Online Course: "Reps of Lie Algs" by Vyacheslav Futorny

- PBW Thm:
  If $e_1, e_2, \cdots, e_n$ basis of $\mathfrak{g}$, then
  $\{e_1^{i_1} \cdots e_n^{i_n} \mid i_j \in \mathbb{Z}_{\geq 0}, \forall j\}$ is a basis of $\mathcal{U}(\mathfrak{g})$

- $\mathfrak{g}$-module $\cong \mathcal{U}(\mathfrak{g})$-module

- sl_2-module construction
  1) $V_n \cong \mathbb{C}[x]_{\leq n} = \{f(x) \mid \partial f \leq n\}$
  $e \mapsto \frac{d}{dx}$, $f \mapsto -x^2\frac{d}{dx} + nx$, $h \mapsto -2x\frac{d}{dx} + n$

  2) $V_n \cong \mathbb{C}[x,y]_n^h$ homogeneous polynomial of degree n.
  $e \mapsto x\frac{\partial}{\partial y}$; $h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$; $f \mapsto y\frac{\partial}{\partial x}$

- Cartan subalg. of $\mathfrak{g}$ is a maximal toral Lie alg. (consisting of semi-simple elements respect to adjoint rep)

- $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(N_-) \otimes \mathcal{U}(H) \otimes \mathcal{U}(N_+)$
  Where $\mathfrak{g} = N_- \oplus H \oplus N_+$

- Simple Lie algs $\longleftrightarrow$ Dynkin diagrams $\longleftrightarrow$ Cartan matrices

$A = (a_{ij})_{i,j=1}^r$ is a Cartan matrix if

- A is indecomposable
- $a_{ii} = 2, \forall i$
- $a_{ij} = 0 \Rightarrow a_{ji} = 0$
- $a_{ij} \in \mathbb{Z}_{\leq 0}$
- $\exists$ a diagonal D such that DAD^{-1} is symmetric and positive definite.

as a consequence, we get that $a_{ij} \in \{2, 0, -1, -2, -3\}$

- Serre relation. $[e_i, [e_i, e_j]] = [f_i, [f_i, f_j]] = 0$.

- Any submodule and any quotient module of a weight module is a weight module

- Let $\lambda \in H^*$, a $\mathfrak{g}$-module V is a highest weight module with highest weight λ if
  1) $\exists v \in V_\lambda$, such that $V = \mathcal{U}(\mathfrak{g})v$ (generated by v)
  2) $N_+ v = 0$.

- Universal highest weight modules $=$ Verma modules

Denote by $S(\lambda)$ a left ideal of $\mathcal{U}(\mathfrak{g})$ generated by N_+ and elements $h - \lambda(h)$, $\forall h \in H$. Then $S(\lambda)$ is a $\mathfrak{g}$-submod and $S(\lambda) \cap \mathcal{U}(N_-) = 0$

$M(\lambda) = \mathcal{U}(\mathfrak{g})/S(\lambda) \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(N_+ \oplus H)} kv_\lambda \cong \mathcal{U}(N_-)\bar{1}$, where $hv_\lambda = \lambda(h)v_\lambda, \forall h$ and $N_+ v_\lambda = 0$.

• Character of a weight module:

$V=\bigoplus_{\lambda\in H^*} V_\lambda$, $\dim V_\lambda<\infty$, $\forall\lambda$.

$$\underbrace{\mathrm{ch}\,V = \sum_{\lambda\in H^*}(\dim V_\lambda)\, e^\lambda}_{\text{formal character}}$$

e^λ is a formal symbol such that $e^\lambda e^\mu = e^{\lambda+\mu}$

Ex. $\mathrm{ch}(L(\alpha+\beta)) = e^{\alpha+\beta}+e^{\alpha}+e^{\beta}+e^{-\alpha}+e^{-\beta}+e^{-\alpha-\beta}+2e^{0}$

Remark. For λ dominant integral, $\dim L(\lambda)<\infty$ and $\mathrm{ch}\,L(\lambda)$ can be viewed as a $\mathbb{Z}$-valued function on the lattice of integral weight: $\mathrm{ch}\,L(\lambda)(\mu) = \dim L(\lambda)_\mu$. Besides, it has a finite support.

