考虑如下 model

$$\Delta = \Delta_0 + \Delta_1 \cos(\omega t) \quad \boxed{\text{Real}}$$

$$H(t) = \left(\begin{array}{cc|cc} \varepsilon_k + h & x+iy & 0 & \Delta(t) \\ x-iy & \varepsilon_k - h & -\Delta(t) & 0 \\ \hline 0 & -\Delta(t) & -(\varepsilon_k + h) & a(x-iy) \\ \Delta(t) & 0 & (x+iy) & -(\varepsilon_k - h) \end{array}\right) = H_0 + \Delta_1 \cos\omega t \, B$$

- IF $\Delta_1 = 0$ $H(t)$ is a time-independent Equation, thus

$$H(t) = H_0$$

The PH symmetry of this model is

$$\Sigma = \left(\begin{array}{c|c} 0 & I_{2\times2} \\ \hline I_{2\times2} & 0 \end{array}\right) K = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right) K$$

$$= \sigma_x K$$

Check

$$H_0 = \begin{pmatrix} H(k) & \bar{\Delta} \\ \bar{\Delta}^+ & -H^*(-k) \end{pmatrix}$$

Where $\bar{\Delta} = \left(\begin{array}{c|c} & \Delta \\ \hline -\Delta & \end{array}\right) \Rightarrow \boxed{\bar{\Delta}^T = -\bar{\Delta}}$

$\Sigma H(k) \Sigma^{-1}$ ↙ Well-know result in fact.

$$= \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right) \begin{pmatrix} H^*(k) & \bar{\Delta}^* \\ \bar{\Delta}^T & -H(-k) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\Delta}^* & H^*(k) \\ -H(-k) & \bar{\Delta}^T \end{pmatrix}$$

$$= \begin{pmatrix} -H(-k) & \bar{\Delta}^T \\ \bar{\Delta}^* & H^*(k) \end{pmatrix} = -\begin{pmatrix} H(-k) & \Delta \\ \Delta^+ & -H^*(k) \end{pmatrix} = -H(-k)$$

SO : $H_0(k) \Rightarrow$ PH operator is Σ

Note

we expect the whole model after mapping from

time-dependent $\underset{4\times N}{H(t)}$ $\rightarrow$ time-independant $H_{4N\times 4N}$

The PH Operator should be.

$$\Sigma_{4N\times 4N} = \begin{pmatrix} \Sigma & & & & & 0 \\ & \Sigma & & & & \\ & & \Sigma & & & \\ & & & \Sigma & & \\ & 0 & & & \ddots & \\ & & & & & \Sigma \end{pmatrix}$$

$$= \oplus \Sigma = I_{N\times N} \otimes \Sigma$$

<u>Reason</u> : We Can write $H_{4N\times 4N}$ In the following way

$H_{4N\times 4N}$ = diagonal terms + off-diagonal coupling terms.

where diagonal terms = eigenvalue of $(\underline{i\partial_t - H_0})$

So we write

$$H_{4N\times 4N} = H_{diag} + H_{off\text{-}diag}$$

$$\Sigma_{4N\times 4N} H_{4N\times 4N}(k) \Sigma^{-1}_{4N\times 4N} = -H_{4N\times 4N}(k)$$

Requires that

$$\boxed{\begin{aligned} \Sigma_{4N\times 4N} H_{diag}(k) \Sigma^{-1}_{4N\times 4N} &= -H_{diag}(-k) \\ \Sigma_{4N\times 4N} H_{off\text{-}diag}(k) \Sigma^{-1}_{4N\times 4N} &= -H_{off\text{-}diag}(-k) \end{aligned}}$$

we prove this point in ~~the~~ following by using the following model.

$$H(k,t) = \mathcal{H} = H_0 + \Delta_1 \cos(\omega t)\, B \qquad (3)$$

where
$$B = \left(\begin{array}{cc|cc} & 0 & & 1 \\ & & -1 & \\ \hline & -1 & & \\ 1 & & 0 & \end{array}\right)$$

obviously:

~~$\Sigma H_0 \Sigma^{-1}$~~

$$\boxed{\Sigma\, H(k,t)\, \Sigma^{-1} = -H(-k,t) \quad \text{for any } t}$$

~~That~~ is

$$\boxed{\begin{array}{l} \Sigma\, H_0(k)\, \Sigma^{-1} = -H_0(-k) \\ \Sigma\, B\, \Sigma^{-1} = -B \end{array}}$$

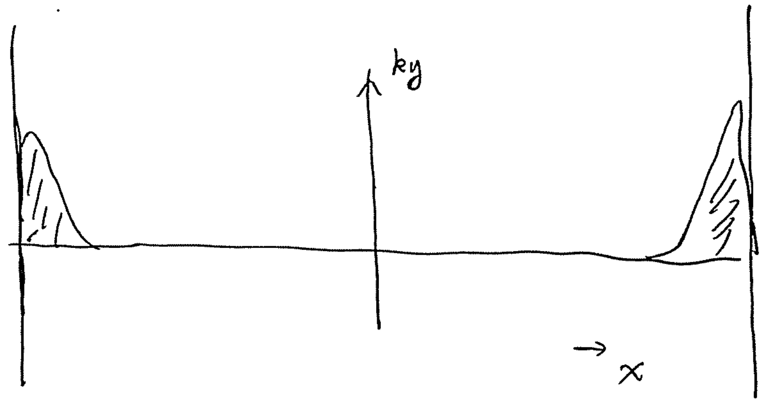
this ~~Equations~~ ensures ~~that~~ $H_{4N\times 4N}$ is always PH symmetric with ~~respect to~~

$$\boxed{\Sigma_{4N\times 4N} = I_{N\times N} \otimes \Sigma}$$

Lin, Can you check this PH operator in you numerical Calculation?

