- 教材讨论
  - TC第29章

问题1：线性规划的standard和slack form

- 什么是一个linear program？
- 它的standard form有哪些特征？
- 如果一个linear program不具备上述特征，如何将其转化为standard form？

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m$$

$$x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n\,.$$

问题1：线性规划的standard和slack form (续)

- slack form有哪些特征？
  如何将standard form转化为slack form？

maximize $\displaystyle\sum_{j=1}^{n} c_j x_j$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m$$

$$x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n\,.$$

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B\,,$$

in which all variables x are constrained to be nonnegative.

问题2：linear program的应用

- Shortest paths
  - 你能解释为什么这样建模是正确的吗？

$$
\begin{aligned}
&\text{maximize} \quad d_t \\
&\text{subject to} \\
&\qquad d_v \le d_u + w(u,v) \quad \text{for each edge } (u,v) \in E\ , \\
&\qquad d_s = 0\ .
\end{aligned}
$$

问题2：linear program的应用 (续)

- 你学会利用linear program来建模实际问题了吗？
  - There are m different types of food, $F_1,...,F_m$, that supply varying quantities of the n nutrients, $N_1,...,N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient N_j . Let b_i be the price per unit of food F_i. Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.
  - There are I persons available for J jobs. The value of person i working a whole day at job j is a_{ij} for i=1,...,I and j=1,...,J. The problem is to choose an assignment of persons to jobs to maximize the total value in one day. (Note: A person can work at different jobs at different times of the day.)

问题2：linear program的应用 (续)

- 你学会利用linear program来建模实际问题了吗？

A company is involved in the production of two items (X and Y). The resources need to produce X and Y are twofold, namely machine time for automatic processing and craftsman time for hand finishing. The table below gives the number of minutes required for each item:

	Machine time	Craftsman time
X	13	20
Y	19	29

The company has 40 hours of machine time available in the next working week but only 35 hours of craftsman time. Machine time is costed at £10 per hour worked and craftsman time is costed at £2 per hour worked. Both machine and craftsman idle times incur no costs. The revenue received for each item produced (all production is sold) is £20 for X and £30 for Y. The company has a specific contract to produce 10 items of X per week for a particular customer.

Formulate the problem of deciding how much to produce per week as a linear program.

问题3：SIMPLEX

- 你能用自己的语言，概述SIMPLEX的思路吗？
  - 关键词1：basic solution
  - 关键词2：pivot

```
SIMPLEX(A, b, c)
(N, B, A, b, c, v) = INITIALIZE-SIMPLEX(A, b, c)
let Δ be a new vector of length m
while some index j ∈ N has c_j > 0
    choose an index e ∈ N for which c_e > 0
    for each index i ∈ B
        if a_ie > 0
            Δ_i = b_i / a_ie
        else Δ_i = ∞
    choose an index l ∈ B that minimizes Δ_l
    if Δ_l == ∞
        return "unbounded"
    else (N, B, A, b, c, v) = PIVOT(N, B, A, b, c, v, l, e)
for i = 1 to n
    if i ∈ B
        x̄_i = b_i
    else x̄_i = 0
return (x̄_1, x̄_2, ..., x̄_n)
```

$$
\begin{array}{llllllll}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \le & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \le & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \le & 36 \\
& x_1, x_2, x_3 & & & & & \ge & 0 .
\end{array}
$$

$$
\begin{aligned}
z &= & & 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 &-& x_1 - x_2 - 3x_3 \\
x_5 &= 24 &-& 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 &-& 4x_1 - x_2 - 2x_3 .
\end{aligned}
$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} .$$

$$
\begin{aligned}
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
&= 30 - \left(9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}\right) - x_2 - 3x_3 \\
&= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} .
\end{aligned}
$$

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} .
\end{aligned}
$$

问题3：SIMPLEX (续)

- 你理解这步初始化了吗？
  - 如何判定 linear program 是否具有可行解？
  - initial basic solution 如果不是feasible，怎么办？

$$
\begin{array}{lrcll}
\text{maximize} & -x_0 & & & \\
\text{subject to} & & & & \\
& \displaystyle\sum_{j=1}^{n} a_{ij}x_j - x_0 & \le & b_i & \text{for } i = 1, 2, \ldots, m\,, \\
& x_j & \ge & 0 & \text{for } j = 0, 1, \ldots, n\,.
\end{array}
$$

```
INITIALIZE-SIMPLEX(A, b, c)
let k be the index of the minimum b_i
if b_k ≥ 0                    // is the initial basic solution feasible?
    return ({1, 2, ..., n}, {n + 1, n + 2, ..., n + m}, A, b, c, 0)
form L_aux by adding −x_0 to the left-hand side of each constraint
    and setting the objective function to −x_0
let (N, B, A, b, c, v) be the resulting slack form for L_aux
l = n + k
// L_aux has n + 1 nonbasic variables and m basic variables.
(N, B, A, b, c, v) = PIVOT(N, B, A, b, c, v, l, 0)
// The basic solution is now feasible for L_aux.
iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
    to L_aux is found
if the optimal solution to L_aux sets x̄_0 to 0
    if x̄_0 is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of L_aux, remove x_0 from the constraints and
        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"
```

问题3：SIMPLEX (续)

- pivot没有变什么？变化了什么？
  这些不变和变化，在SIMPLEX中各有什么用？
- pivot有没有可能永不终止？
  如何判断？为什么可以这样判断？
- 如何避免这种情况？

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3 .
\end{aligned}
$$

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} .
\end{aligned}
$$