

Announcements

- Oct. 28: midterm exam
 - starts at 14:20
 - [Optional] bring a A4 cheat sheet (one side only)
- Midterm 2021 is available on moodle.

Solving linear systems III

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Review

- ***Direct*** methods for solving linear systems
 - Gaussian elimination with *partial pivoting*
 - The $PA = LU$ factorization

Today

- ***Iterative*** methods for solving linear systems
 - Jacobi method
 - Gauss-Seidel method

Solving $\underline{A}\underline{x} = \underline{b}$

- A is given, real, $n \times n$, and \underline{b} is given, real vector.
- Two types of approaches
 - Direct methods: yield exact solution in absence of roundoff error
Example: Gaussian elimination and its variants
 - **Iterative** methods: iterate in a similar fashion to what we do for solving nonlinear equations

Iterative methods for solving $\underline{Ax} = \underline{b}$

- Starting from initial guess \underline{x}_0 , generate iterates $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_k$, hopefully converging to solution \underline{x} .
- But why not simply using a direct method?



Why using iterative methods?

- Can be faster if the input matrix is large
 - One step of an iterative method requires only a fraction of the floating operations of a full LU factorization.
- A good approximation to the solution is already known.
- The input matrix is *sparse*.

Direct vs. iterative linear solvers

- Direct solvers
 - Computation is numerically stable in many relevant cases. 😊
 - Can solve economically for several right-hand sides. 😊
 - But: fill-in limits usefulness (memory). ☹️
- Iterative solvers
 - Only a rough approximation to \underline{x} is required. 😊
 - A good \underline{x}_0 approximating \underline{x} is generally known (warm start). 😊
 - Quality may depend on “right” choice of parameters. ☹️

Typical scenarios

- **Direct** solvers
 - Many linear systems with the same matrix A
 - Applications that require very accurate solutions
- **Iterative** solvers
 - Many linear systems with “slightly changing” matrices
 - Matrix-free applications
 - Very large problems

Iterative methods

- Jacobi method
- Gauss-Seidel method

Jacobi method

- A form of fixed-point iteration
- Example: Solve the linear system

$$\begin{aligned} 3u + v &= 5 \\ u + 2v &= 5 \end{aligned}$$
- The answer is $u = 1, v = 2$.
- First, we have

$$u = \frac{5 - v}{3}$$

$$v = \frac{5 - u}{2}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/2}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-5/6}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{12} \end{bmatrix}$$

⋮

- Start from an initial guess $(0, 0)$

[continue](#)

Jacobi method

- Use the example again, but now the equations are given in the *reverse* order.

$$u + 2v = 5 \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3u + v = 5$$

- The answer is $u = 1, v =$ $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 5 - 2v_0 \\ 5 - 3u_0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

- We have

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5 - 2v_1 \\ 5 - 3u_1 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \end{bmatrix}$$

$$u = 5 - 2v$$

$$v = 5 - 3u$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 - 2(-10) \\ 5 - 3(-5) \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \end{bmatrix}$$

- Start from an initial guess (0, 0)

⋮

The difference?

- $3u + v = 5$
- $u + 2v = 5$

VS.

- $u + 2v = 5$
- $3u + v = 5$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

The Jacobi method converges. 😊

The Jacobi method diverges. ☹️

Strictly diagonally dominant

- **Definition.** The $n \times n$ matrix $A = (a_{ij})$ is **strictly diagonally dominant** if, for each $1 \leq i \leq n$,
 $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$.
- Which matrix is strictly diagonally dominant?

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 8 & 1 \\ 9 & 2 & -2 \end{bmatrix}$$



Theorem. If the $n \times n$ matrix A is **strictly diagonally dominant**, then for every vector \underline{b} and every starting guess, the Jacobi method applied to $A\underline{x} = \underline{b}$ converges to the solution.

The previous example

- $3u + v = 5$
- $u + 2v = 5$

VS.

