计算机问题求解 – 论题3-8
- 单源最短通路算法

2016年10月26日

什么是最短通路问题?

In a ***shortest-paths problem***, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \to \mathbb{R}$ mapping edges to real-valued weights. The ***weight*** $w(p)$ of path $p = \langle v_0, v_1, \ldots, v_k \rangle$ is the sum of the weights of its constituent edges:

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i) .$$

问题1:

输入是什么?输出是什么?

We define the ***shortest-path weight*** $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \overset{p}{\leadsto} v\} & \text{if there is a path from } u \text{ to } v , \\ \infty & \text{otherwise .} \end{cases}$$

A ***shortest path*** from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.

问题2：

为什么说可以将单源最短路问题的解看成一个树？你认为这个树与两种图遍历搜索树相比，更可能象哪一个？

As in breadth-first search, we shall be interested in the ***predecessor subgraph*** $G_\pi = (V_\pi, E_\pi)$ induced by the π values. Here again, we define the vertex set V_π to be the set of vertices of G with non-NIL predecessors, plus the source s:

$$V_\pi = \{v \in V : v.\pi \neq \text{NIL}\} \cup \{s\} \ .$$

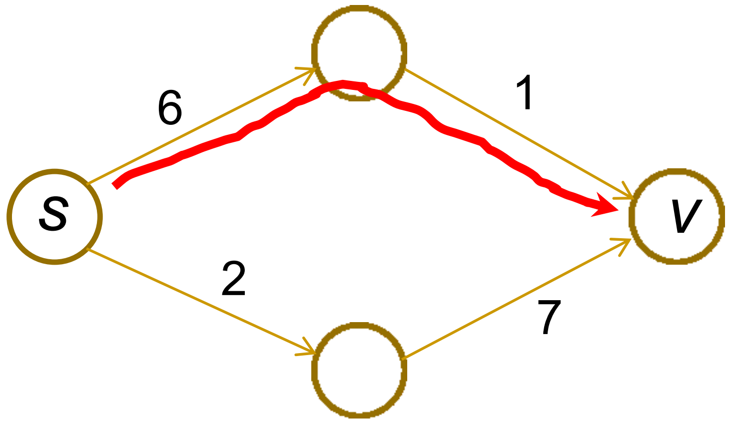
The directed edge set E_π is the set of edges induced by the π values for vertices in V_π:

$$E_\pi = \{(v.\pi, v) \in E : v \in V_\pi - \{s\}\} \ .$$

A ***shortest-paths tree*** rooted at s is a directed subgraph $G' = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$, such that

1. V' is the set of vertices reachable from s in G,
2. G' forms a rooted tree with root s, and
3. for all $v \in V'$, the unique simple path from s to v in G' is a shortest path from s to v in G.

Predecessor-subgraph property (Lemma 24.17)
Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s.



问题3：

能否借助上图说明最简单的greedy策略不一定能正确解决最短通路问题？这是单源最短通路问题具有“最优子结构”矛盾吗？

问题4：

具有负值权的回路对于单源最短通路问题的解有什么影响？非负值权的回路呢？

问题5：

在本章中介绍的算法基本思路是一样的，那是什么？

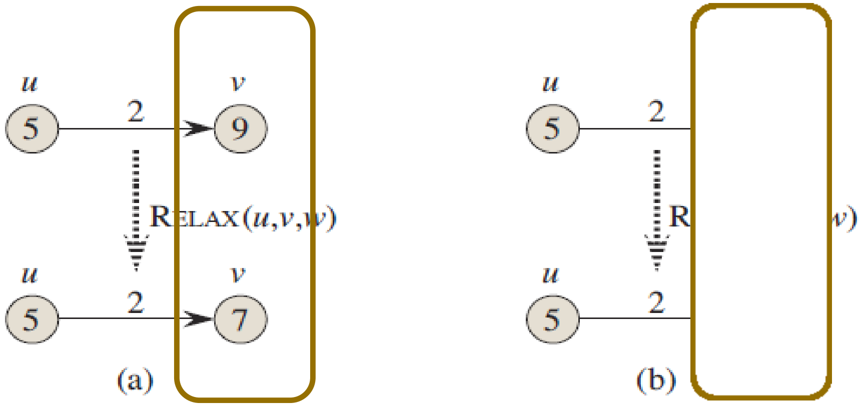
“预估”与“修正”

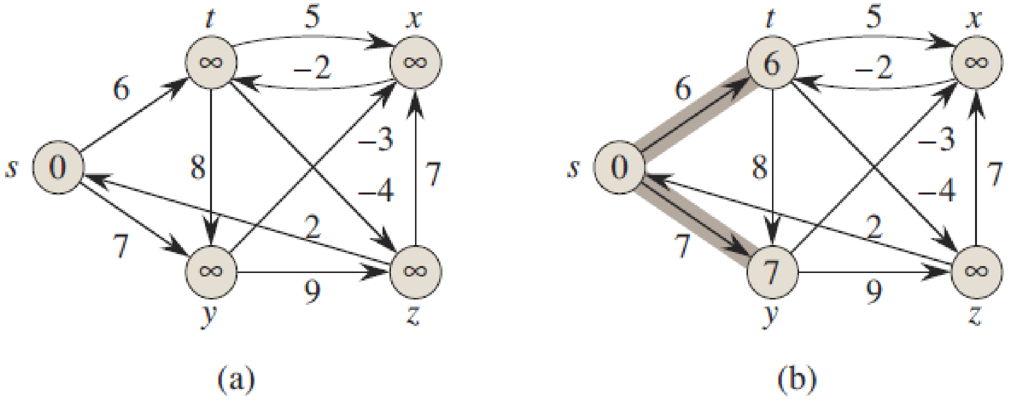
INITIALIZE-SINGLE-SOURCE(G, s)

