

2-5 Recursion

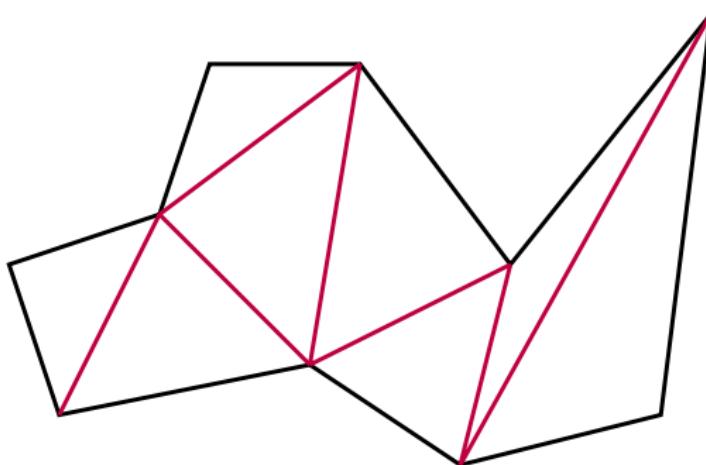
魏恒峰

hfwei@nju.edu.cn

2018 年 04 月 23 日



Triangulating Polygons



The Art Gallery Problem



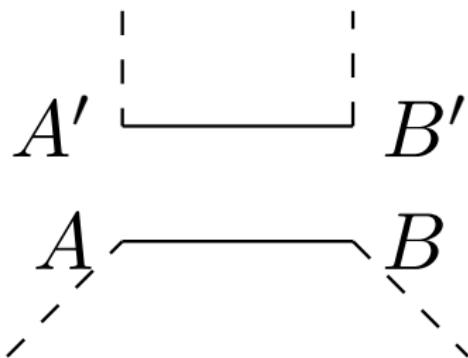
Q : How many “BIG BROs” to hire?

Another Version of the Ear Lemma (Problem 4.1 – 16)

A triangulated polygon is either a triangle with three ears or has at least two ears (which are *not necessarily non-adjacent*).

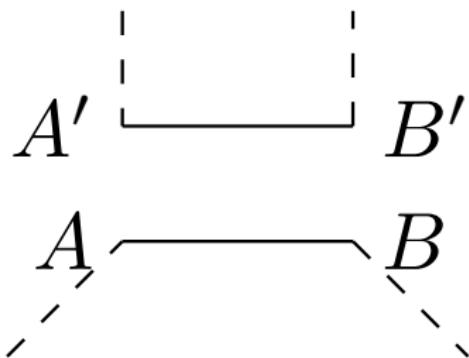
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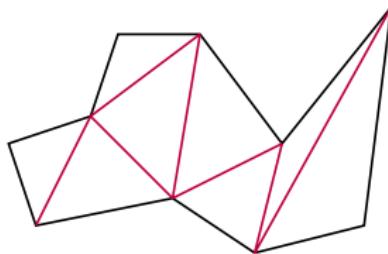
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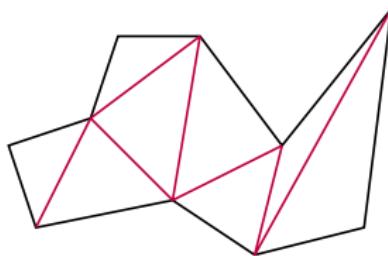
Q : Adjacent ears?

of triangles (Problem 4.1 – 17)

