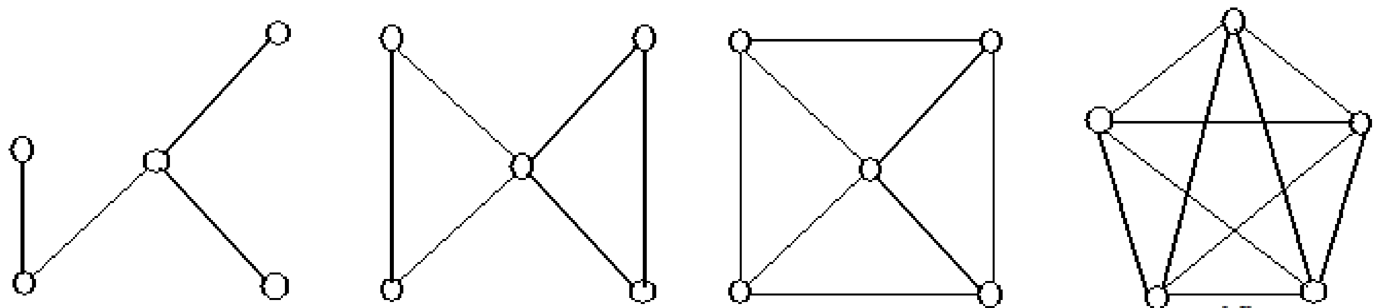
计算机问题求解 – 论题3-10 - 图的连通度

2016年11月9日

预习检查

- Whitney theorem then gives us an alternative method for determining the connectivity of a graph *G*. Not only is *K(G)* the minimum number of vertices whose removal from *G* results in a disconnected or trivial graph but *K(G)* is the maximum positive integer *k* for which every two vertices *u* and *v* in *G* are connected by *k* internally disjoint *u* - *v* paths in *G*.

“连通”并不都是一样的



(a) (b) (c) (d)

问题1:

你能否解释一下它们的“连通”怎么不一样?

割点、割边与回路

问题2：

衡量连通的“牢度”的指标是什么？

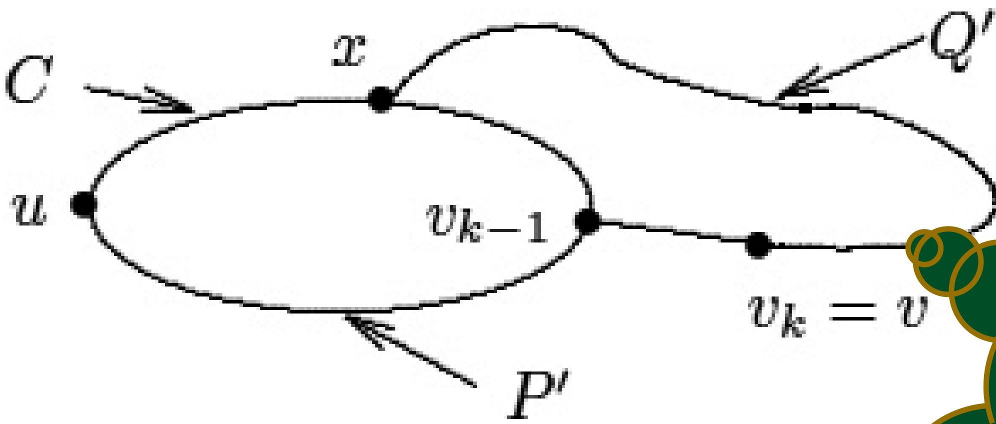
指标1：minimum vertex-cut

By a **vertex-cut** in a graph *G*, we mean a set *U* of vertices of *G* such that *G* - *U* is disconnected. A vertex-cut of minimum cardinality in *G* is called a **minimum vertex-cut**.

问题3：
你能据此解释：割点是什么？
Nonseparable graph是什么？
图的block是什么？

关于block的几个理解

Theorem 5.7 *A graph of order at least* 3 *is nonseparable if and only if every two vertices lie on a common cycle.*



问题4：从这个定理及证明中能否看出，非分离图在外表上具有什么共性？

关于block的几个理解

A maximal nonseparable subgraph of a graph G is called a **block** of G.

极大还是最大？

关于block的几个理解

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Theorem 5.8 *Let R be the relation defined on the edge set of a nontrivial connected graph G by e R f, where e, f* $\in$ *E*(*G*), *if e = f or e and f lie on a common cycle of G. Then R is an equivalence relation*.

用等价关系来描述边之间的关系，到底有何用意？

Each subgraph of G induced by the edges in an equivalence class is in fact a block of G.

