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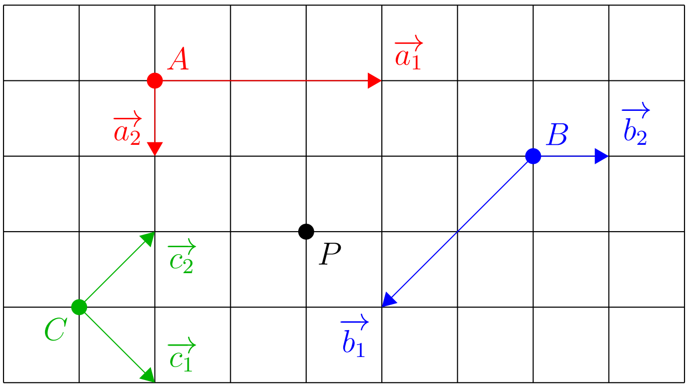
Data Analysis and Visualization

Exercise 1

Exercise A1.1: Point coordinates

Decompose the point P in the coordinate systems $(A, \vec{a_1}, \vec{a_2}, \vec{a_3})$, $(B, \vec{b_1}, \vec{b_2}, \vec{b_3})$, and $(C, \vec{c_1}, \vec{c_2}, \vec{c_3})$.



Solution A1.1:

$$\begin{aligned} P &= A + \frac{2}{3}\vec{a_1} + 2\vec{a_2} \\ &= B + \frac{1}{2}\vec{b_1} - 2\vec{b_2} \\ &= C + \vec{c_1} + 2\vec{c_2} \end{aligned}$$

Exercise A1.2: Base change in two dimensions

Let $(\vec{b_1}, \vec{b_2})$ be the basis of a 2–dimensional vector space. We write $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} := \alpha_1\vec{b_1} + \alpha_2\vec{b_2}$.

Define $\vec{c_1} := \begin{pmatrix}1\\1\end{pmatrix}$ and $\vec{c_2} := \begin{pmatrix}2\\3\end{pmatrix}$.

a) Compute γ_{11} and γ_{21} satisfying $\vec{b_1} = \gamma_{11}\vec{c_1} + \gamma_{21}\vec{c_2}$.

b) Compute γ_{12} and γ_{22} satisfying $\vec{b_2} = \gamma_{12}\vec{c_1} + \gamma_{22}\vec{c_2}$.

c) Write $\begin{pmatrix}u\\v\end{pmatrix}$ as a linear combination of $\vec{c_1}$ and $\vec{c_2}$.

d) What does the matrix $\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix}$ represent?

Solution A1.2:

The definitions of $\overrightarrow{c_1}$ and $\overrightarrow{c_2}$ say

$$\overrightarrow{c_1} = \overrightarrow{b_1} + \overrightarrow{b_2} \tag{1}$$
$$\overrightarrow{c_2} = 2\overrightarrow{b_1} + 3\overrightarrow{b_2}. \tag{2}$$

a) Compute $3 \cdot (1) - (2)$ to obtain $3\overrightarrow{c_1} - \overrightarrow{c_2} = \overrightarrow{b_1}$. Thus, $\gamma_{11} = 3$ and $\gamma_{21} = -1$.

b) Compute $(2) - 2 \cdot (1)$ to obtain $-2\overrightarrow{c_1} + \overrightarrow{c_2} = \overrightarrow{b_2}$. Thus, $\gamma_{21} = -2$ and $\gamma_{22} = 1$.

c) We insert the results of the previous two tasks:

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix} &= u\overrightarrow{b_1} + v\overrightarrow{b_2} \\ &= u(3\overrightarrow{c_1} - \overrightarrow{c_2}) + v(-2\overrightarrow{c_1} + \overrightarrow{c_2}) \\ &= (3u - 2v)\overrightarrow{c_1} + (-u + v)\overrightarrow{c_2}\end{aligned}$$

d) The matrix $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$ converts the column representation with respect to $(\overrightarrow{b_1}, \overrightarrow{b_2})$ into the column representation with respect to $(\overrightarrow{c_1}, \overrightarrow{c_2})$.

Exercise A1.3: Base change in three dimensions

Let $(\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3})$ be the basis of a 3–dimensional vector space and define

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} := \alpha_1\overrightarrow{b_1} + \alpha_2\overrightarrow{b_2} + \alpha_3\overrightarrow{b_3}, \quad \overrightarrow{c_1} := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \overrightarrow{c_2} := \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{c_3} := \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}.$$

Write $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ as a linear combination of $(\overrightarrow{c_1}, \overrightarrow{c_2}, \overrightarrow{c_3})$.

Solution A1.3:

From the definitions of $(\overrightarrow{c_1}, \overrightarrow{c_2}, \overrightarrow{c_3})$, we can read of the base transformation from $(\overrightarrow{c_1}, \overrightarrow{c_2}, \overrightarrow{c_3})$ to $(\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3})$:

$$T_{C\to B} := \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix}.$$

To compute the inverse transformation $T^{-1}_{C\to B}$ with columns a_1, a_2, a_3, we solve these linear systems of equations:

$$T_{C\to B} \cdot a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad T_{C\to B} \cdot a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad T_{C\to B} \cdot a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the linear part is always the same, we can solve all of them simultaneously by `Gauss`ian elimination (without reordering rows to prevent book–keeping).

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 4 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{3}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1/2 & -1/2 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{4}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -1 & -1/2 & -1/2 & 1 \end{array}\right) \tag{5}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \end{array}\right) \tag{6}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 3/2 & 1/2 & -1 \\ 0 & 1 & 0 & 3/2 & 1/2 & -2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \end{array}\right) \tag{7}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -1/2 & 3 \\ 0 & 1 & 0 & 3/2 & 1/2 & -2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \end{array}\right). \tag{8}$$

Thus, we can compute the column representation with respect to $(\overrightarrow{c_1}, \overrightarrow{c_2}, \overrightarrow{c_3})$:

$$T_{C\to B}^{-1} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -3/2 & -1/2 & 3 \\ 3/2 & 1/2 & -2 \\ 1/2 & 1/2 & -1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -3u - v + 6w \\ 3u + v - 4w \\ u + v - 2w \end{pmatrix}.$$

Exercise A1.4: Change of coordinates

Let $(E, \overrightarrow{e_1}, \overrightarrow{e_2})$ be an affine coordinate system in 2 dimensions. Let $(Q, \overrightarrow{q_1}, \overrightarrow{q_2})$ be another affine coordinate system, with extended coordinates (with respect to $(E, \overrightarrow{e_1}, \overrightarrow{e_2})$)

$$Q = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}, \qquad \overrightarrow{q_1} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \qquad \overrightarrow{q_2} = \begin{pmatrix} -1/3 \\ 1/2 \\ 0 \end{pmatrix}.$$

Construct the coordinate transformation to convert $E + x\overrightarrow{e_1} + y\overrightarrow{e_2}$ into the system $(Q, \overrightarrow{q_1}, \overrightarrow{q_2})$.

Solution A1.4:

We proceed in two steps. First, we construct the transformation from $(E, \overrightarrow{e_1}, \overrightarrow{e_2})$ to $(E, \overrightarrow{q_1}, \overrightarrow{q_2})$. Afterwards, we construct the transformation to $(Q, \overrightarrow{q_1}, \overrightarrow{q_2})$.
The first transformation is a change of basis from $(\overrightarrow{e_1}, \overrightarrow{e_2})$ to $(\overrightarrow{q_1}, \overrightarrow{q_2})$. The transformation from $(\overrightarrow{q_1}, \overrightarrow{q_2})$ to $(\overrightarrow{e_1}, \overrightarrow{e_2})$ is given by

$$\begin{pmatrix} 2 & -1/3 \\ -1 & 1/2 \end{pmatrix}. \tag{9}$$

To compute the inverse transformation, we employ the following fact, that can be proven by simple calculation: If $ad - bc \neq 0$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{10}$$

In our case, the transformation from $(\overrightarrow{e_1}, \overrightarrow{e_2})$ to $(\overrightarrow{q_1}, \overrightarrow{q_2})$ is

$$\begin{pmatrix} 2 & -1/3 \\ -1 & 1/2 \end{pmatrix}^{-1} = \frac{1}{1-\frac{1}{3}} \begin{pmatrix} 1/2 & 1/3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/2 \\ 3/2 & 3 \end{pmatrix}. \tag{11}$$

Written in extended coordinates, the coordinate transformation from $(E, \overrightarrow{e_1}, \overrightarrow{e_2})$ to $(E, \overrightarrow{q_1}, \overrightarrow{q_2})$ can thus be represented by the matrix

$$T := \begin{pmatrix} 3/4 & 1/2 & 0 \\ 3/2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

To shift from $(E, \overrightarrow{q_1}, \overrightarrow{q_2})$ to $(Q, \overrightarrow{q_1}, \overrightarrow{q_2})$, we need to translate by $-\overrightarrow{EQ}$. Since the extended coordinates of Q are written with respect to $(E, \overrightarrow{e_1}, \overrightarrow{e_2})$, a conversion with T is necessary. The final transformation is thus

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto T \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} - T \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/2 & -9/4 \\ 3/2 & 3 & 3/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{13}$$

Exercise A1.5: Transformations in three dimensions

Let $(E, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ be a 3–dimensional affine coordinate system.

a) What is the transformation matrix for a translation by $\frac{5}{4}\overrightarrow{e_1} - \frac{2}{9}\overrightarrow{e_2} + \frac{3}{7}\overrightarrow{e_3}$ in extended coordinates?

b) Assume that $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ is right–handed and orthogonal. Compute the linear term of the rotation with angle ψ around $\overrightarrow{e_2}$ and $\overrightarrow{e_3}$.

Solution A1.5:

The first task has the solution

$$\begin{pmatrix} 1 & 0 & 0 & 5/4 \\ 0 & 1 & 0 & -2/9 \\ 0 & 0 & 1 & 3/7 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

For the second task, we use `Rodrigues` rotation formula from the lecture: with axis $\overrightarrow{n}$ and angle ψ, we obtain

$$R_{\overrightarrow{n},\psi} = \cos(\psi) \cdot I_3 + (1 - \cos(\psi)) \cdot \overrightarrow{n}\,\overrightarrow{n}^t + \sin(\psi) \cdot R_{\overrightarrow{n}}. \tag{15}$$

Inserting $\overrightarrow{n} = \overrightarrow{e_2}$ gives

$$\cos(\psi) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1 - \cos(\psi)) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sin(\psi) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\psi) & 0 & \sin(\psi) \\ 0 & 1 & 0 \\ -\sin(\psi) & 0 & \cos(\psi) \end{pmatrix}.$$

For $\overrightarrow{n} = \overrightarrow{e_3}$, we obtain

$$\cos(\psi) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1 - \cos(\psi)) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin(\psi) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise A1.6: Transformations in two dimensions

Let $(E, \overrightarrow{e_1}, \overrightarrow{e_2})$ be a 2–dimensional affine coordinate system. Compute the following transformation matrices:

a) Uniform scaling by 3 around the point $(2, 2, 1)^t$.

b) Non–uniform scaling around the origin, by a factor of 3 in direction of $(1, 1, 0)^t$ and by a factor of $1/2$ in direction of $(1, -1, 0)^t$.

Solution A1.6:

The lecture gives the scaling matrix for scalings around the origin along the coordinate axes. To obtain the desired scalings, we need to transform the coordinate system first, perform the "standard" scaling second, and finally transform the coordinate system back.

a) To switch the coordinate origin to $(2, 2, 1)^t$ (while fixing all points!), we use the transformation matrix

$$T := \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

The combined transformation can then be computed by matrix multiplication (recall that the right–most matrix acts first!):

$$T^{-1} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -4 \\ 0 & 3 & -4 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Like in the first case, we first change coordinate axes, then apply a non–uniform scaling, and finally change back the axes. The coordinate transformation from the intermediate system to $(E, \overrightarrow{e_1}, \overrightarrow{e_2})$ is given by

$$T := \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

The combined transformation can then be computed by matrix multiplication (recall that the right–most matrix acts first!):

$$T \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} T^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7/4 & 5/4 & 0 \\ 5/4 & 7/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Exercise 2

Exercise A2.1: Homogeneous points

Find all $k, l \in \mathbb{R}$ with $[1, 0, k, 2]^t = [-2, l, 1, -4]^t$.

Solution A2.1:

The homogeneous points are equal if there is an $s \in \mathbb{R}_{\neq 0}$ with

$$s \begin{pmatrix} 1 \\ 0 \\ k \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ l \\ 1 \\ -4 \end{pmatrix}.$$

We can immediately read off $s = -2$, which gives $l = 0$ and $k = -1/2$.

Exercise A2.2: Projective Transformations

Find all projective 3–dimensional transformations (as matrices in $\mathbb{R}^{4\times 4}$) that fix the homogeneous points

$$[0,0,0,1]^t, \qquad [1,0,0,1]^t, \qquad [0,1,0,1]^t, \qquad [0,0,1,1]^t.$$

Hint: What do you know about the columns of these matrices?
How many of these transformations also fix $[1, 1, 1, 1]^t$?

Solution A2.2:

Let $M \in \mathbb{R}^{4\times 4}$ be a matrix satisfying the requirements. Since it has to fix $[0, 0, 0, 1]^t$, there has to be an $\alpha \in \mathbb{R}_{\neq 0}$ such that

$$M \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{pmatrix}$$

holds. Thus, up to the parameter α, we know the fourth column of M. Each of the other points gives us another column. With parameters $\beta, \gamma, \delta \in \mathbb{R}_{\neq 0}$, the matrix has to be a multiple (recall: matrices of projective transformations are only defined up to a scaling factor!) of

$$\begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \delta & 0 \\ \beta-\alpha & \gamma-\alpha & \delta-\alpha & \alpha \end{pmatrix}. \tag{1}$$

The image of $[1, 1, 1, 1]^t$ is $[\beta, \gamma, \delta, \beta + \gamma + \delta - 2\alpha]^t$. These can only be equal if $\beta = \gamma = \delta$ and $\beta = \alpha$ hold. Thus, up to scaling, exactly one projective transformation fixes these 5 homogenous points.

