Interpolation – Part 2

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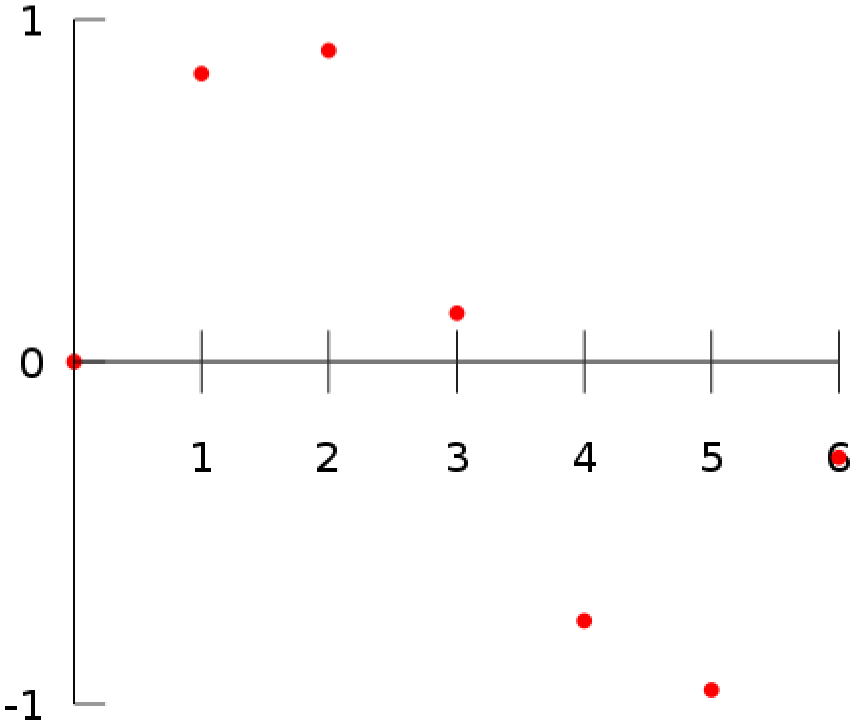
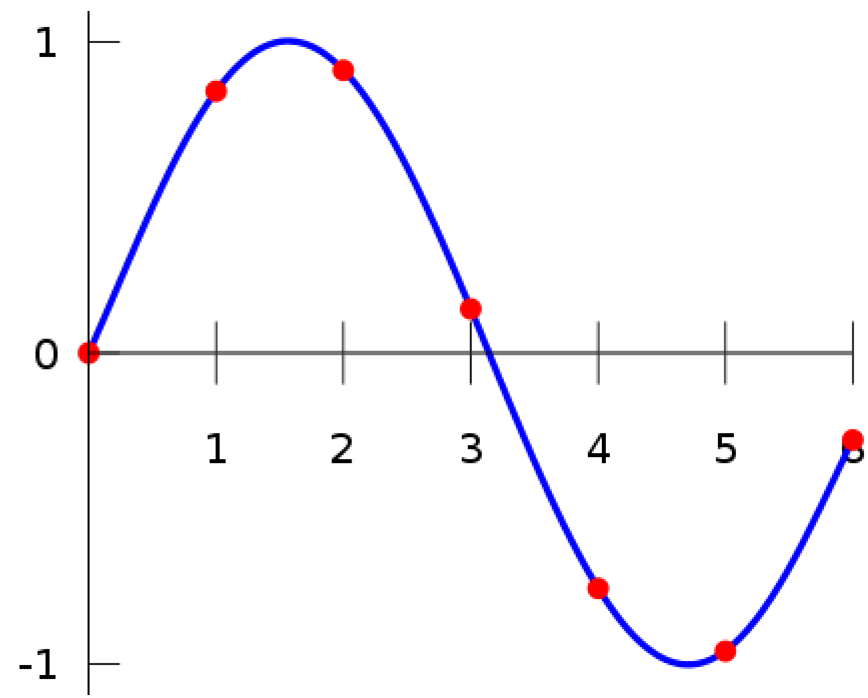
Review: Interpolating data

- Given a collection of data samples $\{(x_i, y_i)\}_{i=1}^{n}$, we want to find a function $P(x)$ which can be used to estimate samples for any $x \neq x_i$.





