- 作业讲解
  - TC第26.1节练习1、2、6、7
  - TC第26.2节练习2、6、8、10、12、13
  - TC第26.3节练习3
  - TC第26章问题1、2

TC第26.1节练习1

26.1-1

Show that splitting an edge in a flow network yields an equivalent network. More formally, suppose that flow network G contains edge (u, v), and we create a new flow network G' by creating a new vertex x and replacing (u, v) by new edges (u, x) and (x, v) with $c(u, x) = c(x, v) = c(u, v)$. Show that a maximum flow in G' has the same value as a maximum flow in G.

- 怎么证明“相等”？
  - ≥：原来的流量可以保持
  - ≤：不会形成更大的流

TC第26.1节练习2

26.1-2

Extend the flow properties and definitions to the multiple-source, multiple-sink problem. Show that any flow in a multiple-source, multiple-sink flow network corresponds to a flow of identical value in the single-source, single-sink network obtained by adding a supersource and a supersink, and vice versa.

- 看清题意
  - Extend the flow properties and definitions…
  - … by adding a supersource and a supersink

TC第26.2节练习10

26.2-10
Show how to find a maximum flow in a network $G = (V, E)$ by a sequence of at most $|E|$ augmenting paths. (*Hint:* Determine the paths *after* finding the maximum flow.)

1. 以最大流的f为边权，新建一个图。
2. 不断找s-t路（相当于augmenting path），将其包含的最小边权从所有边权中扣除，则至少1条边的权扣为0。
3. 因此，上述步骤可重复至多|E|次。

TC第26.2节练习12

26.2-12

Suppose that you are given a flow network G, and G has edges entering the source s. Let f be a flow in G in which one of the edges (v, s) entering the source has $f(v, s) = 1$. Prove that there must exist another flow f' with $f'(v, s) = 0$ such that $|f| = |f'|$. Give an $O(E)$-time algorithm to compute f', given f, and assuming that all edge capacities are integers.

- 证明：
  - f(v,s)=1，则v有入流，则存在一条s到v的流路径
  - 这条路径以及(v,s)边上的f各减1，得f'
- 算法：
  - 通过BFS找到流路径……

TC第26.2节练习13

26.2-13

Suppose that you wish to find, among all minimum cuts in a flow network G with integral capacities, one that contains the smallest number of edges. Show how to modify the capacities of G to create a new flow network G' in which any minimum cut in G' is a minimum cut with the smallest number of edges in G.

- $c' = c + \varepsilon$，其中$0 < \varepsilon < 1/|E|$
  - 新的最小割仍对应某个旧的最小割
  - 并且是其中边最少的

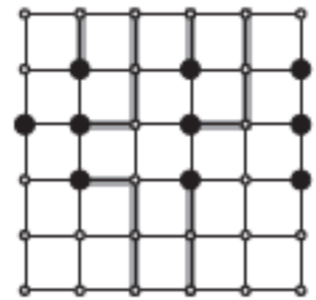
TC第26章问题1(b)

26-1 Escape problem

An $n \times n$ ***grid*** is an undirected graph consisting of n rows and n columns of vertices, as shown in Figure 26.11. We denote the vertex in the ith row and the jth column by (i, j). All vertices in a grid have exactly four neighbors, except for the boundary vertices, which are the points (i, j) for which $i = 1$, $i = n$, $j = 1$, or $j = n$.

Given $m \le n^2$ starting points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ in the grid, the ***escape problem*** is to determine whether or not there are m vertex-disjoint paths from the starting points to any m different points on the boundary. For example, the grid in Figure 26.11(a) has an escape, but the grid in Figure 26.11(b) does not.

b. Describe an efficient algorithm to solve the escape problem, and analyze its running time.



- 转化为求最大流
  - （边界上的starting point从图中去除）
  - 源点集：所有不在边界上的starting point
  - 汇点集：所有边界上的非starting point点
  - 点容量、边容量：1

- 教材讨论
  - TC第28章

问题1：线性方程组求解

- 如何将线性方程组表示为矩阵形式？
- LUP分解的矩阵表示是什么？
  - L、U、P分别是怎样的矩阵？
- 如何用它来改写线性方程组的矩阵表示？

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 ,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 ,\\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n .
\end{aligned}
$$

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\end{aligned}
$$

$Ax = b$ $\quad$ $PA = LU$ $\quad \Rightarrow \quad$ $LUx = Pb$

L： unit lower-triangular matrix
U： upper-triangular matrix
P： permutation matrix

问题1：线性方程组求解 (续)

