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Topology and geometry in quantum condensed matter

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- a mainly theoretical partial survey of surprises involving topology and geometry in condensed matter problems

- Berry phase, Berry curvature, and the associated quantum metric

"Topological matter"

- Integer quantum Hall effect (2D Landau levels)1981
- Fractional quantum Hall effect (2D Landau levels) 1983
- "Haldane gap" quantum spin chains (1D) 1985
- quantum anomalous Hall effect (Chern Insulators) (theory only, 1988)
- quantum spin Hall effect (2d topological insulator) 2006
- 3d topological insulator 2007
- topological superconductor (theory only 2009)
- Fractional Chern Insulator (theory only 2011)

Unexpected Geometry

- Anomalous Hall effect (theory 1954, experiment. 2004)
- Fractional quantum Hall effect (geometric picture 2011)

From geometry to topology: Gauss-Bonnet

- Gaussian Curvature (2d surface)

genus 1

$$K = \frac{1}{R_1 R_2}$$

principal
radii of curvature

$$\boxed{\int_{\mathcal{M}} d^2\boldsymbol{r}\, K(\boldsymbol{r}) = 2\pi\chi_{\mathcal{M}}}$$

Euler characteristic

$2(1 - \text{ genus }) = V + F - E$

compact 2D surface (no edges)

Gauss-Bonnet was generalized in more abstract ways, and is a special case of a more general structure that includes "Chern classes";

$$\int d^2x\,(\text{curvature}) = \text{integer topological invariant}$$

- curvature is now "Berry curvature" of a continuously-variable quantum state defined on a manifold, for example:

  - coherent states of quantum spins parallel to an arbitrary axis (manifold is sphere)
  - periodic parts of electronic Bloch states that vary with 2D Bloch vector k. (manifold is Brillouin zone)

$\mathrm{U}(1)$ Gauge ambiguity of quantum mechanics

- quantum states are defined in Hilbert space

$$|\Psi\rangle = \sum_{i=1}^{n} \Psi_i |i\rangle \qquad \langle i|j\rangle = \delta_{ij}$$

fixed orthonormal basis

- However, physical properties depend only on expectation values $\langle\Psi|\hat{O}|\Psi\rangle$

invariant under

$$|\Psi\rangle \mapsto e^{i\chi}|\Psi\rangle$$

$|\Psi\rangle$ and $e^{i\chi}|\Psi\rangle$
are physically equivalent

“gauge fixing” (avoid doing it!)

- “gauge-dependent” $\equiv$ non-physical !

One philosophy about gauge fixing is:
“choose a gauge that is convenient for calculations”

- This is dangerous, because it hides the distinction between physically-meaningful and physically-meaningless quantities

distance measures in a metric space:

(a space in which a distance between points is defined)

- $d_{ij} = 0$ iff $i = j$ identity of indiscernables
- $d_{ij} = d_{ji}$ symmetry
- $d_{ij} + d_{jk} \geq d_{ij}$ triangle inequality

"Trivial distance" in a discrete set:

$$d_{ij} = 1 - \underset{\text{Kronecker symbol}}{\delta_{ij}} \begin{cases} = 0 & (i = j) \\ = 1 & (i \neq j) \end{cases}$$

- To establish a physically-meaningful distance measure between quantum states in a Hilbert space, physically-equivalent states must be treated as "indiscernables"
- "Hilbert space distance" between normalized state $|\Psi_i\rangle$ and $|\Psi_j\rangle$ must be a function of the absolute value of the overlap $|\langle\Psi_i|\Psi_j\rangle|$

a family of "quantum distance" measures:

$$(d_{ij}(\lambda))^2 = 1 - |\langle\Psi_i|\Psi_j\rangle|^\lambda \quad \lambda \geq 1$$

$\lambda = 1$	$\lambda = 2$	$\lambda \to \infty$
sine of "Bargmann angle", Bures distance	Hilbert-Schmidt distance	trivial (classical) distance

- The fact that the $d_{ij}(\lambda)$ with $\lambda > 1$ are distance measures follows from the fact that $d_{ij} \equiv d_{ij}(1)$ is a distance (obeys the triangle inequality).

- The original measure is due to Bargmann who showed that the angle θ_B was a distance, where

$$\cos\theta_B = |\langle\Psi_1|\Psi_2\rangle|$$

$$d_{12} = \sin\theta_B$$

is proportional to the pure-state limit of the Bures distance (a distance between density matrices)

$\langle\Psi_1|\Psi_2\rangle$ not gauge invariant

$$\boxed{\langle\Psi_1|\Psi_2\rangle\langle\Psi_2|\Psi_1\rangle = \left(1-(d_{12})^2\right)^2 \text{ gauge-invariant, but real positive}}$$

$$\langle\Psi_1|\Psi_2\rangle\langle\Psi_2|\Psi_3\rangle\langle\Psi_3|\Psi_1\rangle = \left(1-(d_{12})^2\right)\left(1-(d_{23})^2\right)\left(1-(d_{31})^2\right)e^{i\theta_{123}}$$

gauge-invariant, with a complex phase factor

- amplitudes for "closed paths" are gauge invariant

$$|\Psi_1\rangle \to |\Psi_2\rangle \to \ldots \to |\Psi_n\rangle \to |\Psi_1\rangle$$

$$W = \langle\Psi_1|\Psi_2\rangle\langle\Psi_2|\Psi_3\rangle\ldots\langle\Psi_{n-1}|\Psi_n\rangle\langle\Psi_n|\Psi_1\rangle$$

In 1982, Berry's "geometric phase" exposed issues in "adiabatic quantum mechanics" that had previously been hidden by implicit gauge-fixing.

- From a modern perspective, we do not need to restrict attention to "adiabatic quantum mechanics", and will treat the Berry phase more generally as a property of a map from Hilbert space to a manifold where each point is associated with a quantum state (or, more generally, a multiplet of quantum states)

The original Berry problem:

- a quantum spin-*S* with a Zeeman coupling to a slowly-time-dependent magnetic field

$$H(t) = -\vec{h}(t) \cdot \vec{S} \qquad \vec{S} \cdot \vec{S} = S(S+1)$$

$$\vec{h}(t) = |h(t)|\hat{\Omega}(t)$$

slowly-changing unit vector

- adiabatic assumption: state remains close to instantaneous ground state of $H(t)$:

$$\hat{\Omega} \cdot \vec{S}|\Psi(t)\rangle = S|\Psi(t)\rangle$$

coherent state of spin

- from a modern viewpoint, just consider the two-dimensional compact manifold defined by the coherent states of a quantum spin

$$\hat{\Omega}\cdot\vec{S}|\hat{\Omega}\rangle = S|\hat{\Omega}\rangle$$

$$|\langle\hat{\Omega}_1|\hat{\Omega}_2\rangle|^2 = \left(\tfrac{1+\hat{\Omega}_1\cdot\hat{\Omega}_2}{2}\right)^{2S}$$

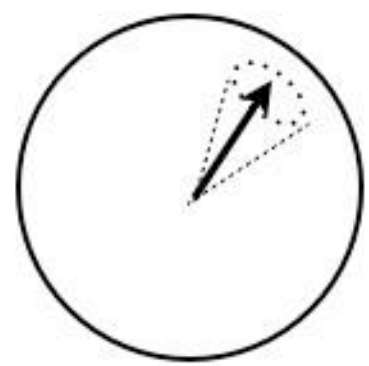
coherent state = " most classical" state. For a spin this is the state most aligned along a given direction

- in the orthonormal basis of S_z eigenstates

$$|\hat{\Omega}\rangle = \sum_m \left(\frac{2S!}{S+m)!(S-m)!}\right)^{1/2} u^{S+m}v^{S-m}|m\rangle$$

$$(u,v) = e^{i\chi}\left(\cos\tfrac{1}{2}\theta e^{\frac{1}{2}i\phi}, \sin\tfrac{1}{2}\theta e^{-\frac{1}{2}i\phi}\right)$$

$$\hat{\Omega} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$



- Let a sequence of states $\{|\hat{\Omega}_1\rangle, |\hat{\Omega}_2\rangle, \ldots |\hat{\Omega}_n\rangle = |\hat{\Omega}_1\rangle\}$ become a closed continuous path $|\hat{\Omega}(s)\rangle$ on the 2-sphere

$$W \to \prod_i e^{-(ds_i)^2} e^{id\phi_i}$$

"quantum length" of step i

$$\prod_i e^{id\phi_i} \to e^{i\phi_\Gamma}$$

total Berry phase accumulated along path is well-defined provided path is closed

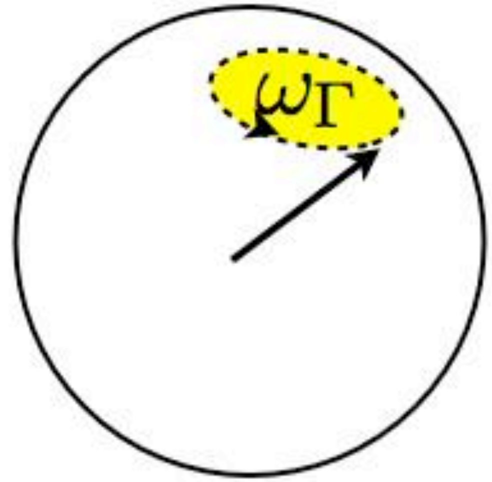
$$\sum_i ds_i \to \ell_\Gamma$$

Total quantum path length is well-defined whether or not path is closed.

- For a coherent state of a spin S the Berry phase of a closed path is

$$e^{iS\omega_\Gamma}$$

solid angle subtended by the (directed) path Γ on the 2-sphere (only defined modulo 4π)



- For a coherent state of a spin S the Berry phase of a closed path is the only place in "semiclassical" calculations that the **quantized** (integer or half-integer) value of S enters (as opposed to factors $1/S$)

- vector coherent states are not the only possible "almost classical" states a spin can have. Integer spins can be in a time-reversal invariant "nematic" state

directrix

$$\hat{\Omega}\cdot\vec{S}|\hat{\Omega}\rangle = 0 \qquad |\langle -\hat{\Omega}|\hat{\Omega}\rangle| = \langle\hat{\Omega}|\hat{\Omega}\rangle$$

- The Berry phase for a path starting at $|\hat{\Omega}\rangle$ and finishing at the the equivalent state $|-\hat{\Omega}\rangle$ is

$$e^{i\phi_\Gamma} = (-1)^S$$

distinguishes even and odd integer spins!

note that if time-reversal-invariance is maintained along a path $e^{i\phi_\Gamma} = \pm 1$

Action for a quantum spin

- (vector) coherent state path integral:

$$\langle\Omega(t_f)|T_t e^{-i\int \frac{dt H(t)}{\hbar}}|\Omega(t_i)\rangle \;=\; \sum_{\hat{\Omega}(t)} e^{i\mathcal{S}/\hbar}$$

$$\boxed{S = \int_{t_i}^{t_f} dt\, \hbar S \vec{\mathcal{A}}(\hat{\Omega}(t)) \cdot \partial_t \hat{\Omega}(t) - H(\hat{\Omega}(t), t)}$$

formally, this is a Dirac monopole vector potential

$$\delta\mathcal{A}_i(\hat{\Omega}) = \epsilon_{ijk}\Omega_j\delta\Omega_k$$

one gauge choice:

$$\vec{\mathcal{A}}(\hat{\Omega}) = \frac{\hat{z}\times\hat{\Omega}}{1+\hat{z}\cdot\hat{\Omega}}$$

The Berry connection and the quantum metric

- generally, we are not interested in arbitrary states in the Hilbert space, but a subset of states with some defining physical property such as the coherent spin states

- A continuously-variable family of such states defines a manifold of dimension d:

$$|\Psi(\boldsymbol{x})\rangle = \sum_i \Psi_i(\boldsymbol{x})|i\rangle$$

$$\boldsymbol{x} = \{x^\mu, \mu = 1, \ldots d\}$$

parameterization of the manifold

$$\langle i|j\rangle = \delta_{ij}$$

fixed orthonormal basis in Hilbert space

(manifold = the 2-sphere in the case of the coherent spin states)

- derivatives of the state on the manifold

$$|\partial_\mu \Psi(\boldsymbol{x})\rangle = \sum_i \frac{\partial \Psi_i(x)}{\partial x^\mu} |i\rangle$$

- effect of a gauge transformation

$$|\Psi(\boldsymbol{x})\rangle \to e^{i\chi(\boldsymbol{x})}|\Psi(\boldsymbol{x})\rangle$$

$$|\partial_\mu \Psi(\boldsymbol{x})\rangle \to e^{i\chi(\boldsymbol{x})}\left(|\partial_\mu \Psi(\boldsymbol{x})\rangle + i\partial_\mu \chi(\boldsymbol{x})|\Psi(\boldsymbol{x})\rangle\right)$$

- gauge-covariant derivative:

now orthogonal to $|\Psi(\boldsymbol{x})\rangle$

$$|D_\mu \Psi(\boldsymbol{x})\rangle = |\partial_\mu \Psi(\boldsymbol{x})\rangle - |\Psi(\boldsymbol{x})\rangle\langle \Psi(x)|\partial_\mu \Psi(\boldsymbol{x})\rangle$$

$$|D_\mu \Psi(\boldsymbol{x})\rangle \to e^{i\chi(\boldsymbol{x})}|D_\mu \Psi(\boldsymbol{x})\rangle \longleftarrow$$ (same as $|\Psi(\boldsymbol{x})\rangle$)

- The gauge-covariant derivative obeys the gauge-invariant parallel-transport condition

$$\langle \Psi(x) | D_\mu \Psi(\boldsymbol{x}) \rangle = 0$$

- It can be written as

$$|D_\mu \Psi(\boldsymbol{x})\rangle = |\partial_\mu \Psi(\boldsymbol{x})\rangle - i\mathcal{A}_\mu(\boldsymbol{x})|\Psi(\boldsymbol{x})\rangle$$

$$\mathcal{A}_\mu(\boldsymbol{x}) = -i\langle \Psi(\boldsymbol{x}) | \partial_\mu \Psi(\boldsymbol{x}) \rangle$$

is the "Berry connection" (essentially a $U(1)$ gauge potential (vector potential) on the manifold)

compare to $$p_i - eA_i(\boldsymbol{r}) = -i\hbar\left(\frac{\partial}{\partial x_i} - i\frac{eA_i(\boldsymbol{r})}{\hbar}\right)$$

- with the analogy between the quantum mechanics of charged particles in 3D space with a magnetic field, the Berry phase becomes analogous to the Bohm-Aharonov phase:

$$e^{i\phi_\Gamma} = e^{i\oint_\Gamma \mathcal{A}_\mu(\boldsymbol{x})dx^\mu}$$

integral around a 1d closed path on the d-dimensional manifold.

