HW4 Solutions

November 12, 2019

1. First, let $y \in \mathcal{N}(A)$, $\forall z = A^T x \in \mathcal{R}(A^T)$.

$$\begin{aligned} \langle y, z \rangle &= \left\langle y, A^T x \right\rangle \\ &= \langle Ay, x \rangle \\ &= \langle 0, x \rangle \\ &= 0. \end{aligned}$$

   $\Rightarrow y \in \mathcal{R}(A^T)^\perp$.

   $\therefore \mathcal{N}(A) \subseteq \mathcal{R}(A^T)^\perp$.

   Second, let $y \in \mathcal{R}(A^T)^\perp$.

   Thus

$$\begin{aligned} &\left\langle y, A^T x \right\rangle = 0, \quad \forall x \\ \Rightarrow\ &\langle Ay, x \rangle = 0, \quad \forall x \\ \Rightarrow\ &Ay = 0. \end{aligned}$$

   $\Rightarrow y \in \mathcal{N}(A)$.

   $\therefore \mathcal{R}(A^T)^\perp \subseteq \mathcal{N}(A)$.

   Combining two steps, $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$.

2. We verify that x^* satisfies the optimality condition. The gradient of the objective function at x^* is

$$\nabla f_0(x^*) = (-1, 0, 2).$$

   Therefore the optimality condition is that

$$\nabla f_0(x^*)^T (y - x) = -1(y_1 - 1) + 2(y_2 + 1) \geq 0$$

   for all y satisfying $-1 \leq y_i \leq 1$, which is clearly true.

3. (a) Suppose that x is feasible in the original problem. Define $t = 1/(c^T x + d)$, $y = x/(c^T x + d)$. Then $t > 0$ and it is easily verified that t, y are feasible in the transformed problem, with the objective value $g_0(y,t) = f_0(x)/(c^T x + d)$.

   Conversely, suppose that y, t are feasible for the transformed problem. We must have $t > 0$, by definition of the domain of the perspective function. Let $x = y/t$. We have $x \in \mathbf{dom} f_i$ for $i = 0, ..., m$. x is feasible in the original problem, because

$$f_i(x) = g_i(y,t)/t \le 0, \quad i = 1, ..., m, \qquad Ax = A(y/t) = b.$$

   From the last equality, $c^T x + d = (c^T y + dt)/t = 1/t$, and hence,

$$t = 1/(c^T x + d), \qquad f_0(x)/(c^T x + d) = t f_0(x) = g_0(y,t).$$

   Therefore x is feasible in the original problem, with objective value $g_0(y,t)$.

   In conclusion, from any feasible point of the problem we can derive a feasible point of the other problem, with the same objective value.

   (b) We must prove that the objective function, $f_0(x)/(c^T x + d)$ is convex function.

   Suppose x_1, x_2 are feasible in the problem, and $0 \le \theta \le 0$.

$$\begin{aligned}
\frac{f_0(\theta x_1 + (1-\theta)x_2)}{c^T(\theta x_1 + (1-\theta)x_2) + d} &\le \frac{\theta f_0(x_1) + (1-\theta) f_0(x_2)}{c^T(\theta x_1 + (1-\theta)x_2) + d} \qquad (f_0(x) \textit{ is convex}) \\
&= \frac{\theta f_0(x_1) + (1-\theta) f_0(x_2)}{\theta(c^T x_1 + d) + (1-\theta)(c^T x_2 + d)} \\
&= \mu \frac{f_0(x_1)}{(c^T x_1 + d)} + (1-\mu) \frac{f_0(x_2)}{(c^T x_2 + d)},
\end{aligned}$$

   where $\mu = \frac{\theta(c^T x_1 + d)}{\theta(c^T x_1 + d) + (1-\theta)(c^T x_2 + d)}$, and $0 \le u \le 1$.

   Because $f_i(x)$ are convex functions, $Ax - b$ is affine, this is a convex optimization problem.

4. This can be formulated as the LP

$$\begin{aligned}
&\textit{minimize} \quad C = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
&\textit{subject to} \quad b_i - \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} x_{ji} = 0, \quad i = 1, ..., n \\
&\qquad \sum_{i=1}^{n} b_i = 1 \\
&\qquad l_{ij} \le x_{ij} \le u_{ij}.
\end{aligned}$$

5. We make a change of variables

$$y = A^{1/2}(x - x_c), \qquad x = A^{-1/2} y + x_c,$$

and consider the problem

$$\begin{aligned} &\textit{minimize} \quad c^T A^{-1/2} y + c^T x_c \\ &\textit{subject to} \quad y^T y \le 1. \end{aligned}$$

The objective function that we minimize is a linear function over the unit ball with

$$-\nabla f(y) = -(A^{-1/2} c).$$

Then the solution is

$$y^* = -(1/||A^{-1/2}c||_2)A^{-1/2}c, \qquad x^* = x_c - (1/||A^{-1/2}c||_2)A^{-1}c.$$