



# Definition of a Lie algebra

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Def. Let  $K$  be a field, and let  $A$  be a vector space over  $K$  equipped with multiplication

- $\cdot : A \times A \rightarrow A$  such that  $(A, \cdot)$  is a ring, then we say  $A$  is a  $K$ -algebra if  $\forall x, y, z \in A, k \in K$
- 1)  $(x+y) \cdot z = x \cdot z + y \cdot z \quad \Rightarrow \quad z \cdot (xy) = z \cdot x + z \cdot y$
- 2)  $(kx)y = k(xy) = x(ky)$ ,

In particular, if  $A$  is commutative (associative), we say  $A$  is a commutative (associative)  $K$ -algebra

Ex. 1<sup>o</sup>  $A = \text{Mat}_n(\mathbb{C})$ .  $A$  is an associative  $\mathbb{C}$ -algebra

$$2^o A = \mathbb{C}G = \left\{ \sum_{g \in G} kg : g \in G, \text{ only finitely many } kg \neq 0 \right\}$$

(over  $\mathbb{C}$ )

Def. A Lie algebra is a vector space  $g$ , with a Lie bracket map  $g \times g \rightarrow g$ , written  $(x, y) \mapsto [x, y]$ , satisfying

- 1) the Lie bracket is bilinear,
  - 2) the Lie bracket is skew-symmetric. i.e.  $[x, y] = -[y, x], \forall x, y \in g$
  - 3) the Lie bracket satisfies the Jacobi identity. i.e.  $\forall x, y, z \in g$
- $$[[x, y], z] + [y, [z, x]] + [z, [x, y]] = 0$$

Ex. 1<sup>o</sup>  $gl_n = \text{Mat}_n(\mathbb{C})$  with Lie bracket  $[x, y] = xy - yx$ .

$$\Rightarrow [xy] = xy - yx = -(yx - xy) = -[y, x]$$

$$\begin{aligned} 2) [ax+by, z] &= (ax+by)z - z(ax+by) = a(xz - bz) + b(yz - zy) \\ &= a[x, z] + b[y, z] \end{aligned}$$

$$\begin{aligned} 3) [[x, y], z] + [y, [z, x]] + [z, [x, y]] \\ &= x(yz - zy) - (yz - zy)x + y(zx - xz) - (zx - xz)y + z(xy - yx) - (xy - yx)z \\ &= 0 \end{aligned}$$

2<sup>o</sup>  $sl_n = \{x \in \text{Mat}_n(\mathbb{C}) : \text{tr}x = 0\}$ . Lie bracket is denoted by  $[xy] = xy - yx$

3<sup>o</sup>  $so_n = \{x \in \text{Mat}_n(\mathbb{C}) : x^t = -x\}$ . Lie bracket is denoted by  $[xy] = xy - yx$



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${}^4 \text{ gl}(V) = \{ \text{ linear transformations of } V \}$ . Lie bracket is denoted by  $[x,y] = xy - yx$

$\Leftrightarrow \text{End}(V)$

Prop I. Any associative  $\mathbb{C}$ -algebra  $A$  becomes a Lie algebra if we use the commutator  $xy - yx$  as the Lie bracket.

- $\mathfrak{sl}_2$

$$\mathfrak{sl}_2 = \{ x \in \text{Mat}_2(\mathbb{C}) : \text{tr } x = 0 \} .$$

$$= \text{span} \{ e_{11} - e_{22}, e_{12}, e_{21} \}$$

$$\left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$$

Denote  $e_{12} = e$ ,  $e_{21} = f$ ,  $e_{11} - e_{22} = h$

$$\mathfrak{sl}_2 = \text{span} \{ e, f, h \} .$$

$$[e, f] = e_{12} e_{21} - e_{21} e_{12} = e_{11} - e_{22} = h$$

$$[e, h] = e_{12} (e_{11} - e_{22}) - (e_{11} - e_{22}) e_{12} = e_{11} - e_{12} = -2e$$

$$[f, h] = e_{21} (e_{11} - e_{22}) - (e_{11} - e_{22}) e_{21} = e_{21} + e_{21} = 2f$$

$[.,.]$	$e$	$f$	$h$
$e$	0	$h$	$-2e$
$f$	$-h$	0	$2f$
$h$	$2e$	$2f$	0

Def. A Lie subalgebra of a Lie algebra  $\mathfrak{g}$  is a subspace  $\mathfrak{h}$  that is closed under the Lie bracket, i.e. that satisfies  $\forall x, y \in \mathfrak{h}$ ,  $[x, y] \in \mathfrak{h}$

$\mathfrak{sl}_n$  is a Lie subalgebra of  $\text{gl}_n$

Def. If  $\mathfrak{g}$  and  $\mathfrak{g}'$  are Lie algebras, a homomorphism  $\Phi: \mathfrak{g} \rightarrow \mathfrak{g}'$  is a linear map such that  $\Phi([x, y]) = [\Phi(x), \Phi(y)] \quad \forall x, y \in \mathfrak{g}$ . In particular,  $\Phi$  is an isomorphism if it is bijective.



