Introduction to Mathematics

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CHAPTER 1

Schubert Calculus

Schubert calculus

In mathematics, Schubert calculus is a branch of algebraic geometry introduced in the nineteenth century by Hermann Schubert, in order to solve various counting problems of projective geometry.

Theorem 1.1: Pieri's formula

Let σ_λ be a special Schubert cycle and suppose σ_μ is any Schubert cycle with $\mu = \{\mu_1, \ldots, \mu_k\}$. Then we have

$$\sigma_\lambda \cdot \sigma_\mu = \sum \sigma_\nu$$

where the sum is over all ν such that $\mu_\ell \leqslant \nu_\ell \leqslant \mu_{\ell-1}$ and $\sum \nu_\ell - \mu_\ell = \lambda$.

Proof:

Simply invoke the duality theorem. □

Remark:

Theorem 1.1 is an import theorem[a]. Let's look at an example.

[a]theorem reference testing.

Example:

Click here for details.

Now let's look at a new theorem!

Theorem 1.2

For a fixed k, as $n \to \infty$, the following asymptotic holds:

$$\log \operatorname{edge} \ G(k,n) = kn \log \left(\frac{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \right) - \mathcal{O}(\log n)$$

A general, guiding theme of our work is to compare the number of complex solutions with the expected number of real solutions. With this regard, we obtain:

Corrollary 1.3

For fixed $k \geq 2$, we have $\operatorname{edge} G(k,n) = \deg G_{\mathbb{C}}(k,n)^{\frac{1}{2}\varepsilon_k + o(1)}$ for $n \to \infty$, where ε_k is monotonically decreasing with $\lim_{k\to\infty} = 1$.

Remark:

This means that for large n, the expected degree of the real Grassmannian exceeds the square root of the degree of the corresponding complex Grassmannian...

Lemma 1.4

Let $K \subseteq \mathbb{R}^d$ be a convex body containing the origin. Then the radial function $_K$ and the support function h_K of K have the same maximum on S^{d-1} . Moreover, a direction $u \in S^{d-1}$ is maximizing for r_K if and only u is maximizing for h_K.

Proof:

Let $\overline{r}$ denote the maximum of r_K. Then $K \subseteq B(0,\overline{r})$ and hence $h_K \leq h_{B(0,\overline{r})}$, which implies $\max h_K \leq \max h_{B(0,\overline{r})} = \overline{r}$. For the other direction let $u \in S^{d-1}$. Then ... The claim about the maximizing directions follows easily by tracing our argument. □

Now let's look at a proposition.

Proposition 1.5

The set $\operatorname{Sing}(S)$ is semialgebraic and $\dim \operatorname{Sing}(S) < \dim S$. In particular, $S \neq \operatorname{Sing}(S)$.

Remark:

In the sequel it will be convenient to say that generic points of a semialgebraic set S satisfy a certain property if this property is satisfied by all points except in a semialgebraic subset of positive codimension in S. (For example, in view of the previous proposition, generic points of a semialgebraic set S are regular points.)

CHAPTER 2

General Relativity

Now let's apply our math knowledge to physics!

Moving from math to physics involves the introduction of dynamical equations which relate matter and energy to the curvature of spacetime. In GR, the "equation of motion" for the metric is the famous **Einstein equation**:

Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.1}$$

Kruskal coordinates

We can demonstrate this by making a transformation to what are known as **Kruskal coordinates**, defined by

$$\begin{aligned} u &= \left(\frac{r}{2Gm} - 1\right)^{1/2} e^{r/4Gm} \cosh(t/4Gm) \\ v &= \left(\frac{r}{2Gm} - 1\right)^{1/2} e^{r/4Gm} \sinh(t/4Gm) \end{aligned} \tag{2.2}$$

Remark:

The useful thing about the Schwarzschild solution is that it describes both mundane things like the solar system, and more exotic objects like black holes. To get a feel for it, let's look at how particles move in a Schwarzschild geometry. It turns out that we can cast the problem of a particle moving in the plane $\frac{\theta=\pi}{2}$ as a one-dimensional problem for the radial coordinate $r = r(\tau)$.

2.1 Cosmology

Just as we were able to make great strides with the Schwarzschild metric on the assumption of sperical symmetry, we can make similar progress in cosmology by assuming that the Universe is homogeneous and isotropic. That is to say, we assume the existence of a "rest frame for the Universe," which defines a universal time coordinate, and singles out three-dimensional surfaces perpendicular to this time coordinate. (In the real Universe, this rest frame is the one in which galaxies are at rest and the microwave background is isotropic.) "Homogeneous" means that the curvature of any two points at a given time t is the same. "Isotropic" is trickier, but basically means that the universe looks the same in all directions. Thus, the surface of a cylinder is homogeneous (every point is the same) but not isotropic (looking along the long axis of the cylinder is a preferred direction); a cone is isotropic around its vertex, but not homogeneous.

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