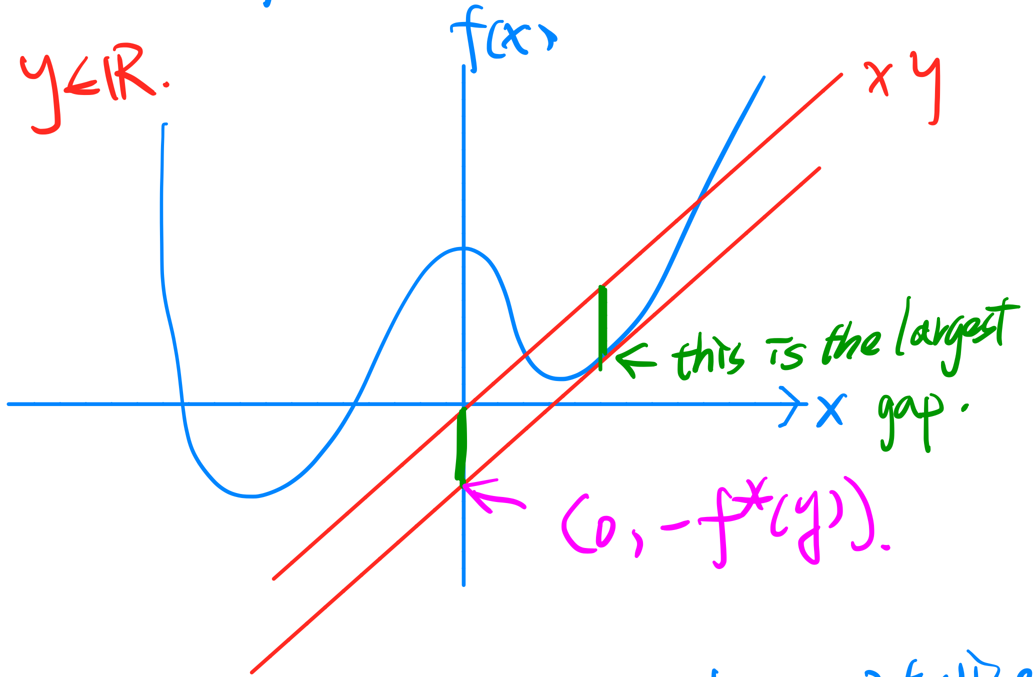
The conjugate function

Let $f: \mathbb{R}^n \to \mathbb{R}$. The conjugate of f is

$$f^*(y) = \sup_{x \in \operatorname{dom} f} \left(\underline{y^T x} - \underline{f(x)} \right)$$

For a fixed $y \in \mathbb{R}$.

$y > 0$.



- f^* is convex since it is the pointwise supremum of a family of affine functions of y.

  (Note f is not necessary convex).

- useful in Ch.5.

Ex. $f(x) = ax+b$.

$$f^*(y) = \sup_x \{ yx - ax - b \}.$$

$$= \sup_x \{ (y-a)x \} - b$$

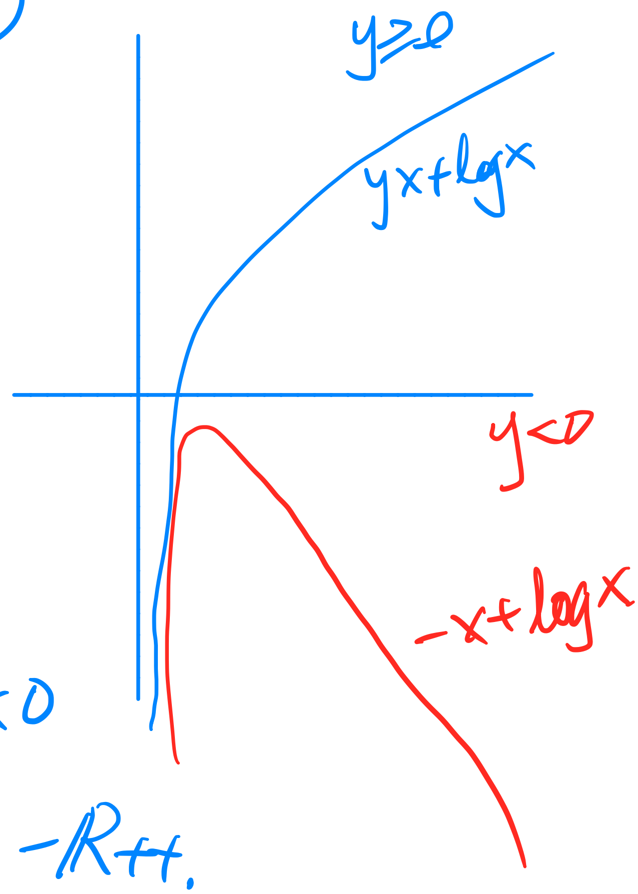
If $y \neq a$, $(y-a)\cdot x$ is unbounded.

$\therefore$ $f^*(y) = -b$ for $y = a$.

$\operatorname{dom} f^* = \{ a \}$.

