- 教材讨论
  - TC第34章

问题1：判定问题和优化问题

- Although P, NP, and NP-complete problems are confined to the realm of decision problems, we can take advantage of a convenient relationship between optimization problems and decision problems.
  - 你怎么理解这句话？
  - 以下这些优化问题对应的判定问题分别是什么？
    - traveling salesperson problem
    - set cover problem
    - knapsack problem
  - 优化问题和对应的判定问题哪个更难？

问题1：判定问题和优化问题 (续)

- 优化问题有自己的“NP”和“P”，你理解了吗？

Definition 2.3.3.21. NPO *is the class of optimization problems, where* $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal) \in \mathrm{NPO}$ *if the following conditions hold:*

(i) $L_I \in \mathrm{P}$,
(ii) there exists a polynomial p_U *such that*
  a) for every $x \in L_I$, *and every* $y \in \mathcal{M}(x)$, $|y| \leq p_U(|x|)$, *and*
  b) there exists a polynomial-time algorithm that, for every $y \in \Sigma_O^*$ *and every* $x \in L_I$ *such that* $|y| \leq p_U(|x|)$, *decides whether* $y \in \mathcal{M}(x)$, *and*
(iii) the function cost is computable in polynomial time.

Informally, we see that an optimization problem U is in NPO if

(i) one can efficiently verify whether a string is an instance of U,
(ii) the size of the solutions is polynomial in the size of the problem instances and one can verify in polynomial time whether a string y is a solution to any given input instance x, and
(iii) the cost of any solution can be efficiently determined.

Definition 2.3.3.23. PO *is the class of optimization problems* $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal)$ *such that*

(i) $U \in \mathrm{NPO}$, *and*
(ii) there is a polynomial-time algorithm that, for every $x \in L_I$, *computes an optimal solution for* x.

问题2：P

- 关于P，你如何理解这几句话：
  - Very few practical problems require time on the order of a high-degree polynomial.
  - For many reasonable models of computation, a problem that can be solved in polynomial time in one model can be solved in polynomial time in another.
    - 特别地，对于parallel computer，为什么要强调the number of processors grows polynomially with the input size
  - The class of polynomial-time solvable problems has nice closure properties.

问题2：P (续)

- 为什么关于复杂性的讨论会涉及到编码？你能举个例子吗？
- 如何将编码从复杂性的讨论中隔离出去？
  - Rule out “expensive” encodings.
  - If two encodings e_1 and e_2 of an abstract problem are polynomially related, whether the problem is polynomial-time solvable or not is independent of which encoding we use.

问题2：P (续)

- decide in polynomial time和accept in polynomial time的区别是什么？
  - To accept a language, an algorithm need only produce an answer when provided a string in L, but to decide a language, it must correctly accept or reject every string in {0, 1}*.

- 联系又是什么？为什么会有这种联系？

$$\mathrm{P} = \{L \subseteq \{0,1\}^* : \text{there exists an algorithm } A \text{ that decides } L \text{ in polynomial time}\} .$$

$$\mathrm{P} = \{L : L \text{ is accepted by a polynomial-time algorithm}\} .$$

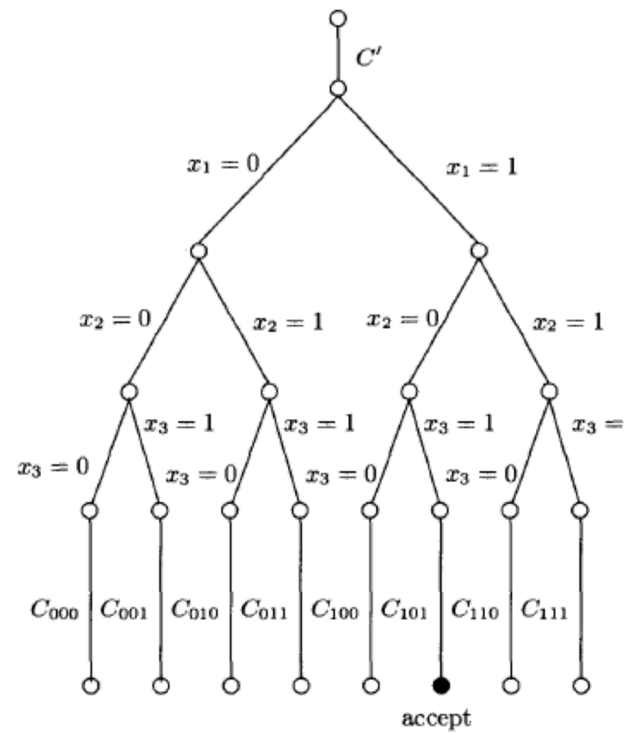
- 在模拟接受算法的判定算法中，如果接受算法不停机，判定算法可以在什么时候停下来？