Review: Lagrange interpolation

- $P(x)$: a polynomial in this form:

$$\boxed{P(x) = y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x)}$$

$$L_k(x) = \frac{(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

- For example, given 3 points:

$$P_2(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Review: Lagrange interpolation

- $P(x)$: a polynomial in this form:

$$\boxed{P(x) = y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x)}$$

$$L_k(x) = \frac{(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

- The degree of $P(x)$ is ? at most $n - 1$

$$\boxed{\begin{aligned} P(x) &= y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x) \\ &= a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \end{aligned}}$$

Theorem. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be n points on a plane with distinct x_i. Then there exists **one and only one** polynomial P of degree $n-1$ or less that satisfies $P(x_i) = y_i$ for $i = 1, \ldots, n$.

Today

- Interpolation error
- Runge phenomenon
- Newton’s divided differences

Interpolation error

- Assume we have $y_i = f(x_i)$, $i = 0, 1, \ldots, n$ and an interpolating polynomial $P(x)$. The interpolation error at x is

$$f(x) - P(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{n!} f^{(n)}(c)$$

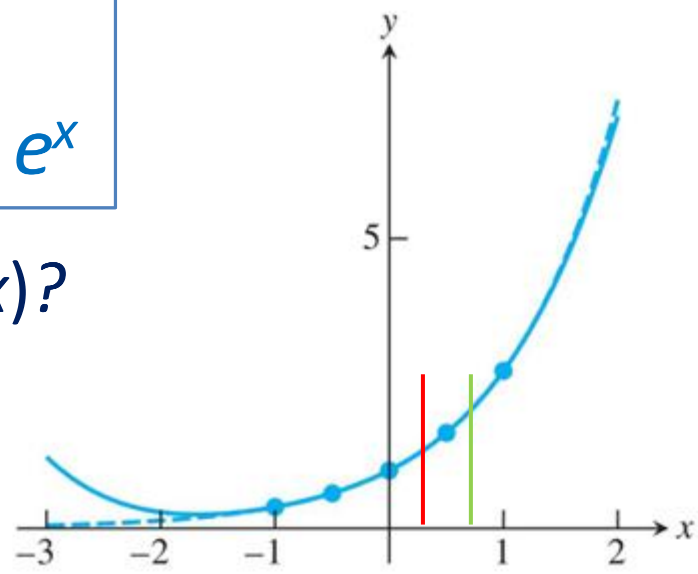
c lies between the smallest and largest of x_i.

Example

- Find an upper bound for the difference at $x = 0.25$ and $x = 0.75$ between $f(x) = e^x$ and the polynomial that interpolates it at the points -1, -0.5, 0, 0.5, 1.

solid: $P(x)$
dash: $f(x) = e^x$

degree of $P(x)$?

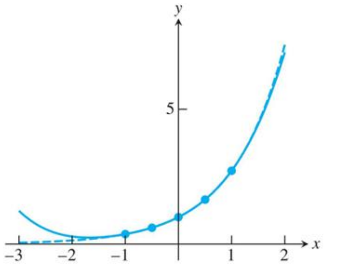


x_i = -1, -0.5, 0, 0.5, 1,
$f(x) = e^x \rightarrow f'(x) = e^x$

$$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(c)$$

$$= \frac{(x+1)\left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right)(x-1)}{5!} f^{(5)}(c)$$

- Range of c?   −1 to 1
- Maximal value of $|f^{(5)}(c)|$ on [-1, 1]? e^1



$$|\, f(x) - P(x)| \leq \frac{(x+1)\left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right)(x-1)}{5!} e$$

$$| f(x) - P(x) | \leq \frac{(x+1)\left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right)(x-1)}{5!}e$$

