We conclude that

$$\begin{aligned} \mathrm{E}[M] &\le k_0 \cdot 1 + n \cdot (1/n) \\ &= k_0 + 1 \\ &= O(\lg n / \lg\lg n) . \end{aligned}$$

Solution to Problem 11-3

a. From how the probe-sequence computation is specified, it is easy to see that the probe sequence is $\langle h(k), h(k)+1, h(k)+1+2, h(k)+1+2+3, \ldots, h(k)+1+2+3+\cdots+i, \ldots\rangle$, where all the arithmetic is modulo m. Starting the probe numbers from 0, the ith probe is offset (modulo m) from $h(k)$ by

$$\sum_{j=0}^{i} j = \frac{i(i+1)}{2} = \frac{1}{2}i^2 + \frac{1}{2}i .$$

Thus, we can write the probe sequence as

$$h'(k,i) = \left(h(k) + \frac{1}{2}i + \frac{1}{2}i^2\right) \bmod m ,$$

which demonstrates that this scheme is a special case of quadratic probing.

b. Let $h'(k,i)$ denote the ith probe of our scheme. We saw in part (a) that $h'(k,i) = (h(k) + i(i+1)/2) \bmod m$. To show that our algorithm examines every table position in the worst case, we show that for a given key, each of the first m probes hashes to a distinct value. That is, for any key k and for any probe numbers i and j such that $0 \le i < j < m$, we have $h'(k,i) \ne h'(k,j)$. We do so by showing that $h'(k,i) = h'(k,j)$ yields a contradiction.

Let us assume that there exists a key k and probe numbers i and j satsifying $0 \le i < j < m$ for which $h'(k,i) = h'(k,j)$. Then

$$h(k) + i(i+1)/2 \equiv h(k) + j(j+1)/2 \pmod{m} ,$$

which in turn implies that

$$i(i+1)/2 \equiv j(j+1)/2 \pmod{m} ,$$

or

$$j(j+1)/2 - i(i+1)/2 \equiv 0 \pmod{m} .$$

Since $j(j+1)/2 - i(i+1)/2 = (j-i)(j+i+1)/2$, we have

$$(j-i)(j+i+1)/2 \equiv 0 \pmod{m} .$$

The factors $j-i$ and $j+i+1$ must have different parities, i.e., $j-i$ is even if and only if $j+i+1$ is odd. (Work out the various cases in which i and j are even and odd.) Since $(j-i)(j+i+1)/2 \equiv 0 \pmod{m}$, we have $(j-i)(j+i+1)/2 = rm$ for some integer r or, equivalently, $(j-i)(j+i+1) = r \cdot 2m$. Using the assumption that m is a power of 2, let $m = 2^p$ for some nonnegative integer p, so that now we have $(j-i)(j+i+1) = r \cdot 2^{p+1}$. Because exactly one of the factors $j - i$ and $j + i + 1$ is even, 2^{p+1} must divide one of the factors. It cannot be $j - i$, since $j - i < m < 2^{p+1}$. But it also cannot be $j + i + 1$, since $j + i + 1 \le (m - 1) + (m - 2) + 1 = 2m - 2 < 2^{p+1}$. Thus we have derived the contradiction that 2^{p+1} divides neither of the factors $j - i$ and $j + i + 1$. We conclude that $h'(k, i) \ne h'(k, j)$.