

# 3-11 Matchings and Factors

## (Part II: Perfect Matchings)

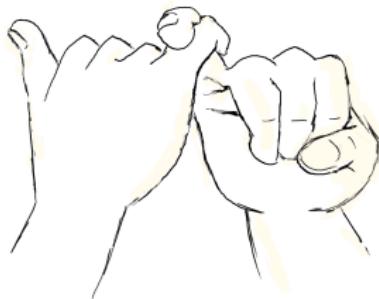
Hengfeng Wei

hfwei@nju.edu.cn

December 24, 2018

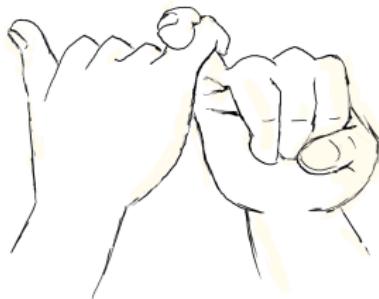


# Chinese Postman Problem (CPP)



Shortest Simple Path in Undirected Graphs

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## (Postman Tour Problem, Route Inspection Problem)





管梅谷(1934-)

第 10 卷 第 3 期  
1960 年 12 月

数学学报  
ACTA MATHEMATICA SINICA

Vol. 10, No. 3  
Dec., 1960

## 奇偶点图上作业法\*

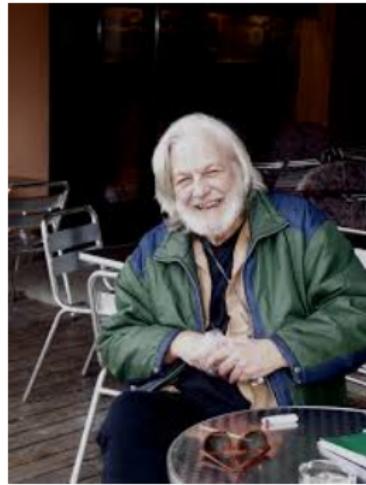
管梅谷  
(山东师范学院)

### §1. 問題的提出

在邮局搞綫性规划时,发现了下述問題:“一个投递員每次上班,要走遍他負責送信的段<sup>1</sup>,然后回到邮局,問應該怎样走才能使所走的路程最短。”

《奇偶点图上作业法》, 1960

Translated into English in 1962



Jack Edmonds (1934-)

## MATCHING, EULER TOURS AND THE CHINESE POSTMAN

Jack EDMONDS

*University of Waterloo, Waterloo, Ontario, Canada*

and

Ellis L. JOHNSON

*IBM Watson Research Center, Yorktown Heights, New York, U.S.A.*

Received 20 May 1972

Revised manuscript received 3 April 1973

The solution of the Chinese postman problem using matching theory is given. The convex hull of integer solutions is described as a linear programming polyhedron. This polyhedron is used to show that a good algorithm gives an optimum solution. The algorithm is a specialization of the more general  $b$ -matching blossom algorithm. Algorithms for finding Euler tours and related problems are also discussed.

“Matching, Euler Tours and the Chinese Postman”, 1973 (1965)

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to find the shortest tour such that each edge is traversed at least once.

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*Q* : What is the relation between Postman Tour and Eulerian Tour?