- At $x = 0.25$

$$| f(x) - P(x) | \leq \frac{(1.25)(0.75)0.25(-0.25)(-0.75)}{5!}e$$

$$\approx 0.000995$$

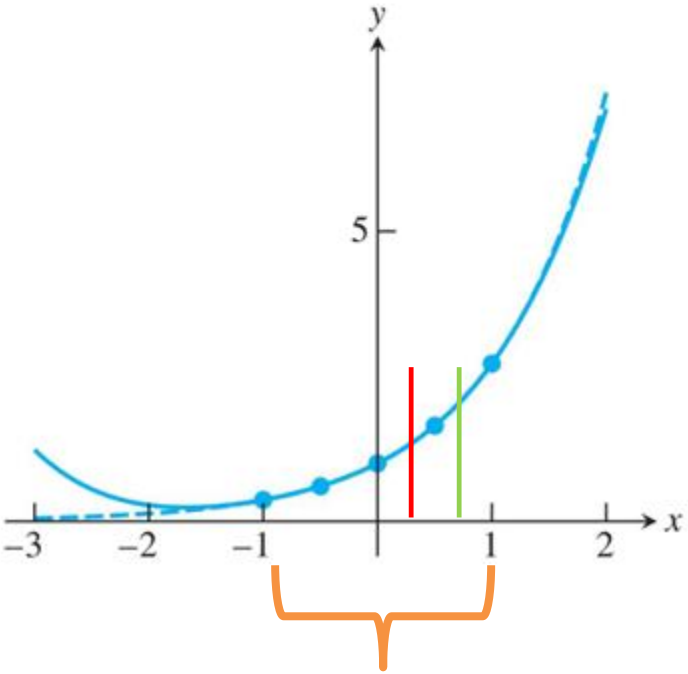
- At $x = 0.75$

$$| f(x) - P(x) | \leq \frac{(1.75)(1.25)0.75(0.25)(-0.25)}{5!}e$$

$$\approx 0.002323$$

- error = 0.000995 at x = 0.25
- error = 0.002323 at x = 0.75

The interpolation error will be smaller close to the center of the interpolation interval.



