Problem 9

We give the example that $A_n = \{x|x \in (0, \frac{1}{2^n})\},\ n = 1, 2, ...$

proof:

Since

$$\frac{1}{2^{n+1}} < \frac{1}{2^n}$$

We know

$$(0, \frac{1}{2^{n+1}}) \subseteq (0, \frac{1}{2^n})$$

So,

$$A_{n+1} \subseteq A_n,\ n = 1, 2, ...$$

We assume that $\bigcap_{n=1}^{\infty} A_n \neq \phi$, so there is one real number $y \in \bigcap_{n=1}^{\infty} A_n$.It is for any n, $y \in A_n$.

If $y \leqslant 0$, $y \notin (0, \frac{1}{2})$, so $y \notin A_1$, the **assumption** is not valid.

If $y > 0$,

we set $n_0 = \lceil \log_2^{\frac{1}{y}} \rceil + 1$, Then,

$$\log_2^{\frac{1}{y}} < n_0$$

$$\frac{1}{y} < 2^{n_0}$$

$$\frac{1}{2^{n_0}} < y$$

$$y \notin (0, \frac{1}{2^{n_0}})$$

$$y \notin A_{n_0}$$

So the **assumption** is not valid.

It means that there is not one real number $x \in \bigcap_{n=1}^{\infty} A_n$, and $\bigcap_{n=1}^{\infty} A_n = \phi$

The proof is complete.

The exmple $A_n = \{x|x \in (0, \frac{1}{2^n})\},\ n = 1, 2, ...$ has the stated property.