- continuing with gauge-invariant quantities

$$\langle D_\mu \Psi(\boldsymbol{x}) | D_\nu \Psi(\boldsymbol{x}) \rangle = \tfrac{1}{2} \left(\mathcal{G}_{\mu\nu}(\boldsymbol{x}) + i \mathcal{F}_{\mu\nu}(\boldsymbol{x}) \right)$$

positive Hermitian	real symmetric	imaginary antisymmetric

Berry curvature

$$\mathcal{F}_{\mu\nu}(\boldsymbol{x}) = \partial_\mu \mathcal{A}_\nu(\boldsymbol{x}) - \partial_\nu \mathcal{A}_\mu(\boldsymbol{x})$$

(analog of magnetic flux density)

quantum distances between points near $\boldsymbol{x}$

$$d(\boldsymbol{x} + d\boldsymbol{x}_1, \boldsymbol{x} + d\boldsymbol{x}_2) = \tfrac{1}{4} \mathcal{G}_{\mu\nu}(\boldsymbol{x}) \left(dx_1^\mu - dx_2^\mu \right) \left(dx_1^\nu - dx_2^\nu \right)$$

- The Berry curvature and the quantum metric are related by the inequality

$$\mathcal{G}_{\mu\mu}\mathcal{G}_{\nu\nu} - (\mathcal{G}_{\mu\nu})^2 \geq (\mathcal{F}_{\mu\nu})^2.$$

- The quantum metric $\mathcal{G}_{\mu\nu}(\boldsymbol{x})$on the manifold is the metric induced by the quantum metric in the full Hilbert space (also called the Fubini-Study metric).

- on a 2-d manifold $\mathcal{M}$ with boundaries $\partial\mathcal{M}_i$

$$\boxed{\exp\left(i\int_{\mathcal{M}} \mathcal{F}_{\mu\nu}dx^{\mu}\wedge dx^{\nu}\right) = \prod_i \exp i\oint_{\partial\mathcal{M}_i} \mathcal{A}_{\mu}dx^{\mu}}$$

Integrated Berry curvature("flux") in interior

product of Berry phase factors on boundaries

"punctures"

Stokes theorem

Topological invariants

$$e^{i \oint_\Gamma \mathcal{A}_\mu(\boldsymbol{x}) dx^\mu} = e^{i \int_S dx^\mu \wedge dx^\nu \mathcal{F}_{\mu\nu}(\boldsymbol{x})}$$

integral of the Berry connection around a closed $1\mathrm{d}$ loop Γ

integral of the Berry curvature over a 2d surface S bounded by Γ

- First Chern class (the "Chern number")

<u>If a 2D manifold is compact (has no boundaries)</u>

$$e^{i \int_S dx^\mu \wedge dx^\nu \mathcal{F}_{\mu\nu}(\boldsymbol{x})} = 1$$

$$\int_S dx^\mu \wedge dx^\nu \mathcal{F}_{\mu\nu}(\boldsymbol{x}) = 2\pi \times (\text{ an integer})$$

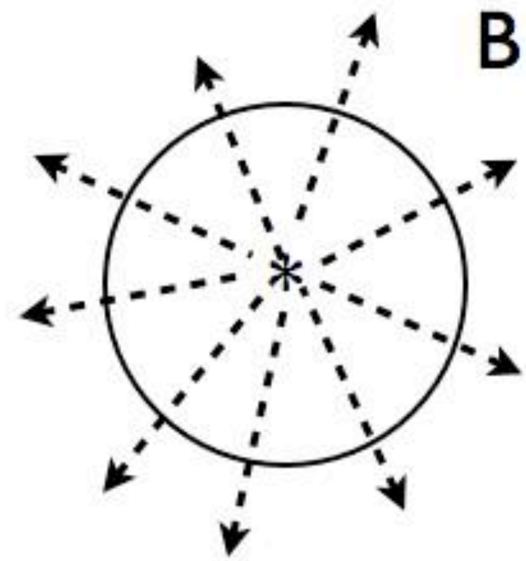
the Chern number

- This is the equivalent of the quantization of the Dirac magnetic monopole in the electromagnetic analogy

$$\int_S dx^\mu \wedge dx^\nu \mathcal{F}_{\mu\nu}(\boldsymbol{x}) = 2\pi \times (\text{ an integer})$$

$$\int_S d^2\boldsymbol{r} \frac{e\boldsymbol{B}\cdot\hat{\boldsymbol{n}}}{\hbar} = 2\pi \times (\text{an integer})$$

closed 2D surface enclosing the magnetic monopole



- For the case of the 2-sphere on which the coherent spin states are defined

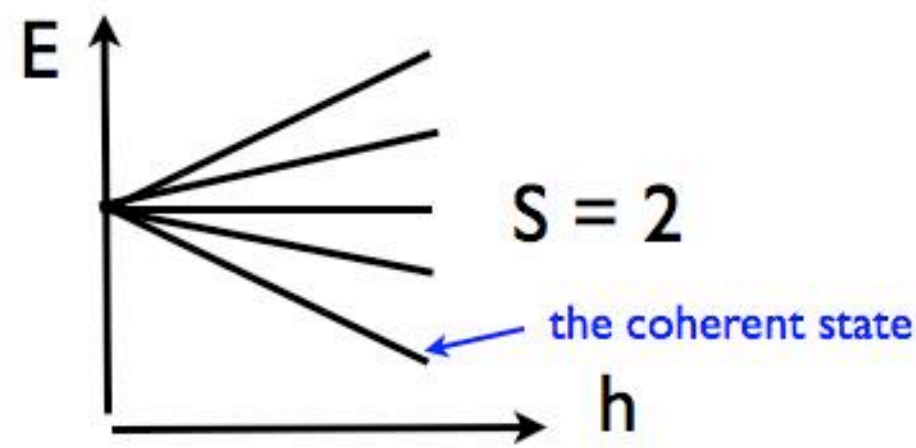
$$\int d^2\hat{\Omega}\,\mathcal{F}(\hat{\Omega}) = 4\pi S$$ Chern number 2S

$$H = -h\hat{\Omega}\cdot\vec{S}$$

ground state of H is a non-degenerate coherent state when $h > 0$

monopole singularity at the "center of the sphere" is the degeneracy when h = 0

h is the "radial coordinate"



examples of compact 2d manifolds with Chern numbers

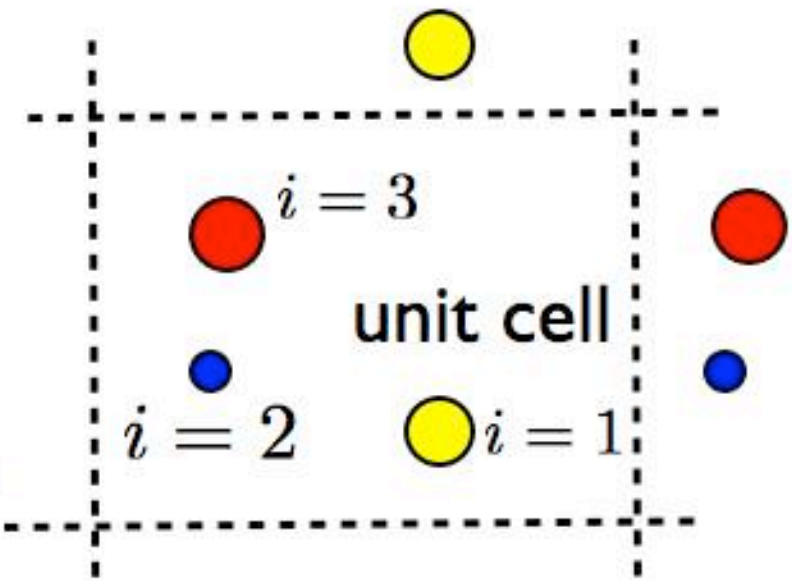
- 2d Brillouin zone of a "Chern insulator"
- Fermi surface of a 3D metal surrounding a "Dirac" (Weyl) point.
- 2d Landau level "compactified" on a torus or a sphere

Berry curvature and quantum metric of Bloch states

- one-electron bands on periodic lattice
- consider a tight-binding band-structure with more than one orbital in the unit cell
- $$|\boldsymbol{R}, i\rangle$$

labels unit cells

labels orbitals in cell



band index

Bloch vector

$$|\Psi_n(\boldsymbol{k})\rangle = \sum_i u_{ni}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{R}+\boldsymbol{a}_i)} |\boldsymbol{R}, i\rangle$$

$$H|\Psi_n(\boldsymbol{k})\rangle = \varepsilon_n(\boldsymbol{k})|\Psi_n(\boldsymbol{k})\rangle$$

$$T(\boldsymbol{R})|\Psi_n(\boldsymbol{k})\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{R}}|\Psi_n(\boldsymbol{k})\rangle$$

Bloch states

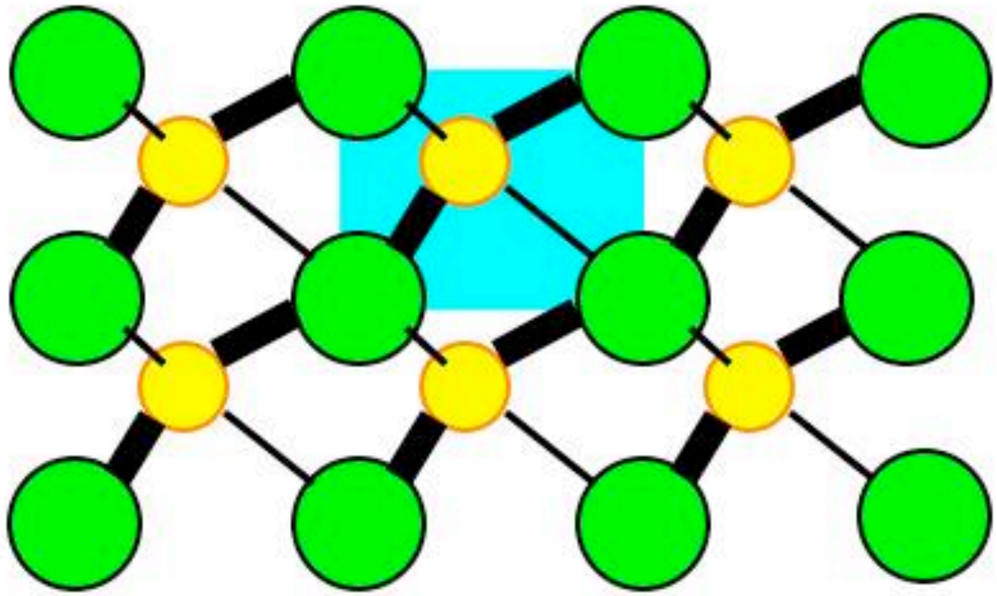
$$|\Phi_n(\boldsymbol{k})\rangle = U(-\boldsymbol{k})|\Psi_n(\boldsymbol{k})\rangle$$

$$T(\boldsymbol{R})|\Phi_n(\boldsymbol{k})\rangle = |\Phi_n(\boldsymbol{k})\rangle$$

periodic states

$$\boxed{U(\boldsymbol{k}) = \sum_{\boldsymbol{R},i} e^{i\boldsymbol{k}\cdot(\boldsymbol{R}+\boldsymbol{a}_i)}|\boldsymbol{R},i\rangle\langle\boldsymbol{R},i|}$$

spatial embedding of orbital i



- the Bloch Hamiltonian only “knows” about the “hopping matrix elements” between orbitals, but **not** how the orbitals are embedded in space

- the Berry curvature in k-space of $|\Phi_n(\boldsymbol{k}, \{\boldsymbol{a}_i\}\rangle$ “knows” about the relative spatial locations of the orbitals, and allows the effect of perturbation by uniform electric and magnetic fields to be described

- The Bloch eigenstates of the Hamiltonian cannot be directly compared, because for different Bloch vectors they belong to different Hilbert subspaces.

- The states that can be compared are periodic states

$$|\Phi_n(\boldsymbol{k})\rangle = U(-\boldsymbol{k}; \{\boldsymbol{a}_i\})|\Psi_n(\boldsymbol{k})\rangle$$

These depend on how the orbitals are embedded in Euclidean space

$\mathcal{F}_n^{ab}(\boldsymbol{k}; \{\boldsymbol{a}_i\})$ Berry curvature in Bloch space

$\mathcal{G}_n^{ab}(\boldsymbol{k}; \{\boldsymbol{a}_i\})$ quantum metric in Bloch space

k_a components of Bloch vector

- Physical consequences of the Berry curvature

semiclassical equation of motion

(response to quasi-uniform electric and magnetic fields that do not vary on the scale of the unit cell)

kinetic energy potential energy

$$H = \varepsilon_n(\boldsymbol{k}) + V(\boldsymbol{r})$$

$$\hbar \begin{pmatrix} F_{ab}(\boldsymbol{r}) & -\delta_a^b \\ \delta_b^a & -\mathcal{F}_n^{ab}(\boldsymbol{k}) \end{pmatrix} \begin{pmatrix} \partial_t r^b \\ \partial_t k_b \end{pmatrix} = \begin{pmatrix} \frac{\partial V(\boldsymbol{r}}{\partial r^a} \\ \frac{\partial \varepsilon_n(\boldsymbol{k})}{\partial k_a} \end{pmatrix} \leftarrow$$

derivatives of scalar potentials in real and k-space

$$F_{ab}(\boldsymbol{r}) = \frac{e}{\hbar}\left(\frac{\partial A_b(\boldsymbol{r})}{\partial r_a} - \frac{\partial A_a(\boldsymbol{r})}{\partial r_b}\right) \qquad \mathcal{F}_n^{ab}(\boldsymbol{k})$$

magnetic flux in real space Berry curvature in k-space

- The Berry curvature term restores complete symmetry between real space and k-space, with the Karplus-Luttinger "anomalous velocity" term:

$$\frac{\partial r^a}{\partial t} = \frac{1}{\hbar}\frac{\partial \varepsilon_n}{\partial k_a} + \mathcal{F}_n^{ab}(\boldsymbol{k})\frac{\partial k_b}{\partial t}$$

k-space analog of $e\vec{E}(\boldsymbol{r}) \longrightarrow$ group velocity

"anomalous velocity" $\longleftarrow$ k-space analog of $e\frac{d\vec{r}}{dt} \times \vec{B}(\boldsymbol{r})$

The anomalous velocity represents the current that flows within the unit cell as the charge is redistributed between the orbitals as k changes

- Karplus and Luttinger (1954) found their "anomalous" term from spin-orbit coupling in a relativistic band structure calculation

- it gives an antisymmetric (Hall) contribution to the conductivity

$$\sigma_H^{ab} = \frac{e^2}{\hbar} \sum_{n\boldsymbol{k}} \mathcal{F}_n^{ab}(\boldsymbol{k}) n(\varepsilon_n(k))$$

Fermi-Dirac occupation function

- It describes an intrinsic "anomalous" Hall effect in a clean metal that is <u>**not due to the Lorentz force**</u>

- It was dismissed as "obviously wrong", "counter to basic physical principles", and only recognized as the Berry curvature by Sundaram and Niu in1999!

anomalous Hall effect

$$\sigma_H^{ab} = \frac{e^2}{\hbar} \sum_{n\boldsymbol{k}} \mathcal{F}_n^{ab}(\boldsymbol{k}) n(\varepsilon_n(k))$$

- The Karplus-Luttinger theory was widely dismissed as "obviously wrong", "contrary to basic physical principles", etc. and only recognized as the Berry curvature by Sundaram and Niu in 1999!

$$\rho^{xy} = \frac{\sigma^{xy}}{(\sigma^{xx})^2 + (\sigma^{xy})^2} \approx (\rho^{xx}(T))^2 \sigma^{xy} \qquad \propto T^4$$

$$\rho^{xx} \propto T^2$$

(clean metal,
Fermi liquid)

- Karplus/Luttinger is now recognized as the dominant contribution to the anomalous Hall effect in ferromagnetic metals in certain temperature ranges