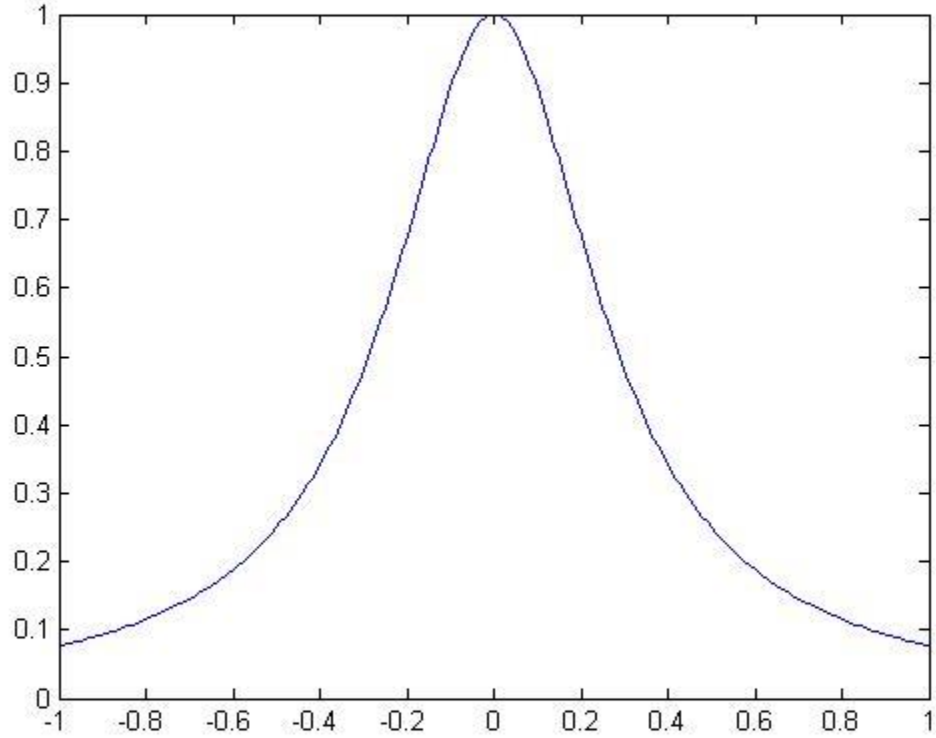
Today

- Interpolation error
- Runge phenomenon
- Newton's divided differences

Runge phenomenon

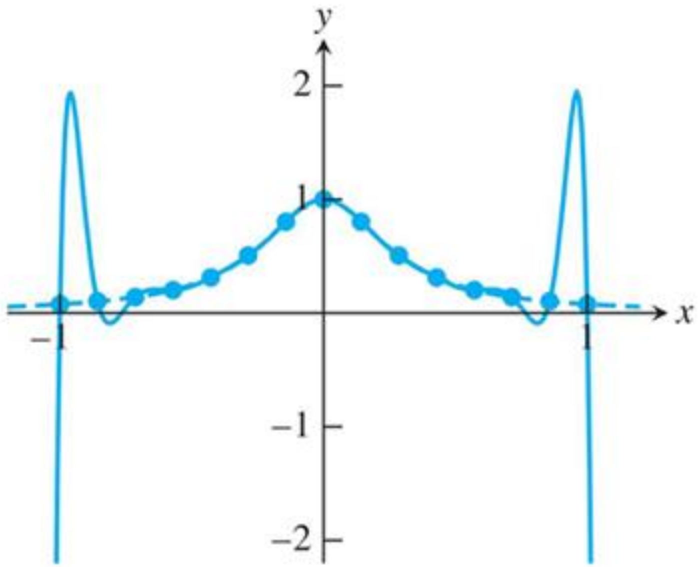
- Polynomials can fit any set of data points.
- But, there are some shapes that polynomials prefer over others.
- Example
  - Interpolate $f(x) = 1/(1+12x^2)$ at evenly spaced points in [-1, 1].
  - The shape $f(x)$?

$f(x) = 1/(1+12x^2)$

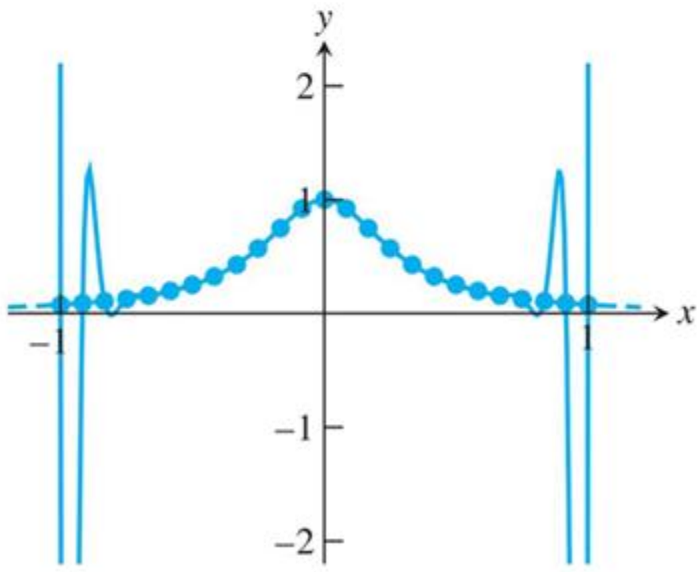


Polynomial interpolation of *f*(*x*)

15 base points



25 base points



Runge phenomenon: polynomial wiggle near the ends of the interpolation interval.

Today

- Interpolation error
- Runge phenomenon
- Newton’s divided differences

Newton's divided differences

- $P(x)$: a polynomial in this form: (given n points)

$$\boxed{\begin{gathered} P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) \\ + \cdots + c_{n-1}(x - x_1)\cdots(x - x_{n-1}) \end{gathered}}$$

- Example, given 3 points:

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2)$$

- Degree of P? at most $n - 1$
- How to compute c_i?

