SOME COMPOSITIONS OF PICTURE FUZZY RELATIONS

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ABSTRACT *— Relations are a suitable tool for describing correspondences between objects. Fuzzy relation and intuitionistic fuzzy relation, which were defined based on concepts of fuzzy set and intuitionistic fuzzy set, are generalizations of crisp relation. Fuzzy relation and intuitionistic fuzzy relation have being applied in many areas, such as: fuzzy reasoning, fuzzy control, fuzzy diagnosis ,…. Very recently, Bui Cong Cuong and Vladik Kreinovich defined a novel notion of picture fuzzy set as an extension of fuzzy set and intuitionistic fuzzy set. Picture fuzzy relation was also defined as one of the first concepts when constructing the desired picture fuzzy set theory. In this paper, some properties of compositions of picture fuzzy relations are examined. Then, a new approach for medical diagnosis using composition of picture fuzzy relations is proposed.*

Key words *— Picture fuzzy sets, max – min composition, medical diagnosis.*

TÓM TẮT *— Quan hệ là một công cụ thích hợp để mô tả sự tương ứng giữa các đối tượng. Quan hệ mờ và quan hệ mờ trực cảm, được định dựa trên khái niệm tập mờ và tập mờ trực cảm, là những khái quát của quan hệ rõ. Quan hệ mờ và quan hệ mờ trực cảm đã được áp dụng trong nhiều lĩnh vực như: suy diễn mờ, điều khiển mờ, chẩn đoán mờ, ... Gần đây, Bùi Công Cường và Vladik Kreinovich định nghĩa tập mờ bức tranh như một mở rộng của tập mờ và tập mờ trực cảm. Quan hệ mờ bức tranh cũng được định nghĩa và là một trong những khái niệm đầu tiên khi xây dựng lý thuyết tập mờ bức tranh. Trong bài báo này, một số tính chất của phép hợp thành quan hệ mờ bức tranh được kiểm tra. Sau đó, chúng tôi đề xuất một cách tiếp cận mới trong chẩn đoán y khoa sử dụng phép hợp thành quan hệ mờ trực cảm.*

Từ khóa *— Tập mờ bức tranh, phép hợp thành max – min, chẩn đoán y khoa.*

I. INTRODUCTION

A. *Picture fuzzy sets*

Since Zadeh introduced fuzzy sets (FS) in 1965 [11], many theories treating imprecision and uncertainty have been introduced. Some of these are extensions of fuzzy set theory [1], [12], [6]. Intuitionistic fuzzy sets were introduced by Atanassov [1] constitute a generalization of the notion of a fuzzy set. When fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give a degree of membership and a degree of non-membership.

Definition I.1. [1] *An intuitionistic fuzzy set (IFS)* A *on a universe* X *is an object of the form*

$$A=\left\{\left(x,\mu_A\left(x\right),\nu_A\left(x\right)\right)\middle|x\in X\right\},$$

where $\mu_A\left(x\right)\in\left[0,1\right]$ *is called the "degree of membership of* x *in* A*",* $\nu_A\left(x\right)\in\left[0,1\right]$ *is called the "degree of non-membership of* x *in* A*"* *, and* μ_A *and* ν_A *satisfy the following condition:*

$$\mu_A\left(x\right)+\nu_A\left(x\right)\le 1,\ \textit{for all}\ x\in X.$$

A generalization of fuzzy sets and intuitionistic fuzzy sets are the following notion of picture fuzzy sets.

Definition I.2. [3], [4] *A picture fuzzy set (PFS)* A *on a universe* X *is an object of the form*

$$A=\left\{\left(x,\mu_A\left(x\right),\eta_A\left(x\right),\nu_A\left(x\right)\right)\middle|x\in X\right\},$$

where $\mu_A\left(x\right)\in\left[0,1\right]$ *is called the "degree of positive membership of* x *in* A*",* $\eta_A(x)\in\left[0,1\right]$ *is called the "degree of neutral membership of* x *in* A*", and* $\nu_A\left(u\right)\in\left[0,1\right]$ *is called the "degree of negative membership of* x *in* A*", and* μ_A*,* η_A *and* ν_A *satisfy the following condition:*

$$\mu_A(x)+\mu_A(x)+\nu_A(x)\le 1,\ \textit{for all}\ x\in X.$$

In above definition, for all $x\in X$, $\pi(x)=1-\left[\mu_A(x)+\eta_A(x)+\nu_A(x)\right]$ could be called the "degree of refusal membership of x in A".