- 接下来如何分两步求解？ $LUx = Pb$
- 分两步看起来更复杂了，能换来什么好处？

问题1：线性方程组求解 (续)

- 接下来如何分两步求解？ $LUx = Pb$
- 分两步看起来更复杂了，能换来什么好处？

$$LUx = Pb$$

$$\Downarrow$$

$$y = Ux$$

$$Ly = Pb$$

$$\Downarrow$$

$$Ux = y$$

问题1：线性方程组求解 (续)

- 你理解LUP-SOLVE了吗？

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.6 & 0.5 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 & 3 \\ 0 & 0.8 & -0.6 \\ 0 & 0 & 2.5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 1.4 \\ 1.5 \end{pmatrix}$$

$$\begin{array}{lcl} y_1 & = & b_{\pi[1]} , \\ l_{21}y_1 + y_2 & = & b_{\pi[2]} , \\ l_{31}y_1 + l_{32}y_2 + y_3 & = & b_{\pi[3]} , \\ & \vdots & \\ l_{n1}y_1 + l_{n2}y_2 + l_{n3}y_3 + \cdots + y_n & = & b_{\pi[n]} . \end{array}$$

$$\begin{array}{rcl} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1,n-2}x_{n-2} + u_{1,n-1}x_{n-1} + u_{1n}x_n & = & y_1 , \\ u_{22}x_2 + \cdots + u_{2,n-2}x_{n-2} + u_{2,n-1}x_{n-1} + u_{2n}x_n & = & y_2 , \\ & \vdots & \\ u_{n-2,n-2}x_{n-2} + u_{n-2,n-1}x_{n-1} + u_{n-2,n}x_n & = & y_{n-2} , \\ u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n & = & y_{n-1} , \\ u_{n,n}x_n & = & y_n . \end{array}$$

LUP-SOLVE(L, U, π, b)

```
n = L.rows
let x and y be new vectors of length n
for i = 1 to n
    y_i = b_π[i] − Σ_{j=1}^{i−1} l_ij y_j
for i = n downto 1
    x_i = (y_i − Σ_{j=i+1}^{n} u_ij x_j) / u_ii
return x
```

问题1：线性方程组求解 (续)

- 你理解LU-DECOMPOSITION了吗？

$$
\begin{aligned}
A &= \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ v & A' \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ v/a_{11} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ 0 & A' - vw^{\mathrm{T}}/a_{11} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ v/a_{11} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ 0 & L'U' \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ v/a_{11} & L' \end{pmatrix} \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ 0 & U' \end{pmatrix} \\
&= LU \,,
\end{aligned}
$$

```
LU-DECOMPOSITION(A)
n = A.rows
let L and U be new n × n matrices
initialize U with 0s below the diagonal
initialize L with 1s on the diagonal and 0s above the diagonal
for k = 1 to n
    u_kk = a_kk
    for i = k + 1 to n
        l_ik = a_ik/a_kk          // a_ik holds v_i
        u_ki = a_ki               // a_ki holds w_i
    for i = k + 1 to n
        for j = k + 1 to n
            a_ij = a_ij − l_ik u_kj
return L and U
```

2	3	1	5
6	13	5	19
2	19	10	23
4	10	11	31

(a)

2	3	1	5
3	4	2	4
1	16	9	18
2	4	9	21

(b)

2	3	1	5
3	4	2	4
1	4	1	2
2	1	7	17

(c)

2	3	1	5
3	4	2	4
1	4	1	2
2	1	7	3

(d)

问题1：线性方程组求解 (续)

- 为什么要permutation？
- 你能解释这些步骤吗？

$$
\begin{aligned}
QA &= \begin{pmatrix} a_{k1} & w^{\mathrm{T}} \\ v & A' \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ v/a_{k1} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{k1} & w^{\mathrm{T}} \\ 0 & A' - vw^{\mathrm{T}}/a_{k1} \end{pmatrix}
\end{aligned}
$$

$$P'(A' - vw^{\mathrm{T}}/a_{k1}) = L'U'$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & P' \end{pmatrix} Q$$

$$
\begin{aligned}
PA &= \begin{pmatrix} 1 & 0 \\ 0 & P' \end{pmatrix} QA \\
&= \begin{pmatrix} 1 & 0 \\ 0 & P' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v/a_{k1} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{k1} & w^{\mathrm{T}} \\ 0 & A' - vw^{\mathrm{T}}/a_{k1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ P'v/a_{k1} & P' \end{pmatrix} \begin{pmatrix} a_{k1} & w^{\mathrm{T}} \\ 0 & A' - vw^{\mathrm{T}}/a_{k1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ P'v/a_{k1} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{k1} & w^{\mathrm{T}} \\ 0 & P'(A' - vw^{\mathrm{T}}/a_{k1}) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ P'v/a_{k1} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{k1} & w^{\mathrm{T}} \\ 0 & L'U' \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ P'v/a_{k1} & L' \end{pmatrix} \begin{pmatrix} a_{k1} & w^{\mathrm{T}} \\ 0 & U' \end{pmatrix} \\
&= LU \,,
\end{aligned}
$$