推广到 $H(t) = H_0 + \sum_i \Delta_i \cos(\omega_i t)\, B_i$ 也是一样的.

Hard wall boundary:



At $k_y = 0$, $\quad H(\vec{k}) \longrightarrow H(k_x) = H(-i\partial_x)$

is a one-dimensional model

For the state $E=0 \Rightarrow$ Majorana Fermions..

$$\boxed{\Sigma H = -H\Sigma}$$

The MFs at the same edge at the same edge have the same chirality defined as

$$\Sigma\psi = \eta\psi, \quad \text{where } \eta = \pm 1$$

$\boxed{\text{Why?}}$

For ~~~~ PH symmetry

$$\underbrace{H\psi = \varepsilon\psi}_{\boxed{\text{Particle}}} \rightarrow \underbrace{H\Sigma^{-1}\psi = -\varepsilon\Sigma^{-1}\psi}_{\boxed{\text{Hole}}}$$

~~Now~~

Now assume $\varepsilon = 0$ thus ψ & $\Sigma^{-1}\psi$ are degenerate

$\Rightarrow$ $\Phi_\pm = \psi \pm \Sigma^{-1}\psi$

Then we find that $\boxed{\Sigma\Phi_\pm = \pm\Phi_\pm}$

a lot of MFS Here, we need to show that the robustness of those edge modes requires that they have the same chirality.

For any potential V_d with PH symmetry $\Sigma V_d \Sigma^{-1} = -V_d$.

$$\langle \psi_i | V_d | \psi_j \rangle$$

$$= \eta_i \eta_j \langle \psi_i | \underbrace{\Sigma^{\dagger} V_d \Sigma^{-1}}_{-V_d} | \psi_j \rangle$$

$$= -\eta_i \eta_j \langle \psi_i | V_d | \psi_j \rangle$$

$$\boxed{\text{Thus when } \eta_i \eta_j = +1 \Rightarrow \langle \psi_i | V_d | \psi_j \rangle = 0}$$

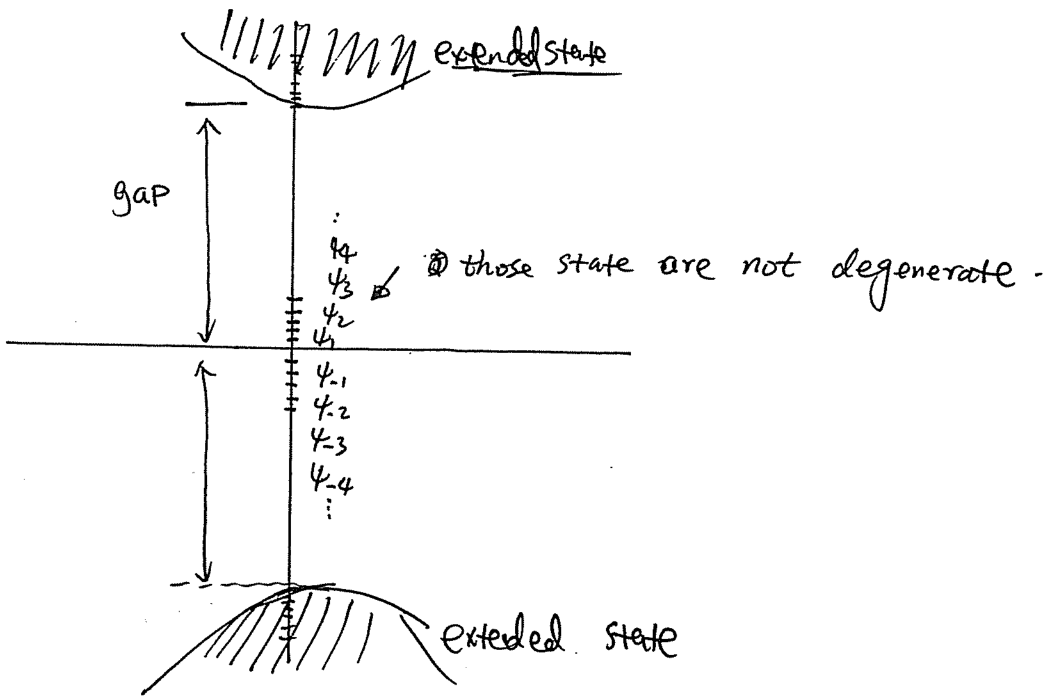
The above result show ~~that~~ the edge state should have the same chirality to be robust.