Basically, picture fuzzy sets based models may be adequate in situations when human opinions involve types of answer: yes, abstain, no, refusal.

In this paper, $IFS(X)$ and $PFS(X)$ denote the set of all intuitionistic fuzzy set and the set of all intuitionistic fuzzy set on the universe X.

In [3], [4], some set operators on picture fuzzy sets are also given.

Definition I.3. [3], [4] *For every two PFSs A and B. Operators union, intersection and complement are defined as following, respectively:*

$$A\cup B=\left\{\left(x,\max\left\{\mu_A(x),\mu_B(x)\right\},\min\left\{\eta_A(x),\eta_B(x)\right\},\min\left\{\nu_A(x),\nu_B(x)\right\}\right)\middle| x\in X\right\};$$

$$A\cup B=\left\{\left(x,\min\left\{\mu_A(x),\mu_B(x)\right\},\min\left\{\eta_A(x),\eta_B(x)\right\},\max\left\{\nu_A(x),\nu_B(x)\right\}\right)\middle| x\in X\right\};$$

$$coA=A^c=\left\{\left(x,\nu_A(x),\eta_A(x),\mu_A(x)\right)\middle| x\in X\right\}.$$

B. Intuitionistic fuzzy relations

In 1995 [2], Burillo and Bustince introduced concepts of intuitionistic fuzzy relation and a composition of intuitionistic fuzzy relations using four triangular norms or conorms where triangular norm and triangular conorm are notions used in the framework of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic.

Definition I.4. [2] *An intuitionistic fuzzy relation (IFR) R between X and Y ($R\in IFR(X\times Y)$) is defined as an intuitionistic fuzzy subset of $X\times Y$, that is, R is given by*

$$R=\left\{\left\langle (x,y),\mu_R(x,y),\nu_R(x,y)\right\rangle\middle| (x,y)\in X\times Y\right\},$$

where $\mu_R,\nu_R: X\times Y\to[0,1]$ satisfy the condition $\mu_R(x,y)+\nu_R(x,y)\le 1$ for every $(x,y)\in X\times Y$.

Definition I.5. 1) *A triangular norm (t-norm) is a commutative, associative, increasing mapping $T: [0,1]^2\to[0,1]$ satisfying $T(x,1)=x$, for all $x\in[0,1]$.*

2) A triangular conorm (t-conorm) is a commutative, associative, increasing mapping $S: [0,1]^2\to[0,1]$ satisfying $S(0,x)=x$, for all $x\in[0,1]$.

In [2], composition of two intuitionistic fuzzy relations is also defined using four t-norm or t-conorm.

Definition I.6. [2] *Let α, β, λ, ρ be four t-norms or t-conorm, $R\in IFR(X\times Y)$, $P\in IFR(Y\times Z)$. Composed relation $P\overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}}R\in IFR(X\times Z)$ is the one defined by*

$$P\overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}}R=\left\{\left\langle (x,z),\mu_{P\overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}}R}(x,z),\nu_{P\overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}}R}(x,z)\right\rangle\middle| (x,z)\in X\times Z\right\},$$

where

$$\mu_{P\overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}}R}(x,z)=\underset{y}{\alpha}\left\{\beta\left[\mu_R(x,y),\mu_P(y,z)\right]\right\},\ \nu_{P\overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}}R}(x,z)=\underset{y}{\lambda}\left\{\rho\left[\nu_R(x,y),\nu_P(y,z)\right]\right\},$$

and α, β, λ, ρ must be chosen such that:

$$\mu_{\underset{\lambda,\rho}{\overset{\alpha,\beta}{P \circ R}}}(x,z)+\nu_{\underset{\lambda,\rho}{\overset{\alpha,\beta}{P \circ R}}}(x,z)\le 1, \textit{ for all } (x,z)\in X\times Z .$$

C. Medical diagnosis

One of important application of fuzzy relation and intuitionistic relation is medical diagnosis model, which first proposed by Sanchez [8], [9]. In this model, suppose S is a set of symptoms, D a set of diagnoses, and P a set of patients. The methodology involves mainly the following three jobs:

1. **Determining relation between patients and symptoms:** this relation expresses the correspondence between patients and symptoms.