问题2：矩阵求逆

- 你能简要描述利用LUP分解求逆矩阵的思路吗？

- The proof of Theorem 28.2 suggests a means of solving the equation Ax=b by using LU decomposition without pivoting, so long as A is nonsingular.
  你理解这种新方法了吗？

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- 你能简要描述利用LUP分解求逆矩阵的思路吗？

  $AX = I_n$

  $AX_i = e_i$

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  你理解这种新方法了吗？
  1. $(A^T A)x = A^T b$
  2. Factor the symmetric positive-definite matrix A^TA by computing an LU decomposition.
  3. Use forward and back substitution to solve for x with the right-hand side A^Tb.

问题3：求行列式

- 我们还可以利用LUP分解来求方阵的行列式，你能想到吗？

问题3：求行列式

- 我们还可以利用LUP分解来求方阵的行列式，你能想到吗？

$$A = P^{-1}LU$$

$$\det(A) = \det(P^{-1})\det(L)\det(U) = (-1)^S \left(\prod_{i=1}^{n} l_{ii}\right)\left(\prod_{i=1}^{n} u_{ii}\right).$$

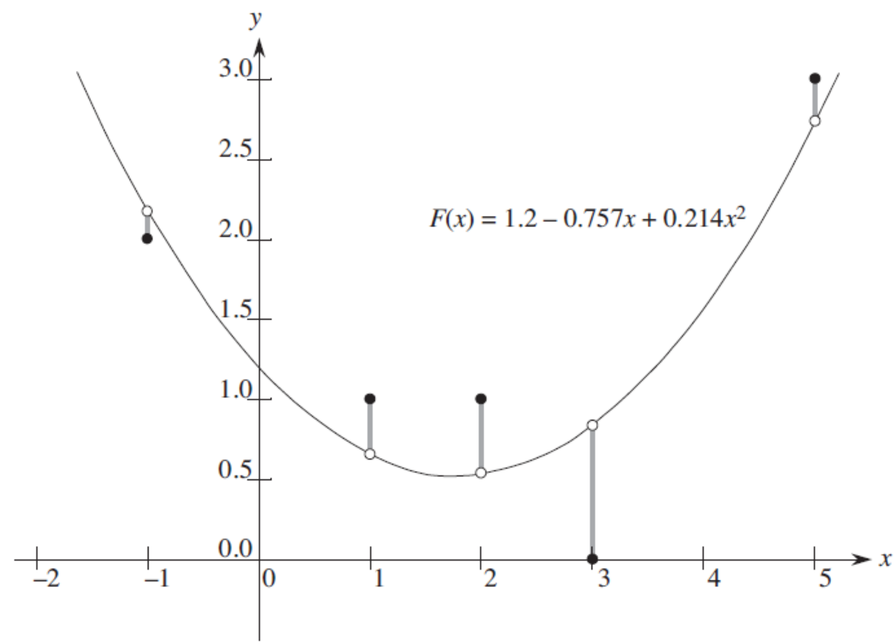
问题4：最小二乘法

- 最小二乘法想要解决的是一个什么问题？

$$(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$$

$$\eta_i = F(x_i) - y_i$$

$$\|\eta\| = \left(\sum_{i=1}^{m} \eta_i^2\right)^{1/2}$$



问题4：最小二乘法 (续)

- 你能解释这些步骤吗？

$$\|\eta\| = \left(\sum_{i=1}^{m} \eta_i^2\right)^{1/2}$$

$$\|\eta\|^2 = \|Ac - y\|^2 = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}c_j - y_i\right)^2$$

$$\frac{d\,\|\eta\|^2}{dc_k} = \sum_{i=1}^{m} 2\left(\sum_{j=1}^{n} a_{ij}c_j - y_i\right) a_{ik} = 0$$

$$(Ac - y)^{\mathrm{T}} A = 0$$

$$A^{\mathrm{T}}(Ac - y) = 0$$

$$A^{\mathrm{T}} Ac = A^{\mathrm{T}} y$$

$$c \;=\; \left((A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}}\right) y$$