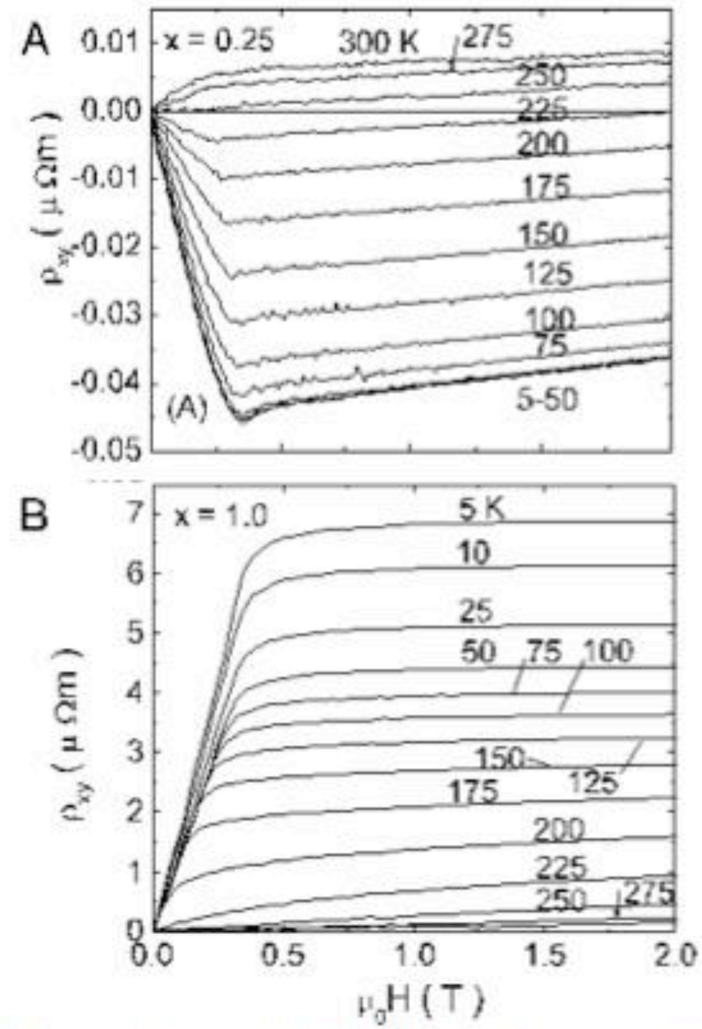


Figure 2 The observed Hall resistivity r_{xy} in two samples of $CuCr_2Se_{4-x}Br_x$, with the Br content x = 0.25 (Panel A (Panel B). In Panel A, the AHE signal is negative at low temperatures and saturates to the value ~ 40 nWm at 5 K. x = 1.0 has about 25 times smaller carrier concentration (Panel B). Its AHE signal is positive and rises to 7 mWm a the largest AHE signals ever recorded in any ferromagnet) [from Wei Li Lee *et al*. (Ref. 1)].

- For a 2D band insulator, the Karplus-Luttinger formula reduces to the TKNN formula, the total Chern number of occupied bands, and the anomalous Hall effect is quantized.

$$\sigma_H = \frac{e^2}{2\pi\hbar}\underbrace{\overbrace{\left(\sum_n \frac{1}{2\pi}\int_{\mathrm{BZ}} d^2\boldsymbol{k}\mathcal{F}_n(\boldsymbol{k})\right)}^{\text{an integer}}}_{\text{Brillouin zone = Torus}}$$

- For a 2D metal, the non-quantized part is determined at the Fermi surface(s), which are oriented 1D paths in 2D k-space:

$$\sigma_H = \frac{e^2}{2\pi\hbar}\left(n + \sum_\alpha \frac{\phi_\alpha}{2\pi}\right)$$

Berry phases for going around each "loop" of Fermi surface

- 3D case: also get nice geometrical/ topological formulas

$$\sigma_H^{ab} = \frac{e^2}{2\pi\hbar}\epsilon^{abc}\frac{k_c}{2\pi}$$

- For a 3D "Chern insulator" $\boldsymbol{k} = \boldsymbol{G}$, a (quantized) lattice vector (Kohmoto +Halperin), associated with edge states

- For a 3D metal, non-quantized (bulk) $\boldsymbol{k}$ = is an averaged Fermi vector, weighted by the Berry curvature on the Fermi surface (FDMH 2004)

- This is defined modulo a reciprocal vector

- It is natural that the Berry curvature in k-space that appears in the semiclassical equation of motion depends on the relative position of the orbitals in the unit cell, as well as the band structure. (potential differences induced by the electric field depend on relative positions)

- Only the topological invariant Chern number is independent of the embedding:

$$\frac{1}{2\pi}\int_{\mathrm{BZ}} d^2\boldsymbol{k}\,\mathcal{F}_n(\boldsymbol{k};\{\boldsymbol{a}_i\}) \qquad \text{(= integer)}$$

does not depend on the embedding, and is a pure band structure property

- what about the metric? Less is known
- For Bloch states, there are two relevant metrics:

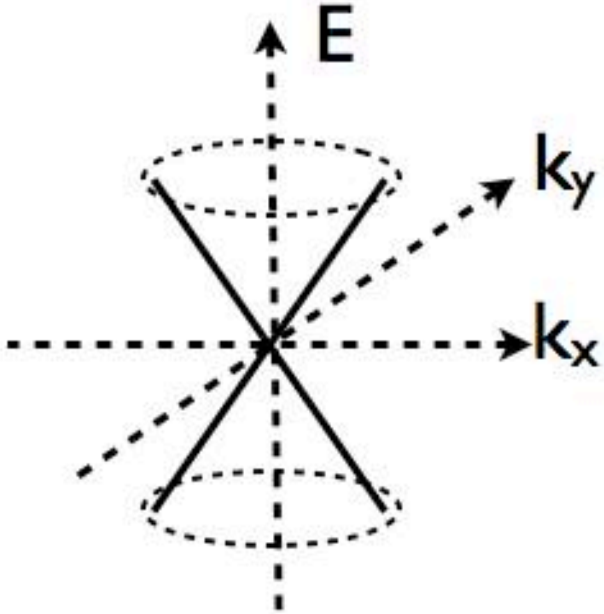
(a) Euclidean distances in k-space

(b) quantum distances in Hilbert space

A slowly-varying impurity potential (quasi-uniform on the scale of the unit cell, with only small Fourier components) can only scatter though small displacements in k-space

It can also only scatter between states that are close in Hilbert space. Usually closeness in k-space implies closeness in Hilbert space. An exception is near Dirac points (e.g. graphene).

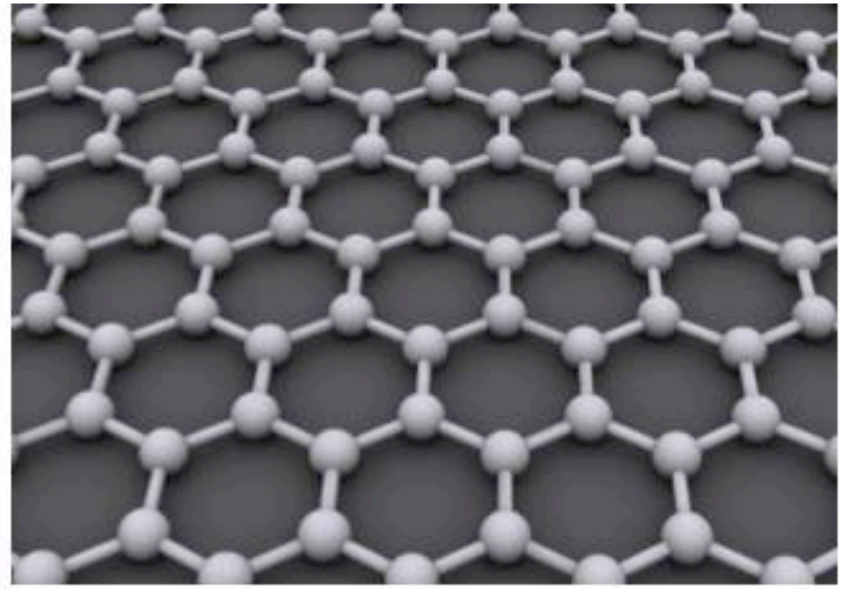
“Dirac/Weyl Points”



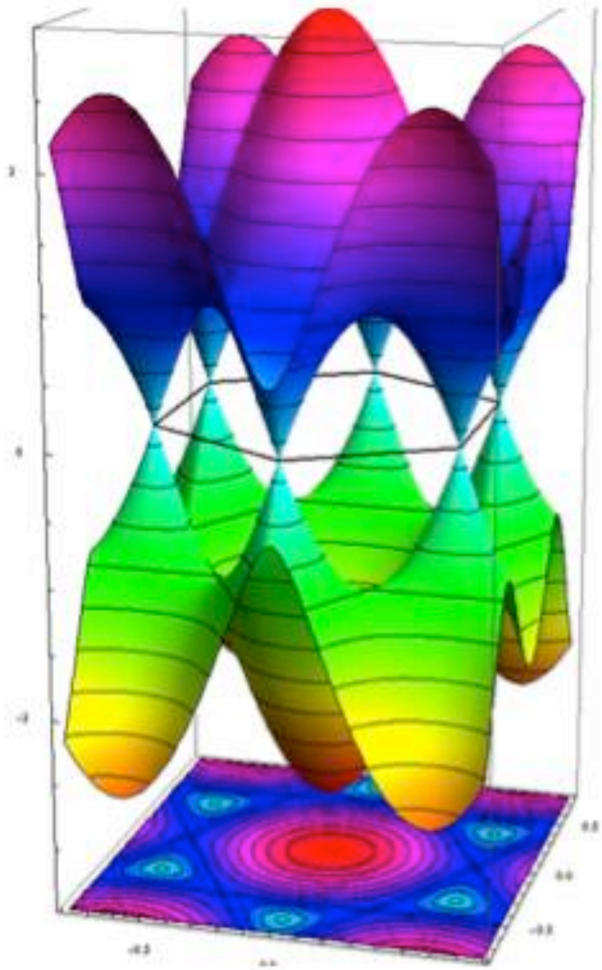
2D Dirac point

- Bands touch conically
- In 2D, need both inversion and time-reversal symmetry
- In 3D, a Fermi surface surrounding a Dirac Point has a non-zero Chern number

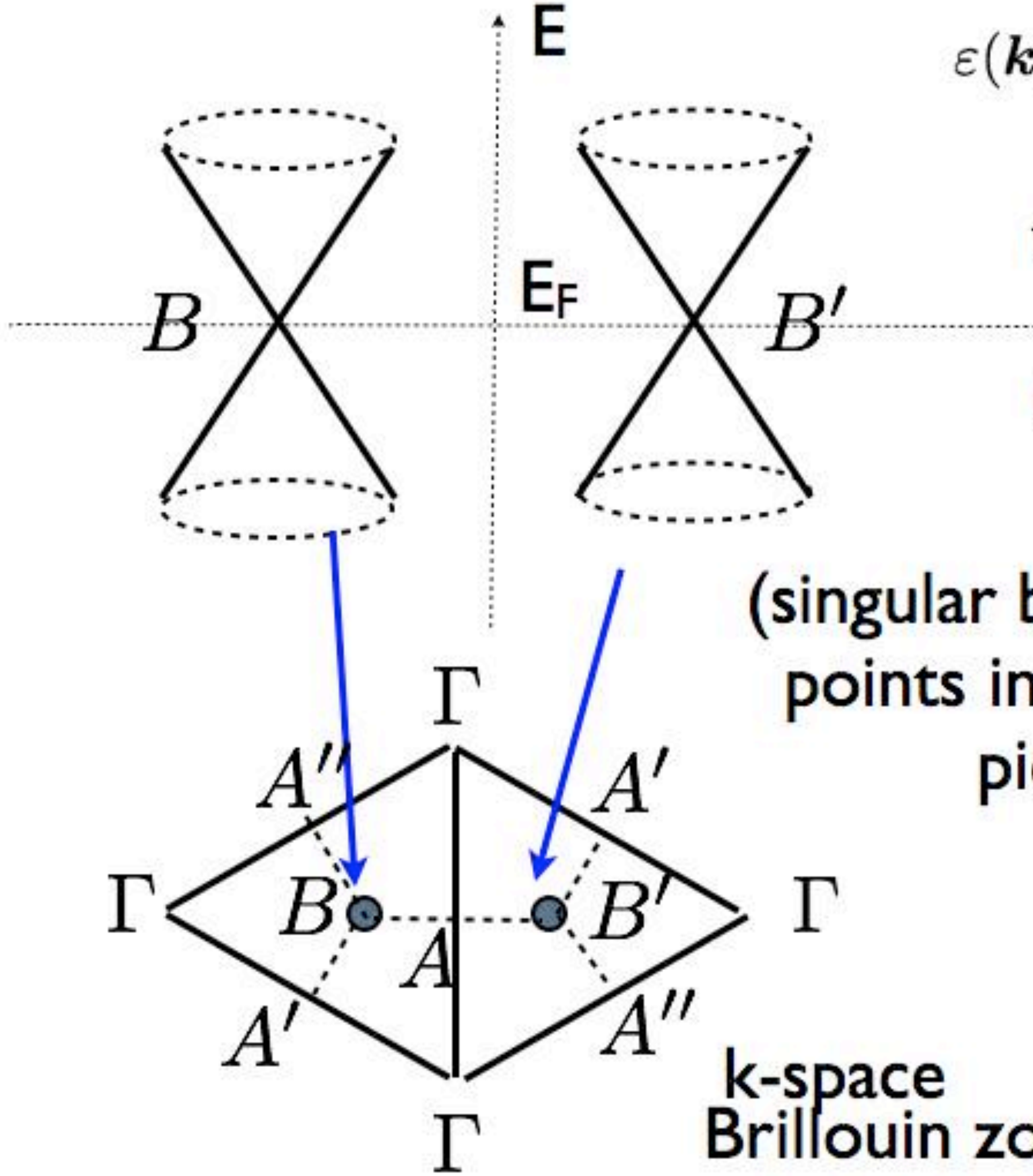
2D Graphene:



- Dirac points (2 valleys)



$$\varepsilon(\boldsymbol{k}) = \pm t \left| \sum_{i=1}^{3} e^{i\boldsymbol{k}\cdot a_i} \right|$$



two Dirac points of Graphene

(singular band-touching points in the k-space picture)

k-space Brillouin zone

Graphene:

