

# Homework 4

STA-360-602

Total points: 10 (reproducibility) + 10 (Q1) + 25 (Q2) = 45 points.

1. (10 points, 5 points each) Hoff, 3.10 (Change of variables).

(a)  $\theta \sim \text{beta}(a, b)$ ,  $\psi = \log[\theta/(1 - \theta)]$ . Obtain the form of  $p_\psi$  and plot it for the case that  $a = b = 1$ .

Q1a Ans:

$$\psi = \log[\theta/(1 - \theta)]$$

$$e^\psi = \theta/(1 - \theta)$$

$$(1 - \theta)e^\psi = \theta$$

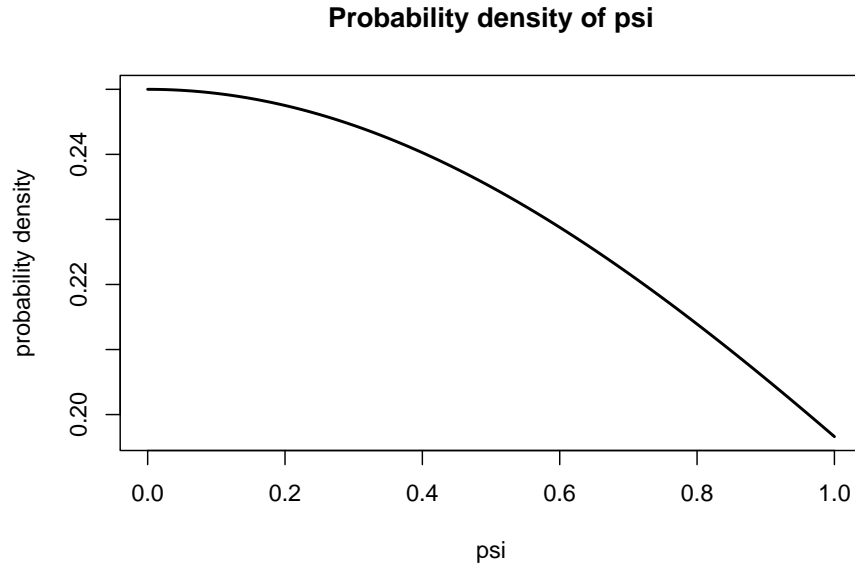
$$e^\psi = \theta + \theta e^\psi$$

$$\theta = \frac{e^\psi}{1 + e^\psi} = h(\psi)$$

Note that,  $0 < h(\psi) = \frac{e^\psi}{1 + e^\psi} < 1$ .

Then,

$$\begin{aligned} p_\psi(\psi) &= p_\theta(h(\psi)) \times \left| \frac{dh}{d\psi} \right| \\ &= \text{Beta}(h(\psi)|a, b) \times \left| \frac{d(\frac{e^\psi}{1+e^\psi})}{d\psi} \right| \\ &= \frac{1}{B(a, b)} h(\psi)^{a-1} (1 - h(\psi))^{b-1} I(0 < h(\psi) < 1) \times \left| \frac{e^\psi}{1 + e^\psi} - \frac{e^{2\psi}}{(1 + e^\psi)^2} \right| \\ &= \frac{1}{B(a, b)} \frac{e^{(a-1)\psi}}{(1 + e^\psi)^{(a-1)}} \frac{1}{(1 + e^\psi)^{(b-1)}} \left| \frac{(1 + e^\psi)e^\psi - e^{2\psi}}{(1 + e^\psi)^2} \right| \\ &= \frac{1}{B(a, b)} \frac{e^{(a-1)\psi}}{(1 + e^\psi)^{(a+b-2)}} \left| \frac{e^\psi}{(1 + e^\psi)^2} \right| \\ &= \frac{1}{B(a, b)} \frac{e^{a\psi}}{(1 + e^\psi)^{(a+b)}} \end{aligned}$$



- (b)  $\theta \sim \text{gamma}(a, b)$ ,  $\psi = \log \theta$ . Obtain the form of  $p_\psi$  and plot it for the case that  $a = b = 1$ .

Q1b Ans:

$$\psi = \log \theta$$

$$e^\psi = \theta = h(\psi)$$

Note that,  $h(\psi) = e^\psi > 0$ .

Then,

$$\begin{aligned}
 p_\psi(\psi) &= p_\theta(h(\psi)) \times \left| \frac{dh}{d\psi} \right| \\
 &= \text{Gamma}(h(\psi)|a, b) \times \left| \frac{d(e^\psi)}{d\psi} \right| \\
 &= \frac{b^a}{\Gamma(a)} h(\psi)^{a-1} e^{-bh(\psi)} I(h(\psi) > 0) \times |e^\psi| \\
 &= \frac{b^a}{\Gamma(a)} e^{(a-1)\psi} e^{-b \exp(\psi)} e^\psi \\
 &= \frac{b^a}{\Gamma(a)} e^{a\psi - b \exp(\psi)}
 \end{aligned}$$

2. *Lab component* (25 points total) Please refer to lab 4 and complete tasks 4—5.

(a) (10) Task 4 [Q2a Ans:](#)

(b) (15) Task 5 [Q2b Ans:](#)