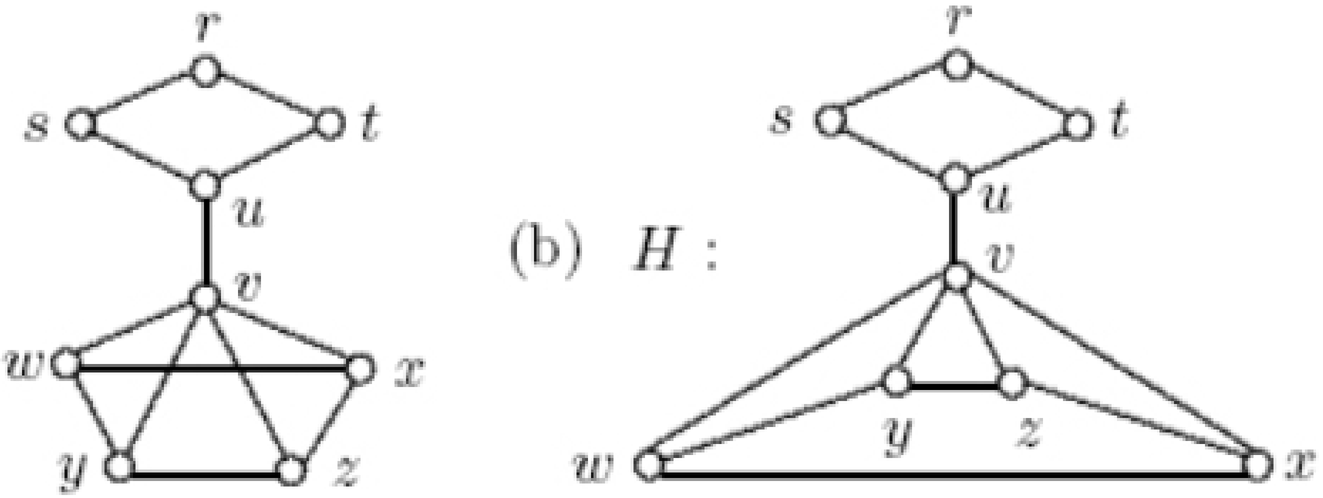
计算机问题求解 – 论题3-14
- 图论中的其它专题

2016年12月06日

平面图



问题1：

一个图是否为平面图与其
如何画法有关吗？

Theorem 9.1 (The Euler Identity) *If G is a connected plane graph of order n, size m and having r regions, then $n - m + r = 2$.*

证明要点：

1，树是满足欧拉公式

2，反证法：如果除树外，欧拉公式不成立

2.1 找一个边最小的图

2.1.1 必定存在回路

2.2 去除回路上一条边，构造新图

2.2.1 新图有n个点，m-1条边，r-1个区域

2.3 新图满足欧拉公式（原图是最小的）

2.3.1 $n-(m-1)+(r-1) = 2$

2.3.2 $n-m+r=2$

2.4 矛盾

3，欧拉公式满足

如果我们用归纳法证明该定理，该如何归纳？

问题2:

你是否在哪里里还见过“这个”欧拉公式?

问题3：

必要条件可以判断什么？欧拉公式在实际判断平面图时有用吗？

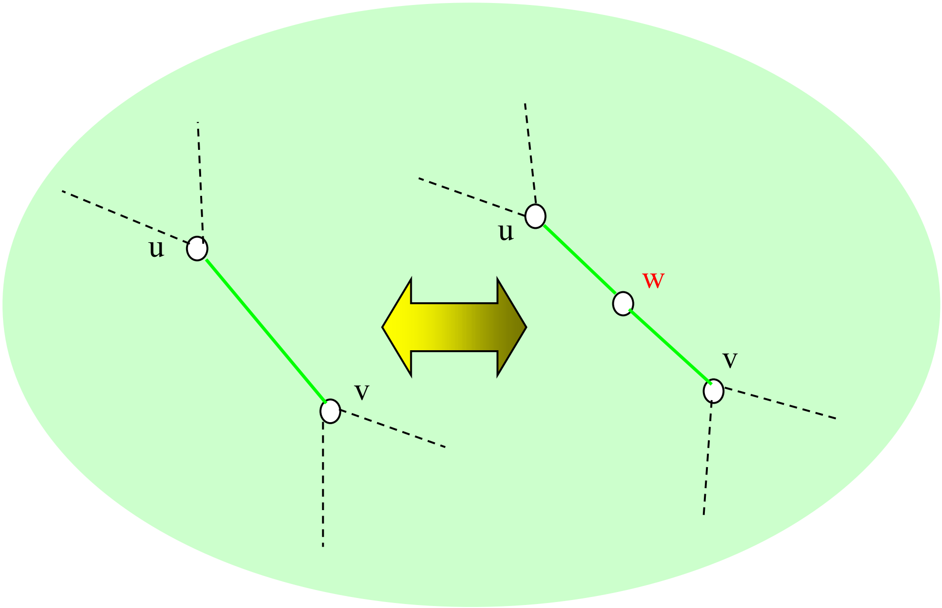
简单连通平面图的必要条件

- ***欧拉公式的推论***：若G是简单连通平面图(n≥3)，则**m≤3n-6**。
  - 证明：至少3个顶点的简单图G中，**面的最小度数是3**，∴3r≤2m，根据欧拉公式：3r=3m+6-3n，∴3m+6-3n≤2m，即：m≤3n-6。
- K_5不是平面图：在K_5中，n=5, m=10, 3n-6=9。

问题4：

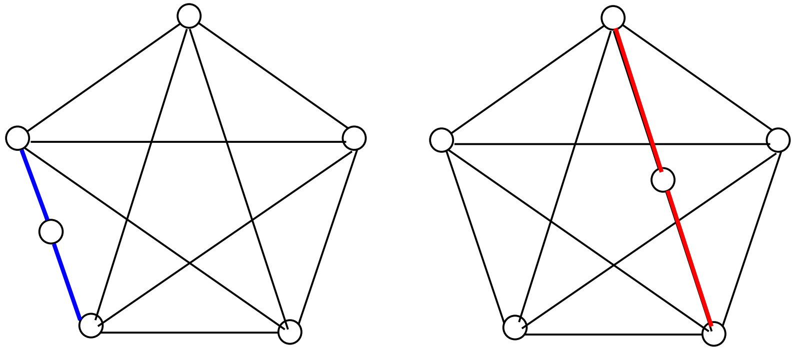
上述推论不能证明$K_{3,3}$是非平面图，你能推出一个类似的推论用于$K_{3,3}$吗？

同胚图



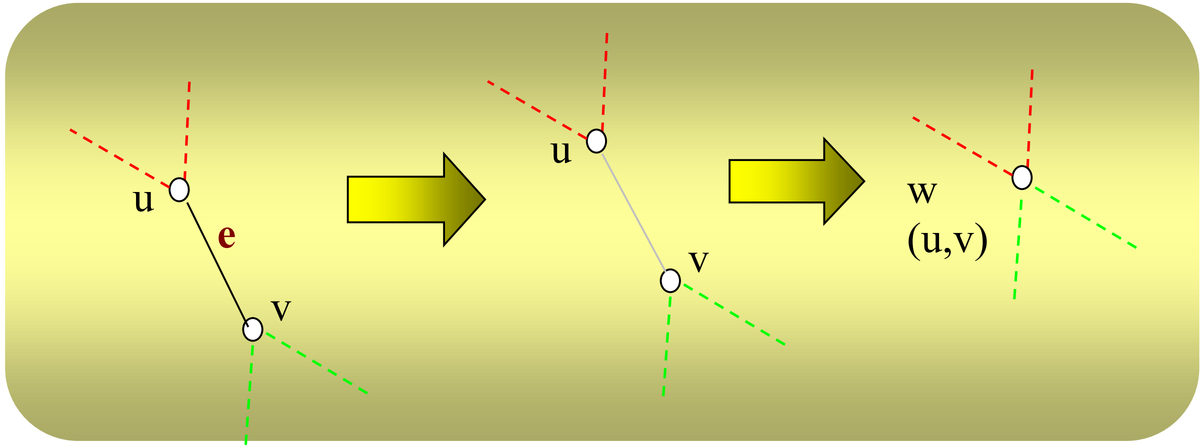
- 基本动作：
  - 二次顶点的插入和消去

- 如果图G_1和G_2经过反复的插入和消去二次顶点，可达到同构，则G_1和G_2是同胚图。

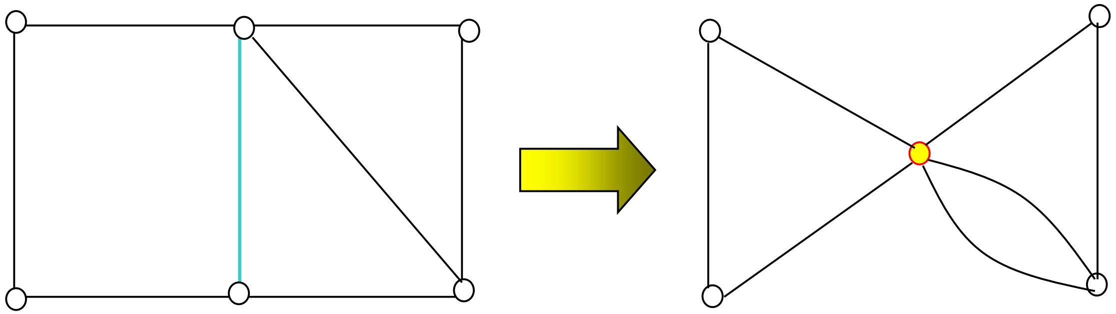


图的收缩

- 基本动作：



- 图的收缩

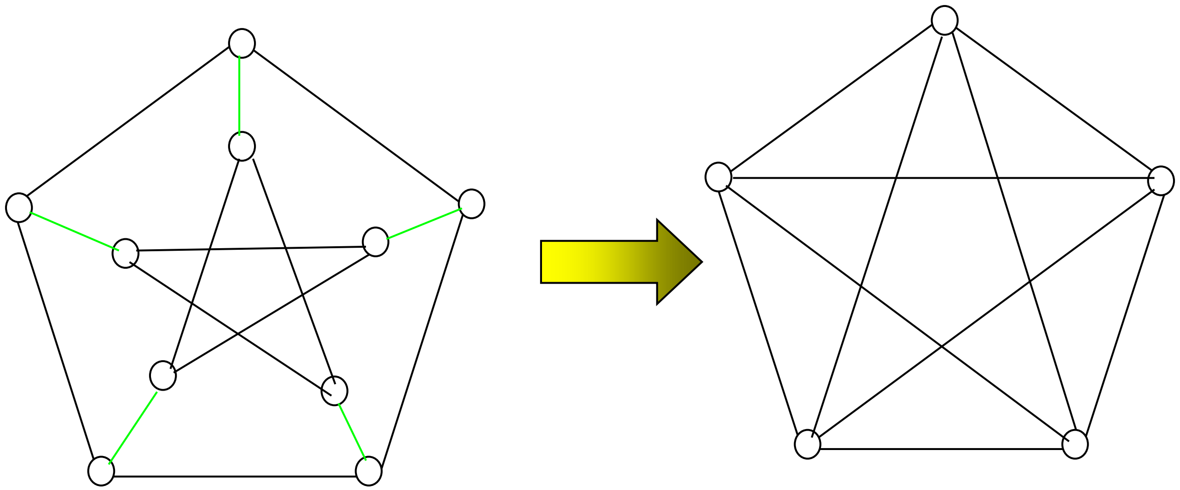


平面图的充分必要条件

- Kuratowsky定理
  - 图G是平面图当且仅当G中不含与K_5或者$K_{3,3}$同胚的子图,也不含可以收缩到K_5或者$K_{3,3}$的子图。

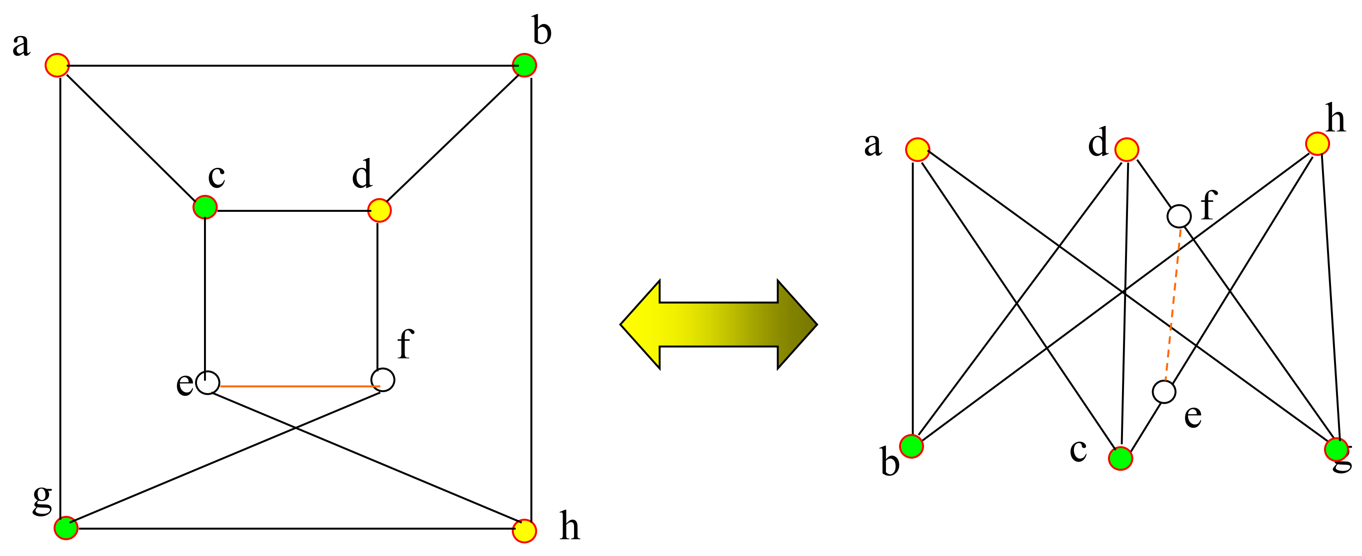
Kuratowsky定理的应用 (1)

- Petersen图不是平面图
  - 它可以收缩到K_5



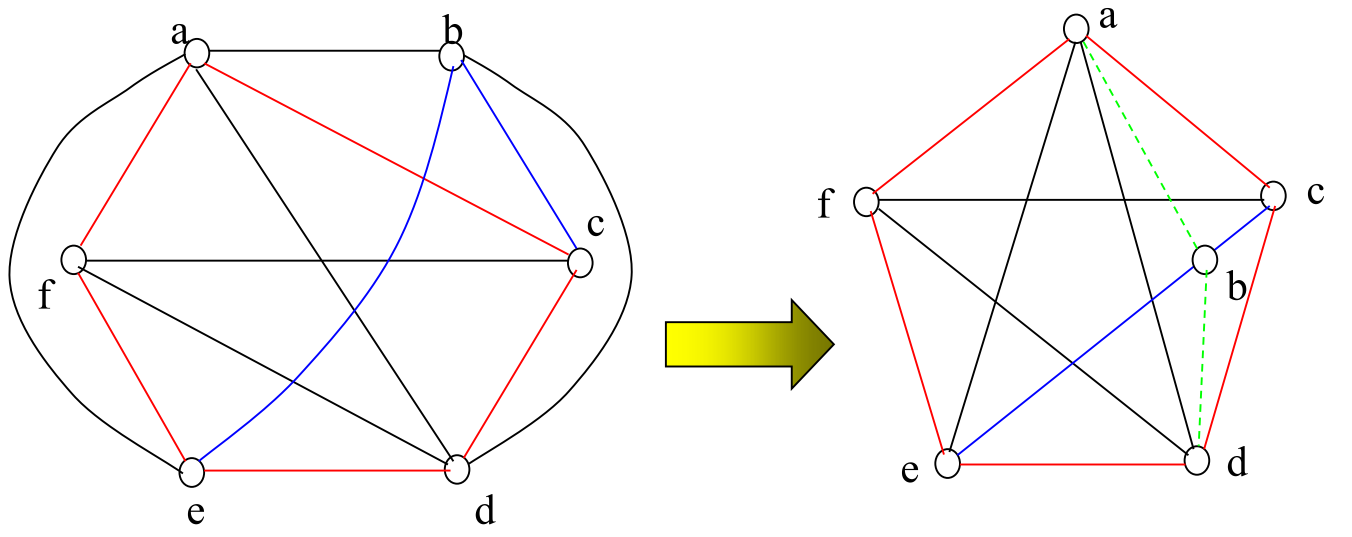
Kuratowsky定理的应用 (2)

- 左图的一个子图与$K_{3,3}$同胚。因此它是***非平面图***。

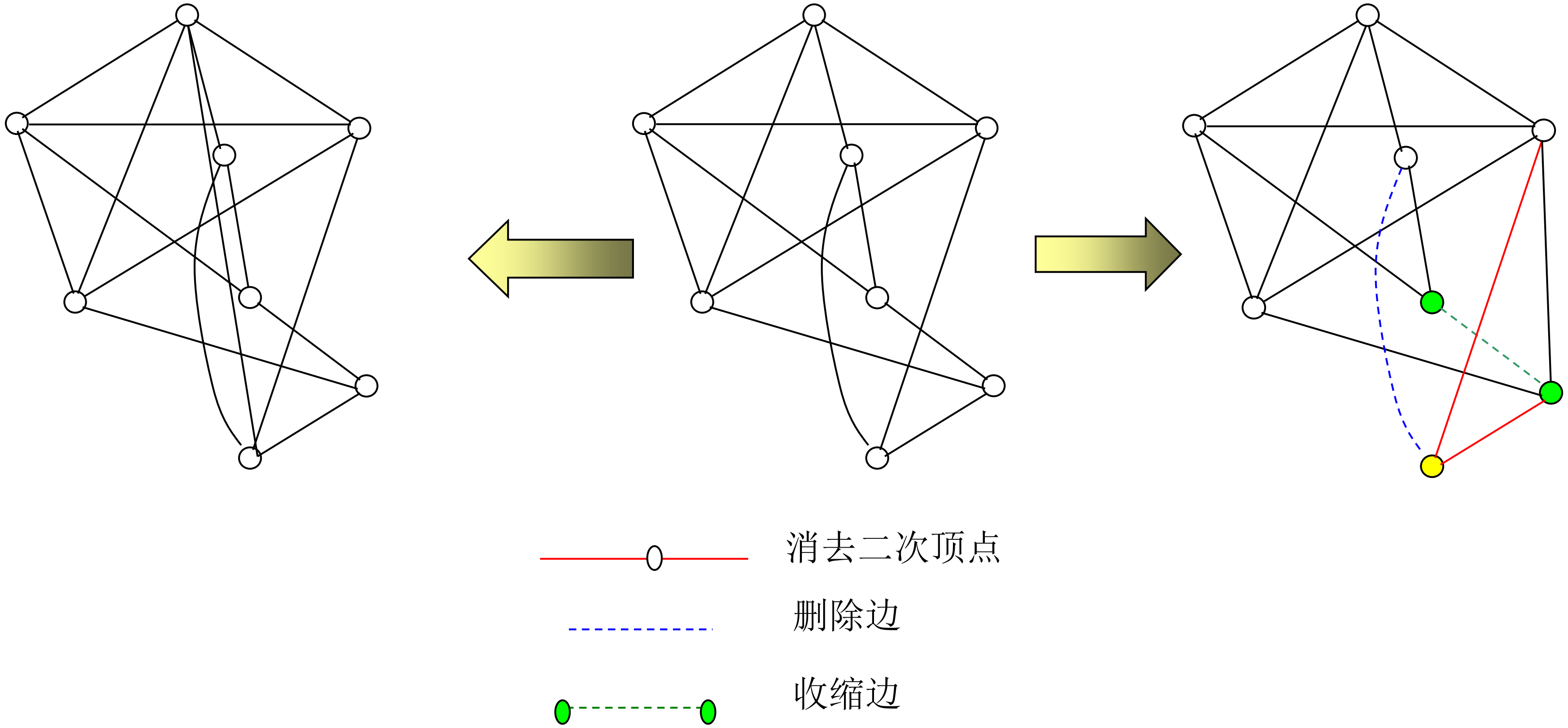


Kuratowsky定理的应用 (3)

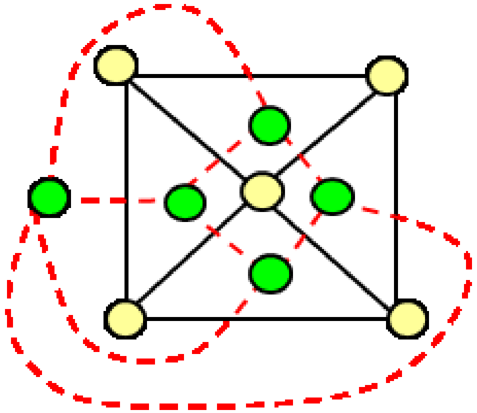
- 左图含有与K_5同胚的子图, 因此它也是非平面图。



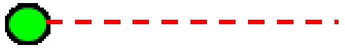
Kuratowsky定理的应用 (4)



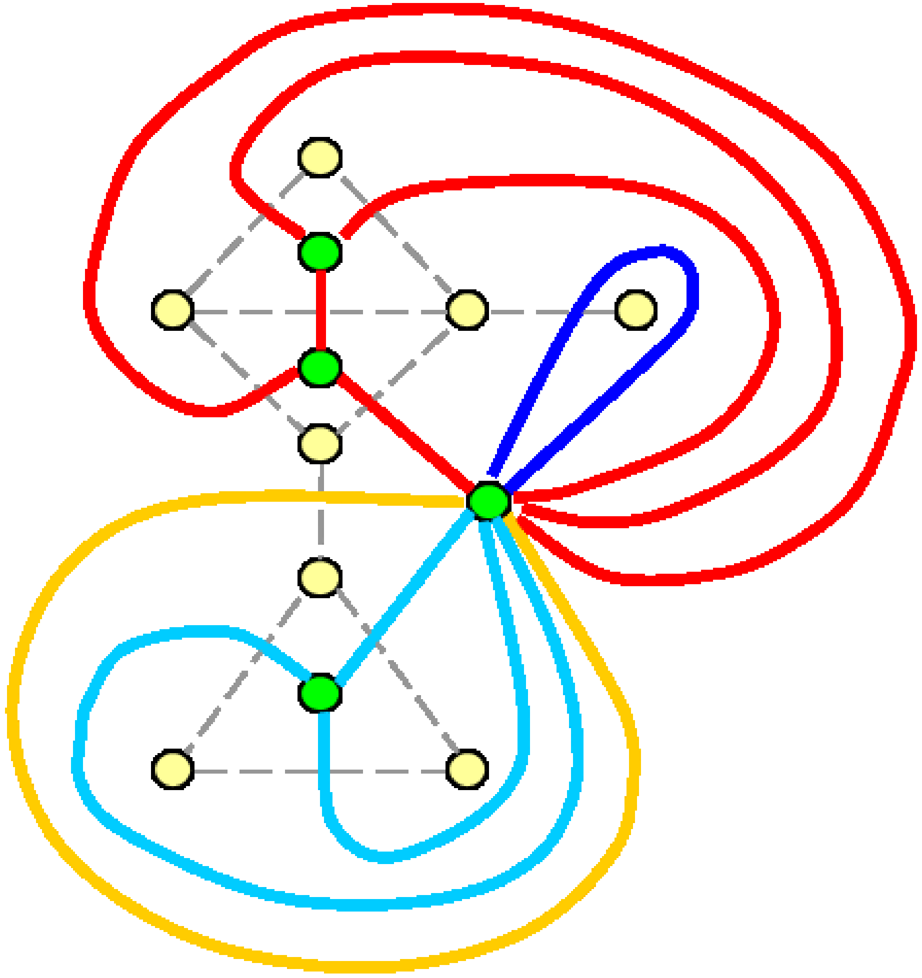
平面图及其对偶图



图*G*中的点和边



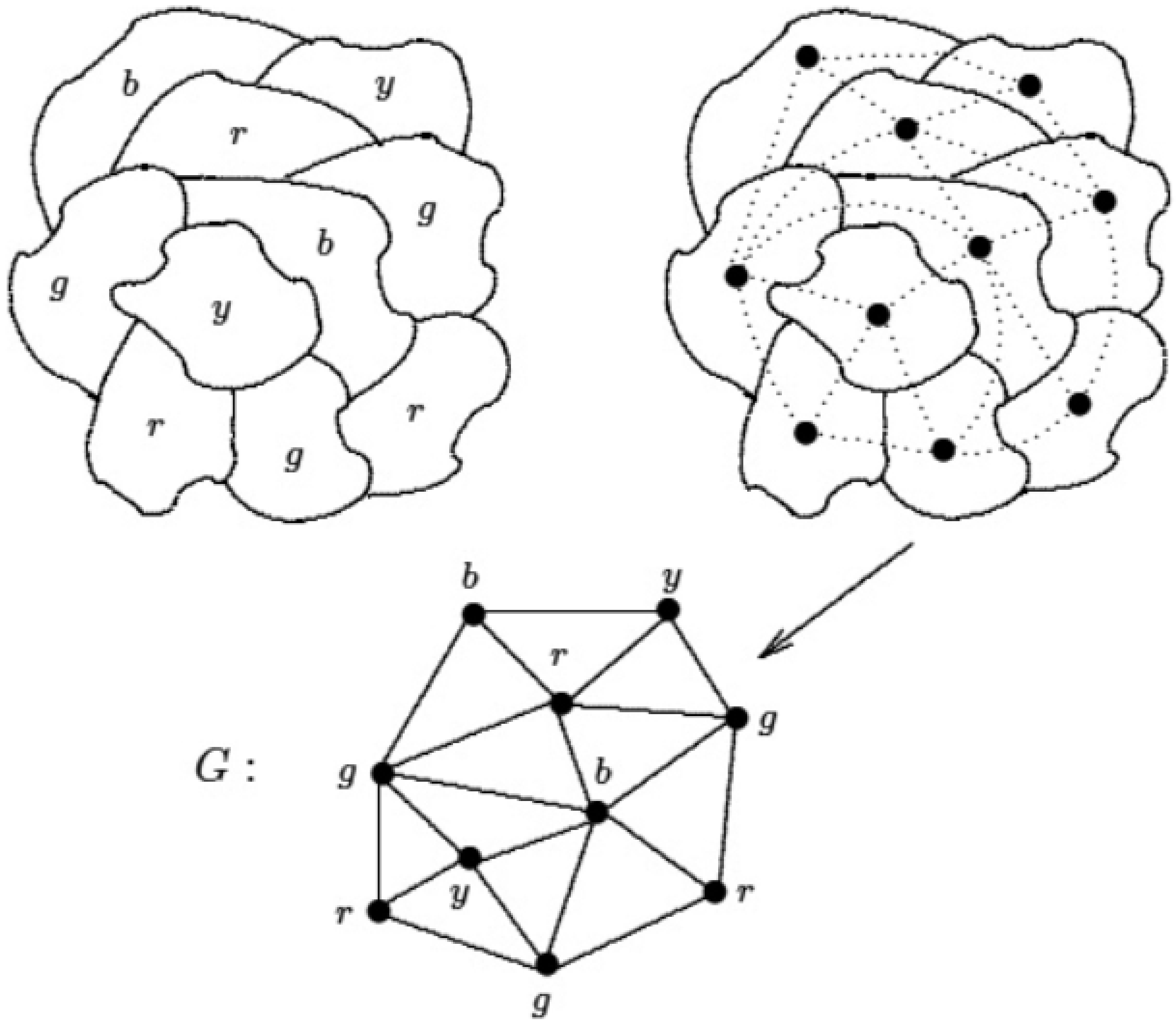
*G*的对偶图*G'*中的点和边



问题5：

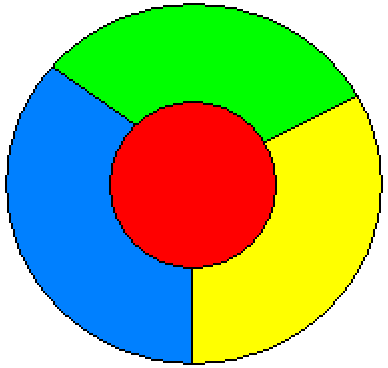
你能严格定义一个平面图的对偶图吗？

地图及其对偶图



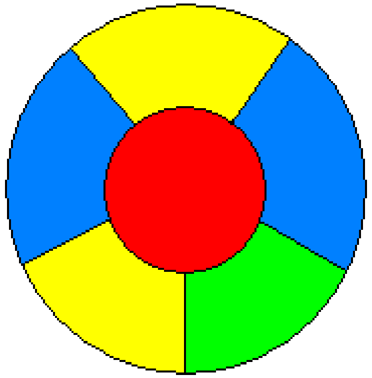
Francis Guthrie的猜想

给地图的每个区域着色，保持任何有公共边界的区域使用不同颜色，只要四种颜色就够了。



很容易证明，三种颜色是不够的

- Guthrie, de Morgan



问题6：这是为什么？

可以证明：任何地图不会有这样的构型：5个区域，每个均与其它4个相邻。

误认为这蕴涵四色猜想导致许多错误证明

四色地图猜想的另一种描述

- 每个平面图是4-可着色的当且仅当每个平面图是4-区域可着色的
  - 只需考虑连通的情况
  - 如果*G*是任意平面图，考虑它的对偶图*G'*删除环以及多余的多重边后得到的简单图*G**。*G*的任意两个区域相邻当且仅当*G**中对应的两顶点相邻，因此只要*G**可4-着色，*G*就可以4-区域着色。
  - 反之，*G的*对偶图的每个区域中只会包含*G*中唯一的顶点。于是，对偶图的任意两个区域相邻当且仅当他们所包含的两个*G*中顶点在*G*中相邻，因此只要那个对偶图是4-区域可着色的，*G*就是4-可着色的。

Computer Mathematics Comes of Age

From Keith Devlin: Mathematics: the New Golden Age

In 1976, two mathematicians at the University of Illinois, Kenneth Appel and Wolfgang Haken, announced that they had solved a century-old problem to do with the colouring of maps. They had, they said, proved the *four-colour conjecture*. This in itself was a newsworthy event. The four-colour problem was, after Fermat's last theorem (see Chapter 8), probably the second most famous unsolved problem in mathematics. But for mathematicians the really dramatic aspect of the whole affair was the way the proof had been achieved. Large and crucial parts of their argument were carried out by a computer, using ideas which had themselves been formulated as a result of computer-based evidence. So great was the amount of computing required that it was not feasible for a human mathematician to check every step. This meant that the whole concept of a 'mathematical proof' had suddenly changed. Something that had been threatening to occur ever since electronic computers were first developed in the early 1950s had finally happened; the computer had taken over from the human mathematician part of the construction of a real mathematical proof.

图的顶点着色

- 给定简单图G及一有限集合$C=\{c_1,c_2,\ldots,c_k\}$, 用C中元素给G中每个顶点指定一个标号，使得任何相邻顶点的标号不相同，则称为给G做一个点着色。诸c_i $(i=1,2,\ldots,k)$称为“颜色”。
- 如果图G可以用k种颜色着色，则G称为**k-可着色**的。
- 使G是k-可着色的最小的k称为G的***着色数***，记为$\chi(G)$。
- 若$\chi(G)=k$，则称G是**k色图**。

确定一个图的着色数是非常困难的，我们常常只能借助一些下界/上界观察：

Theorem 10.5 *For every graph G of order n,*

$$\chi(G) \geq \omega(G) \quad \textit{and} \quad \chi(G) \geq \frac{n}{\alpha(G)}.$$

Theorem 10.7 *For every graph G,*

$$\chi(G) \leq 1 + \Delta(G).$$

问题7：

Δ(G)+1这个上界是tight吗？

Theorem 10.8 (Brooks' Theorem) *For every connected graph G that is not an odd cycle or a complete graph,*

$$\chi(G) \leq \Delta(G).$$

五色定理

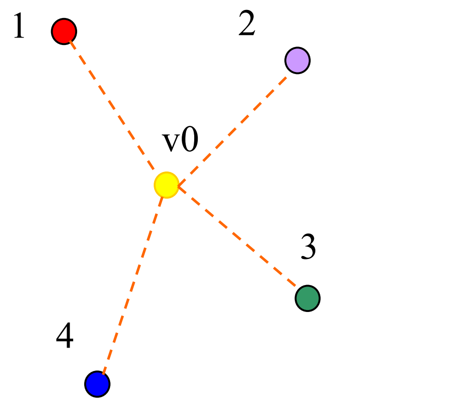
- 为了给*任何*一张平面地图着色，保证有公共边界的区域颜色不同，最多只需要五种颜色就够了。

问题8：为什么？

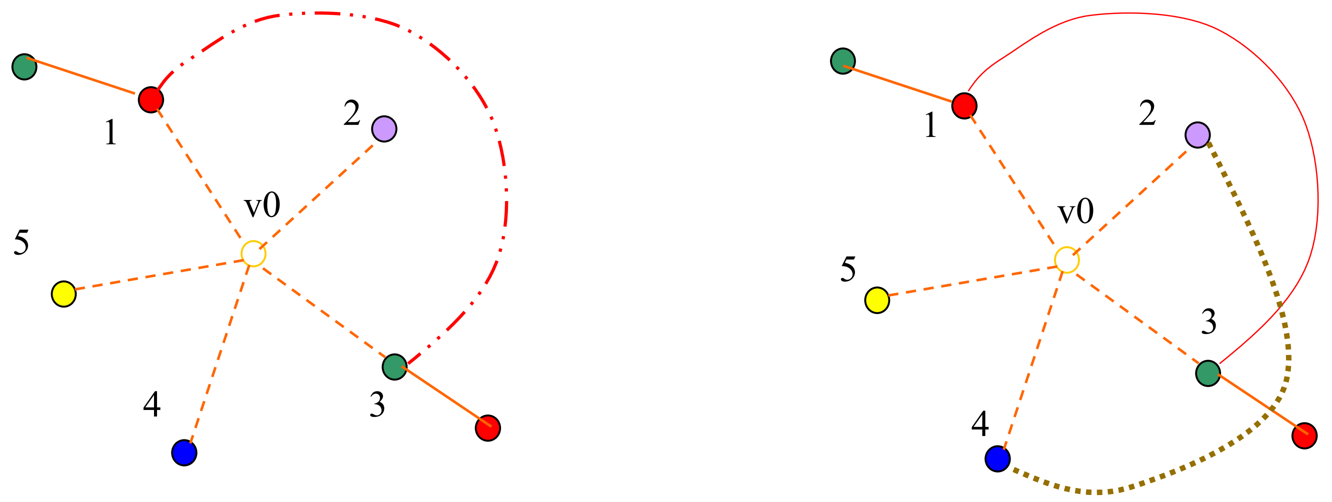
  - 证明要点：

在简单平面图中，一定存在度数不大于5的顶点。

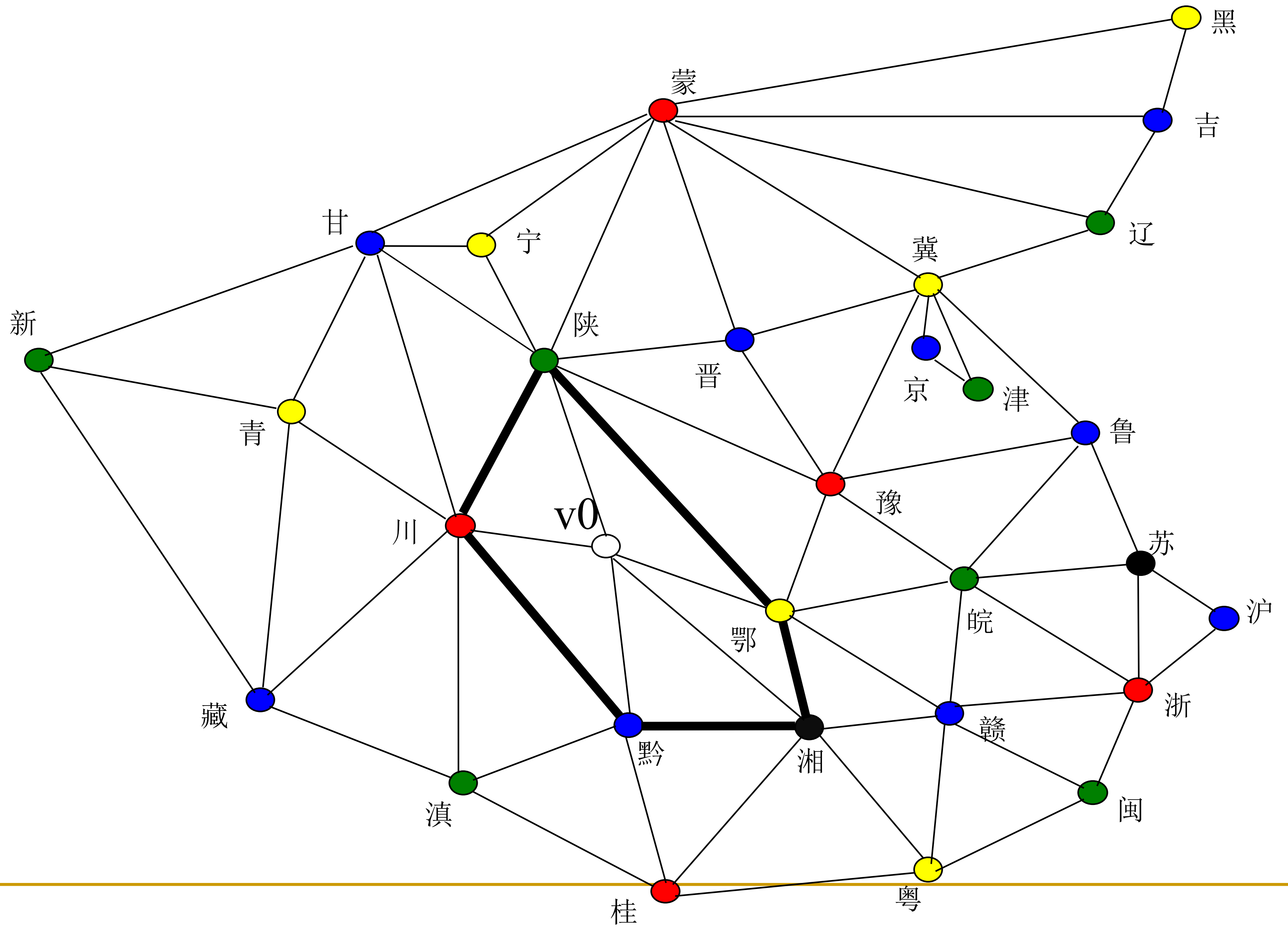
对顶点个数归纳: (当n≤5, 结论显然成立); 归纳的图示如下：



V0的度数小于等于4

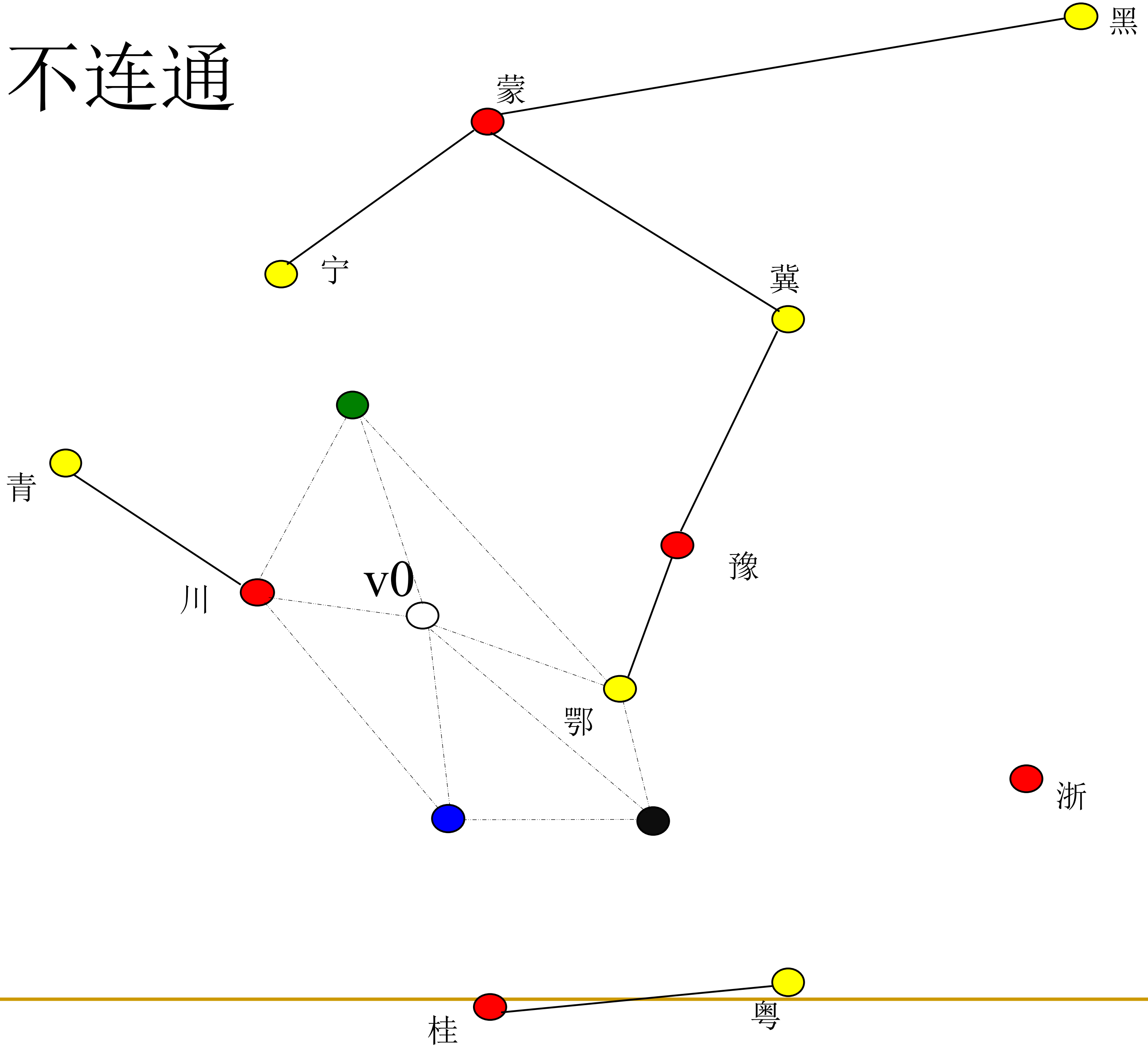


V0的度数等于5

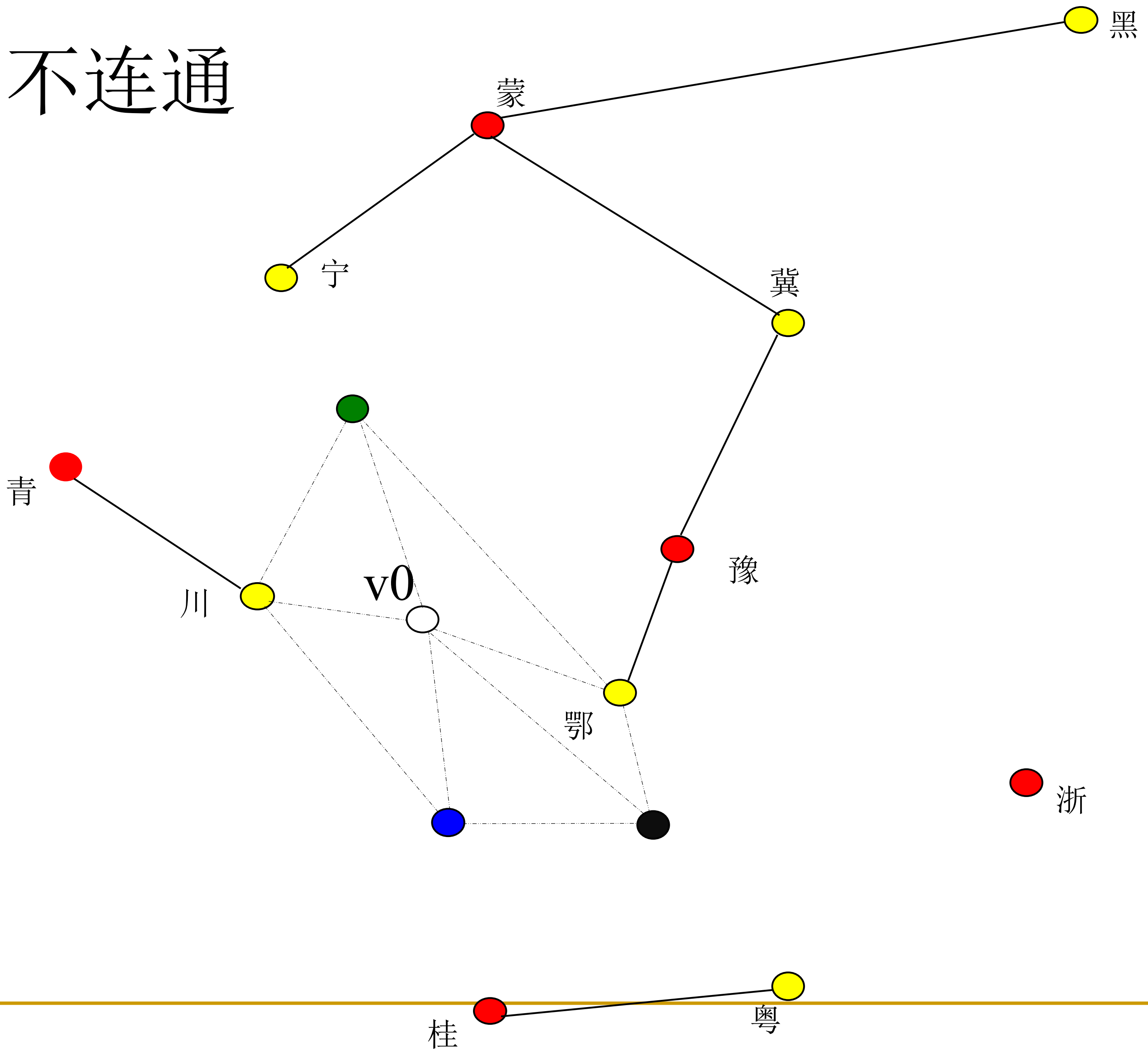


黑
蒙
吉
甘
宁
辽
冀
新
陕
晋
京
津
青
鲁
豫
v0
川
苏
沪
皖
鄂
浙
藏
赣
黔
湘
滇
闽
桂
粤

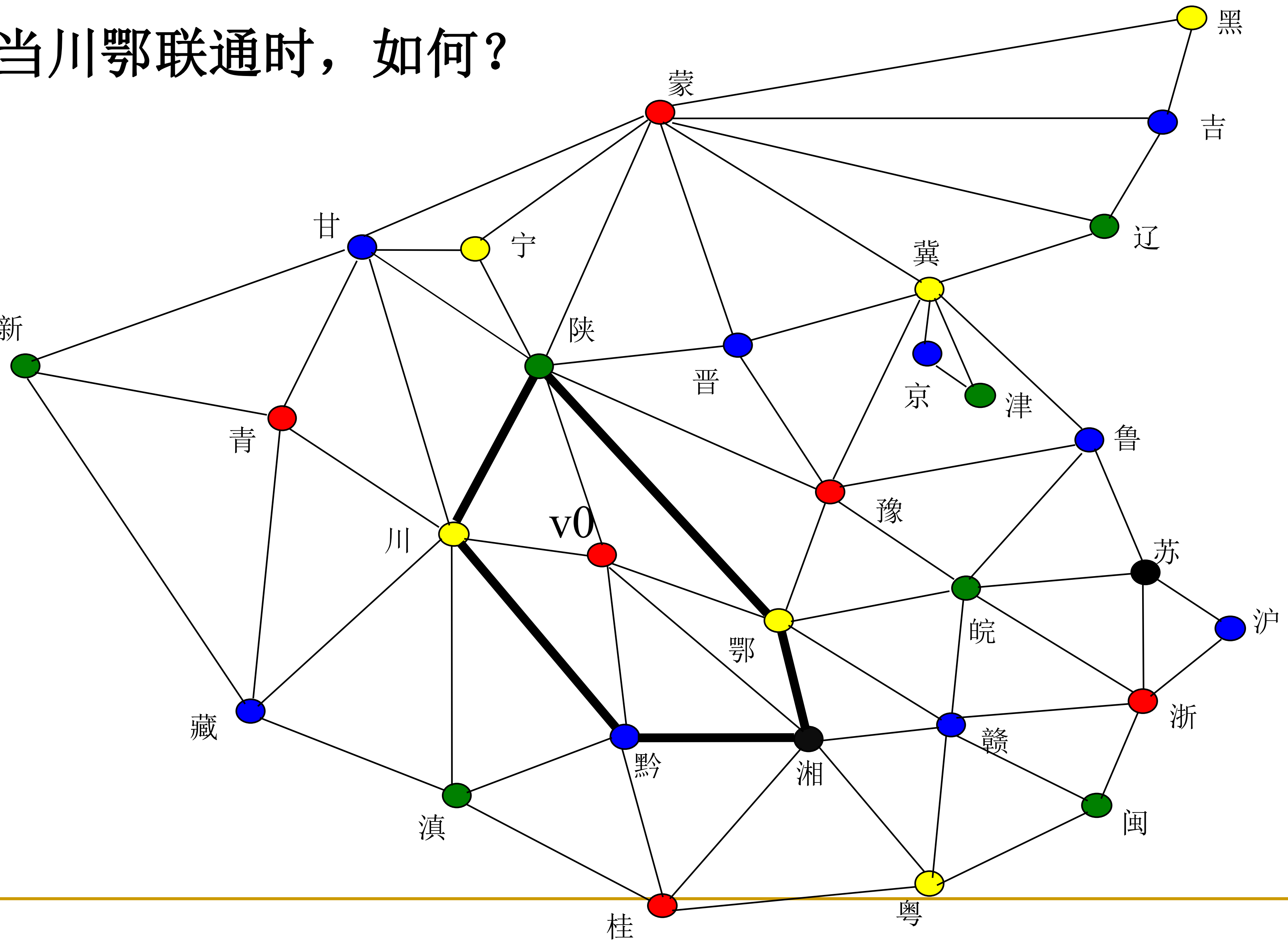
$G_{川鄂}$中，川鄂不连通

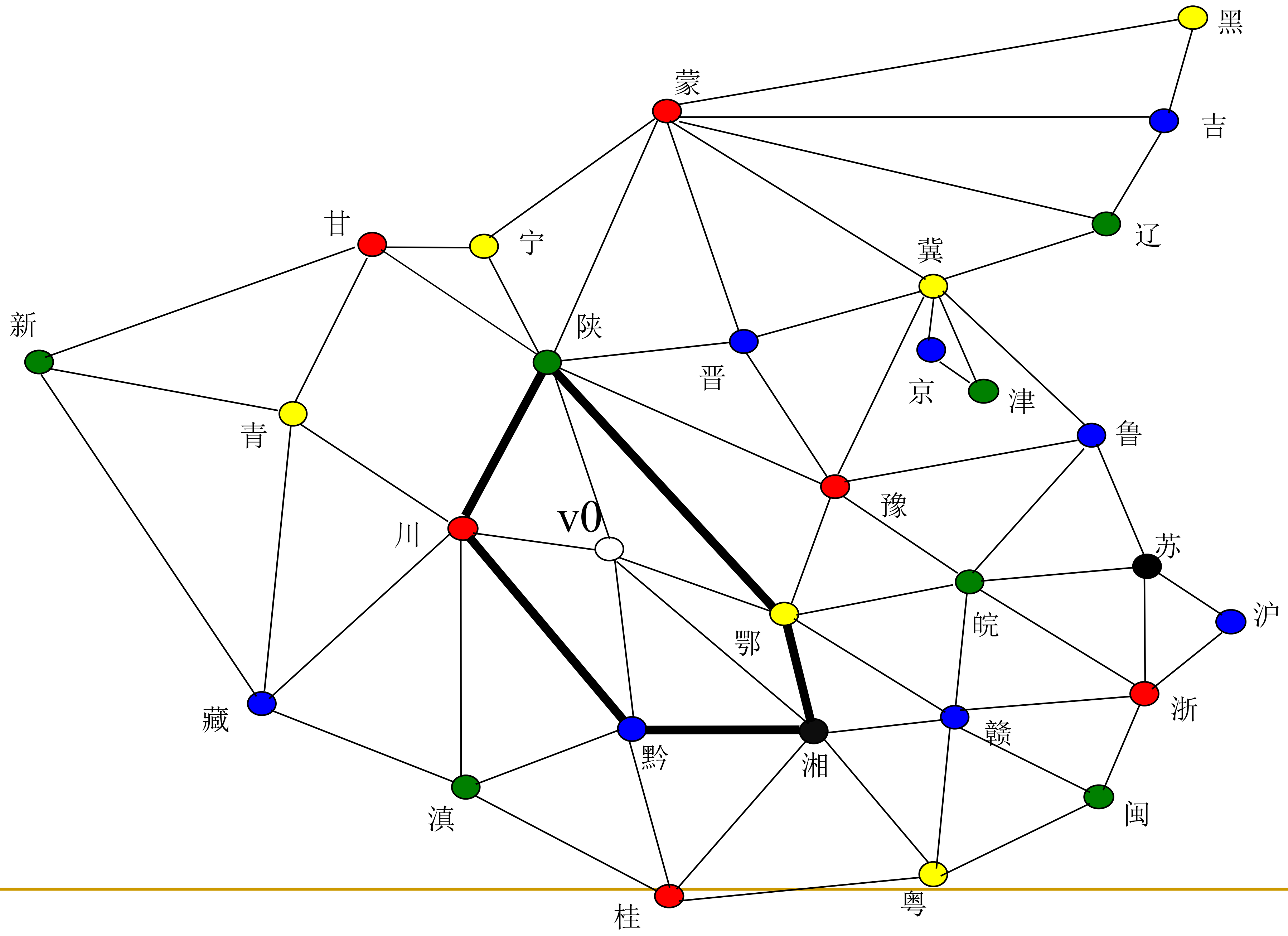


$G_{川鄂}$中，川鄂不连通



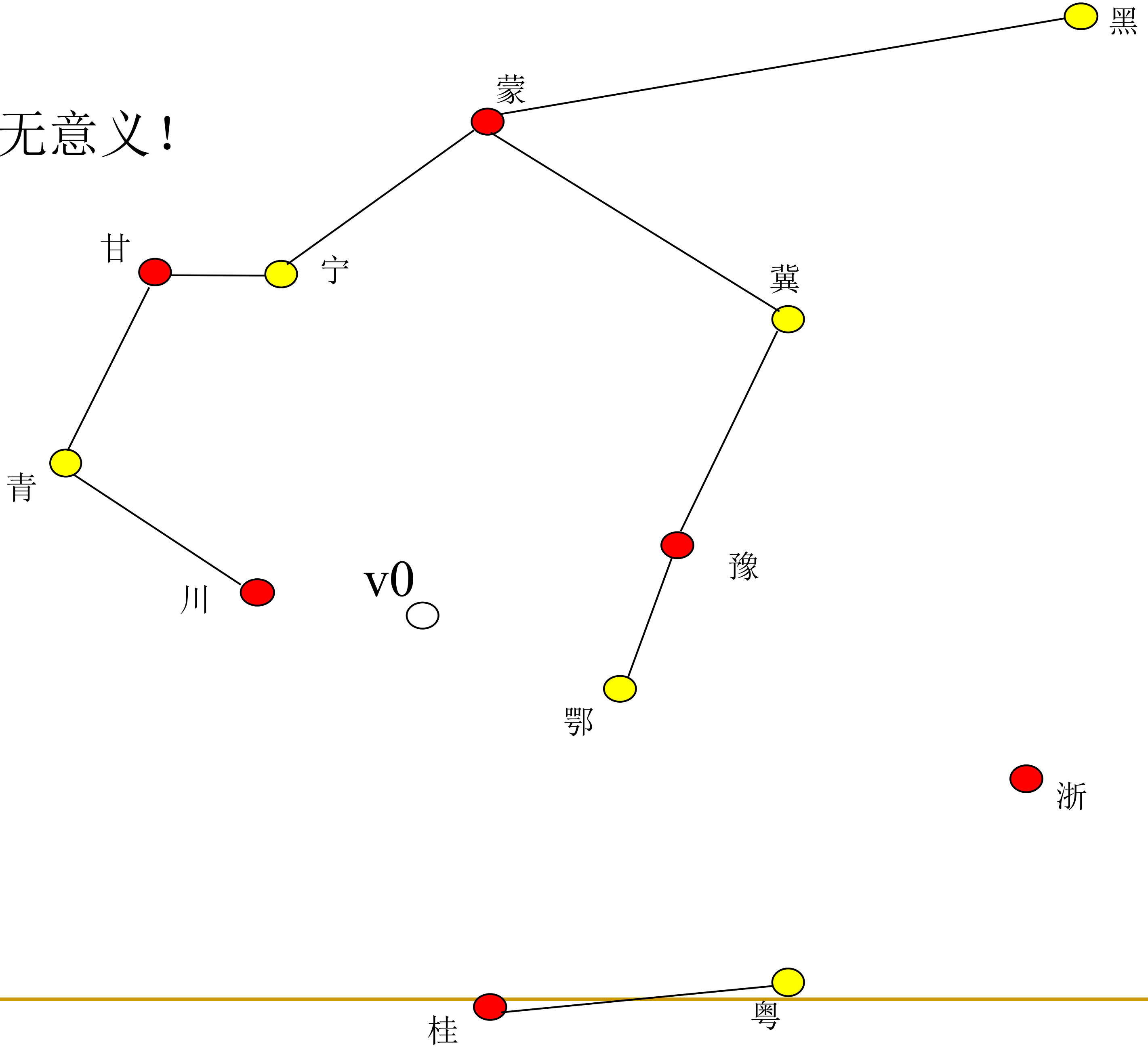
但是，当川鄂联通时，如何？



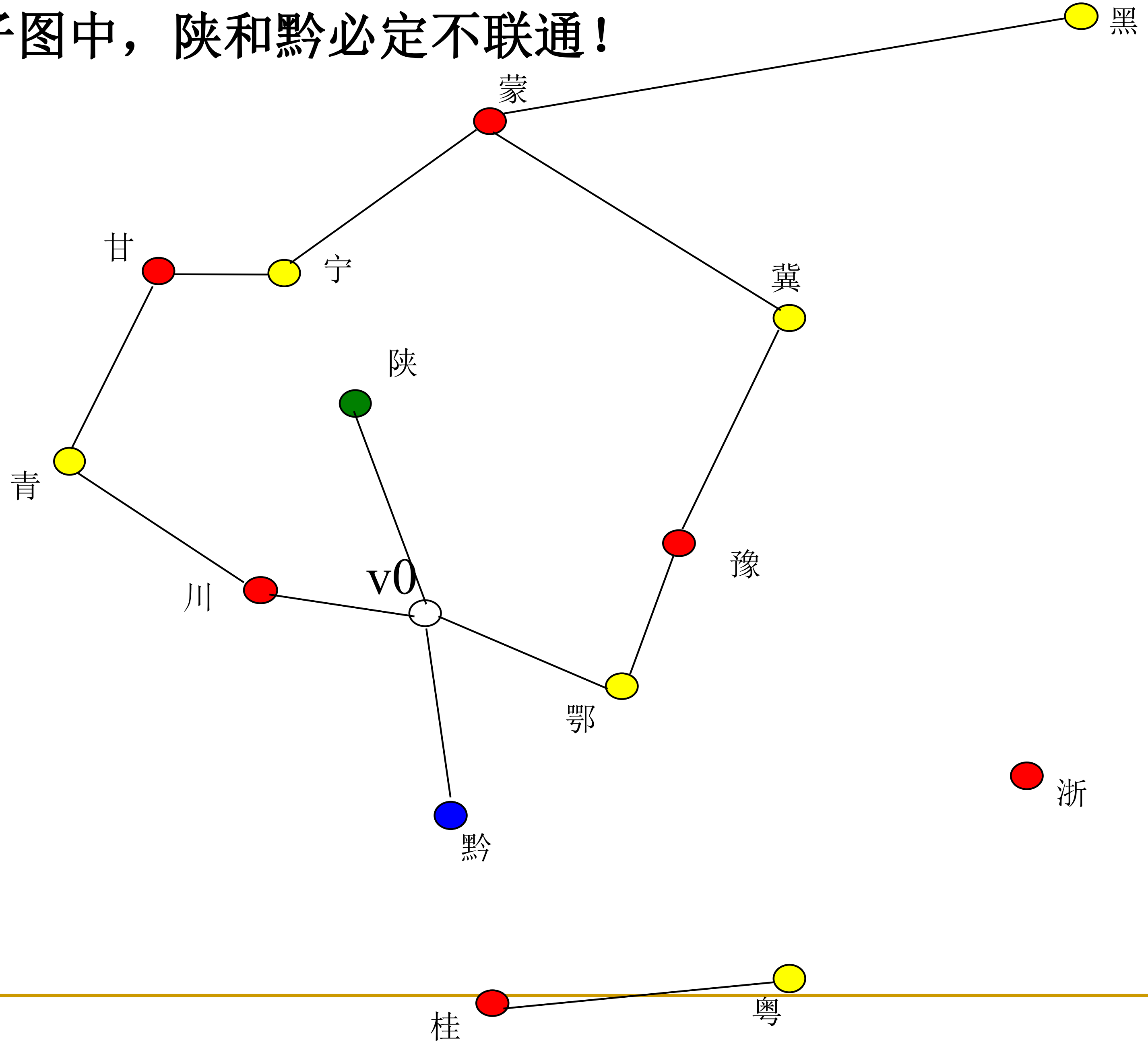


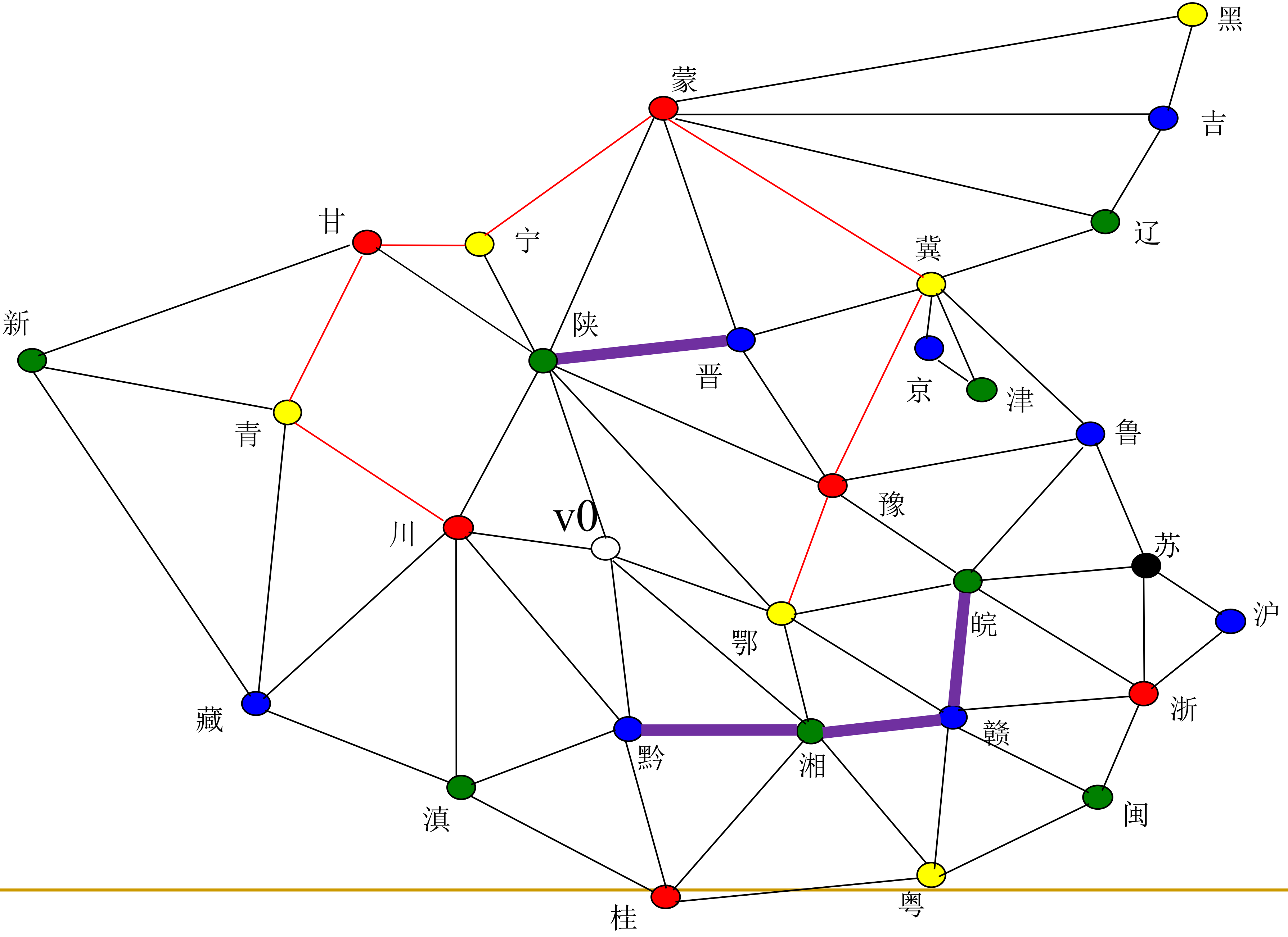
黑
蒙
吉
甘
宁
辽
冀
新
陕
晋
京
津
鲁
青
豫
v0
川
苏
沪
皖
鄂
浙
藏
赣
黔
湘
闽
滇
粤
桂

此时交换红黄颜色无意义！



此时，陕黔子图中，陕和黔必定不联通！





黑
蒙
吉
甘
宁
辽
冀
新
陕
晋
京
津
青
鲁
豫
v0
川
苏
沪
皖
鄂
浙
藏
赣
黔
湘
闽
滇
粤
桂

Open topics

- 请证明Brooks定理
- 欧拉公式在非连通图中，是什么样子的？

课外作业

- DW Ex.5.1: 1, 3, 11, 15, 29, 38, 46
- DW Ex.6.1: 1, 6, 9, 10, 35
- DW Ex.6.2: 1, 2, 4, 5
- DW Ex.7.2: 3, 4, 8, 10, 12, 17

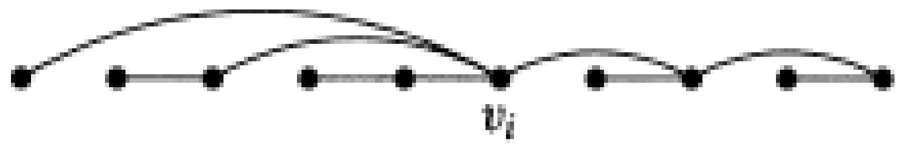
Brooks定理：case 1

5.1.22. Theorem. (Brooks [1941]) If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \le \Delta(G)$.

Proof: Let G be a connected graph, and let $k = \Delta(G)$. We may assume that $k \ge 3$, since G is a complete graph when $k \le 1$, and G is an odd cycle or is bipartite when $k = 2$, in which case the bound holds.

Our aim is to order the vertices so that each has at most $k-1$ lower-indexed neighbors; greedy coloring for such an ordering yields the bound.

When G is not k-regular, we can choose a vertex of degree less than k as v_n. Since G is connected, we can grow a spanning tree of G from v_n, assigning indices in decreasing order as we reach vertices. Each vertex other than v_n in the resulting ordering $v_1, \ldots, v_n$ has a higher-indexed neighbor along the path to v_n in the tree. Hence each vertex has at most $k-1$ lower-indexed neighbors, and the greedy coloring uses at most k colors.



Brooks定理：case 2

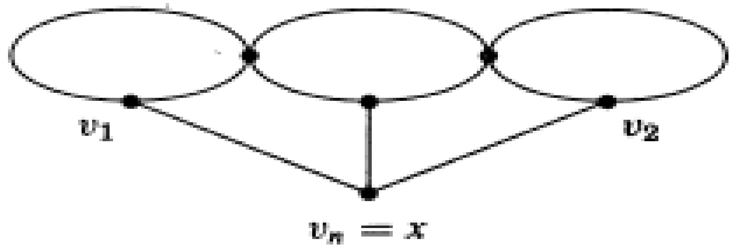
In the remaining case, G is k-regular. Suppose first that G has a cut-vertex x, and let G' be a subgraph consisting of a component of $G - x$ together with its edges to x. The degree of x in G' is less than k, so the method above provides a proper k-coloring of G'. By permuting the names of colors in the subgraphs resulting in this way from components of $G - x$, we can make the colorings agree on x to complete a proper k-coloring of G.

We may thus assume that G is 2-connected. In every vertex ordering, the last vertex has k earlier neighbors. The greedy coloring idea may still work if we arrange that two neighbors of v_n get the same color.

In particular, suppose that some vertex v_n has neighbors v_1, v_2 such that $v_1 \not\leftrightarrow v_2$ and $G - \{v_1, v_2\}$ is connected. In this case, we index the vertices of a spanning tree of $G - \{v_1, v_2\}$ using $3, \dots, n$ such that labels increase along paths to the root v_n. As before, each vertex before v_n has at most $k - 1$ lower indexed neighbors. The greedy coloring also uses at most $k - 1$ colors on neighbors of v_n, since v_1 and v_2 receive the same color.

Hence it suffices to show that every 2-connected k-regular graph with $k \geq 3$ has such a triple v_1, v_2, v_n. Choose a vertex x. If $\kappa(G - x) \geq 2$, let v_1 be x and let v_2 be a vertex with distance 2 from x. Such a vertex v_2 exists because G is regular and is not a complete graph; let v_n be a common neighbor of v_1 and v_2.

If $\kappa(G - x) = 1$, let $v_n = x$. Since G has no cut-vertex, x has a neighbor in every leaf block of $G - x$. Neighbors v_1, v_2 of x in two such blocks are nonadjacent. Also, $G - \{x, v_1, v_2\}$ is connected, since blocks have no cut-vertices. Since $k \geq 3$, vertex x has another neighbor, and $G - \{v_1, v_2\}$ is connected. ■

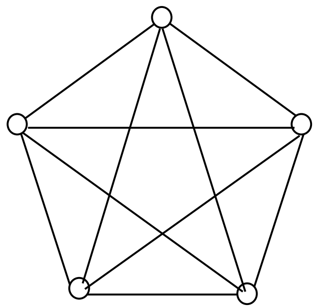


典型的非平面图

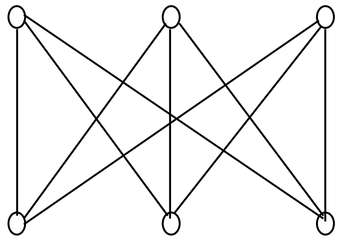
6.1.2. Proposition. K_5 and $K_{3,3}$ cannot be drawn without crossings.

Proof: Consider a drawing of K_5 or $K_{3,3}$ in the plane. Let C be a spanning cycle. If the drawing does not have crossing edges, then C is drawn as a closed curve. Chords of C must be drawn inside or outside this curve. Two chords conflict if their endpoints on C occur in alternating order. When two chords conflict, we can draw only one inside C and one outside C.

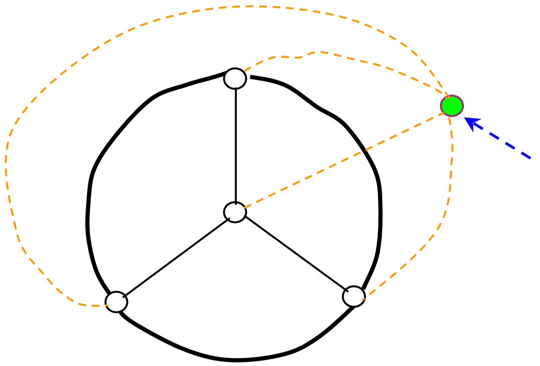
A 6-cycle in $K_{3,3}$ has three pairwise conflicting chords. We can put at most one inside and one outside, so it is not possible to complete the embedding. When C is a 5-cycle in K_5, at most two chords can go inside or outside. Since there are five chords, again it is not possible to complete the embeddings. Hence neither of these graphs is planar. ∎



K_5

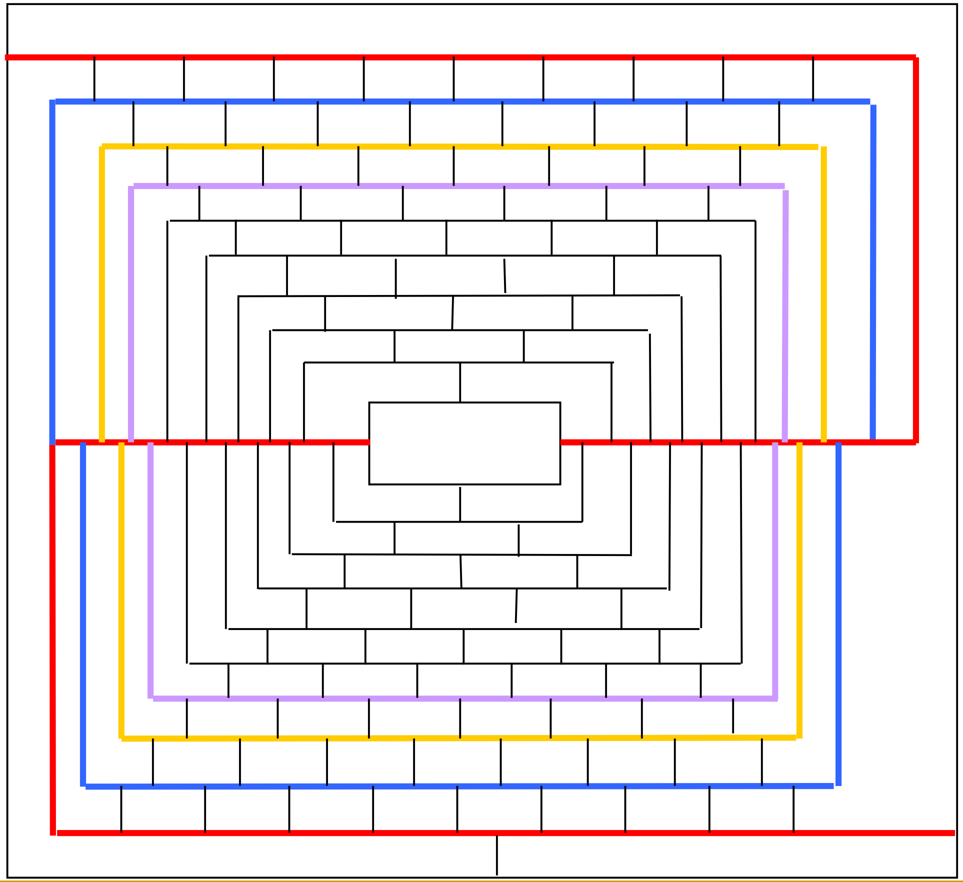


$K_{3,3}$



不管这第5个顶点加在何处，都出现Jordan条件

Martin Gardner的愚人节礼物



四色地图猜想
的一个反例
？

Scientific American
Fool’s Day of 1975