2. **Formulating relation between symptoms and diagnoses:** this roles medical knowledge base in medical diagnosis.

3. **Determining of diagnoses for all patient on the basis of composition of relations:** the relation between patients and diagnoses is composed relation of two above relations.

Relations in above methodology can be fuzzy relations as in [9], interval valued relations as in [7], or intuitionistic fuzzy relations as in [10].

II. SOME COMPOSITIONS OF PICTURE FUZZY RELATIONS

Definition II.1. [3], [4] *A picture fuzzy relation* R *is a picture fuzzy subset on* $X\times Y$ *, i.e.,* R *is given by*

$$R=\left\{\left((x,y),\mu_R(x,y),\eta_R(x,y),\nu_R(x,y)\right)\middle|(x,y)\in X\times Y\right\}$$

where $\mu_R : X\times Y\to[0,1]$, $\eta_R : X\times Y\to[0,1]$, $\nu_R : X\times Y\to[0,1]$ *satisfy the condition*

$$\mu_R(x,y)+\eta_R(x,y)+\nu_R(x,y)\le 1, \textit{ for every}(x,y)\in X\times Y .$$

We will represent by $PFR(X\times Y)$ the set of all the picture fuzzy relations on $X\times Y$.

Definition II.2. [3], [5] *Let* $E\in PFR(X\times Y)$ *and* $P\in PFR(Y\times Z)$**.** *We will call* $\max - \min$ *composed relation* $P\mathbb{C}_1E$ *to the one defined by*

$$P\mathbb{C}_1E=\left\{\left((x,z),\mu_{P\mathbb{C}_1E}(x,z),\eta_{P\mathbb{C}_1E}(x,z),\nu_{P\mathbb{C}_1E}(x,z)\right)\middle|(x,z)\in X\times Z\right\},$$

where

$$\mu_{P\mathbb{C}_1E}(x,z)=\bigvee_y\left\{\mu_E(x,y)\wedge\mu_P(y,z)\right\},$$

$$\eta_{P\mathbb{C}_1E}(x,z)=\bigwedge_y\left\{\eta_E(x,y)\wedge\eta_P(y,z)\right\},$$

$$\nu_{P\mathbb{C}_1E}(x,z)=\bigwedge_y\left\{\nu_E(x,y)\vee\nu_P(y,z)\right\}.$$

Proposition I.1. *If* $E\in PFR(X\times Y)$ *and* $P\in PFR(Y\times Z)$, *then* $P\mathbb{C}_1E\in PFR(X\times Z)$.

Proof. For all $(x,z)\in X\times Z$, let examine

$$\mu_{P\mathbb{C}_1E}(x,z)+\eta_{P\mathbb{C}_1E}(x,z)+\nu_{P\mathbb{C}_1E}(x,z)\le 1 .$$

For all $\varepsilon>0$, there exists $y^*\in Y$:

$$\mu_{P\mathbb{C}_1E}(x,z)<\mu_E(x,y^*)\wedge\mu_P(y^*,z)+\varepsilon . \tag{II.1}$$

It is easily seen that

$$\eta_{P\mathbb{C}_1E}(x,z)\le\eta_E(x,y^*)\wedge\eta_P(y^*,z), \tag{II.2}$$

and

$$\nu_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right) \le \nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right) \vee \nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right). \tag{II.3}$$

By (II.1)-(II.3),

$$\begin{aligned}&\mu_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)+\eta_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)+\nu_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)\\ &\qquad < \mu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\mu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\eta_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\eta_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\vee\nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\varepsilon.\end{aligned}$$

Let consider following cases:

- Case 1: $\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\vee\nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)=\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)$. Then

$$\begin{aligned}&\mu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\mu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\eta_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\eta_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\vee\nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\varepsilon\\ &=\mu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\mu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\eta_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\eta_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)+\varepsilon\\ &\le\mu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)+\eta_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)+\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)+\varepsilon\le 1+\varepsilon.\end{aligned}$$

- Case 2: $\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\vee\nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)=\nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)$. Then

$$\begin{aligned}&\mu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\mu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\eta_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\eta_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\vee\nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\varepsilon\\ &=\mu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\mu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\eta_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\wedge\eta_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\varepsilon\\ &\le\mu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\eta_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\nu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\varepsilon\le 1+\varepsilon.\end{aligned}$$

Then $\mu_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)+\eta_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)+\nu_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)<1+\varepsilon$ for all $\varepsilon>0$.

