

Homework 5

STA-360-602

Total points: 10 (reproducibility) + 15 (Q1) + 25 (Q2) = 50 points.

1. (15 points, 5 points each) Hoff, 3.12 (Jeffrey's prior).
 - (a) $Y \sim \text{binomial}(n, \theta)$. Obtain Jeffreys' prior distribution $p_J(\theta)$ for this model.

Q1a Ans:

$$\begin{aligned}
I(\theta) &= -E_\theta[\partial^2 \log p(Y|\theta)/\partial\theta^2] \\
&= -E_\theta \left[\frac{\partial^2 \log \left(\binom{n}{Y} \theta^Y (1-\theta)^{n-Y} \right)}{\partial\theta^2} \right] \\
&= -E_\theta \left[\frac{\partial^2 (\log \binom{n}{Y} + Y \log(\theta) + (n-Y) \log(1-\theta))}{\partial\theta^2} \right] \\
&= -E_\theta \left[\frac{\partial \left(\frac{Y}{\theta} - \frac{n-Y}{1-\theta} \right)}{\partial\theta} \right] \\
&= -E_\theta \left[\frac{\partial \left(\frac{Y-Y\theta-n\theta+Y\theta}{\theta(1-\theta)} \right)}{\partial\theta} \right] \\
&= -E_\theta \left[\frac{\partial \left(\frac{Y-n\theta}{\theta(1-\theta)} \right)}{\partial\theta} \right] \\
&= -E_\theta [(Y-n\theta)\theta^{-1}(-1)(-1)(1-\theta)^{-2} + (Y-n\theta)(-1)\theta^{-2}(1-\theta)^{-1} + (-n)\theta^{-1}(1-\theta)^{-1}] \\
&= -E_\theta \left[\frac{(Y-n\theta)\theta - (Y-n\theta)(1-\theta) - n\theta(1-\theta)}{\theta^2(1-\theta)^2} \right] \\
&= -E_\theta \left[\frac{Y\theta - n\theta^2 - Y + Y\theta + n\theta - n\theta^2 - n\theta + n\theta^2}{\theta^2(1-\theta)^2} \right] \\
&= -E_\theta \left[\frac{Y\theta - n\theta^2 - Y + Y\theta + n\theta - n\theta^2 - n\theta + n\theta^2}{\theta^2(1-\theta)^2} \right] \\
&= E_\theta \left[\frac{n\theta^2 + Y - 2Y\theta}{\theta^2(1-\theta)^2} \right] \\
&= \frac{n\theta^2 + (n\theta) - 2(n\theta)\theta}{\theta^2(1-\theta)^2} \\
&= \frac{n\theta - n\theta^2}{\theta^2(1-\theta)^2} \\
&= \frac{n}{\theta(1-\theta)} \\
&\propto \frac{1}{\theta(1-\theta)}
\end{aligned}$$

$$\text{Hence, } p_J(\theta) \propto \sqrt{I(\theta)} \propto \sqrt{\frac{1}{\theta(1-\theta)}}$$

- (b) Reparameterize with $\psi = \log[\theta/(1-\theta)]$ so that $p(y|\psi) = \binom{n}{y} e^{\psi y} (1+e^\psi)^{-n}$.
Obtain Jeffreys' prior distribution $p_J(\psi)$ for this model.

Q1b Ans:

$$\begin{aligned}
 I(\psi) &= -E_\psi [\partial^2 \log p(y|\psi) / \partial \psi^2] \\
 &= -E_\psi \left[\frac{\partial^2 \log(\binom{n}{y} e^{\psi y} (1+e^\psi)^{-n})}{\partial \psi^2} \right] \\
 &= -E_\psi \left[\frac{\partial^2 (\log \binom{n}{y} + \psi y - n \log(1+e^\psi))}{\partial \psi^2} \right] \\
 &= -E_\psi \left[\frac{\partial (y - n(1+e^\psi)^{-1} e^\psi)}{\partial \psi} \right] \\
 &= -E_\psi [-n(-1)(1+e^\psi)^{-2} e^\psi e^\psi - n(1+e^\psi)^{-1} e^\psi] \\
 &= nE_\psi \left[\frac{e^\psi}{(1+e^\psi)} - \frac{e^{2\psi}}{(1+e^\psi)^2} \right] \\
 &= nE_\psi \left[\frac{e^\psi}{(1+e^\psi)^2} \right] \\
 &\propto \frac{e^\psi}{(1+e^\psi)^2}
 \end{aligned}$$

$$\text{Hence, } p_J(\psi) \propto \sqrt{I(\psi)} \propto \sqrt{\frac{e^\psi}{(1+e^\psi)^2}}$$

- (c) Take the prior distribution from a) and apply the change of variables formula from Exercise 3.10 to obtain the induced prior density on ψ . This density should be the same as the one derived in part b) of this exercise. This consistency under reparameterization is the defining characteristic of Jeffrey's' prior.