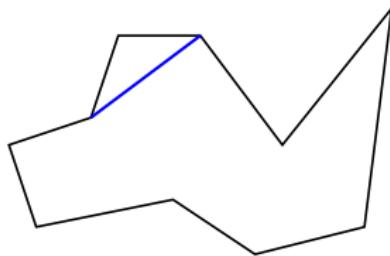


$$T(n) = n - 2$$

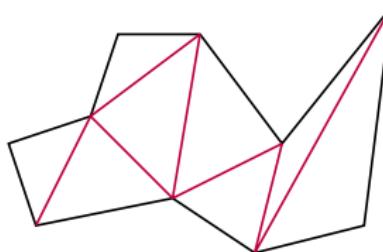
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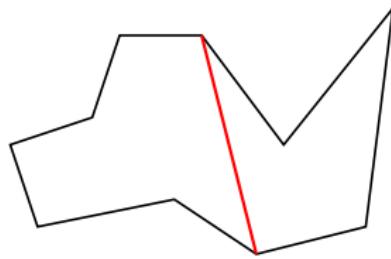
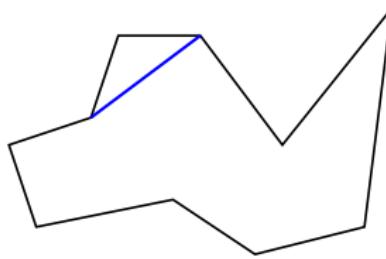
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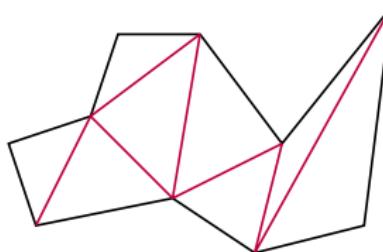
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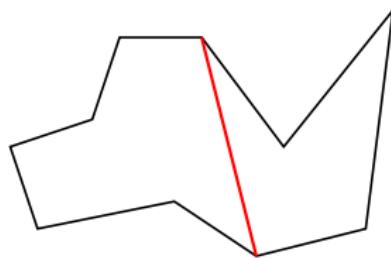
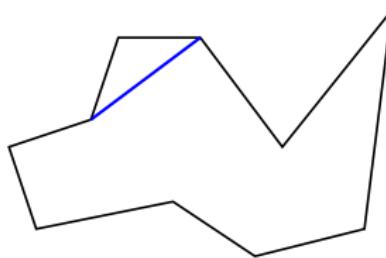
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Q : Existence of diagonals?

Lemma (Ear Lemma)

A triangle has 3 ears, and a larger *triangulated* polygon has at least 2 non-adjacent ears.

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Q : Can every polygon be triangulated?

Theorem (Existence of Triangulation)

Any polygon can be triangulated.

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Theorem (Existence of Diagonal)

Every polygon with $n > 3$ has a diagonal.

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Definition (Convex Vertex)

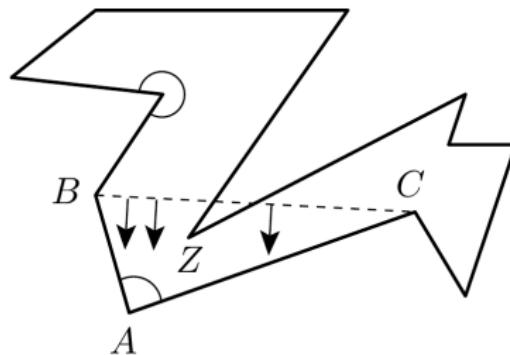
A vertex v is **convex** if the *interior* angle at v is less than 180° .

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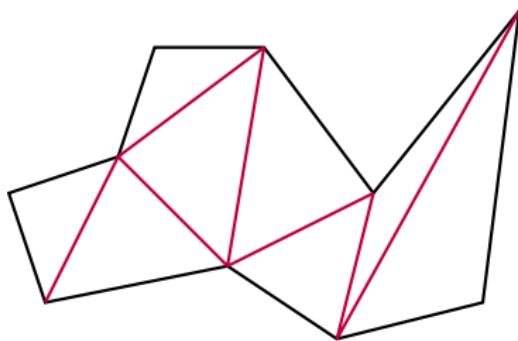
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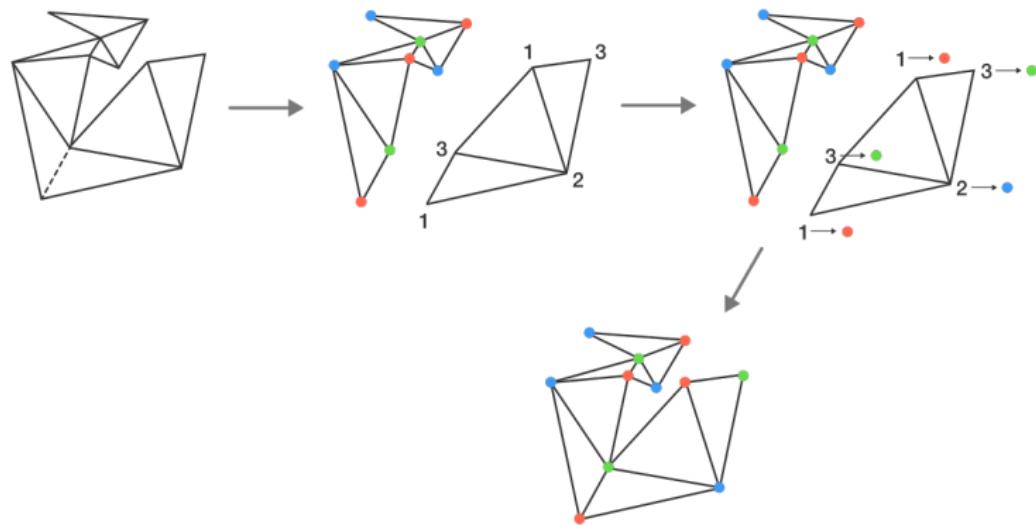
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Any triangulated polygon polygon is 3-colorable.