Hence, $\mu_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)+\eta_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)+\nu_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)\le 1$. □

Definition II.3. [3], [5] *Let* $\mathrm{E}\in\mathrm{PFR}\left(\mathrm{X}\times\mathrm{Y}\right)$ *and* $\mathrm{P}\in\mathrm{PFR}\left(\mathrm{Y}\times\mathrm{Z}\right)$. *We will call* $\max\text{-}\mathrm{prod}$ *composed relation* $\mathrm{P}\mathbb{C}_2\mathrm{E}$ *to the one defined by*

$$\mathrm{P}\mathbb{C}_2\mathrm{E}=\left\{\left(\left(\mathrm{x},\mathrm{z}\right),\mu_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right),\eta_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right),\nu_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)\right)\middle|\left(\mathrm{x},\mathrm{z}\right)\in\mathrm{X}\times\mathrm{Z}\right\},$$

where

$$\mu_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)=\bigvee_{\mathrm{y}}\left\{\mu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}\right)\cdot\mu_{\mathrm{P}}\left(\mathrm{y},\mathrm{z}\right)\right\},$$

$$\eta_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)=\bigwedge_{\mathrm{y}}\left\{\eta_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}\right)\cdot\eta_{\mathrm{P}}\left(\mathrm{y},\mathrm{z}\right)\right\},$$

$$\nu_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)=\bigwedge_{\mathrm{y}}\left\{\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}\right)+\nu_{\mathrm{P}}\left(\mathrm{y},\mathrm{z}\right)-\nu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}\right)\cdot\nu_{\mathrm{P}}\left(\mathrm{y},\mathrm{z}\right)\right\}.$$

Proposition I.2. *If* $\mathrm{E}\in\mathrm{PFR}\left(\mathrm{X}\times\mathrm{Y}\right)$ *and* $\mathrm{P}\in\mathrm{PFR}\left(\mathrm{Y}\times\mathrm{Z}\right)$, *then* $\mathrm{P}\mathbb{C}_2\mathrm{E}\in\mathrm{PFR}\left(\mathrm{X}\times\mathrm{Z}\right)$.

Proof. For all $\left(\mathrm{x},\mathrm{z}\right)\in\mathrm{X}\times\mathrm{Z}$, let examine

$$\mu_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)+\eta_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)+\nu_{\mathrm{P}\mathbb{C}_2\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)\le 1.$$

For all $\varepsilon>0$, there exists $\mathrm{y}^*\in\mathrm{Y}$:

$$\mu_{\mathrm{P}\mathbb{C}_1\mathrm{E}}\left(\mathrm{x},\mathrm{z}\right)<\mu_{\mathrm{E}}\left(\mathrm{x},\mathrm{y}^*\right)\cdot\mu_{\mathrm{P}}\left(\mathrm{y}^*,\mathrm{z}\right)+\varepsilon. \tag{II.4}$$

It is easily seen that

$$\eta_{P\mathbb{C}_1E}(x,z) \le \eta_E(x,y^*)\cdot\eta_P(y^*,z), \tag{II.5}$$

and

$$\nu_{P\mathbb{C}_1E}(x,z) \le \nu_E(x,y^*)+\nu_P(y^*,z)-\nu_E(x,y^*)\cdot\nu_P(y^*,z). \tag{II.6}$$

By (II.4)-(II.6),

$$\begin{aligned}&\mu_{P\mathbb{C}_1E}(x,z)+\eta_{P\mathbb{C}_1E}(x,z)+\nu_{P\mathbb{C}_1E}(x,z)\\ &<\mu_E(x,y^*)\cdot\mu_P(y^*,z)+\eta_E(x,y^*)\cdot\eta_P(y^*,z)+\left[\nu_E(x,y^*)+\nu_P(y^*,z)-\nu_E(x,y^*)\cdot\nu_P(y^*,z)\right]+\varepsilon.\end{aligned}$$

Let consider following cases and notice that $\nu_E(x,y^*)+\nu_P(y^*,z)-\nu_E(x,y^*)\cdot\nu_P(y^*,z)\le\max\left\{\nu_E(x,y^*),\nu_P(y^*,z)\right\}$:

