3-7 Relax! We are SSSP Algorithms.

Hengfeng Wei

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Definition (Shortest Path)

$$G = (V, E, w) : \text{ weighted digraph}$$

$$\delta(u, v) = \begin{cases} \min \{w(p) : u \overset{p}{\rightsquigarrow} v\} & \text{if } u \rightsquigarrow v \\ \infty & \text{o.w.} \end{cases}$$

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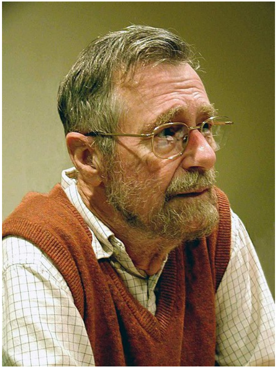
Path *vs.* Simple path



For fundamental contributions to programming as a high, intellectual challenge;

for eloquent insistence and practical demonstration that programs should be composed correctly, not just debugged into correctness;

for illuminating perception of problems at the foundations of program design.

— *Turing Award*, 1972

```
procedure DIJKSTRA(G, w, s)
    INIT-SINGLE-SOURCE(G, s)
    S = ∅
    Q = G.V
    while Q ≠ ∅ do
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        S ← S ∪ {u}
        for v ∈ G.Adj[u] do
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Array: $O(n^2)$

Min-heap: $O(E \log V)$

Fib-heap: $O(V \log V + E)$

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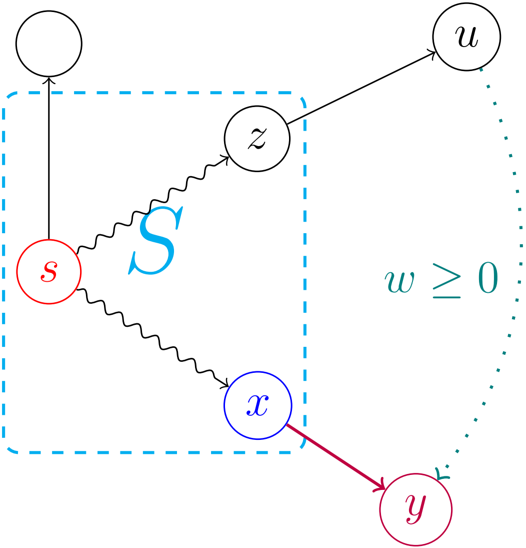
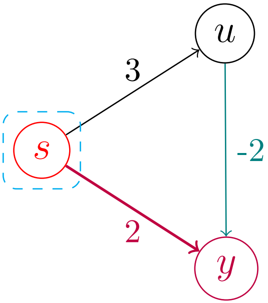

u
z
S
s
w ≥ 0
x
y

Negative-weight Edges for Dijkstra's Algorithm (Problem 24.3-2)



Negative-weight Edges for Dijkstra's Algorithm (Additional Problem 24.3-10)

- All negative-weight egdes are from s
- No negative-weight cycles

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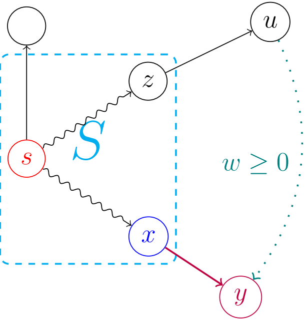

Checking Output of Dijkstra's Algorithm (Problem 24.3-4)

$$\forall v \in V : v.\pi, v.d$$

To check whether π and d match some shortest-paths tree?

$$O(V + E)$$

(1) π forms a tree

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(2) $s.d = 0$

$$(1)\ \pi \text{ forms a tree}$$

$$(2)\ s.d = 0$$

$$\boxed{u \triangleq v.\pi}$$

$$(3)\ \forall v \in V : v.d = u.d + w(u, v)$$

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$$(3)\ \forall v \in V: v.d = u.d + w(u,v)$$

$$(4)\ \forall v \in V: u.d + w(u,v) = \min_{(v',v)\in E} \left\{ v'.d + w(v',v) \right\}$$

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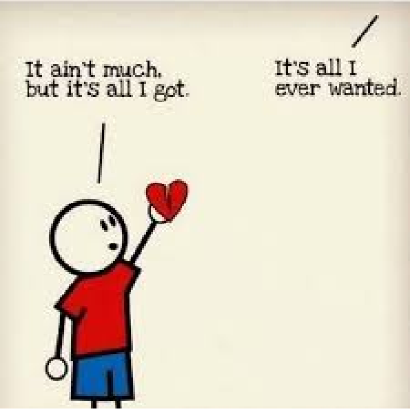
$$(4)\ \forall (v', v) \in E : v'.d + w(v', v) \geq v.d$$


It ain't much,
but it's all I got.
It's all I
ever wanted.

$$\boxed{\forall v \in V : v.d = \delta(s, v)}$$

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$$\exists v \in V : v.d \neq \delta(s, v)$$

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$$v.d < \delta(s, v)$$

$$\begin{aligned} v.d &= u.d + w(u, v) \\ &< \delta(s, v) \\ &\leq \delta(s, u) + w(u, v) \end{aligned}$$

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$$\exists v \in V : v.d \neq \delta(s, v)$$

$v.d < \delta(s, v)$ $\qquad$ $v.d > \delta(s, v)$

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$$\begin{aligned} v.d &= u.d + w(u, v) \\ &< \delta(s, v) \\ &\leq \delta(s, u) + w(u, v) \end{aligned}$$

$$v.d = u.d + w(u, v) > \delta(s, v)$$

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$$\begin{cases} u.d = \delta(s, u) & \boxed{v.\pi} \\ \boxed{u.d > \delta(s, u)} \end{cases}$$

Lawler's Algorithm on DAG

$$\Longleftarrow$$

Dijkstra's Algorithm on Digraph with Nonnegative-weight Edges

$$\Longrightarrow$$

Bellman-Ford Algorithm on Digraph with Negative-weight Edges