Exercise A2.3: Frustum transformation

In this exercise, we derive the projective transformation that transforms the viewing frustum to the cube.

a) Construct all projective transformations T that satisfy:

$$\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \qquad \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \qquad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}0\\1\\0\\0\end{bmatrix}.$$

Hint: The resulting matrix has 9 guaranteed zeros.

b) Characterize those T from part (a) that map $[l, t, -n, 1]^t$ to $[-1, 1, -1, 1]^t$.

Hint: Eliminate the parameters in the third column.

c) Assuming the frustum transformation has the form of part (b), compute the value of its free parameters from

$$\begin{bmatrix}r\\t\\-n\\1\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \qquad \begin{bmatrix}l\\t\\-n\\1\end{bmatrix} \mapsto \begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}, \qquad \begin{bmatrix}{}^{lf}/_{n}\\{}^{tf}/_{n}\\-f\\1\end{bmatrix} \mapsto \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

and the convention that the entry in row 4 and column 3 should be normalized to -1.

Hint: Check the images of the other points to check the correctness of your solution.

Solution A2.3:

a) From the three given images, we obtain three columns of the matrix T. Thus, the most general form of T is

$$\begin{pmatrix} x & 0 & \alpha & 0 \\ 0 & y & \beta & 0 \\ 0 & 0 & \gamma & w \\ 0 & 0 & \delta & 0 \end{pmatrix}, \tag{2}$$

where $x, y, w \in \mathbb{R}_{\neq 0}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

b) The image condition is equivalent to $T \cdot (l, t, -n, 1)^t = k \cdot (-1, 1, -1, 1)^t$ for some $k \in \mathbb{R}_{\neq 0}$. Evaluating gives

$$\begin{pmatrix} lx - n\alpha \\ ty - n\beta \\ -n\gamma + w \\ -n\delta \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} -k \\ k \\ -k \\ k \end{pmatrix}. \tag{3}$$

Solving for the greek letters results in the matrix

$$\begin{pmatrix} x & 0 & \frac{lx+k}{n} & 0 \\ 0 & y & \frac{ty-k}{n} & 0 \\ 0 & 0 & \frac{w+k}{n} & w \\ 0 & 0 & -\frac{k}{n} & 0 \end{pmatrix}. \tag{4}$$

c) Evaluating the first image condition gives

$$\begin{pmatrix} rx - lx - k \\ ty - ty + k \\ -w - k + w \\ k \end{pmatrix} = \begin{pmatrix} (r-l)x - k \\ k \\ -k \\ k \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} L \\ L \\ -L \\ L \end{pmatrix} \tag{5}$$

for some $L \in \mathbb{R}_{\neq 0}$. Thus, $L = k$ and $x = \frac{2k}{r-l}$. Similarly, we obtain $y = \frac{2k}{t-b}$. From the final image, we obtain the equation

$$-f\frac{w+k}{n} + w = \frac{kf}{n}, \tag{6}$$

with solution $w = \frac{2fk}{n-f}$. Inserting these results into the matrix gives

$$\begin{pmatrix} \frac{2k}{r-l} & 0 & \frac{k}{n}\frac{r+l}{r-l} & 0 \\ 0 & \frac{2k}{t-b} & \frac{k}{n}\frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{k}{n}\frac{f+k}{f-n} & -\frac{2fk}{f-n} \\ 0 & 0 & -\frac{k}{n} & 0 \end{pmatrix}. \tag{7}$$

The normalization condition implies $k = n$ and gives the complete frustum transformation:

$$\begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{8}$$

Exercise A2.4: Bresenham Algorithm

Consider a line from $(1, 2)$ to $(4, 4)$ on a 2–dimensional grid.

a) Compute the implicit line function $F(x, y)$.

b) What are the fragments returned by the `Bresenham` algorithm?

Solution A2.4:

a) Since $\Delta x = 3$ and $\Delta y = 2$, we obtain

$$F(x, y) = (\Delta y, -\Delta x)\begin{pmatrix} x \\ y \end{pmatrix} - (\Delta y, -\Delta x)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2x - 3y + 4. \tag{9}$$

b) Since $0 \leq \Delta y \leq \Delta x$, the `Bresenham` algorithm is directly applicable. The first fragment is $(x_0, y_0) := (1, 2)$. To obtain the next fragment, we compute

$$F(x_0 + 1, y_0 + 1/2) = F(2, 2.5) = 0.5 > 0.$$

Thus, the next fragment is $(x_1, y_1) := (2, 3)$. For the next fragment, we perform the same test:

$$F(x_1 + 1, y_1 + 1/2) = F(2, 2.5) + 2 - 3 = -0.5 < 0.$$

Thus, the next fragment is $(x_2, y_2) := (3, 3)$. The final fragment is the end of the line $(x_3, y_3) := (4, 4)$.

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Exercise 3

Exercise A3.1: Backface Culling

a) What is the angle between two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$?

b) Given a tetrahedron with vertices

$$P_1 := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad P_2 := \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \qquad P_3 := \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \qquad P_4 := \begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix},$$

and a viewer in the origin, which faces are removed by backface culling?

Solution A3.1:

For the first part, we have

$$\cos(\phi) = \frac{\overrightarrow{v}^t \overrightarrow{w}}{\| \overrightarrow{v} \| \, \| \overrightarrow{w} \|}. \tag{1}$$

For the second one, we have to consider each of the four faces individually and determine the direction of the normal. Let V be the origin.

- $\{P_1, P_2, P_3\}$: We can determine a normal vector by using the cross product:

$$\overrightarrow{n}_{123} := (P_2 - P_1) \times (P_3 - P_1) = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}.$$

Unfortunately, we still have to determine whether this normal vector points *away* from the tetrahedron. We check this with the scalar product and the fourth vertex:

$$\overrightarrow{n}^t_{123}(P_4 - P_1) = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}^t \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = 6 + 4 = 10 > 0.$$

Thus, $-\overrightarrow{n}_{123}$ points outside. For backface culling, we need to form the scalar product with the view vector:

$$-\overrightarrow{n}^t_{123}(V - P_1) = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}^t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 4 > 0.$$

Thus, this face is *not* removed.

- $\{P_1, P_2, P_4\}$: We obtain a normal vector:

$$\overrightarrow{n}_{124} := (P_2 - P_1) \times (P_4 - P_1) = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix}.$$

We perform the orientation check:

$$\overrightarrow{n}^t_{124}(P_3 - P_1) = \begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix}^t \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = -4 - 6 = -10 < 0.$$

Thus, $\overrightarrow{n}_{124}$ is outward–facing. Finally, the view vector check

$$\overrightarrow{n}^t_{124}(V - P_1) = \begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix}^t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 6 > 0$$

shows that this face is *not* removed.

- $\{P_1, P_3, P_4\}$: We obtain a normal vector:

$$\overrightarrow{n}_{134} := (P_3 - P_1) \times (P_4 - P_1) = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}.$$

We perform the orientation check:

$$\overrightarrow{n}^t_{134}(P_2 - P_1) = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}^t \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = 10 > 0.$$

Thus, $-\overrightarrow{n}_{134}$ is outward–facing. Finally, the view vector check

$$-\overrightarrow{n}^t_{134}(V - P_1) = \begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix}^t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -10 < 0$$

shows that this face is removed.

- $\{P_2, P_3, P_4\}$: We obtain a normal vector:

$$\overrightarrow{n}_{234} := (P_3 - P_2) \times (P_4 - P_2) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}.$$

We perform the orientation check:

$$\overrightarrow{n}^t_{234}(P_1 - P_2) = \begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}^t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = -10 < 0.$$

Thus, $\overrightarrow{n}_{234}$ is outward–facing. Finally, the view vector check

$$\overrightarrow{n}^t_{234}(V - P_2) = \begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}^t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -10 < 0$$

shows that this face is removed.

Exercise A3.2: Barycentric coordinates

Let $A := (0,0)$, $B := (6,0)$, and $C := (0,4)$ be points in the plane. We denote the coefficients of the barycentric combination $\alpha A + \beta B + \gamma C$ as the row (α, β, γ).

a) Compute and plot the points associated to these coefficients:

$$\begin{array}{ccccc} (1/3, 1/3, 1/3) & (0, 1/2, 1/2) & (2/3, 1/3, 0) & (5/6, 1/6, 0) & (11/12, 1/12, 0) \\ (1/2, 1/4, 1/4) & (-1, 1, 1) & (1, 1, -1) & (1, -1, 1) & (3, -1, -1) \end{array}$$

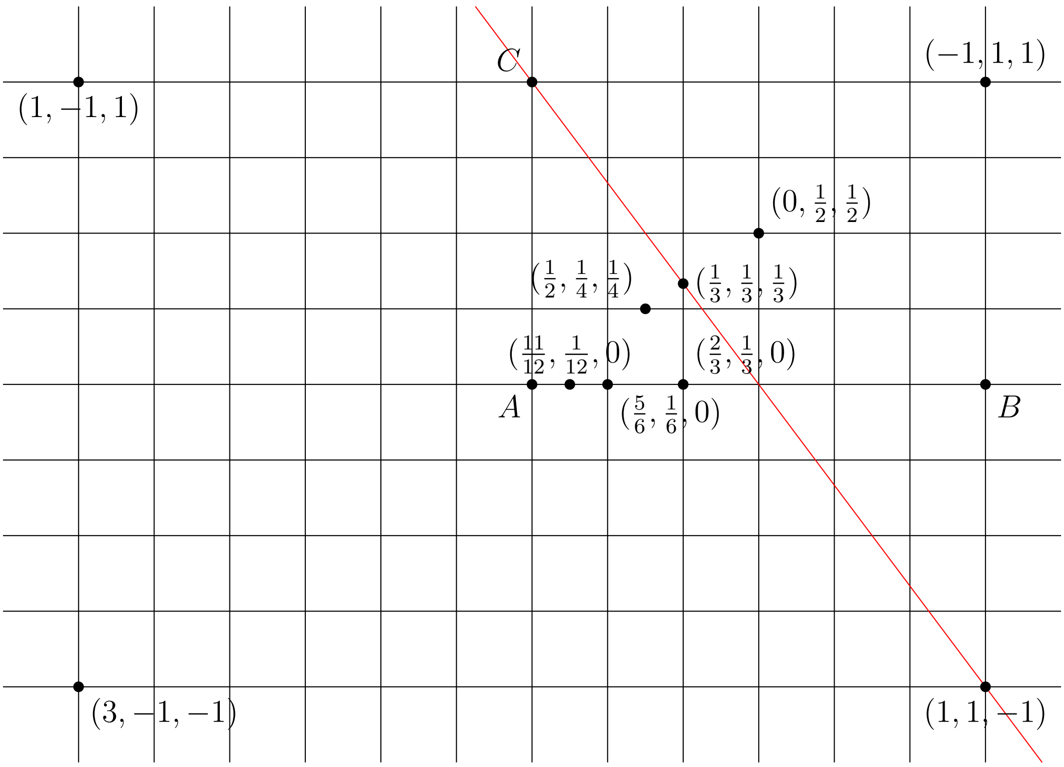
b) Compute the barycentric coefficients for any point (x, y). How can these coefficients be computed for generic points A, B, and C?

c) Assume A has brightness 1, B has brightness 0, and C has brightness $1/2$. Using `Gouraud` Shading, what is the brightness of the point (x, y)?

d) With the assumptions from (c), plot the region of points with brightness $1/2$.

Solution A3.2:

a) Drawing gives:



b) In extended coordinates, we have

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix},$$

which gives $\alpha = 1 - \beta - \gamma$, $\beta = \frac{x}{6}$, and $\gamma = \frac{y}{4}$. In general, we can obtain these coefficients by solving a system of linear equations.

c) `Gouraud` Shading interpolates the brightness directly, so we obtain with the previous result:

$$(1 - \frac{x}{6} - \frac{y}{4}) \cdot 1 + \frac{x}{6} \cdot 0 + \frac{y}{4} \cdot \frac{1}{2} = 1 - \frac{x}{6} - \frac{y}{8}.$$

d) The points with brightness $1/2$ are those satisfying $x/6 + y/8 = 1/2$. This is the equation of a line. Thus, we need only two points to plot it. We already know that C lies on this line. We further know (since A has brightness 1 and B has brightness 0) that a point between A and B has brightness $1/2$. It's easy to see that $(3, 0)$ is that point.

Exercise A3.3: Scanlines

Let $A := (0, 0)$, $B := (2, 0)$, $C := (2, 2)$, and $D := (0, 2)$ be the corners of a polygon in the plane. The brightness of A and C is 0 and the brightness of B and D is 1.

a) Using scanlines parallel to $\overrightarrow{AC}$, compute the brightness of all points within the polygon (using `Gouraud` Shading).

b) Perform the same calculation as in (a), using scanlines parallel to $\overrightarrow{BD}$.

Solution A3.3:

a) The scanlines parallel to $\overrightarrow{AC}$ satisfy the implicit equation $x - y = c$ for some $c \in \mathbb{R}$. This line only intersects the polygon if $-2 \leq c \leq 2$. We distinguish two cases:

- $-2 \leq c \leq 0$. The line intersects the polygon edges AD and CD in $(0, -c)$ and $(c + 2, 2)$, respectively. Written in barycentric line coordinates, these intersections become

$$\begin{pmatrix} 0 \\ -c \end{pmatrix} = (1 + \frac{c}{2})A - \frac{c}{2}D \qquad \begin{pmatrix} c+2 \\ 2 \end{pmatrix} = \frac{c+2}{2}C - \frac{c}{2}D.$$

  Thus, both have the same brightness $-c/2$. Thus, all points (x, y) on this line in the polygon have the same brightness $\frac{y-x}{2}$.

- $0 \leq c \leq 2$. This line intersects the polygon edges AB and BC in $(c, 0)$ and $(2, 2-c)$, respectively. With a similar calculation as before, the point (x, y) has brightness $\frac{x-y}{2}$.

In total: The brightness of a point (x, y) is $\frac{|x-y|}{2}$.

b) The scanlines parallel to $\overrightarrow{BD}$ satisfy $x+y = c$ for some $c \in \mathbb{R}$. They intersect the polygon for $0 \leq c \leq 4$.

- For $0 \leq c \leq 2$, the edges AB and AD are intersected in $(c, 0)$ and $(0, c)$, respectively. Both have the same brightness $\frac{c}{2}$.
- For $2 \leq c \leq 4$, the edges BC and DC are intersected in $(2, c - 2)$ and $(c - 2, 2)$, respectively. Both have the same brightness $2 - \frac{c}{2}$.