- Case 1: $\nu_E(x,y^*)\vee\nu_P(y^*,z)=\nu_E(x,y^*)$. Then

$$\begin{aligned}&\mu_E(x,y^*)\cdot\mu_P(y^*,z)+\eta_E(x,y^*)\cdot\eta_P(y^*,z)+\left[\nu_E(x,y^*)+\nu_P(y^*,z)-\nu_E(x,y^*)\cdot\nu_P(y^*,z)\right]+\varepsilon\\ &=\mu_E(x,y^*)\cdot\mu_P(y^*,z)+\eta_E(x,y^*)\cdot\eta_P(y^*,z)+\nu_E(x,y^*)+\varepsilon\\ &\le\mu_E(x,y^*)+\eta_E(x,y^*)+\nu_E(x,y^*)+\varepsilon\le 1+\varepsilon.\end{aligned}$$

- Case 2: $\nu_E(x,y^*)\vee\nu_P(y^*,z)=\nu_P(y^*,z)$. Then

$$\begin{aligned}&\mu_E(x,y^*)\cdot\mu_P(y^*,z)+\eta_E(x,y^*)\cdot\eta_P(y^*,z)+\left[\nu_E(x,y^*)+\nu_P(y^*,z)-\nu_E(x,y^*)\cdot\nu_P(y^*,z)\right]+\varepsilon\\ &=\mu_E(x,y^*)\cdot\mu_P(y^*,z)+\eta_E(x,y^*)\cdot\eta_P(y^*,z)+\nu_P(y^*,z)+\varepsilon\\ &\le\mu_P(y^*,z)+\eta_P(y^*,z)+\nu_P(y^*,z)+\varepsilon\le 1+\varepsilon.\end{aligned}$$

Then $\mu_{P\mathbb{C}_1E}(x,z)+\eta_{P\mathbb{C}_1E}(x,z)+\nu_{P\mathbb{C}_1E}(x,z)<1+\varepsilon$ for all $\varepsilon>0$.

Hence, $\mu_{P\mathbb{C}_1E}(x,z)+\eta_{P\mathbb{C}_1E}(x,z)+\nu_{P\mathbb{C}_1E}(x,z)\le 1$. □

In following definition, a more general composition, which uses two t-norm, is given.

Definition II.4. *Let* $E\in PFR(X\times Y)$, $P\in PFR(Y\times Z)$, β_1,β_2 *be two* t*-norm. We define composed relation* $P\mathbb{C}_3E$ *to the one defined by*

$$P\mathbb{C}_3E=\left\{\left((x,z),\mu_{P\mathbb{C}_3E}(x,z),\eta_{P\mathbb{C}_3E}(x,z),\nu_{P\mathbb{C}_3E}(x,z)\right)\middle|(x,z)\in X\times Z\right\},$$

where

$$\mu_{P\mathbb{C}_3E}(x,z)=\bigvee_y\left\{\beta_1\left[\mu_E(x,y),\mu_P(y,z)\right]\right\},$$

$$\eta_{P\mathbb{C}_3E}(x,z)=\bigwedge_y\left\{\beta_2\left[\eta_E(x,y),\eta_P(y,z)\right]\right\},$$

$$\nu_{P\mathbb{C}_3E}(x,z)=\bigwedge_y\left\{\nu_E(x,y)\vee\nu_P(y,z)\right\}.$$

Validation of $P\mathbb{C}_3E$ will be examined using the importance properties of t-norm: if β is a t-norm, then $\beta(x,y)\le\min(x,y)$ for all x, $y\in[0,1]$.

Proposition I.3. *If* $E\in PFR(X\times Y)$ *and* $P\in PFR(Y\times Z)$, *then* $P\mathbb{C}_3E\in PFR(X\times Z)$.

