Ch. 5. duality

An optimization problem in standard form

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \; i = 1, \dots, m \\ & h_j(x) = 0, \; j = 1, \dots, q \end{aligned} \qquad \underline{\underline{(1)}}$$

- $x \in \mathbb{R}^n$ is the optimization valuable.
- $D = \left(\bigcap_{i=0}^{m} \operatorname{dom} f_i \right) \cap \left(\bigcap_{j=1}^{q} \operatorname{dom} h_j \right)$

△ Lagragian $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}$

domain L : $D \times \mathbb{R}^m \times \mathbb{R}^q$

$$L(x, \lambda, \nu) = f_0(x) + \underbrace{\sum_{i=1}^{m} \lambda_i f_i(x)}_{\text{inequality}} + \underbrace{\sum_{j=1}^{q} \nu_j h_j(x)}_{\text{equality}}.$$

- $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \in \mathbb{R}^m$, $\nu = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_q \end{pmatrix} \in \mathbb{R}^q$

are called dual variables or Lagrange multiplier vectors associated with (1).

Lagrange dual function: $g: \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}$.

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$$
$$= \inf_{x \in D} \left\{ f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{q} \nu_j h_j(x) \right\}.$$

— g is concave in λ and ν since it is the pointwise infimum of a family of affine functions of (λ, ν).

for a fixed x. $f_0(x) + \sum_i \lambda_i f_i(x) + \sum_j \nu_j h_j(x)$ is affine in λ & ν.

— Let p^* denote the optimal value of (1).

Lower bound:
For $\lambda \geq 0$, and any ν, $g(\lambda, \nu) \leq p^*$.

Proof: If $\tilde{x}$ is feasible. $\lambda \geq 0$, then

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})$$

$$L(\tilde{x}, \lambda, \nu) = \underbrace{f_0(\tilde{x}) + \sum \underbrace{\lambda_i}_{\geq 0} \underbrace{f_i(\tilde{x})}_{\leq 0} + \sum \nu_j \underbrace{h_j(\tilde{x})}_{= 0}}_{\leq 0}$$

Then minimizing over feasible $\tilde{x}$, we get $g(\lambda, \nu) \leq p^*$.

Ex. minimize x^Tx

s.t. $Ax=b$. $A \in \mathbb{R}^{q\times n}$. $b \in \mathbb{R}^q$.

equality constraint $\Rightarrow \nu$

— The Lagrangian is

$$L(x,\nu) = x^Tx + \nu^T(Ax-b)$$

with domain $\mathbb{R}^n \times \mathbb{R}^q$.

— To minimize L over x, set the gradient equal to zero:

$$0 = \nabla_x L(x,\nu) = 2x + A^T\nu \Rightarrow x = -\frac{1}{2}A^T\nu$$

So the dual function is

$$g(\nu) = L\left(-\frac{1}{2}A^T\nu, \nu\right) = -\frac{1}{4}\nu^TAA^T\nu - b^T\nu$$

which is a concave function of ν.

— The lower bound property:

$$-\frac{1}{4}\nu^TAA^T\nu - b^T\nu \le p^* \quad \forall \nu.$$

Ex. Standard form LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \Rightarrow Ax - b = 0. \\ & x \geq 0 \Rightarrow -x \leq 0. \end{aligned}$$

- The Lagrangian is

$$L(x, \lambda, \nu) = c^T x + \nu^T (Ax - b) + \lambda^T (-x)$$

$$= -b^T \nu + \boxed{(c + A^T \nu - \lambda)^T x}.$$

take inf

this is a linear function of x.

- The dual function is

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$$

$$= \begin{cases} -b^T \nu, & \text{if } c + A^T \nu - \lambda = 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

$$g(\lambda, \nu) = -b^T \nu \quad \text{if } c + A^T \nu - \lambda = 0.$$

$$\Updownarrow$$

$$c + A^T \nu = \lambda \geq 0$$

$$-b^T \nu \leq p^* \text{ if } c + A^T \nu \geq 0.$$

lower bound property.