2-7 Discrete Probability

"Life is a school of probability — Walter Bagehot"

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May 07, 2018

Two Extra Tasks

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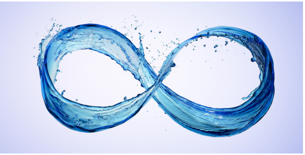
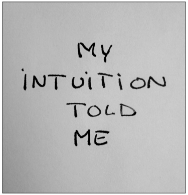
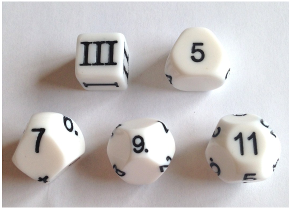
Q : What is probability?

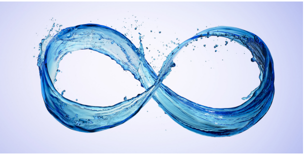


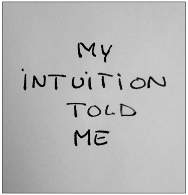


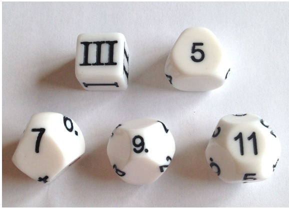


"... and the many paradoxes show clearly that we, as humans, lack a well grounded intuition in this matter."

— "The Art of Probability", Richard W. Hamming







"...and the many paradoxes show clearly that we, as humans, lack a well grounded intuition in this matter."

— *"The Art of Probability", Richard W. Hamming*

"When called upon to judge probability, people actually judge something else and **believe** *they have judged probability."*

— *"Thinking, Fast and Slow", Daniel Kahneman*

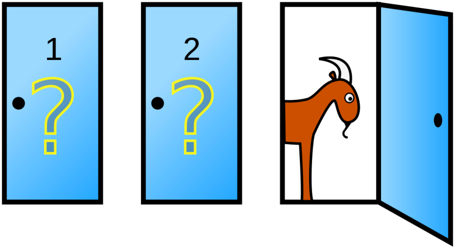
Let us calculate [calculemus].

(a) Monty Hall problem

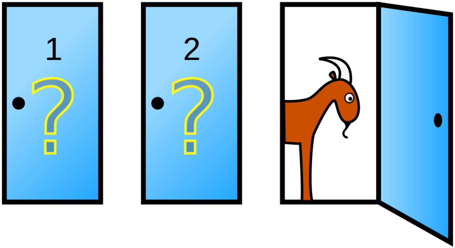
(b) Boy or Girl paradox

(c) Searching unsorted array

The Monty-Hall Problem



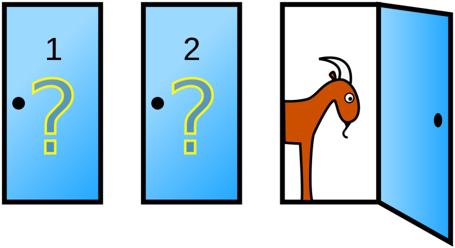
The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

Q : Do you want to switch to door 2?

C_i : The car is behind door i $(i = 1, 2, 3)$

$$\boxed{\Pr\{C_i\} = \frac{1}{3}}$$

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ASSUMPTION: The car is initially hidden randomly behind the doors.

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Y_1 : you initially pick door 1

$$\boxed{\Pr\{Y_1\} = \frac{1}{3}}$$

C_i : The car is behind door i $(i = 1, 2, 3)$

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ASSUMPTION: The car is initially hidden randomly behind the doors.

Y_1 : you initially pick door 1

$$\boxed{\Pr\{Y_1\} = \frac{1}{3}}$$

ASSUMPTION: Your initial choice is random.

I_3 : I open door 3

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ASSUMPTION: If you initially pick the car, then I open a door randomly.

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ASSUMPTION: I know what's behind the doors.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

ASSUMPTION: I always open a door to reveal a goat and never the car.

$$I_3 : \text{I open door } 3$$

ASSUMPTION: I know what's behind the doors.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

ASSUMPTION: I always open a door to reveal a goat and never the car.