# Representations of a Lie Algebra

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Def. A representation of a Lie algebra  $g$  is a Lie algebra homomorphism  $\Psi: g \rightarrow gl(V)$ , where  $V$  is a vectorspace and  $gl(V)$  denotes the Lie algebra of linear transformations of  $V$ . The dimension of  $\Psi$  is denoted by  $\dim V$ .

Ex. 1) Let  $V = g$ .  $\forall x, y \in g$ ,  $y \in V$ , define  $\text{ad}x(y) = [x, y]$  i.e.  $\Psi(x) = [x, \cdot]$

$$\text{ad}[x_1, x_2](y) = [[x_1, x_2], y] = [x_1, [y, x_2]] + [x_2, [x_1, y]]$$

$$[\text{ad}(x_1), \text{ad}(x_2)](y) = \text{ad}(x_1)\text{ad}(x_2)y - \text{ad}(x_2)\text{ad}(x_1)y = [x_1, [x_2, y]] - [x_2, [x_1, y]]$$

$\text{ad}$  is called the adjoint representation of  $g$

In particular,  $V = g = sl_2$ . Take basis  $\{e, f, h\}$ .

$$\text{ad}(e)(e) = [e, e] = 0 \quad \text{ad}(e)(f) = [e, f] = h \quad \text{ad}(e)(h) = -2e$$

$$\text{ad}(f)(e) = [f, e] = -h \quad \text{ad}(f)(f) = 0 \quad \text{ad}(f)(h) = 2f$$

$$\text{ad}(h)(e) = [h, e] = 2e \quad \text{ad}(h)(f) = -2f \quad \text{ad}(h)(h) = 0$$

$$\text{ad}(e) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{ad}(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{ad}(h) = \begin{pmatrix} 2 & & \\ & -2 & \\ & & 0 \end{pmatrix}$$

2)  $V = \mathbb{C}^2$ ,  $g = sl_2$ .  $\forall x \in sl_2$ ,  $v \in \mathbb{C}^2$ ,  $\Psi(x)(v) = xv$

$$\Psi([x, y])(v) = [x, y]v = xyv - yvx = ([\Psi(x), \Psi(y)]v)$$

Def. Let  $g$  be a Lie algebra. A  $g$ -module is a vectorspace  $V$  equipped with a bilinear map  $g \times V \rightarrow V: (x, v) \mapsto xv$ , called the  $g$ -action, which is assumed to satisfy  $[x, y]v = x(yv) - y(xv) \quad \forall x, y \in g, v \in V$

Prop 2.  $\Psi: g \rightarrow gl(V)$  is a rep of  $g$  if and only if  $V$  is a  $g$ -module

Pf. Assume that  $\Psi: g \rightarrow gl(V)$  is a rep of  $g$ . We can define a bilinear map  $gxV \rightarrow V: (x, v) \mapsto \Psi(x)(v)$ . Since  $[x, y]v = \Psi([x, y])(v) = [\Psi(x), \Psi(y)](v)$



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$$= \varphi(x_0)\varphi(y)V - \varphi(y_0)\varphi(x)V = xyV - yxV$$

Conversely, if there is a  $g$ -action  $\varphi: g \times V \rightarrow V: (x, v) \mapsto xv$

We can define a linear map  $\Phi: g \rightarrow \text{gl}(V)$  by  $x \mapsto \varphi_x: v \mapsto xv$

$$\Phi([x,y])(v) = [xv,y]v = xyv - yxv = \varphi_{xy}(\varphi_y(v)) - \varphi_{yx}(\varphi_y(v)) = [\varphi_x, \varphi_y](v)$$

Def. Let  $V, W$  be  $g$ -modules. A  $g$ -module homomorphism or  $g$ -linear map from  $V$  to  $W$  is a linear map  $\psi: V \rightarrow W$  such that  $x\psi(v) = \psi(xv)$ ,  $\forall x \in g$ . If  $\psi$  is bijective, we say  $V$  is isomorphic to  $W$  and  $\psi$  is a isomorphism.