```
procedure DAG-SSSP(G, w, s)
    INIT-SINGLE-SOURCE(G, s)
    TOPO-SORT(G)
    for u ∈ V in topo. order do
        for v ∈ G.Adj[u] do
            RELAX(u, v, w)
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    INIT-SINGLE-SOURCE(G, s)
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$$\Theta(V + E)$$

procedure DAG-SSSP(G, w, s)
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procedure Dijkstra(G, w, s)
  Init-Single-Source(G, s)
  $Q = G.V$
  while $Q \neq \emptyset$ **do**
    $u \leftarrow$ Extract-Min(Q)
    for $v \in G.Adj[u]$ **do**
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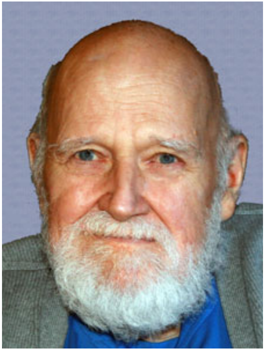
procedure DIJKSTRA(G, w, s)
    INIT-SINGLE-SOURCE(G, s)
    Q = G.V
    while Q ≠ ∅ do
        u ← EXTRACT-MIN(Q)
        for v ∈ G.Adj[u] do
            RELAX(u, v, w)
```

Q : Why is $\delta(s, u)$ determined right now?

Little Modification to DAG-SSSP (Problem 24.2-2)

procedure DAG-SSSP(G, w, s)
  INIT-SINGLE-SOURCE(G, s)
  TOPO-SORT(G)
  for the first $|V| - 1$ vertices $u \in V$ in topo. order **do**
    for $v \in G.Adj[u]$ **do**
      RELAX(u, v, w)

Richard Bellman (1920—1984)



Lester Randolph Ford Jr. (1927—2017)

```
procedure BELLMAN-FORD(G, w, s)
    INIT-SINGLE-SOURCE(G, s)

    for i ← 1 to |V| − 1 do
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    for (u, v) ∈ E do
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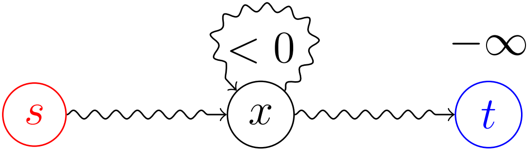
            return FALSE

