2-7 Discrete Probability

"Life is a school of probability — Walter Bagehot"

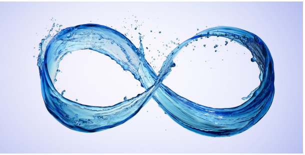
Hengfeng Wei

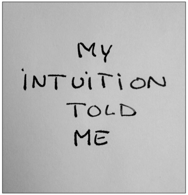
hfwei@nju.edu.cn

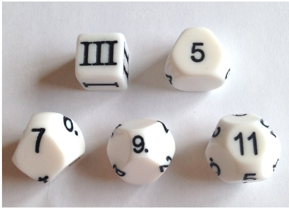
May 07, 2018

Two Extra Tasks

Q : What is probability?







"...and the many *paradoxes* *show clearly that we, as humans,* *lack a well grounded intuition* *in this matter."*

— "The Art of Probability", Richard W. Hamming

"When called upon to judge probability, people actually judge something else and ***believe*** *they have judged probability."*

— "Thinking, Fast and Slow", Daniel Kahneman

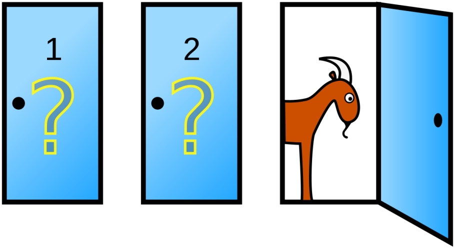
Let us calculate [calculemus].

(a) Monty Hall problem

(b) Boy or Girl paradox

(c) Searching unsorted array

The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

Q : Do you want to switch to door 2?

C_i : The car is behind door i $(i = 1, 2, 3)$

$$\boxed{\Pr\{C_i\} = \frac{1}{3}}$$

ASSUMPTION: The car is initially hidden randomly behind the doors.

Y_1 : you initially pick door 1

$$\boxed{\Pr\{Y_1\} = \frac{1}{3}}$$

ASSUMPTION: Your initial choice is random.

I_3 : I open door 3

ASSUMPTION: I know what's behind the doors.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

ASSUMPTION: I always open a door to reveal a goat and never the car.

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} = \frac{\Pr\{I_3, Y_1 \mid C_2\} \Pr\{C_2\}}{\Pr\{I_3 \mid Y_1\} \Pr\{Y_1\}}$$
$$= \frac{\Pr\{I_3, Y_1 \mid C_2\}}{\Pr\{I_3 \mid Y_1\}}$$

$$\Pr\{I_3, Y_1 \mid C_2\} = \Pr\{I_3 \mid C_2, Y_1\} \Pr\{Y_1 \mid C_2\}$$
$$= \frac{1}{3} \Pr\{I_3 \mid C_2, Y_1\}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3 \Pr\{I_3 \mid Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3\Pr\{I_3 \mid Y_1\}}$$

$$\begin{aligned}
\Pr\{I_3 \mid Y_1\} &= \Pr\{I_3 \mid C_1, Y_1\}\Pr\{C_1 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_3, Y_1\}\Pr\{C_3 \mid Y_1\} \\
&= \frac{1}{3}\Big(\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}\Big)
\end{aligned}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\boxed{\Pr\{I_3 \mid C_3, Y_1\} = 0}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{C_1 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_1, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\frac{\Pr\{C_2 \mid I_3, Y_1\}}{\Pr\{C_1 \mid I_3, Y_1\}} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

Q : Switching *vs.* Choosing between the two remaining doors randomly?

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

Always Switch!

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

Assumption: I don't know either.

Opens one randomly that happens not to reveal the car.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = \frac{1}{2}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{2}$$

Monty Hall problem (wiki)

The Boy/Girl Puzzle

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls,

(a) given that one of the children is a girl?

(b) given that the older child is a girl?

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\
&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr\{G_1 \mid G_1 \vee G_2\}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge (G_1 \vee G_2)\}}{\Pr\{G_1 \vee G_2\}} = \frac{\Pr\{G_1\}}{\Pr\{G_1 \wedge G_2\}} = \frac{2}{3}$$


Easy!


KEEP
CALM
AND
WE ARE NOT
FINISHED YET

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has two girls,

(a) given that one of the children is a girl?

