Lemma. *The expected time r_i for a random walk starting at node i to return to i is the reciprocal of the stationary probability of i. That is*

$$r_i = \frac{1}{\pi_i}.$$

- Intuitively
  - A long walk always ends up in stationary distribution $\boldsymbol{\pi}$
  - Suppose the walk length is T, then the expected number of times it visits i is $\pi_i \cdot T$
  - The average length between two visits is $\frac{T}{\pi_i \cdot T} = \frac{1}{\pi_i}$
  - A rigorous proof requires the Strong Law of Large Numbers

Theorem. *The probability that a random walk starting at node i visits j before returning to i, which equals* $\mathrm{ep}(i \to j)$, *satisfies*

$$\mathrm{ep}(i \to j) c(i,j) = \frac{1}{\pi_i},$$

where $c(i,j)$ is the commute time between i and j.

- Proof
  - Consider a random walk w starting at i, and random variables
    - $X =$ the first time w returns to i
    - $Y =$ the first time w returns to i after visiting j
  - By definition $E(X) = \frac{1}{\pi_i}$ and $E(Y) = c(i,j)$
  - Clearly $X \leq Y$, and $\Pr[X = Y] = p \triangleq \mathrm{ep}(i \to j)$
    - $E(Y - X) = p \cdot 0 + (1-p) \cdot E(Y) = (1-p)c(i,j)$
  - Also $E(Y - X) = E(Y) - E(X) = c(i,j) - \frac{1}{\pi_i}$
- $\mathrm{ep}(i \to j) + \mathrm{ep}(j \to i) = \frac{1}{c(i,j)} \left(\frac{1}{\pi_i} + \frac{1}{\pi_j} \right)$
- Recall that $h(i,j)$ is small whenever π_j is large, which is bad for personalization
  - Sarkar et al. (2008) alleviate this by restricting the length of random walk
  - Tong et al. (2007) alleviate this by reducing the dependence on stationary distribution