- Dirac points are protected against mass generation because:
- They are at different k-points
- Spin-orbit coupling is absent (negligible), and both time reversal and inversion symmetry are present:

$$\boxed{\mathcal{F}_n(\boldsymbol{k}) = \mathcal{F}_n(-\boldsymbol{k})}$$

inversion

$+$

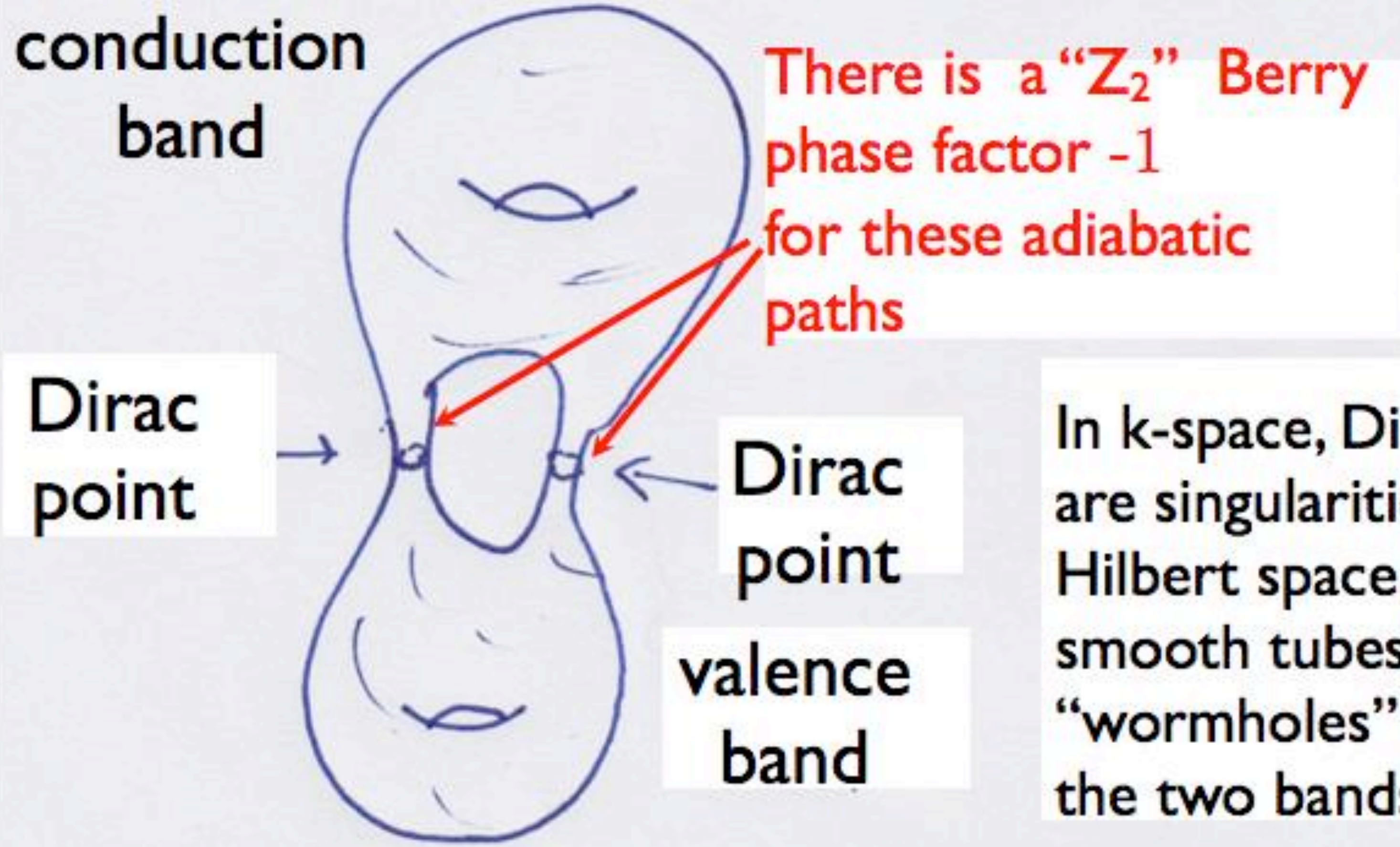
$$\boxed{\mathcal{F}_n(\boldsymbol{k}) = -\mathcal{F}_n(-\boldsymbol{k})}$$

Time-reversal

Berry curvature vanishes because both inversion and time-reversal symmetry are present with no spin-orbit coupling $\mathcal{F}_n(\boldsymbol{k}) = 0$

Hilbert-Space picture of the geometry of the graphene valence and conduction band

(Brillouin zone of an isolated 2D band is a simple torus with genus 1)

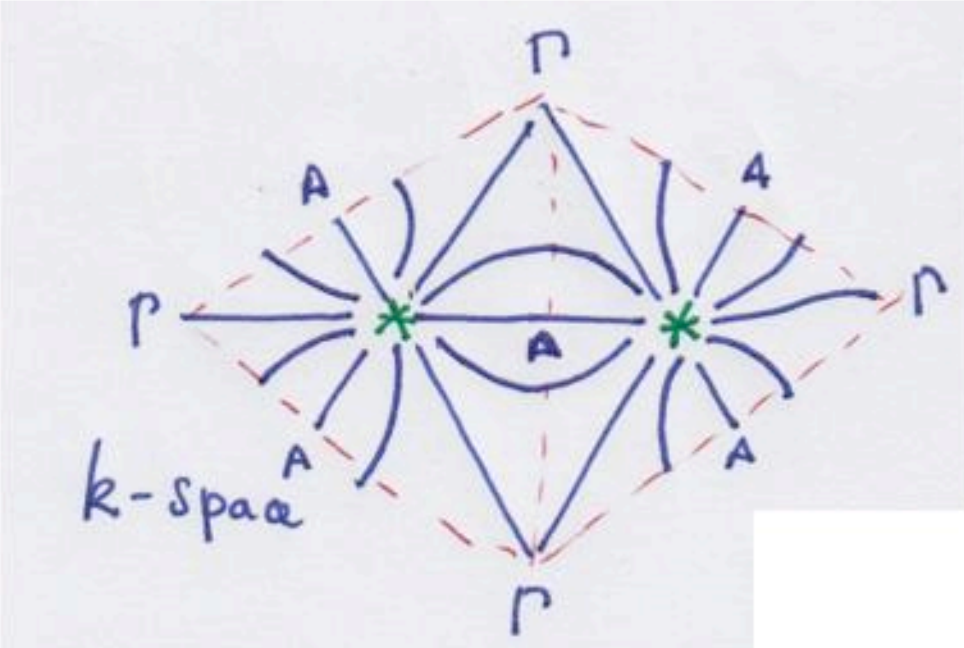


In k-space, Dirac points are singularities, in Hilbert space, they are smooth tubes or "wormholes" that join the two bands

The two bands joined at the Dirac points form a Genus-3 manifold

"Pseudospin conservation" and the "quantum distance" (Fubini-Study metric)

- Since the Berry curvature field vanishes,in graphene, the only remaining ingredient of the "quantum geometry" is the metric $\mathcal{G}$ which measures quantum distance.



The simple 2-band model has null geodesics emanating from the Dirac points

when more bands are included, geodesics are no longer null, metric becomes positive definite

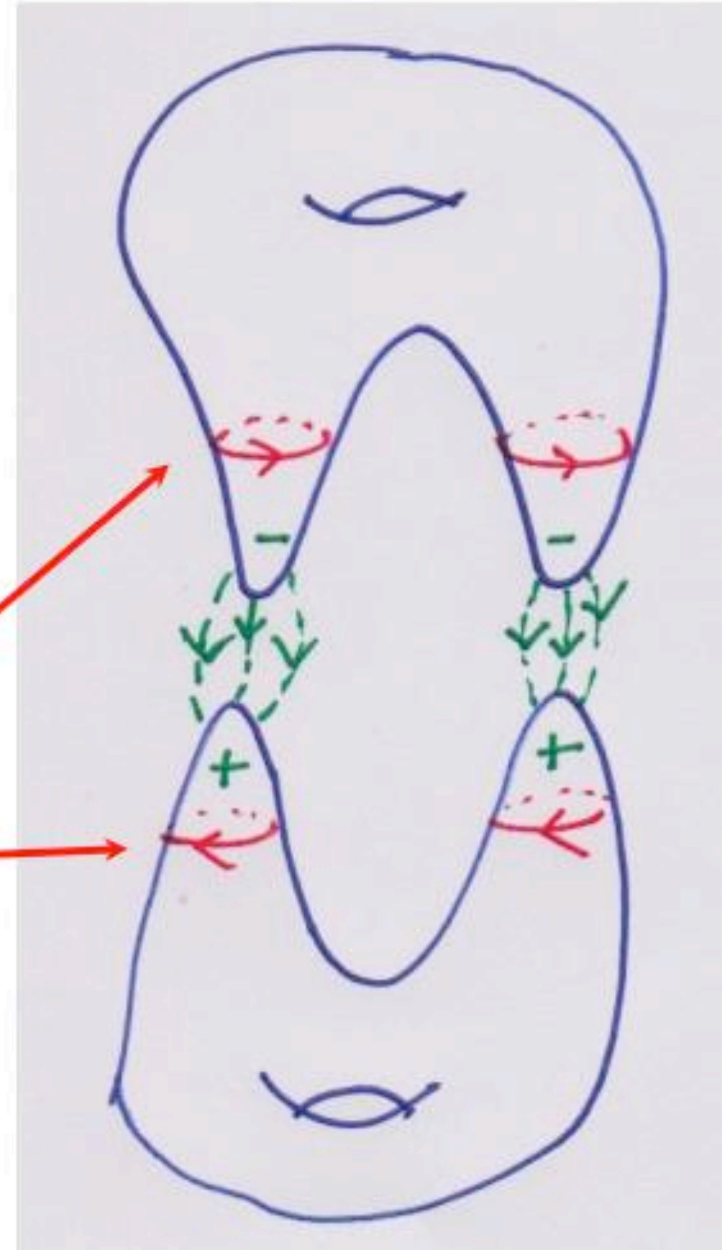
- Significance: semiclassical (long-wavelength) scattering processes are **suppressed when there is a large "quantum distance" between initial and final states.** THIS IS THE "PSEUDOSPIN CONSERVATION" near the Dirac points.

- Slowly-varying potentials can only scatter electrons through small distances in k-space.
- Usually, Hilbert space distance between states in the same band, with similar Bloch vectors is small.
- Exception: near topologically-singular points, there can be nearby points in <u>**k-space**</u> with a large Hilbert-space separation. Slowly-varying potentials can also only scatter through short distances in <u>**Hilbert space**</u>!

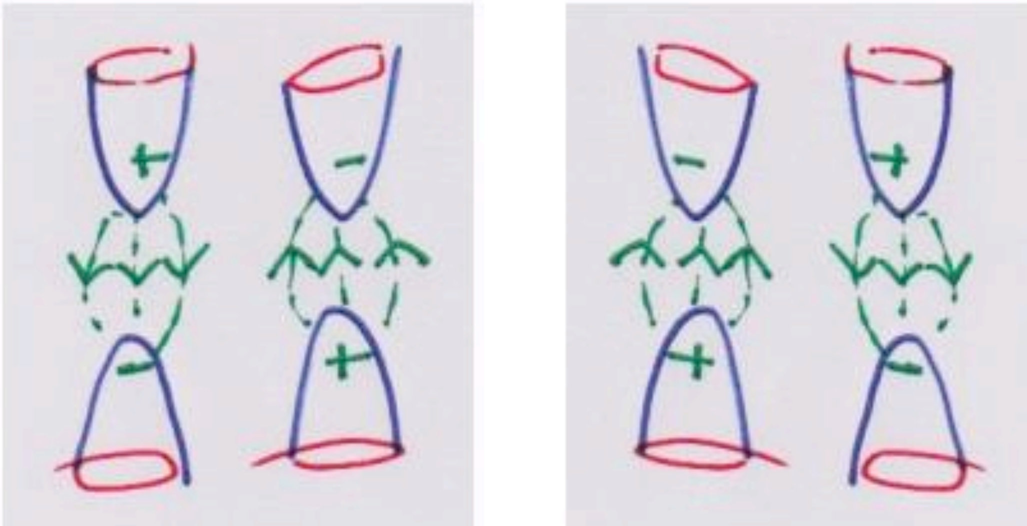
Extra approximate conservation laws: "Pseudospin conservation"

- Breaking either Inversion or time-reversal allows a local Berry-curvature at each point in the Hilbert space.
- The Dirac points lose topological protection and split

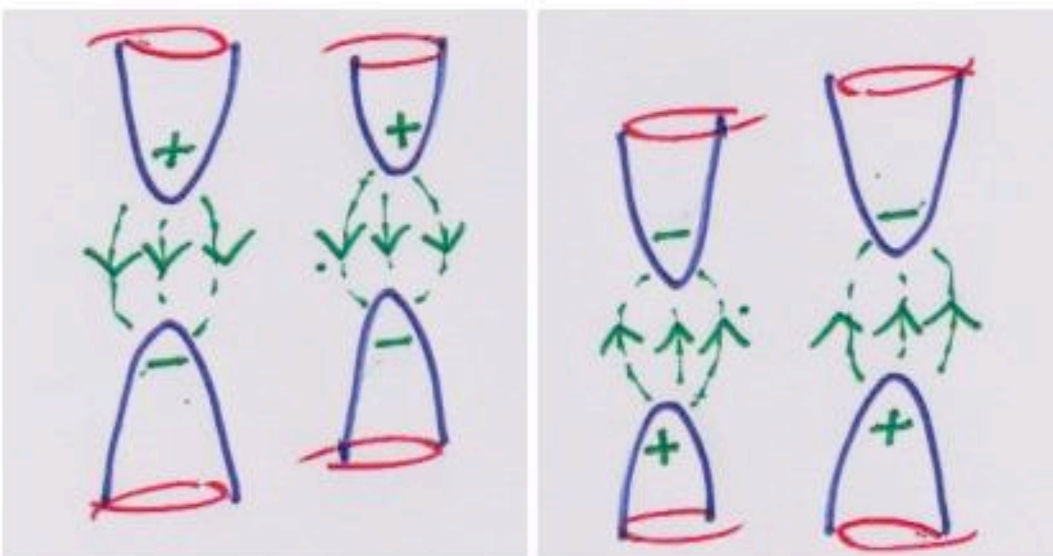
Berry "flux" of close to π flows through these paths. It must emerge or enter near the former Dirac points



four possibilities



Broken inversion symmetry, no net Berry flux flows between bands



Broken time reversal symmetry, net Berry flux 2π flows between bands.   bands have Chern numbers $+1$, -1

The original "Topological Insulator" model

2D zero-field Quantized Hall Effect

FDMH, Phys. Rev. Lett. 61, 2015 (1988).

- 2D quantized Hall effect: $\sigma^{xy} = \nu e^2/h$. In the absence of interactions between the particles, ν must be an integer. There are no current-carrying states at the Fermi level in the interior of a QHE system (all such states are localized on its edge).

- The 2D integer QHE does NOT require Landau levels, and can occur if time-reversal symmetry is broken even if there is no net magnetic flux through the unit cell of a periodic system. (This was first demonstrated in an explicit "graphene" model shown at the right.).

- Electronic states are "simple" Bloch states! (real first-neighbor hopping t_1, complex second-neighbor hopping $t_2e^{i\varphi}$, alternating onsite potential M.)

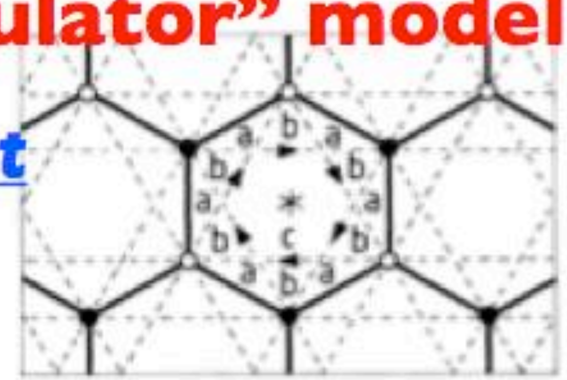


FIG. 1. The honeycomb-net model ("2D graphite") showing nearest-neighbor bonds (solid lines) and second-neighbor bonds (dashed lines). Open and solid points, respectively, mark the A and B sublattice sites. The Wigner-Seitz unit cell is conveniently centered on the point of sixfold rotation symmetry (marked "*") and is then bounded by the hexagon of nearest-neighbor bonds. Arrows on second-neighbor bonds mark the directions of positive phase hopping in the state with broken time-reversal invariance.

