高代习题课

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Part I. 习题讲解

A.5.8. $\forall x\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$，　$x\in\frac{1}{2}\mathbb{Z}\Rightarrow\exists k\in\mathbb{Z}$ s.t. $x=\frac{k}{2}$.

$x\notin\mathbb{Z}\Rightarrow\frac{k}{2}\notin\mathbb{Z}\Rightarrow$ k 为奇数 $\Rightarrow\exists l\in\mathbb{Z}$, s.t. $k=2l+1$.

$\Rightarrow x=\frac{2l+1}{2}=l+\frac{1}{2}$，$l\in\mathbb{Z}\Rightarrow x\in\frac{1}{2}+\mathbb{Z}$

$\forall x\in\frac{1}{2}+\mathbb{Z}$，　$\exists m\in\mathbb{Z}$, s.t. $x=\frac{1}{2}+m=\frac{1+2m}{2}\in\frac{1}{2}\mathbb{Z}$

若 x 为整数，则 $\frac{1}{2}$ 为整数，故矛盾.

$\Rightarrow x\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$

$\Rightarrow\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}=\frac{1}{2}+\mathbb{Z}$

A.5.9. 对任意 $x\in X$，$y\in Y$，

若 $(x,y)\in(A\cap B)\times(C\cap D)$，则 $(x\in A$ 且 $x\in B)$ 且 $(y\in C$ 且 $y\in D)$

即 $(x\in A$ 且 $y\in C)$ 且 $(x\in B$ 且 $y\in D)$

即 $(x,y)\in(A\times C)\cap(B\times D)$

反过来，若 $(x,y)\in(A\times C)\cap(B\times D)$，只需将上述过程倒过来写即可证明. $(A\times C)\cap(B\times D)\subset(A\cap B)\times(C\cap D)$

综上，两集合相等.

A.5.10　1. 正确

2. 不正确：考虑 $X=\{1,2,3\}$，$A=\{1\}$，$B=\{2\}$，$C=\{1,3\}$

则 $(A\cap B)\cup C=\phi\cup\{1,3\}=\{1,3\}$

而 $A\cap(B\cup C)=\{1\}\cap\{1,2,3\}=\{1\}$

3. 正确.

A.5.12. $\# \mathcal{P}(X) = 2^{|X|}$

A.5.14. 定义映射 $\sigma_Q : X \to \{0,1\}$, $\sigma_Q(x) = \begin{cases} 0 & , x \notin Q \\ 1 & , x \in Q \end{cases}$ 示性函数

对于 $Q \in \mathcal{P}(X)$ 事实上, $\sigma_Q \in Map(X, \{0,1\})$, 所以我们可以

定义 $\sigma : \mathcal{P}(X) \to Map(X, \{0,1\}) : Q \mapsto \sigma_Q$

下验证其为双射

1. 单射: 若 $\sigma_Q = \sigma_P$, 其中 $Q, P \in \mathcal{P}(X)$,

$\forall x \in Q$, $\sigma_P(x) = \sigma_Q(x) = 1 \Rightarrow x \in P$

$\forall x \in P$, $\sigma_Q(x) = \sigma_P(x) = 1 \Rightarrow x \in Q$

$\Rightarrow Q = P$ 得

2. 满射: 对任意 $\alpha \in Map(X, \{0,1\})$, 考虑集合

$N = \{x \in X : \alpha(x) = 1\}$, $M = \{x \in X : \alpha(x) = 0\}$

M, N 为 X 的子集 $\Rightarrow N \in \mathcal{P}(X)$. 且 $N \cup M = X$

$(\sigma_N - \alpha)(x) = \begin{cases} 1 - 1 = 0 & x \in N \\ 0 - 0 = 0 & x \in M \end{cases}$

$\Rightarrow \sigma_N = \alpha$

A.5.17.

1. f, g 为单射. $g \circ f : X \to Z$, $\forall x_1, x_2 \in X$, 若有

$g \circ f(x_1) = g \circ f(x_2)$

则 $g(f(x_1)) = g(f(x_2))$. 由 g 为单射, $f(x_1) = f(x_2)$.

又有 f 为单射, 于是 $x_1 = x_2$. 故 $g \circ f$ 为单射.

2. ~~$g \circ f$ 的陪域为 $g(f(X))$, 对任意~~

对任意 $z \in Z$, 由 g 是满射, $\exists y \in Y$, s.t. $g(y) = z$.

又由 f 是满射, $\exists x \in X$ s.t. $f(x) = y$, 即 $g(f(x)) = g \circ f(x) = z$

故 $g \circ f$ 为满射.

3. 由 1, 2 可知, 若 f, g 为双射, 则 $g \circ f$ 是双射.

$(f^{-1} \circ g^{-1})(g \circ f) = 1_X$, $\cancel{(g \circ f)(f^{-1} \circ g^{-1}) = 1_Z}$ $\Rightarrow (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

A.5.18. 1. 不正确：$f:\{1\}\to\mathbb{Z}$，$g:\mathbb{Z}\to\{1\}$，其中 $f(1)=1$

则 $g\circ f:\{1\}\to\{1\}$ 是单射且是满射

然而 f 不满且 g 不单

2. 不正确：反例如上.

3. 不正确：反例如上.

A.5.24. $f\circ i(x)=f(x)=\sin\pi x=0 \quad \forall x\in\mathbb{Z}$

$\Rightarrow g=f\circ i$

A.5.26. ① $\forall x\in X$，$g(x)=x^3=x$. 所以有

$f\circ g=f$，故图交换

②

$\{1,-1\}\xrightarrow{f}\{1\}$

$\{0\}\longrightarrow\{0\}$

$g_1=id_X$，$g_2:\begin{array}{l}1\mapsto 1\\-1\mapsto 1\\0\mapsto 0\end{array}$ $g_3:\begin{array}{l}1\mapsto -1\\-1\mapsto 1\\0\mapsto 0\end{array}$ $g_4:\begin{array}{l}1\mapsto -1\\-1\mapsto -1\\0\mapsto 0\end{array}$

Part II. 补充内容

Peano Axioms: Start at the Begining.

- Peano (1852-1932) from Italy. A mathematician and glottologist.

1) Peano Axioms: 1889

a formal foundation for the collection of natural numbers

2) Peano Curve: 1890

the first example of space filling curve.

Hook: The Secret Number (《隐匿的数字》):

"There is a number between 3 & 4."

- "Characteristic" of the natural numbers:
  ① with a start number
  ② successive incremend
  ③ not wrap-around
- IMPORTANT! Set aside, for the moment, everything you know about the natural numbers, Forget how to count, to add, to multiply.
- Two foundamental concepts: 0 and increment operation (successor operation)

Axiom 1. 0 is a natural number.

Axiom 2. If n is a natural number, then $n{+}{+}$ is also a natural number.

Axiom 3. 0 is not the successor of any natural number.

Axiom 4. Different natural numbers must have different successors.

Axiom 5. (Principle of mathematical induction). Let $P(n)$ be any property pertaining to a natural number n. Suppose $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.

- There is a number system $\mathbb{N}$, whose elements we will call natural numbers, for which Axioms 1-5 are true.

- Denote $0{+}{+}$ by 1, $1{+}{+}$ by 2, $\cdots$. Then $\mathbb{N} = \{0, 1, 2, \cdots\}$

Exercise. Prove

1) 3 is a natural number 2) $4 \neq 0$ 3) $6 \neq 2$.

Refences: 1. Analysis I by Terence Tao, chapter 2

2. 陶哲轩的实分析，第2章.