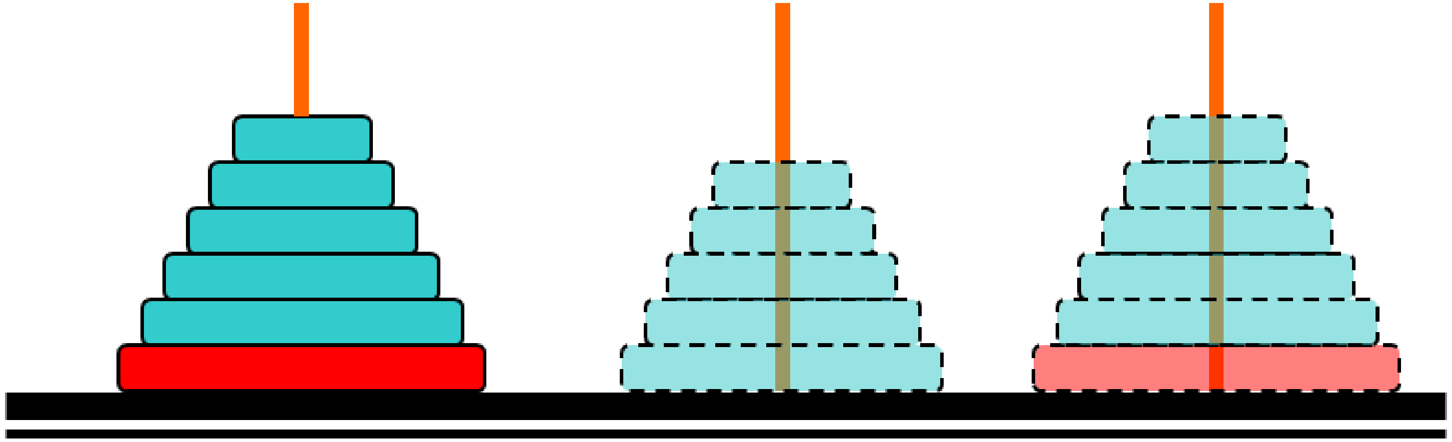
计算机问题求解 – 论题2-6

- 递归及其数学基础

2016年03月31日

问题1:

以Hanoi Tower为例，说说你对“递归思想”、“递归过程”、“递归式”的理解。



最简单的解递归的方法 – 回朔

$T(1) = 1$

$T(n) = 2T(n-1) + 1$

$$T(n) = 2T(n-1) + 1$$

$$2T(n-1) = 4T(n-2) + 2$$

$$4T(n-2) = 8T(n-3) + 4$$

$$\cdots\cdots$$

$$2^{n-2}T(2) = 2^{n-1}T(1) + 2^{n-2}$$

$$\Rightarrow \quad T(n) = 2^n-1$$

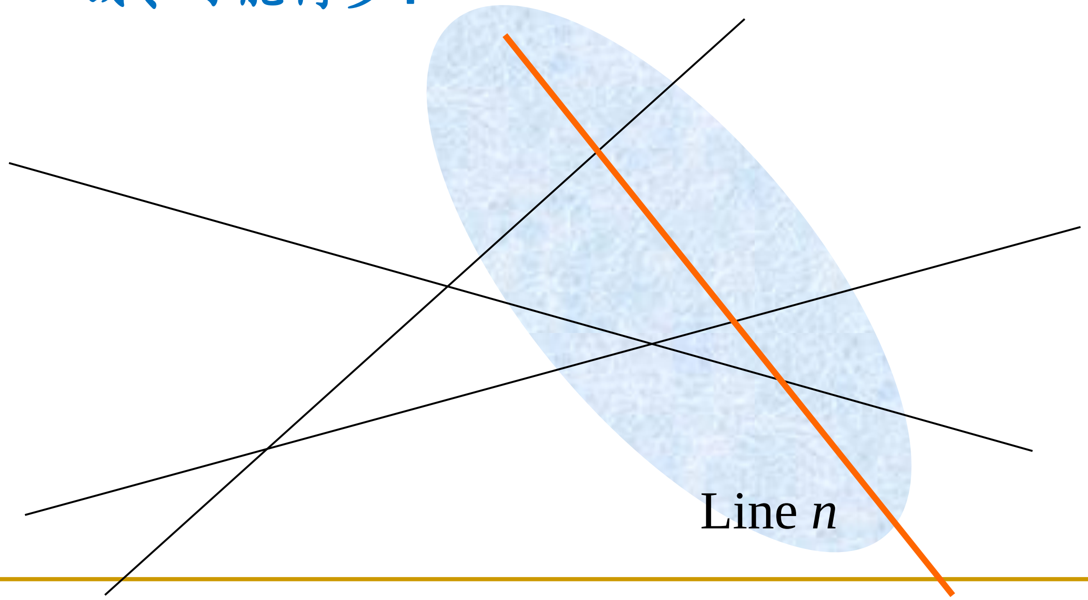
问题2:

这里的解递归式和前一次讨论中的解递归式有什么区别?