Proof. For all $(x,z)\in X\times Z$, let proof

$$\mu_{P\mathbb{C}_3E}(x,z)+\eta_{P\mathbb{C}_3E}(x,z)+\nu_{P\mathbb{C}_3E}(x,z)\le 1.$$

For all $\varepsilon>0$, there exists $y^*\in Y$:

$$\mu_{P\mathbb{C}_3E}(x,z)<\beta_1\left[\mu_E(x,y^*),\mu_P(y^*,z)\right]+\varepsilon. \tag{II.7}$$

It is easily seen that

$$\eta_{P\mathbb{C}_3E}(x,z)\le\beta_2\left[\eta_E(x,y^*),\eta_P(y^*,z)\right], \tag{II.8}$$

and

$$\nu_{P\mathbb{C}_3E}(x,z)\le\nu_E(x,y^*)\vee\nu_P(y^*,z). \tag{II.9}$$

By (II.7)-(II.9),

$$\begin{aligned}&\mu_{P\mathbb{C}_3E}(x,z)+\eta_{P\mathbb{C}_3E}(x,z)+\nu_{P\mathbb{C}_3E}(x,z)\\&<\beta_1\left[\mu_E(x,y^*),\mu_P(y^*,z)\right]+\beta_2\left[\eta_E(x,y^*),\eta_P(y^*,z)\right]+\nu_E(x,y^*)\vee\nu_P(y^*,z)+\varepsilon.\end{aligned}$$

- Case 1: $\nu_E(x,y^*)\vee\nu_P(y^*,z)=\nu_E(x,y^*)$. Then

$$\begin{aligned}&\beta_1\left[\mu_E(x,y^*),\mu_P(y^*,z)\right]+\beta_2\left[\eta_E(x,y^*),\eta_P(y^*,z)\right]+\nu_E(x,y^*)\vee\nu_P(y^*,z)+\varepsilon\\&=\beta_1\left[\mu_E(x,y^*),\mu_P(y^*,z)\right]+\beta_2\left[\eta_E(x,y^*),\eta_P(y^*,z)\right]+\nu_E(x,y^*)+\varepsilon\\&\le\mu_E(x,y^*)+\eta_E(x,y^*)+\nu_E(x,y^*)+\varepsilon\le 1+\varepsilon.\end{aligned}$$

- Case 2: $\nu_E(x,y^*)\vee\nu_P(y^*,z)=\nu_P(y^*,z)$. Then

$$\begin{aligned}&\beta_1\left[\mu_E(x,y^*),\mu_P(y^*,z)\right]+\beta_2\left[\eta_E(x,y^*),\eta_P(y^*,z)\right]+\nu_E(x,y^*)\vee\nu_P(y^*,z)+\varepsilon\\&=\beta_1\left[\mu_E(x,y^*),\mu_P(y^*,z)\right]+\beta_2\left[\eta_E(x,y^*),\eta_P(y^*,z)\right]+\nu_P(y^*,z)+\varepsilon\\&\le\mu_P(y^*,z)+\eta_P(y^*,z)+\nu_P(y^*,z)+\varepsilon\le 1+\varepsilon.\end{aligned}$$

Then $\mu_{P\mathbb{C}_3E}(x,z)+\eta_{P\mathbb{C}_3E}(x,z)+\nu_{P\mathbb{C}_3E}(x,z)<1+\varepsilon$ for all $\varepsilon>0$.

Hence, $\mu_{P\mathbb{C}_3E}(x,z)+\eta_{P\mathbb{C}_3E}(x,z)+\nu_{P\mathbb{C}_3E}(x,z)\le 1$. □

Example II.1. Let $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3,y_4\}$, $Z=\{z_1,z_2,z_3\}$, $E\in PFR(X\times Y)$ *and* $P\in PFR(Y\times Z)$ *are given by two following tables.*

Table 1. E is a picture fuzzy relation between X and Y

E	y_1	y_2	y_3	y_4
x_1	$(0.7,0.2,0.1)$	$(0.1,0.05,0.6)$	$(0.02,0.6,0.2)$	$(0.07,0.3,0.4)$
x_2	$(0.5,0.4,0.01)$	$(0.8,0.03,0.05)$	$(0.2,0.25,0.5)$	$(0.7,0.15,0.08)$
x_3	$(0.3,0.5,0.15)$	$(0.9,0.05,0.01)$	$(0.45,0.5,0.01)$	$(0.1,0.1,0.4)$

Table 2. P is a picture fuzzy relation between Y and Z

P	z_1	z_2	z_3
y_1	$(0.75, 0.1, 0.15)$	$(0.5, 0.25, 0.01)$	$(0.45, 0.4, 0.01)$
y_2	$(0.2, 0.4, 0.3)$	$(0.36, 0.6, 0.05)$	$(0.2, 0.2, 0.6)$
y_3	$(0.06, 0.24, 0.4)$	$(0.55, 0.09, 0.3)$	$(0.7, 0.1, 0.1)$
y_4	$(0.3, 0.04, 0.6)$	$(0.4, 0.3, 0.25)$	$(0.4, 0.2, 0.1)$

