Lab 6: Rejection Sampling - Supplementary Notes

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Feb 14, 2022

1. A quick review of rejection sampling

Goal: generate samples from a complicated pdf $f(x)$.

Suppose

$$f(x) = \frac{\tilde{f}(x)}{\alpha},$$

where $\alpha > 0$ is the normalizing constant, and we may not be able to derive its closed-form. Suppose we can evaluate $\tilde{f}(x)$.

Steps:

1. Choose a proposal distribution q and a ceiling constant $c > 0$, such that $cq(x)$ fully envelops $\tilde{f}(x)$, i.e.

$$cq(x) \geq \tilde{f}(x).$$

2. Sample $X \sim q$. Given the value of X we've sampled, sample $Y \sim Unif(0, cq(X))$.
3. If $Y \leq \tilde{f}(X)$, let $Z = X$. If $Y > \tilde{f}(X)$, reject this X and return to step 2.

Finally, Z is the desired sample from f. Repeat this procedure until you get the desired number of samples.

2. Validity of rejection sampling

Target: show that the accepted sample follows the target probability $f(x)$. That is, for any random sample X generated from $q(x)$, if it is accepted (i.e. $Y \leq \tilde{f}(X)$), then X follows $f(x)$:

$$P(X = x | Y \leq \tilde{f}(X)) \stackrel{?}{=} f(x).$$

What we've already known:

- X is generated from $q(x)$. That is, $X \sim q(x)$.
- Given X, then $Y \sim Unif(0, cq(X))$. That is, $Y|X \sim Unif(0, cq(X))$.
- $f(x) = \frac{\tilde{f}(x)}{\alpha}$.

Proof:

$$P(X = x | Y \leq \tilde{f}(X)) \stackrel{\text{Bayes' theorem}}{=} \frac{P(Y \leq \tilde{f}(X) | X = x) P(X = x)}{P(Y \leq \tilde{f}(X))}.$$

See each term:

1. $P(Y \leq \tilde{f}(X) | X = x)$: Since $Y|X \sim Unif(0, cq(X))$, then

$$P(Y \leq \tilde{f}(X) | X = x) = \int_0^{\tilde{f}(x)} \frac{1}{cq(x)} dy = \frac{\tilde{f}(x)}{cq(x)}.$$

2. $P(X = x)$: Since $X \sim q(x)$, then

$$P(X = x) = q(x).$$

3. $P(Y \leq \tilde{f}(X))$: By probability theory,

$$P(Y \leq \tilde{f}(X)) \overset{(1)}{=} E\left[1_{\{Y \leq \tilde{f}(X)\}}\right] \overset{(2)}{=} E\left[E\left(1_{\{Y \leq \tilde{f}(X)\}}|X\right)\right].$$

Here:

(1): The expectation of the indicator function is the probability that the corresponding event happens.

(2): Law of total expectation: $E(A) = E(E(A|B))$.

Then,

$$\begin{aligned}
P(Y \leq \tilde{f}(X)) &= E\left[E\left(1_{\{Y \leq \tilde{f}(X)\}}|X\right)\right] = E\left[P(Y \leq \tilde{f}(X)|X)\right] \\
&= \int P(Y \leq \tilde{f}(X)|X = x) \cdot q(x)dx \\
&= \int \frac{\tilde{f}(x)}{cq(x)} \cdot q(x)dx \\
&= \frac{1}{c} \int \tilde{f}(x)dx \\
&= \frac{\alpha}{c} \qquad (f(x) = \frac{\tilde{f}(x)}{\alpha} \ \Rightarrow \ \int \tilde{f}(x) = \int \alpha f(x) = \alpha).
\end{aligned}$$

Combine all the results,

$$P(X = x|Y \leq \tilde{f}(X)) = \frac{\frac{\tilde{f}(x)}{cq(x)} \cdot q(x)}{\frac{\alpha}{c}} = \frac{\tilde{f}(x)}{\alpha} = f(x).$$

Therefore, we've shown that if X is accepted, then X follows the target probability distribution $f(x)$. In other words, rejection sampling can give us the desired sample from the target distribution.

3. More interpretations

See the conditional probability:

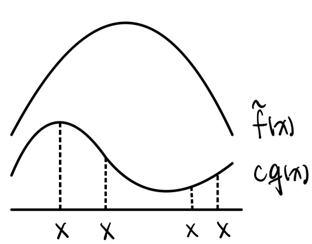
$$P(Y \leq \tilde{f}(X)|X = x) = \frac{\tilde{f}(x)}{cq(x)}.$$

Why generate Y from uniform distribution:

- We generate Y from the uniform distribution in order to control the acceptance ratio. In short, we generate the sample X from q, and accept this sample with the probability $\frac{\tilde{f}(x)}{cq(x)}$.

Why we require that $cq(x)$ fully envelops $\tilde{f}(x)$:

- If otherwise $cq(x) < \tilde{f}(x)$, then the acceptance ratio $\frac{\tilde{f}(x)}{cq(x)} > 1$, which means that the sample $X = x$ is always accepted.



If all samples generated from $q(x)$ are accepted without rejection, then they simply follow $q(x)$ and won't recover $f(x)$.