3-1 Dynamic Programming

(Part II: "Theory")

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Definition (Optimal Substructure)

A problem exhibits ***optimal substructure*** if an optimal solution to the problem contains within it optimal solutions to subproblems.

Who exhibits $\cdots$?

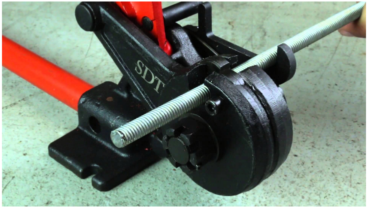
How to prove "YES"?

How to use "Cut-and-Paste"?

How to show "NO"?

$\boxed{\textit{Relative to Subproblems}}$

Rod Cutting

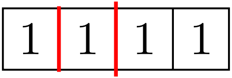


Optimal Substructure of Rod-Cutting (Problem 15.3-5)

Limit l_i : # of pieces of length i, $\quad 1 \le i \le n$

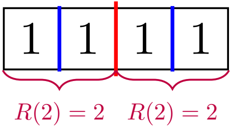
$$n = 4$$

length i	1	2	3	4
price p_i	1	1	1	1



$$R(4) = 3$$

length i	1	2	3	4
limit l_i	2	1	1	1



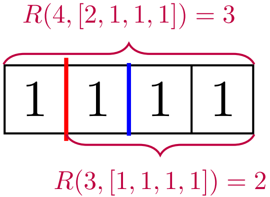
"Show that the optimal-substructure property described in Section 15.1 *no longer holds."*

$R(i, L)$: max *revenue obtainable by cutting up a rod of length* i

with the length limit array L

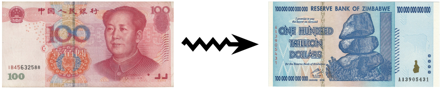
Where is the leftmost cut?

$$R(i, L) = \max_{\substack{1 \le j \le i \\ L_j \ge 1}} \Big(p_j + R\big(i - j, L[j \mapsto L_j - 1]\big) \Big)$$



Currency Exchange (Problem 15.3-6)

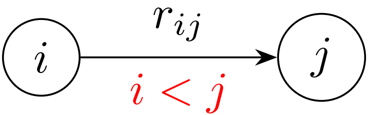
$1, 2, \cdots, n$ currencies



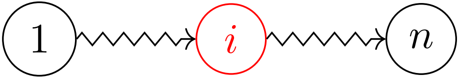
$$i \xrightarrow{r_{ij}} j$$

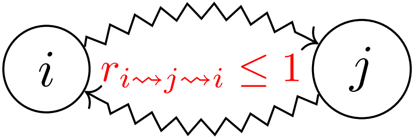
c_k : Commission charged for k trades

$$c_k = 0$$

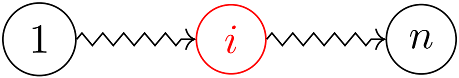


An optimal sequence of trades from 1 *to* n *through* i*:*

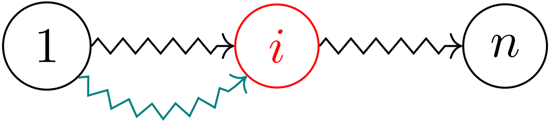




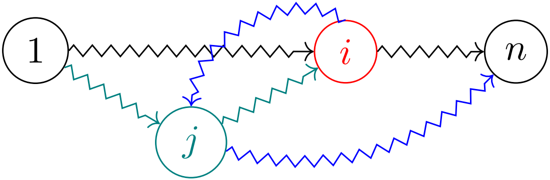
An *optimal* *sequence of trades from* 1 *to* n *through* i*:*



By Contradiction.



$\textsc{Case I}: s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n} = \emptyset$

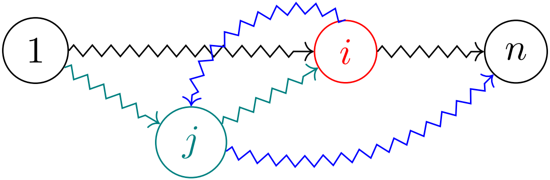


$$\textsc{Case II}: j \in s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n}$$

$$\begin{aligned} & 1 \rightsquigarrow j \rightsquigarrow n \\ \geq\ & 1 \rightsquigarrow j \rightsquigarrow i \rightsquigarrow j \rightsquigarrow n \\ =\ & 1 \rightsquigarrow j \rightsquigarrow i \rightsquigarrow n \\ >\ & 1 \rightsquigarrow i \rightsquigarrow n \end{aligned}$$

Longest Path Problem

To find a *simple* path of maximum length from s to t in a graph.



$$
\begin{aligned}
& 1 \leadsto j \leadsto n \\
\geq\ & 1 \leadsto j \leadsto i \leadsto j \leadsto n \\
\geq\ & 1 \leadsto j \leadsto i \leadsto j \leadsto n \\
=\ & 1 \leadsto j \leadsto i \leadsto n \\
>\ & 1 \leadsto i \leadsto n
\end{aligned}
$$

Does the longest path problem really have no optimal substructure?

WAIT WAIT...
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$L(s,t,A)$: *the longest path from s to t*

through exactly the intermediate vertices in I

$$L(s,t) = \max_{I \subseteq V} L(s,t,I)$$

What is the next vertex from s?

$$L(s,t,I) = 1 + \max_{\substack{(s,x) \in E \\ x \in I}} l(x,t,I \setminus \{x\})$$

$$L(t,t,I) = 0, \qquad L(s,t,\emptyset) = \begin{cases} 1, & \text{if } (s,t) \in E \\ 0, & \text{otherwise} \end{cases}$$

$$L(s,t,I) = 1 + \max_{\substack{(s,x) \in E \\ x \in I}} l(x,t,I \setminus \{x\})$$

$$O(n2^n)$$

Cycle $\implies$ *No Order on Vertices*

DP does not necessarily lead to efficient (polynomial) algorithms.

The (decision version of the) longest path problem is NP-hard!

The Change-Making Problem

Coins values: $x_1, x_2, \cdots, x_n$

Amount: v

Is it possible to make change for v*?*

Without repetition: $0/1$

$C[i, w]$: Make change for w using only values of $x_1 \ldots x_i$?

$$C[n, v]$$

Using value x_i or not?

$$C[i, w] = \underbrace{C[i-1, w]}_{\notin} \lor \big(\underbrace{C[i-1, w - x_i] \land w \geq x_i}_{\in} \big)$$

$$C[i, 0] = \text{true},\ \forall i = 0 \ldots n$$

$$C[0, w] = \text{false},\ \text{if } w > 0$$

$$C[0, 0] = \text{true}$$

The $0/1$ Knapsack Problem (Problem 16.2-2)

Values : v_i Weights : w_i Capacity : W

Taking as valuable a load as possible

$C[i, w]$: Knapsack with capacity w using only items of values $v_1 \ldots v_i$

$\boxed{\textit{Using value } v_i \textit{ or not?}}$

$$C[i, w] = \max \Big(\underbrace{C[i-1, w]}_{\notin}, \big(\underbrace{w \geq x_i \implies C[i-1, w-w_i] + v_i}_{\in} \big) \Big)$$

Unbounded repetition: ∞

$C[i, w]$: Make change for w using only values of $x_1 \dots x_i$?

$$C[n, v]$$

$\boxed{\textit{Using value } x_i \textit{ or not?}}$

$$C[i, w] = \underbrace{C[i-1, w]}_{\notin} \vee \Big(\underbrace{C[i, w - x_i] \wedge w \geq x_i}_{\in}\Big)$$

$$C[i, 0] = \text{true}, \ \forall i = 0 \dots n$$
$$C[0, w] = \text{false}, \ \text{if } w > 0$$
$$C[0, 0] = \text{true}$$

The Change-Making Problem

Unbounded repetition: ∞

$C[w]$: Possible to make change for w?

$$C[v]$$

What is the first coin to use?

$$C[w] = \bigvee_{i:\, x_i \le w} C[w - x_i]$$

$$C[0] = \text{true}$$

$$O(nv)$$

The Change-Making Problem

Unbounded repetition: ∞

$C[i, w]$ *vs.* $C[w]$

$$C[i, w] = C[i-1, w] \vee \left(C[i, w - x_i] \wedge w \geq x_i\right)$$

$$C[w] = \bigvee_{i:\, x_i \leq w} C[w - x_i]$$

The Change-Making Problem (Problem $16\text{-}1$ (d))

Unbounded repetition: ∞

Using the fewest number of coins

$\boxed{\textit{Using value } x_i \textit{ or not?}}$

$$C[i, w] = \min \Big(\underbrace{C[i-1, w]}_{\notin}, \big(\underbrace{w \ge x_i \implies C[i, w - x_i] + 1}_{\in} \big) \Big)$$

The Change-Making Problem (Problem 16-1 (d))

Unbounded repetition: ∞

Using the fewest number of coins

$\boxed{\textit{What is the first coin to use?}}$

$$C[w] = \min_{i:\ x_i \le w} C[w - x_i] + 1$$

The Change-Making Problem

Unbounded repetition with $\leq k$ coins overall

$C[i, w, l]$: Possible to make change for w with $\leq l$ coins of values of $x_1 \dots x_i$

$$C[n, v, k]$$

$\boxed{\textit{Using value } x_i \textit{ or not?}}$

$$C[i, w, l] = \underbrace{C[i-1, w, l]}_{\notin} \lor \big(\underbrace{C[i, w - x_i, l-1] \land w \geq x_i}_{\in}\big)$$

$$C[0, 0, l] = \text{true}, \quad C[0, w, l] = \text{false}, \text{ if } w > 0$$
$$C[i, 0, l] = \text{true}, \quad C[i, w, 0] = \text{false}, \text{ if } w > 0$$

The Change-Making Problem

Unbounded repetition with $\leq k$ coins overall

$C[w, l]$: Possible to make change for w with $\leq l$ coins?

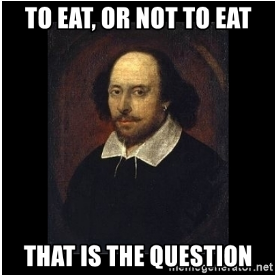
$$C[v, k]$$

What is the first coin to use?

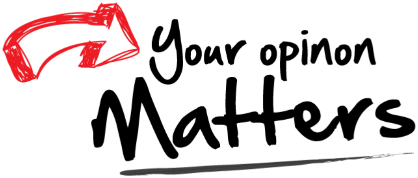
$$C[w, l] = \bigvee_{i:\, x_i \leq w} C[w - x_i, l - 1]$$

$$C[0, l] = \text{true},$$
$$C[w, 0] = \text{false}, \;\text{ if } w > 0$$

Problem (Hungry-Lion Game)




THANK YOU



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