2-5 Recursion

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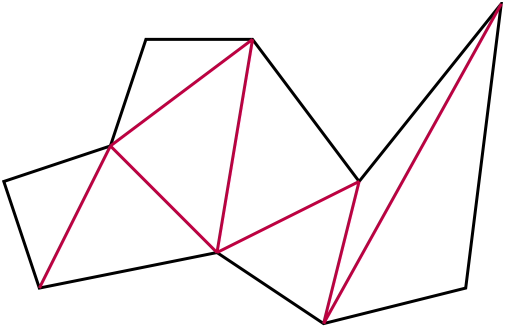
Triangulating Polygons

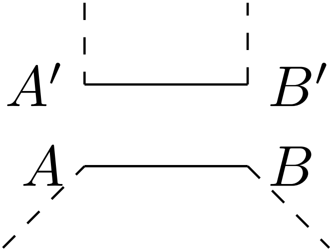


The Art Gallery Problem

Q : How many "BIG BROs" to hire?

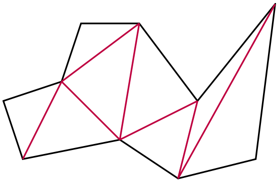
Another Version of the Ear Lemma (Problem $4.1 - 16$)

A triangulated polygon is either a triangle with three ears or has at least two ears (which are *not necessarily non-adjacent*).

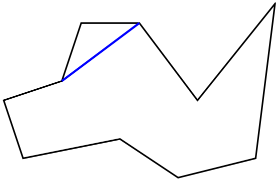


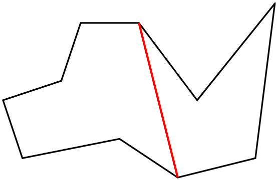
Q : Adjacent ears?

of triangles (Problem $4.1 - 17$)



$$T(n) = n - 2$$





Q : Existence of diagonals?

Lemma (Ear Lemma)

A triangle has 3 ears, and a larger triangulated polygon has at least 2 non-adjacent ears.

Q : *Can every polygon be triangulated?*

Theorem (Existence of Triangulation)

Any polygon can be triangulated.

Proof.

> *"To triangulate a polygon one keeps adding diagonals connecting pairs of vertices until no more diagonals can be added.*
>
> *These diagonals must lie entirely interior to the polygon and are not allowed to intersect.*
>
> *They break the interior of the polygon into a number of triangles, because any larger polygon can be split by adding a diagonal."*

□

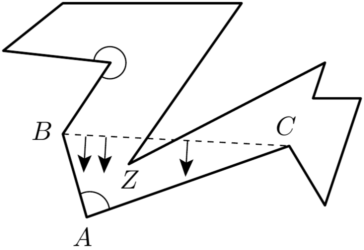
"(This fact is perhaps not obvious,
but we won't get sidetracked by proving it here.)"

Theorem (Existence of Diagonal)

Every polygon with $n > 3$ has a diagonal.

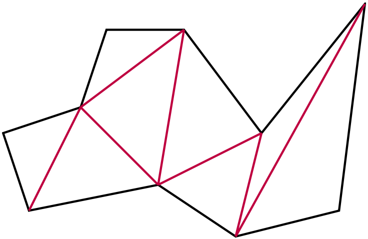
Definition (Convex Vertex)

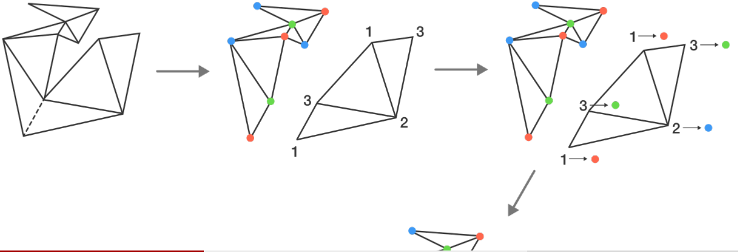
A vertex v is *convex* if the *interior* angle at v is less than 180°.