递归思维：直线划分平面

- 问题：
  - *n* 条直线（无限长）**最多**能将平面分为多少个区域（包括有限与无限区域）？

怎样能使划分的区
域尽可能得多？



$$L(0) = 1$$
$$L(n) = L(n-1) + n$$

用回朔的办法解递归

$$
\begin{aligned}
L(n) &= L(n-1)+n \\
&= L(n-2)+(n-1)+n \\
&= L(n-3)+(n-2)+(n-1)+n \\
&= \ldots\ldots \\
&= L(0)+1+2+\ldots\ldots+(n-2)+(n-1)+n
\end{aligned}
$$

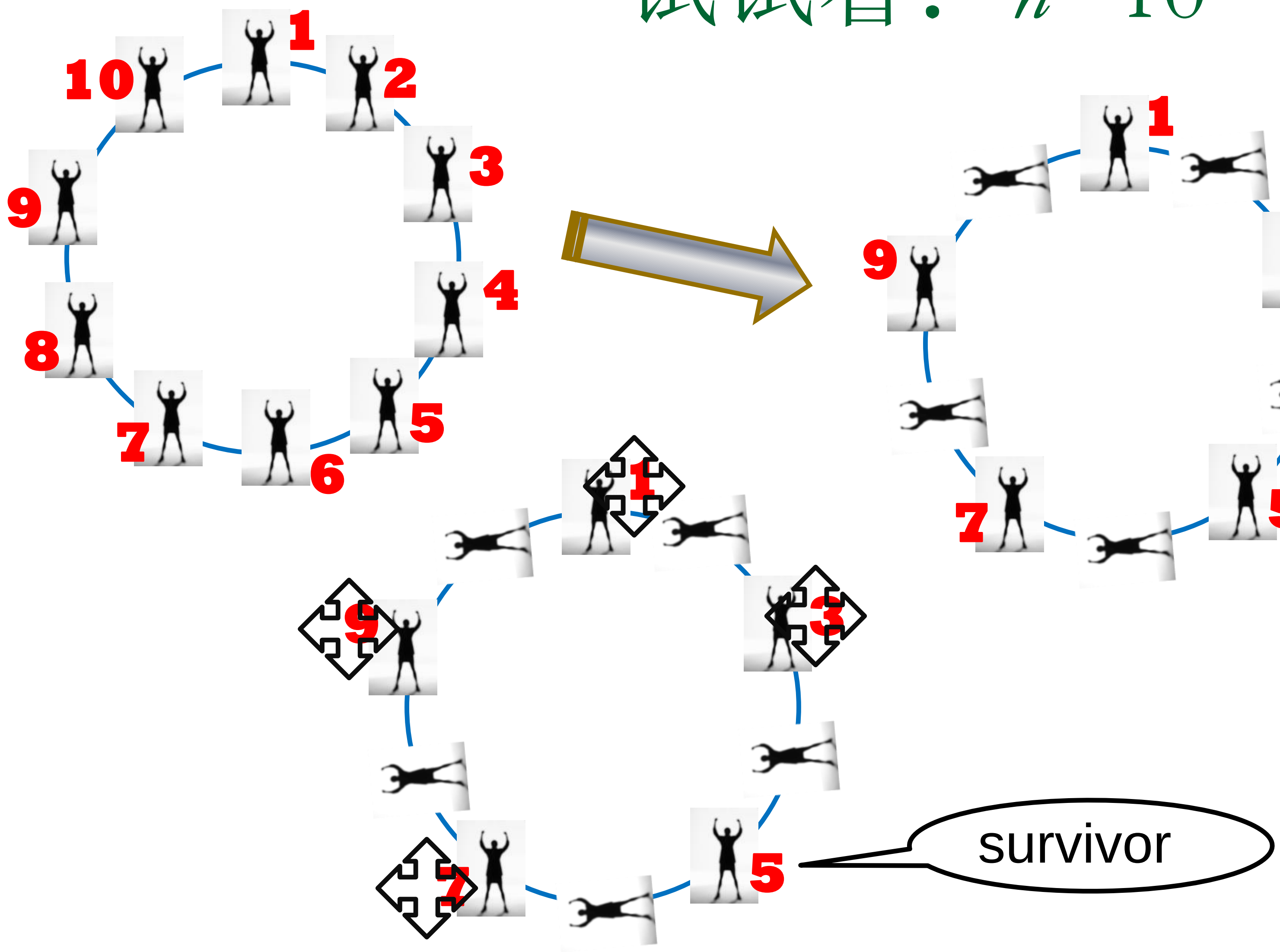
$$\boxed{L(n) = n(n+1)/2 + 1}$$

递归思维：Josephus 问题

- **Live or die, it’s a problem!**
- Legend has it that Josephus wouldn't have lived to become famous without his mathematical