Theorem. (Weyl character formular) For any dominant integral λ.

$$\mathrm{ch}\,L(\lambda) = \frac{\sum_{w\in W}(-1)^{l(w)}\varepsilon_w(\lambda+\rho)}{\sum_{w\in W}(-1)^{l(w)}\varepsilon_w(\rho)}$$

or equivalently

length of w (about simple reflection)

$$\mathrm{ch}\,L(\lambda) * \Big(\sum_{w\in W}(-1)^{l(w)}\varepsilon_w(\rho)\Big) = \sum_{w\in W}(-1)^{l(w)}\varepsilon_w(\lambda+\rho)$$

• If $M(\mu)\subset M(\lambda)$, then $\chi_\mu=\chi_\lambda$ and $\mu\in W\cdot\lambda$.
Converse is not true.

• $M(\lambda)$ has a finite composition series
$M(\lambda)=M_0\supset M_1\supset\cdots\supset M_r=0$, where $M_i/M_{i+1}\cong L(\mu_i)$

The number of times $L(\mu)$ appears as a subquotient in the composition series doesn't depend on the series. It's denoted by $[M(\lambda):L(\mu)]$.

1) $[M(\lambda):L(\lambda)]=1$ (only comes from the highest weight)

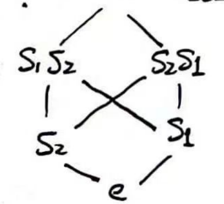
2) If $[M(\lambda):L(\mu)]\neq 0$, then $\mu\in W\cdot\lambda$ (as for Verma submodules above)

3) $\mathrm{ch}\,M(\lambda) = \sum_{\mu\in H^*}[M(\lambda):L(\mu)]\,\mathrm{ch}\,L(\mu) = \sum_{w\in W}[M(\lambda):L(w\cdot\lambda)]\,\mathrm{ch}\,L(w\cdot\lambda)$

• Bruhat order on W: a partial order such that $v\le w$ if any reduced expression for w contains a subexpression which is reduced for v. (reduced expression is the shortest decomposition in the product of simple reflections)

Ex. $\mathfrak{g}=\mathfrak{sl}_3$, $W=S_3=\langle s_1,s_2\rangle=\{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$, $s_2s_1s_2 = s_1s_2s_1$

Bruhat order: $\omega_0 = s_1s_2s_1 = s_2s_1s_2$



• Kazhdan-Lusztig conjecture:

Let $-\lambda$ be integral dominant weight. Then $w\in W$, $\lambda'=\lambda-2\rho$.

1) $\mathrm{ch}\,M(w\cdot\lambda') = \sum_{v\le w} P_{w_0w,\,w_0v}(1)\cdot \mathrm{ch}\,L(v\cdot\lambda')$

where $P_{x,y}$ are certain polynomials (called Kazhdan-Lusztig polynomials). Note that the formular does not depend on λ.

2) $\mathrm{ch}\,L(w\cdot\lambda') = \sum_{v\le w}(-1)^{l(w)-l(v)}P_{v,w}(1)\,\mathrm{ch}\,M(v\cdot\lambda')$

Ex. Apply it on $\mathrm{ch}\,M(\lambda')$, we can get

$$\mathrm{ch}\,M(\lambda') = -\underbrace{P_{e,s_2}(1)}_{1}\,\mathrm{ch}\,M(-n-2) + \underbrace{P_{s_2,s_2}(1)}_{1}\,\mathrm{ch}\,M(n)$$

- Properties of $M(\lambda)$

1) Any highest module with highest weight λ is a homomorphic image of $M(\lambda)$.

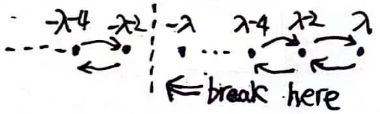
2) $M(\lambda)$ has a unique maximal submodule and hence a unique irreducible quotient $L(\lambda)$

3) Theorem. a) $\dim L(\lambda) < \infty \Leftrightarrow \lambda(h_i) \in \mathbb{Z}_+ .\ \forall i$

b) Any irreducible fin-dim g-module is isomorphic to $L(\lambda)$ for some λ.