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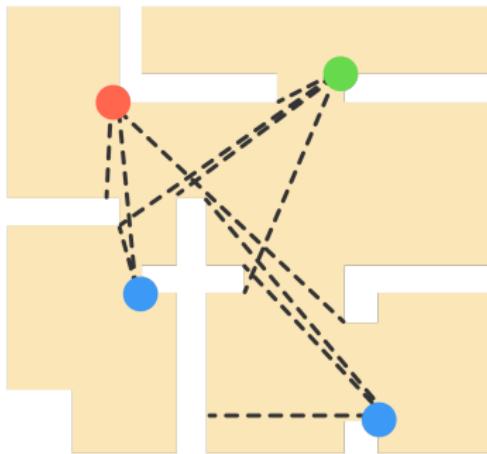


The Art Gallery Problem

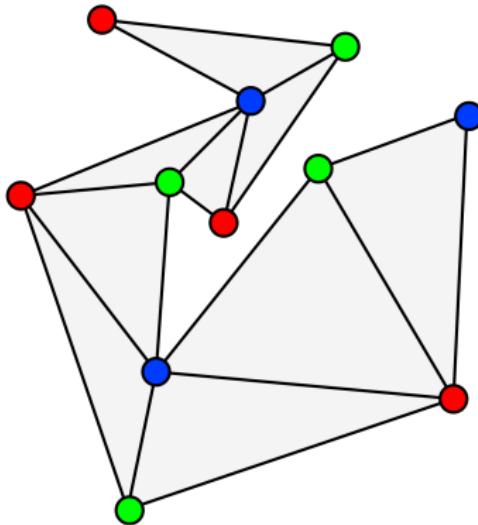


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The Art Gallery Problem



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Theorem (The Art Gallery Theorem (O))

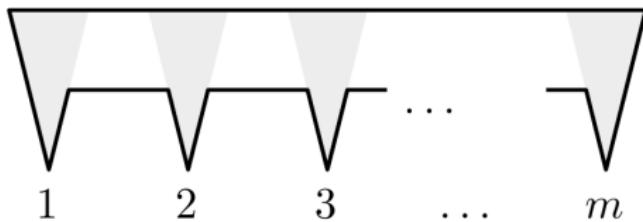
For any art gallery with n walls, $\lfloor \frac{n}{3} \rfloor$ "BIG BROs" suffice.

Theorem (The Art Gallery Theorem (Ω))

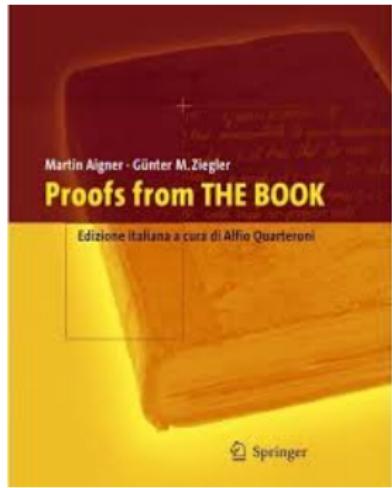
There exists an art gallery with n walls such that $\lfloor \frac{n}{3} \rfloor$ “BIG BROs” are necessary.