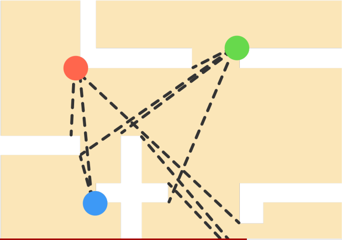
Theorem (Coloring)

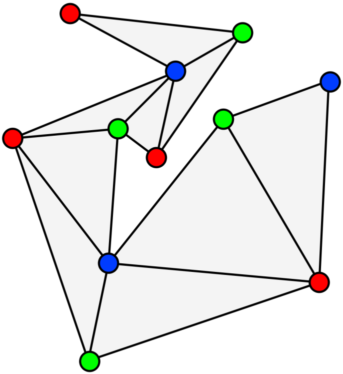
Any triangulated polygon polygon is 3-colorable.





The Art Gallery Problem



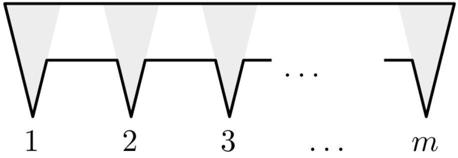


Theorem (The Art Gallery Theorem (O))

For any art gallery with n walls, $\lfloor \frac{n}{3} \rfloor$ "BIG BROs" suffice.

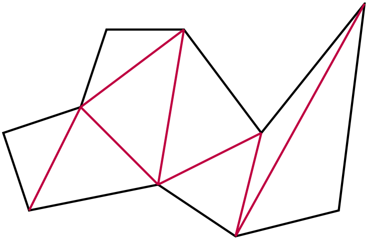
Theorem (The Art Gallery Theorem (Ω))

There exists an art gallery with n walls such that $\lfloor \frac{n}{3} \rfloor$ "BIG BROs" are necessary.



$$n = 3m$$

"How to Guard a Museum?"

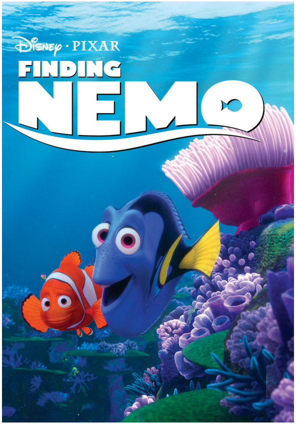


$$O(n \log n)$$

$$O(n \log \log n)$$

$$O(n)$$

Fish Recurrence



Fish Recurrence (Problem $4.2 - 8$)

At the end of each year, a state fish hatchery puts 2000 fish into a lake.

The number of fish in the lake at the beginning of the year doubles by the end of the year due to reproduction.

Give a recurrence for the number of fish in the lake after n years, and solve the recurrence.

$$T(n) = \# \text{ of fish at the end of the } n\text{-th year}$$

$$T(n) = 2T(n-1) + 2000$$

$$T(0) = 2000$$

Solving Recurrence

Q : Recurrences without Base Cases?

"Boundary conditions represent another class of details that we typically ignore."

"We shall generally omit statements of the boundary conditions of recurrences and assume that $T(n)$ is constant for small n."

— "Technicalities in recurrences", Chapter 4, CLRS

Not Exactly!

First-order Linear Recurrence (CS $4.2-11$)

$$T(n) = \begin{cases} 1 & n = 0 \\ 2T(n-1) + n2^n & n > 0 \end{cases}$$

Theorem (First-order Linear Recurrences with *Constant Coefficients* (CS Theorem 4.5))

$$T(n) = \begin{cases} a & n = 0 \\ rT(n-1) + g(n) & n > 0 \end{cases}$$

$$\boxed{T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i)}$$

Theorem (First-order Linear Recurrences)

$$T(n) = x_n T(n-1) + y_n \quad \textit{for } n > 0 \textit{ with } T(0) = 0$$

$$\boxed{T(n) = y_0 + \sum_{1 \le j < n} y_j x_{j+1} x_{j+2} \cdots x_n}$$

Proof.