Example. 1) $g = sl_2$, $\lambda \in \mathbb{C}$. then Verma module $M(\lambda) \cong U(N_-) \cong k[f]$

- If $\lambda \notin \mathbb{Z}_+$, then $L(\lambda) = M(\lambda)$
- If $\lambda \in \mathbb{Z}_+$, then $M(-\lambda-2) \subseteq M(\lambda)$ as a submodule and $L(\lambda) \cong M(\lambda)/M(-\lambda-2) \cong V_n$ (that we construct before)



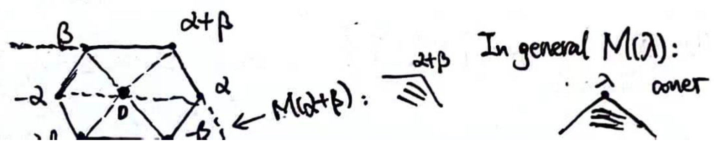
2) $g = sl_3$. Consider adjoint rep $ad: sl_3 \to gl(sl_3)$.

weights of ad $\Leftrightarrow$ {roots of g} $\cup$ {0} (for H)

- Adjoint rep is irr (as sl_3 is simple) of dim 8.

weights: α, β, $\alpha+\beta$, $-\alpha$, $-\beta$, $-\alpha-\beta$, 0

$\uparrow E_{12}$ $\uparrow E_{23}$ $\uparrow E_{13}$ $\uparrow h_1, h_2$



- Since K_g is nondeg, we have an isomorphism $\nu: H \to H^*$: $h \mapsto K_g(h, \cdot)$

This allows to define a non-deg form on H^* : $(\nu(h), \nu(h')) = K_g(h, h')$.

- For any $\alpha \in \Delta$, define a reflection in α : $S_\alpha \in \mathrm{Aut}\, H^*$ such that $S_\alpha(\lambda) = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$ (fixes the hyperplane orthognal to α)

- Further properties of Verma modules

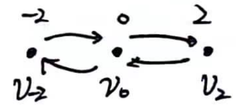
1) $Z(U) \subset U(g)$ the center of $U(g)$ (not g!) action by a scalar!

$\exists$ a homo $\chi_\lambda : Z(U) \to k$ such that $zv = \chi_\lambda(z) v$. $\forall z \in Z(U)$. $v \in M(\lambda)$

χ_λ is the central character of $M(\lambda)$. (or $L(\lambda)$)

Example $g = sl_2$. $z = (h+1)^2 + 4fe$ generates the center $Z(U)$ (Casimir)

Consider V_2 :



$z v_2 = 9 v_2$
$z v_0 = 9 v_0 \Rightarrow \chi_2(z) = 9$
$z v_2 = 9 v_2$.

2) $Z(U) \cong S(H)^W$ (Harish-Chandra isomorphism)

e.g. $g = sl_3$, $H = \mathbb{C}h_1 \oplus \mathbb{C}h_2$, $S(H) \cong \mathbb{C}[h_1, h_2]$, $W = S_2$ (Weyl group)

$S(H)^W = \mathbb{C}[h_1, h_2]^{S_2} = \mathbb{C}[h_1+h_2, h_1h_2] \cong \mathbb{C}[x,y] \cong Z(U)$

- Theorem. (Harish-Chandra) $\chi_\lambda = \chi_\mu \Leftrightarrow \mu \in W\cdot\lambda$. W is the Weyl group where $w\cdot\lambda = w(\lambda+\rho) - \rho$, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$

Category $\mathcal{O}$

Consider of $\mathfrak{g}$-modules satisfying:

1) V is a weighted module
2) V is a finitely generated module
3) $\forall v\in V$, $\dim U(N_+)v<\infty$

then we say $V\in\mathcal{O}$

Ex. $\forall\lambda\in H^*$, $M(\lambda), L(\lambda)\in\mathcal{O}$

Infinite dimensional Lie algs

Ex. (First Witt alg)