问题6：Block的“极大”特性，是否能够被“等价类”保证？

问题7：

两个block的“边界”是什么？

Corollary 5.9 *Every two distinct blocks* B_1 *and* B_2 *in a nontrivial connected graph* G *have the following properties:*

(a) *The blocks* B_1 *and* B_2 *are edge-disjoint.*

(b) *The blocks* B_1 *and* B_2 *have at most one vertex in common.*

(c) *If* B_1 *and* B_2 *have a vertex* v *in common, then* v *is a cut-vertex of* G.

问题8：

从一个k-连通图中删除*k*个点，剩下的图是否一定不连通了？

是否存在一种方法，使得从一个k-连通图中删除*k*个点，剩下的图是否一定不连通？

问题9：

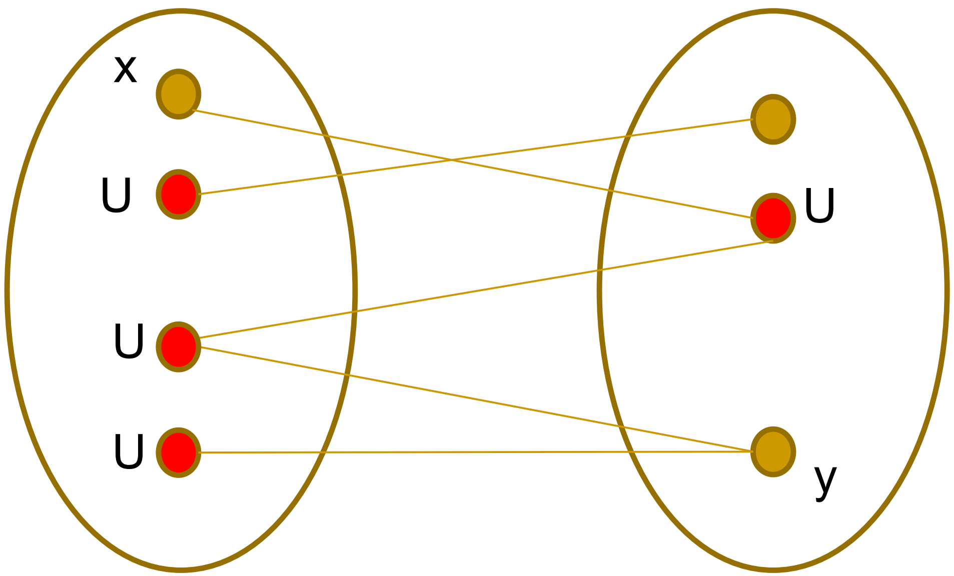
Block和2-连通图是什么关系？

点连通度、边连通度与图的最小度

Theorem 5.11 For every graph G, (By Whitney)

$$\kappa(G) \le \lambda(G) \le \delta(G).$$

证明K小于等于λ的case2的基本思路：在最小边割集的两侧选一对不直接相邻的点，考虑如何“切断”它们之间的通路。



$$\kappa(G) \le |U| \le |X| = \lambda(G).$$

点连通度和图的边点数关系

Theorem 5.13 *If G is a graph of order n and size m* >=*n* - 1, *then*

$$\kappa(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor .$$

问题10：如何理解下文中的bound？

The bound given in Theorem 5.13 is sharp in the sense that for every two integers *n* and m of a graph.