Theorem (The Art Gallery Theorem (Ω))

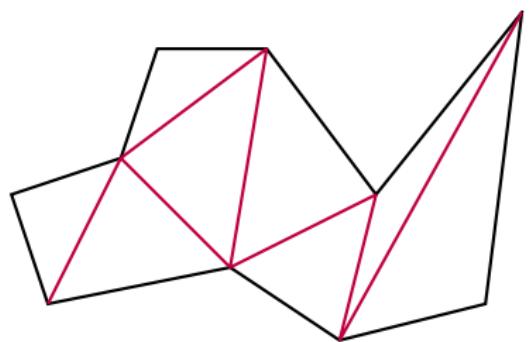
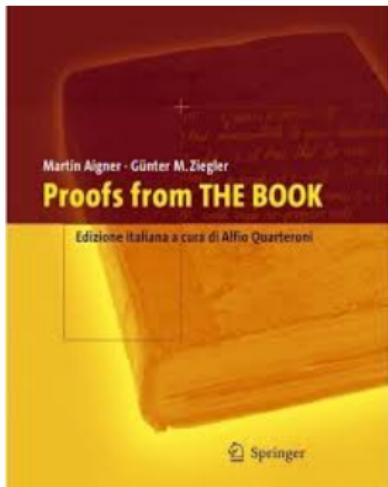
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$$n = 3m$$



“How to Guard a Museum?”



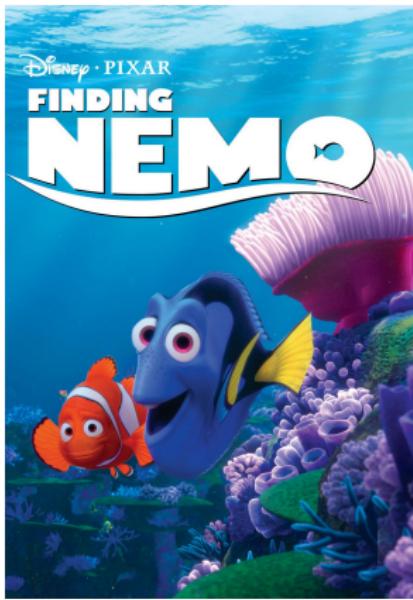
$O(n \log n)$

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$O(n)$

"How to Guard a Museum?"

Fish Recurrence



Fish Recurrence (Problem 4.2 – 8)

At the end of each year, a state fish hatchery puts 2000 fish into a lake.

The number of fish in the lake at the beginning of the year **doubles** by the end of the year due to reproduction.

Give a recurrence for the number of fish in the lake after n years, and solve the recurrence.

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Solving Recurrence





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Not Exactly!

First-order Linear Recurrence (CS 4.2 – 11)

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$$T(n) = r^n a + \sum_{i=1}^n r^{n-i} g(i)$$

Theorem (First-order Linear Recurrences)

$T(n) = \textcolor{red}{x_n} T(n-1) + y_n \quad \text{for } n > 0 \text{ with } T(0) = 0$

$$T(n) = y_0 + \sum_{1 \leq j < n} y_j x_{j+1} x_{j+2} \cdots x_n$$

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$$S(n) \triangleq \frac{T(n)}{x_n x_{n-1} \cdots x_1}$$



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$$T(n) = 2(n+1)\left(H_{n+1} - \frac{3}{2}\right)$$

After-class Exercise

$$T(n) = T(n-1) - \frac{2T(n-1)}{n} + 2 \left(1 - \frac{2T(n-1)}{n}\right), n > 0 \text{ with } T(0) = 0$$

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Theorem (Linear Recurrences with Constant Coefficients)

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$a_0, a_1, \dots, a_{t-1}$$

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$$a_n = \sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \cdots + \sum_{0 \leq j < m_k} c_{kj} n^j \beta_k^n$$

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$$\beta^{n-t} ((n-t) q(\beta) + \beta q'(\beta)) = 0$$



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$$\boxed{x^2 - 5x + 6 = (x - 2)(x - 3) = 0} \implies x = 2, 3$$