Thus, the point (x, y) has brightness $1 - \frac{|x+y-2|}{2}$.

Exercise A3.4: Local Lighting

What are the components of local lighting and what do they model?

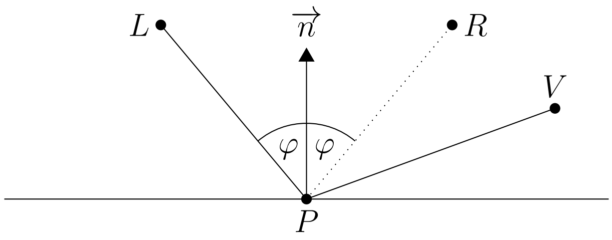
Solution A3.4:

Local lighting has three components:

1. Ambient: Models indirect lighting and multiple light ray interactions by a constant lighting term (which may be precomputed).
2. Diffuse: Models diffuse reflection, e. g. light spread in all directions from an object. It depends on the cosine of the input angle of the light.
3. Specular: Models glossy reflection, e. g. light reflected to output angles that are similar to the input angle. This is responsible for specular highlights.

Exercise A3.5: Specular Terms

We compare the specular terms of the reflection models of `Phong` and `Blinn-Phong`. Let P be a point on a surface with unit normal vector $\overrightarrow{n}$. Let L be the position of the light source and V the position of the viewer.



a) Let R be the reflection of L along $\overrightarrow{n}$ (which lies in the same plane as L, P, and $\overrightarrow{n}$). Show that

$$\overrightarrow{PR} = 2\overrightarrow{n}\,\overrightarrow{n}^t\overrightarrow{PL} - \overrightarrow{PL}. \tag{2}$$

b) The specular term of the `Phong` reflection model is proportional to

$$\left(\frac{\overrightarrow{PR}\cdot\overrightarrow{PV}}{\left\|\overrightarrow{PR}\right\|\cdot\left\|\overrightarrow{PV}\right\|}\right)^s \tag{3}$$

for some $s \in \mathbb{N}$. Why is this a plausible model?

Hint: Interpret the term with an angle.

c) The `Blinn-Phong` reflection model uses the vector $\overrightarrow{h}$ with

$$\overrightarrow{h} = \frac{\omega_L + \omega_V}{\|\omega_L + \omega_V\|}, \qquad \omega_L = \frac{\overrightarrow{PL}}{\left\|\overrightarrow{PL}\right\|}, \qquad \omega_V = \frac{\overrightarrow{PV}}{\left\|\overrightarrow{PV}\right\|}. \tag{4}$$

Interpret this vector geometrically.

d) The specular term of the `Blinn-Phong` reflection model is proportional to $(\overrightarrow{n}^t \cdot \overrightarrow{h})^s$ for some $s \in \mathbb{N}$. Why is this a plausible model?

e) Assuming that $\overrightarrow{PL}$, $\overrightarrow{PR}$, and $\overrightarrow{PV}$ lie in the same plane, how are the angles from `Phong` and `Blinn-Phong` related?

Solution A3.5:

a) We decompose $\overrightarrow{PL}$ into $\overrightarrow{n}$ and a vector $\overrightarrow{u}$ orthogonal to $\overrightarrow{n}$:

$$\overrightarrow{PL} = \alpha\overrightarrow{n} + \beta\overrightarrow{u}. \tag{5}$$

Then, it is easy to see that $\overrightarrow{PR} = \alpha\overrightarrow{n} - \beta\overrightarrow{u}$. We check the proposed term:

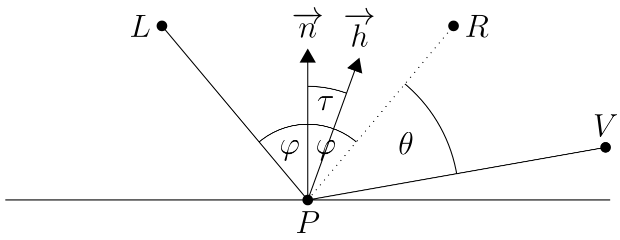
$$\begin{aligned}2\overrightarrow{n}\,\overrightarrow{n}^t\overrightarrow{PL} - \overrightarrow{PL} &= 2\overrightarrow{n}\,\overrightarrow{n}^t(\alpha\overrightarrow{n} + \beta\overrightarrow{u}) - \alpha\overrightarrow{n} - \beta\overrightarrow{u}\\ &= 2\alpha\overrightarrow{n}(\overrightarrow{n}^t\overrightarrow{n}) + 2\beta\overrightarrow{n}(\overrightarrow{n}^t\overrightarrow{u}) - \alpha\overrightarrow{n} - \beta\overrightarrow{u}\\ &= 2\alpha\overrightarrow{n} - \alpha\overrightarrow{n} - \beta\overrightarrow{u}\\ &= \alpha\overrightarrow{n} - \beta\overrightarrow{u}.\end{aligned}$$

b) The proposed term is equal to $(\cos(\theta))^s$, where θ is the angle between $\overrightarrow{PR}$ and $\overrightarrow{PV}$. The cosine is close to 1 for small angles. Thus, this term is close to 1 if $\overrightarrow{PV}$ only deviates a bit from $\overrightarrow{PR}$. Otherwise, the exponent s leads to a very small value. This behaviour is similar to that of glossy reflection.

c) $\overrightarrow{h}$ is the (normalised) vector that bisects the angle between $\overrightarrow{PL}$ and $\overrightarrow{PV}$ at P.

d) The proposed term is equal to $(\cos(\tau))^s$, where τ is the angle between $\overrightarrow{h}$ and $\overrightarrow{n}$. Thus, we only obtain a significant value if $\overrightarrow{h}$ is close to $\overrightarrow{n}$. These vectors are close if and only if $\overrightarrow{PR}$ is close to $\overrightarrow{PV}$. Thus, the behaviour is similar to that of glossy reflection.

e) We draw all angles in the same picture:



This allows us to see that

$$\tau = \frac{2\varphi + \theta}{2} - \varphi = \frac{\theta}{2}.$$

Exercise A3.6: Shading

Which shading models were covered in the lecture and how do they differ?

Solution A3.6:

We covered three shading models in the lecture:

1. Flat Shading: All fragments in a polygon get the same brightness. This looks blocky and produces `Mach` banding.
2. `Gouraud` Shading: Interpolate the brightness of each fragment bilinearly from the brightness of the vertices.
3. `Phong` Shading: Interpolate the normal vector of each fragment bilinearly from the brightness of the corner vertices. Then, compute local lighting with respect to these normal vectors. Since lighting is not linear but angle–dependent, this gives a more realistic brightness gradient.

Data Analysis & Visualization

Virtual Reality

Visual Computing Institute

20.01.2021

Exercise 1

Enumerate and explain the three primary components of virtual reality (as defined by Burdea and Coiffet in the book Virtual Reality Technology).

Answer of exercise 1

The three primary component are: Immersion, interaction, imagination:

- **Immersion** refers to the perception of being physically present in a non-physical world.
- **Interaction** refers to the requirement that the virtual environment should not be static instead be navigable and the objects within the environment should be manipulable.
- **Imagination** refers to human mind's capability to perceive nonexistent phenomena and believe in auditory/visual/haptic illusions (a non-technical aspect).

Exercise 2

Name and describe three

a) psychological cues that provide depth perception in traditional computer graphics.

b) physiological cues that provide depth perception in virtual reality.

Answer of exercise 2

a) **Psychological cues** that provide depth perception in traditional computer graphics:

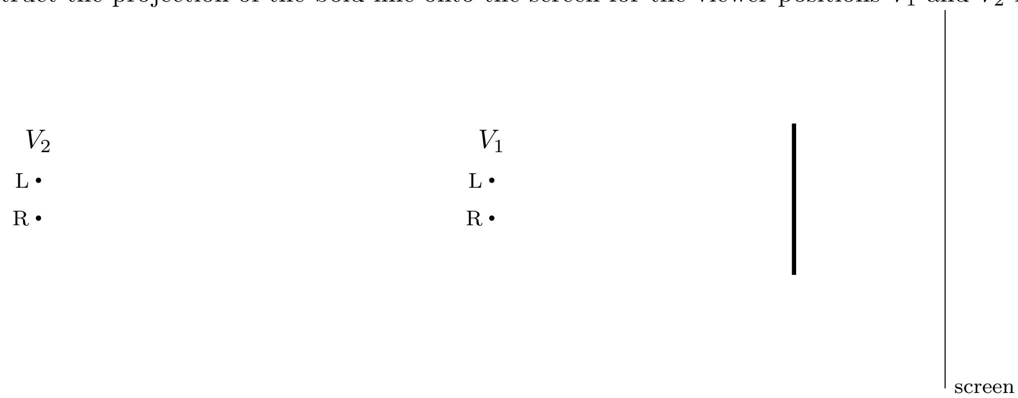
- **Perspective shortening**: The apparent size of objects with known size reveal information on their spatial configuration.
- **Occlusion**: The relative occlusion of objects reveal information on their spatial configuration.
- **Lights/shadows**: The gradient coloring of objects due to lights and the shadows they cast reveal information on their spatial configuration.
- **Texture gradients**: The distortion of regularly spaced objects reveal information on their spatial configuration.
- **Atmospheric perspective**: The decrease in visual clarity of objects due to extinction (absorption/scattering) of light reveal information on their spatial configuration.

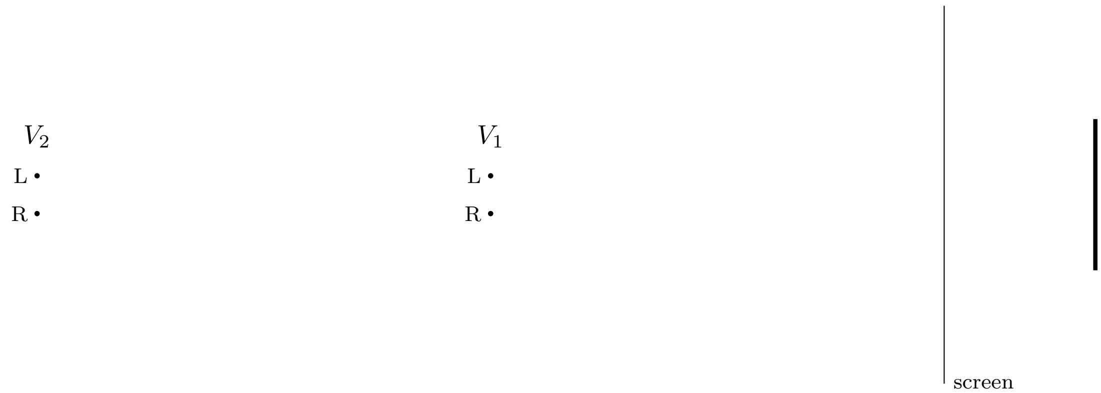
b) physiological cues that provide depth perception in virtual reality:

- **Stereopsis**: The horizontal disparity of images created by binocular vision reveal information on the relative depth of objects.
- **Ocular motor factors (accomodation/convergence)**: Accomodation refers to the adjustment of the lens to maintain a clear image or focus on an object as its distance varies. Convergence refers to the simultaneous inward movement of the eyes when focusing to an object. Both reveal information on an object's configuration depending on their magnitude.
- **Motion parallax**: As the viewer is in motion, objects further from the viewer move slower than objects closer to the viewer which reveal information on objects' relative configuration.

Exercise 3

Construct the projection of the bold line onto the screen for the viewer positions V_1 and V_2 in the following two settings.





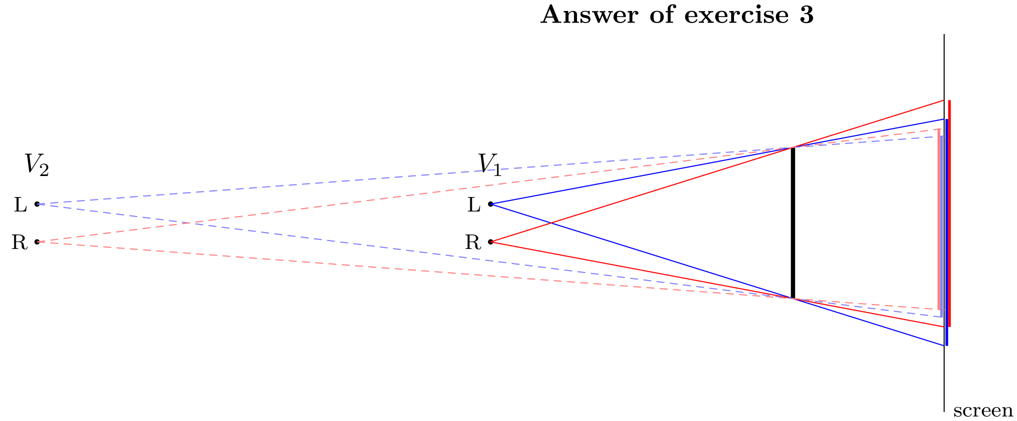
Answer the following for each setting:

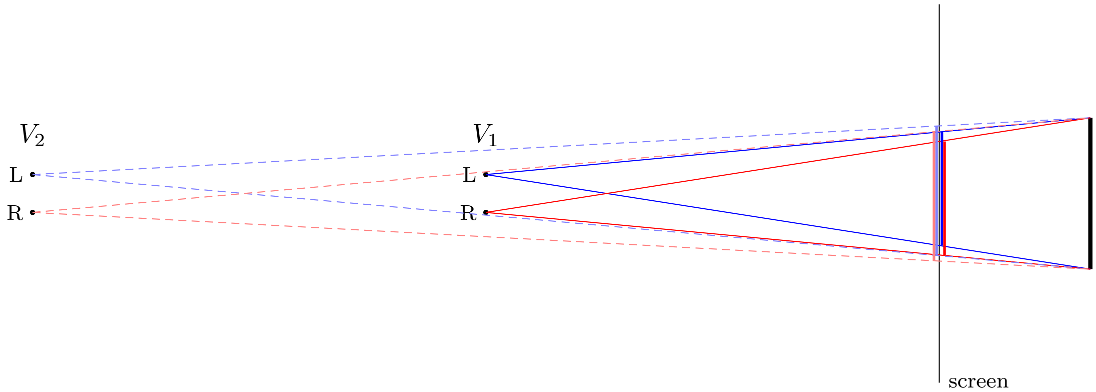
a) In which direction do the images of the left and right eyes shift?

b) How does the distance of the viewer to the object effect the apparent size of the object?

c) How does the distance of the viewer to the object effect the magnitude of the shift?