talents. During the Jewish Roman war, he was among a band of 41 Jewish rebels trapped in a cave by the Romans. Preferring suicide to capture, the rebels decided to form a circle and, proceeding around it, to kill every third remaining person until no one was left. But Josephus, along with an unindicted co-conspirator, wanted none of this suicide nonsense; so he quickly calculated where he and his friend should stand in the vicious circle.

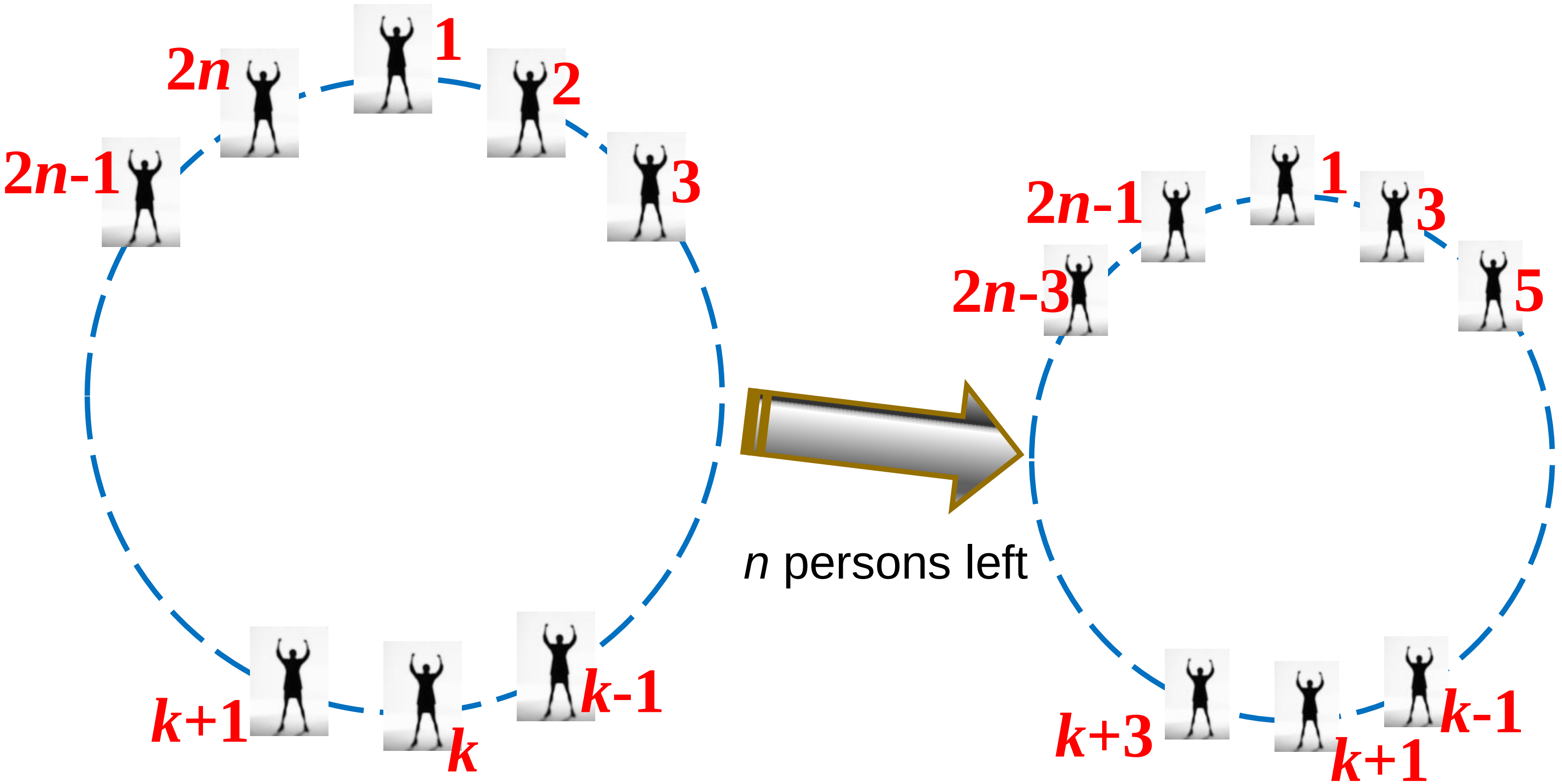
We use a simpler version: “every second...”