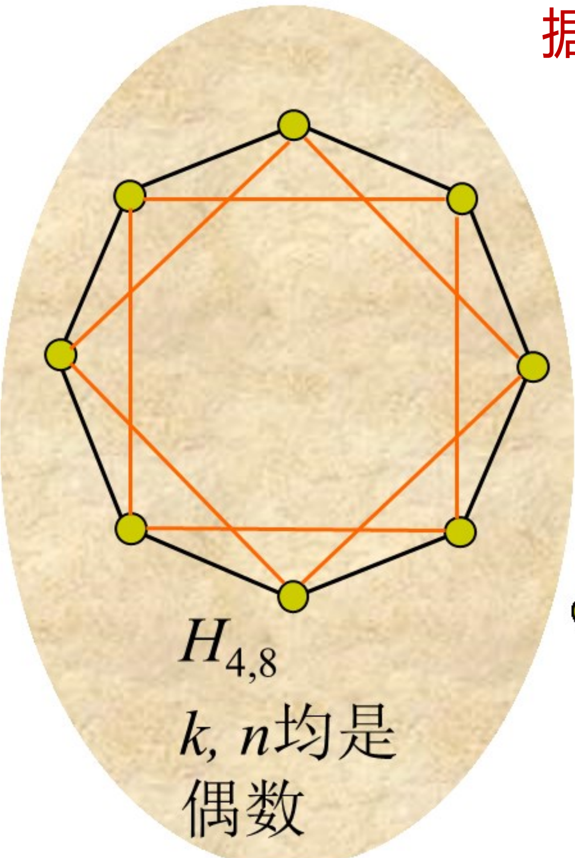
计算机网络－一个应用的例子

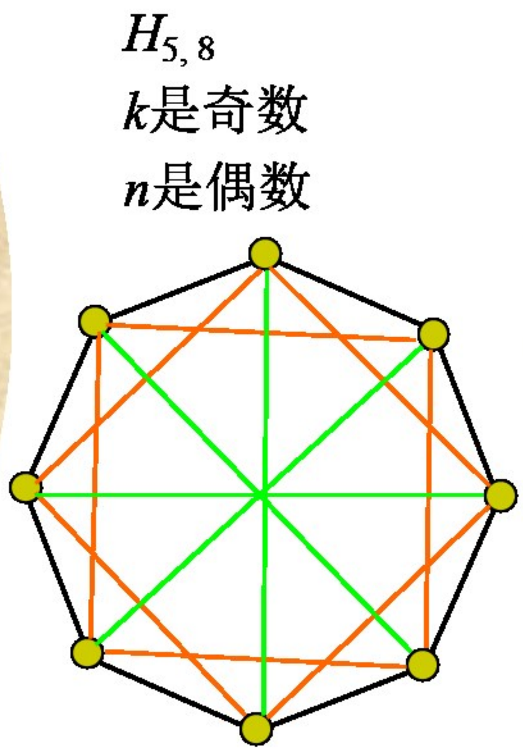
- 问题: 将n个计算机连成一个通信网络以共享资源，如果要以最小的代价（假设以链路条数计）保证在故障节点少于k个的条件下所有计算机能保持互连，网络应该如何连接？
- 数学模型：找出n个结点的完全图的一个边最少的k-连通子图。

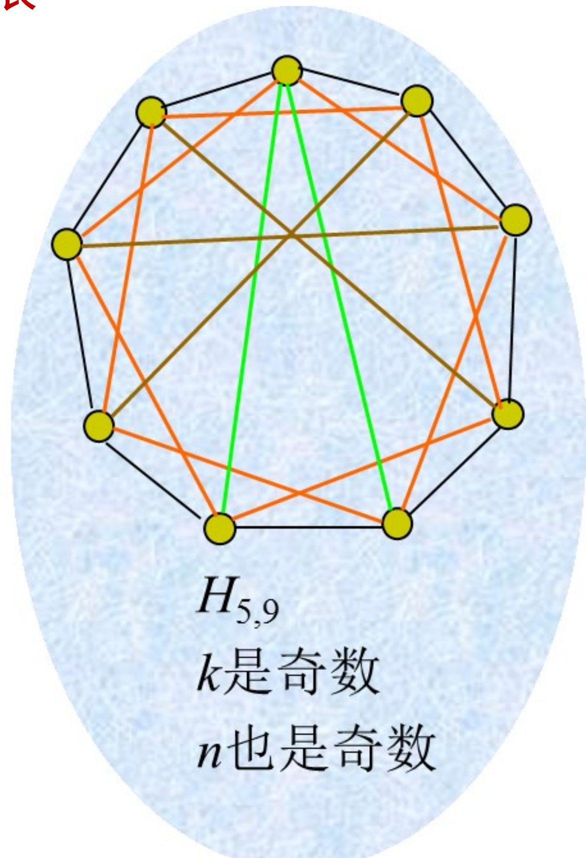
（注意：含n个顶点的k-连通图至少有$nk/2$条边，因为该图中最小顶点次数不能小于k）

