Generating Functions for solving recurrences

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Outline

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Generating function

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Visually,(we only talk about OGF)

For a sequence $\{a_0, a_1, ..., a_n, ...\}$, we generate a function $G(x)$ with $G(x) = a_0x^0 + a_1x^1 + ... + a_nx^n + ...$, namely $G(x) = \sum_{k=0}^{\infty} a_kx^k$

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Comments

the most useful but most difficult to understand method (for counting) –Stanley

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We can note that the coefficient of $\boldsymbol{x^k}$ is the number of ***k***-subsets of a n-element set. (Why?)

Fibonacci numbers

We know that

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if n} \geq 2 \\ 1 & \text{if n=1} \\ 0 & \text{if n=0} \end{cases}$$

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We generate a function G(x) from the sequence of Fibonacci numbers, namely $G(x) = \sum_{n\geq 0} F_n x^n$.

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Base on homework, we deduce this: $G(x) = \frac{x}{1-x-x^2}$ and the value of F_n is the coefficient of x^n in the Taylor series for this formular.

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Steps for solving recurrences

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2. Mutiply both sides of the equation by x^n and sum over all n. This gives the generating function :

$$G(x) = \sum_{n\geq 0} a_n x^n = \sum (a_{n-1} + a_{n-2})x^n$$

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And manipulate the right hand side of the equation so that it becomes some other expression involving $G(x)$.

$$G(x) = x + (x + x^2)G(x)$$

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$$G(x) = \frac{x}{1 - x - x^2}$$
4. Expand $G(x)$ into a power series and read off the coefficient

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Let $G(x) = \sum_{n\geq 0} g_n x^n$ and $F(x) = \sum_{n\geq 0} f_n x^n$

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- convolution: $F(x)G(x) = \sum_{n\geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$

Algebraic operations on generating functions

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- addition: $F(x) + G(x) = \sum_{n\geq 0}(f_n + g_n)x^n$
- convolution: $F(x)G(x) = \sum_{n\geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$
- differentiation: $G'(x) = \sum_{n\geq 0}(n+1)g_{n+1}x^n$

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- Taylor expansion

Expanding generating functions

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- Geometric sequence $\frac{1}{1-x} = \sum_{n\geq 0} x^n$

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- Geometric sequence $\frac{1}{1-x} = \sum_{n\geq 0} x^n$
- Binomial theorem(Newton's formular(generaized binomial theorem))

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- wiki 上关于 generating functions 的条目
- 尹一通老师的讲义