Quad-Rotor Dynamical Model with Euler-Lagrange Approach

Satoshi Nakano

May 7th,2013

FL13-3-2

Euler-Lagrange equation

Euler Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]-\frac{\partial L}{\partial q}=\mathbf{Q}'$$

L :lagrangian
$\boldsymbol{q}$:generalized coordinate
$\mathbf{Q}'$:generalized force

$$M(\boldsymbol{\eta})\ddot{\boldsymbol{\eta}}+C(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}}=\mathbf{Q}'$$

Lagrangian

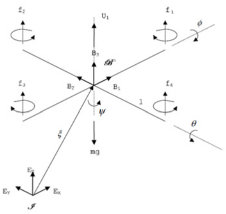
$$L(\boldsymbol{q},\dot{\boldsymbol{q}})=T-U$$

T :kinetic energy
U :potential energy

Lagrangian

Generalized coordinate

$$\boldsymbol{q}=(x,y,z,\phi,\theta,\psi)^T\in\mathbb{R}^6$$
$$\boldsymbol{\xi}=(x,y,z)^T\in\mathbb{R}^3$$
$$\boldsymbol{\eta}=(\phi,\theta,\psi)^T\in\mathbb{R}^3$$



Translational kinetic energy

$$T_{trans}=\frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}}$$

m:the mass of the quad-rotor

Rotational kinetic energy

$$T_{rot}=\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{I}\boldsymbol{\Omega}$$

$\boldsymbol{\Omega}$:angular velocity $I=\begin{bmatrix}I_{xx}&0&0\\0&I_{yy}&0\\0&0&I_{zz}\end{bmatrix}$:inertia matrix

Potential energy

$$U=mgz$$

g:acceleration of gravity z:rotorcraft altitude

$$L(\boldsymbol{q},\dot{\boldsymbol{q}})=\frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}}+\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{I}\boldsymbol{\Omega}-mgz$$

https://blog.csdn.net/a735148617/article/details/116740453

Angular velocity

$$\boldsymbol{\Omega}=\boldsymbol{W}_\eta\dot{\boldsymbol{\eta}}\qquad \boldsymbol{W}_\eta=\begin{bmatrix}1&0&-\sin\theta\\0&\cos\phi&\cos\theta\sin\phi\\0&-\sin\phi&\cos\theta\cos\phi\end{bmatrix}\qquad \mathbb{J}(\boldsymbol{\eta})=\boldsymbol{W}_\eta^T\boldsymbol{I}\boldsymbol{W}_\eta$$

$$T_{rot}=\frac{1}{2}\dot{\boldsymbol{\eta}}^T\mathbb{J}\dot{\boldsymbol{\eta}}$$

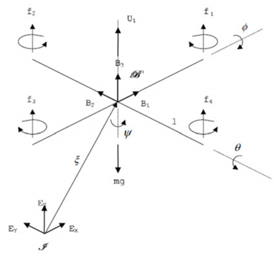
Generalized force

$$\mathbf{Q}'=\begin{bmatrix}\boldsymbol{F}_\xi\\\boldsymbol{\tau}\end{bmatrix}$$

$$\boldsymbol{F}_\xi=\boldsymbol{R}\hat{F}\in\mathbb{R}^3\qquad \hat{F}=\begin{bmatrix}0\\0\\\sum_{i=1}^4 f_i\end{bmatrix}$$

$$\boldsymbol{R}=\begin{bmatrix}C\psi C\theta & C\psi S\theta S\phi-S\psi C\phi & C\psi S\theta C\phi+S\psi S\phi\\ S\psi C\theta & S\psi S\theta S\phi+C\psi C\phi & S\psi S\theta C\phi-C\psi S\phi\\ -S\theta & C\theta S\phi & C\theta C\phi\end{bmatrix}$$

$$\boldsymbol{\tau}=\begin{bmatrix}\tau_\phi\\\tau_\theta\\\tau_\psi\end{bmatrix}\triangleq\begin{bmatrix}(f_3-f_1)l\\(f_2-f_4)l\\\sum_{i=1}^4\tau_{M_i}\end{bmatrix}$$



Euler-Lagrange equation

Translational lagrangian

$$L_{trans}=\frac{m}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}}-mgz$$

