Solving nonlinear equations in one variable

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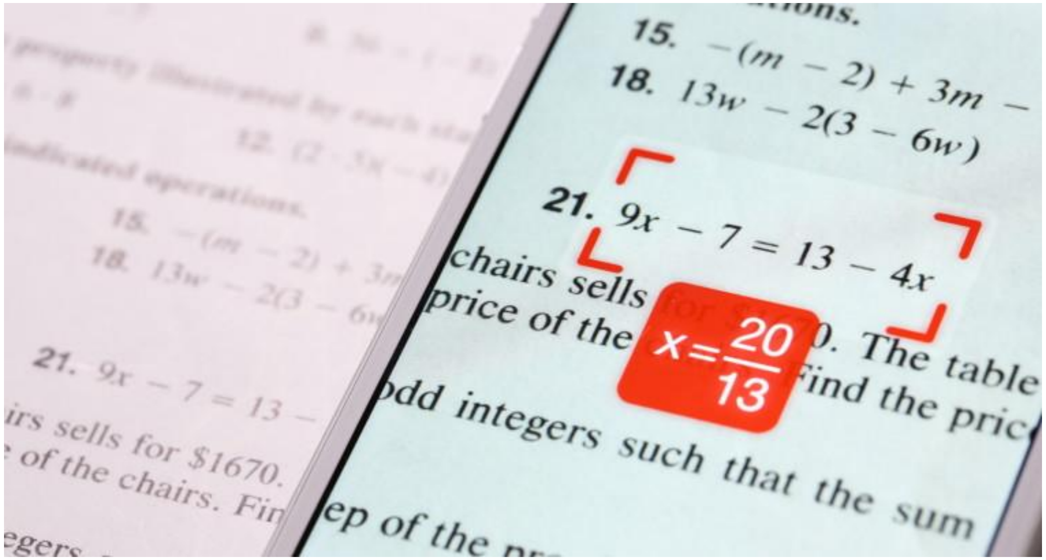
Many slides are from Gerald W. Recktenwald (PSU) and Peter Arbenz (ETH)

PhotoMath

- https://www.youtube.com/watch?v=XlbVB50mIh4

Today

- Evaluating a polynomial (warm-up!)
- Solving an equation with one variable



What is an efficient way to compute

$$P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$$

$$P\left(\frac{1}{2}\right) = ?$$

How many operations in total?

Method 1

$$\boxed{P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1}$$

$$P\left(\frac{1}{2}\right) = 2*\frac{1}{2}*\frac{1}{2}*\frac{1}{2}*\frac{1}{2} + 3*\frac{1}{2}*\frac{1}{2}*\frac{1}{2} - 3*\frac{1}{2}*\frac{1}{2} + 5*\frac{1}{2} - 1$$

- Number of multiplications? **10**
- Number of additions? **4**

Method 2

$$P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$$

$$\frac{1}{2} * \frac{1}{2} = \left(\frac{1}{2}\right)^2 \quad \left(\frac{1}{2}\right)^2 * \frac{1}{2} = \left(\frac{1}{2}\right)^3 \quad \left(\frac{1}{2}\right)^3 * \frac{1}{2} = \left(\frac{1}{2}\right)^4$$

$$P\left(\frac{1}{2}\right) = 2 * \left(\frac{1}{2}\right)^4 + 3 * \left(\frac{1}{2}\right)^3 - 3 * \left(\frac{1}{2}\right)^2 + 5 * \frac{1}{2} - 1$$

- Number of multiplications? **7**
- Number of additions? **4**

$\boxed{14 \downarrow 11}$

fewer *operations?*

Nested multiplication (Horner's method)

$$P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$$
$$= -1 + 5x - 3x^2 + 3x^3 + 2x^4$$
$$= -1 + x\left(5 - 3x + 3x^2 + 2x^3\right)$$
$$= -1 + x\left(5 + x\left(-3 + 3x + 2x^2\right)\right)$$
$$= -1 + x\left(5 + x\left(-3 + x\left(3 + 2x\right)\right)\right)$$