Answer of exercise 3





Answer the following for each setting:

a) In which direction do the images of the left and right eyes shift?

   For an object in front of the screen, the left eye image is shifted to the right, and the right eye image is shifted to the left.
   For an object behind the screen, the left eye image is shifted to the left, and the right eye image is shifted to the right.

b) How does the distance of the viewer to the object effect the apparent size of the object?

   For an object in front of the screen, closer the viewer, larger the image.
   For an object behind the screen, closer the viewer, smaller the image.

c) How does the distance of the viewer to the object effect the magnitude of the shift?

   For both settings, closer the viewer, larger the shift.

Exercise 4

Discuss the advantages and disadvantages of *head-mounted displays* in comparison to *room-mounted displays* with respect to:

a) field of view
b) field of regard
c) resolution and screen-door effect
d) cyber-sickness
e) navigation volume

Answer of exercise 4

	head-mounted displays	room-mounted Displays
field of view	usually way smaller than FOV of human eye	up to the full FOV of the human eye; may depend on distance to projection surface
field of regard	360° if proper tracking is available	depends on arrangement of projection surfaces; 360° only in fully-surrounding projections
resolution and screen-door effect	fixed resolution per eye; screen door effect depends on ratio of FOV/resolution	fixed resolution for projection surfaces; screen door effect dependent on distance to projection surface
cyber-sickness	often strong or easily occurring due to limited resolution, translational *and* rotational latency, etc.	usually less than with HMDs, because of higher resolution and the lack of rotational latency
navigation volume	varies very strongly; can reach from only small head movements to more than 1000 m^2; not dependent on the display, but instead on the tracking system and area	usually dependent on the display size, since only the area in front of the projection screens is tracked; in CAVE-like systems, usually exactly matches the interior

Exercise 5

Discuss the advantages and disadvantages of *anaglyph*, *polarized*, *shutter* and *infitec* stereo techniques with respect to:

a) image quality
b) synchronization requirements
c) cost

Answer of exercise 5

	anaglyph	polarized	shutter	infitec
image quality	strong ghosting, bad colors	ghosting may occur, good colors	no ghosting, good colors	(nearly) no ghosting, mediocre colors
synchronization requirements	none	buffer swap	buffer swap, glasses, projectors	buffer swap
costs	very cheap (only glasses)	mediocre (relatively cheap glasses, one standard projector + filter per eye, special screen material needed)	expensive (expensive glasses, fast projectors (120HZ))	mediocre (expensive glasses and filters, one standard projector per eye)

Data Analysis & Visualization

Immersive and High-Performance Visualization

Visual Computing Institute

27.01.2021

Exercise 1

Redirected walking is a navigation technique which enables the user to explore virtual worlds much larger than the tracking area.

a) Within this context, describe what "gain" means. Explain the three gains mentioned in the class.

b) Design an infinitely long virtual path and show that by using Redirected Walking, the path can be traversed using a finite real tracking area. Use the following:

- Use only rotation gains between 0.8 and 1.2 (no translation or curvature gains).
- Fit the path into a real area of 4m×4m.
- The rotation gain can be changed instantly and as often as necessary.
- Choose the starting point within the real tracking space freely.

Sketch both the virtual and the real path. Stop as soon as a repetition becomes clear.

c) Indicate at least two problems that can be expected to arise with your plan in practice.

Answer of exercise 1

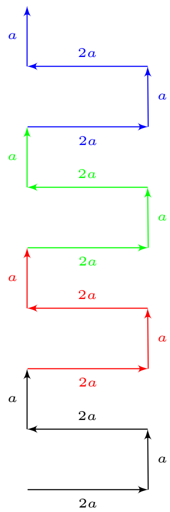
a) Gain refers to scaling of movement in the physical tracking area as it is being mapped to the virtual coordinate system:

- **Translation gain:** Amplifies or reduces the movements of the user.
- **Rotation gain:** Amplifies or reduces the head rotations of the user around the yaw axis.
- **Curvature gain:** Applies a rotation around the yaw axis based on forward translation, which enables seamless infinite walking (e.g. walking a straight road in circles).

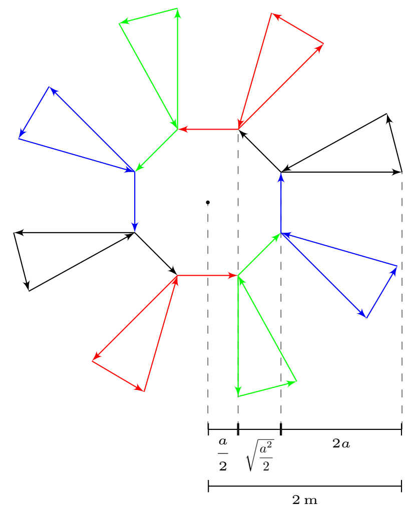
For more reading, click here.

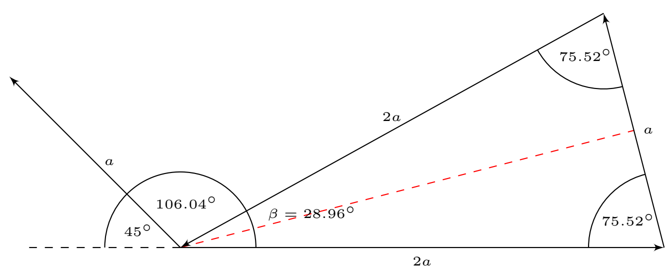
b)

Virtual path:



Real path:





Use only rotation gains between 0.8 and 1.2:

$\Rightarrow 90° \cdot 0.8 \leq angle \leq 90° \cdot 1.2$

$\approx 72° \leq angle \leq 108°$

Determine angles of triangle:

$sin(\frac{\beta}{2}) = \frac{0.5a}{2a} = 0.25$

$\Rightarrow \beta = 2 \cdot sin^{-1}\left(\frac{0.5a}{2a}\right)$

$\Rightarrow \beta \approx 28.96°$

$\Rightarrow 75.52° = (180° - 28.96°) \cdot 0.5$

Choose 45° and determine other angle:

$\Rightarrow 106.04° = 180° - 28.96° - 45°$

Choose a accordingly:

$\Rightarrow \frac{a}{2} + \sqrt{\frac{a^2}{2}} + 2a = 2\,m$

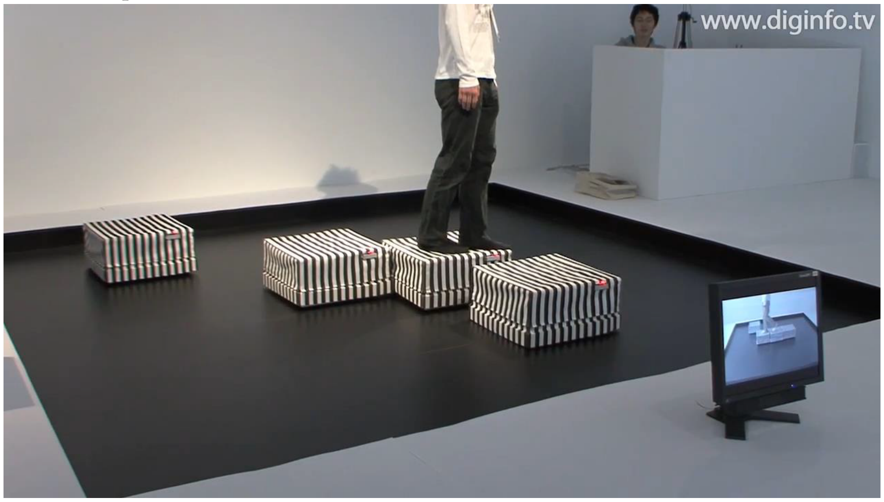
$\Rightarrow a \approx 0.6\,m$

c)
- If the user's path is known (or predetermined by the application), real users would still not traverse the path perfectly. For example, they can be expected to round corners and deviate from the prescribed path due to gait imperfections, inattentiveness and other variations that are different for each user. Even over short times, this would induce significant variations from the intended path, requiring a larger tracking area in practice.
- Changing the rotation gain instantly (instead of over time) can increase the noticeability of Redirected Walking being used and may require smaller gains.

Exercise 2

We presented different walking technology in the lecture to you that try to modify real walking such that it can be applied in finite spaces. In the following sub-tasks discuss the arising problems of these technologies and why they are not widely in use.

a) Moveable plates.



`https://www.youtube.com/embed/qXORobSAT44?start=2`

b) Redirected walking.

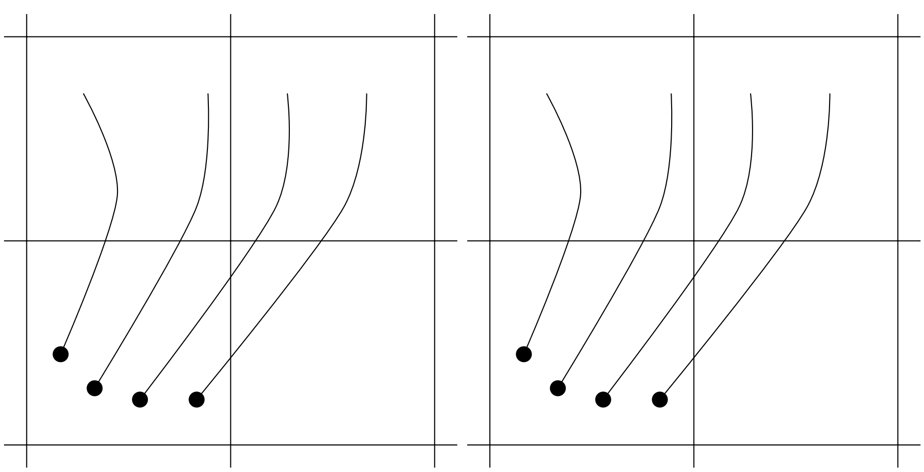
Answer of exercise 2

a)
- Slow
- Susceptibility to mechanical and software failures
- Not well able to react to spontaneous direction changes

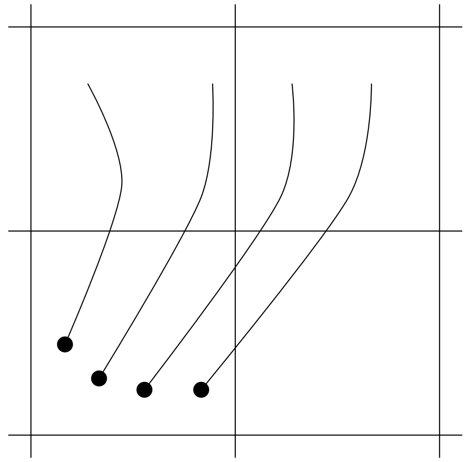
b)
- In a real application, the path a user is going to take is usually not known in advance, preventing fixed path designs.
- Users can be sensitive to gains. Limits for deviation: $\sim 0.7 - \sim 1.2$. Limits are also dependent on kind of gain.
- Hard to modify in runtime to enable free walking and discovering of scene.

Exercise 3

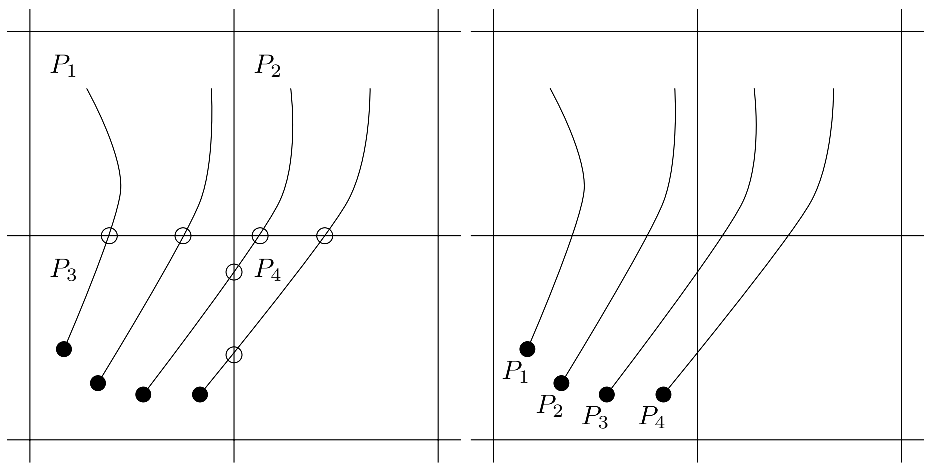
Assign the processes P_1, P_2, P_3, P_4 to the following particle advection scenarios in such a way that they achieve (a) data-parallelism (b) task-parallelism. Mark inter-process communication points when necessary.

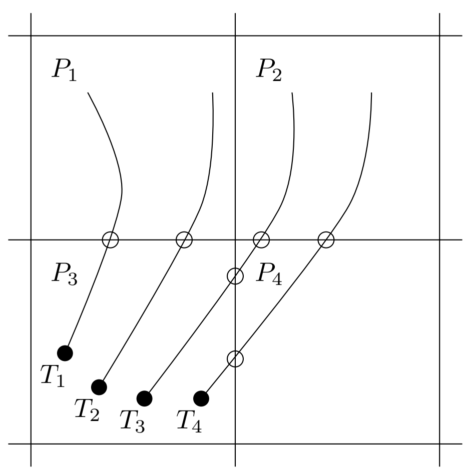


Each process now has four threads T_1, T_2, T_3, T_4. Achieve hybrid parallelism, by combining the two approaches.



Answer of exercise 3





Although we learned them within the specific context of particle advection, these parallelization approaches apply to all data analysis and visualization algorithms today. Hybrid parallelism (e.g. MPI+TBB or MPI+CUDA or HPX) is the de-facto standard to high-performance visualization today, varying from marching cubes to topological analysis.

Exercise 4

State the major difference between post-hoc and in-situ visualization. Describe three motives for in-situ visualization.

Answer of exercise 4

In post-hoc (after the fact) visualization, the data is written to disk during simulation/acquisition and later retrieved to memory for analysis/visualization, whereas in-situ (in the situation) analysis/visualization is performed in conjunction with the simulation, often in the same memory space.