问题3：NP

- 你怎么理解certificate？
- 这些NP问题中的certificate是什么？如何来验证？
  - longest simple path
  - Hamiltonian cycle
  - 3-CNF satisfiability
  - circuit satisfiability
  - formula satisfiability
  - clique problem
  - vertex cover problem
  - traveling salesman problem
  - subset sum problem
  - primality testing
    - http://en.wikipedia.org/wiki/Primality_certificate

问题3：NP (续)

- 你能结合这个例子，从以下两个方面来解释P和NP吗？
  - 可解 vs. 可验证
  - 确定性 vs. 非确定性
- 因此，为什么P⊆NP？

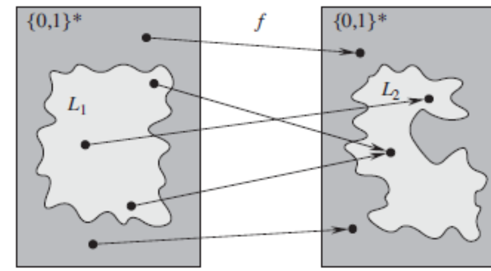


问题4：NP-hard与NPC

- 你怎么理解L_1 is polynomial-time reducible to L_2？

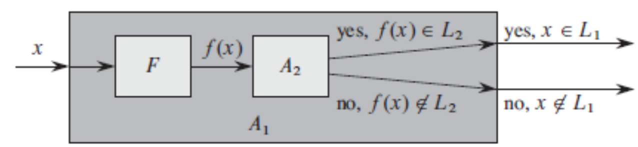
  if there exists a polynomial-time computable function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ such that for all $x \in \{0,1\}^*$,

  $x \in L_1$ if and only if $f(x) \in L_2$.



- 作为一种二元关系，$\le_P$具有什么性质？
- 为什么这条引理成立？

  Lemma 34.3
  If $L_1, L_2 \subseteq \{0,1\}^*$ are languages such that $L_1 \le_P L_2$, then $L_2 \in$ P implies $L_1 \in$ P.



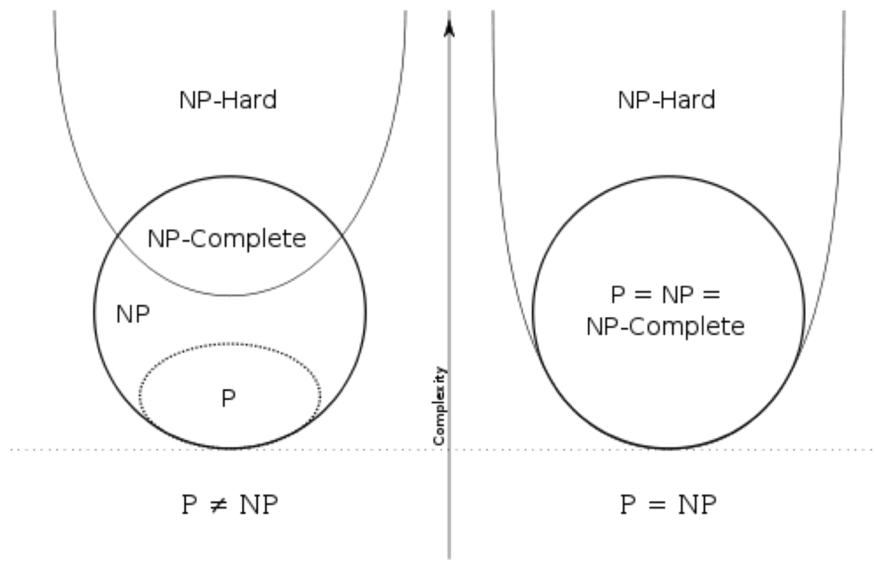
- 如何基于$\le_P$来定义NP-hard？

  $L' \le_P L$ for every $L' \in$ NP

- NPC又是怎么定义的？为什么这条定理成立？

  Theorem 34.4
  If any NP-complete problem is polynomial-time solvable, then P = NP.