- Number of multiplications? **4**
- Number of additions? **4**

11→8

程式練習(HW#0)

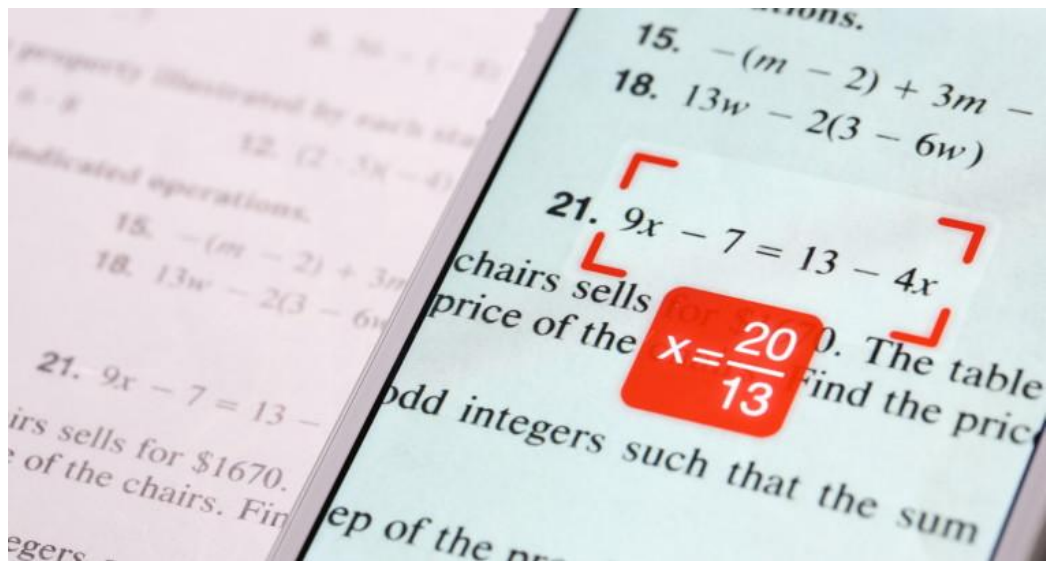
- 請使用Horner’s method計算多項式的值
- 請用你的程式計算

$$P(x) = 1 + x + \cdots + x^{50}$$

$$P(1.0001) = ?$$

Today

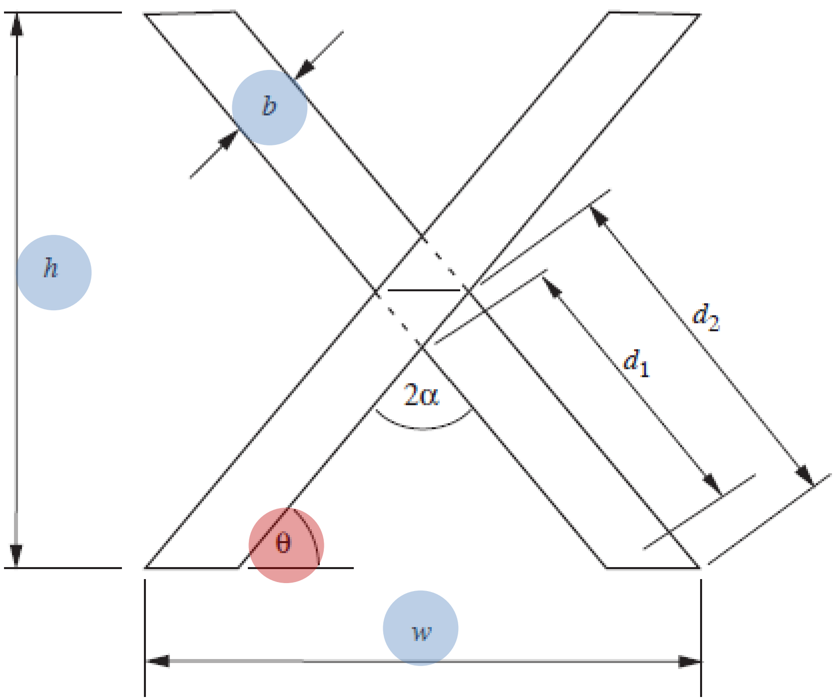
- Evaluating a polynomial (warm-up!)
- Solving an equation with one variable



