minimize $f_0(x)$ (Affine equality constraint)

s.t. $Ax = b$. where $A \in \mathbb{R}^{m\times r}$. $b \in \mathbb{R}^r$.

$\boxed{\begin{array}{l} \triangleright\ x \text{ is optimal if and only if } x \in \mathbf{dom} f_0 \\ Ax = b \text{, and there exists } \nu \in \mathbb{R}^r \text{ s.t.} \\ \nabla f_0(x) + A^T \nu = 0. \quad \nabla f_0(x) = -A^T\nu \end{array}}$

$\in \mathcal{R}(A^T)$

proof. $x$ is optimal $\Leftrightarrow$ $\forall y$ s.t. $Ay = b$

$\nabla f_0(x)^T (y - x) \geq 0.$

If $x$ is feasible, a feasible $y$ is of the form $y = x + \nu$ for $\nu \in \mathcal{N}(A)$.

$(Ay = Ax + A\nu = Ax = b.)$

The optimality condition becomes

$\nabla f_0(x)^T \nu \geq 0 \quad \forall \nu \in \mathcal{N}(A)$

$-\nu \in \mathcal{N}(A) \Rightarrow \nabla f_0(x)^T(-\nu) \geq 0.$

$\nabla f_0(x)^T \nu = 0 \quad \forall \nu \in \mathcal{N}(A).$

$\| $

$\langle \nabla f_0(x), \nu \rangle.$

$\therefore \underline{\nabla f_0(x) \in \mathcal{N}(A)^\perp} = \{x : \langle x, \nu\rangle = 0\ \forall \nu \in \mathcal{N}(A)\}$,

Fact: $\mathcal{N}(A)^\perp = \mathcal{R}(A^T)$ ← verify it.

$\therefore \nabla f_0(x) \in \mathcal{R}(A^T)$. ※.

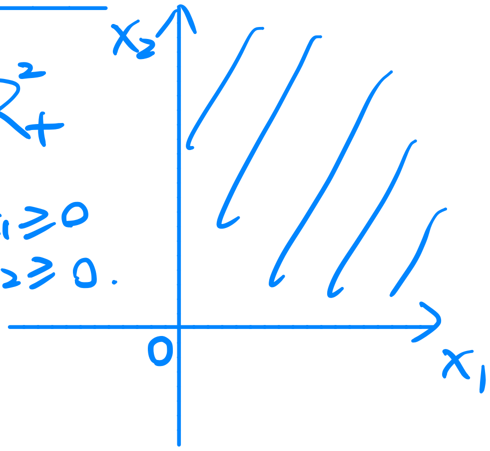
# Minimization over nonnegative orthant.

minimize $f_0(x)$

s.t. $x \succeq 0$

$\mathbb{R}^2_+$

$x_1 \geq 0$
$x_2 \geq 0$.




▷ $x$ is optimal if and only if

$x \in \mathrm{dom}\, f_0$, $x \succeq 0$, and

$$\begin{cases} (\nabla f_0(x))_i \geq 0 & \text{if } x_i = 0. \\ (\nabla f_0(x))_i = 0 & \text{if } x_i > 0. \end{cases}$$

(complementarity), $(\nabla f_0(x))_i \cdot x_i = 0$

Proof.

$x$ is optimal $\Leftrightarrow$ $x \succeq 0$. $\boxed{\nabla f_0(x)^T (y-x) \geq 0}$ $\forall y \succeq 0$ feasible

$\nabla f_0(x)^T y$ is linear function of $y$

and is unbounded below on $y \succeq 0$ unless

$\nabla f_0(x) \succeq 0$

( If $(\nabla f_0(x))_i < 0$, choose $y_i \to \infty$.
$(\nabla f_0(x))_i \cdot y_i \to -\infty$. ),

Choose $y = 0$. The optimality condition $y = 0$.

becomes $-\nabla f_0(x)^T x \geq 0$. $(x \succeq 0)$.