for each vertex $v \in G.V$
    $v.d = \infty$
    $v.\pi = \text{NIL}$
$s.d = 0$

RELAX(u, v, w)

if $v.d > u.d + w(u, v)$
    $v.d = u.d + w(u, v)$
    $v.\pi = u$



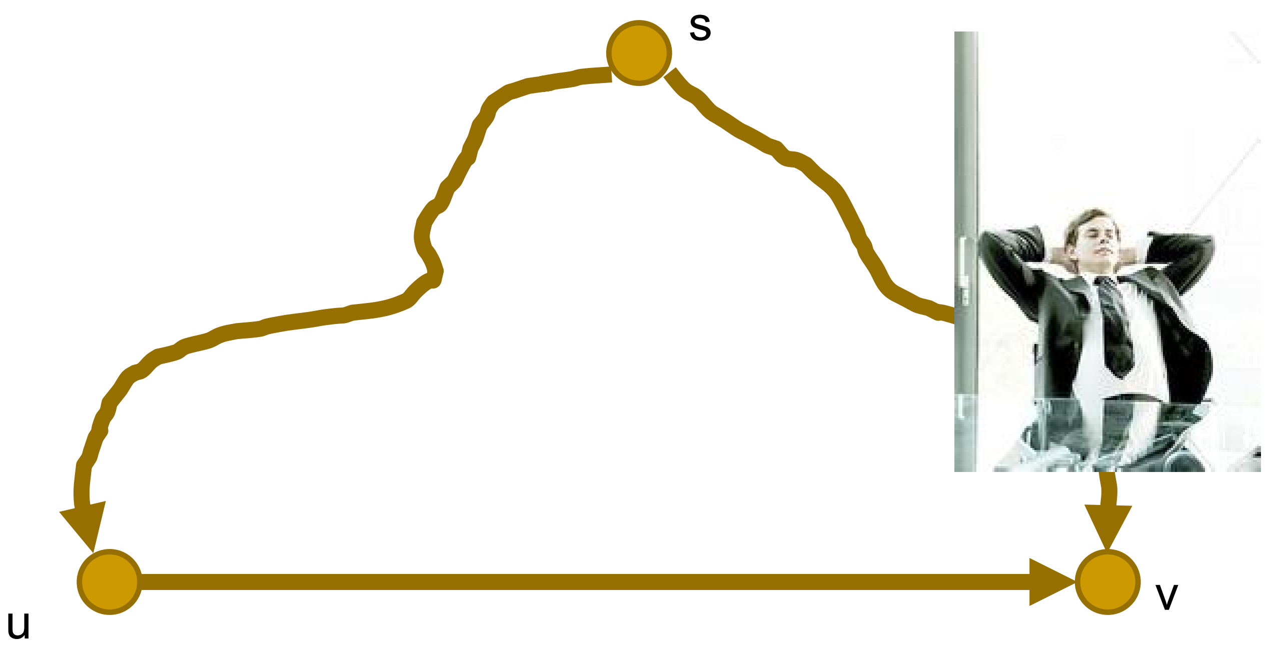


$(t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)$

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| - 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

问题6：

Relax中的“修正”到底在干什么？



当我们有u.d这么一个预估值后，v.d这个预估值必须小于u.d+w(u,v)**(三角不等式),**
如果relax时不小于，修正v.d为u.d+w(u,v)
修正后的v.d满足三角不等式的可能性大大提高

Lemma 24.10 (Triangle inequality)

Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$ and source vertex s. Then, for all edges $(u, v) \in E$, we have

$$\delta(s, v) \le \delta(s, u) + w(u, v) .$$

Proof Suppose that p is a shortest path from source s to vertex v. Then p has no more weight than any other path from s to v. Specifically, path p has no more weight than the particular path that takes a shortest path from source s to vertex u and then takes edge (u, v).

Lemma 24.11 (Upper-bound property)

Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$. Let $s \in V$ be the source vertex, and let the graph be initialized by INITIALIZE-SINGLE-SOURCE(G, s). Then, $v.d \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps on the edges of G. Moreover, once $v.d$ achieves its lower bound $\delta(s, v)$, it never changes.

Proof We prove the invariant $v.d \geq \delta(s, v)$ for all vertices $v \in V$ by induction over the number of relaxation steps.

For the basis, $v.d \geq \delta(s, v)$ is certainly true after initialization, since $v.d = \infty$ implies $v.d \geq \delta(s, v)$ for all $v \in V - \{s\}$, and since $s.d = 0 \geq \delta(s, s)$ (note that $\delta(s, s) = -\infty$ if s is on a negative-weight cycle and 0 otherwise).