Harary的解：$H_{k,n}$

将所有顶点排成一圈，根据n和k的奇偶性加边。







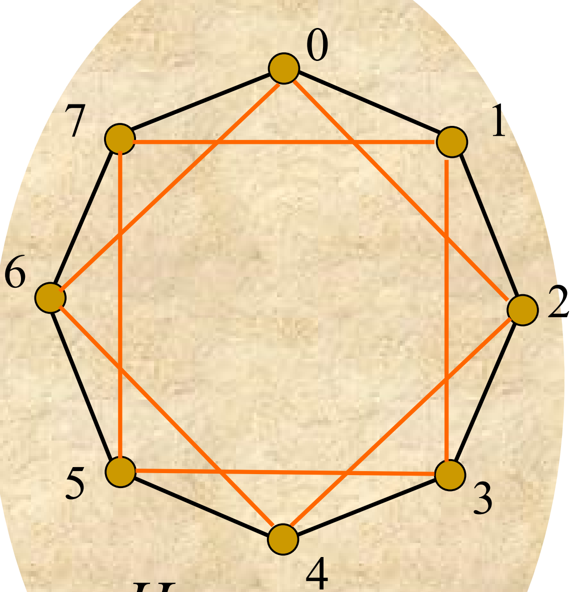
问题11：

你能够看出加边的连接方法与图的连通度有什么关系吗？

证明的思路：只讨论k为偶数

1. 前已说明：含n个顶点的k-连通图至少有$nk/2$条边
2. 左边的解恰好是$nk/2$条边
3. 因此，只须证明，这图是k一连通的.

以这一较简单的情况为例



$H_{k,n}$

k,n均是偶数

令k=2r（r是整数）

Harary的解法实际上是对任意顶点i，让它与满足下述条件的顶点j 相连：

$$j \geq (i-r) \bmod n \text{ 或 } j \leq (i+r) \bmod n$$

于是，如果两点取模差不大于r, 则相连.

假设从图中删除少于 2 r个顶点（构成子集V'），图就不连通了，删除后，顶点i,j属不同的分支.

考虑两个子集合(这里的序号对n 取模)：

$S=\{i,i+1, \ldots, j-1, j\}$; $T=\{j, j+1, \ldots, i-1, i\}$。

由于V'中总点数小于2r, **这两集合中至少有一个含V'中的点少于r 个，则此集合中删除V'后仍构成一 ij-通路**，矛盾。

指标2：multiplicity of alternative paths

How many internally disjoint paths are there to link any pair of vertex u,v in graph G?

两种指标本质上是一样的。

对图G中任意两点u,v，如果点不相交的uv–通路有k条，显然，要使u,v不连通，至少须删除k个顶点。

Menger theorem

Let u and v be ***nonadjacent*** *vertices in a graph G. The minimum number of vertices in a* ***u-v separating set*** *equals the maximum number of* ***internally disjoint u-v paths*** *in G.*