    return TRUE
```

Deal with Negative-weight Cycles (Problem 24.1-4)



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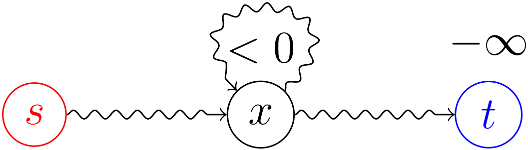

---

procedure BELLMAN-FORD-NC-WRONG(G, w, s)
    INIT-SINGLE-SOURCE(G, s)
    for $i \leftarrow 1$ **to** $|V| - 1$ **do**
        for $(u, v) \in E$ **do**
            RELAX(u, v, w)
    for $(u, v) \in E$ **do**
        if $v.d > u.d + w(u, v)$ **then**
            $v.d = -\infty$

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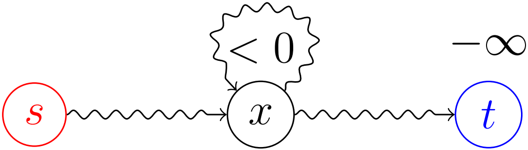


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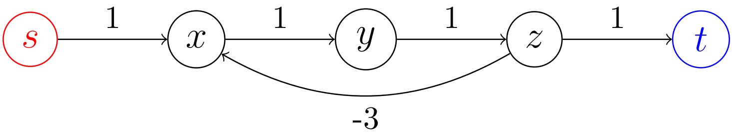
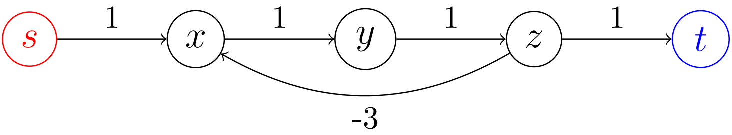
```


s
1
x
1
y
1
z
1
t
-3

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    for i ← 1 to |V| do                                ▷
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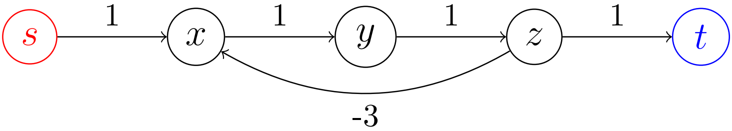

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    for $i \leftarrow 1$ **to** $|V|$ **do** ▷ $\Theta(VE)$
        for $(u, v) \in E$ **do**
            if $v.d > u.d + w(u, v)$ **then**
                $v.d = -\infty$

$$O(V + E)$$

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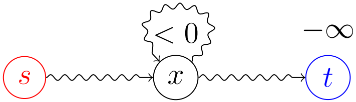

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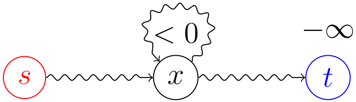
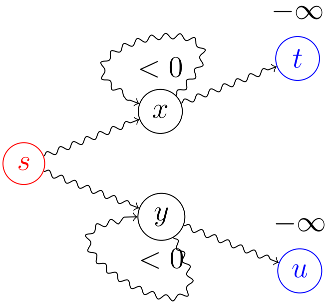
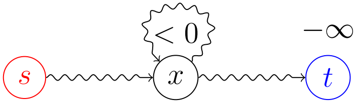
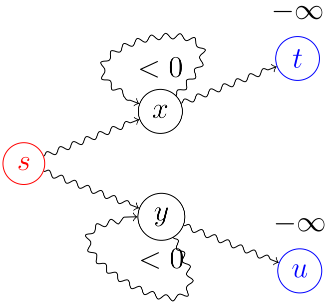




$$O(V+E)$$





Theorem (The $|V|$-th Pass of Bellman-Ford Algorithm)

For every reachable negative-weight cycle, at least one edge of it has been relaxed in the $|V|$-th pass.

Terminate Early in Bellman-Ford Algorithm (Problem 24.1-3)

$$G = (V, E) \text{ without negative-weight cycles}$$

$$m \triangleq \min_{v \in V} \left\{ \text{Len}\big(\delta(s, v)\big) \right\} \text{ (Unknown!)}$$