Let $T_\chi : [0,1]^2 \to [0,1]$ is a t-norm defined by

$$T(x,y) = \begin{cases} 0 & \text{if } x+y \le 1 \\ x+y-1 & \text{if } x+y > 1 \end{cases}, \text{ for all } (x,y) \in [0,1]^2 .$$

Then, composed relation $P\mathbb{C}_3E$, where $\beta_1 = T_\chi$, $\beta_2 = \wedge$, is given as below.

Table 3. $P\mathbb{C}_3E$ relation with $\beta_1 = T_\chi$, $\beta_2 = \wedge$

$P\mathbb{C}_3E$	z_1	z_2	z_3
x_1	$(0.45, 0.04, 0.15)$	$(0.2, 0.05, 0.1)$	$(0.15, 0.05, 0.1)$
x_2	$(0.25, 0.03, 0.15)$	$(0.15, 0.03, 0.01)$	$(0.1, 0.03, 0.01)$
x_3	$(0.1, 0.04, 0.15)$	$(0.25, 0.05, 0.05)$	$(0.15, 0.05, 0.1)$

III. APPLICATION

In this section we present an application of picture fuzzy relation in Sanchez's approach [8], [9] for medical diagnosis. In this approach, S denotes a set of symptoms, D denotes a set of diagnoses, and P denotes a set of patients.

We define picture medical knowledge as a picture fuzzy relation R between the set of symptoms S and the set of diagnoses D which reveals the degree of positive association, neutral – association, and the degree of negative – association between symptoms and the diagnosis.

Now let us discuss picture fuzzy medical diagnosis. As a similarity of traditional approach, the methodology also involves following three jobs:

1. Determination of symptoms.
2. Formulation of medical knowledge based on picture fuzzy relations.
3. Determination of diagnosis on the basis of composition of picture fuzzy relations.

Let $R \in PFR(P \times S)$ and $Q \in PFS(D \times S)$, clearly, the composition T of R and Q ($T = R \circ Q$) describes the state of patients in terms of the diagnosis. For sample, the state of patients can be define as a $\max - \min$ composed relation T from P to D:

$$\mu_T(p,d) = \bigvee_{s \in S} \{\mu_Q(p,s) \wedge \mu_R(s,d)\};$$

$$\eta_T(p,d) = \bigwedge_{s \in S} \{\eta_Q(p,s) \wedge \eta_R(s,d)\};$$

$$\nu_T(p,d) = \bigwedge_{s \in S} \{\nu_Q(p,s) \wedge \nu_R(s,d)\} \quad \forall p_i \in P, d \in D.$$

Example II.2. Let consider four patients p_1, p_2, p_3 and p_4. Their symptoms are temperature, headache, stomach pain, cough, and chest pain. Then, the set of patients is $P=\{p_1,p_2,p_3,p_4\}$ and the set of symptoms is $S=\{\text{temperature, headache, cough, chest pain}\}$. The picture fuzzy relation $Q\in PFS(P\times S)$ is hypothetical given as in Table 4.

Let the set of diagnoses be $D=\{\text{viral fever, malaria, typhoid, stomach problem, heart problem}\}$. The picture fuzzy relation $R\in PFS(S\times D)$ is given as in Table 3. Therefore composed relation $T=R\circ Q$ is as given in Table 5.

The correspondence between patient p and diagnosis d is expressed as a triple containing $\mu_T(p,d)$, $\eta_T(p,d)$, $\nu_T(p,d)$. For each $(p,d)\in P\times D$, we calculate $S_T(p,d)$ as below:

$$S_T(p,d)=\mu_T(p,d)-\nu_T(p,d)\pi_T(p,d),$$

where $\pi_T(p,d)=1-[\mu_T(p,d)+\eta_T(p,d)+\nu_T(p,d)]$.

It is easily seen that if $\mu_T(p,d)+\eta_T(p,d)+\nu_T(p,d)=1$, then $S_T(p,d)=\mu_T(p,d)$. If $S_T(p,d)\geq 0.5$, then the patient p is said to be suffered from illness d. So, From Table 7, it is obvious that, if the doctor agrees, then p_1, p_3 and p_4 suffer from Malaria, p_1 and p_3 suffer from Typhoid whereas p_2 faces Stomach problem.