For the inductive step, consider the relaxation of an edge (u, v). By the inductive hypothesis, $x.d \geq \delta(s, x)$ for all $x \in V$ prior to the relaxation. The only d value that may change is $v.d$. If it changes, we have

$$\begin{aligned} v.d &= u.d + w(u, v) \\ &\geq \delta(s, u) + w(u, v) && \text{(by the inductive hypothesis)} \\ &\geq \delta(s, v) && \text{(by the triangle inequality)} , \end{aligned}$$

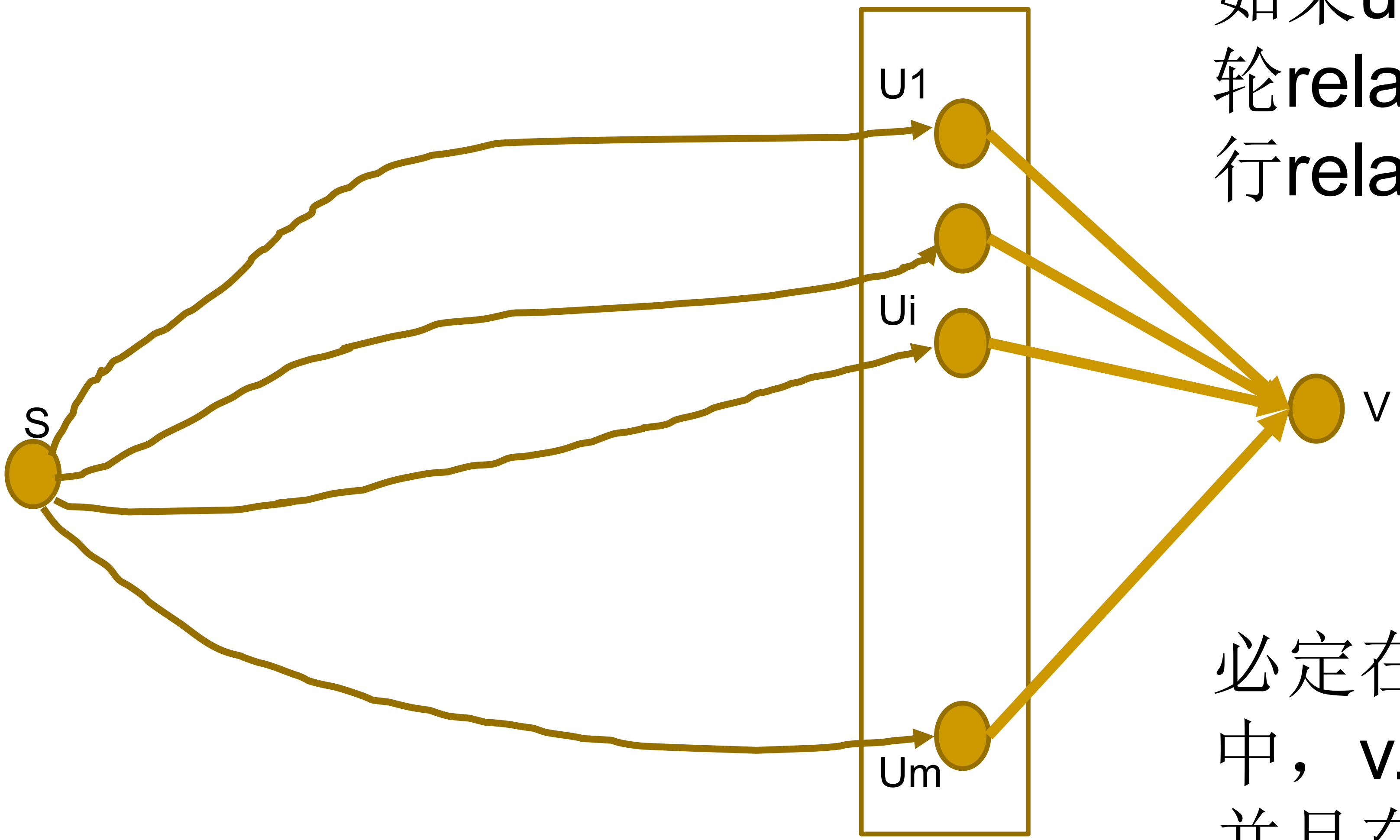
and so the invariant is maintained.

To see that the value of $v.d$ never changes once $v.d = \delta(s, v)$, note that having achieved its lower bound, $v.d$ cannot decrease because we have just shown that $v.d \geq \delta(s, v)$, and it cannot increase because relaxation steps do not increase d values. ■

既不会再减小，也不会增大。

问题6：

“修正”最终“可能”导致 $v.d=\delta(s,v)$。但“可能”怎么能变成“一定”？



如果ui.d=δ(s,ui),我们在某一轮relax中对所有(ui,v)边进行relax,会有什么结果?