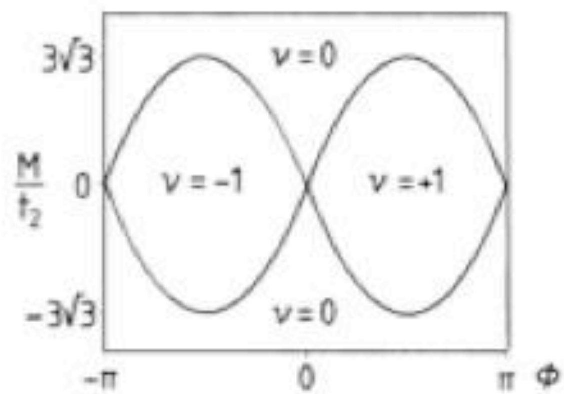
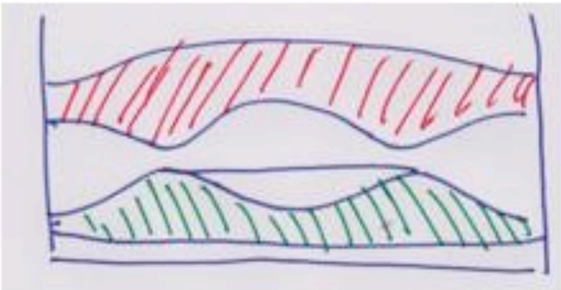
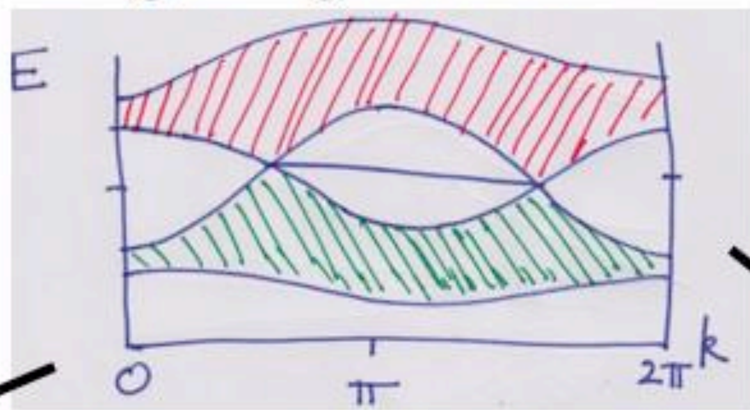


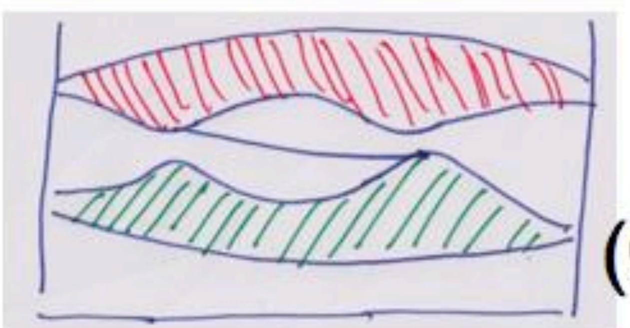
FIG. 2. Phase diagram of the spinless electron model with $|t_2/t_1| < \frac{1}{3}$. Zero-field quantum Hall effect phases ($\nu = \pm 1$, where $\sigma^{xy} = \nu e^2/h$) occur if $|M/t_2| < 3\sqrt{3}|\sin\phi|$. This figure assumes that t_2 is positive; if it is negative, ν changes sign. At the phase boundaries separating the anomalous and normal ($\nu = 0$) semiconductor phases, the low-energy excitations of the model simulate undoubled massless chiral relativistic fermions.



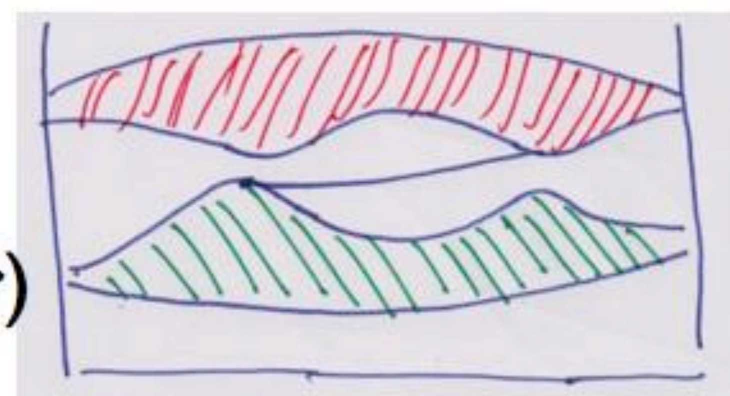
Broken
inversion

- gapless graphene "zig-zag" edge modes





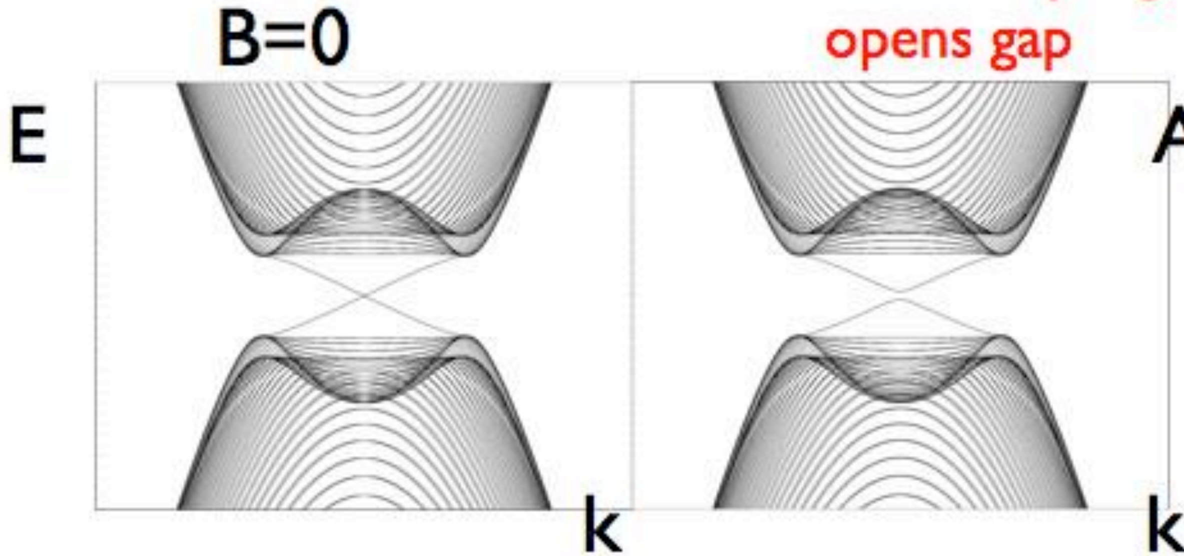
Broken
time-reversal
(Chern insulator)



Kane and Mele 2005

- Two conjugate copies of the 1988 spinless graphene model, one for spin-up, other for spin-down

Zeeman coupling opens gap



At edge, spin-up moves one way, spin-down the other way

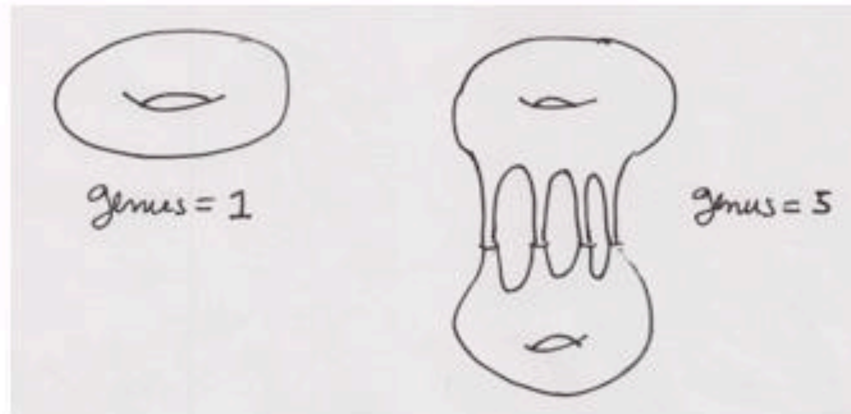
If the 2D plane is a plane of mirror symmetry, spin-orbit coupling preserves the two kind of spin. Occupied spin-up band has chern number +1, occupied spin-down band has chern-number -1.

- This looks "trivial", but Kane and Mele found that the gapless "helical" edge states were still there when Rashba spin-orbit coupling that mixed spin-up and spin-down was added.
- They found a new "Z2" topological invariant of 2D bands with time-reversal symmetry that takes two values, +1 or -1. The invariant derives from **Kramers degeneracy** of fermions with time-reversal symmetry.
- This launched the new "topological insulator" revolution when an experimental realization was demonstrated.

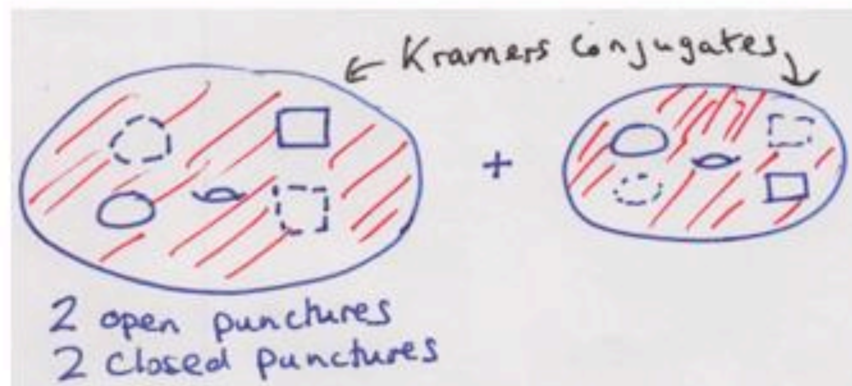
An explicitly gauge-invariant rederivation of the Z2 invariant

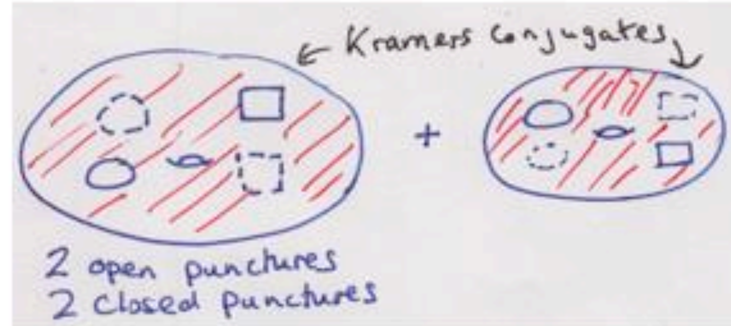
FDMH unpub.

- If inversion symmetry is absent, 2D bands with SOC split except at the four points where the Bloch vector is 1/2 x a reciprocal vector. The generic single genus-1 band becomes a pair of bands joined to form a genus-5 manifold



- This manifold can be cut into two Kramers conjugate parts, each is a torus with two pairs of matched punctures. In each pair, one puncture boundary is open one is closed.





- on a punctured 2-manifold

$$\exp i \int d^2\boldsymbol{k}\, \mathcal{F}^{12}(\boldsymbol{k}) = \prod_i e^{i\phi_i}$$

product of Berry phase-factors of puncture boundaries

- without punctures,

$$\int d^2\boldsymbol{k}\, \mathcal{F}^{12}(\boldsymbol{k}) = 2\pi C$$

- punctures come in Kramers pairs: $\displaystyle\prod_{i=1}^{2n} e^{i\phi_i} = \left(\prod_{i=1}^{n} e^{i\phi_i}\right)^2$

$$\boxed{\left(\exp i\frac{1}{2}\int d^2\boldsymbol{k}\, \mathcal{F}^{12}(\boldsymbol{k})\right)\prod_{i=1}^{n} e^{-i\phi_i} = \pm 1}$$

a perfect square, so we can take a square root!

- If inversion symmetry is present, the bands are unsplit and doubly-degenerate at all points in k-space, so the Berry curvature is undefined.

- Fu and Kane found a beautiful formula

$$\prod_{n} \prod_{k^*} I_{n,k^*} = \pm 1$$ = the Z_2 invariant

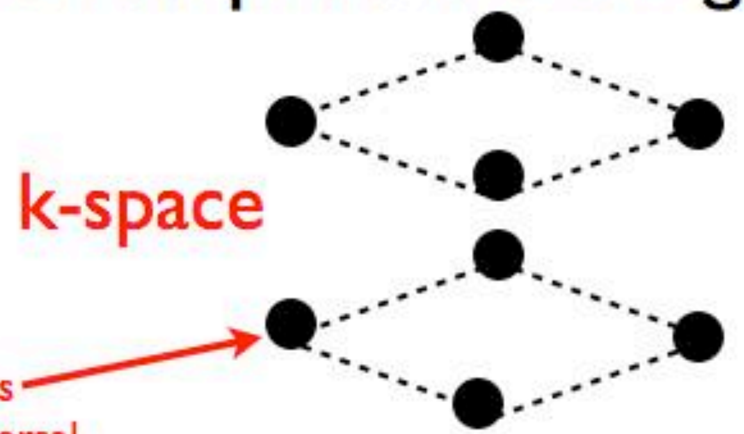
occupied bands

T+I-invariant k-points

Inversion quantum number ± 1 (about any inversion center)

A further surprise: the 3D generalization:

Moore and Balents
Roy
Fu and Kane



- In the 3D BZ there are 8 T-invariant points, which can be decomposed in many ways to two sets of 4 coplanar T-invariant points on two parallel T-invariant planes in the BZ
- For all such decompositions, the product $Z_1 Z_2$ has the same value, +1 or -1.
- Strong TI's have $Z_1 Z_2 = -1$. Weak TI's (+1) are stacks of 2D TI's

Strong TI's have an odd number of 2D Dirac points on any facet of a crystal

- Dirac points can only annihilate each other in pairs
- Disorder cannot destroy the metallic properties of the 2D surface state of a 3D strong TI, providing time-reversal symmetry remains unbroken. (Topological stability)

Geometry and Incompressibility in the Fractional Quantum Hall effect

F. D. M. Haldane, Princeton University

- The previously-overlooked collective degree of freedom in the FQHE is a dynamical internal geometry associated with incompressibility
- This geometry can be found in the Laughlin state (and other model states), but was hidden for the last 30 years because of misinterpretation of the meaning of the "Laughlin wavefunction"
- Leads to a quantitive description of FQHE incompressibility explaining earlier results of Girvin, MacDolnald and Platzman (1985)
- A new (2D) "guiding-center spin" characterizes FQH incompressibility