- $u + 2v = 5$
- $3u + v = 5$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

strictly diagonally dominant

The Jacobi method converges. 😊

NOT strictly diagonally dominant

The Jacobi method diverges. ☹️

A form of fixed-point iteration $\underline{x} = g(\underline{x})$

- Start with $A\underline{x} = \underline{b}$

$$(D + L + U)\underline{x} = \underline{b}$$

$$D\underline{x} = \underline{b} - (L + U)\underline{x}$$

$$\underline{x} = D^{-1}(\underline{b} - (L + U)\underline{x})$$

$$\underbrace{\quad}_{g(\underline{x})}$$

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix}$$

L
D
U

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 6 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Jacobi Method

x_0 = initial vector

$$x_{k+1} = D^{-1}(b - (L + U)x_k) \text{ for } k = 0, 1, 2, \dots$$

Example: Jacobi method

- Jacobi method

x_0 = initial vector

$$x_{k+1} = D^{-1}(b - (L + U)x_k) \text{ for } k = 0, 1, 2, \dots$$

- The previous example

$$-3u + v = 5$$

$$-u + 2v = 5$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} \right) = \begin{bmatrix} (5 - v_k)/3 \\ (5 - u_k)/2 \end{bmatrix}$$

$$\mathbf{D}^{-1} \quad \underline{\mathbf{b}} \quad \mathbf{L+U} \quad \underline{\mathbf{x}}_k$$

check

Iterative methods

- Jacobi method
- Gauss-Seidel method

Gauss-Seidel method

- Uses **the most recently updated** values of the knowns, even if the updating occurs in the current step!

• The previous example

$$\begin{aligned} 3u + v &= 5 \\ u + 2v &= 5 \end{aligned}$$

$$u = \frac{5 - v}{3}$$

$$v = \frac{5 - u}{2}$$

$$\begin{aligned} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} &= \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \end{bmatrix} \\ \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} &= \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-10/9}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{35}{18} \end{bmatrix} \\ \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} &= \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-35/18}{3} \\ \frac{5-55/54}{2} \end{bmatrix} = \begin{bmatrix} \frac{55}{54} \\ \frac{215}{108} \end{bmatrix} \\ &\vdots \end{aligned}$$

A form of fixed-point iteration $\underline{x} = g(\underline{x})$

- Start with $A\underline{x} = \underline{b}$

$$(L+D+U)\underline{x} = \underline{b}$$

$$(L+D)\underline{x}_{k+1} = \underline{b} - U\underline{x}_k$$

$$D\underline{x}_{k+1} = \underline{b} - U\underline{x}_k - L\underline{x}_{k+1}$$

$$\underline{x}_{k+1} = \underbrace{D^{-1}(\underline{b} - U\underline{x}_k - L\underline{x}_{k+1})}_{g(\underline{x})}$$

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix}$$

L
D
U

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 6 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Gauss-Seidel Method

x_0 = initial vector

$$x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1}) \text{ for } k = 0, 1, 2, \dots$$

- Gauss-Seidel Method

x_0 = initial vector

$$x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1}) \text{ for } k = 0, 1, 2, \dots$$

- Example

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_{k+1} \\ v_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \left(\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} u_{k+1} \\ v_{k+1} \\ w_{k+1} \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \left(\begin{bmatrix} 4 - v_k + w_k \\ 1 - w_k - 2u_{k+1} \\ 1 + u_{k+1} - 2v_{k+1} \end{bmatrix} \right)$$

$$3u + v - w = 4 \Rightarrow u = \frac{1}{3}(4 - v + w) \quad -u + 2v + 5w = 1 \Rightarrow w = \frac{1}{5}(1 + u - 2v)$$

$$2u + 4v + w = 1 \Rightarrow v = \frac{1}{4}(1 - w - 2u)$$

Jacobi vs. Gauss-Seidel

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/2}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-5/6}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{12} \end{bmatrix}$$

$$\underline{x}_{k+1} = D^{-1}(\underline{b} - (L+U)\underline{x}_k)$$

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-10/9}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{35}{18} \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-35/18}{3} \\ \frac{5-55/54}{2} \end{bmatrix} = \begin{bmatrix} \frac{55}{54} \\ \frac{215}{108} \end{bmatrix}$$

$$\underline{x}_{k+1} = D^{-1}(\underline{b} - U\underline{x}_k - L\underline{x}_{k+1})$$

Next week

- Solving nonlinear systems
- Will use your code that solves a linear system in the next assignment!

程式練習

And, please upload your program on moodle.

- Please use Jacobi or Gauss-Seidel to solve the following system of 10 equations in 10 unknowns:

$$a_{ii} = -4$$

$$a_{ij} = 2 \text{ if } |i - j| = 1$$

$$a_{ij} = 0 \text{ if } |i - j| \geq 2 \text{ where } i, j = 1, 2, \dots, 10$$

$$\underline{b}^T = [2 \ 3 \ 4 \ \dots \ 11]$$