Since $x \succeq 0$. $\nabla f_0(x) \succeq 0$, $\nabla f_0(x)^T x = 0$

$$\nabla f_0(x)^T x = \sum_{i=1}^{n} \underbrace{(\nabla f_0(x))_i}_{(\geq 0)} \cdot \underbrace{x_i}_{(\geq 0)} \Rightarrow (\nabla f_0(x))_i x_i = 0$$

# Linear Program (LP)

$$\begin{aligned} \text{minimize} \quad & c^T x + d \\ \text{s.t.} \quad & \boxed{\begin{aligned} Gx &\le h \\ Ax &= b. \end{aligned}} \end{aligned}$$

a polyhedron.

- affine objective & constraint functions

- $c^T x$ is linear.
  level curves of the objective
  are hyperplanes orthogonal to $c$.

△ $f_0(x)$ is decreasing along the direction $-\nabla f_0(x)$

$f_0(x) = c^T x$

$-\nabla f_0(x) = -c$

$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

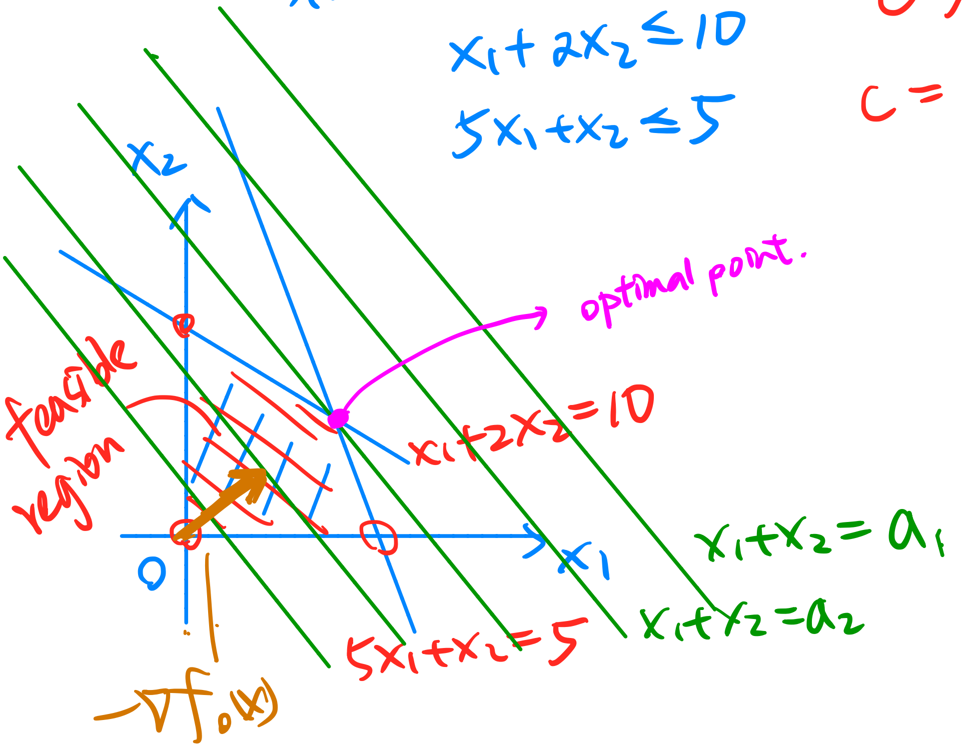
maximize $x_1 + x_2$

Ex. minimize $-x_1 - x_2 = -(1 \;\; 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

s.t. $x \ge 0$ $\quad = c^T x$.

$x_1 + 2x_2 \le 10$

$5x_1 + x_2 \le 5$ $\quad c = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.



Example. Diet Problem

- choose quantities $x_1, \ldots, x_n$ of $n$ foods.
- One unit of food $j$ costs $c_j$, contains amount $a_{ij}$ of nutrient $i$.
- Healthy diet requires nutrient $i$ in quantity at least $b_i$.
- To find the cheapest healthy diet.

$$\text{minimize} \quad \sum_{j=1}^{n} c_j \cdot x_j = c^T x$$

$$\text{s.t.} \quad \underline{\sum_j a_{ij} x_j \geq b_i} \quad \text{for } i = 1, \ldots, m$$

$$Ax \geq b.$$

Ex. Piecewise-linear minimization

$$\text{minimize} \quad f_0(x) = \max_{i=1,\ldots,m} \left( a_i^T x + b_i \right).$$