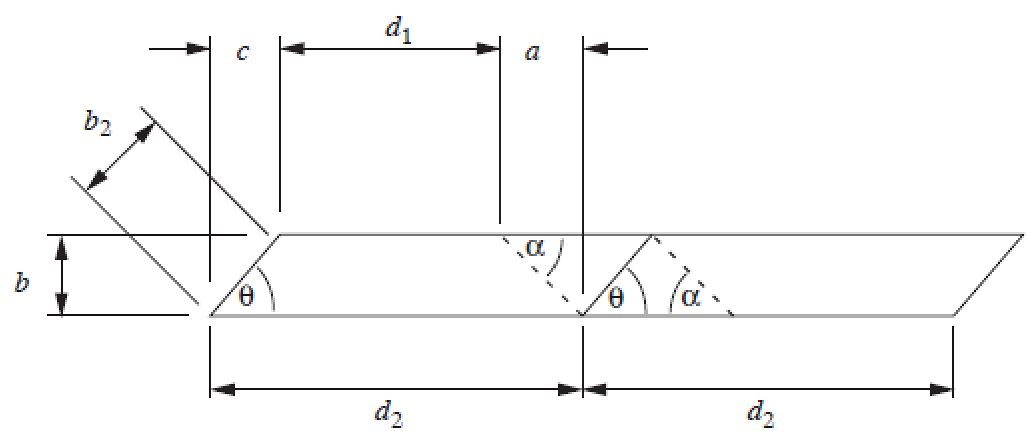
Example 1: Picnic Table Leg

- Computing the dimensions of a picnic table leg involves a root-finding problem.

Leg assembly



Detail of one leg



Slide credit: Gerald W. Recktenwald

- Dimensions of a the picnic table leg satisfy

$$\boxed{w\sin\theta = h\cos\theta + b}$$

- Given *w*, *h*, and *b*, what is the value of θ?
- An analytical solution for $\theta = f(w, h, b)$ exists, but is not obvious.
- Use a numerical root-finding procedure to find the value of θ that satisfies

$$f(\theta) = w\sin\theta - h\cos\theta - b = 0$$

$$\boxed{\rightarrow \text{ 方程式求根問題}}$$

Example 2: Kepler's equation

(計算行星的軌道)

$$\boxed{x - a\sin x = b}$$

- Given *a* and *b*, what is the value of *x*?
- $a = 0.2$, $b = \pi/3$, $x = ?$
- A numerical approach:

