2-8 Probabilistic Analysis

"No Expectation, No Disappointment."

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Definition (Expectation)

$$\mathbb{E}[X] = \sum_{x} x \Pr(X = x)$$

Theorem (Computing Expectation)

Let X be a discrete random variable that takes on only nonnegative integer values $\mathbb{N}$.

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} i \Pr(X = i) = \sum_{i=1}^{\infty} \Pr(X \geq i)$$

Proof.

$$\sum_{j=1}^{\infty} \sum_{i=1}^{j} \Pr(X = j) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr(X = j)$$

□

Searching an Unsorted Array (CLRS Problem $5-2\ (f)$)

procedure DETERMINISTIC-SEARCH($A[1 \cdots n], x$)
  $i \gets 1$
  while $i \le n$ **do**
    if $\boxed{A[i] = x}$ **then**
      return *true*
    $i \gets i + 1$
  return *false*

(e)

$$\exists!\ i : A[i] = x$$

(f)

$$\exists!_k\ i : A[i] = x$$

$$\exists!_k \; i : A[i] = x$$

Y : # of comparisons

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n-k+1} i \Pr\{i \text{ is the first index among } k \text{ indices } \textit{s.t. } A[i] = x\} \\
&= \sum_{i=1}^{n-k+1} i \frac{\binom{n-i}{k-1}}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \cdots \\
&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
\end{aligned}$$

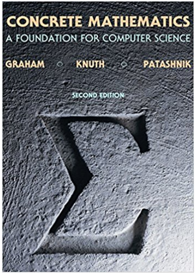
$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$

Summation by parts (Abel transformation; wiki)

How Did I (an ant) Evaluate this Summation:

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$



$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

$$
\begin{aligned}
\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} &= \sum_{i=0}^{n-k}(i+1)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}\left((n+1)-(n-i)\right)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}(n+1)\binom{n-i-1}{k-1} - \sum_{i=0}^{n-k}(n-i)\binom{n-i-1}{k-1} \\
&= (n+1)\sum_{i=0}^{n-k}\binom{n-i-1}{k-1} - k\sum_{i=0}^{n-k}\binom{n-i}{k} \\
&= (n+1)\sum_{m=k-1}^{n-1}\binom{m}{k-1} - k\sum_{m=k}^{n}\binom{m}{k} \\
&= (n+1)\binom{n}{k} - k\binom{n+1}{k+1} = \binom{n+1}{k+1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} \Pr\{Y \geq i\} \\
&= \sum_{i=1}^{n-k+1} \frac{\binom{n-i+1}{k}}{\binom{n}{k}} \\
&= \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i+1}{k} \\
&= \frac{1}{\binom{n}{k}} \sum_{r=k}^{n} \binom{r}{k} \\
&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
\end{aligned}
$$

$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

Y : # of comparisons

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \Pr\{I_i = 1\}$$

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

$$\mathbb{E}[Y] = \sum_{i=1}^{n} \Pr\{I_i = 1\} = k \cdot \frac{1}{k} + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

$$i = 1 \implies \Pr\{I_1 = 1\} = 1$$

$$i = n \implies \Pr\{I_n = 1\} = 0$$

Hat-check Problem (CLRS Problem $5.2-4$)

X : # of customers who get back their own hat $\mathbb{E}[X]$

$$I_i = \begin{cases} 1 & \text{customer } c_i \text{ gets back his/her hat} \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{i=1}^{n} I_i \qquad \mathbb{E}[I_i] = \Pr\Big(c_i \text{ gets back his/her hat}\Big) = \frac{1}{n}$$

Inversions (CLRS Problem $5.2-5$)