必定在某边(Ui,V)的relax中，v.d=δ(s,v)
并且在之后轮次relax中，v.d不会改变

“一定”何时会发生

Lemma 24.14 (Convergence property)

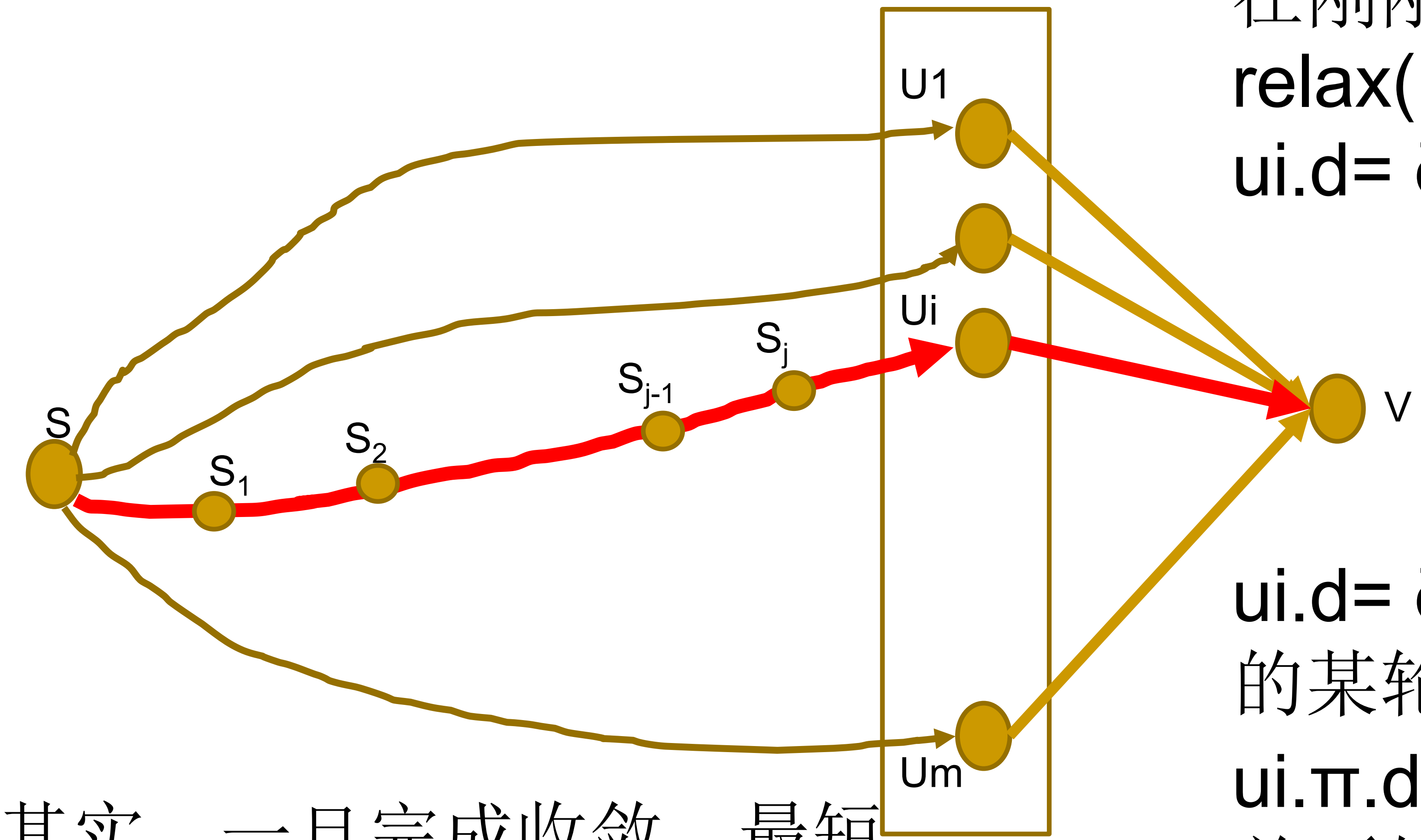
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, let $s \in V$ be a source vertex, and let $s \leadsto u \to v$ be a shortest path in G for some vertices $u, v \in V$. Suppose that G is initialized by INITIALIZE-SINGLE-SOURCE(G, s) and then a sequence of relaxation steps that includes the call RELAX(u, v, w) is executed on the edges of G. If $u.d = \delta(s, u)$ at any time prior to the call, then $v.d = \delta(s, v)$ at all times after the call.

Proof By the upper-bound property, if $u.d = \delta(s, u)$ at some point prior to relaxing edge (u, v), then this equality holds thereafter. In particular, after relaxing edge (u, v), we have

$$\begin{aligned} v.d &\boxed{\leq} u.d + w(u, v) && \text{(by Lemma 24.13)} \\ &= \delta(s, u) + w(u, v) \\ &= \delta(s, v) && \text{(by Lemma 24.1)}\ . \end{aligned}$$

By the upper-bound property, $v.d \boxed{} \delta(s, v)$, from which we conclude that $v.d \boxed{} \delta(s, v)$, and this equality is maintained thereafter. ■

