ECM5901 Midterm Exam (sampple) ID: Name:

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1. (a) Let $A \in \mathbb{R}^{p \times n}$. Suppose $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. Show that
   i. $\dfrac{\partial Ax}{\partial x} = A$; (2%)
   ii. $\dfrac{\partial y^T Ax}{\partial y} = x^T A^T$. (2%)

   (b) Consider the following primal problem with variable $x \in \mathbb{R}^n$:

$$\begin{array}{ll} \text{minimize} & x^T P x \\ \text{subject to} & Ax \leq b \end{array}$$

   where $P > 0$ and $A \in \mathbb{R}^{p \times n}$. What is the corresponding dual problem? (8%)

   (c) Show that $f(A) = x^T Ax$ is convex on $A \in \mathbb{S}^n_+$. (6%)

2. A matrix $X \in \mathbb{S}^n$ is called copositive if $z^T X z \geq 0$ for all $z \geq 0$.

   (a) Verify that the set of copositive matrices is a proper cone. (A proper cone $K \subset \mathbb{R}^n$ has the following properties: (1) K is convex. (2) K is closed. (3) K is solid; K has nonempty interior. (4) K is pointed; $x, -x \in K \Rightarrow x = 0$.) (8%)

   (b) Find its dual cone. (4%)

3. (a) Show that $\{X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2_+\}$ is a convex cone. (2%)

   (b) Is $f(x_1, x_2) = x_1 x_2$ convex or concave? (5%)

   (c) Is $f(x_1, x_2) = x_1 x_2$ quasiconvex? (7%) (Do not use a plot here.)

4. Consider the parabola $\mathcal{P} = \{(x, y) : x^2 - 4y = 0\}$ on $\mathbb{R}^2$. We would like to find the shortest distance from a point $(0, b)$ on the y-axis to $\mathcal{P}$.

   (a) Formulate the question as an unconstrained optimization problem. (Specify the variables and the objective function.) (2%)

   (b) Reformulate the optimization problem with one equality constraint. (Specify the variables, the objective function, and the constraint.) (2%)

   (c) What is the Lagrangian function in (b)? (2%)

   (d) What is the shortest distance from $(0, b)$ to $\mathcal{P}$? (10%)

5. (a) If $f : \mathbb{R}^n \to \mathbb{R}$, the perspective of f is $g : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $g(x, t) = tf(x/t)$, with domain $\operatorname{dom} g = \{(x, t) : x/t \in \operatorname{dom} f, t > 0\}$. Show that if f is convex, then g is convex. (3%)

   (b) Show that $t \log t$ is convex. (Hint: use (a) with an appropriate f.) (5%)

   (c) (Capacity of a communication channel.) We consider a communication channel, with input $X \in \{1, \ldots, n\}$, and output $Y \in \{1, \ldots, m\}$. The relation between the input and the output is given statistically:

$$p_{ij} = prob(Y = i | X = j), \quad i = 1, \ldots, m; \quad j = 1, \ldots, n.$$

   The matrix $P \in \mathbb{R}^{m \times n}$ is called the channel transition matrix. Assume that X has a probability distribution denoted $x \in \mathbb{R}^n$, i.e.,

$$x_j = prob(X = j), \quad j = 1, \ldots, n$$

   and $x \geq 0$, $\sum_j x_j = 1$. Y and its distribution $y \in \mathbb{R}^m$ are defined similarly, and $y = Px$. The mutual information between X and Y is given by

$$I(X; Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j p_{ij} \log \frac{p_{ij}}{\sum_{k=1}^{n} x_k p_{ik}}.$$

The Shannon capacity of the channel is

$$C = \sup_x I(X;Y).$$

Show that this channel capacity can be formulated as a **convex** optimization problem with variables x and y. (8%)

6. Find the Lagrange dual problem of the following convex optimization problem with variable $x \in \mathbb{R}^n$:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{n} x_k \log(x_k/y_k) \\ \text{subject to} & Ax = b, \\ & \sum_{k=1}^{n} x_k = 1 \end{array}$$

where $y \in \mathbb{R}^n_{++}$, $A = [a_1 \ a_2 \ \cdots \ a_n] \in \mathbb{R}^{m\times n}$, $a_k \mathbb{R}^m$, and $b \in \mathbb{R}^m$.

(a) By introducing the dual variables $z \in \mathbb{R}^m$ and $u \in \mathbb{R}$, show that the dual problem is

$$\text{maximize} \qquad b^T z + u - \sum_{k=1}^{n} y_k e^{a_k^T z + u - 1}$$

(up to minus signs in z or u). (5%)

(b) Show that the problem can be further reduced to

$$\text{maximize} \qquad b^T z - \log \sum_{k=1}^{n} y_k e^{a_k^T z}.$$

(5%)

7. Consider the following linear program:

$$\begin{array}{ll} \text{minimize} & 5x_1 + 16x_2 + 3x_3 \\ \text{subject to} & \begin{bmatrix} -1 & -6 & 1 \\ -1 & -2 & 7 \\ 0 & 3 & -10 \\ 1 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} -6 \\ 4 \\ -7 \\ 7 \end{bmatrix}. \end{array}$$

(a) What is the dual program (with dual variable λ)? (3%)

(b) An optimal solution to the primal problem is $x^* = (1,1,1)^T$. Use the property of **complementary slackness** to find a certificate that x^* is optimal. (7%)

(c) Show that x^* is the **unique** optimal solution. (4%)