An equivalent LP with variable $t$ and $x$:

$$\text{minimize} \quad t$$

$$\text{s.t.} \quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m.$$

Linear-fractional Programming

minimize $f_0(x) = \dfrac{c^Tx+d}{e^Tx+f}$ ← (quasiconvex).

s.t. $Gx \le h$

$Ax = b.$ (3)

$e^Tx+f > 0.$

Equivalent LP with variables $y \in \mathbb{R}^n$, $z \in \mathbb{R}$.

minimize $c^Ty + dz$

s.t. $Gy - hz \le 0$ (4)

$Ay - bz = 0$

$e^Ty + fz = 1. \to e^T\dfrac{y}{z} + f = \dfrac{1}{z}$

proof: ① If $x$ is feasible in (3).

$\left( y = \dfrac{x}{e^Tx+f},\ z = \dfrac{1}{e^Tx+f} \right)$ is feasible

in (4) with the same objective value

$c^Ty + dz = f_0(x).$

$\Rightarrow$ opt. in (3) $\ge$ opt. in (4).

② If $(y, z)$ feasible in (4)

- $z \ne 0$, $x = y/z$ feasible in (3)

- $z = 0$. Suppose $x_0$ feasible in (3)

Let $x = x_0 + ty$ for $t \ge 0$. $x$ is feasible in (3)

$(Ay = 0.\ Gy \le 0.\ \underline{e^Ty = 1})$,

$\lim_{t\to\infty} f_0(x+ty)$

$= \lim_{t\to\infty} \dfrac{c^Tx + tc^Ty + d}{e^Tx + te^Ty + f}$

$= c^Ty + dz$

$\quad \underset{0}{\overset{\|}{}}$

$(z=0).$

# Generalized inequality constraints

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \preceq_{K_i} 0, \quad i = 1, \dots, m. \\ & Ax = b. \end{aligned}$$

where $f_0(x): \mathbb{R}^n \to \mathbb{R}$ is convex. .

$K_i \subseteq \mathbb{R}^{k_i}$ are proper cones

and $f_i: \mathbb{R}^n \to \mathbb{R}^{k_i}$ are $K_i$-convex.

– $\mathbb{R}^n_+$ . $S^n_+$

△ Many of the results for ordinary convex optimization problems still hold here.

- The feasible set, any sublevel set, and the optimal set are convex.

- Any point that is locally optimal for the problem is globally optimal.

- The optimality conditions for differentiable $f_0$ hold without any change.

Ex. Conic-form Problem.

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & Fx + g \preceq_K 0 \\ & Ax = b. \end{aligned}$$

When $K = \mathbb{R}_+^n$, it reduces to an LP.

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## Semidefinite Program (SDP)

When $K = S_+^m$, the associated conic form problem is called an SDP.

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b. \end{aligned}$$

linear matrix inequality.

where $F_i, G \in S^m$, and $A \in \mathbb{R}^{r \times n}$, $b \in \mathbb{R}^r$.

Ex. Let $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$,

where $A_i \in S^k$.

$$\text{minimize} \quad \lambda_{max}(A(x))$$

- An equivalent SDP.

$$\begin{aligned} &\text{minimize} && t \\ &\text{s.t.} && A(x) \preceq t \mathbb{1}. \end{aligned}$$

$$A(x) \preceq \lambda_{max}(A)\, \mathbb{1}$$

$$\overset{\parallel}{U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^T} \preceq U \begin{pmatrix} \lambda_{max} & & \\ & \ddots & \\ & & \lambda_{max} \end{pmatrix} U^T = \lambda_{max} \mathbb{1}.$$

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| Standard form LP | Standard form SDP |
|---|---|
| minimize $c^T x$ | minimize $\operatorname{tr} CX$ |
| s.t. $Ax = b$ | s.t. $\operatorname{tr} A_i X = b_i$ |
| $x \geq 0$. | $i = 1, \ldots, m$. |
| | $X \succeq 0$. |

Quadratic Programming (QP)

Quadratically-constrained QP. (QCQP).

Second-order cone Programming (SOCP).