$A[1 \cdots n]$ of n distinct numbers

(i, j) is an inversion of $A : i < j \wedge A[i] > A[j]$

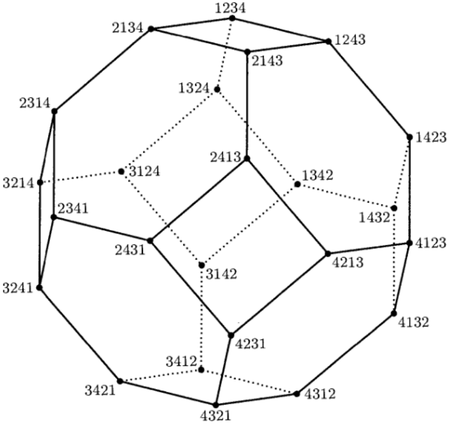
X : # of inversions in A

$\mathbb{E}[X]$ (A is randomly ordered)

$$I_{ij} = \begin{cases} 1 & (A[i], A[j]) \text{ is an inversion} \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{i=1}^{n-1} \sum_{j>i}^{n} I_{ij} \qquad \mathbb{E}[I_{ij}] = \Pr\Big((i, j) \text{ is an inversion}\Big) = \frac{1}{2}$$

Q : Average $\#$ of swaps (comparisons) of INSERTION-SORT?



$$\langle 3214 \rangle \sim \langle 4123 \rangle$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_{x} x \Pr(X = x \mid E)$$

$$\Big(\mathbb{E}[X] = \sum_{x} x \Pr(X = x)\Big)$$

Theorem (The Law of Total Expectation (CS Theorem 5.23))

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

Proof.

By definition.

$$\sum_{x} x \sum_{i=1}^{n} \Pr(X = x, E_i) = \sum_{i=1}^{n} \sum_{x} x \Pr(X = x, E_i)$$

□

($\sharp$) Rational Person Playing a Card Game (CS Problem $5.6 - 4$)

A : \$1.00; Repeat

J : \$2.00; End

K : \$3.00; End

Q : \$4.00; End

Conditioning on the first draw c

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q]\Big)$$

$$\mathbb{E}[X \mid c = A] = \mathbb{E}[X] + 1$$

$$4 * A + 1 * Q = \$8.00$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

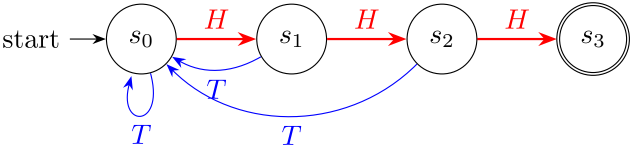
Conditioning on the first 3 tosses

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$\boxed{X : \# \text{ of tosses to get } HHH}$

$$T, \quad HT, \quad HHT, \quad HHH$$

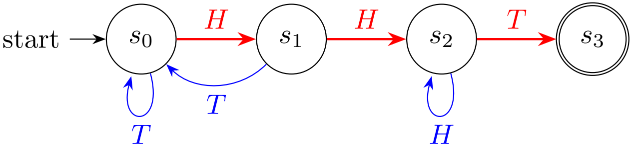


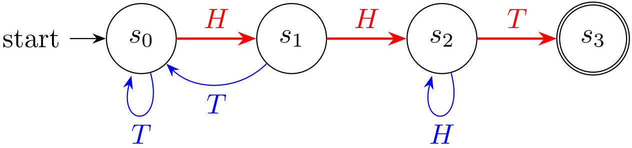
$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$\boxed{X : \# \text{ of tosses to get } HHT}$

$$T, \quad HT, \quad HHH, \quad HHT$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$





S_i : Expected number of tosses from state s_i to reach state s_n

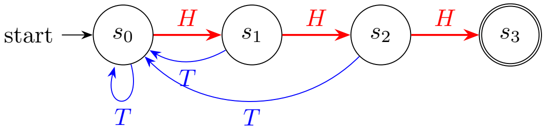
$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$