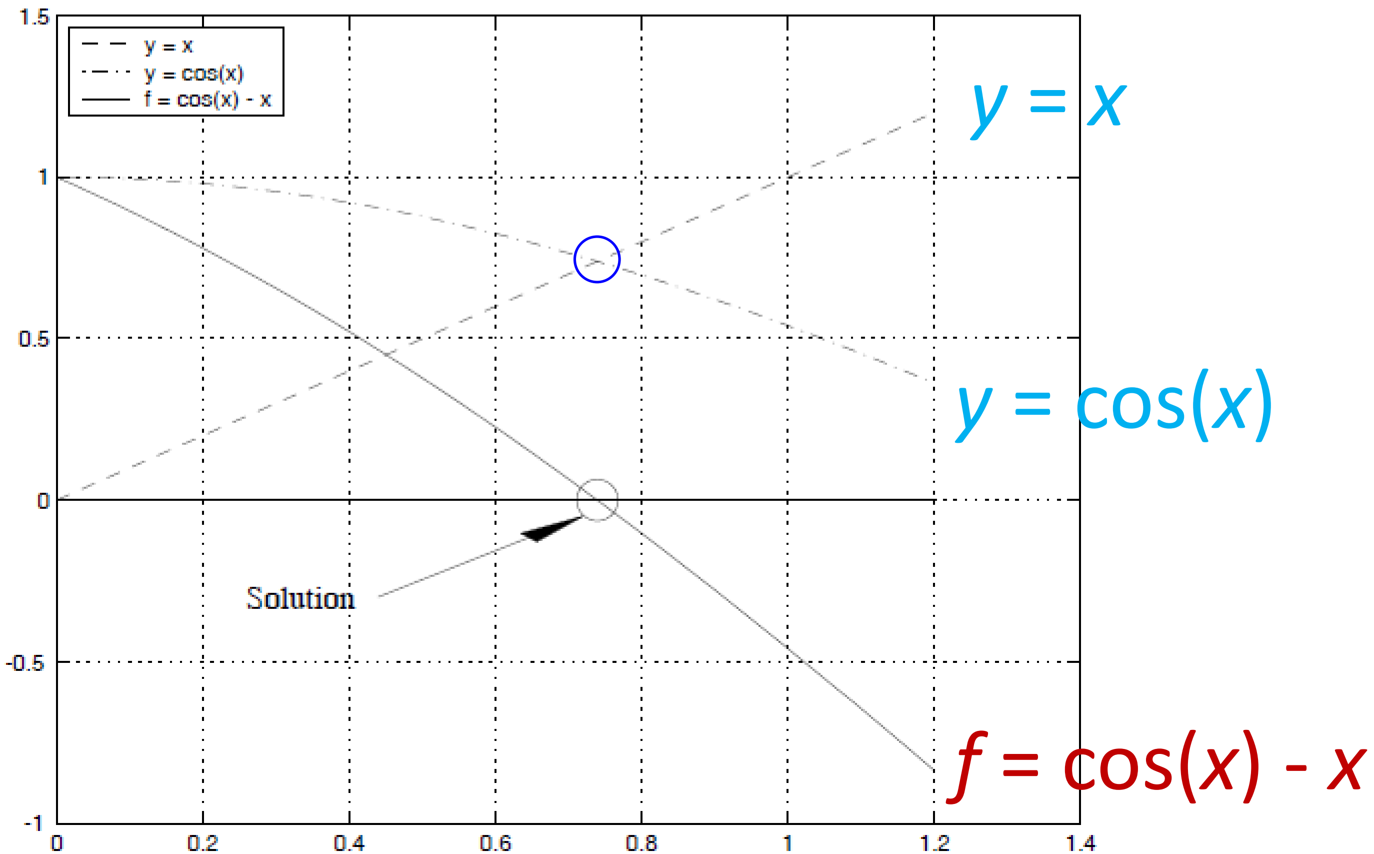
solve

$$f(x) = x - a\sin x - b = 0$$

Roots of $f(x) = 0$

- Any function of one variable x can be put in the form $f(x) = 0$? ***Yes!***
- Example:
  - To find x that satisfies $\cos(x) = x$,
  - Find the zero crossing of $f(x) = \cos(x) - x = 0$.

$\cos(x) = x, \quad x = ?$



y = x
y = cos(x)
f = cos(x) - x
Solution
$y = x$
$y = \cos(x)$
$f = \cos(x) - x$
1.5
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Number of Roots

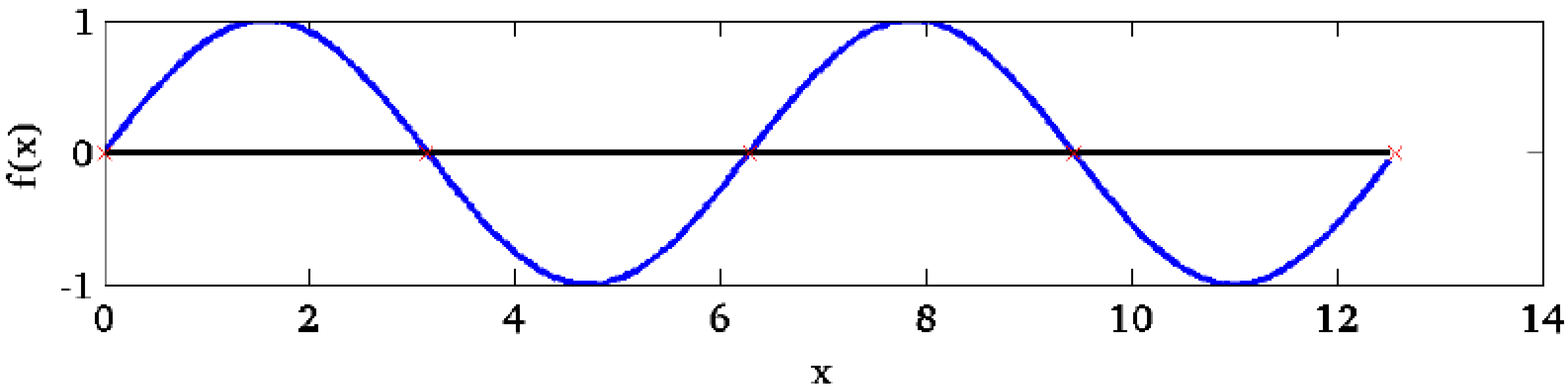
- In contrast to scalar linear equations