Def. Let  $V$  be a  $g$ -module. A  $g$ -submodule or an invariant subspace is a subspace  $W \subseteq V$  such that  $xW \subseteq W$ ,  $\forall x \in g$ .

Def. Let  $V$  be a  $g$ -module, if  $V$  has no submodule except trivial submodules  $\{0\}$  and  $V$ , we say  $V$  is irreducible or simple. If  $V$  does have a nontrivial submodule, we say  $V$  is reducible. If  $V$  can be written as a direct sum of irreducible module, we say  $V$  is semisimple or completely reducible.

Rank. Any  $g$ -module of dimension 1 is irreducible.

$\Phi: g \rightarrow \text{gl}(V)$	$ $	$V$ $g$ -module
subrep $\psi: g \rightarrow \text{gl}(W)$	$ $	$g$ -submodule $W$
irreducible rep $\Phi: g \rightarrow \text{gl}(V)$	$ $	irreducible $g$ -module $V$
completely reducible rep. $\Phi = \Phi_1 \oplus \dots \oplus \Phi_n$	$ $	completely reducible $g$ -module $V = V_1 \oplus V_2 \oplus \dots \oplus V_n$



# The theory of $\mathfrak{sl}_2$ -modules

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$\cdot \mathfrak{sl}_2$ -module

Thm. Every  $\mathfrak{sl}_2$ -module is completely reducible.

Example 1.  $\mathbb{C}$  is an  $\mathfrak{sl}_2$ -module,  $x \cdot c = 0 \quad \forall x \in \mathfrak{sl}_2, c \in \mathbb{C}$ .

$\mathbb{C}$  is simple for  $\dim \mathbb{C} = 1$ .

2.  $\text{ad}: \mathfrak{sl}_2 \rightarrow \mathfrak{gl}(\mathfrak{sl}_2) \Rightarrow \mathfrak{sl}_2$  is an  $\mathfrak{sl}_2$ -module.

$\forall x \in \mathfrak{sl}_2, y \in \mathfrak{sl}_2, xy \stackrel{?}{=} [x, y]$

We claim  $\mathfrak{sl}_2$  is a simple  $\mathfrak{sl}_2$ -module.

If  $V \neq \{0\}, V \subseteq \mathfrak{sl}_2$  is a submodule, there is  $x = a_1e + a_2f + a_3h \in V, x \neq 0$

$$1^{\circ} a_3 \neq 0, ex = a_3[e, f] + a_3[h, f] = a_3h - 2a_3f,$$

$$h(ex) = -2a_3[h, f] = 4a_3f$$

$$e(hex) = 4a_3[e, f] = 4a_3h \Rightarrow heV \Rightarrow V = \mathfrak{sl}_2$$

$$2^{\circ} a_3 = 0, x = a_1e + a_2f$$

$$hx = a_1[h, e] + a_2[h, f] = 2a_1e \Rightarrow a_1e$$

$$2x + hx = 4a_1e \neq 0 \quad \text{or} \quad 2x - hx = 4a_2f \neq 0$$

$$\Rightarrow e \in V \text{ or } f \in V \Rightarrow V = \mathfrak{sl}_2$$

3.  $\mathbb{C}^2$  is an  $\mathfrak{sl}_2$ -module,  $x \cdot v = xv \leftarrow \text{mult. of matrices}$ .

We claim that  $\mathbb{C}^2$  is a simple  $\mathfrak{sl}_2$ -module.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \Rightarrow \text{if } \begin{pmatrix} a \\ b \end{pmatrix} \in V, \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix} \in V$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix} \Rightarrow \mathbb{C}^2 \text{ has no nontrivial submodule.}$$

4.  $\mathfrak{gl}_2$  is an  $\mathfrak{sl}_2$ -module,  $x \cdot v = [x, v] \quad \forall x \in \mathfrak{sl}_2, v \in \mathfrak{gl}_2$

$$x \cdot V = [x, V] = 2V - Vx \in \mathfrak{sl}_2, \text{ since } \text{tr}(xV - Vx) = 0.$$

$\mathfrak{gl}_2 = \mathfrak{sl}_2 \oplus \mathbb{C}1_2$  as a vector space.

$$\text{① } \mathfrak{sl}_2 \cap \mathbb{C}1_2 = \{0\} \quad \text{② } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \text{ad} & 0 \\ 0 & \text{ad} \end{pmatrix} + \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \quad \text{③ } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}_2$$

Both  $\mathfrak{sl}_2$  and  $\mathbb{C}1_2$  are irreducible modules.