Computing c_i

- $P(x)$: a polynomial in this form:

$$\boxed{\begin{gathered} P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) \\ + \cdots + c_{n-1}(x - x_1)\cdots(x - x_{n-1}) \end{gathered}}$$

- $P(x_1) = c_0 = y_1$
- $P(x_2) = c_0 + c_1(x_2 - x_1) = y_2 \quad \Rightarrow \quad c_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$
- $P(x_3) = c_0 + c_1(x_3 - x_1) + c_2(x_3 - x_1)(x_3 - x_2) = y_3$

$$\Rightarrow \quad c_2 = \ ?$$

“Divided Difference”

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) \\ + \cdots + c_{n-1}(x - x_1)\cdots(x - x_{n-1})$$

$$\Vert$$

$$\begin{aligned} P(x) = \overset{c_0}{f[x_1]} &+ \overset{c_1}{f[x_1\ x_2]}(x - x_1) \\ &+ \overset{c_2}{f[x_1\ x_2\ x_3]}(x - x_1)(x - x_2) \\ &+ \overset{c_3}{f[x_1\ x_2\ x_3\ x_4]}(x - x_1)(x - x_2)(x - x_3) \\ &+ \cdots \\ &+ \overset{c_{n-1}}{f[x_1 \cdots x_n]}(x - x_1)\cdots(x - x_{n-1}). \end{aligned}$$

Denoted by $f[x_1 \ldots x_n]$ the coefficient of the x^{n-1} term

$f[x_1 \ldots x_n] \equiv c_{n-1}$

Newton's divided differences

- 0-th divided difference

$$f[x_i] = P(x_i) = y_i$$

$$\boxed{f[x_1] = y_1}$$

- 1-st divided difference

$$f[x_i\ x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$\boxed{f[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}}$$

- *k*-th divided difference

$$f[x_i \cdots x_{i+k}] = \frac{f[x_{i+1} \cdots x_{i+k}] - f[x_i \cdots x_{i+k-1}]}{x_{i+k} - x_i}$$

For example,

- The 0-th divided difference is $f[x_1] = P(x_1) = y_1$
- The 1-st divided difference is

$$f[x_1\ x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

- The 2-nd divided difference is

$$f[x_1\ x_2\ x_3] = \frac{f[x_2\ x_3] - f[x_1\ x_2]}{x_3 - x_1}$$

- The 3-rd divided difference is

$$f[x_1\ x_2\ x_3\ x_4] = \frac{f[x_2\ x_3\ x_4] - f[x_1\ x_2\ x_3]}{x_4 - x_1}$$

Computing the divided differences

- The divided differences are defined ***recursively***; they are ratios of differences of previously computed ratios.

Newton's divided differences

- The divided differences are best arranged in a triangular array:

$$
\begin{array}{c|llll}
x_1 & f[x_1] & & & \\
 & \quad y_1 & f[x_1\, x_2] & & \\
x_2 & f[x_2] & & f[x_1\, x_2\, x_3] & \\
 & \quad y_2 & f[x_2\, x_3] & & f[x_1\, x_2\, x_3\, x_4] \\
x_3 & f[x_3] & & f[x_2\, x_3\, x_4] & \\
 & \quad y_3 & f[x_3\, x_4] & & \\
x_4 & f[x_4] & & & \\
 & \quad y_4 & & &
\end{array}
$$

$$
\begin{aligned}
P(x) = f[x_1] &+ f[x_1\ x_2](x - x_1) \\
&+ f[x_1\ x_2\ x_3](x - x_1)(x - x_2) \\
&+ f[x_1\ x_2\ x_3\ x_4](x - x_1)(x - x_2)(x - x_3) \\
&+ \cdots \\
&+ f[x_1 \cdots x_n](x - x_1)\cdots(x - x_{n-1}).
\end{aligned}
$$