When $\eta_i \eta_j = -1$, the above result always holds for any V_d and $i \neq j$ $\Rightarrow$ we can choose some V_d to make $\langle \psi_i | V_d | \psi_j \rangle \neq 0$ $\Rightarrow$ can be gapped out.

How to define it in numerical Calculation?

$\boxed{A}$: In a finite chain, $\mathcal{L}$ is not required to be very large.
for example $\mathcal{L} = 100/k_F$.

Then at $k_y = 0$, we have the following spectrum



Now we have $\psi_{-1} = \Sigma^{-1}\psi_1 = \Sigma\psi_1$ (Note $\Sigma^{-1} = \Sigma$ since $\Sigma^2 = 1$)

Thus the edge state can be well-defined using

$$\begin{cases} \phi_{\pm 1} = \psi_1 \pm \Sigma\psi_1 \\ \phi_{\pm 2} = \psi_2 \pm \Sigma\psi_2 \\ \phi_{\pm 3} = \psi_3 \pm \Sigma\psi_3 \\ \vdots \end{cases} \Rightarrow \boxed{\Sigma\phi_\pm = \pm\phi_\pm}$$

Lin, you can check ~~that~~ those WFs $\phi_{\pm 1}, \phi_{\pm 2}, \cdots$ are localized at the ends.

$\boxed{\text{How } \cancel{\text{about}} \ \mathcal{L} = 200/k_F\ ?}$

In this case $\varepsilon_i \sim 0$ ($|\varepsilon_i| < 10^{-7}$)

of course, we can still do ~~that~~ ~~if~~ the energy levels are not degenerate.

For degenerate energy levels, any superpositions of the wavefunctions are also the eigenvector ~~of~~ the model and Lapack can't automatically ensures ~~that~~ they are ~~the solution~~ localized edge states.

So, I suggest to study a relative smaller system $\mathcal{L} = 100/k_F$ to study the edge states and their chirality.

MFS:

$$\begin{cases} \gamma = \sum_i \left(u_i c_i + v_i c_i^{+} \right) \\ \gamma^{+} = \sum_i \left(u_i^{*} c_i^{+} + v_i^{*} c_i \right) \end{cases}$$

$$\Rightarrow \quad \boxed{V_i = u_i^{*}\ , \quad u_i = V_i^{*}}$$

check that the wavefunctions have the above feature when $|\varepsilon| < 10^{-7}$.

Here $\boxed{C_i \Rightarrow \phi_{n\sigma} \sim \sin\left(\frac{n\pi x}{L}\right)\sigma}$

σ is spin.

$\boxed{\text{Pfaffian}}$

$$\Sigma H(k) \Sigma^{-1} = - H(-k)$$

Thus $\quad \Sigma H(0) \Sigma^{-1} = - H(0)$

defines $\quad \Sigma = \Lambda K =$

$\Rightarrow \cancel{\Lambda H^*}$

$$\Sigma H \Sigma^{-1} = \Lambda K H K^{-1} \Lambda^{-1} = \Lambda H^* \Lambda^{-1} = - H$$

$$\Rightarrow \quad \Lambda H^* = - H \Lambda = - W$$

$$W^T = (H \Lambda)^T = \Lambda^T H^T = \Lambda^T H^* = \Lambda H^*$$

because $\quad H^\dagger = H^\dagger, \quad \Lambda^T = \Lambda$

$$\Rightarrow \quad \boxed{W = - W^T}$$

W is a skew matrix.

$$pf^2(W) = pf^2(H\Lambda) = \det(H\Lambda) = \det(H) \cdot \det(\Lambda)$$
$$= \det(H)$$

because $\quad \det(\Lambda) = 1$

$$\Rightarrow \quad pf(W) = \pm \sqrt{\det(H)}$$

$\boxed{\nu = - \text{sign}(pf(w))}$ is topological index, which will not change sign upon on deformation when the gap is not closed, that is, ~~det~~ $\det(H) \neq 0$.

$\boxed{\text{method to Calculate pfaffian}}$

First, $\det(H) = \prod_{i=1}^{N} \lambda_i$ is impossible to be calculated numerically

because when N is very big ($N = 200$, say)

$$\det(H) = \begin{cases} \to 0 \\ \to +\infty \end{cases}$$

Computer Can't store this data.

The following have definite paffian.

$$W = \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & a_2 & & O & \\ & -a_2 & 0 & a_3 & & \\ & & -a_3 & 0 & \ddots & \\ & O & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

then $\mathrm{pf}(W) = a_1 a_2 \cdots a_N$

$\nu = \mathrm{Sign}(\mathrm{pf}(W)) = \prod_i \mathrm{Sign}(a_i) \Rightarrow$ well-defined.
in numerical Calculation.

This Can be done using the following Eq.

$$\boxed{A = B T B^T}$$

$$\underline{\mathrm{Pf}(A) = \mathrm{pf}(B T B^T) = \mathrm{pf}(T) . \det(B)}$$

$$= \mathrm{pf}(T) \quad \text{if } \det(B) = 0$$

phase transition $\Rightarrow$ $\boxed{\text{Pfaffian change sign}}$