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$$a_0 = 0 = c_0 + c_1$$

$$a_1 = 1 = 3c_0 + 2c_1$$

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$$a_n = c_1 \cdot 1^n + c_2 \cdot 2^{\textcolor{red}{n}} + c'_2 \cdot \textcolor{red}{n}2^n$$

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}, n \geq 3 \quad (a_0 = 0, a_1 = 1, a_2 = 4)$$

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$$1, \textcolor{red}{0}, \textcolor{blue}{-1}, \textcolor{red}{0}, \textcolor{blue}{1}, \textcolor{red}{0}, \textcolor{blue}{-1}, \textcolor{red}{0}, \textcolor{blue}{1}, \textcolor{red}{0}, \textcolor{blue}{-1}, \textcolor{red}{0} \dots$$

After-class Exercise

To give initial conditions a_0, a_1 , and a_2 such that the growth rate of the solution to

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \quad n > 2$$

is (1) constant; (2) exponential; (3) fluctuating in sign.

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First-order Linear Non-homogeneous Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + \textcolor{red}{r} \quad \text{for } n \geq t$$

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7. 费波那契数列的定义如下： $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ ($n \geq 3$)。如果用下面的函数计算费波那契数列的第 n 项，则其时间复杂度为（ ）。

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int F(int n)
{
    if (n <= 2)
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    else
        return F(n - 1) + F(n - 2);
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```

- A. $O(1)$ B. $O(n)$ C. $O(n^2)$ D. $O(F_n)$

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$$T(n, k) = \begin{cases} 0, & k = 0 \vee n = k \\ T(n - 1, k) + T(n - 1, k - 1) + c, & \text{o.w.} \end{cases}$$

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$$a_0 = 0 = c_0 + c_1 + 1$$

$$a_1 = 1 = 3c_0 + 2c_1 + 1$$

$$a_n = 2 \cdot 3^n - 3 \cdot 2^n + 1$$

More Issues about Linear Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + g(n) \quad \text{for } n \geq t$$

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + g(n) \quad \text{for } n \geq t$$

$$a_n = a_n^h + a_n^p$$

*How to Find a **Particular Solution** for a Non-homogeneous Recurrence Relation?*

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$t \geq 5$$

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

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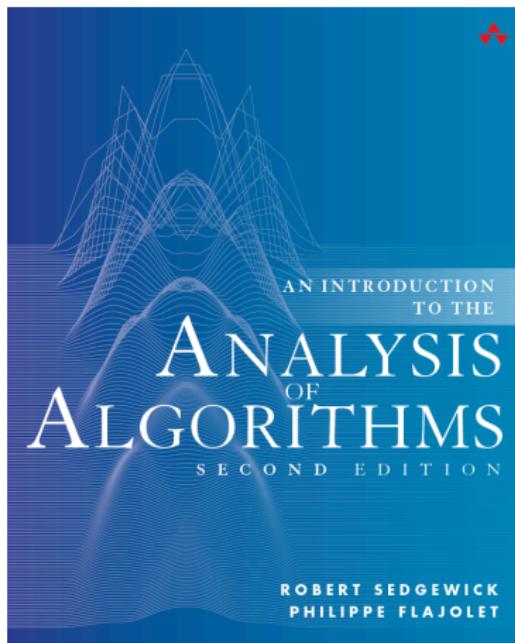
$$t \geq 5$$

Generating Functions and Asymptotic Analysis

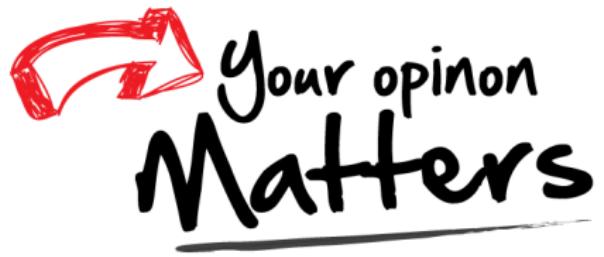
$$a_n = f_1(n)a_{n-1} + f_2(n)a_{n-2} + \cdots + f_t(n)a_{n-t} \quad \text{for } n \geq t$$

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Generating Functions



Thank You!



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