在刚刚这一轮中，我们relax(ui,v)，得到v.d=δ(s,v)，ui.d= δ(s,ui)何时得到？



ui.d= δ(s,ui)必定在前面的某轮relax中完成

ui.π.d= δ(s, ui.π)必定在更前面的某轮relax中完成

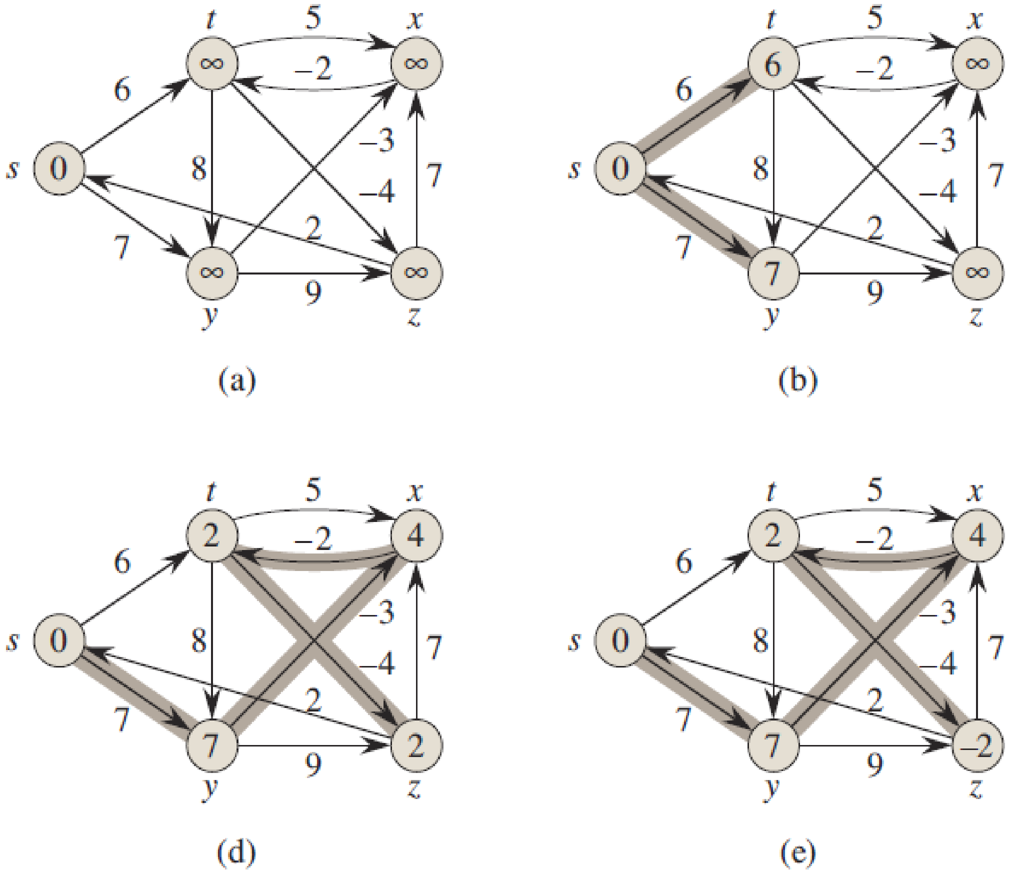
其实，一旦完成收敛，最短路已经形成

Lemma 24.15 (Path-relaxation property)

Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, and let $s \in V$ be a source vertex. Consider any shortest path $p = \langle v_0, v_1, \ldots, v_k \rangle$ from $s = v_0$ to v_k. If G is initialized by INITIALIZE-SINGLE-SOURCE(G, s) and then a sequence of relaxation steps occurs that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of the edges of p.

Proof We show by induction that after the ith edge of path p is relaxed, we have $v_i.d = \delta(s, v_i)$. For the basis, $i = 0$, and before any edges of p have been relaxed, we have from the initialization that $v_0.d = s.d = 0 = \delta(s, s)$. By the upper-bound property, the value of $s.d$ never changes after initialization.

For the inductive step, we assume that $v_{i-1}.d = \delta(s, v_{i-1})$, and we examine what happens when we relax edge (v_{i-1}, v_i). By the convergence property, after relaxing this edge, we have $v_i.d = \delta(s, v_i)$, and this equality is maintained at all times thereafter. ■




```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| - 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

$(t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)$

这会影响什么？

Bellman-Ford算法的“部分”正确性

Lemma 24.2

Let $G = (V, E)$ be a weighted, directed graph with source s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then, after the $|V| - 1$ iterations of the **for** loop of lines 2–4 of BELLMAN-FORD, we have $v.d = \delta(s, v)$ for all vertices v that are reachable from s.

Proof We prove the lemma by appealing to the path-relaxation property. Consider any vertex v that is reachable from s, and let $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = s$ and $v_k = v$, be any shortest path from s to v. Because shortest paths are simple, p has at most $|V| - 1$ edges, and so $k \le |V| - 1$. Each of the $|V| - 1$ iterations of the **for** loop of lines 2–4 relaxes all $|E|$ edges. Among the edges relaxed in the ith iteration, for $i = 1, 2, \ldots, k$, is (v_{i-1}, v_i). By the path-relaxation property, therefore, $v.d = v_k.d = \delta(s, v_k) = \delta(s, v)$. ■

为什么？

