2. (a).

$$p(x_{1:n} \mid \theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i} I(x_i > 0) = \theta^n e^{-\theta \Sigma x_i} I(x_i > 0) \text{ for all } i.$$

$$p(\theta \mid x_{1:n}) = \frac{p(x_{1:n} \mid \theta) \cdot p(\theta)}{p(x)} \propto p(x_{1:n} \mid \theta) \cdot p(\theta)$$

$$p(x_{1:n} \mid \theta) \cdot p(\theta) = \underbrace{\frac{b^a}{\Gamma(a)}}_{\text{constant}} \theta^{n+a-1} e^{-\theta(b+\Sigma x_i)} I(\theta > 0).$$

$\therefore p(\theta \mid x_{1:n}) \sim \text{Gamma}(n+a, b+\Sigma x_i)$

(b) Since $a+n>0$, $b+\Sigma x_i > 0$ by given prior and distn of X.

(c)

(d)

3.

(a) . $\theta \sim$ Galenshore (c, d).

(b). $p(y_{1:n} \mid \theta) = \prod_{i=1}^{n} \frac{2}{\Gamma(a)} \theta^{2a} y_i^{2a-1} e^{-\theta^2 y_i^2}$

$\propto (\theta^{2a})^n e^{-\theta^2 \Sigma y_i^2}$

$p(\theta \mid c, d) = \frac{2}{\Gamma(c)} d^{2c} \underbrace{\theta^{2c-1} e^{-d^2\theta^2}} \propto \theta^{2c-1} e^{-d^2\theta^2}$

$p(\theta \mid y_{1:n}) \propto p(y_{1:n} \mid \theta)\, p(\theta)$

$\propto \theta^{2(na+c)-1} e^{-\theta^2(\Sigma y_i^2 + d^2)}$.

which is the kernel of a density from the same parametric family with posterior parameters $na+c$ and $\sqrt{\Sigma y_i^2 + d^2}$

(c) $\frac{p(\theta_a \mid y_{1:n})}{p(\theta_b \mid y_{1:n})} = \frac{p(y_{1:n} \mid \theta_a)\, p(\theta_a)}{p(y_{1:n} \mid \theta_b)\, p(\theta_b)}$

$= \frac{\left(\frac{2}{\Gamma(a)}\right)^n \cdot \frac{2}{\Gamma(c)} d^{2c} \theta_a^{2(an+c)-1} e^{-\theta_a^2(d^2+\Sigma y_i^2)}}{\left(\frac{2}{\Gamma(a)}\right)^n \frac{2}{\Gamma(c)} d^{2c} \theta_b^{2(an+c)-1} e^{-\theta_b^2(d^2+\Sigma y_i^2)}}$

$= \left(\frac{\theta_a}{\theta_b}\right)^{2(an+c)-1} e^{(\theta_b^2 - \theta_a^2)(d^2 + \Sigma y_i^2)}$

(d) Since $\theta \mid y_{1:n} \sim$ Galenshore$(an+c, \sqrt{\Sigma y_i^2 + d^2})$

$E(\theta \mid y_{1:n}) = \Gamma(an+c+\frac{1}{2}) \Big/ \sqrt{\Sigma y_i^2 + d^2}\, \Gamma(an+c)$

(e). $p(y_{n+1} \mid y_{1:n}) = \int p(y_{n+1} \mid \theta)\, p(\theta \mid y_{1:n})\, d\theta$

$$= \int \frac{2}{\Gamma(a)} \theta^{2a} y_{n+1}^{2a-1} e^{-\theta^2 y_{n+1}^2} \cdot \frac{2}{\Gamma(an+c)} \left(\sum y_i^2 + d^2\right)^{an+c} \theta^{an+c-1} e^{-\theta^2 (\sum y_i^2 + d^2)}\, d\theta$$

$$= \frac{2}{\Gamma(a)} \cdot \frac{2}{\Gamma(an+c)} \left(\sum y_i^2 + d^2\right)^{an+c} \cdot y_{n+1}^{2a-1} \int \theta^{(an+a+c)-1} e^{-\theta^2(\sum y_i^2 + d^2 + y_{n+1}^2)}\, d\theta$$

$$= \frac{2 \cdot 2 \left(\sum y_i^2 + d^2\right)^{an+c} / \left(\sum (y_i^2) + d^2 + y_{n+1}^2\right)^{an+a+c}}{\Gamma(a)\Gamma(an+c) \cdot 2\, \Gamma(an+a+c)} \cdot y_{n+1}^{2a-1}$$

$$\underbrace{\int \frac{2}{\Gamma(an+a+c)} \left(\sum (y_i^2) + d^2 + y_{n+1}^2\right)^{an+a+c} \theta^{(an+a+c)-1} e^{-\theta^2(\sum y_i^2 + d^2 + y_{n+1}^2)}\, d\theta}_{1}$$

$$= \frac{2 y_{n+1}^{2a-1}\, \Gamma(an+a+c)}{\Gamma(a)\Gamma(an+c)} \frac{\left(d^2 + \sum y_i^2\right)^{an+c}}{\left(d^2 + \sum y_i^2 + y_{n+1}^2\right)^{an+a+c}}$$