So  $\mathfrak{gl}_2 = \mathfrak{sl}_2 \oplus \mathbb{C}1_2$  as a  $\mathfrak{g}$ -module



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5.  $gl_3$  is an  $sl_2$ -module  $\forall v \in gl_3, x \in sl_2, x \cdot v = (x \circ) v$

$$\text{eg. } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} gl_3 &= \begin{pmatrix} sl_2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} C I_2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \\ &\cong sl_2 \oplus C I_2 \oplus C^2 \oplus C^2 \oplus C \quad (sl_2\text{-modules}) \end{aligned}$$

$$x \cdot \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xw \\ 0 & 0 \end{pmatrix}, \Rightarrow \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \cong C^2 \quad (\cong \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix})$$

Given an arbitrary  $sl_2$ -module.  $V$

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_n, \quad V_i \text{ is irr}$$

$$\cong a_1 W_1 \oplus a_2 W_2 \oplus \dots \oplus a_m W_m \oplus \dots, \quad \{W_i\} \text{ is the set of irreducible } sl_2\text{-modules, } a_i \in \mathbb{N} \text{ (in the sense of isomorphism)}$$

1. Find all  $W_i$ .

2. Calculate  $a_i$ .

### Classification of irreducibles.

Let  $V$  be an  $sl_2$ -module,  $h_V: V \rightarrow V: v \mapsto hv$  (left scalar multi.)

$$\Phi: sl_2 \rightarrow gl(V), \quad hv \mapsto \Phi(h)$$

Def. For  $a \in \mathbb{C}$ , the  $a$ -eigenspace of  $h_V$  is written  $V_a$  and called the weight space of weight  $a$ . and any  $a$ -eigenvector is called a weight vector of weight  $a$ .

The eigenvalues of  $h_V$  is called the weights of  $V$ .

Prop 3. For any  $a \in \mathbb{C}$ , we have  $eV_a \subseteq V_{a+2}$ ,  $fV_a \subseteq V_{a-2}$

Pf.  $\forall v \in V_a, \quad hv = av$ .

$$h(ev) = [h, e]v + e(hv) = 2ev + aev = (a+2)(ev) \Rightarrow ev \in V_{a+2}$$

$$h(fv) = [h, f]v + f(hv) = -2fv + afv = (a+2)(fv) \Rightarrow fv \in V_{a-2}$$



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Def. A weight vector  $v$  such that  $ev=0$  is called a highest-weight vector

Rmk. Any nonzero  $\mathfrak{sl}_2$ -module  $V$  contains a highest-weight vector

$hv$  is a linear map  $V \rightarrow V$ .  $hv$  has an eigenspace of some eigenvalue, say  $V_\alpha$ . Since  $hv$  has only finitely many eigenvalues, there exist  $i \in \mathbb{N}$  such that  $V_{\alpha+2i} \neq \{0\}$  and  $V_{\alpha+2i+2} = \{0\}$ .

Prop 4. Let  $V$  be an  $\mathfrak{sl}_2$ -module and suppose  $w_0 \in V$  is a highest weight vector of weight  $m$ . Then,

1) If we define  $w_i = \frac{1}{i!} f^i w_0$ ,  $i \geq 1$ , then  $ew_i = (m-i+1)w_{i-1}$   $\forall i \geq 1$

2)  $m \in \mathbb{N}$ .

3)  $V_{(m)} = \text{span}\{w_0, w_1, \dots, w_m\}$  is an irreducible submodule.

Pf.  $e^i f w_0 = (i+1) w_{i+1}$

We claim that  $ew_i = (m-i+1)w_{i-1}$ ,  $\forall i \geq 1$ .

$$ew_i = ef w_0 = [e, f] w_0 + f ew_0 = hw_0 = mw_0.$$

$$\begin{aligned} \text{By induction, } ew_i &= e[ef w_{i-1}] = \frac{1}{i} [e, f] w_{i-1} + \frac{1}{i} f(e w_{i-1}) \\ &= \frac{1}{i} h w_{i-1} + \frac{1}{i} (m-i+2) f w_{i-2} \\ &= \frac{1}{i} (m-2i+2) w_{i-1} + \frac{1}{i} (m-i+2)(i-1) w_{i-2} \\ &= (m-i+1) w_{i-1} \end{aligned}$$

Since  $hv$  has finitely many eigenvalues, there exist  $j \in \mathbb{N}$ , s.t.  $w_j \neq 0$  and  $w_{j+2} = 0$