Translational equation

$$\frac{d}{dt}\left[\frac{\partial L_{trans}}{\partial\dot{\xi}}\right]-\frac{\partial L_{trans}}{\partial\xi}=\boldsymbol{F}_\xi$$

$$m\ddot{\boldsymbol{\xi}}+mg\boldsymbol{E}_z=\boldsymbol{F}_\xi$$

Rotational lagrangian

$$L_{rot}=\frac{1}{2}\dot{\boldsymbol{\eta}}^T\mathbb{J}\dot{\boldsymbol{\eta}}$$

Rotational equation

$$\frac{d}{dt}\left[\frac{\partial L_{rot}}{\partial\dot{\eta}}\right]-\frac{\partial L_{rot}}{\partial\eta}=\boldsymbol{\tau}$$

$$\frac{\partial L_{rot}}{\partial\dot{\eta}}=\frac{1}{2}\left(\mathbb{J}+\mathbb{J}^T\right)\dot{\boldsymbol{\eta}}=\mathbb{J}\dot{\boldsymbol{\eta}}$$

$$\mathbb{J}\ddot{\boldsymbol{\eta}}+\left(\dot{\mathbb{J}}-\frac{1}{2}\frac{\partial}{\partial\boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^T\mathbb{J}\right)\right)\dot{\boldsymbol{\eta}}=\boldsymbol{\tau}$$

$$C(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})=\dot{\mathbb{J}}-\frac{1}{2}\frac{\partial}{\partial\boldsymbol{\eta}}\left(\dot{\boldsymbol{\eta}}^T\mathbb{J}\right)$$

$$\mathbb{J}\ddot{\boldsymbol{\eta}}+C(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}}=\boldsymbol{\tau}$$

Rotational equation

$$\mathbb{J}=\begin{bmatrix}I_{xx}&0&-I_{xx}S\theta\\0&I_{yy}C^2\phi+I_{zz}S^2\phi&(I_{yy}-I_{zz})C\phi S\phi C\theta\\-I_{xx}S\theta&(I_{yy}-I_{zz})C\phi S\phi C\theta&I_{xx}S^2\theta+I_{yy}S^2\phi C^2\theta+I_{zz}C^2\phi C^2\theta\end{bmatrix}$$

$$C(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})=\begin{bmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\\c_{31}&c_{31}&c_{33}\end{bmatrix}$$

$$c_{11}=0$$
$$c_{12}=(I_{yy}-I_{zz})(\dot{\theta}C\phi S\phi+\dot{\psi}S^2\phi C\theta)+(I_{zz}-I_{yy})\dot{\psi}C^2\phi C\theta-I_{xx}\dot{\psi}C\theta$$
$$c_{13}=(I_{zz}-I_{yy})\dot{\psi}C\phi S\phi C^2\theta$$
$$c_{21}=(I_{zz}-I_{yy})(\dot{\theta}C\phi S\phi+\dot{\psi}S^2\phi C\theta)+(I_{yy}-I_{zz})\dot{\psi}C^2\phi C\theta+I_{xx}\dot{\psi}C\theta$$
$$c_{22}=(I_{zz}-I_{yy})\dot{\phi}C\phi S\phi$$
$$c_{23}=-I_{xx}\dot{\psi}S\theta C\theta+I_{yy}\dot{\psi}S^2\phi C\theta S\theta+I_{zz}\dot{\psi}C^2\phi S\theta C\theta$$
$$c_{31}=(I_{yy}-I_{zz})\dot{\psi}C\phi S\phi C^2\theta-I_{xx}\dot{\theta}C\theta$$
$$c_{32}=(I_{zz}-I_{yy})(\dot{\theta}C\phi S\phi S\theta+\dot{\phi}S^2\phi C\theta)+(I_{yy}-I_{zz})\dot{\phi}C^2\phi C\theta$$
$$+I_{xx}\dot{\psi}C\theta S\theta-I_{yy}\dot{\psi}S^2\phi C\theta S\theta-I_{zz}\dot{\psi}C^2\phi C\theta S\theta$$
$$c_{33}=(I_{yy}-I_{zz})\dot{\phi}C\phi S\phi C^2\theta-I_{yy}\dot{\theta}S^2\phi C\theta S\theta-I_{zz}\dot{\theta}C^2\phi C\theta S\theta+I_{xx}\dot{\theta}C\theta S\theta$$