证明要点：

1，对图的边数(size)进行归纳；

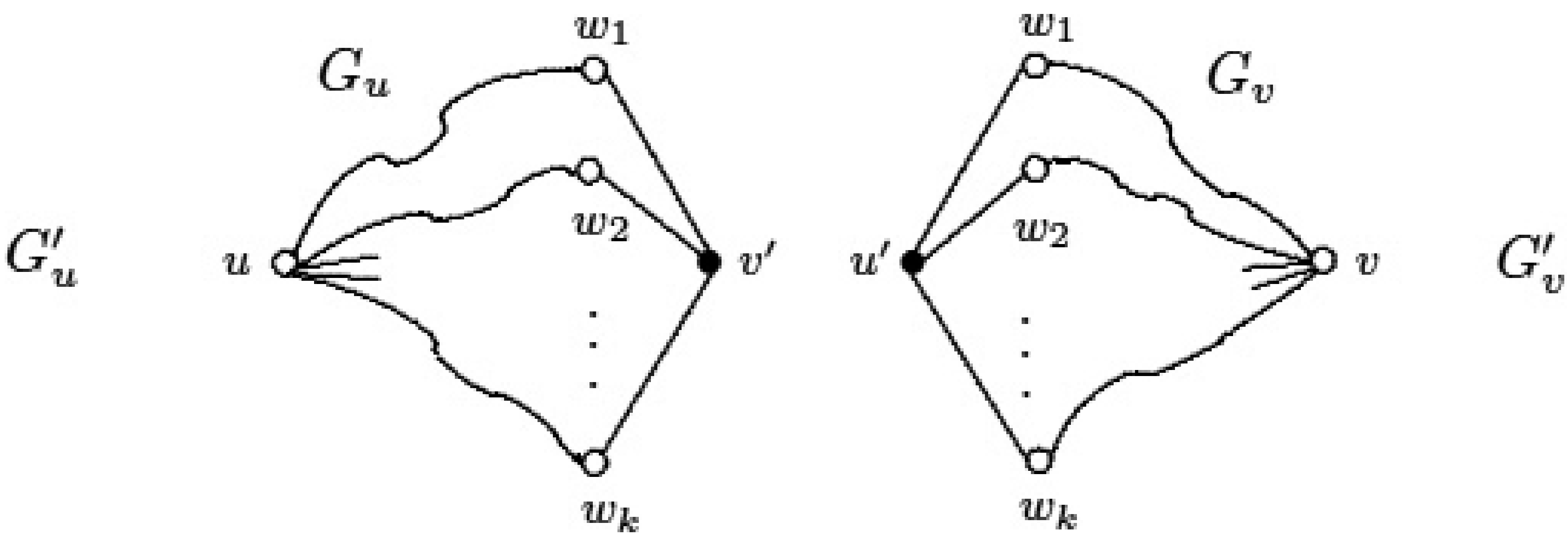
2，对“分离点集”中的点的特殊性进行分别分析：

1）**存在**和uv直接相邻的点；

2）**同时存在**一个不和u相邻的点以及存在一个不和v相邻的点

3）**所有的**点要么和u相邻，要么和v相邻

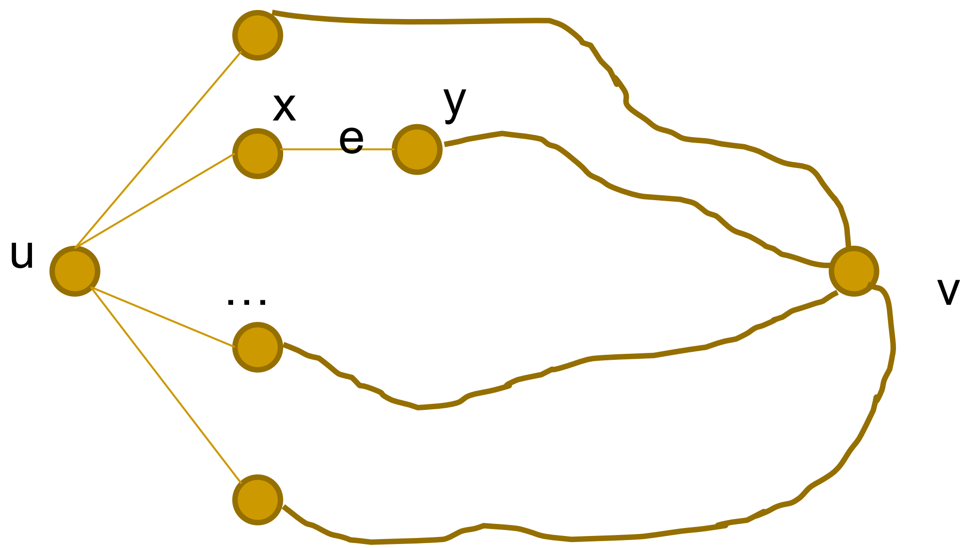
Case2的图例和证明：



要点：Since *W* contains a vertex that is not adjacent to *u* and a vertex that is not adjacent to *v*, **the size of each of the graphs G'_u and G'_v is less than m.**

Case3的图例和证明：

所有的点要么和u相邻，要么和v相邻



考察G-e：

1，适用归纳假设；

2，G-e图中的最小分割集也一定是K个元素，否则可以证明y也是某个最小分割点集的点，进而y也和u相邻，uxy…v就不是最短

3，G-e中最小分割集大小为k，**存在k条不相交uv路**

2-连通图的特征

4.2.4. Theorem. For a graph G with at least three vertices, the following conditions are equivalent (and characterize 2-connected graphs).

A) G is connected and has no cut-vertex.

B) For all $x, y \in V(G)$, there are internally disjoint x, y-paths.

C) For all $x, y \in V(G)$, there is a cycle through x and y.

D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

这里，最有意思的是如何推导出D。

With the aid of Menger's theorem, Hassler Whitney was able to present a characterization of k-connected graphs.

Theorem 5.17 *A nontrivial graph G is k-connected for some integer $k \geq 2$ if and only if for each pair u, v of distinct vertices of G there are at least k internally disjoint $u - v$ paths in G.*

We have introduced two measures of good connection: invulnerability to deletions and multiplicity of alternative paths. Extending Whitney's Theorem, we show that these two notions are the same, for both vertex deletions and edge deletions, and for both graphs and digraphs.

Open Topics:

P1:

4.2.4. Theorem. For a graph G with at least three vertices, the following conditions are equivalent (and characterize 2-connected graphs).

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D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

P2:

The **edge-connectivity** version of Menger's theorem is as follows:

Let G be a finite undirected graph and x and y two distinct vertices. Then the theorem states that the size of the minimum edge cut for x and y (the minimum number of edges whose removal disconnects x and y) is equal to the maximum number of pairwise edge-independent paths from x to y.

课外作业

- CZ：5.4，5.8
- CZ：5.10，5.12
- CZ：5.18，5.22，5.26
- CZ：5.34
- 证明PPT第21页的定理