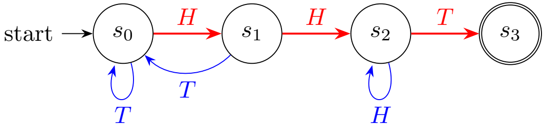
$$S_1 = \frac{1}{2}(1 + S_0 + 1 + S_2)$$

$$S_2 = \frac{1}{2}(1 + S_2 + 1 + S_3)$$

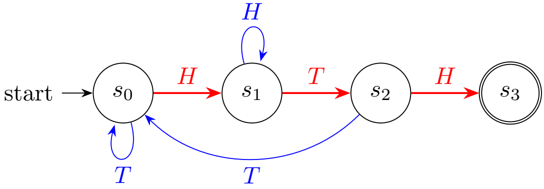
$$S_3 = 0$$

$$S_0 = 8$$

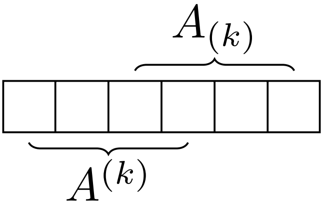




$$\mathbb{E}[X_{HHH}] = 14 > \mathbb{E}[X_{HHT}] = 8$$



$$A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)$$



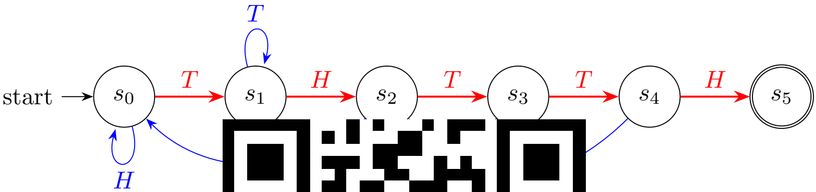
$$A = THTTH \qquad \mathbb{E}[X_A] = 2(2^1 + 2^4) = 36$$

$$\boxed{A : A = \sum_{k=1}^{n} 2^{k-1}[A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)}$$

$$A = HHH \qquad A = HHT$$

$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

$$\mathbb{E}[X_{H^{n-1}T}] = 2(2^{n-1}) = 2^n$$



$$S_0 = \frac{1}{2}(1 + S$$

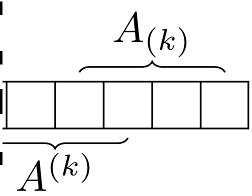
$$S_1 = \frac{1}{2}(1 + S$$

$$S_2 = \frac{1}{2}(1 + S$$

$$S_3 = \frac{1}{2}(1 + S_2 + 1 + S_4)$$

$$S_4 = \frac{1}{2}(1 + S_1 + 1 + S_5)$$

$$S_5 = 0$$



$$2 \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}]$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

Notation:

$$\mathbb{E}[X \mid Y](y) = \mathbb{E}[X \mid Y = y]$$

Theorem

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \sum_y \mathbb{E}[X \mid Y = y] \Pr(Y = y)$$


TREASURE
HUNT
on foot


Hints
&
Tips


Help
Wanted

There are n bins labelled with the numbers $1, 2, \cdots, n$. Balls are placed in these bins one after the other, with the bin into which a ball is placed being independent random variables that assume the value k with probability p_k. Let X be the number of balls placed so that there is at least one ball in every bin.

(a) Assume that $p_k = \frac{1}{n}$. What is the expectation of X?

(b) Assume that $p_k = \frac{1}{n}$. What is the probability distribution of X?

(c) Prove that $\Pr(X > n \ln n + cn) \leq e^{-c},\ \Pr(X < n \ln n - cn) \leq e^{-c}$.

(d) Redo (a) and (b) without the assumption $p_k = \frac{1}{n}$.

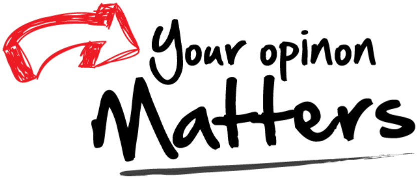
(e) Given a deck of n cards, each time you take the top card from the deck, and insert it into the deck at one of the n distinct possible places, each of them with probability $\frac{1}{n}$. What is the expected times for you to perform the procedure above until the bottom card rises to the top?

The Coupon Collector's Problem

Shuffling Cards

Chapter "Shuffling Cards" of "Proofs from THE Book"

Thank You!



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