- Capability: In cases where human intervention is necessary to the simulation as it is running.
- Capacity : In cases where the data can't be written to the disk due to sizes exceeding IO capacity.
- Economy : In cases where a separate analysis/visualization infrastructure (e.g. GPUs, IO fabric) is not affordable.

For more reading, click here.

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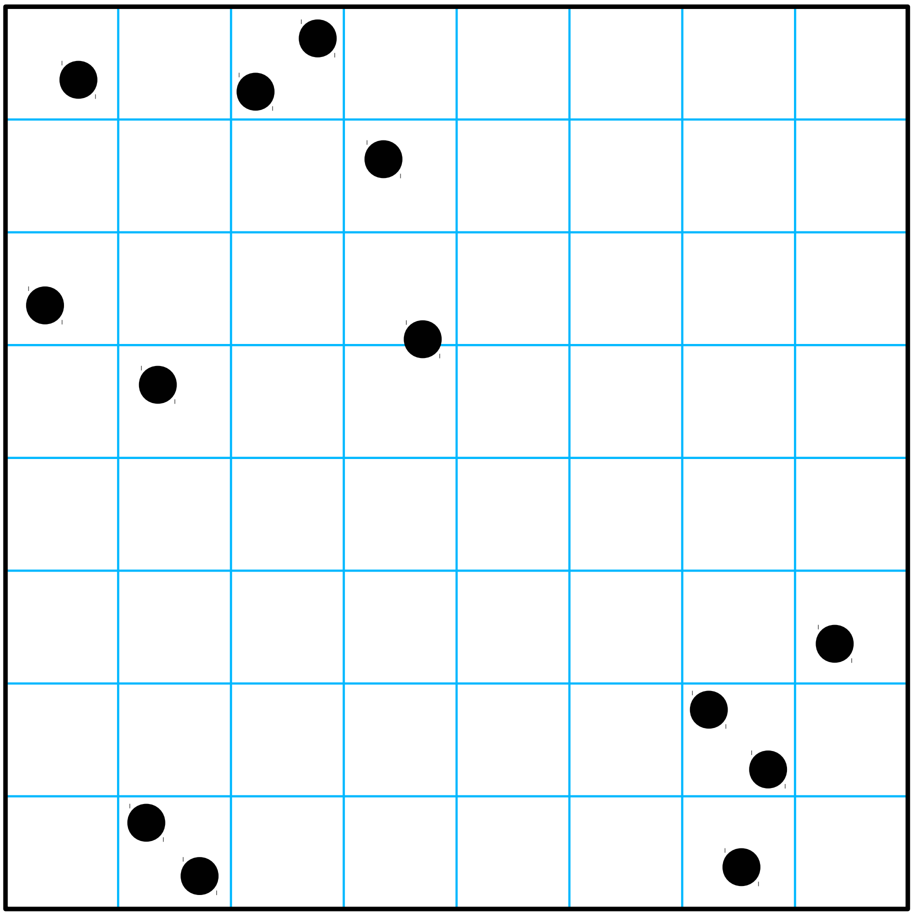
RWTH Aachen

Data Analysis and Visualization

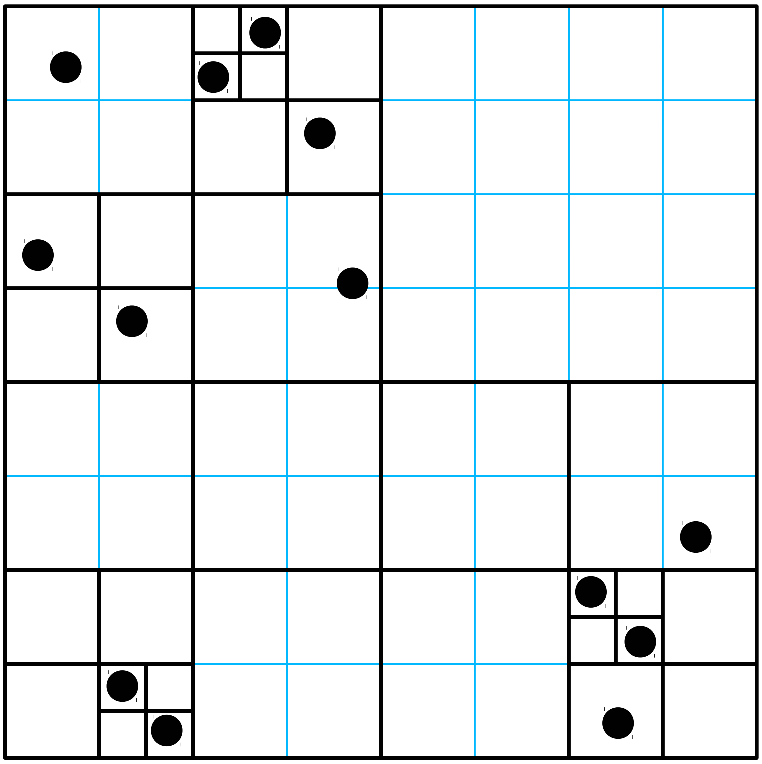
Exercise 4

Exercise A4.1: Volumetric Representations

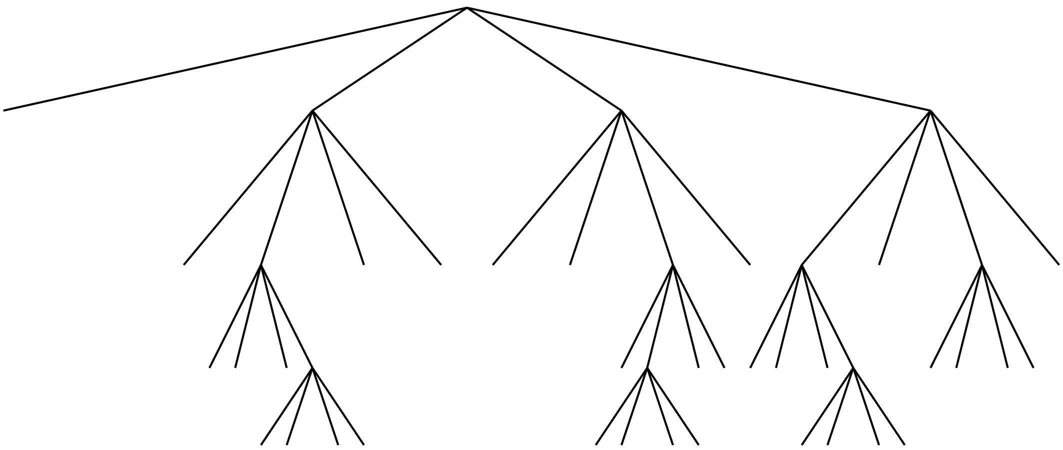
Given is a two-dimensional scene with 13 data points.



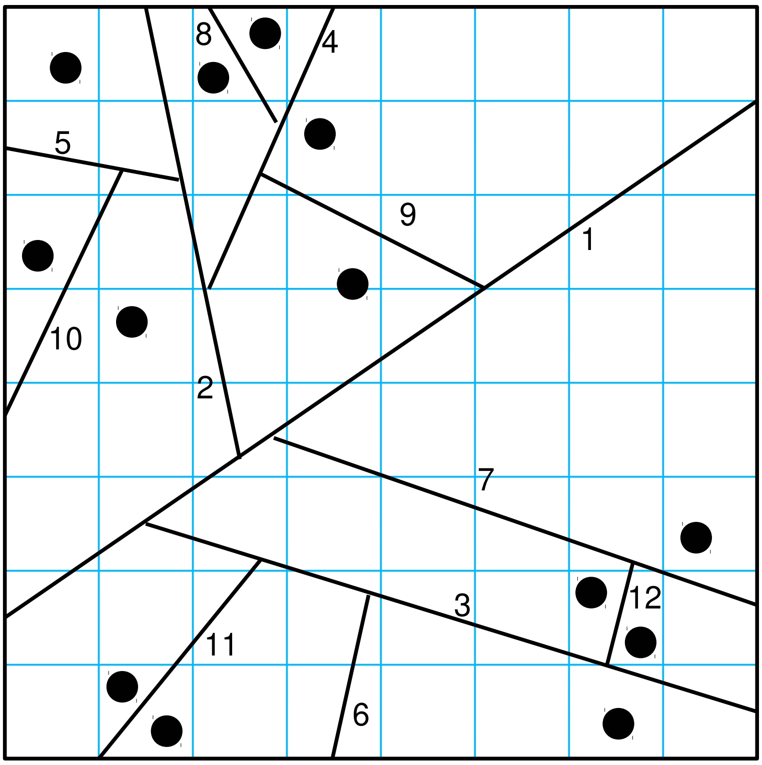
1. Partition the scene using an *adaptive quadtree* (2D version of the adaptive octree) until all data points lie within separate cells. Use the minimum number of partition steps. **Solution A4.1:**



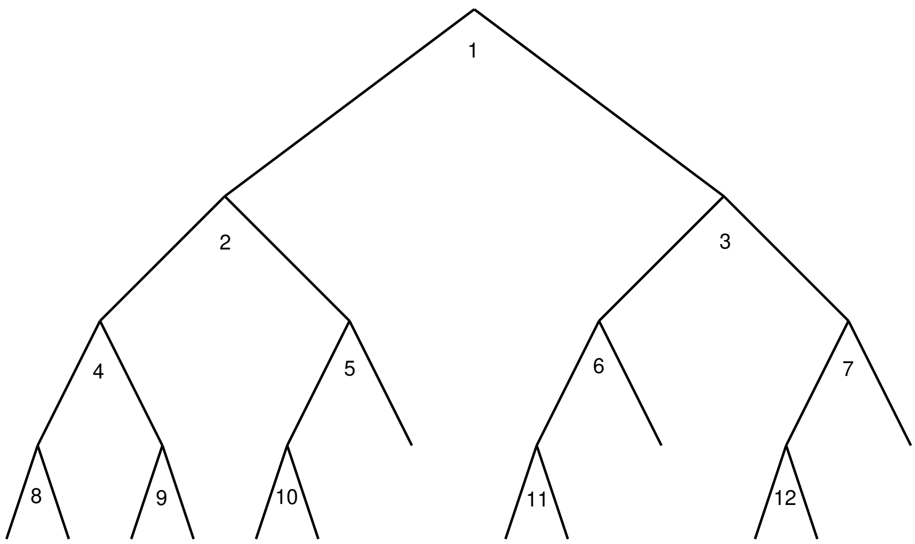
2. Sketch the corresponding quadtree. **Solution A4.1:**



3. Partition the scene using *Binary Space Partitioning* (BSP) until all data points lie within separate cells. Use the minimum number of partition steps. **Solution A4.1:**