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} = \frac{\Pr\{I_3, Y_1 \mid C_2\} \Pr\{C_2\}}{\Pr\{I_3 \mid Y_1\} \Pr\{Y_1\}}$$

$$
\begin{aligned}
\Pr\{C_2 \mid I_3, Y_1\} &= \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} = \frac{\Pr\{I_3, Y_1 \mid C_2\}\Pr\{C_2\}}{\Pr\{I_3 \mid Y_1\}\Pr\{Y_1\}} \\
&= \frac{\Pr\{I_3, Y_1 \mid C_2\}}{\Pr\{I_3 \mid Y_1\}}
\end{aligned}
$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} = \frac{\Pr\{I_3, Y_1 \mid C_2\} \Pr\{C_2\}}{\Pr\{I_3 \mid Y_1\} \Pr\{Y_1\}}$$

$$= \frac{\Pr\{I_3, Y_1 \mid C_2\}}{\Pr\{I_3 \mid Y_1\}}$$

$$\Pr\{I_3, Y_1 \mid C_2\} = \Pr\{I_3 \mid C_2, Y_1\} \Pr\{Y_1 \mid C_2\}$$

$$
\begin{aligned}
\Pr\{C_2 \mid I_3, Y_1\} &= \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} = \frac{\Pr\{I_3, Y_1 \mid C_2\}\Pr\{C_2\}}{\Pr\{I_3 \mid Y_1\}\Pr\{Y_1\}} \\
&= \frac{\Pr\{I_3, Y_1 \mid C_2\}}{\Pr\{I_3 \mid Y_1\}}
\end{aligned}
$$

$$
\begin{aligned}
\Pr\{I_3, Y_1 \mid C_2\} &= \Pr\{I_3 \mid C_2, Y_1\}\Pr\{Y_1 \mid C_2\} \\
&= \frac{1}{3}\Pr\{I_3 \mid C_2, Y_1\}
\end{aligned}
$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3\Pr\{I_3 \mid Y_1\}}$$

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$\Pr\{I_3 \mid Y_1\}$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3 \Pr\{I_3 \mid Y_1\}}$$

$$\begin{aligned}\Pr\{I_3 \mid Y_1\} = &\Pr\{I_3 \mid C_1, Y_1\} \Pr\{C_1 \mid Y_1\} \\ &+ \Pr\{I_3 \mid C_2, Y_1\} \Pr\{C_2 \mid Y_1\} \\ &+ \Pr\{I_3 \mid C_3, Y_1\} \Pr\{C_3 \mid Y_1\}\end{aligned}$$

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$$\begin{aligned}
\Pr\{I_3 \mid Y_1\} &= \Pr\{I_3 \mid C_1, Y_1\}\Pr\{C_1 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_3, Y_1\}\Pr\{C_3 \mid Y_1\} \\
&= \frac{1}{3}\Big(\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}\Big)
\end{aligned}$$

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It depends on how I choose the door to open!

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\Pr\{C_1 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_1, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

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$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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Q : Switching *vs.* Choosing between the two remaining doors randomly?

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

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$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

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Always Switch!

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

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Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

Assumption: I don't know either.

Opens one randomly that happens not to reveal the car.

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

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Monty Hall problem (wiki)

The Boy/Girl Puzzle

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls,

(a) given that one of the children is a girl?

(b) given that the older child is a girl?

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\}$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}}$$
$$= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\
&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}$$

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$$
\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
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\end{aligned}
$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} = \frac{1/4}{3/4} = \frac{1}{3}$$

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\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
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$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

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$$\Pr\{G_1 \land G_2 \mid G_1 \lor G_2\} = \frac{1}{3}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

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$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

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$$\Pr\{G_1 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge (G_1 \vee G_2)\}}{\Pr\{G_1 \vee G_2\}}$$

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Easy!


Easy!


KEEP
CALM
AND
WE ARE NOT
FINISHED YET

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls,

(a) given that one of the children is a girl?

Both Girls (CS Problem $5.3 - 12$)

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Q : **How** do you know that "one of the children is a girl"?

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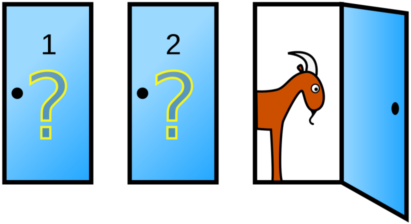
(I) I KNOW them well and I tell you that "one of the children is a girl".

Q : **How** do you know that "one of the children is a girl"?

(I) I KNOW them well and I tell you that "one of the children is a girl".

(II) I DON'T KNOW them. I just open a room door and see a girl.

The Monty-Hall Problem Comes Back



Q : **How** do you know that "one of the children is a girl"?

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Q : **How** do you know that "one of the children is a girl"?

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$$\Pr\{G_1 \wedge G_2 \mid g\} = \frac{\{G_1 \wedge G_2 \wedge g\}}{\Pr\{g\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

After-class Exercise:

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?

Boy or Girl paradox (wiki)

Q : What is probability?

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(Objective? Subjective? Neutral?)

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Kolmogorov axioms

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(Objective? Subjective? Neutral?)