Newton's divided differences

- The divided differences are best arranged in a triangular array:

x_1	$f[x_1]$			
	y_1	$\frac{f[x_2]-f[x_1]}{x_2-x_1}$		
x_2	$f[x_2]$		$\frac{f[x_2\ x_3]-f[x_1\ x_2]}{x_3-x_1}$	
	y_2	$\frac{f[x_3]-f[x_2]}{x_3-x_2}$		$\frac{f[x_2\ x_3\ x_4]-f[x_1\ x_2\ x_3]}{x_4-x_1}$
x_3	$f[x_3]$		$\frac{f[x_3\ x_4]-f[x_2\ x_3]}{x_4-x_2}$	
	y_3	$\frac{f[x_4]-f[x_3]}{x_4-x_3}$		
x_4	$f[x_4]$			
	y_4			

$$
\boxed{\begin{aligned}
P(x) = f[x_1] &+ f[x_1\ x_2](x-x_1) \\
&+ f[x_1\ x_2\ x_3](x-x_1)(x-x_2) \\
&+ f[x_1\ x_2\ x_3\ x_4](x-x_1)(x-x_2)(x-x_3) \\
&+ \cdots \\
&+ f[x_1 \cdots x_n](x-x_1)\cdots(x-x_{n-1}).
\end{aligned}}
$$

Example 1

- Use Newton's divided differences to find the interpolating polynomial passing through (0,1), (2,2), (3,4).

x	y		
0	1		
		$\dfrac{2-1}{2-0} = \dfrac{1}{2}$	
2	2		$\dfrac{2-\frac{1}{2}}{3-0} = \dfrac{1}{2}$
		$\dfrac{4-2}{3-2} = 2$	
3	4		

$$P(x) = 1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2)$$
$$= \frac{1}{2}x^2 - \frac{1}{2}x + 1$$

Example 2

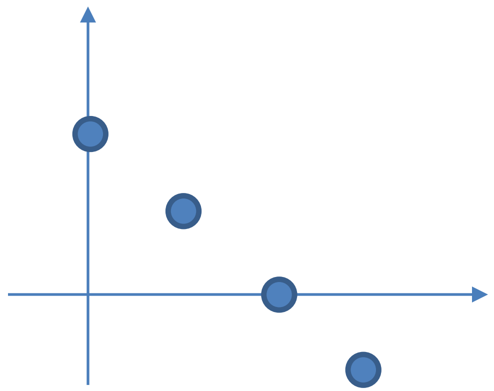
- Use Newton's divided differences to find the interpolating polynomial passing through (0,1), (2,2), (3,4), (1,0).

0	1			
		$\frac{2-1}{2-0}=\frac{1}{2}$		
2	2		$\frac{2-\frac{1}{2}}{3-0}=\frac{1}{2}$	
		$\frac{4-2}{3-2}=2$		$\frac{0-\frac{1}{2}}{1-0}=-\frac{1}{2}$
3	4		$\frac{2-2}{1-2}=0$	
		$\frac{0-4}{1-3}=2$		
1	0			

$$P(x)=1+\frac{1}{2}(x-0)+\frac{1}{2}(x-0)(x-2)-\frac{1}{2}(x-0)(x-2)(x-3)$$

Example 3

- Use Newton's divided differences to find the interpolating polynomial passing through (0,2), (1,1), (2,0), (3,-1).



0	2			
		-1		
1	1		0	
		-1		0
2	0		0	
		-1		
3	-1			

$$P(x) = 2 - 1(x-0) = 2 - \mathrm{x}$$

Evaluating Newton's polynomial

- Newton's polynomial

$$\boxed{\begin{gathered} P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) \\ + \cdots + c_{n-1}(x - x_1) \cdots (x - x_{n-1}) \end{gathered}}$$

$$P(z) = \ ?$$

$$\begin{aligned} P(z) = c_0 + (z - x_1)(c_1 + (z - x_2)(c_2 + \cdots \\ + (z - x_{n-1})(c_{n-1}) \cdots)) \end{aligned}$$

a procedure similar to Horner's rule

程式練習

And, please upload your program on moodle.

- Use Newton's divided differences to find the degree 4 interpolating polynomial for the data,

(0.6, 1.433329)
(0.7, 1.632316)
(0.8, 1.896481)
(0.9, 2.247908)
(1.0, 2.718282)

- Please calculate $P(0.82)$ and $P(0.98)$.

  Bonus! Using Horner's method to evaluate the *z* values!