Practice Exercise 1

Exercise

We write $X \sim \text{Poisson}(\theta)$ if X has the Poisson distribution with rate $\theta > 0$, that is, its p.m.f. is

$$p(x|\theta) = \text{Poisson}(x|\theta) = e^{-\theta}\theta^x/x!$$

for $x \in \{0, 1, 2, \dots\}$ (and is 0 otherwise). Suppose $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$ given θ, and your prior is

$$p(\theta) = \text{Gamma}(\theta|a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}\mathbb{1}(\theta > 0).$$

What is the posterior distribution on θ?

Solution

Since the data is independent given θ, the likelihood factors and we get

$$
\begin{aligned}
p(x_{1:n}|\theta) &= \prod_{i=1}^{n} p(x_i|\theta) \\
&= \prod_{i=1}^{n} e^{-\theta}\theta^{x_i}/x_i! \\
&\underset{\theta}{\propto} e^{-n\theta}\theta^{\sum x_i}.
\end{aligned}
$$

Thus, using Bayes' theorem,

$$
\begin{aligned}
p(\theta|x_{1:n}) &\propto p(x_{1:n}|\theta)p(\theta) \\
&\propto e^{-n\theta}\theta^{\sum x_i}\theta^{a-1}e^{-b\theta}\mathbb{1}(\theta > 0) \\
&\propto e^{-(b+n)\theta}\theta^{a+\sum x_i - 1}\mathbb{1}(\theta > 0) \\
&\propto \mathrm{Gamma}\big(\theta \mid a + \textstyle\sum x_i,\ b+n\big).
\end{aligned}
$$

Therefore, since the posterior density must integrate to 1, we have

$$
p(\theta|x_{1:n}) = \mathrm{Gamma}\big(\theta \mid a + \textstyle\sum x_i,\ b+n\big).
$$