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procedure BELLMAN-FORD(G, w, s)
    INIT-SINGLE-SOURCE(G, s)

    f ← FALSE
    for i ← 1 to |V| − 1 do
        for (u, v) ∈ E do
            if v.d > u.d + w(u, v) then
                v.d = u.d + w(u, v)
                f ← TRUE

        if f = FALSE then
            return
```

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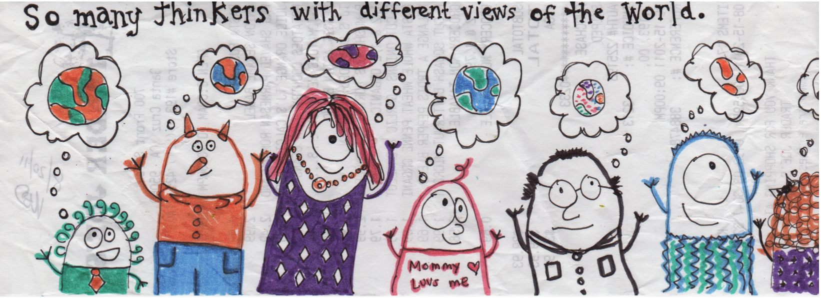
  Init-Single-Source(G, s)
  $f \leftarrow$ False
  for $i \leftarrow 1$ **to** $|V| - 1$ **do**
    for $(u, v) \in E$ **do**
      if $v.d > u.d + w(u, v)$ **then**
        $v.d = u.d + w(u, v)$
        $f \leftarrow$ True
    if $f =$ False **then**
      return

Two Different Views of Bellman-Ford Algorithms



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```

Bellman-Ford Algorithm $\equiv$ Dijkstra's Algorithm with QUEUE

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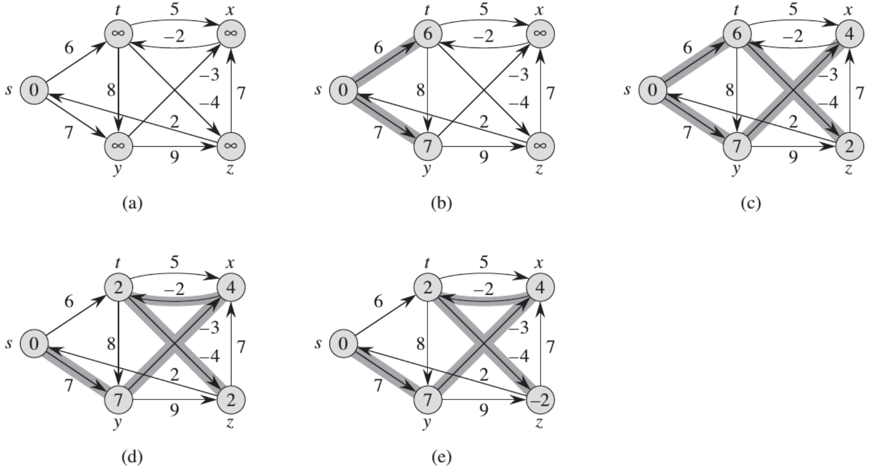

(a) (b) (c) (d) (e)

$\boxed{\text{Bellman-Ford Algorithm is a DP Algorithm.}}$

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$d[i, v]$: the length of the shortest path $s \rightsquigarrow v$ consisting of $\leq i$ edges

Bellman-Ford Algorithm is a DP Algorithm.

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$$d[i, v] = \begin{cases} 0 & i = 0 \land v = s \\ \infty & i = 0 \land v \neq s \\ \min\Big\{ d[i-1, v], \min\limits_{(u,v) \in E} \{ d[i-1, u] + w(u, v) \} \Big\} & \text{o.w.} \end{cases}$$

```
procedure BELLMAN-FORD-DP(G, w, s)
    d[0, s] ← 0
    for (v ≠ s) ∈ V do
        d[0, v] ← ∞

    for i ← 1 to |V| − 1 do
        for v ∈ V do
            d[i, v] = d[i − 1, v]
            for (u, v) ∈ E do
                if d[i − 1, v] > d[i − 1, u] + w(u, v) then
                    d[i, v] = d[i − 1, u] + w(u, v)
```

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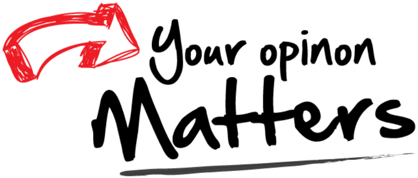
        for v ∈ V do
            d[i, v] = d[i − 1, v]
            for (u, v) ∈ E do
                if d[i − 1, v] > d[i − 1, u] + w(u, v) then
                    d[i, v] = d[i − 1, u] + w(u, v)
```

$$Q : d[i, v] \implies d[v]?$$


THANK YOU



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