试试看： n=10



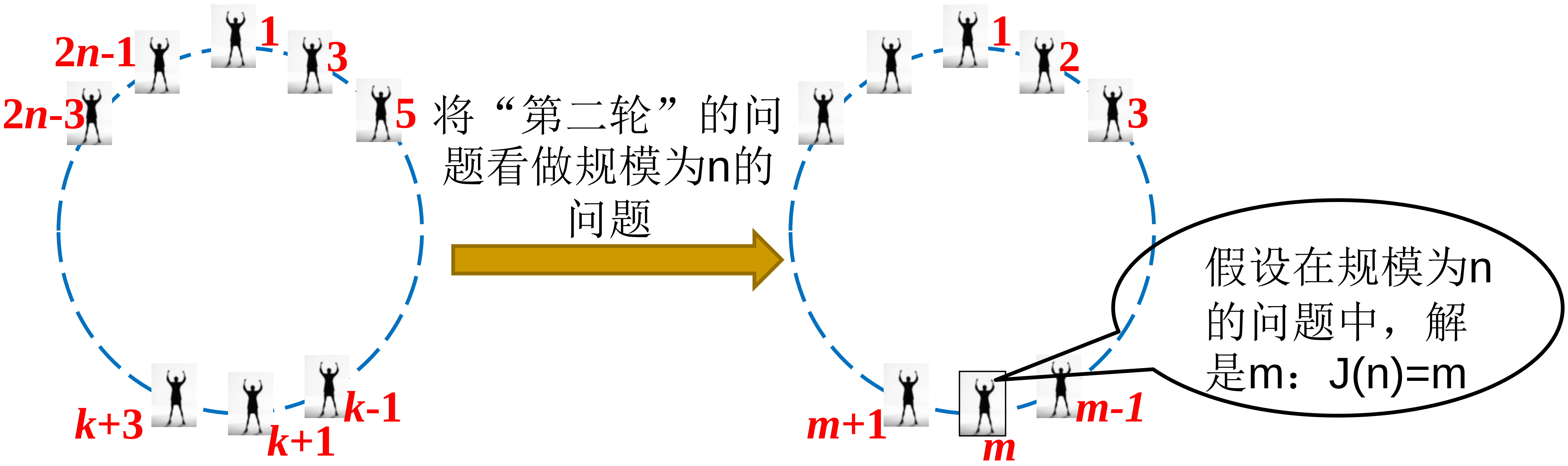
J(10)=5

For 2n Persons (n=1,2,3,...)



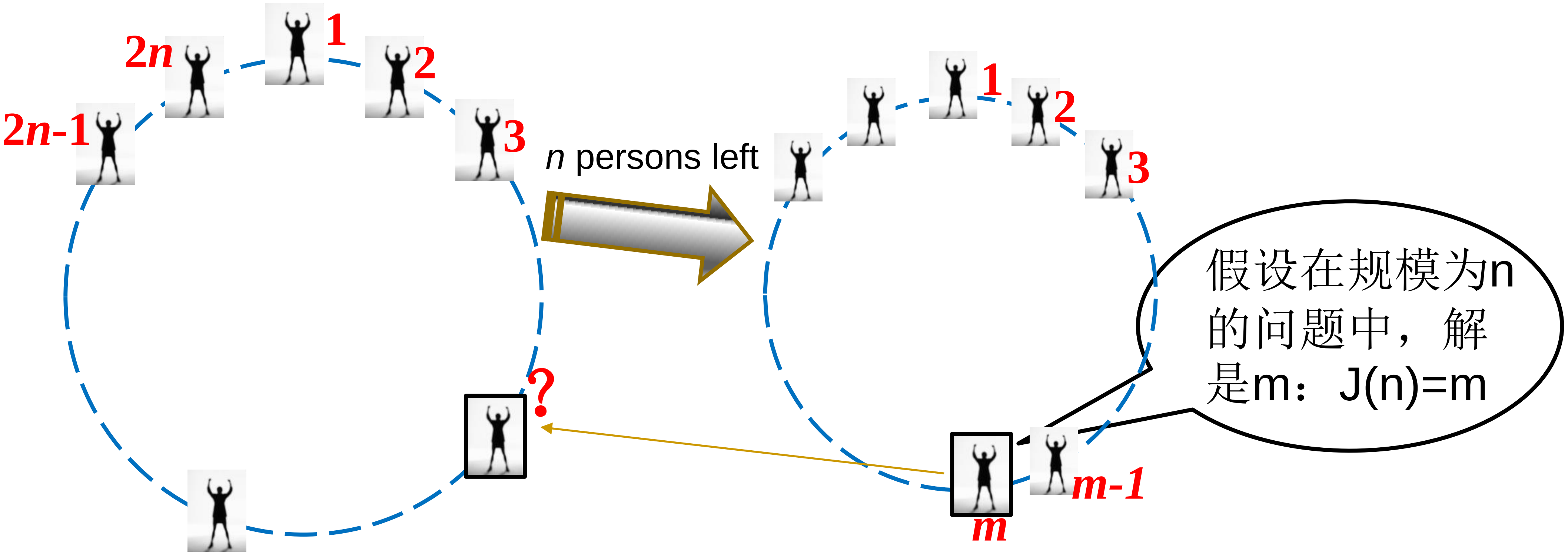
下一轮将如何进行？它和上一轮有什么相同之处？

For 2*n* Persons (*n*=1,2,3,...)



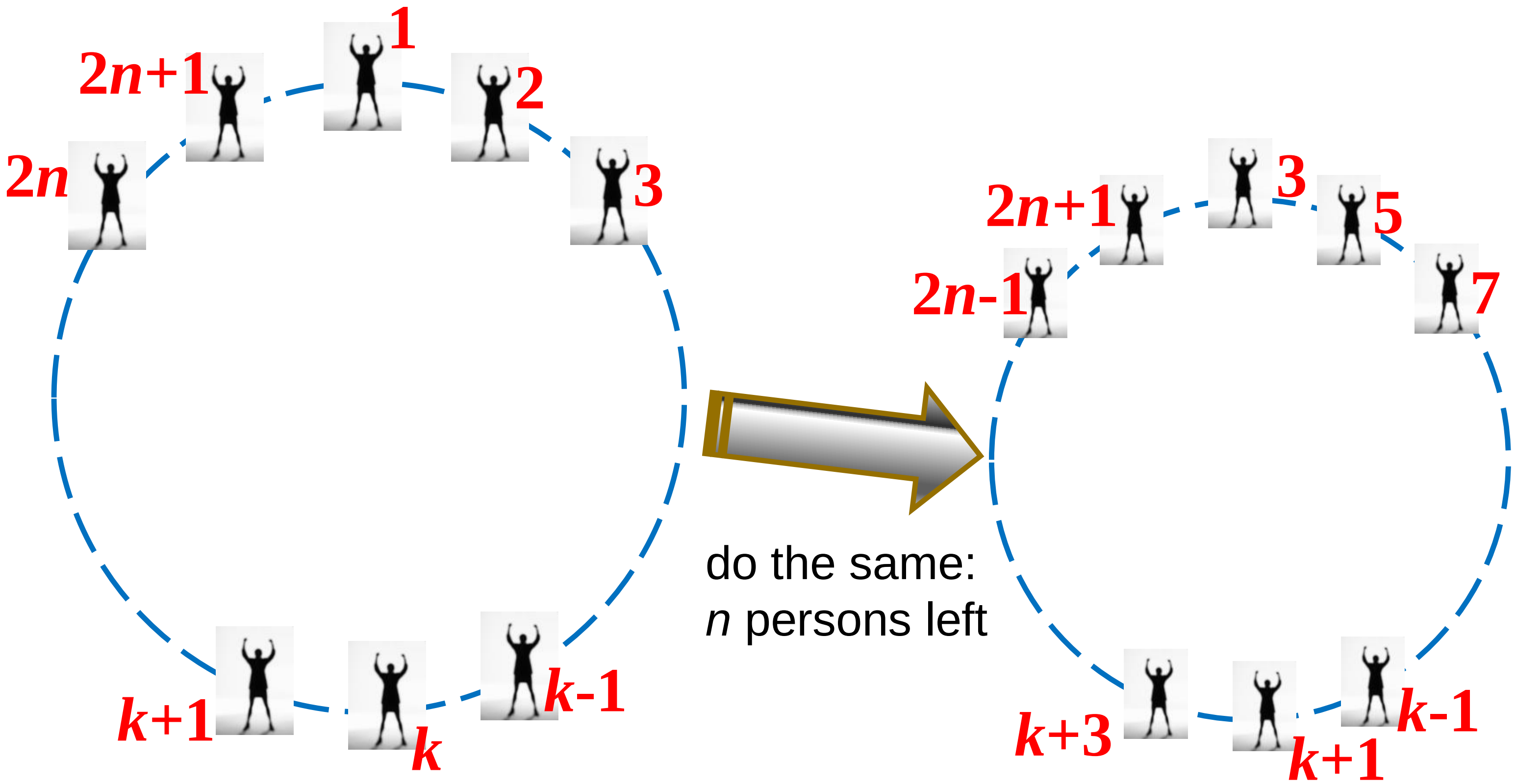
规模为n的问题中，解是m，这个m在2n规模问题中，位置在哪里？

For 2*n* Persons (*n*=1,2,3,...)