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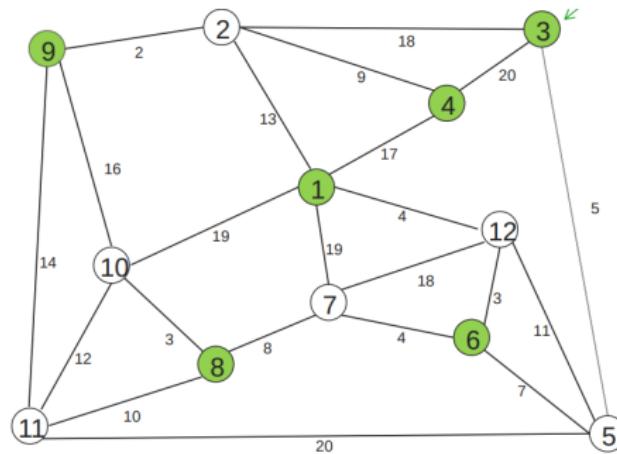
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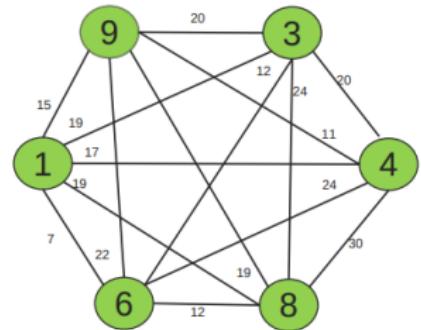
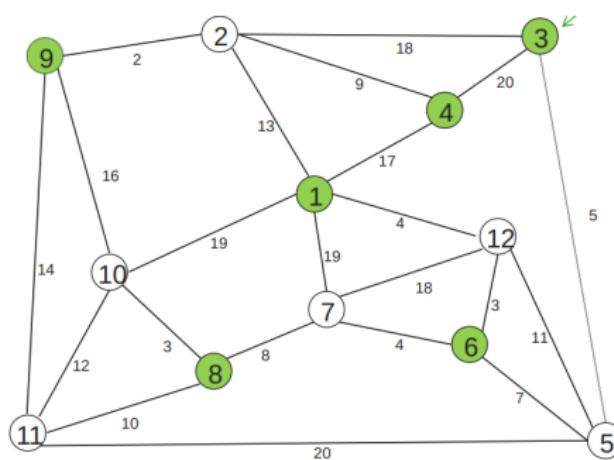
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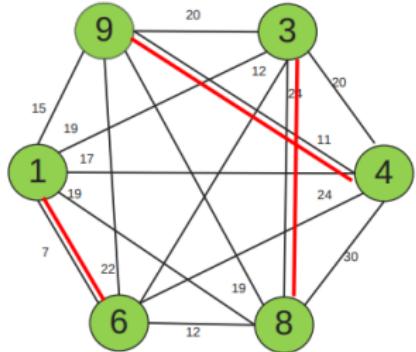
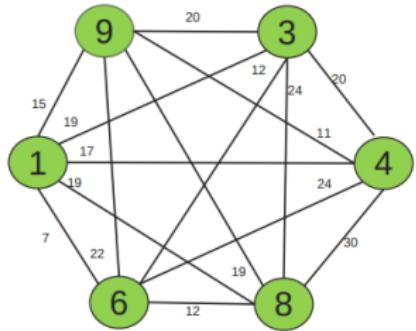
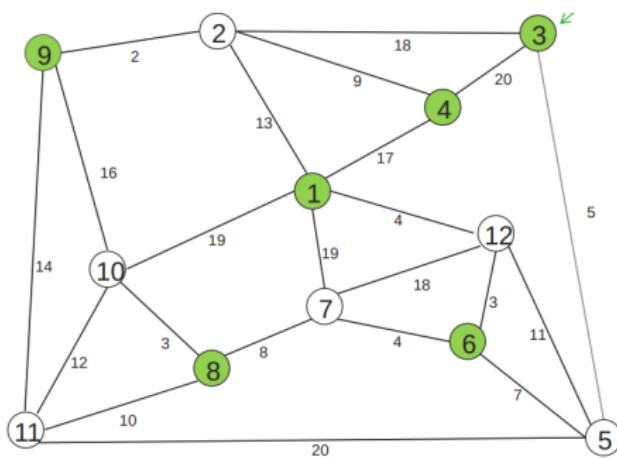
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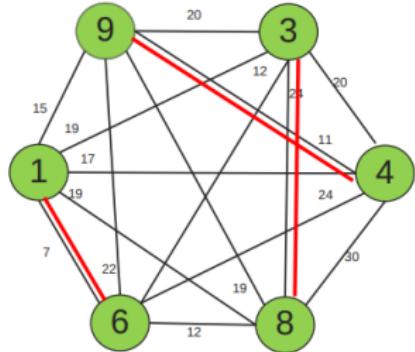
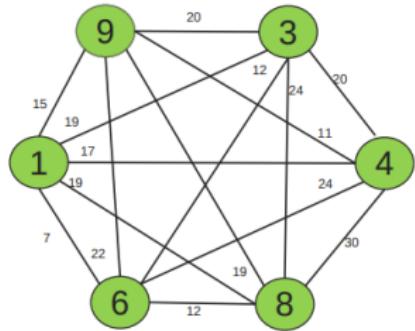
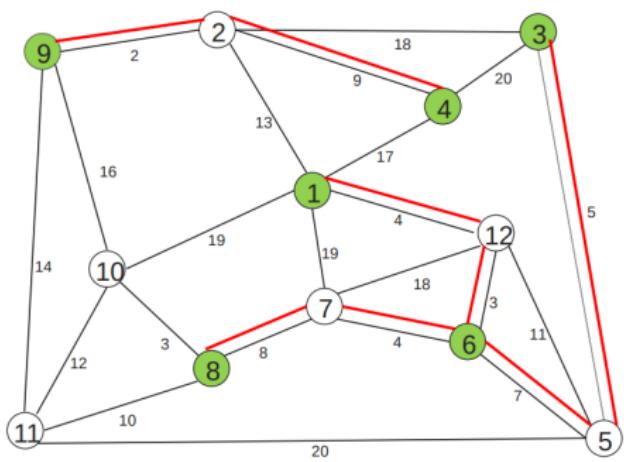
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*Q* : What if some edge  $e \in E(G)$  is in two shortest paths corresponding to (two) matching edges of  $G_p$ ?