问题4：NP-hard与NPC (续)

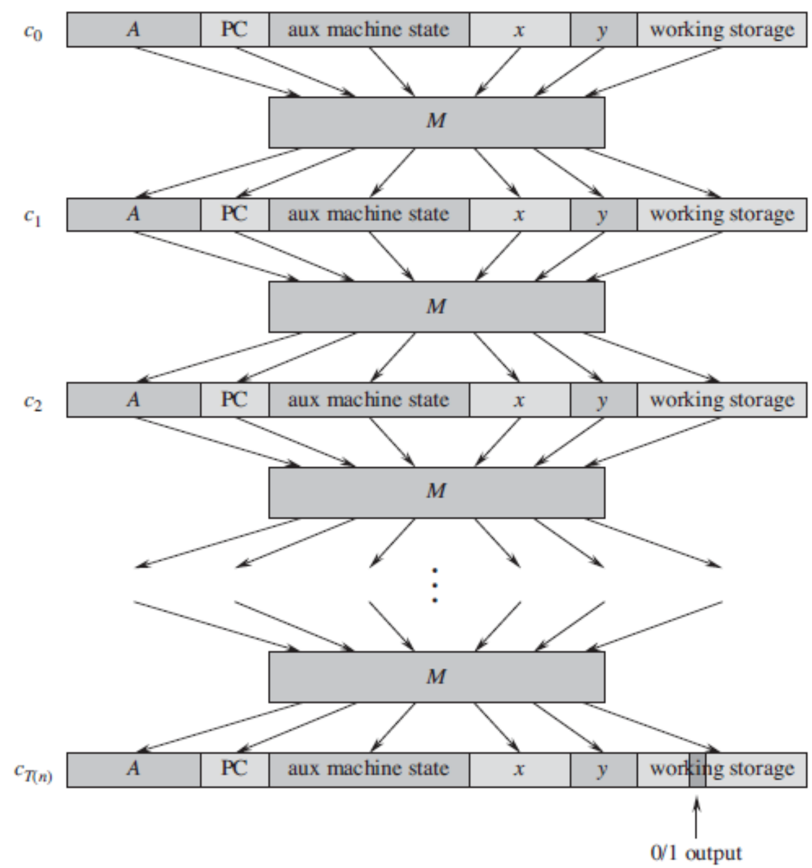


问题4：NP-hard与NPC (续)

- 你理解这条引理的证明了吗？

  Lemma 34.6
  The circuit-satisfiability problem is NP-hard.

  – 如何建立reduction？
    - 电路的内部结构是什么？
    - 电路的输入输出是什么？
  – 如何证明它是polynomial-time computable？

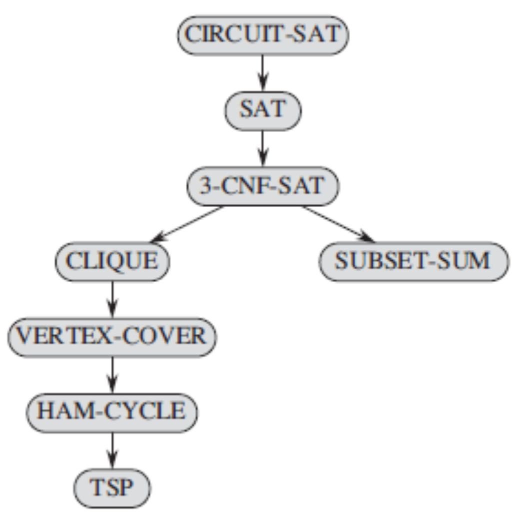


问题4：NP-hard与NPC (续)