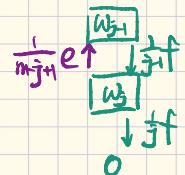
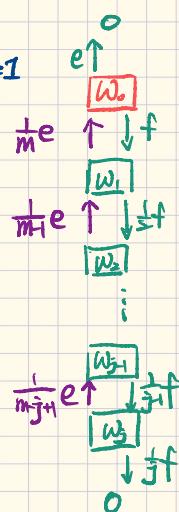
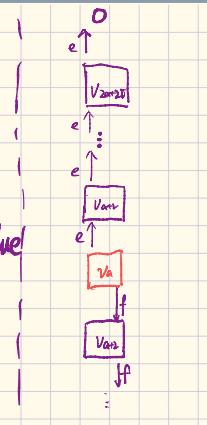
$$0 = ew_{j+1} = (m-j) w_j \Rightarrow m=j$$

3) Since  $\{e, f, h\}$  is a basis of  $\mathfrak{sl}_2$ ,  $\forall x \in \mathfrak{sl}_2$ ,  $xV_{(m)} \subseteq V_{(m)}$ ; this is  $V_{(m)}$  is a submodule.

Assume  $0 \neq U \subseteq V_{(m)}$  is a submodule of  $V_{(m)}$ , let  $v = \sum_{i=0}^m a_i w_i \in U$ ,  $a_i \in \mathbb{C}$ ,

let  $k$  be the maximal index with  $a_k \neq 0$ ,

$$e^K v = \sum_{i=0}^k a_i e^K w_i = a_k e^K w_k = b_K w_0, b_K = \frac{m!}{(m-k)!} a_k.$$

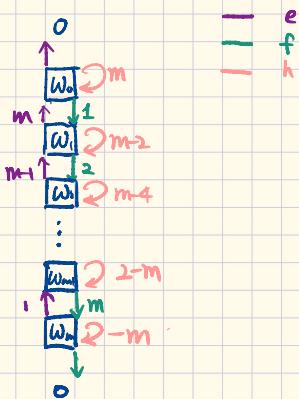




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Thus  $w_0 \in U \Rightarrow w_0, w_1, \dots, w_m \in U \Rightarrow U = V(m)$

$w_0$  is a highest-weight vector of weight  $m$ .  $V(m)$



Prop 5. Every irr  $\mathfrak{sl}_n$ -module  $V$  is isomorphic to  $V(m)$ , where  $m = \dim V - 1$ .

Pf. Firstly, we claim that  $V$  has a highest weight vector.  $h_v$  has an eigenvector  $w$ ,  $hw = aw$ .  $w \in V_a$ .  $e^k w \in V_{a+k}$ . It follows that  $\{e^k w : k \geq 0, e^k w \neq 0\}$  are linearly independent. There are  $n \in \mathbb{N}$  such that  $e^n w \neq 0$  &  $e^{n+1} w = 0$ . Then  $e^n w \in V_{a+n}$  is the highest weight vector.

Let  $w_0 \in V$  be the highest vector, then  $\{w_0, w_1, \dots, w_m\} \subseteq V \Rightarrow V(m)$  is a submodule of  $V$ . Thus,  $V(m) = V$   
 $\dim V(m) = m+1 \Rightarrow m = \dim V - 1$

Ex. For  $m \in \mathbb{N}$ , define  $S^m$  as a vector space of homogeneous polynomials  $p$  of degree  $m$  in two indeterminates  $x$  and  $y$ . Define

$$e = x \frac{\partial}{\partial y}, \quad f = y \frac{\partial}{\partial x}, \quad h = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

Then  $S^m$  is a  $\mathfrak{sl}_2$ -module and  $S^m \cong V(m)$

$$\begin{array}{ccccccc} w_0 & \xrightarrow{f} & w_0 & & & & \\ x^m & x^{m-1}y & \cdots & x^1y & y^m & & \\ \uparrow & \uparrow & & \uparrow & \uparrow & & \\ v_m & v_{m-1} & \cdots & v_1 & v_0 & & \end{array} \quad \text{a basis of } S^m.$$

$$\begin{aligned} h(x^{m-1}y) &= x \frac{\partial}{\partial x}(x^{m-1}y) \\ &\quad - y \frac{\partial}{\partial y}(x^{m-1}y) \\ &= (m-1)x^{m-2}y - x^{m-1}y \\ &= (m-2)x^{m-1}y \end{aligned}$$

Thus,  $x^m$  is a highest weight vector with weight  $m$ .