## Theorem (Edge-disjointness of Shortest Paths)

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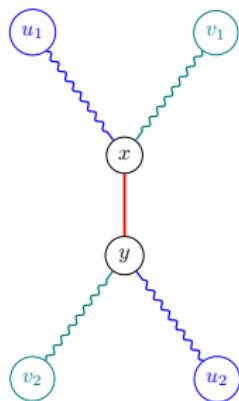
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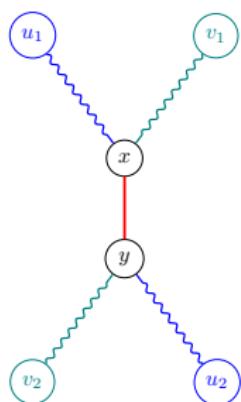
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Contradiction:

$u_1 \rightsquigarrow v_1, u_2 \rightsquigarrow v_2 \implies$  smaller perfect matching

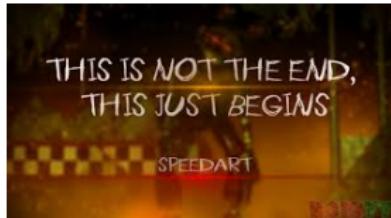


## Theorem (Property of CHINESE-POSTMAN)

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Proof.

A collection of edge-disjoint paths connecting pairs of odd vertices.  $\square$

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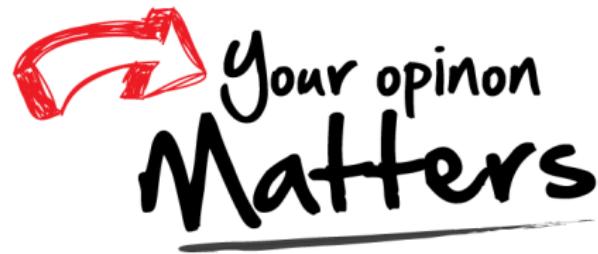
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