$$\underbrace{\frac{T(n)}{x_n x_{n-1} \cdots x_1}}_{\text{summation factor}} = \frac{T(n-1)}{x_{n-1} \cdots x_1} + \frac{y_n}{x_n x_{n-1} \cdots x_1}$$

$$S(n) \triangleq \frac{T(n)}{x_n x_{n-1} \cdots x_1}$$

□

$$T(n) = (1 + \frac{1}{n})T(n-1) + 2 \quad \text{for } n > 1 \text{ with } T(1) = 0$$

$$x_n = 1 + \frac{1}{n} \implies x_n x_{n-1} \cdots x_1 = n + 1$$

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2}{n+1} \quad \text{for } n > 1$$

$$\frac{T(n)}{n+1} = \frac{T(1)}{2} + 2 \sum_{3 \le k \le n+1} \frac{1}{k}$$

$$\boxed{T(n) = 2(n+1)(H_{n+1} - \frac{3}{2})}$$

After-class Exercise

$$T(n) = T(n-1) - \frac{2T(n-1)}{n} + 2\left(1 - \frac{2T(n-1)}{n}\right), n > 0 \text{ with } T(0) = 0$$

Theorem (Linear Recurrences with Constant Coefficients)

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \textit{for } n \geq t$$

$$a_0, a_1, \cdots, a_{t-1}$$

$$\boxed{q(x) \equiv x^t - r_1 x^{t-1} - r_2 x^{t-2} - \cdots - r_t}$$

$$\beta_1\ (m_1), \beta_2\ (m_2), \cdots, \beta_i\ (m_i), \cdots, \beta_k\ (m_k)$$

$$m_1 + m_2 + \cdots + m_k = t$$

$$\boxed{a_n = \sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \cdots + \sum_{0 \leq j < m_k} c_{kj} n^j \beta_k^n}$$

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$\boxed{q(x) \equiv x^t - r_1 x^{t-1} - r_2 x^{t-2} - \cdots - r_t}$$

Proof.

$$\beta \ (m = 2)$$

$$\beta^n = r_1 \beta^{n-1} + r_2 \beta^{n-2} + \cdots + r_t \beta^{n-t} \quad \text{for } n \geq t$$

$$\beta^{n-t} q(\beta) = 0$$

$$n\beta^n = r_1 (n-1) \beta^{n-1} + r_2 (n-2) \beta^{n-2} + \cdots + r_t (n-t) \beta^{n-t} \quad \text{for } n \geq t$$

$$\beta^{n-t} \left((n-t)\, q(\beta) + \beta q'(\beta) \right) = 0$$

□

$$a_n = 5a_{n-1} - 6a_{n-2},\ n \geq 2\ (a_0 = 0, a_1 = 1)$$

$$\boxed{x^2 - 5x + 6 = (x-2)(x-3) = 0} \implies x = 2, 3$$

$$\boxed{a_n = c_0 3^n + c_1 2^n}$$

$$a_0 = 0 = c_0 + c_1$$
$$a_1 = 1 = 3c_0 + 2c_1$$

$$\boxed{a_n = 3^n - 2^n}$$

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}, n \geq 3 \; (a_0 = 0, a_1 = 1, a_2 = 4)$$

$$\boxed{x^3 - 5x^2 + 8x - 4 = 0}$$

$$(x-1)(x-2)^2 = 0 \implies x_1 = 1, x_2 = 2, x_2' = 2$$

$$a_n = c_1 \cdot 1^n + c_2 \cdot 2^n + c_2' \cdot n2^n$$

$$\begin{aligned} a_0 &= 0 = c_1 + c_2 \\ a_1 &= 1 = c_1 + 2c_2 + 2c_2' \\ a_2 &= 4 = c_1 + 4c_2 + 8c_2' \end{aligned}$$

$$\boxed{a_n = n2^{n-1}}$$

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3},\ n \geq 3\ (a_0 = 1, a_1 = 0, a_2 = -1)$$