```
BELLMAN-FORD(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  for i = 1 to |G.V| − 1
3      for each edge (u, v) ∈ G.E
4          RELAX(u, v, w)
5  for each edge (u, v) ∈ G.E
6      if v.d > u.d + w(u, v)
7          return FALSE
8  return TRUE
```

只要没有源点可达的负权值回路，这个条件一定不会满足。
只要有源点可达的负权值回路，这个条件一定会满足。

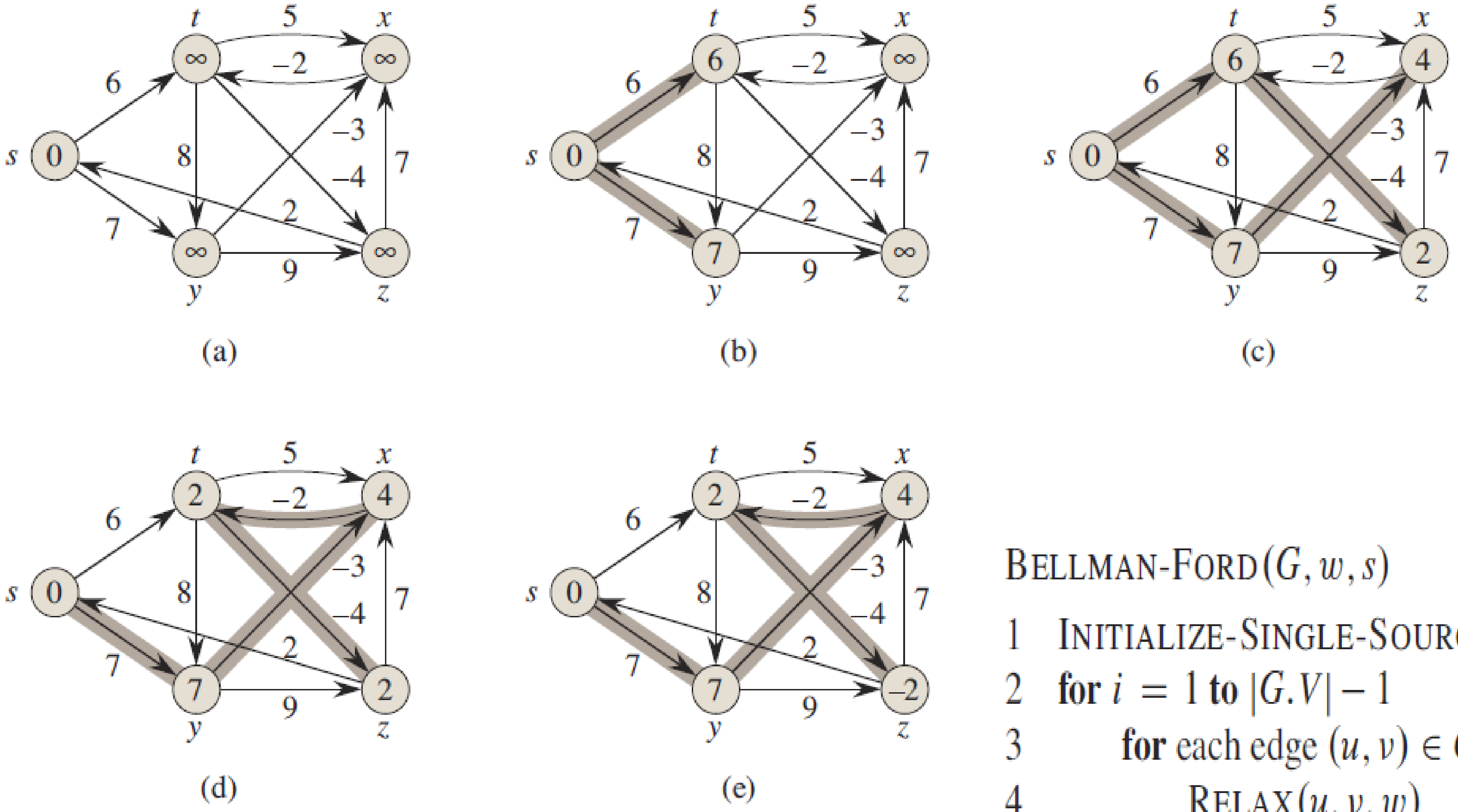
Suppose that graph G contains no negative-weight cycles that are reachable from the source s. We first prove the claim that at termination, $v.d = \delta(s, v)$ for all vertices $v \in V$. If vertex v is reachable from s, then Lemma 24.2 proves this claim. If v is not reachable from s, then the claim follows from the no-path property. Thus, the claim is proven. The predecessor-subgraph property, along with the