Kolmogorov axioms

Probability interpretations (wiki)

Searching an Unsorted Array (CLRS Problem $5-2$ (f))

```
procedure DETERMINISTIC-SEARCH(A[1···n], x)
    i ← 1
    while i ≤ n do
        if A[i] = x then
            return true
        i ← i + 1
    return false
```

Searching an Unsorted Array (CLRS Problem $5-2$ (f))

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        i ← i + 1
    return false
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(e)

$$\exists!\ i : A[i] = x$$

(f)

$$\exists!_k\ i : A[i] = x$$

$$\exists!\ i : A[i] = x$$

Y : # of comparisons

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$$\mathbb{E}[Y] = \sum_{i=1}^{n} i \Pr\{Y = i\}$$

$$\exists!\ i : A[i] = x$$

Y : # of comparisons

$$\mathbb{E}[Y] = \sum_{i=1}^{n} i \Pr\{Y = i\} = \sum_{i=1}^{n} i \Pr\{A[i] = x\}$$

$$\exists!\ i : A[i] = x$$

Y : # of comparisons

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\
&= \frac{1}{n} \sum_{i=1}^{n} i
\end{aligned}$$

$$\exists!\ i : A[i] = x$$

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$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\
&= \frac{1}{n} \sum_{i=1}^{n} i \\
&= \frac{n+1}{2}
\end{aligned}$$

$$\exists!_k \ i : A[i] = x$$

Y : # of comparisons

$$\exists!_k \; i : A[i] = x$$

Y : # of comparisons

$$\mathbb{E}[Y] = \sum_{i=1}^{n-k+1} i \Pr\{Y = i\}$$

$$\exists!_k \ i : A[i] = x$$

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$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n-k+1} i \Pr\{i \text{ is the first index among } k \text{ indices } \textit{s.t.} \ A[i] = x\}
\end{aligned}$$

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&= \sum_{i=1}^{n-k+1} i \frac{\binom{n-i}{k-1}}{\binom{n}{k}}
\end{aligned}$$

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&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
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\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\
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&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
\end{aligned}$$

$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$

Summation by parts (Abel transformation; wiki)

After-class Exercise:

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$

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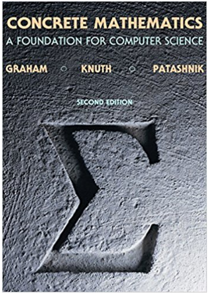
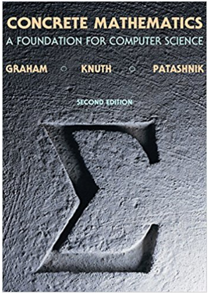


Chapter 5: Binomial Coefficients

After-class Exercise:

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$



$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0\le k\le n}\binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

Indicator Random Variables

Indicator Random Variables

$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

Indicator Random Variables

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$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \Pr\{I_i = 1\}$$

Indicator Random Variables

$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

Y : # of comparisons

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \Pr\{I_i = 1\}$$

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

Indicator Random Variables

$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

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$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \Pr\{I_i = 1\}$$

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

$$\mathbb{E}[Y] = \sum_{i=1}^{n} \Pr\{I_i = 1\} = k \cdot \frac{1}{k} + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$


Wrong!

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

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NOT IID

(Independent and Identically Distributed)

$$
\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n-k+1} \Pr\{Y \geq i\}
\end{aligned}
$$

$$
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$$
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\end{aligned}
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$$
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\end{aligned}
$$


TREASURE
HUNT
on foot


TREASURE
HUNT
on foot


Hints
&
Tips


Help
Wanted

There are n bins labelled with the numbers $1, 2, \cdots, n$. Balls are placed in these bins one after the other, with the bin into which a ball is placed being independent random variables that assume the value k with probability p_k. Let X be the number of balls placed so that there is at least one ball in every bin.

(a) Assume that $p_k = \frac{1}{n}$. What is the expectation of X?

(b) Assume that $p_k = \frac{1}{n}$. What is the probability distribution of X?

(c) Prove that $\Pr(X > n \ln n + cn) \le e^{-c}$, $\Pr(X < n \ln n - cn) \le e^{-c}$.

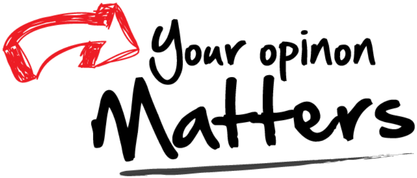
(d) Redo (a) and (b) without the assumption $p_k = \frac{1}{n}$.

(e) Given a deck of n cards, each time you take the top card from the deck, and insert it into the deck at one of the n distinct possible places, each of them with probability $\frac{1}{n}$. What is the expected times for you to perform the procedure above until the bottom card rises to the top?

The Coupon Collector's Problem

Shuffling Cards


Thank
You!



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