Ex. Negative logarithm $f(x) = -\log x$ $\operatorname{dom} f = \mathbb{R}_{++}$.

$$f^*(y) = \sup_{x \in \mathbb{R}_{++}} (xy + \log x)$$



$xy + \log x$ is unbounded above if $y \geq 0$

$$0 = \frac{\partial}{\partial x}(xy + \log x) = y + \frac{1}{x}.$$

$$\Rightarrow \quad x = -1/y$$

$f^*(y) = -1 - \log(-y)$, $y < 0$

$\operatorname{dom} f^* = \{ y : y < 0 \} = -\mathbb{R}_{++}$.

Ex. $f(x) = \frac{1}{2} x^T Q x$, $Q > 0$. $x \in R^n$

$$f^*(y) = \sup_x \left(y^T x - \frac{1}{2} x^T Q x \right)$$

Recall from the first order condition of convex functions

$g(x)$ is convex (concave) $\Leftrightarrow$ $g(y) \geq g(x) + \nabla g(x)^T (y-x)$ ($\leq$) $\forall y \in \operatorname{dom} g$.

if $\nabla g(x) = 0$.

$\Rightarrow g(y) \leq g(x) \ \forall y$

$\Rightarrow x$ is a global maximizer

$$\underbrace{\underbrace{y^T x}_{\text{linear.}} - \underbrace{\frac{1}{2} x^T Q x}_{\text{concave}}}_{\text{concave.}}$$

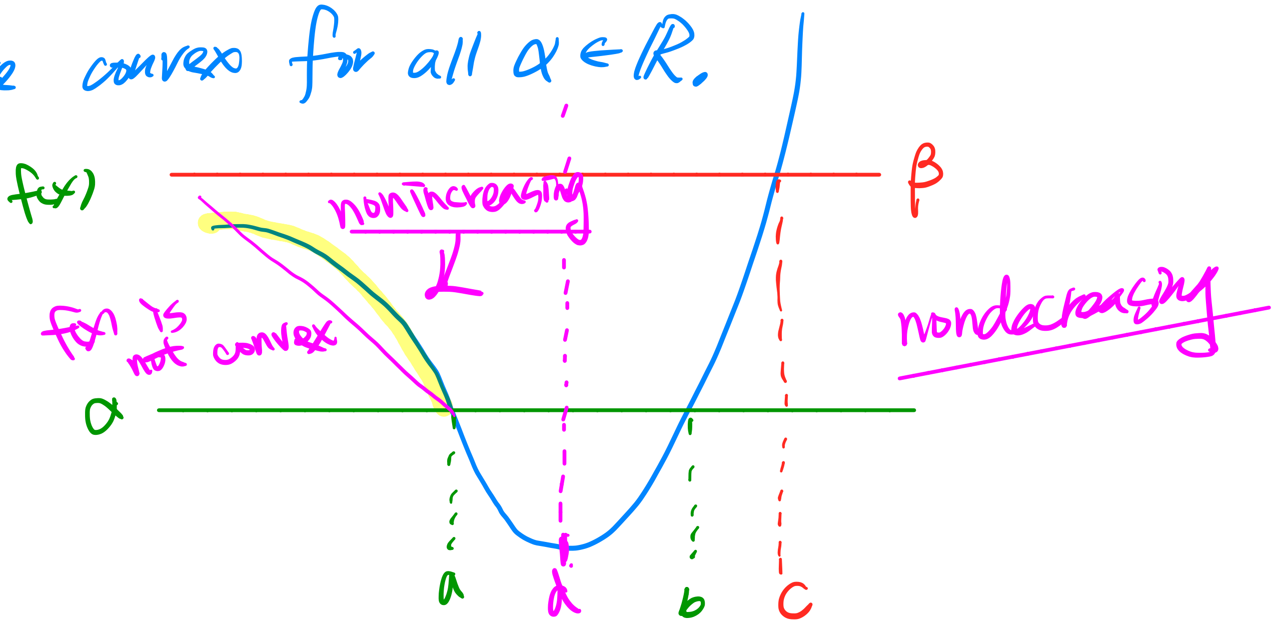
$$0 = \nabla \left(\qquad \right) = y - \underline{\underline{Q}} x \Rightarrow x = Q^{-1} y.$$

$Q > 0$

$$f^*(y) = \frac{1}{2} y^T Q y \quad \#.$$

Quasiconvex functions

▷ $f: \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if $\operatorname{dom} f$ is convex and the sublevel sets $S_\alpha = \{x \in \operatorname{dom} f : f(x) \le \alpha\}$ are convex for all $\alpha \in \mathbb{R}$.



$S_\alpha = [a, b]$. $S_\beta = (-\infty, c]$.

▷ f is quasiconcave if $-f$ is quasiconvex.
f is quasilinear if f is both quasiconvex quasiconcave.

▷ A continuous function $f: \mathbb{R} \to \mathbb{R}$ is quasiconvex if and only if at least one of the following conditions hold:

- f is nondecreasing
- f is nonincreasing
- there is a point $d \in \operatorname{dom} f$ such that for $t \ge d$, $t \in \operatorname{dom} f$, f is nonincreasing; $t \le d$ nondecreasing.