claim, implies that G_π is a shortest-paths tree. Now we use the claim to show that BELLMAN-FORD returns TRUE. At termination, we have for all edges $(u, v) \in E$,

$$\begin{aligned} v.d &= \delta(s, v) \\ &\le \delta(s, u) + w(u, v) \quad \text{(by the triangle inequality)} \\ &= u.d + w(u, v)\,, \end{aligned}$$

and so none of the tests in line 6 causes BELLMAN-FORD to return FALSE. Therefore, it returns TRUE.

问题7:

Bellman-Ford算法的复杂度是$O(VE)$，你是否觉得**relax**操作太多了一些？有什么办法吗？

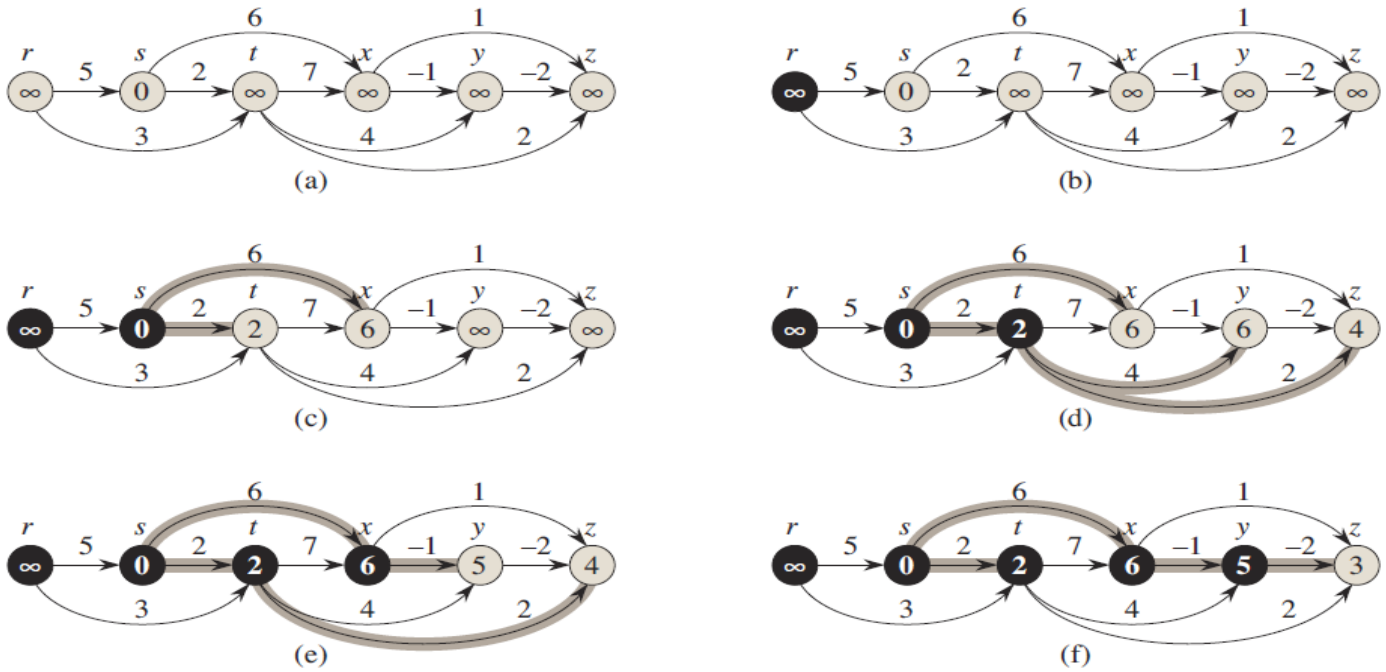


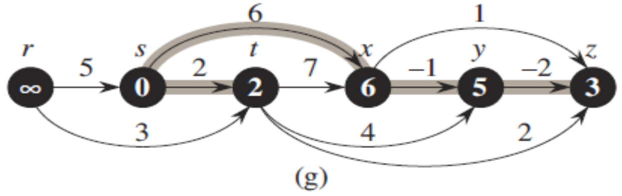

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| − 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

$(t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)$

换一种边的顺序，可能减少边的relax次数！

如果没有回路…