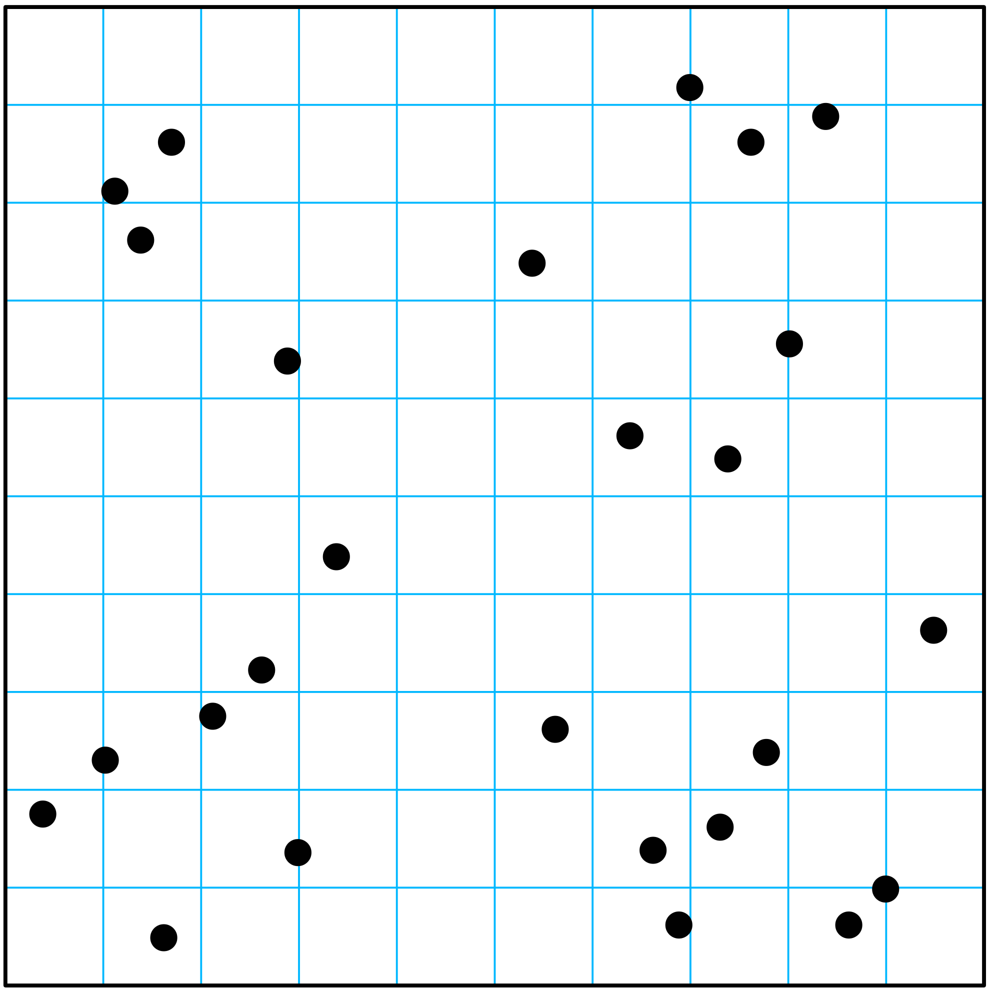


4. Sketch the corresponding binary tree. **Solution A4.1:**

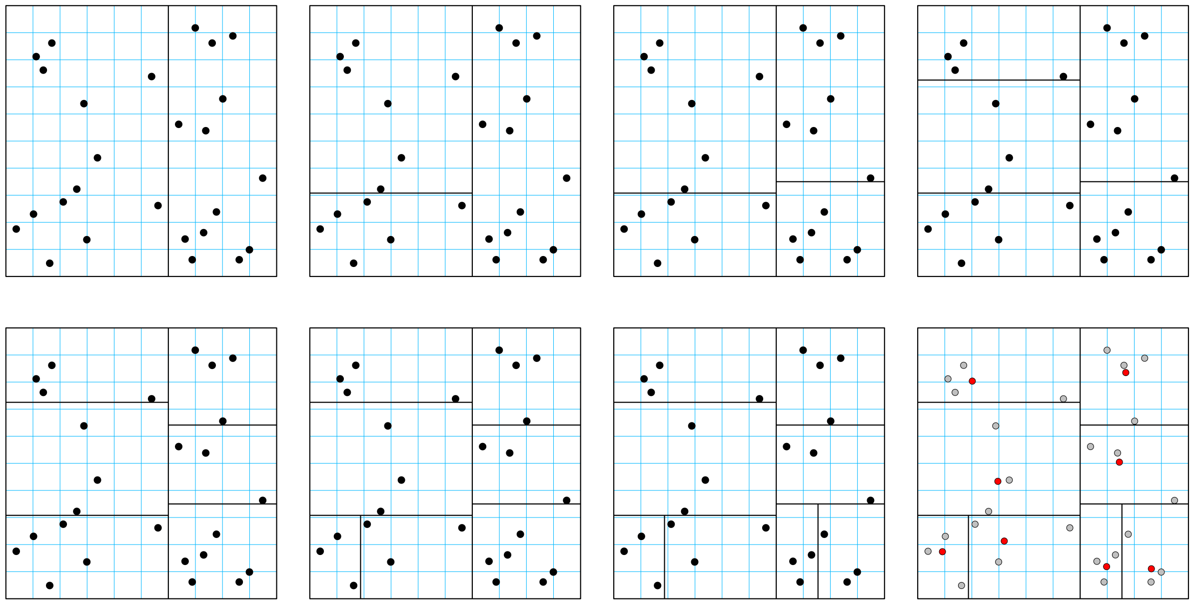


Exercise A4.2: Quantization

The Median Cut algorithm is an application of the k-d tree, where the space is partitioned at each median point along the longest dimension. This can be used to select a set of points that represents the whole data.
Given the following *2D* setting, apply as many steps of the Median Cut algorithm as necessary in order to represent the given setting with *eight* partitions. As a cell's representative, use the center of gravity of all points within the cell.



Solution A4.2:



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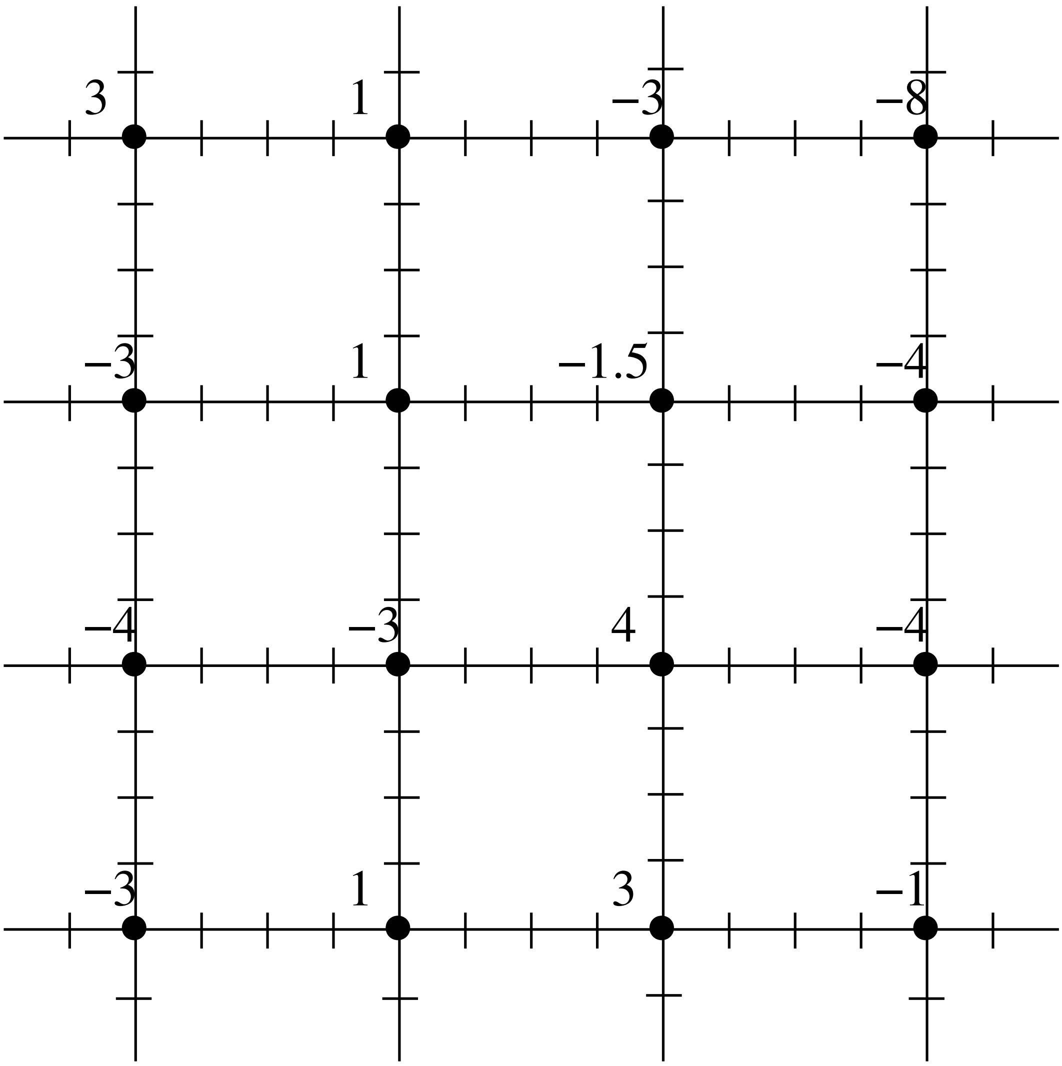
Data Analysis and Visualization

Exercise 5

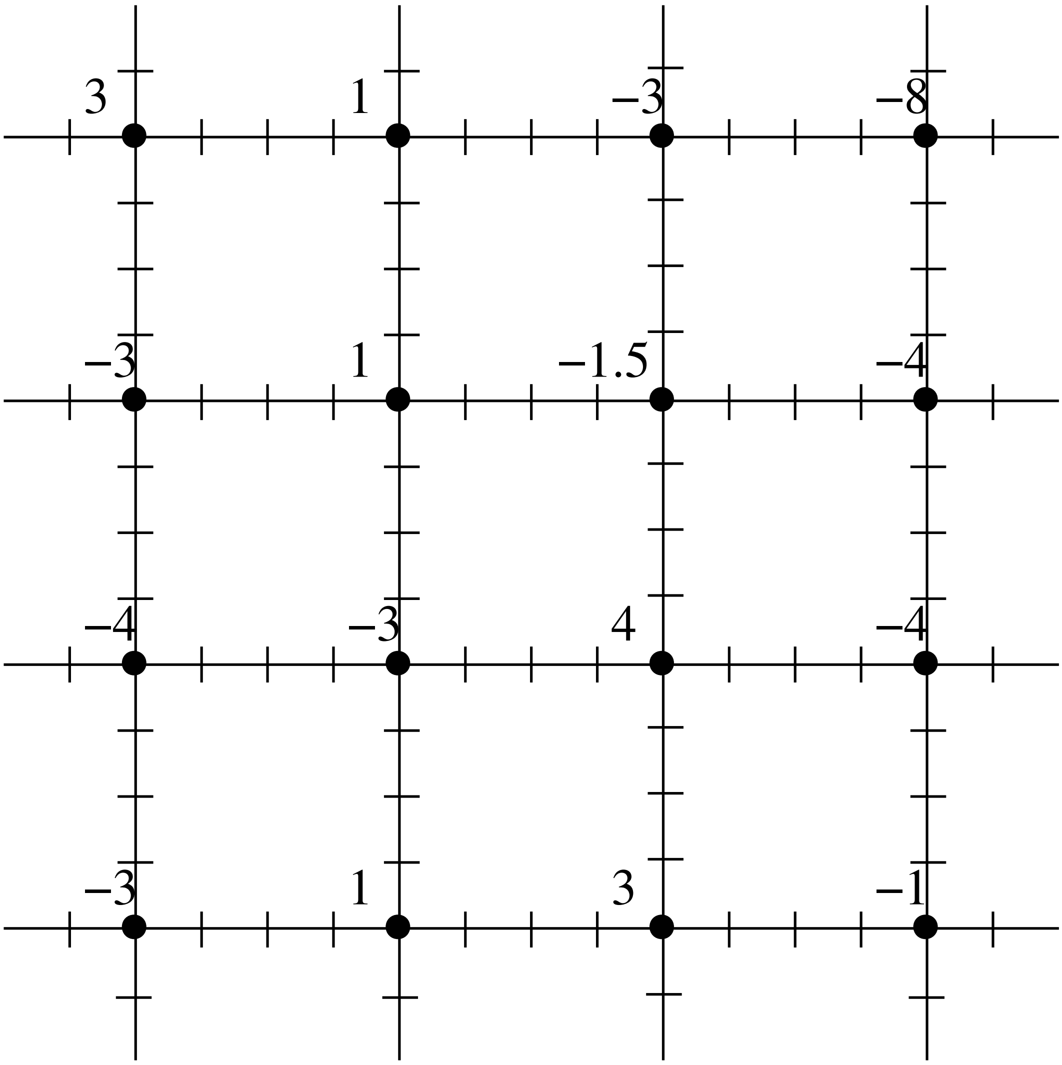
Exercise A5.1: Marching Squares

Use the Marching Squares algorithm to extract the zero-isolines of the given scalar field $f(x, y)$. Scalar values at arbitrary points are computed using bilinear interpolation of the explicitly given values of their surrounding grid points. In case of topological ambiguouity, use

1. the midpoint decider.



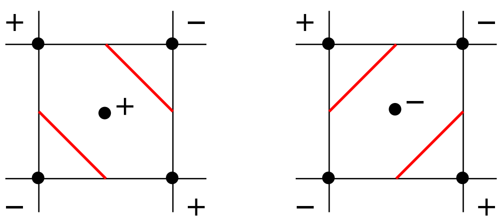
2. the asymptotic decider.



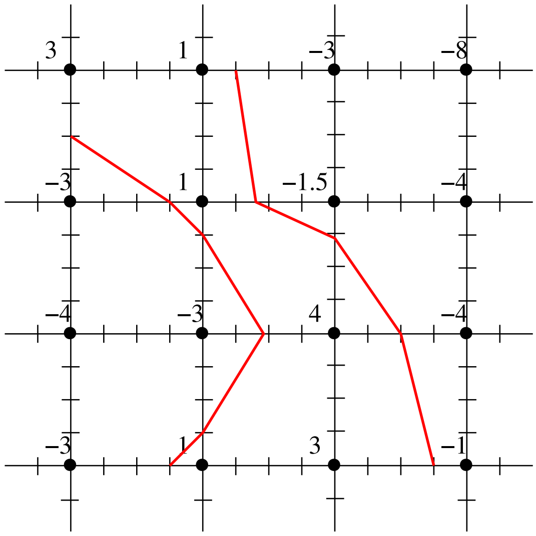
Solution A5.1:

1. Midpoint decider:

$$f_{center} = \frac{1}{4}(f_{i,j} + f_{i+1,j} + f_{i,j+1} + f_{i+1,j+1})$$



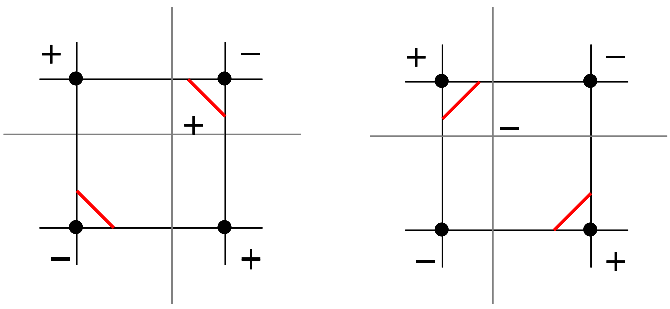
$f_{center} \geq c$ $\qquad$ $f_{center} < c$



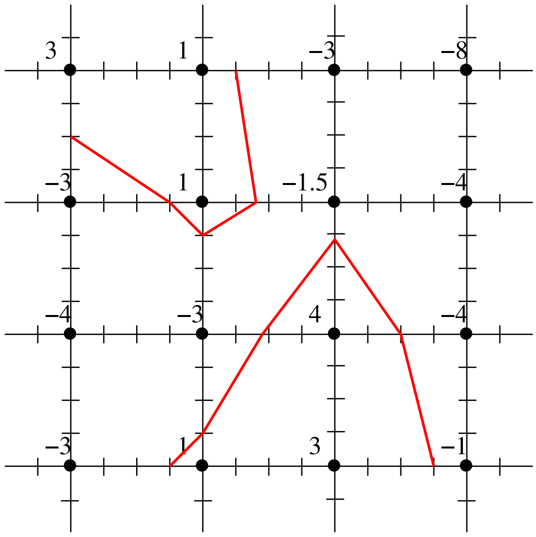
Evaluate f_{center} in center cell: $f_{center} = \frac{1}{4}(-3 + 4 + 1 - 1.5) = \frac{1}{8} > 0$

2. Asymptotic decider:

$$f_{asymptotic} = \frac{f_{i,j} \cdot f_{i+1,j+1} + f_{i+1,j} \cdot f_{i,j+1}}{f_{i,j} + f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j}}$$



$f_{asymptotic} \geq c$ $\quad$ $f_{asymptotic} < c$



Evaluate $f_{asymptotic}$ in center cell: $f_{asymptotic} = \frac{(-3)\cdot(-1.5)+1\cdot 4}{-3-1{,}5-1-4} = -\frac{8.5}{9.5} < 0$

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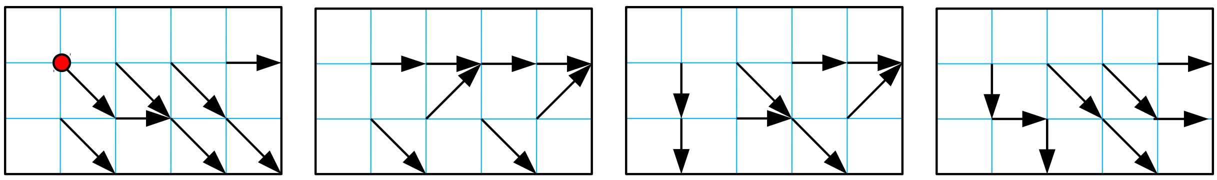
Data Analysis and Visualization

Exercise 6

Exercise A6.1: Characteristic Lines

Given is the following non-stationary flow field at time steps t_0, t_1, t_2 and t_3. Particles are released into the flow field at the marked position at all times.