$$\boxed{x^3 - 2x^2 + x - 2} = (x^2 + 1)(x - 2) = 0 \implies x = 2, i, -i$$

$$\boxed{a_n = c_1 2^n + c_2 i^n + c_3 (-i)^n}$$

$$\begin{aligned} a_0 &= 1 = c_1 + c_2 + c_3 \\ a_1 &= 0 = 2c_1 + c_2 i - c_3 i \\ a_2 &= -1 = 4c_1 - c_2 - c_3 \end{aligned}$$

$$\boxed{a_n = \frac{1}{2} i^n \left(1 + (-1)^n\right)}$$

$$1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0 \cdots$$

After-class Exercise

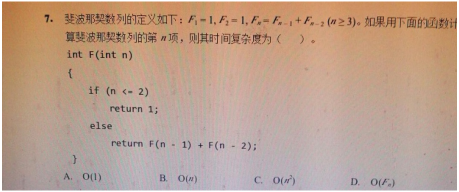
To give initial conditions a_0, a_1, and a_2 such that the growth rate of the solution to

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \ n > 2$$

is *(1) constant*; *(2) exponential*; *(3) fluctuating in sign*.

First-order Linear Non-homogeneous Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + r \quad \text{for } n \geq t$$



7. 斐波那契数列的定义如下：$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}\ (n \geq 3)$。如果用下面的函数计算斐波那契数列的第 n 项，则其时间复杂度为（ ）。

```
int F(int n)
{
    if (n <= 2)
        return 1;
    else
        return F(n - 1) + F(n - 2);
}
```

A. O(1) B. $O(n)$ C. $O(n^2)$ D. $O(F_n)$

$$F(n) = F(n-1) + F(n-2) + 2, \ n \geq 3 \ (F(1) = F(2) = 0)$$

$$\boxed{T(n,k) = \begin{cases} 0, & k = 0 \lor n = k \\ T(n-1,k) + T(n-1,k-1) + c, & \text{o.w.} \end{cases}}$$

$$a_n = 5a_{n-1} - 6a_{n-2} + 2,\ n \geq 2\ (a_0 = 0, a_1 = 1)$$

$$\boxed{a_n = c_0 3^n - c_1 2^n + c_2}$$

$$\boxed{c_2 = 5c_2 - 6c_2 + 2} \implies c_2 = 1$$

$$a_0 = 0 = c_0 + c_1 + 1$$
$$a_1 = 1 = 3c_0 + 2c_1 + 1$$

$$\boxed{a_n = 2 \cdot 3^n - 3 \cdot 2^n + 1}$$

More Issues about Linear Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + g(n) \quad \text{for } n \geq t$$

$$a_n = a_n^h + a_n^p$$

How to Find a Particular Solution for a Non-homogeneous Recurrence Relation?

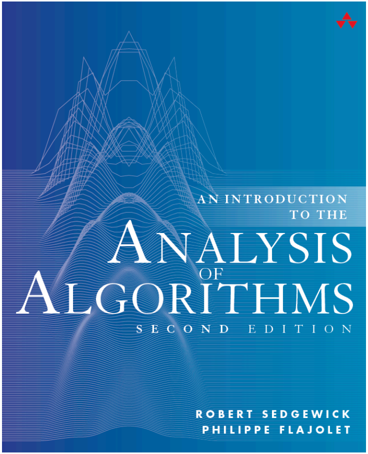
$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$t \geq 5$$

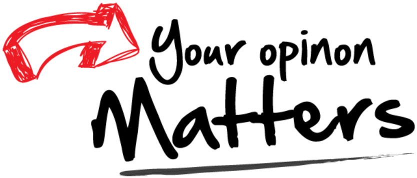
Generating Functions and Asymptotic Analysis

$$a_n = f_1(n)a_{n-1} + f_2(n)a_{n-2} + \cdots + f_t(n)a_{n-t} \quad \text{for } n \geq t$$

Generating Functions


AN INTRODUCTION
TO THE
ANALYSIS
OF
ALGORITHMS
SECOND EDITION
ROBERT SEDGEWICK
PHILIPPE FLAJOLET


Thank You!



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