$$mx - n = 0 \Rightarrow x = \frac{n}{m},$$

nonlinear equations have an undetermined number of zeros.

Number of Roots

- $f(x) = \sin(x)$
- On $[a, b] = [0, 4\pi]$ there are **???** roots.



Finding roots

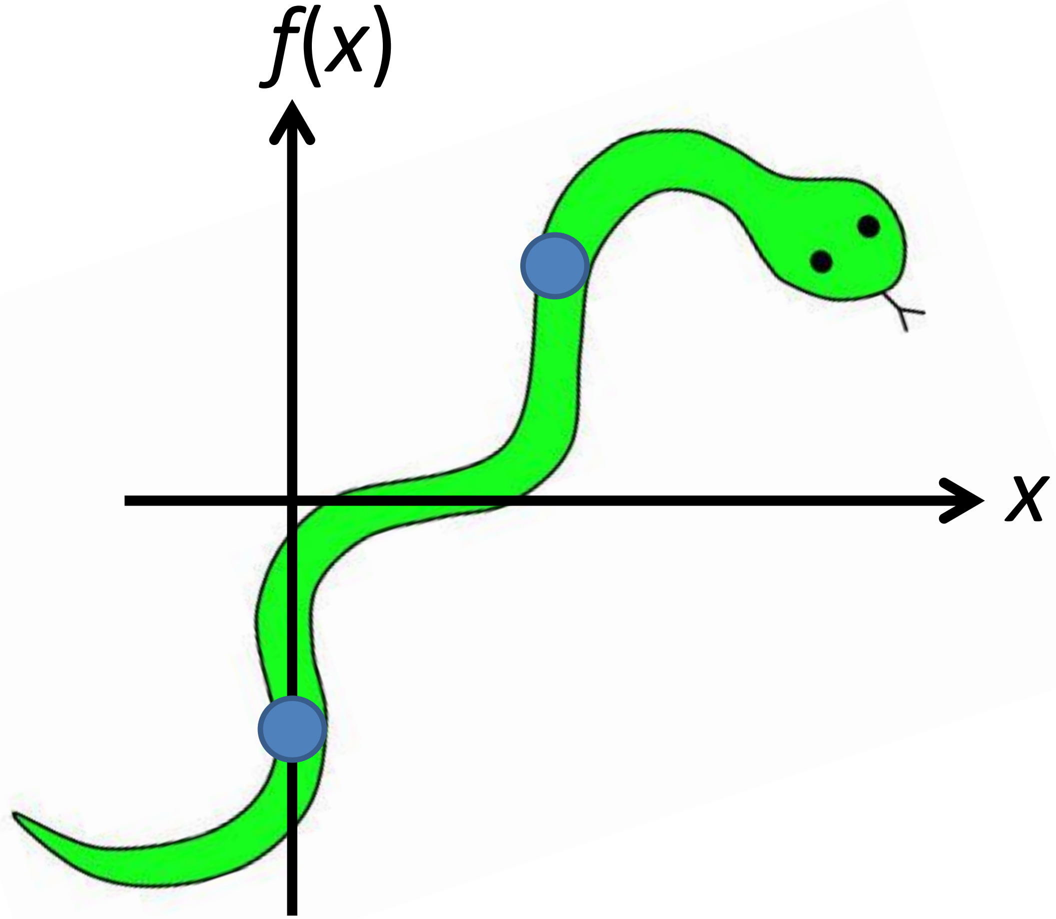
$$f(x) = x^3 + x - 1 = 0$$

$$x = ?$$

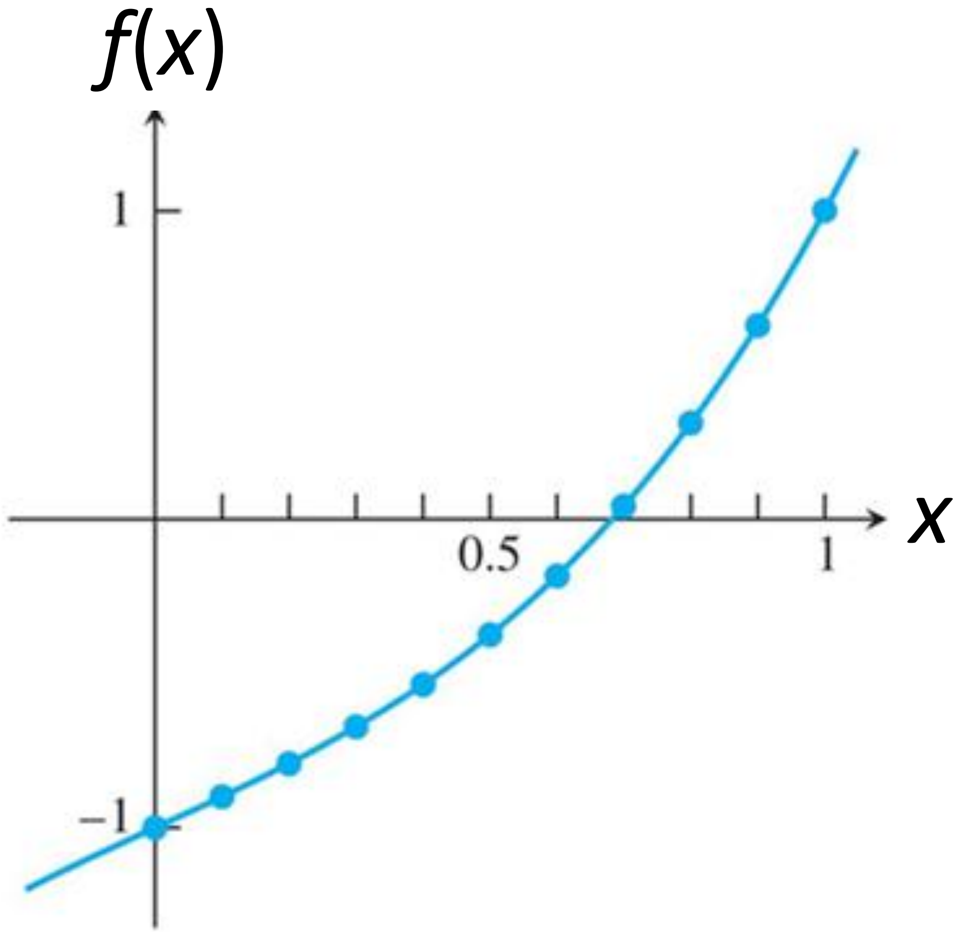
幾個解?
範圍為?