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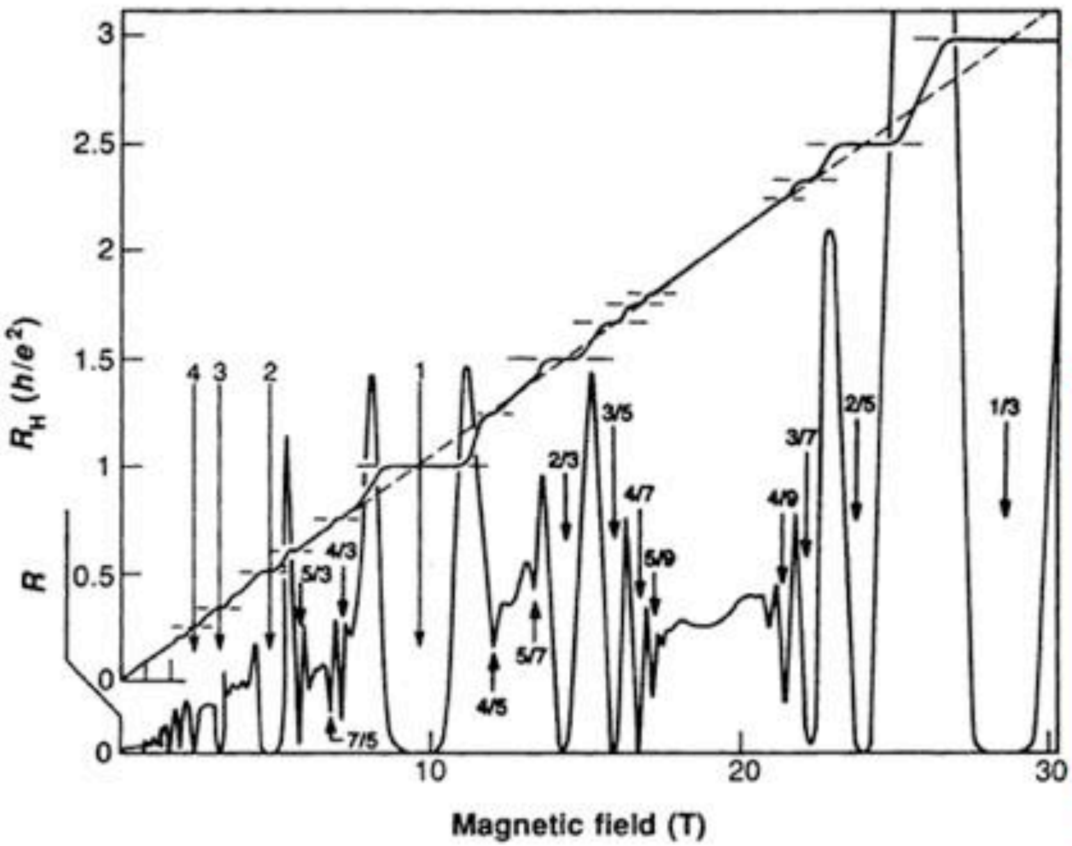
- Fractional quantum Hall effect in 2D electron gas in high magnetic field (filled Landau levels)

$$\Psi_L^{1/3} = \prod_{i<j}(z_i - z_j)^3 \prod_i e^{-|z_i|^2/4\ell_B^2}$$

$\nu = \frac{1}{3}$



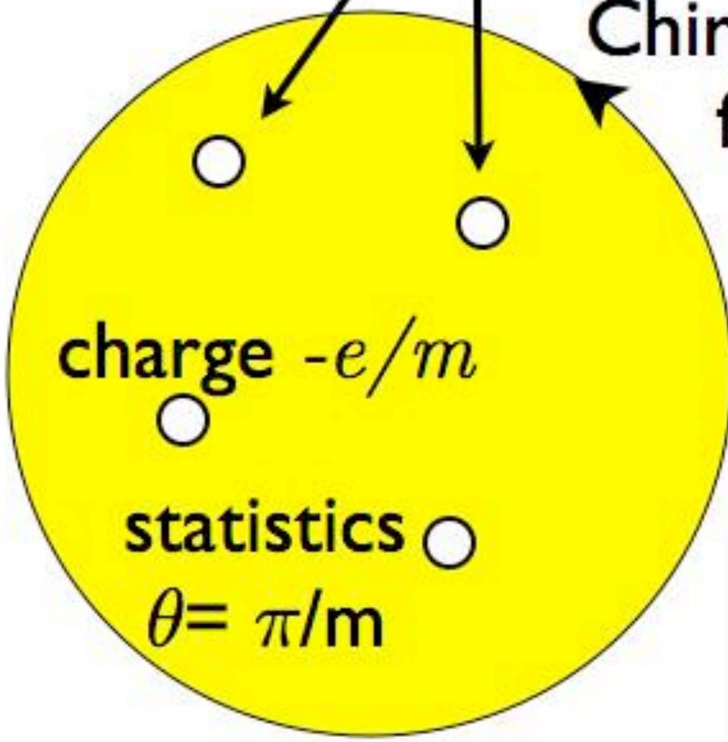
- Laughlin (1983) found the wavefunction that correctly describes the 1/3 FQHE , and got Nobel prize,

- Its known that it works, (tested by finite-size numerical diagonalization) but **WHY** it works has never really been satisfactorily explained!

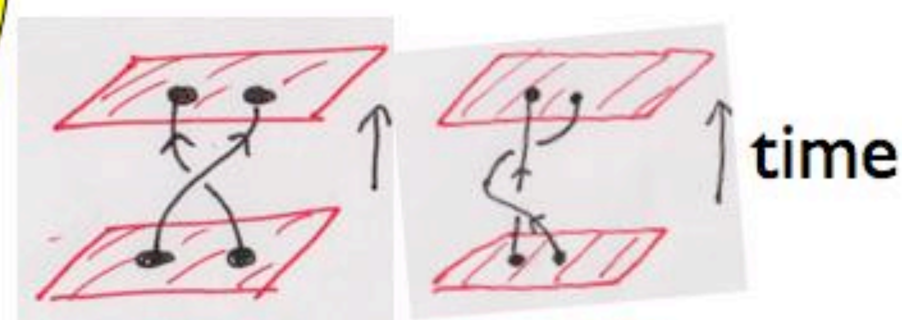
fractional-charge, fractional statistics vortices

$$\Psi = \prod_{i,\alpha}(z_i - w_\alpha)\prod_{i<j}(z_i - z_j)^m \prod_i e^{-\frac{1}{2}z_i^* z_j}$$

Chiral edge states at edge of finite droplet of fluid (Halperin, Wen)



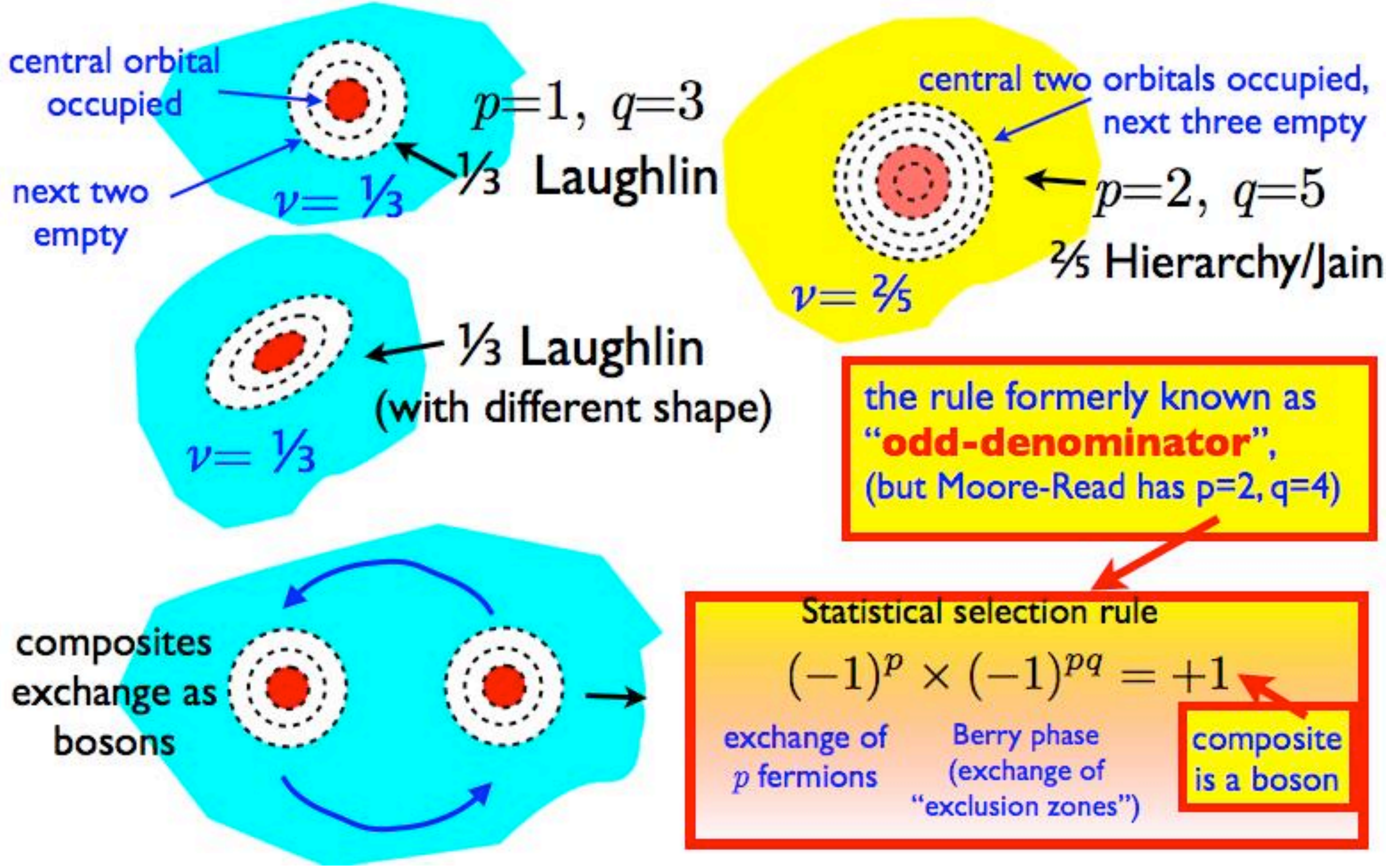
e.g.,
m=3



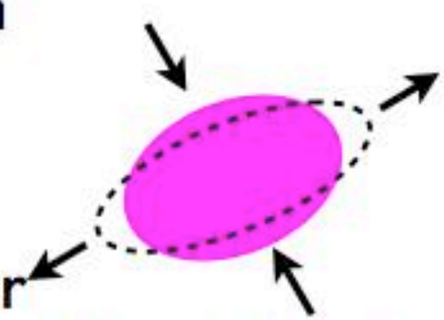
$e^{i\theta}$ $\quad$ $e^{2i\theta}$

- New feature is similar to FQH ferromagnet, where electrons couple to a combination of magnetic flux and Berry curvature of the ferromagnetic order parameter(Skyrmions)
- The electron density is no longer rigidly tied to the magnetic flux density, it can deviate from it at the expense of paying the correlation energy cost for geometric distortion.
- Old results of Girvin, Macdonald and Platzman (O(q^4) "guiding-center structure factor") get a simple explanation as zero-point fluctuations of the geometry

- elementary unit of the FQHE fluid with $\nu = p/q$ is a "**composite boson**" of p electrons that <u>**exclude**</u> other electrons from a region with q London (h/e) flux quanta



- The metric (shape of the composite boson) has a preferred shape that minimizes the correlation energy, but fluctuates around that shape
- The zero-point fluctuations of the metric are seen as the $O(q^4)$ behavior of the "guiding-center structure factor" (Girvin et al, (GMP), 1985)



$\delta E \propto (\text{distortion})^2$

- The metric has a companion "guiding center spin" that is topologically quantized in incompressible states.

configuration of "elementary droplet" (composite boson)

subtract total L (=L_{ref}) of reference configuration (uniform occupation p/q)

⅓ Laughlin

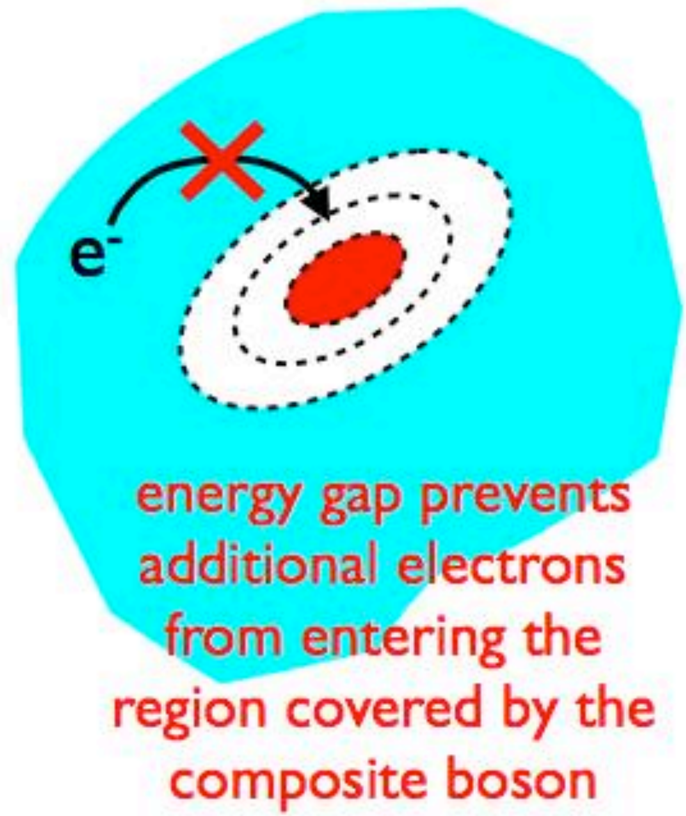
total L

$L=$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	total L
	1	0	0	$\frac{1}{2}$
	⅓	⅓	⅓	$\frac{3}{2}$

$$s = \left(\tfrac{1}{2} - \tfrac{3}{2}\right) = -1$$

- Origin of FQHE incompressibility is analogous to origin of **Mott-Hubbard gap** in lattice systems.
- There is an energy gap for putting an **extra particle** in a quantized region that is **already occupied**