？ =J(2n) =2m-1=2J(n)-1

And What about $2n+1$ Persons (n=1,2,3,...)



The smaller problem's solution is: J(n)

And the solution of the original problem 2n+1 is **$2J(n)+1$**

递归方程：奇偶数分情况列出

$$J(1) = 1;$$
$$J(2n) = 2J(n) - 1, \qquad \text{for } n \geq 1;$$
$$J(2n+1) = 2J(n) + 1, \qquad \text{for } n \geq 1.$$

问题3：

你能解出这个递归式吗？

你能看出什么规律吗？

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3

Eureka!

If we write n in the form $n = 2^m + l$,
(where 2^m is the largest power of 2 not exceeding n and where l is what's left),
the solution to our recurrence seems to be:

$$J(2^m + l) = 2l + 1, \qquad \text{for } m \geq 0 \text{ and } 0 \leq l < 2^m.$$

As an example: J(100) = J(64+36) = 36*2+1 = 73

问题4：

你能否想出一种简单易行的计算办法？

用二进制表示

- Suppose *n*'s binary expansion is :

$$n = (b_m\, b_{m-1} \ldots b_1\, b_0)_2$$

- then:

$$\begin{aligned}
n &= (1\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2\,, \\
l &= (0\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2\,, \\
2l &= (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, 0)_2\,, \\
2l+1 &= (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, 1)_2\,, \\
J(n) &= (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, b_m)_2
\end{aligned}$$

例如： $100 = (1100100)_2 \to (1001001)_2 = 73$

关于递归式的几个“性质”

- 线性递归式
- 一阶递归式
- K阶递归式
- 齐次递归式

$$a_n = a_{n-1} + 3$$

$$g_n = g_{n-1}^2 + g_{n-2}$$

$$c_n = (-2)c_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

一阶线性递归式的通解

A recurrence of the form $T(n) = f(n)T(n-1) + g(n)$ is called a **first-order linear recurrence**. When $f(n)$ is a constant, such as r, the general solution is almost as easy to write as in Theorem 4.1. Iterating the recurrence

If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1-r^n}{1-r} \tag{4.12}$$

for all nonnegative integers n.

问题5：如何从定理4.1得到通解？

- $T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0, \\ a & \text{if } n = 0, \end{cases}$ 和$T(n) = \begin{cases} rT(n-1) + b & if\ n > 0 \\ a & if\ n = 0 \end{cases}$
- 本质上，定理4.1是这样的：
- $T(n) = rT(n-1) + b = r^n a + b \sum_{i=0}^{n-1} r^i$
- 所以，通解似乎可以是：
- $\mathrm{T(n)} = rT(n-1) + g(n) = r^n a + \sum_{i=0}^{n-1} r^i g(n-i)$

问题6：这个递归式以及通解和我们看到的分治法递归式及其通解本质上有区别吗？

线性齐次递归式及其通解

$$t_n = r_1 t_{n-1} + r_2 t_{n-2} + \cdots + r_k t_{n-k}$$

$$t_1 = a_1; t_2 = a_2; \ldots; t_k = a_k$$

is called linear homogeneous relation of degree *k*.

$$c_n = (-2)c_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

Yes

$$a_n = a_{n-1} + 3$$

$$g_n = g_{n-1}^2 + g_{n-2}$$

No

线性齐次递归式的特征方程

- For a linear homogeneous recurrence relation of degree *k*

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

the polynomial of degree *k*

$$x^k = r_1 x^{k-1} + r_2 x^{k-2} + \cdots + r_k$$

is called its characteristic equation.

- The characteristic equation of linear homogeneous recurrence relation of degree 2 is:

$$\boxed{x^2 - r_1 x - r_2 = 0}$$

二阶线性齐次递归式的特征方程

解线性齐次递归式 – 解代数方程

- If the characteristic equation $x^2 - r_1 x - r_2 = 0$ of the recurrence relation has two distinct roots s_1 and s_2, then $a_n = r_1 a_{n-1} + r_2 a_{n-2}$

$$a_n = u s_1^n + v s_2^n$$

  where *u* and *v* depend on the initial conditions, is the explicit formula for the sequence.