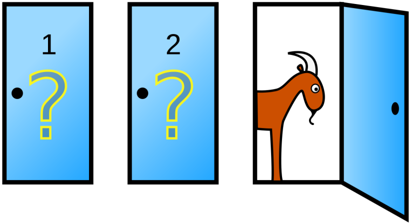
Q : **How** do you know that "one of the children is a girl"?

Q : **How** do you know that "one of the children is a girl"?

(I) I KNOW them well and I tell you that "one of the children is a girl".

(II) I DON'T KNOW them. I just open a room door and see a girl.

The Monty-Hall Problem Comes Back



Q : **How** do you know that "one of the children is a girl"?

(II) g : I DON'T KNOW them. I just open a room door and see a girl.

$$\Pr\{G_1 \land G_2 \mid g\} = \frac{\{G_1 \land G_2 \land g\}}{\Pr\{g\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

After-class Exercise:

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?

Boy or Girl paradox (wiki)

Q : What is probability?

(Objective? Subjective? Neutral?)

Kolmogorov axioms

Probability interpretations (wiki)

Searching an Unsorted Array (CLRS Problem $5-2$ (f))

---

procedure DETERMINISTIC-SEARCH($A[1 \cdots n], x$)
    $i \leftarrow 1$
    while $i \leq n$ **do**
        if $A[i] = x$ **then**
            return *true*
        $i \leftarrow i + 1$
    return *false*

(e)

$$\exists!\ i : A[i] = x$$

(f)

$$\exists!_k\ i : A[i] = x$$

$$\exists!\ i : A[i] = x$$

Y : # of comparisons

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\
&= \frac{1}{n} \sum_{i=1}^{n} i \\
&= \frac{n+1}{2}
\end{aligned}$$

$$\exists!_k \; i : A[i] = x$$

Y : # of comparisons

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n-k+1} i \Pr\{i \text{ is the first index among } k \text{ indices } \textit{s.t. } A[i] = x\} \\
&= \sum_{i=1}^{n-k+1} i \frac{\binom{n-i}{k-1}}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} \\
&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
\end{aligned}$$

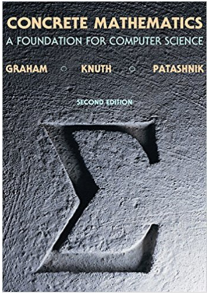
$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$

Summation by parts (Abel transformation; wiki)

After-class Exercise:

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$



$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

Indicator Random Variables

$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

Y : # of comparisons

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \Pr\{I_i = 1\}$$

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

$$\mathbb{E}[Y] = \sum_{i=1}^{n} \Pr\{I_i = 1\} = k \cdot \frac{1}{k} + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

$$i = 1 \implies \Pr\{I_1 = 1\} = 1$$

$$i = n \implies \Pr\{I_n = 1\} = 0$$

NOT IID

(Independent and Identically Distributed)

$$
\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n-k+1} \Pr\{Y \geq i\} \\
&= \sum_{i=1}^{n-k+1} \frac{\binom{n-i+1}{k}}{\binom{n}{k}} \\
&= \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i+1}{k} \\
&= \frac{1}{\binom{n}{k}} \sum_{r=k}^{n} \binom{r}{k} \\
&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
\end{aligned}
$$


TREASURE
HUNT
on foot


Hints
&
Tips


Help
Wanted

There are n bins labelled with the numbers $1, 2, \cdots, n$. Balls are placed in these bins one after the other, with the bin into which a ball is placed being independent random variables that assume the value k with probability p_k. Let X be the number of balls placed so that there is at least one ball in every bin.

(a) Assume that $p_k = \frac{1}{n}$. What is the expectation of X?

(b) Assume that $p_k = \frac{1}{n}$. What is the probability distribution of X?

(c) Prove that $\Pr(X > n \ln n + cn) \leq e^{-c}$, $\Pr(X < n \ln n - cn) \leq e^{-c}$.

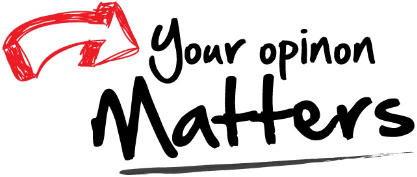
(d) Redo (a) and (b) without the assumption $p_k = \frac{1}{n}$.

(e) Given a deck of n cards, each time you take the top card from the deck, and insert it into the deck at one of the n distinct possible places, each of them with probability $\frac{1}{n}$. What is the expected times for you to perform the procedure above until the bottom card rises to the top?

The Coupon Collector's Problem

Shuffling Cards


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn