harary graph $H_{k,n}$ is k-connected

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Outline

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Definition of Harary Graph

Definition (Harary Graph)

The Harary graph $H_{k,n}$ is a particular example of a k-connected graph with n graph vertices having the smallest possible number of edges.

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In the second book on graph theory ever written, Berge' lists 14 unsolved problems, one of which is the following: "11. Queule est la connexite maximum d'un graphe de n sommets et de m arêtes? L'interît de ce problbme est analogue A celui de trouver le diambtre minimum d'un graphe." The purpose of this note is to solve the problem.

TO FIND THE MAXIMUM CONNECTIVITY OF A GRAPH

Origin of Harary Graph

Theorem

Among all p, q graphs, the maximum connectivity is 0 when $q < p - 1$ and is $\lceil 2q/p \rceil$ when $q > p - 1$.

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- There exists a p, q graph H whose connectivity is $\lceil 2q/p \rceil$.

Theorem

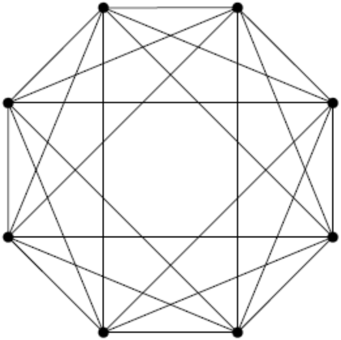
Among all p, q graphs, the maximum connectivity is 0 *when* $q < p - 1$ *and is* $\lceil 2q/p \rceil$ *when* $q > p - 1$.

- The connectivity of a p, q graph cannot exceed $\lceil 2q/p \rceil$. (Let some vertex be isolated.)
- There exists a p, q graph H whose connectivity is $\lceil 2q/p \rceil$. (Construct some graphs that satisfy the conditions——**the Harary Graph**)

Outline

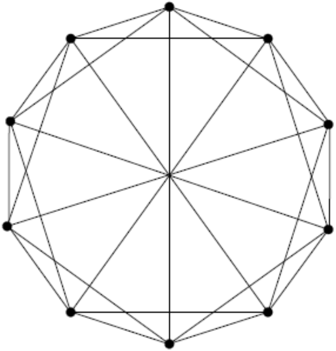
$k = 2r$ is even

- two vertices v_i and v_j are linked if and only if $i - r \leqslant j \leqslant i + r$;



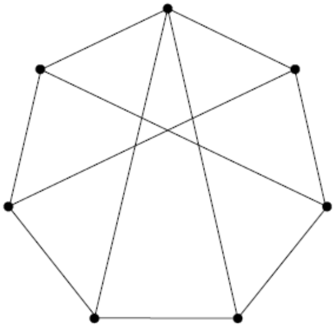
$k = 2r + 1$ is odd, and n is even

- $H_{k,n}$ is obtained by joining v_i and $v_{i+\frac{n}{2}}$ in $H_{2r,n}$ for every $i \in [0, \frac{n}{2} - 1]$;



$k = 2r + 1$ and n are both odd

- $H_{k,n}$ is obtained from $H_{2r,n}$ by first linking v_0 to both $v_{\lfloor \frac{n}{2} \rfloor}$ and $v_{\lceil \frac{n}{2} \rceil}$, and then each vertex v_i to $v_{i+\lceil \frac{n}{2} \rceil}$ for every $i \in [1, \lfloor \frac{n}{2} \rfloor - 1]$;



Outline

1 Harary Graph
- Definition
- Origin

2 Construction Method

3 Proof
- Idea of Proof
- Classified Discussion

Outline

To prove $\kappa(H_{k,n}) = k$. (CZ: Theorem 5.15)

- $\kappa(H_{k,n}) \leqslant k$
- $\kappa(H_{k,n}) \geqslant k$

Outline

reduction to absurdity:
Suppose that V is a vertex cut with $|V| < 2r$:

1. Choose vertices i and j in different components of $H_{2r,n} - V$. Divide the vertices into the following two parts:
   - $S = \{i, i+1, \ldots, j-1, j\}$
   - $T = \{j, j+1, \ldots, i-1, i\}$

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2. Since $|V| < 2r$, assume that $|V \cap S| < r$. Find that in $S \setminus V$, the difference between any two consecutive terms is at most r.

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2. Since $|V| < 2r$, assume that $|V \cap S| < r$. Find that in $S \setminus V$, the difference between any two consecutive terms is at most r.
3. Since each vertex of $H_{2r,n}$ is adjacent with the near $2r$ vertices, any two consecutive terms in $S \setminus V$ are always adjacent.

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2. Since $|V| < 2r$, assume that $|V \cap S| < r$. Find that in $S \setminus V$, the difference between any two consecutive terms is at most r.
3. Since each vertex of $H_{2r,n}$ is adjacent with the near $2r$ vertices, any two consecutive terms in $S \setminus V$ are always adjacent.
4. Thus, we have a path from i to j, which is a contradiction. i.e. $\kappa(H_{k,n}) \geqslant 2r = k$

review the proof just now, we find that:

Lemma

In the case of $H_{2r,n}$, it is necessary (and sufficient) to remove two separate subsets of r consecutive vertices each, along the circumference of the polygon.

1. Similar to the case $k = 2r$, we have that $\kappa(H_{k,n}) \geqslant 2r$.

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2. When $|V| = 2r$, we must remove r consecutive vertices from S according to the above lemma.

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1. Similar to the case $k = 2r$, we have that $\kappa(H_{k,n}) \geqslant 2r$.
2. When $|V| = 2r$, we must remove r consecutive vertices from S according to the above lemma.
3. Since we also join pairs of vertices which are diametrically opposite, at least one more vertex must also be removed to break the diameteric connection.
   i.e. $\kappa(H_{k,n}) \geqslant 2r + 1 = k$

$k = 2r + 1$ and n are both odd

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1. The proof is similar to the case that $k = 2r + 1$ is odd and n is even.
2. In the last case, we need to remove one extra vertex to break the diameteric connection.
   While in this case, we need to remove one extra vertex to break the connection between V_i and $V_{i+\lfloor \frac{n}{2} \rfloor}$.
3. So, we can also get that $\kappa(H_{k,n}) \geqslant 2r + 1 = k$

From the proof, we sense that the construction method can be more universe.

Lemma

To constuct an $H_{2r+1,n}$, we only need to adjoin some random edges in one $H_{2r,n}$, so that each vertex in the graph is adjacent some vertex that is not its near $2r$ vertices.

Reference Material

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- Olivier Baudon, Julien Bensmail, Eric Sopena. Partitioning Harary graphs into connected subgraphs containing prescribed vertices.
- J. A. Bondy, U. S. R. Murty. Graph theory with application.