Q1c Ans:

Since $\psi = \log[\theta/(1 - \theta)]$,

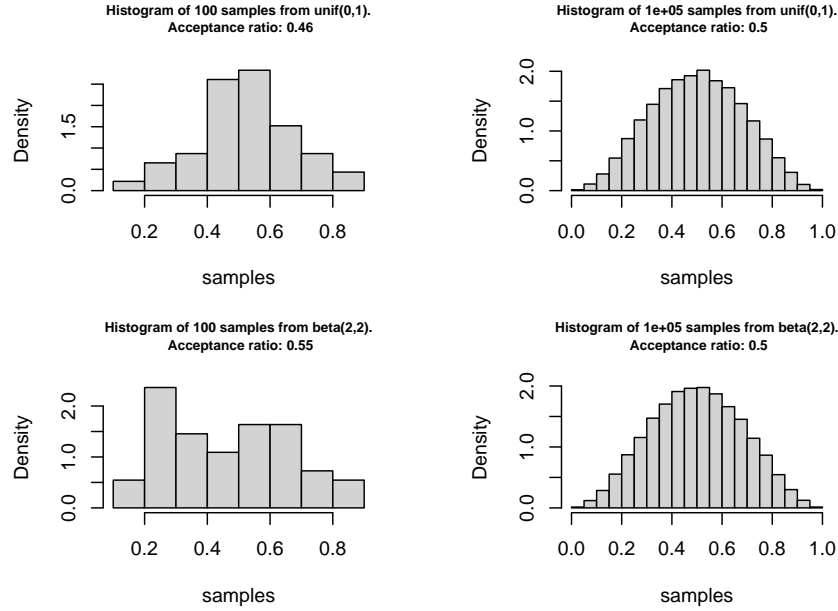
$$\theta = \frac{e^\psi}{1 + e^\psi} = h(\psi)$$

Based on 3.10,

$$\begin{aligned} p_{J,\psi}(\psi) &= p_{J,\theta}(\theta) = p_{J,\theta}(h(\psi)) \times \left| \frac{dh}{d\psi} \right| \\ &\propto \sqrt{\frac{1}{h(\psi)(1-h(\psi))}} \times |e^\psi(1+e^\psi)^{-1} + e^\psi(-1)(1+e^\psi)^{-2}e^\psi| \\ &= \sqrt{\frac{1}{\frac{e^\psi}{1+e^\psi}(1-\frac{e^\psi}{1+e^\psi})}} \times \left| \frac{e^\psi + e^{2\psi} - e^{2\psi}}{(1+e^\psi)^2} \right| \\ &= \sqrt{\frac{1}{\frac{e^\psi}{(1+e^\psi)^2}}} \times \left| \frac{e^\psi}{(1+e^\psi)^2} \right| \\ &= \sqrt{\frac{e^\psi}{(1+e^\psi)^2}} \end{aligned}$$

2. *Lab component* (25 points total) Please refer to lab 5 and complete tasks 4—5.

- (a) (10) Task 4



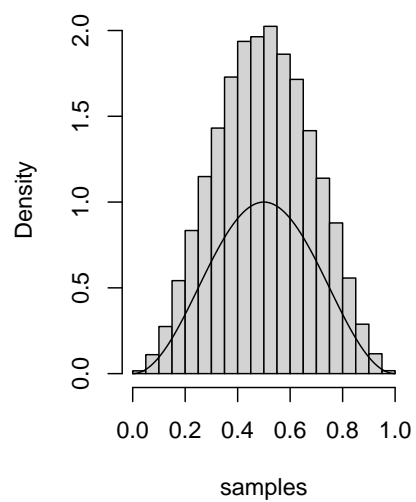
- With only 100 samples, it appeared that the $\text{Beta}(2,2)$ enveloping proposal worked better than $\text{Unif}(0,1)$ with a slightly higher acceptance ratio of 0.55, in comparison to 0.46. However, $\text{Beta}(2,2)$ generated an approximation that is skewed, unlike what was generated using $\text{Unif}(0,1)$ and the actual $f(x)$.
- The above mentioned differences in acceptance ratios and skewness went away when the number of simulations was increased to 100,000 and both enveloping proposals generated essentially the same histogram which approximates $f(x)$ with the same acceptance ratio of 0.5.

(b) (15) Task 5

(c) They were about the same when the number of simulations is sufficiently high (100,000), so neither is better.

(ii)

Histogram of 1e+05 samples from unif(0,1).
Acceptance ratio: 0.5



Histogram of 1e+05 samples from beta(2,2).
Acceptance ratio: 0.5