- 证明NPC的一般方法是什么？

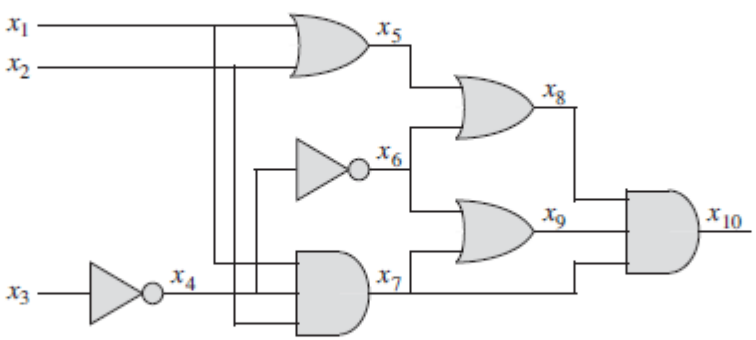
1. Prove $L \in \text{NP}$.
2. Select a known NP-complete language L'.
3. Describe an algorithm that computes a function f mapping every instance $x \in \{0,1\}^*$ of L' to an instance $f(x)$ of L.
4. Prove that the function f satisfies $x \in L'$ if and only if $f(x) \in L$ for all $x \in \{0,1\}^*$.
5. Prove that the algorithm computing f runs in polynomial time.

- 这张图是什么含义？



问题4： NP-hard与NPC (续)

- 如何将circuit satisfiability归约到formula satisfiability，从而证明后者是NP-hard？

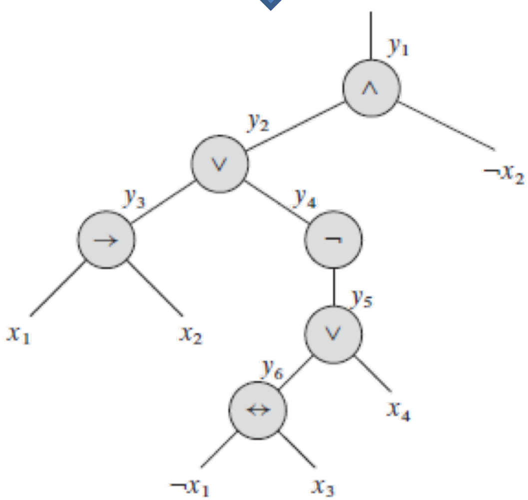


$$
\begin{aligned}
\phi = x_{10} &\wedge (x_4 \leftrightarrow \neg x_3) \\
&\wedge (x_5 \leftrightarrow (x_1 \vee x_2)) \\
&\wedge (x_6 \leftrightarrow \neg x_4) \\
&\wedge (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)) \\
&\wedge (x_8 \leftrightarrow (x_5 \vee x_6)) \\
&\wedge (x_9 \leftrightarrow (x_6 \vee x_7)) \\
&\wedge (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9)) .
\end{aligned}
$$

问题4： NP-hard与NPC (续)

- 如何将formula satisfiability归约到3-CNF satisfiability，从而证明后者是NP-hard？

$$\phi = ((x_1 \to x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2 .$$



$$\begin{aligned}
\phi' \ = \ & y_1 \wedge (y_1 \leftrightarrow (y_2 \wedge \neg x_2)) \\
& \wedge (y_2 \leftrightarrow (y_3 \vee y_4)) \\
& \wedge (y_3 \leftrightarrow (x_1 \to x_2)) \\
& \wedge (y_4 \leftrightarrow \neg y_5) \\
& \wedge (y_5 \leftrightarrow (y_6 \vee x_4)) \\
& \wedge (y_6 \leftrightarrow (\neg x_1 \leftrightarrow x_3)) .
\end{aligned}$$

y_1	y_2	x_2	$(y_1 \leftrightarrow (y_2 \wedge \neg x_2))$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