$W_1=\mathrm{Der}\,\mathbb{C}[t,t^{-1}]$, has basis $d_n=t^{n+1}\frac{d}{dt}$, $n\in\mathbb{Z}$. and $[d_n,d_m]=(n-m)d_{n+m}$

W_1-module V is called weight if d_0 is diagonalizable on V. ($\underset{\text{irr}}{\mathbb{C}}$ & $\underset{\text{not irr.}}{\mathbb{C}[t,t^{-1}]}$)

and $\mathbb{C}\subset\mathbb{C}[t,t^{-1}]$ as a submodule. & $\mathbb{C}[t^{-1},t]/\mathbb{C}$ is irr.

Highest Weight modules:

$W_1=W_1^-\oplus\mathbb{C}d_0\oplus W_1^+$ ($d_n\in W_1^+\Leftrightarrow n>0$)

Let $a\in\mathbb{C}$ and $\mathbb{C}v_a$ is a 1-dim module over $\mathbb{C}d_0\oplus W_1^+$: $\begin{cases} d_0\cdot v_a=a v_a\\ W_1^+v_a=0\end{cases}$. Then

Define $M^+(a)=U(W_1)\otimes_{U(\mathbb{C}d_0\oplus W_1^+)}\mathbb{C}v_a\cong U(W_1^-)$ as vector space, as the Verma module.

Notice that $d_0(d_n v_a)=(a-n)d_n v_a$, so $M^+(a)=\sum_{k\geq0}M^+(a)_{a-k}$.

Prop 1) $\mathrm{ch}\,M^+(a):=\sum_{n\geq0}(\dim M^+(a)_{a-n})t^n=\prod_{j\geq1}(1-t^j)^{-1}$

2) $M^+(a)$ is irr $\Leftrightarrow a\neq\frac{m^2-1}{24}$, $\forall m\in\mathbb{Z}$

Remark: 1). Similarly, we can define the lowest weight modules $M^-(a)$. where $W_1^-v_a=0$.

2). Virasoro alg: $Vir=W_1\oplus\mathbb{C}c$ with $[d_n,d_m]=(m-n)d_{m+n}+\delta_{n,-m}\frac{m^3-m}{12}c$

3) W_1 is isomorphic to the Lie alg of vector field on circle S^1.

Modules of Intermediate series = Kaplansky - Santharoubane modules.

$\forall\alpha,\beta\in\mathbb{C}$, define $T(\alpha,\beta)=\sum_{k\in\mathbb{Z}}\mathbb{C}v_{k+\beta}$, where $d_n v_{k+\beta}=(k+\beta+\alpha n)v_{k+n+\beta}$ $\forall n$.

1) Check that $T(\alpha,\beta)$ is a W_1-module 2) SES: $0\to\mathbb{C}\to T(0,0)\to T(1,0)\to\mathbb{C}\to0$

Since $T(0,0)\cong\mathbb{C}[t,t^{-1}]$, $T(0,0)/\mathbb{C}\cong\mathbb{C}[t,t^{-1}]/\mathbb{C}$, this SES contributes 2 irr modules ($\mathbb{C}$ & $\mathbb{C}[t,t^{-1}]/\mathbb{C}$)

Theorem. $T(\alpha,\beta)$ is irr unless $\alpha=0,1$ and $\beta\in\mathbb{Z}$.

Remark. Let $\mathcal{A} = \mathbb{C}[t, t^{-1}]$. Consider $\mathcal{A} \otimes \mathcal{A}$ is a left $\mathcal{A}$-module: $a(b \otimes c) = ab \otimes c$.

Let I be a submodule generated by $1 \otimes ab - a \otimes b - b \otimes a$ and $\Omega^1_{\mathcal{A}} = \mathcal{A} \otimes \mathcal{A} / I$ (differential 1-forms, $a \otimes b := a\,db$) also a W_1-module.