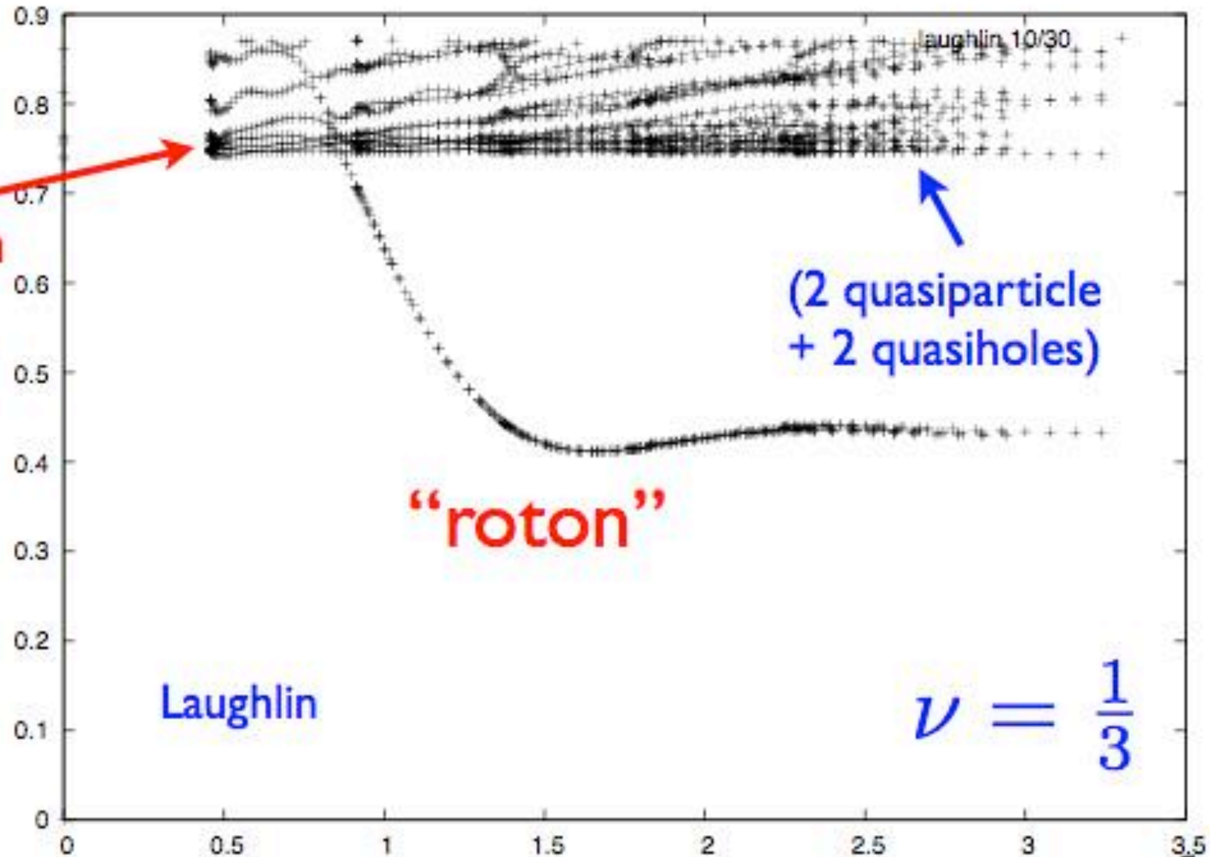
- **On the lattice** the "quantized region" is an atomic orbital with a fixed shape
- **In the FQHE** only the **area** of the "quantized region" is fixed. The **shape** must adjust to minimize the correlation energy.



unfortunately, long-wavelength limit of "graviton" collective mode is hidden in "two-roton continuum"

ΔE

goes into continuum



numerical finite-size diagonalization

$k\ell_B$

gap $\longleftrightarrow$ incompressibility

$$E(\boldsymbol{q})s(\boldsymbol{q}) \leq \tfrac{1}{2}G^{abcd}q_a q_b q_c q_d \ell_B^2.$$

- Previous theoretical treatments of FQHE have missed the fundamental geometrical degree of freedom associated with incompressibility
- Indeed, **no** viable theoretical explanation of FQHE incompressibility was previously found. By focussing on topological quantum field theory which ASSUMES the (unexplained) existence of incompressibility, most theorists have avoided the issue.
- In fact, the geometric degree of freedom can already be found in the Laughlin wavefunction, but deeply hidden during 30 years of its misinterpretation. HOW DID THIS HAPPEN?

The "Laughlin wavefunction"

- originally proposed as a "lowest-Landau-level Schrödinger wavefunction" to explain the 1/3 FQHE state

$$\boxed{\Psi_L^{1/q}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \prod_{i<j}(z_i - z_j)^q \prod_i e^{-\frac{1}{2}z_i^* z_i}}$$

$$z_i = \frac{x_i + iy_i}{\sqrt{2}\ell_B} \qquad \ell_B = \left(\frac{\hbar}{|eB|}\right)^{1/2}$$

"magnetic length"

- physical significance of z: n'th Landau orbit has (semiclassical) shape $|z|^2 = n + \frac{1}{2}$

- Landau level raising- and lowering-operators:

$$H = \frac{1}{2m}\left(\pi_x^2 + \pi_y^2\right)$$

$$\downarrow$$

$$H = \frac{1}{2}\hbar\omega_c\left(a^\dagger a + a a^\dagger\right)$$

$$\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{r})$$

$$[\pi_x, \pi_y] = i\hbar e B^z$$

$$a = \frac{(\pi_x + i\pi_y)\ell_B}{\sqrt{2\hbar}}$$

$$[a, a^\dagger] = 1$$

- Schrödinger representation:

$$a = \tfrac{1}{2}z + \partial_{z^*}$$

$$a^\dagger = \tfrac{1}{2}z^* - \partial_z$$

$$\partial_z f(z, z^*) \equiv \frac{\partial}{\partial z} f(z, z^*)\Big|_{z^*}$$

- holomorphic lowest Landau-Level wavefunctions:

$$a\Psi(z, z^*) = \left(\tfrac{1}{2}z + \partial_{z^*}\right)\Psi(z, z^*) = 0$$

- solution :

$$\boxed{\Psi(z, z^*) = f(z)e^{-\frac{1}{2}z^* z}}$$

single-particle LLL wavefunction

holomorphic function

- Physicists seem to have been "mesmerized" by the mathematical beauty of the LLL wavefunctions.........

- The Vandermonde determinant provides a natural "precursor" to the Laughlin state

$$\Psi = \prod_{i<j}(z_i - z_j)\prod_i e^{-\frac{1}{2}z_i^* z_i}$$ Filled N-particle LLL droplet

Laughlin wavefunction

Laughlin's "little" modification!

$$\Psi_L = \prod_{i<j}(z_i - z_j)^{q}\prod_i e^{-\frac{1}{2}z_i^* z_i}$$

- In 1983, while "mainstream" Condensed Matter Physicists (like me) were struggling with fancy second-quantized formalisms to try to understand the FQHE, Bob Laughlin arrived from on high carrying the solution......

- 25 years after BCS theory of superconductivity, which had consigned Schrödinger wavefunctions to the "dustbin of (Condensed Matter theory) history", they were back!

- **And** they were providing insight **unimaginable** in second-quantization/Feynman-diagram formulations!

- In retrospect, Laughlin's solution seemed so evidently correct and complete that it appears to have **frozen** the development of any understanding of the origin of FQHE incompressibility for almost 30 years.

- The Laughlin wavefunction for the strongest (1/3) FQHE state works, but why?

- In the attempt to "explain" **why** the Laughlin wavefunction works, theorists have been reduced to making nice-sounding pronouncements like

"The holomorphic wavefunction cleverly puts its zeroes on top of the other particles to lower the Coulomb energy"

Various attempts to explain FQHE incompressibility:

- Ginzburg-Landau superfluidity with a Higgs-like effect due to a Chern-Simons term
- Filling of "effective Landau levels" by "composite fermions"
- "Hamiltonian theory" of composite fermions
- non-commutative Chern-Simons theory with diffeomorphism invariance

- There is no doubt that **Chern-Simons** theories as Topological Quantum Field theories capture the essential **<u>topological</u>** features of FQHE liquids, but TQFT provides NO information about **<u>energy gaps</u>** and **<u>incompressibility</u>**: because ......

TQFT has an effective Hamiltonian: H=0!

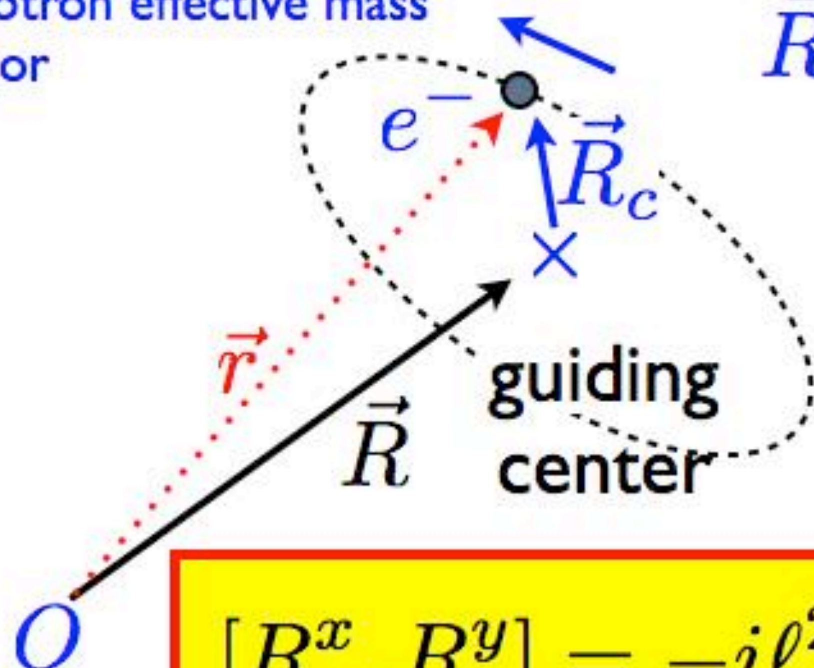
- None of these essentially "verbal" explanations has provided a viable quantitative theory of FQHE incompressibility.
- They attempt in various ways to provide an after-the-fact explanation of the success of the Laughlin wavefunction

Why all these attempts fail*, is because they did not incorporate what turns out to have been the **only** fundamental post-Laughlin result on FQHE incompressibility, the 1985 results on the "guiding-center structure factor" of Girvin, MacDonald, and Platzman

*The CF picture due to Jain provides good model wavefunctions and a description of the compressible Fermi-liquid-like state at $\nu = 1/2$ but no non-verbal explanation of incompressibility.

- Non-commutative geometry of Landau-orbit guiding centers

shape of orbit around guiding center is fixed by the cyclotron effective mass tensor



$\vec{r}$ displacement of electron from origin

$\vec{R}$ displacement of guiding center from origin

$\vec{R}_c$ displacement of electron relative to guiding center of Landau orbit

$$\vec{r} = \vec{R} + \vec{R}_c$$

$$[r^x, r^y] = 0$$

classical geometry

$$[R^x, R^y] = -i\ell_B^2$$

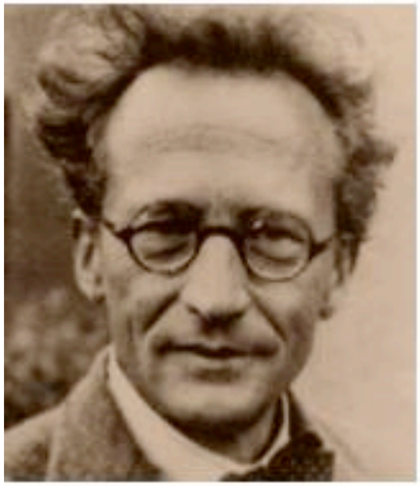
quantum geometry

$$[R_c^x, R_c^y] = +i\ell_B^2$$

Landau orbit (harmonic oscillator)

guiding centers commute with Landau radii

$$[R^a, R_c^b] = 0 \quad (a, b \in \{x, y\})$$



Schrödinger vs Heisenberg

- Schrödinger's picture describes the system by a **wavefunction** $\psi(r)$ in real space
- Heisenberg's picture describes the system by a **state** $|\,\psi\rangle$ in Hilbert space
- They are only equivalent if the basis $|\boldsymbol{r}\rangle$ of states in real-space are orthogonal:

$$\boxed{\psi(\boldsymbol{r}) = \langle \boldsymbol{r}|\psi\rangle}$$

requires

$$\boxed{\begin{array}{c}\langle \boldsymbol{r}|\boldsymbol{r}'\rangle = 0 \\ (\boldsymbol{r} \neq \boldsymbol{r}')\end{array}}$$

this fails in a quantum geometry

classical electron coordinate $\vec{r} = \vec{R} + \vec{R}_c$

The one-particle Hilbert-space factorizes

$$\mathscr{H} = \bar{\mathscr{H}}_{\rm GC} \otimes \mathscr{H}_c$$

space isomorphic to phase space in which the guiding-centers act

$$[R^x, R^y] = -i\ell_B^2$$

space isomorphic to phase space in which the Landau orbit radii act

$$[R_c^x, R_c^y] = +i\ell_B^2$$

- FQHE physics is *COMPLETELY* defined in the many-particle generalization (coproduct) of $\mathscr{H}_{\rm GC}$

Once $\mathscr{H}_{\rm C}$ is discarded, the Schrödinger picture is no longer valid!

Q: **When is a "wavefunction" NOT a wavefunction?**

A: **When it describes a "quantum geometry"**

- In this case space is "fuzzy"(non-commuting components of the coordinates), and the Schrödinger description in real space (i.e., in "classical geometry") fails, though the Heisenberg description in Hilbert space survives
- The closest description to the classical-geometry Schrödinger description is in a non-orthogonal overcomplete **coherent-state** basis of the quantum geometry.

Previous hints that the Laughlin "wavefunction" should not be interpreted as a wavefunction:

- Laughlin states also occur in the second Landau level, and in graphene, and more recently in simulations of "flat-band" Chern insulators

These don't fit into the original paradigm of the Galileian-invariant Landau level

- First, translate Laughlin to the Heisenberg picture:

$$a^\dagger = \tfrac{1}{2}z^* - \partial_z$$

$$a = \tfrac{1}{2}z + \partial_{z^*}$$

Landau-level ladder operators

$$z \leftrightarrow \bar{z}$$

$$\bar{a}^\dagger = \tfrac{1}{2}\bar{z}^* - \partial_{\bar{z}}$$

$$\bar{a} = \tfrac{1}{2}\bar{z} + \partial_{\bar{z}^*}$$

Guiding-center ladder operators

usual identification is

$$\bar{z} = z^*$$

Gaussian lowest-weight state

$$\psi_0(z, z^*) = e^{-\frac{1}{2}z^* z}$$

$$\bar{a}\psi_0(z, z^*) = 0$$

$$a\psi_0(z, z^*) = 0$$

action of guiding-center raising operators on LLL states

$$\bar{a}^\dagger = \tfrac{1}{2}z - \partial_{z^*} \qquad \bar{a} = \tfrac{1}{2}z^* + \partial_z$$

$$\bar{a}^\dagger f(z)\Psi_0(z, z^*) = z f(z)\Psi_0(z, z^*)$$

- Heisenberg form of Laughlin state (not "wavefunction")

$$\bar{a}_i|\bar{\Psi}_0\rangle = 0 \qquad\qquad \cancel{a_i|\Psi_0\rangle = 0}$$

$$|\Psi_L^{1/q}\rangle = \left(\prod_{i<j}(\bar{a}_i^\dagger - \bar{a}_j^\dagger)^q|\bar{\Psi}_0\rangle\right) \otimes \cancel{\left(|\Psi_0\rangle\right)}$$

$$\in \mathscr{H}_{\mathrm{GC}} \equiv \bar{\mathscr{H}} \qquad\qquad \cancel{\in \mathscr{H}_c \equiv \mathscr{H}}$$

Guiding-center factor (keep)

Landau-orbit factor (discard)

- At this point we discard the Landau-orbit Hilbert space.
- The only "memory" of the shape of the Landau orbits is "hidden" in the definition of $\bar{a}$

The "purified" Laughlin state

$$|\Psi_L^{1/q}\rangle = \prod_{i<j}(\bar{a}_i^\dagger - \bar{a}_j^\dagger)^q|\bar{\Psi}_0\rangle \qquad \bar{a}_i|\bar{\Psi}_0\rangle = 0$$

- This is now defined in the many-particle guiding-center Hilbert space, without reference to any Landau-level structure
- What defines $\bar{a}_i^\dagger$?

$$[L(g), \bar{a}_i^\dagger(g)] = \bar{a}_i^\dagger(g)$$

$$\boxed{L(g) = \frac{g_{ab}}{2\ell_B^2}\sum_i R_i^a R_i^b}$$

It is the raising operator for the "guiding-center spin" $L(g)$ of particle i

g_{ab} is a 2x2 positive-definite unimodular (det = 1) 2D spatial metric tensor

- The Laughlin state has suddenly revealed its well-kept secret- a hidden geometric degree of freedom! It is parameterized by a unimodular metric g_{ab}!

$$|\Psi_L^{1/q}(g)\rangle = \prod_{i<j}(\bar{a}_i^\dagger(g) - \bar{a}_j^\dagger(g))^q|\bar{\Psi}_0(g)\rangle$$

$$\bar{a}_i(g)|\bar{\Psi}_0(g)\rangle = 0$$

- In the naive LLL wavefunction picture, the unimodular metric g_{ab} is fixed to be proportional to the cyclotron effective mass tensor ${m^*}_{ab}$.
- In the reinterpretation it is a **free parameter**.