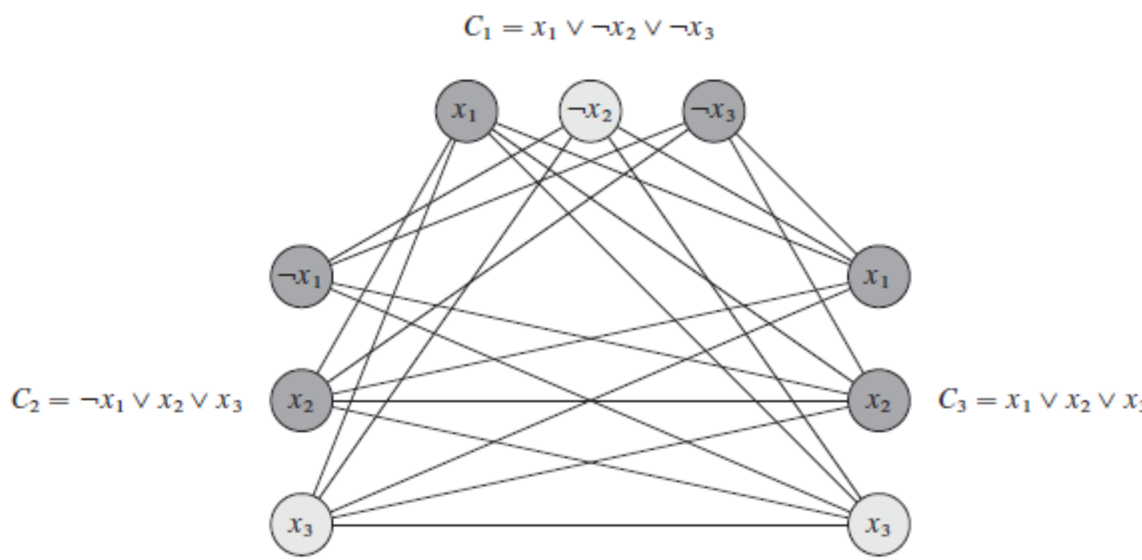
$$\begin{aligned}
\phi''_1 \ = \ & (\neg y_1 \vee \neg y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_2) \\
& \wedge (\neg y_1 \vee y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee x_2) ,
\end{aligned}$$

If C_i has 2 distinct literals, that is, if $C_i = (l_1 \vee l_2)$, where l_1 and l_2 are literals, then include $(l_1 \vee l_2 \vee p) \wedge (l_1 \vee l_2 \vee \neg p)$ as clauses of ϕ'''.
If C_i has just 1 distinct literal l, then include $(l \vee p \vee q) \wedge (l \vee p \vee \neg q) \wedge (l \vee \neg p \vee q) \wedge (l \vee \neg p \vee \neg q)$ as clauses of ϕ'''.

问题4：NP-hard与NPC (续)

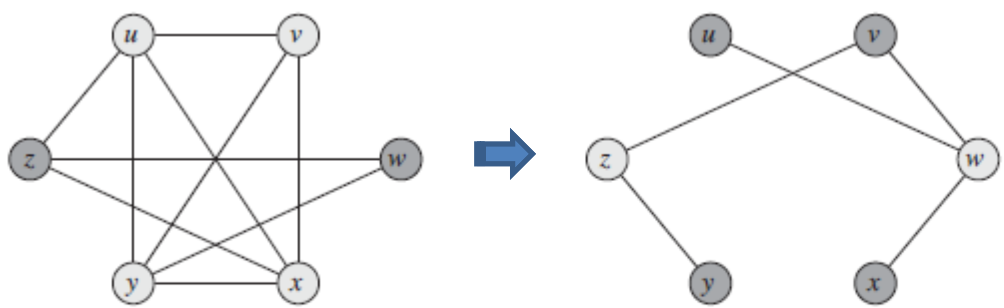
- 如何将3-CNF satisfiability归约到clique problem，从而证明后者是NP-hard？

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$



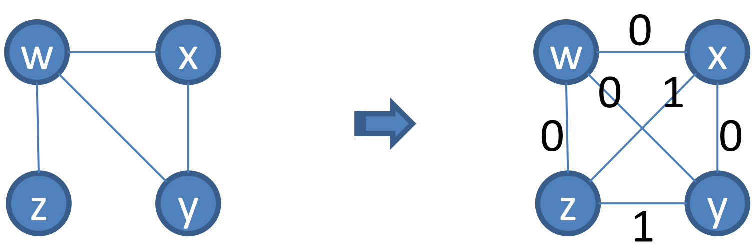
问题4： NP-hard与NPC (续)

- 如何将clique problem归约到vertex cover problem，从而证明后者是NP-hard？



问题4：NP-hard与NPC (续)

- 如何将hamiltonian cycle归约到traverling salesman problem，从而证明后者是NP-hard？



问题4： NP-hard与NPC (续)

- 如何将3-CNF satisfiability归约到subset sum problem，从而证明后者是NP-hard？

$\phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$, where $C_1 = (x_1 \vee \neg x_2 \vee \neg x_3)$, $C_2 = (\neg x_1 \vee \neg x_2 \vee \neg x_3)$, $C_3 = (\neg x_1 \vee \neg x_2 \vee x_3)$, and $C_4 = (x_1 \vee x_2 \vee x_3)$.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4