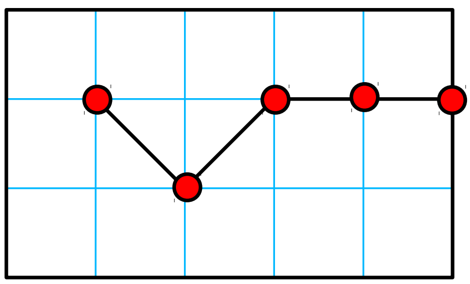


t_0 t_1 t_2 t_3

Sketch the following characteristic lines:

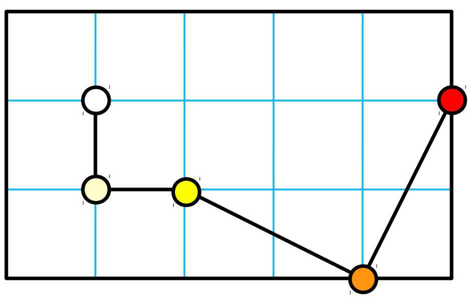
1. The **path line** of a single particle after time step t_3 that is released into the flow at time step t_0 at the marked position.

   Solution A6.1:



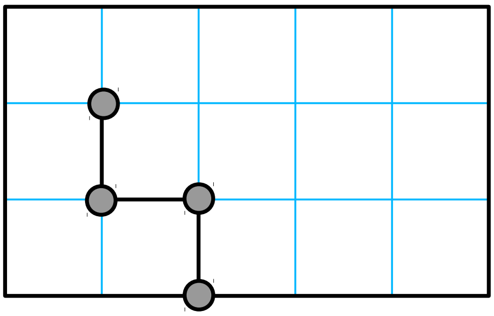
2. The **streak line** after time step t_3 when particles are continuously released into the flow at the marked position.

   Solution A6.1:



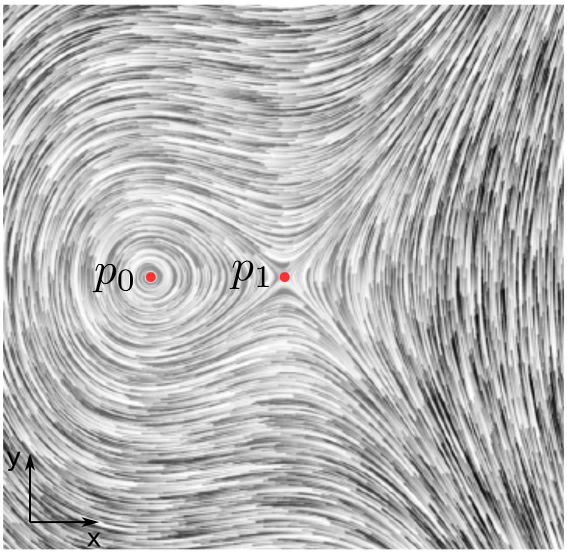
3. The **stream line** at time step t_3 for the position that is marked in the left image.

Solution A6.1:

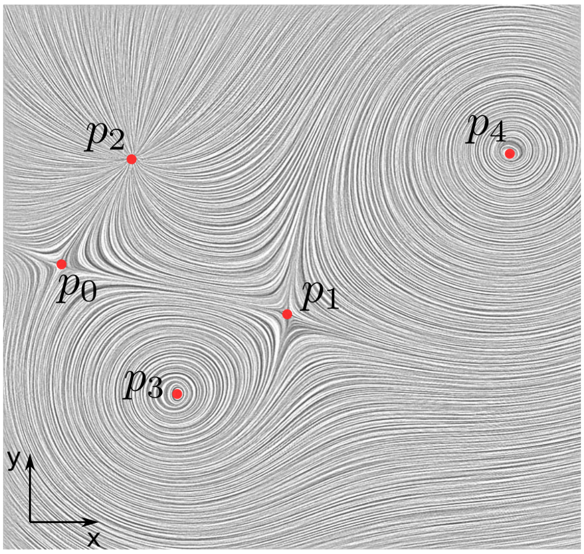


Exercise A6.2: Topological Skeleton

In the two vector fields given below, draw the topological skeleton of separatrices by connecting the marked critical points. Add an arrow indicating the flow direction to each of the separatrices.



(a)



(b)

At the critical points p_i, the Eigenvectors e_i^0, e_i^1 of the Jacobian matrix with corresponding Eigenvalues λ_i^0, λ_i^1 were computed as follows.

(a)

$$e_0^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \lambda_0^0 = i \qquad e_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_0^1 = -i$$

$$e_1^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_1^0 = -0.5 \qquad e_1^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \lambda_1^1 = 0.5$$

(b)

$$e_0^0 = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} \qquad \lambda_0^0 = 1 \qquad e_0^1 = \begin{pmatrix} 0.8 \\ -0.2 \end{pmatrix} \qquad \lambda_0^1 = -1$$

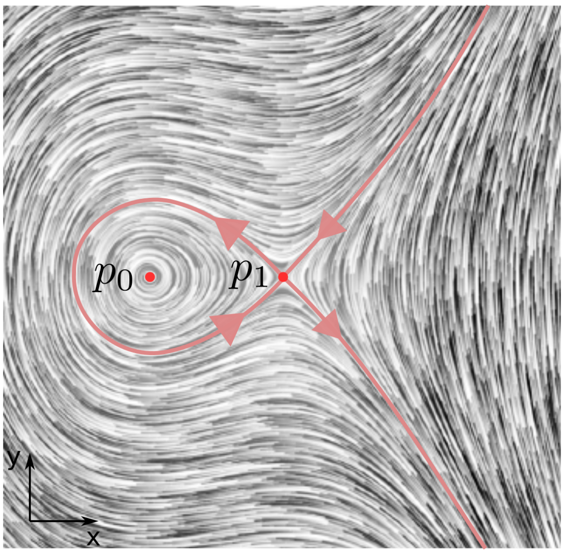
$$e_1^0 = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} \qquad \lambda_1^0 = -0.5 \qquad e_1^1 = \begin{pmatrix} 0.7 \\ -0.3 \end{pmatrix} \qquad \lambda_1^1 = 1.5$$

$$e_2^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_2^0 = -2 \qquad e_2^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \lambda_2^1 = -3$$

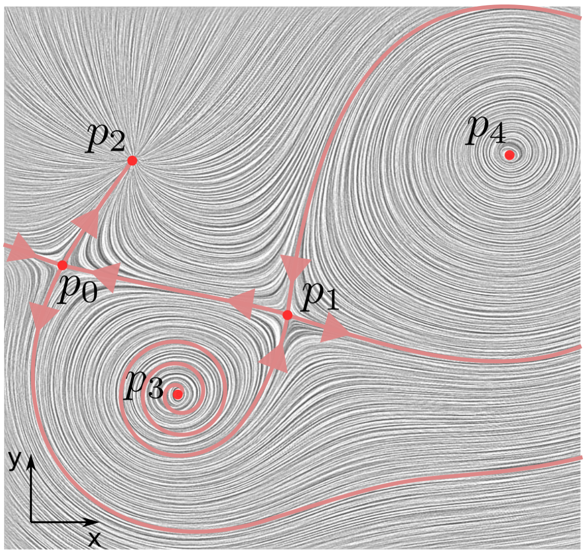
$$e_3^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \lambda_3^0 = 1+i \qquad e_3^1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \lambda_3^1 = 1-i$$

$$e_4^0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \lambda_0^0 = \frac{i}{2} \qquad e_4^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_0^1 = -\frac{i}{2}$$

Solution A6.2:



(a)



(b)

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Data Analysis and Visualization

Exercise 7

Exercise A7.1: Feature Vector Comparison

Let the feature vectors $f_1, f_2 \in \mathbb{R}^5$ be given as

$$f_1 = \begin{pmatrix} 3.3 \\ 2.4 \\ 1.9 \\ 5.0 \\ 3.8 \end{pmatrix} \quad f_2 = \begin{pmatrix} 2.8 \\ 2.4 \\ 1.1 \\ 6.2 \\ 4.3 \end{pmatrix}$$

1. Compute a dissimilarity measure $\delta(f_1, f_2)$ using the Minkowski distance for $p = 1, 2, \infty$.
2. For each result, turn $\delta(f_1, f_2)$ into a similarity measure using the
   - logarithmic
   - exponential

   similarity functions.
3. Compute the direct similarity measures
   - dot-product measure ${f_1}^T f_2$
   - cosine measure $\frac{{f_1}^T f_2}{\|f_1\| \|f_2\|}$

Solution A7.1:

1. $$f_\Delta = f_2 - f_1 = \begin{pmatrix} -0.5 \\ 0.0 \\ -0.8 \\ 1.2 \\ 0.5 \end{pmatrix}$$
   - $\|f_\Delta\|_1 = 3$
   - $\|f_\Delta\|_2 = 1.6062$
   - $\|f_\Delta\|_\infty = 1.2$
2. logarithmic: $s_{log}(f_1, f_2) = 1 - \log(1 + \delta(f_1, f_2))$
   - $\|f_\Delta\|_1 \rightarrow s_{log} = -0.38629$

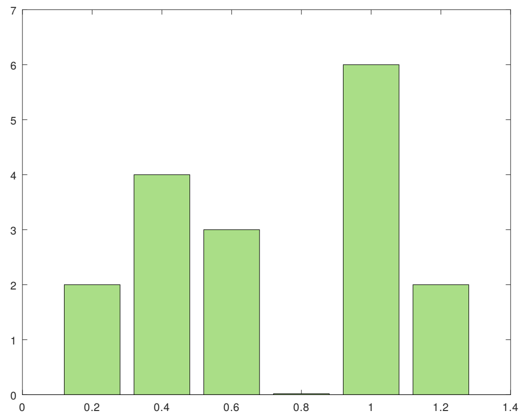
- $\|f_\Delta\|_2 \rightarrow s_{log} = 0.042107$
- $\|f_\Delta\|_\infty \rightarrow s_{log} = 0.21154$

exponential: $s_{exp}(f_1, f_2) = e^{-\delta(f_1, f_2)}$

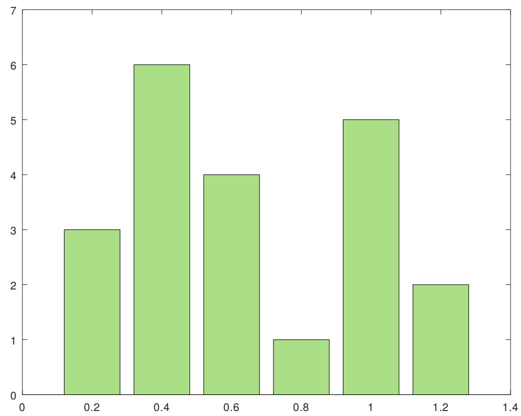
- $\|f_\Delta\|_1 \rightarrow s_{exp} = 0.049787$
- $\|f_\Delta\|_2 \rightarrow s_{exp} = 0.20065$
- $\|f_\Delta\|_\infty \rightarrow s_{exp} = 0.30119$

3.
- ${f_1}^T f_2 = 64.43$
- $\frac{{f_1}^T f_2}{\|f_1\|\|f_2\|} = 0.98451$

Exercise A7.2: Feature Histogram Comparison



(a) h_1



(b) h_2

For the feature histograms h_1, h_2 compute the dissimilarity measures

- Hamming distance
- Minkowski distance for $p = 1, 2, \infty$
- χ^2 distance
- Histogram Intersection distances $HI(h_1, h_2)$ and $HI(h_2, h_1)$
- Histogram Intersection over Union distance

Solution A7.2:

$$h_\Delta = h_2 - h_1 = \begin{pmatrix} 3 \\ 6 \\ 4 \\ 1 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

- Hamming distance:

  Count number of differing bins: 5

- Minkowski distance:

  $p = 1$: $|1| + |2| + |1| + |1| + |-1| = 6$

  $p = 2$: $\sqrt{|1|^2 + |2|^2 + |1|^2 + |1|^2 + |-1|^2} = 2^{\frac{3}{2}} = 2.8284$

  $p = \infty$: $\max\limits_i |h_{\Delta,i}| = 2$

- χ^2 distance:

  $\frac{|1|^2}{5} + \frac{|2|^2}{10} + \frac{|1|^2}{7} + \frac{|1|^2}{1} + \frac{|-1|^2}{11} = \frac{1}{5} + \frac{4}{10} + \frac{1}{7} + \frac{1}{1} + \frac{1}{11} = 1.8338$

- Histogram Intersection:

  $HI(h_1, h_2)$: $\frac{\sum\limits_i \min(h_{1,i}, h_{2,i})}{\sum\limits_i h_{2,i}} = \frac{2+4+3+0+5+2}{3+6+4+1+5+2} = \frac{16}{21} = 0.76190$

  $HI(h_2, h_1)$: $\frac{\sum\limits_i \min(h_{1,i}, h_{2,i})}{\sum\limits_i h_{1,i}} = \frac{2+4+3+0+5+2}{2+4+3+0+6+2} = \frac{16}{17} = 0.94118$

- Histogram Intersection over Union:

  $IoU(h_1, h_2)$: $\frac{\sum\limits_i \min(h_{1,i}, h_{2,i})}{\sum\limits_i \max(h_{1,i}, h_{2,i})} = \frac{2+4+3+0+5+2}{3+6+4+1+6+2} = \frac{16}{22} = 0.72727$

Exercise A7.3: Earth Mover's distance

Given are the following (not necessarily optimal) transport plans. The horizontal histogram is called X and the vertical one Y. Which constraints for the EMD are violated?

$$\begin{array}{cc} \begin{array}{ccc} 0.5 & 0.2 & 0.3 \end{array} & \\ \begin{pmatrix} 0.2 & 0.1 & 0 \\ 0.2 & 0 & 0.3 \\ 0.1 & 0.1 & 0 \end{pmatrix} & \begin{array}{c} 0.3 \\ 0.3 \\ 0.4 \end{array} \end{array}$$

$$\begin{array}{cc} \begin{array}{ccc} 0.5 & 0.2 & 0.3 \end{array} & \\ \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.3 & 0 & 0.1 \end{pmatrix} & \begin{array}{c} 0.3 \\ 0.3 \\ 0.4 \end{array} \end{array}$$

$$\begin{array}{cc} \begin{array}{ccc} 0.5 & 0.2 & 0.3 \end{array} & \\ \begin{pmatrix} 0.3 & 0.1 & 0 \\ -0.2 & 0.1 & 0.3 \\ 0.4 & 0 & 0 \end{pmatrix} & \begin{array}{c} 0.3 \\ 0.3 \\ 0.4 \end{array} \end{array}$$

Solution A7.3:

1. The capacity of Y is exceeded (the rows don't sum up to Y)
2. No constraints are violated (this is a valid transport plan).
3. The entries of the transport plan have to be ≥ 0. The capacity of Y is also exceeded.