- $T(1,0) \cong \Omega^1_{\mathcal{A}}$, $T(0,0) \cong \Omega^0_{\mathcal{A}} \cong \mathcal{A}$

Theorem. (O. Mathieu, Martin-Piard, 1982)

Irreducible weight W_1-modules are:

1) $\mathbb{C}[t, t^{-1}]/\mathbb{C}$; 2) Highest/lowest weight

3) Intermediate series $T(\alpha, \beta)$

- Let $W_n = \mathrm{Der}\, \mathbb{C}[t_1^{\pm}, t_2^{\pm}, \cdots t_n^{\pm}]$. If $W_1 \simeq \mathrm{Vect}\, S^1$, then $W_n \cong$? In fact, W_n = polynomial vector fields on the torus $\mathbb{T}^n$.

__Generalization.__ Let $X \subset \mathbb{A}^n$ an affine variety defined by an ideal $I \subset k[x_1, \cdots, x_n]$ and $\underbrace{\mathcal{A} = k[x_1, \cdots, x_n]/I}_{\text{ring of functions on } X}$.

The Lie alg of polynomials vector fields on X: $V(X) = \mathrm{Der}\, \mathcal{A}$.

__Theorem__ Let k algebraically closed and $\mathrm{char}\, k = 0$, If X is irr then $V(X)$ is simple if and only if X is smooth. (Jordan, Siebert).

Remark. 1) X is irreducible $\Leftrightarrow \mathcal{A}$ has no zero divisors (I is prime)

2) X is smooth if the Jacobian matrix of I has the maximal possible rank in every point of X

__Examples.__ 1) $V(\mathbb{A}^n) = W_n^+ = \mathrm{Der}\, k[x_1, \cdots, x_n]$.

2) Consider an elliptic curve $C: y^2 = x^3 + 1$, then $V(C) = A\xi$, where $\xi = y\frac{d}{dx} + \frac{3}{2}x^2\frac{d}{dy}$. Here $A = k[x,y]/(y^2 - x^3 - 1)$.

Clearly, ξ is a vector field on $\mathbb{A}^2$. Also, $\xi(y^2 - x^3 - 1) = -3x^2y + 3x^2y = 0$ and hence ξ is a vector field on A.

If $f = y^2 - x^3 - 1$, then $J(f) = (2y \ -3x^2)$, $0 \notin C$

3) $X = S^2$, $A = k[x,y,z]/(\underbrace{x^2 + y^2 + z^2 - 1}_{f})$.

$\Rightarrow J(f) = (2x \ 2y \ 2z)$

$\Delta_{xy} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, $\Delta_{xy}(f) = 0 \Rightarrow \Delta_{xy} \in V(S^2)$. Similarly, Δ_{xz}, Δ_{yz}.

- $V(S^2) = A\Delta_{xy} + A\Delta_{yz} + A\Delta_{xz}$ as an A-module. But $x\Delta_{yz} + y\Delta_{zx} + z\Delta_{xy} = 0$.
$\Rightarrow$ not a free module.

- Kac-Moody algs.

Ex. Loop alg: $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t] = L(\mathfrak{g})$

Consider Ω^1_A and $\underbrace{\Omega^1_A / dA}_{\text{Kähler differentials}}$ (1-forms module exact forms), there is a quotient

center of $\hat{\mathfrak{g}}$, with $[x \otimes a, y \otimes b] = [x,y] \otimes ab + K_{\mathfrak{g}}(x,y)\,\overline{b\,da}$

__Theorem.__ (Kassel) $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \Omega^1_A / dA$ is the universal center extension of $\tilde{\mathfrak{g}}$

- $\hat{\mathfrak{g}}$ is untwisted Affine Lie alg

- Let $\sigma \in \mathrm{Aut}\, \mathfrak{g}$, $\sigma^\ell = 1$. Consider $\mu \in \mathbb{C}$, $\mu^\ell = 1$, Extend σ to an automorphism $\hat{\sigma}: L(\mathfrak{g}) \to L(\mathfrak{g})$: $\hat{\sigma}(x \otimes t^n) = \mu^n \sigma(x) \otimes t^n$. Then $L(\mathfrak{g})^{\hat{\sigma}}$ is twisted affine Lie alg.