- As a variational parameter of the Laughlin state, g_{ab} must be chosen to minimize the correlation energy.
- Unless there is rotational symmetry around the normal to the 2D "Hall surface", it will not be congruent to the cyclotron effective mass tensor, and

$$\bar{z} \neq z^*$$

$$\begin{pmatrix} \bar{z} \\ \bar{z}^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} z^* \\ z \end{pmatrix}$$

$$\alpha^* \alpha - \beta^* \beta = 1$$

A Bogoliubov transformation

- If we reconstruct a wavefunction in the full Hilbert space, the naive statement about the zeroes of the wavefunction (as a function of any one particle coordinate) coinciding with the positions of the other particles is only true if

$$\bar{z} = z^*$$

- guiding-center Coherent states (single particle)

$$\bar{a}(g)|\Psi_g(0)\rangle = 0$$

$$|\Psi_g(\bar{z})\rangle = e^{\bar{z}a^\dagger(g) - \bar{z}^* a(g)}|\Psi_g(0)\rangle$$

- This is a non-orthogonal overcomplete basis

$$S(\bar{z}, \bar{z}') = \langle \Psi_g(\bar{z})|\Psi_g(\bar{z}')\rangle$$

- non-zero eigenvalues of the positive Hermitian overlap function are holomorphic!

$$\int \frac{d\bar{z}' d\bar{z}'^*}{2\pi} S(\bar{z}, \bar{z}')\Psi(\bar{z}', \bar{z}'^*) = \Psi(\bar{z}, \bar{z}^*)$$

$$\Psi(\bar{z}, \bar{z}^*) = f(\bar{z}^*)e^{-\frac{1}{2}\bar{z}^*\bar{z}}$$

- The “wavefunction” reappears as a guiding-center coherent-state representation of the Laughlin state!

$$|\Psi_L(g)\rangle = \prod_i \int \frac{d^2 \boldsymbol{r}_i}{2\pi\ell_B^2} \left(\prod_{i<j} (\bar{z}_i^* - \bar{z}_j^*)^q \prod_i e^{-\frac{1}{2}\bar{z}_i^* \bar{z}_i} \right) |\Psi_g(\{\boldsymbol{r}_i\})\rangle$$

same as “Laughlin wavefunction”, but with $z_i \to \bar{z}_i^*$

Many-particle coherent state, parametrized by g

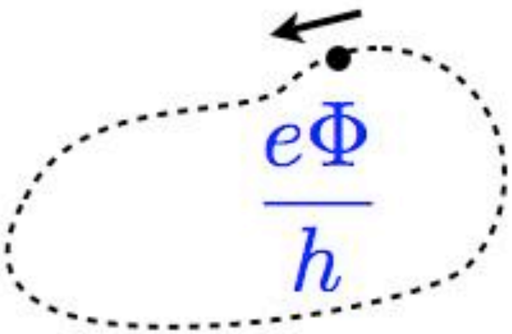
- The “wavefunction” has reappeared, but is not a wavefunction! This will apply in general to model states (Moore-Read, etc,obtained as CFT correlators)

Quantum Geometry

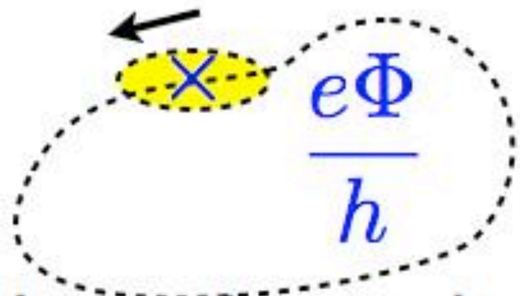
- A much more powerful result follows from the realization that the metric g_{ab} is not just a "variational parameter of the Laughlin state" but the fundamental dynamical degree of freedom of the FQHE.
- Once the connection to quantum geometry is made many things follow.

“Bohm-Aharonov becomes Berry”

- How does a “fuzzy electron” experience a Bohm-Aharonov phase?



point particle going around a loop picks up the Bohm-Aharonov phase

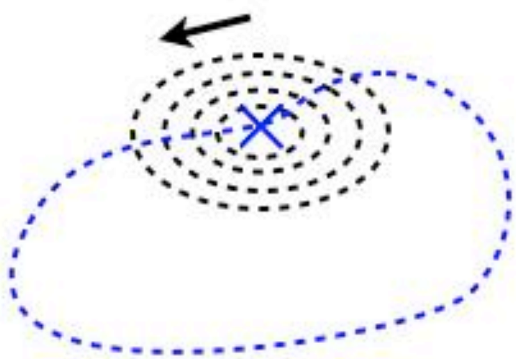


“fuzzy” charged particle is transported as a guiding-center coherent state with a center that traces out a closed path

Berry phase = BA phase of path

- The state that is transported doesn't have to be the central coherent state of the basis

$$L(g)|\Psi_{g,m}(0)\rangle = s_m|\Psi_{g,m}(0)\rangle$$
$$s_m = m + \tfrac{1}{2}, \quad m = 0, 1, 2, \ldots$$



Provided the coherent-state metric g_{ab} does not change during the adiabatic transportation process, the Berry phase equals the BA phase of the path, independent of m

If the metric does vary, the Berry phase turns out to be

$$\frac{e\Phi_B}{\hbar} + s_m \times (\text{geodesic curvature of metric } g_{ab} \text{ around path})$$

Which leads directly to the guiding-center spin coupling to Gaussian curvature!

- Geometric action

(after Chern-Simons fields are integrated over)

electromagnetic gauge potentials

spin connection of metric

$$S = \int d^3x \mathcal{L}_0 - \mathcal{H}_0$$

$$\mathcal{L}_0 = \frac{1}{4\pi pq\hbar}\epsilon^{\mu\nu\lambda}\left(peA_\mu - s\Omega^g_\mu\right)\partial_\nu\left(peA_\lambda - \hbar s\Omega^g_\lambda\right)$$

(reduces to electromagnetic Chern-Simons action when s = 0 (integer QHE))

$$\mathcal{H}_0 = J^0 U(J^0 g) \qquad J^0 = \frac{1}{2\pi pq\hbar}\left(peB - \hbar s J^0_g\right)$$

energy function

Gaussian curvature

$$J^\mu_g = \epsilon^{\mu\nu\lambda}\partial_\nu\Omega^g_\lambda$$

correlation energy density

composite-boson density

Geometric distortion energy

correlation energy density $$\mathcal{H}_0 = (\det G)^{1/2} U(G) = J^0 U(J^0 g)$$

geometric chemical potential (of composite bosons)

$$\boxed{\mu_g = U(G) + G_{ab}\frac{\partial U}{\partial G_{ab}}}$$

shear-stress tensor (traceless)

$$\boxed{\sigma^a_b = 2G_{bc}\frac{\partial U}{\partial G_{ac}} - \delta^a_b G_{cd}\frac{\partial U}{\partial G_{cd}}}$$

$$\boxed{\sigma^a_a = 0}$$

$$\sigma^a_c(x)\epsilon^{bc} = \sigma^{bc}(x)\epsilon^{ac}$$

$$\sigma^a_c(x)g^{bc}(x) = \sigma^{bc}(x)g^{ac}(x)$$

both expressions are symmetric in $a \leftrightarrow b$

Stress tensor is traceless because the gapped quantum incompressible fluid does not transmit pressure

(unlike incompressible limit of classical incompressible fluid, which has speed of sound $v_s \to \infty$

Euler equation

- action is minimized by Hall viscosity condition

$$\boxed{J^0\sigma^a_b(G) = \eta^{ac}_{bd}(G)\nabla^g_c J^d}$$

composite boson current

Traceless stress-tensor

covariant spatial gradient of $J^a = J^0 v^a$

fluid flow-velocity

Hall viscosity

$$\eta^{ac}_{bd}(G) = \tfrac{1}{2}\hbar s \epsilon_{be}\epsilon_{df} J^0 \Gamma^{aecf}_H(g)$$

$$\Gamma^{abcd}_H(g) = \tfrac{1}{2}\left(\epsilon^{ac}g^{bd} + \epsilon^{ad}g^{bc} + \epsilon^{bc}g^{ad} + \epsilon^{bd}g^{ac}\right)$$

$\boxed{\eta^{ac}_{bd} = -\eta^{ca}_{db}}$ dissipationless

$\boxed{\eta^{ab}_{ac} = \eta^{ba}_{ca} = 0}$ incompressible

- composite boson current

Gaussian curvature density and current

$$J^0 = \frac{1}{2\pi pq\hbar}\left(\epsilon^{ab} peB - \hbar s J_g^0\right)$$

$$J^a = \frac{1}{2\pi pq\hbar}\left(\epsilon^{ab}(peE_b - \partial_b \mu_g) - \hbar s J_g^a\right)$$

responds to gradient of geometric chemical potential as well as electric field

$$peE_a J^a \neq 0$$

Energy flow from electromagnetic field to FQH fluid

$$pe(J^0 E_a + \epsilon_{ab} J^a B) \neq 0$$

tangential momentum flow from electromagnetic field to FQH fluid

- Action gives gapped spin-2 (graviton-like) collective mode that coincides at long wavelengths with the "single-mode approximation" of Girvin-MacDonald and Platzman.

- charge fluctuations relative to the back-ground charge density fixed by the magnetic flux are given by the Gaussian curvature

$$J_g^0 = -\frac{1}{2}\partial_a\partial_b g^{ab} + \frac{1}{8}g_{ac}\epsilon_{bd}\epsilon^{ef}\left(\partial_e g^{ab}\right)\left(\partial_f g^{cd}\right)$$

$$\delta J_e^0 = \frac{e^* s}{2\pi} J_g^0$$

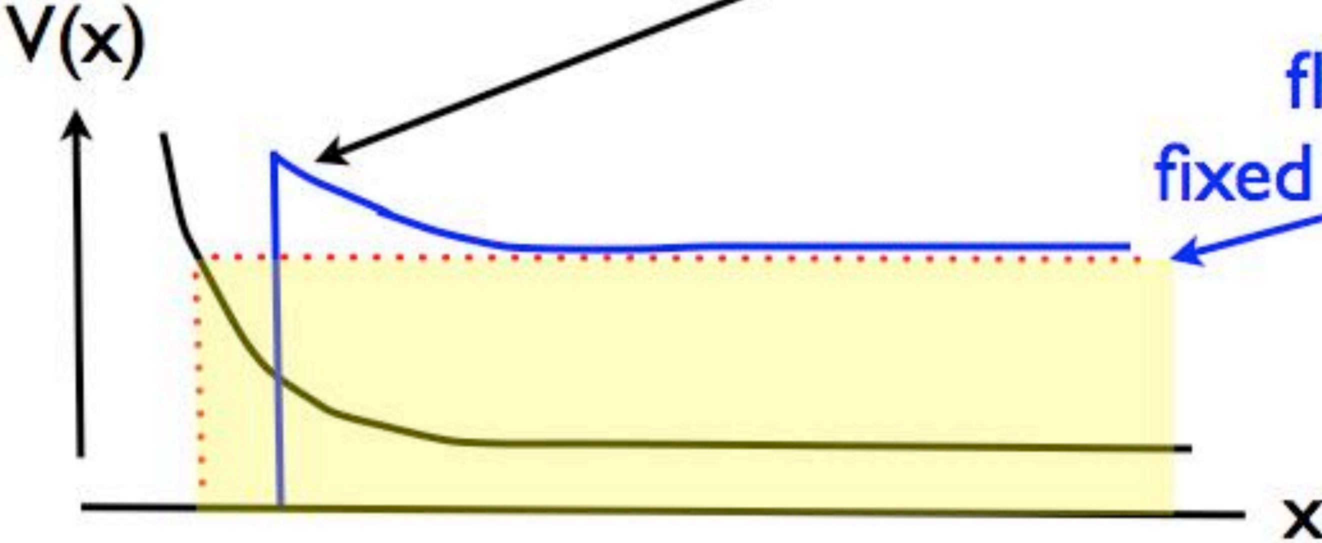
second derivative of metric

zero-point fluctuations of gaussian curvature give quantitatively correct $O(q^4)$ structure factor

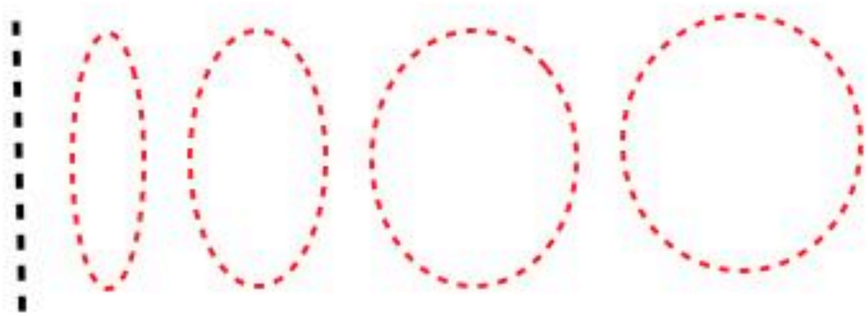
- near edges:

fluid is compressed at edges
by creating Gaussian
curvature

$$\delta J^0_e = \frac{e^* s}{2\pi} J^0_g$$



fluid density
fixed by flux density



$$g = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \frac{1}{\alpha(x)} \end{pmatrix}$$

$$J^0_g = -\frac{1}{2}\frac{d^2}{dx^2}\frac{1}{\alpha(x)}$$

For larger s, fluid becomes more compressible
(less distortion needed for a given density change)

SUMMARY

- New collective geometric degree of freedom leads to a description of the origin of incompressibility in FQHE in a continuum "geometric field theory"
- many new relations: guiding-center spin characterizes coupling to Gaussian curvature of intrinsic metric, stress in fluid, guiding-center structure-factors, etc.

http://wwwphy.princeton.edu/~haldane

Can be also be accessed through Princeton University Physics Dept home page (look for Research:condensed matter theory)

also see arXiv (search for author=haldane)