- If the equation has a single root *s*, then, both s_1 and s_2 in the formula above are replaced by *s*

Proof of the Solution

Remember equation : $x^2 - r_1 x - r_2 = 0$

We need prove that : $us_1^n + vs_2^n = r_1 a_{n-1} + r_2 a_{n-2}$

$$
\begin{aligned}
us_1^n + vs_2^n &= us_1^{n-2}s_1^2 + vs_2^{n-2}s_2^2 \\
&= us_1^{n-2}(r_1 s_1 + r_2) + vs_2^{n-2}(r_1 s_2 + r_2) \\
&= r_1 us_1^{n-1} + r_2 us_1^{n-2} + r_1 vs_2^{n-1} + r_2 vs_2^{n-2} \\
&= r_1(us_1^{n-1} + vs_2^{n-1}) + r_2(us_1^{n-2} + vs_2^{n-2}) \\
&= r_1 a_{n-1} + r_2 a_{n-2}
\end{aligned}
$$

?

问题7：你能解决这个问号吗？

Fibonacci Sequence

$f_1=1$

$f_2=1$

$f_n= f_{n-1}+ f_{n-2}$

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Explicit formula for Fibonacci Sequence

The characteristic equation is x^2-x-1=0, which has roots:

$$\underbrace{s_1=\frac{1+\sqrt{5}}{2}}_{\phi} \quad and \quad s_2=\frac{1-\sqrt{5}}{2}$$

Note: (by initial conditions) $f_1=us_1+vs_2=1 \ \ and \ \ f_2=us_1{}^2+vs_2{}^2=1$

which results: $$f_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$f_1=1$

$f_2=1$

$f_n=f_{n-1}+f_{n-2}$

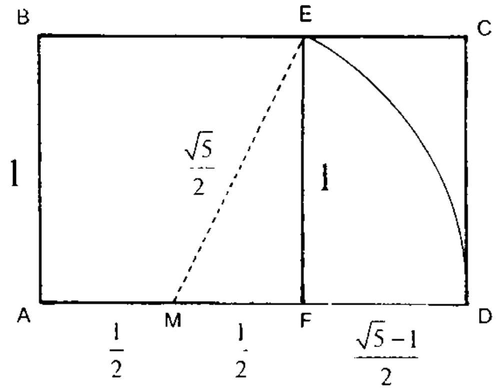
1, 1, 2, 3, 5, 8, 13, 21, 34, ...

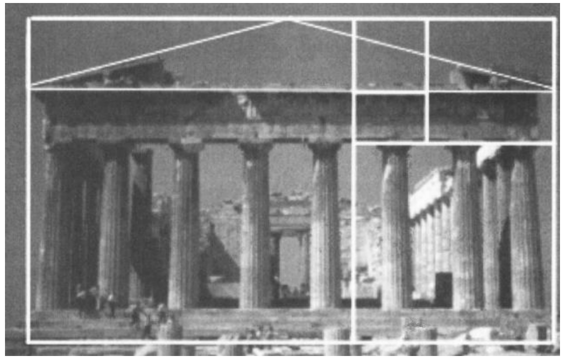
问题8:

这是Fibonacci序列，你知道它为什么那么出名吗？

$\frac{F_{n+1}}{F_n}$	
$\frac{1}{1} = 1.000000000$	$\frac{55}{34} = 1.617647059$
$\frac{2}{1} = 2.000000000$	$\frac{89}{55} = 1.6182181618$
$\frac{3}{2} = 1.500000000$	$\frac{144}{89} = 1.617977528$
$\frac{5}{3} = 1.666666667$	$\frac{233}{144} = 1.618055556$
$\frac{8}{5} = 1.600000000$	$\frac{377}{233} = 1.618025751$
$\frac{13}{8} = 1.625000000$	$\frac{610}{377} = 1.618037135$
$\frac{21}{13} = 1.615384615$	$\frac{987}{610} = 1.618032787$
$\frac{34}{21} = 1.619047619$	

黄金分割





PHIDIAS (460-430 BC)
OR
ϕ-BONACCI ?

