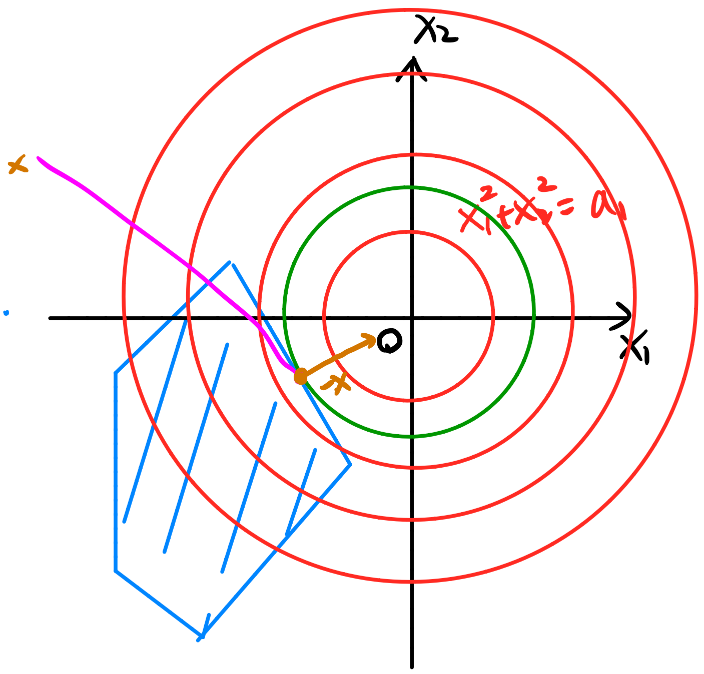
# Quadratic Programming (QP)

minimize $\boxed{\frac{1}{2}x^T P x} + q^T x + r$ , $P \in S_+^n$.

s.t. $\begin{cases} Gx \preceq h \\ Ax = b \end{cases}$ a polyhedron.

- $\nabla f_0(x) = Px + q$

$-\nabla f_0(x)$




Ex. $n=2$. $P = 2\mathbb{1}$, $q=0$.

minimize $x^T x = x_1^2 + x_2^2$

s.t. $Ax=b$, $Gx \preceq h$.

$-\nabla f_0(x) = -2x$.

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# Quadratically-constrained quadratic program (QCQP)

minimize $\frac{1}{2}x^T P x + q^T x + r$.

s.t. $\boxed{\frac{1}{2}x^T P_i x} + q_i^T x + r_i \leq 0$. $i=1,\dots,m$.

$Ax = b$.

where $P, P_i \in S_+^n$. ellipsoids if $P_i \succ 0$.

Example : Least - squares.

minimize $\| Ax - b \|^2 = x^T A^T A x - 2 b^T A x + b^T b$

differentiable.

- unconstrained QP.

- analytic solution.

$$0 = \nabla f_0(x) = 2A^T A x - 2A^T b$$

$$x^* = (A^T A)^{-1} A^T b.$$ (See A.5.4).

# Second-order cone programming (SOCP).

The second-order cone

$$C_{se} = \{ (x,t) \in \mathbb{R}^{n+1} : \|x\|_2 \le t \}$$

$$= \{ (x,t) \in \mathbb{R}^{n+1} : (x \ \ t) \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \le 0, \ t \ge 0 \}$$

is a convex set.

$$\begin{aligned} \text{minimize} \quad & f^T x \\ \text{s.t.} \quad & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \dots, m \\ & Fx = g. \end{aligned}$$

$x \in \mathbb{R}^n$. $A_i \in \mathbb{R}^{m \times m}$, $b_i \in \mathbb{R}^m$. $C_i \in \mathbb{R}^n$. $d_i \in \mathbb{R}$.

$$\begin{cases} A_i = 0 \Rightarrow LP \\ c_i = 0 \Rightarrow QCQP. \end{cases}$$

Example: (Robust LP).

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \le b_i, \quad i = 1, \dots, m. \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & \bar{a}_i^T x + \|P_i x\|_2 \le b_i \\ & i = 1, \dots, m. \end{aligned}}$$

where $a_i$, $b_i$, $c$ are uncertain.

$a_i \in \mathcal{E}_i = \{ \bar{a}_i + P_i u : \|u\|_2 \le 1 \}$. $\bar{a}_i \in \mathbb{R}^n$.

$P_i \in S_{++}^n$

$$\sup \{ a_i^T x : a_i \in \mathcal{E}_i \} \le b_i$$

$$\| $$

$$\bar{a}_i^T x + \sup \{ u^T P_i x : \|u\|_2 \le 1 \}$$

$$\| $$

$$\bar{a}_i^T x + \|P_i x\|_2$$

$$\sup \{ x^T a \} = \frac{\|a\|_2^2}{\|a\|_2}$$

$$x = \frac{a}{\|a\|_2}$$

Ex. LP with random constraints.

Assume that $\underbrace{a_i}_{\text{vector} \in \mathbb{R}^n} \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$,

$\bar{a}_i \in \mathbb{R}^n$, $\Sigma_i$: convariance.

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & \Pr\{a_i^T x \le b_i\} \ge \eta_i, \quad i=1,\dots,m. \end{aligned}$$

where $\eta_i \ge 0.5$.

— $a_i^T x \sim \mathcal{N}(\bar{a}_i^T x, \ x^T \Sigma_i x)$

$\mu = \mathbb{E}\{a_i^T x\}$. $\quad \mathbb{E}\{(a_i^T x - \mu)(a_i^T x - \mu)^T\}$.

Hence $\Pr\{a_i^T x \le b_i\} = \Phi\left(\dfrac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2}\right)$,

where $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-t^2/2}\, dt$.

$\Phi\left( \qquad \right) \ge \eta_i \Rightarrow \bar{a}_i^T x + \Phi^{-1}(\eta_i) \|\Sigma_i^{1/2} x\|_2 \le b_i$

— SOCP can be formulated as SDP