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Data Analysis and Visualization

Exercise 7

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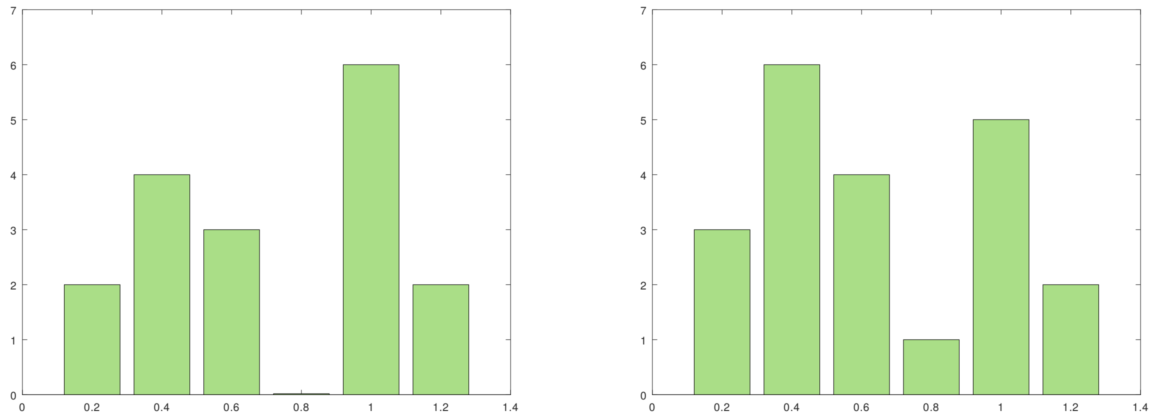

(a) h_1 (b) h_2

For the feature histograms h_1, h_2 compute the dissimilarity measures

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- Minkowski distance for $p = 1, 2, \infty$
- χ^2 distance
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  $p = \infty$: $\max_i |h_{\Delta,i}| = 2$

- χ^2 distance:

  $\frac{|1|^2}{5} + \frac{|2|^2}{10} + \frac{|1|^2}{7} + \frac{|1|^2}{1} + \frac{|-1|^2}{11} = \frac{1}{5} + \frac{4}{10} + \frac{1}{7} + \frac{1}{1} + \frac{1}{11} = 1.8338$

- Histogram Intersection:

  $HI(h_1, h_2)$: $\frac{\sum\limits_i \min(h_{1,i}, h_{2,i})}{\sum\limits_i h_{2,i}} = \frac{2+4+3+0+5+2}{3+6+4+1+5+2} = \frac{16}{21} = 0.76190$

  $HI(h_2, h_1)$: $\frac{\sum\limits_i \min(h_{1,i}, h_{2,i})}{\sum\limits_i h_{1,i}} = \frac{2+4+3+0+5+2}{2+4+3+0+6+2} = \frac{16}{17} = 0.94118$

- Histogram Intersection over Union:

  $IoU(h_1, h_2)$: $\frac{\sum\limits_i \min(h_{1,i}, h_{2,i})}{\sum\limits_i \max(h_{1,i}, h_{2,i})} = \frac{2+4+3+0+5+2}{3+6+4+1+6+2} = \frac{16}{22} = 0.72727$

Exercise A7.3: Earth Mover's distance

Given are the following (not necessarily optimal) transport plans. The horizontal histogram is called X and the vertical one Y. Which constraints for the EMD are violated?

$$\begin{array}{cc} \begin{array}{ccc} 0.5 & 0.2 & 0.3 \end{array} & \\ \begin{pmatrix} 0.2 & 0.1 & 0 \\ 0.2 & 0 & 0.3 \\ 0.1 & 0.1 & 0 \end{pmatrix} & \begin{array}{c} 0.3 \\ 0.3 \\ 0.4 \end{array} \end{array}$$

$$\begin{array}{cc} \begin{array}{ccc} 0.5 & 0.2 & 0.3 \end{array} & \\ \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.3 & 0 & 0.1 \end{pmatrix} & \begin{array}{c} 0.3 \\ 0.3 \\ 0.4 \end{array} \end{array}$$

$$\begin{array}{cc} \begin{array}{ccc} 0.5 & 0.2 & 0.3 \end{array} & \\ \begin{pmatrix} 0.3 & 0.1 & 0 \\ -0.2 & 0.1 & 0.3 \\ 0.4 & 0 & 0 \end{pmatrix} & \begin{array}{c} 0.3 \\ 0.3 \\ 0.4 \end{array} \end{array}$$

Solution A7.3:

1. The capacity of Y is exceeded (the rows don't sum up to Y)
2. No constraints are violated (this is a valid transport plan).
3. The entries of the transport plan have to be ≥ 0. The capacity of Y is also exceeded.

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Data Analysis and Visualization

Exercise 8

Exercise A8.1: Comparison of Dimensionality Reduction Methods

Given the 3D dataset below, match each of the depicted 2D embeddings to the dimensionality reduction method they were computed with. Please justify your answer.

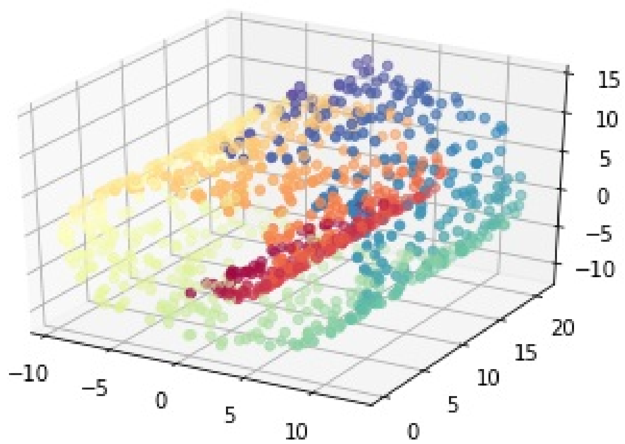
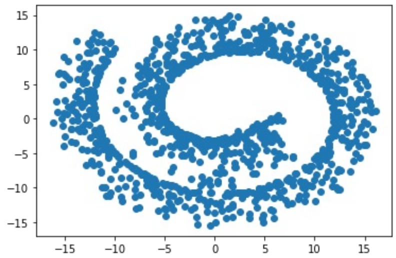
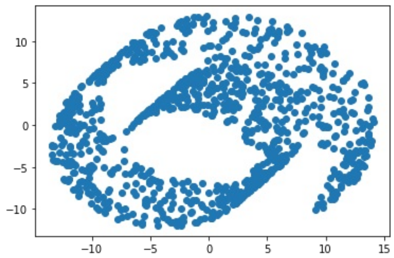


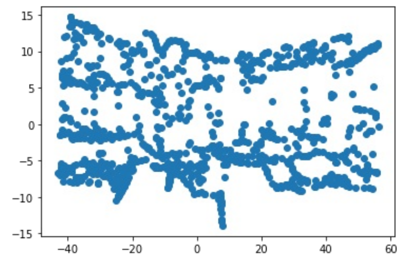
Figure 1: 3D dataset



(a)



(b)



(c)

Solution A8.1:

a) **Classical MDS:** Since Euclidean distances in the 3D dataset are the basis for MDS, the embedding brings points on the nearby windings of the manifold close to each other in an attempt to preserve these distances.

b) **PCA:** Principal Component Analysis determines a orthogonal set of 2 basis vectors, onto which the 3D points are then projected. The figure looks exactly like an orthogonal projection of the dataset onto a plane.

c) **IsoMap:** Distances inside the underlying manifold are approximated by graph distances in the k-nearest neighbors graph. IsoMap thus succeeds in "unwinding" the underlying 2D structure into a planar stripe.

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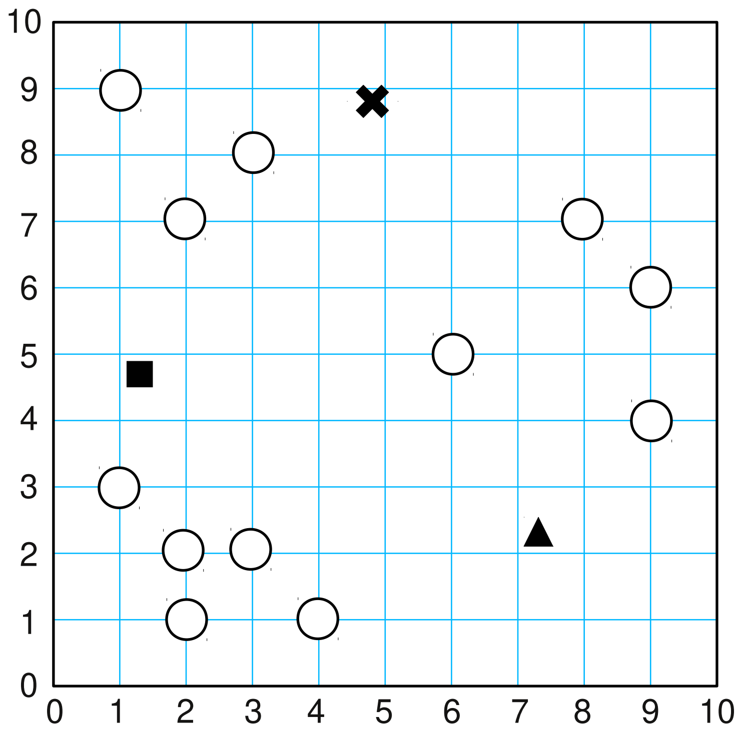
Data Analysis and Visualization

Exercise 9

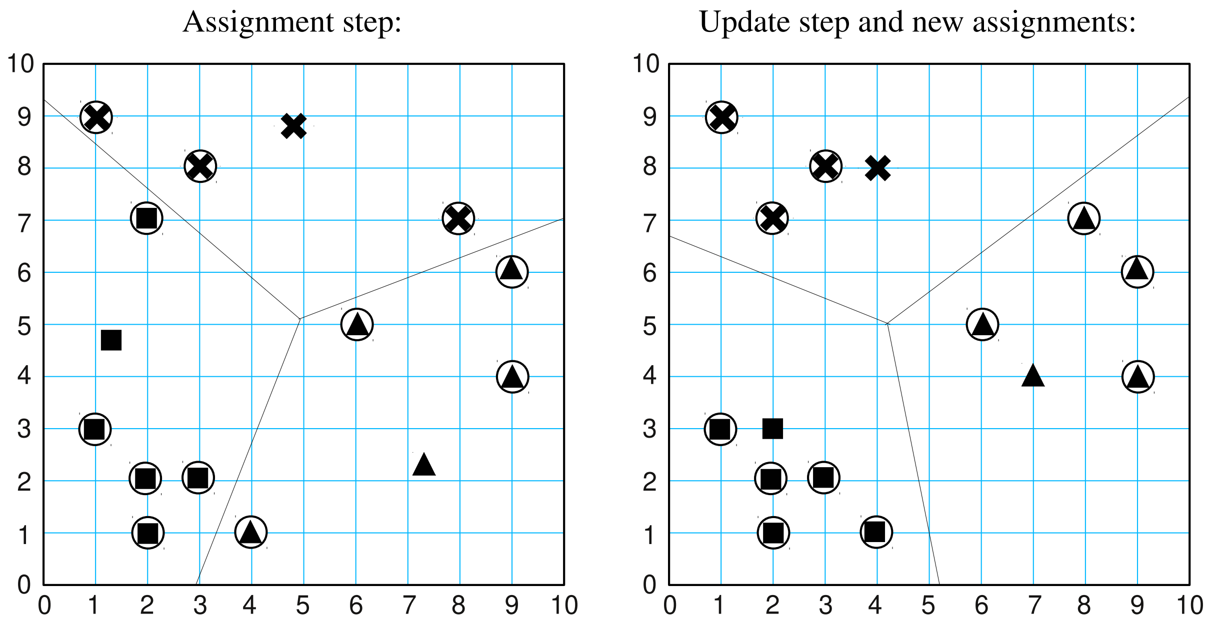
Exercise A9.1: K-Means Clustering

Given is the following set of data samples (circles ◯).



Apply an *assignment, an update, and another assignment step* of the k-means algorithms to assign the data to three clusters. The initial cluster centers are depicted as a cross $\times$, a square ■, and a triangle ▲.
Assign each data sample to a cluster by filling in one of the cluster symbols into the circle representing that sample.

Solution A9.1:



New cluster centers:

Cross $\times$: $\frac{(1,9)+(3,8)+(8,7)}{3} = (12, 24)/3 = (4, 8)$

Square $\blacksquare$: $\frac{(1,3)+(2,1)+(2,2)+(2,7)+(3,2)}{5} = (10, 15)/5 = (2, 3)$

Triangle $\blacktriangle$: $\frac{(4,1)+(6,5)+(9,4)+(9,6)}{4} = (28, 16)/4 = (7, 4)$

Exercise A9.2: K-Means Clustering (Programming)

See IPython notebook.

Exercise A9.3: Expectation Maximization (Programming)

See IPython notebook.

Exercise A9.4: Mean-Shift Clustering (Programming)

See IPython notebook.