最简单的形式

Suppose that we have a recurrence of the form

$$T(n) = aT\left(\frac{n}{2}\right) + n,$$

where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

问题9：

三种情况的本质差别在哪里？

Suppose that we have a recurrence of the form

$$T(n) = aT\left(\frac{n}{2}\right) + n,$$

where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

问题10：

你能解释一下这是如何“推广”的吗？

(Master Theorem, Preliminary Version) Let a be an integer greater than or equal to 1, and let b be a real number greater than 1. Let c be a positive real number, and d, a nonnegative real number. Given a recurrence of the form

$$T(n) = \begin{cases} aT(n/b) + n^c & \text{if } n > 1, \\ d & \text{if } n = 1, \end{cases}$$

in which n is restricted to be a power of b, we get the following:

1. If $\log_b a < c$, then $T(n) = \Theta(n^c)$.
2. If $\log_b a = c$, then $T(n) = \Theta(n^c \log n)$.
3. If $\log_b a > c$, then $T(n) = \Theta(n^{\log_b a})$.

比来比去，也就是比较$n^{\log_b a}$和n^c的渐进增长性

证明的关键

$$\log_b a \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} c \text{ iff. } \left(\frac{a}{b^c}\right) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 1$$

利用递归树容易得到，i层代价之和为：

$$a^i\left(\frac{n}{b^i}\right)^c = n^c\left(\frac{a^i}{b^{ci}}\right) = n^c\left(\frac{a}{b^c}\right)^i .$$

总代价为: $$n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i$$

问题10：

上述Master Theorem的局限性在何处？我们如何进一步推广？

(Master Theorem) Let a and b be positive real numbers, with $a \geq 1$ and $b > 1$. Let $T(n)$ be defined for integers n that are powers of b by

$$T(n) = \begin{cases} aT(n/b) + f(n) & \text{if } n > 1, \\ d & \text{if } n = 1. \end{cases}$$

Then we have the following:

1. If $f(n) = \Theta(n^c)$, where $\log_b a < c$, then $T(n) = \Theta(n^c) = \Theta(f(n))$.
2. If $f(n) = \Theta(n^c)$, where $\log_b a = c$, then $T(n) = \Theta(n^c \log n) = \Theta(f(n) \log n)$.
3. If $f(n) = \Theta(n^c)$, where $\log_b a > c$, then $T(n) = \Theta(n^{\log_b a})$.

其实推广的步子不算大，但再推广证明就比较麻烦了！

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ∎

课外作业

- CS pp.180-: prob.16, 17
- CS pp.197-: prob.8, 11, 17
- CS pp.212-: prob.9, 13, 16
- CS pp.221-: 1, 4, 6
- CS pp.233-: 8-10

问题7-1：

我们在分析分治法效率时，
假定n是b的幂，合理吗？

平滑函数

- Let $f(n)$ be a nonnegative eventually non-decreasing function defined on the set of natural numbers, $f(n)$ is called **smooth** if

$$\boldsymbol{f(2n) \in \Theta(f(n))}.$$

- Note: $\log n$, n, $n\log n$ and n^{α} ($\alpha \geq 0$) are all smooth.
  - For example: $2n\log 2n = 2n(\log n + \log 2) \in \Theta(n\log n)$

比想像的更“平滑”

- Let $f(n)$ be a smooth function, then, for any fixed integer $b \geq 2$, $f(bn) \in \Theta(f(n))$.
- That is, there exist positive constants c_b and d_b and a nonnegative integer n_0 such that

$$d_b f(n) \leq f(bn) \leq c_b f(n) \quad \text{for } n \geq n_0.$$

It is easy to prove that the result holds for $b = 2^k$,

for the second inequality:

$f(2^k n) \leq c_2^k f(n)$ for $k = 1,2,3\ldots$ and $n \geq n_0$.

For an arbitrary integer $b \geq 2$, $2^{k-1} \leq b \leq 2^k$

Then, $f(bn) \leq f(2^k n) \leq c_2^k f(n)$, we can use c_2^k as c_b.

平滑规则

- Let $T(n)$ be an eventually nondecreasing function and $f(n)$ be a smooth function. If $T(n) \in \Theta(f(n))$ for values of n that are powers of $b (b \geq 2)$, then $T(n) \in \Theta(f(n))$.

Just proving the big-Oh part:

By the hypothsis: $T(b^k) \leq cf(b^k)$ for $b^k \geq n_0$.

By the prior result: $f(bn) \leq c_b f(n)$ for $n \geq n_0$.

Let $n_0 \leq b^k \leq n \leq b^{k+1}$,

$$\underbrace{T(n) \leq T(b^{k+1})}_{\text{Non-decreasing}} \leq \underbrace{cf(b^{k+1})}_{\text{hypothesis}} = \underbrace{cf(bb^k) \leq cc_b f(b^k)}_{\text{Prior result}} \leq \underbrace{cc_b f(n)}_{\text{Non-decreasing}}$$

问题7-2：

你能说说Master Theorem的含义与其背后的原理吗？