ECM5901 2019, Assignment #2 due October 15

1. Suppose $C \subset \mathbb{R}^m$ is convex and $f : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function. Show that the inverse image of the convex set C
$$f^{-1}(C) = \{x | f(x) \in C\}$$
is convex. (10%)

2. The second-order cone is the norm cone for the Euclidean norm, i.e.,
$$\begin{aligned} C =& \{(x,t) \in \mathbb{R}^{n+1} : \|x\|_2 \le t\} \\ =& \left\{ \begin{bmatrix} x \\ t \end{bmatrix}^T : \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le 0,\ t \ge 0 \right\}. \end{aligned}$$
Show that C is convex. (10%)

3. The distance between two sets C and D is defined as
$$\inf\{\|u - v\|_2 : u \in C, v \in D\}.$$
What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n : a^T x = b_1\}$ and $\{x \in \mathbb{R}^n : a^T x = b_2\}$ for $a \in \mathbb{R}^n, b_1, b_2 \in \mathbb{R}$? (10%)

4. (Hyperbolic sets.)
   (a) Show that the hyperbolic set $\{x \in \mathbb{R}^2_+ : x_1 x_2 \ge 1\}$ is convex. (5%)
   (b) As a generalization, show that $\{x \in \mathbb{R}^n_+ : \prod_{i=1}^n x_i \ge 1\}$ is convex. (5%)

   (Hint: If $a, b \ge 0$ and $0 \le \theta \le 1$, then $a^\theta b^{1-\theta} \le \theta a + (1-\theta) b$; see §3.1.9.)

5. Show that if S_1 and S_2 are convex sets in $\mathbb{R}^{m+n}$, then so is their partial sum $S = \{(x, y_1 + y_2) : x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}$. (10%)

6. Linear-fractional functions and convex sets. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be the linear-fractional function $f(x) = (Ax + b)/(c^T x + d)$, $\operatorname{dom} f = \{x : c^T x + d > 0\}$. In this problem we study the inverse image of a convex set C under f, i.e.,
$$f^{-1}(C) = \{x \in \operatorname{dom} f : f(x) \in C\}.$$
For each of the following sets $C \subset \mathbb{R}^n$, give a simple description of $f^{-1}(C)$.
   (a) The halfspace $C = \{y : g^T y \le h\}$ (with $g \ne 0 \in \mathbb{R}^n$ and $h \in \mathbb{R}$). (5%)
   (b) The ellipsoid $C = \{y : y^T P^{-1} y \le 1\}$ (where $P > 0$). (5%)

7. Give an example of two closed convex sets that are disjoint but cannot be *strictly* separated. (You have to verify that the two sets you provide are closed and convex.) (10%)

8. Copositive matrices. A matrix $X \in S^n$ is called copositive if $z^T X z \ge 0$ for all $z \ge 0 \in \mathbb{R}^n$. Verify that the set of copositive matrices is a proper cone. Find its dual cone. (10%)

9. Properties of dual cones. Let K^* be the dual cone of a convex cone K, as defined in (2.19) of the textbook by Boyd and Vandenberghe. Prove the following. (25%)
   (a) K^* is indeed a convex cone.
   (b) Two sets $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.
   (c) K^* is closed.
   (d) If K has nonempty interior, then K^* is pointed.
   (e) K^{**} is the closure of K. (Hence if K is closed, $K^{**} = K$.)