Table 4. Q is picture fuzzy relation between the set of patients P and the set of symptoms S

Q	Temperature	Headache	Stomach pain	Cough	Chest pain
p_1	$(0.8,0.03,0.1)$	$(0.7,0.05,0.2)$	$(0.1,0.2,0.6)$	$(0.7,0.15,0.1)$	$(0.2,0.3,0.5)$
p_2	$(0.01,0.2,0.7)$	$(0.5,0.05,0.3)$	$(0.65,0.1,0.1)$	$(0.05,0.2,0.7)$	$(0.07,0.2,0.6)$
p_3	$(0.75,0.15,0.05)$	$(0.8,0.1,0.08)$	$(0.15,0.35,0.5)$	$(0.3,0.05,0.6)$	$(0.1,0.4,0.5)$
p_4	$(0.6,0.25,0.1)$	$(0.4,0.15,0.4)$	$(0.2,0.4,0.3)$	$(0.6,0.2,0.15)$	$(0.35,0.2,0.2)$

Table 5. R is picture fuzzy relation between the set of symptoms S and the set of diagnoses D

S_R	Fever	Malaria	Typhoid	Stomach	Chest problem
Temperature	$(0.4,0.4,0.05)$	$(0.8,0.1,0.1)$	$(0.3,0.3,0.3)$	$(0.15,0.05,0.6)$	$(0.05,0.15,0.7)$
Headache	$(0.4,0.25,0.3)$	$(0.1,0.2,0.6)$	$(0.75,0.05,0.03)$	$(0.3,0.05,0.05)$	$(0.01,0.1,0.8)$
Stomach pain	$(0.1,0.25,0.6)$	$(0.01,0.03,0.9)$	$(0.1,0.2,0.7)$	$(0.8,0.1,0.01)$	$(0.1,0.15,0.75)$
Cough	$(0.45,0.2,0.1)$	$(0.65,0.5,0.05)$	$(0.2,0.15,0.6)$	$(0.25,0.25,0.5)$	$(0.15,0.2,0.7)$
Chest pain	$(0.05,0.25,0.6)$	$(0.03,0.07,0.8)$	$(0.01,0.01,0.85)$	$(0.1,0.1,0.7)$	$(0.9,0.02,0.05)$

Table 7. T is picture fuzzy relation between the set of patients P and the set of diagnoses D

T	Fever	Malaria	Typhoid	Stomach	Chest problem
p_1	$(0.45, 0.03, 0.1)$	$(0.8, 0.03, 0.1)$	$(0.7, 0.01, 0.2)$	$(0.3, 0.03, 0.2)$	$(0.2, 0.02, 0.5)$
p_2	$(0.4, 0.05, 0.3)$	$(0.1, 0.03, 0.6)$	$(0.5, 0.01, 0.3)$	$(0.65, 0.05, 0.1)$	$(0.1, 0.02, 0.5)$
p_3	$(0.4, 0.05, 0.05)$	$(0.75, 0.03, 0.1)$	$(0.75, 0.01, 0.08)$	$(0.3, 0.05, 0.08)$	$(0.15, 0.02, 0.5)$
p_4	$(0.45, 0.15, 0.1)$	$(0.6, 0.03, 0.1)$	$(0.4, 0.01, 0.3)$	$(0.3, 0.05, 0.3)$	$(0.35, 0.02, 0.2)$

Table 6. S_T

S_T	Fever	Malaria	Typhoid	Stomach	Chest problem
p_1	0.408	0.793	0.682	0.206	0.06
p_2	0.325	–0.062	0.443	0.63	–0.09
p_3	0.375	0.738	0.7372	0.2544	–0.015
p_4	0.42	0.573	0.313	0.195	0.264

IV. CONCLUSION

The notion of picture fuzzy set is proposed very recently, it should be more researched on the theoretical as well as practical aspects. Picture fuzzy relation is one of importance notion first studied. In this paper, we first examine the validation of $\max - \min$ composition and $\max - \text{prod}$. Then, a more general composition using two arbitrary t-norm is defined. At last, a practical example is given, in which we use picture fuzzy relations as knowledge representation mean.

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