$$f(x) = x^3 + x - 1 = 0$$

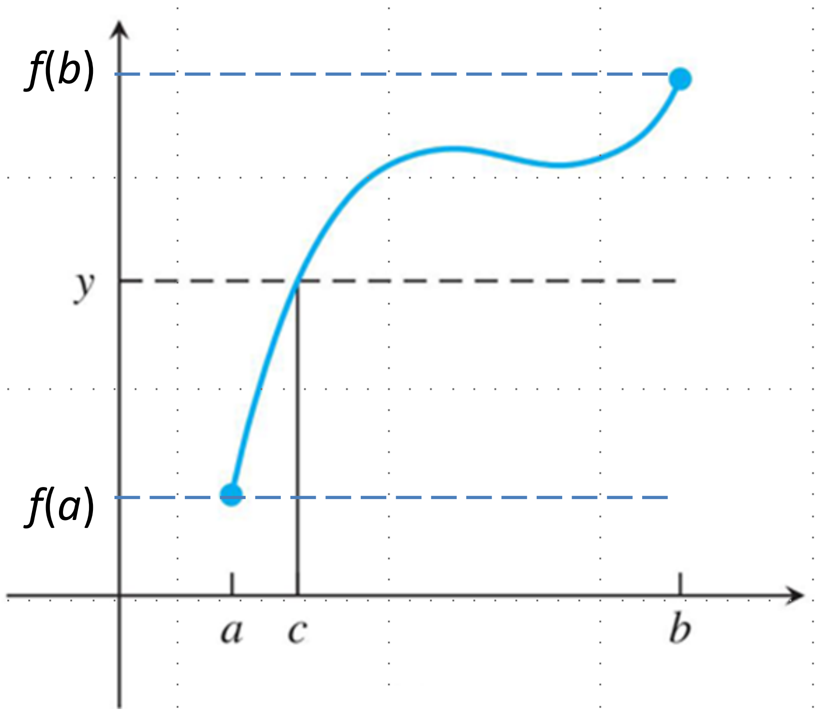
- Must have a root between 0 and 1
  - $f(0) = -1$
  - $f(1) = 1$
  - $f(0)f(1) < 0$



$$f(x) = x^3 + x - 1 = 0$$



Intermediate value theorem



- Let f be a continuous function on the interval $[a, b]$. If y is a number between $f(a)$ and $f(b)$, then there exists a number c with $a \leqq c \leqq b$ such that $f(c) = y$.

猜數字 (1~100)

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Bisection

- 猜數字
- Method for finding a root of a scalar equation $f(x) = 0$ in an interval $[a, b]$
- Assumption: $f(a)\, f(b) < 0$
- Since f is continuous there must be a zero $x^* \in [a, b]$

1. Compute midpoint m of the interval and check $f(m)$
2. Depending on the sign of $f(m)$, we can decide if $x^* \in [a, m]$ or $x^* \in [m, b]$
   - Of course, if $f(m) = 0$ then we are done.

Bisection

- Given $f(\cdot)$ $\quad f(x) = x^3 + x - 1$
- Given a range $[a, b]$ $\quad [0, 1]$
- Determine a stopping condition

$(b - a) < 10^{-6}$ or $f((a+b)/2) \approx 0$

Compute the roots of $f(x) = 0$

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750
0.5	1	0.75	0.1719

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750
0.5	1	0.75	0.1719
0.5	0.75	0.625	-0.1309

Bisection: Example

$f(x) = x^3 + x - 1$

a	*b*	*mid*	*f*(*mid*)
0	1	0.5	-0.3750
0.5	1	0.75	0.1719
0.5	0.75	0.625	-0.1309
0.625	0.75	0.6875	0.0125

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750
0.5	1	0.75	0.1719
0.5	0.75	0.625	-0.1309
0.625	0.75	0.6875	0.0125
0.625	0.6875	0.6563	-0.0611

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0	1	0.5	-0.3750
0.5	1	0.75	0.1719
0.5	0.75	0.625	-0.1309
0.625	0.75	0.6875	0.0125
0.625	0.6875	0.6563	-0.0611
0.6563	0.6875	0.6719	-0.0248

Bisection: Example

$f(x) = x^3 + x - 1$

a	b	mid	$f(mid)$
0.6563	0.6875	0.6719	-0.0248
0.6719	0.6875	0.6797	-0.0063
0.6797	0.6875	0.6836	0.0031
0.6797	0.6836	0.6816	-0.0016
0.6816	0.6836	0.6826	0.0006
⋮	⋮	⋮	⋮

程式練習 (HW#1)

- 請寫一個程式(Bisection)計算方程式的根
- 請用你的程式計算

$$x - x^{1/3} - 2 = 0 \qquad 3 < x < 4$$