```
DAG-SHORTEST-PATHS(G, w, s)
topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex u, taken in topologically sorted order
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

问题8：

为什么不需要做那么多次relax操作了？

关键是被relax的边的顺序。

If the dag contains a path from vertex u to vertex v, then u precedes v in the topological sort.

Theorem 24.5

If a weighted, directed graph $G = (V, E)$ has source vertex s and no cycles, then at the termination of the DAG-SHORTEST-PATHS procedure, $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree.

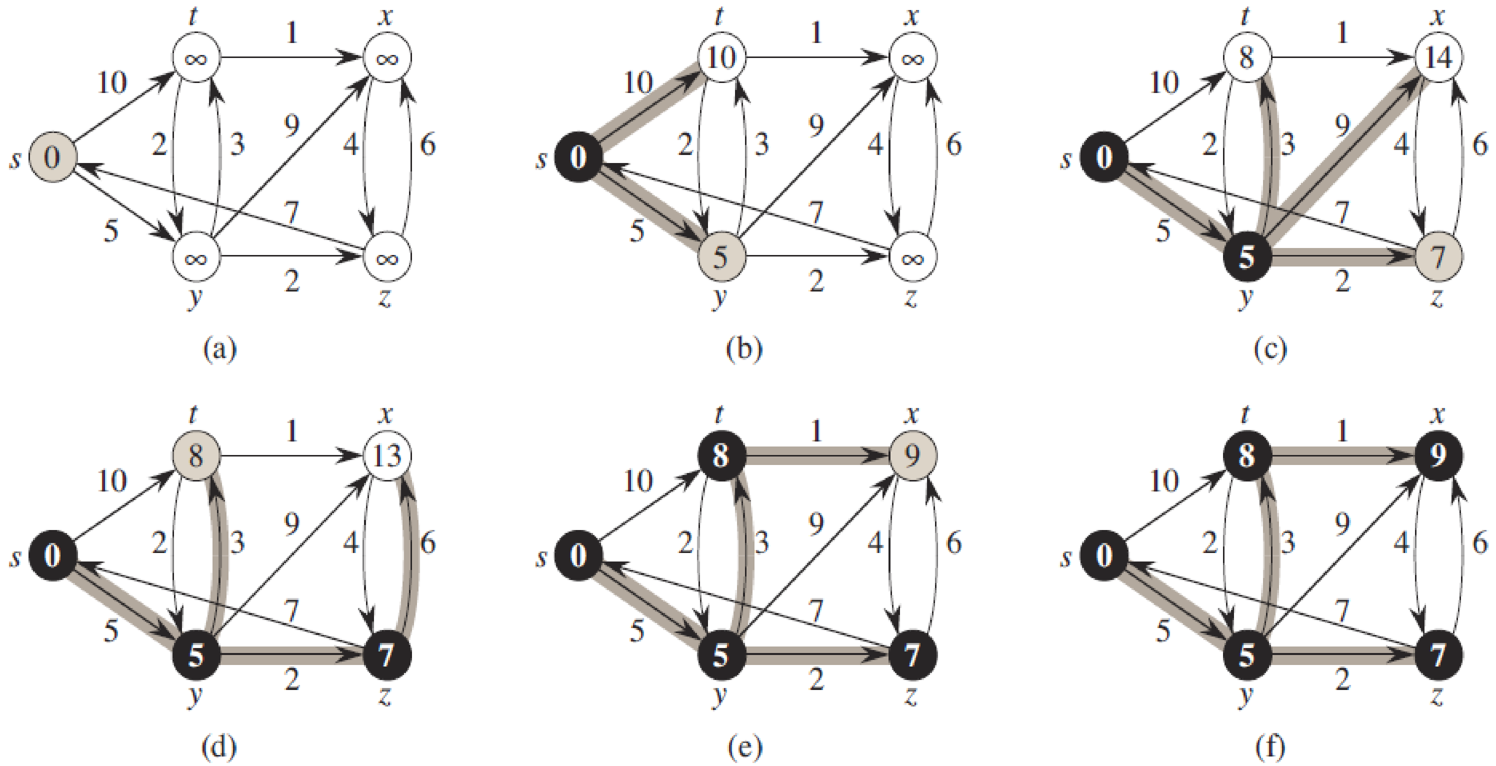
Proof We first show that $v.d = \delta(s, v)$ for all vertices $v \in V$ at termination. If v is not reachable from s, then $v.d = \delta(s, v) = \infty$ by the no-path property. Now, suppose that v is reachable from s, so that there is a shortest path $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = s$ and $v_k = v$. Because we process the vertices in topologically sorted order, we relax the edges on p in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$. The path-relaxation property implies that $v_i.d = \delta(s, v_i)$ at termination for $i = 0, 1, \ldots, k$. Finally, by the predecessor-subgraph property, G_π is a shortest-paths tree. ■

问题9：

没有回路的要求过高了，有什么办法达到类似的效果呢？

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

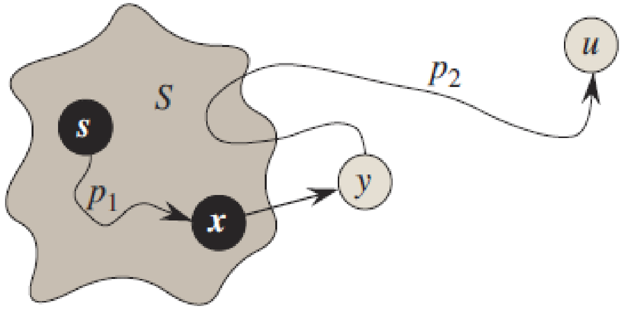
问题10：为什么这被认为是一个Greedy算法？



Dijstra算法的正确性

循环不变式:

At the start of each iteration of the **while** loop of lines 4–8, $v.d = \delta(s,v)$ for each vertex $v \in S$.



用反证法证明关键的一步: 任给一次特定循环, 即将加入*S*的顶点*u*必须满足 $u.d = \delta(s,u)$.

在左图的形势下(*s*到*u*的最短路), *u.d*既不能大于*y.d*(否则不可能选*u*加入*S*), 也不能小于*y.d* (*y.d*=δ(s,*y*)<=&(s,u))。

因此，只能是*u.d*=*y.d*=δ(*s,d*)=&(s,u)

问题11：

Dijstra算法对每条边最多relax一次，而且不要求输入是DAG，它付出的代价是什么？为什么必须如此？

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

显性或者隐性的优先队列操作

问题12：

为什么说Dijstra算法的复杂度与其实现方法有关？

问题13：

你能比较一下Dijstra算法与计算最小生成树的Prim算法吗？Dijstra算法的结果是否一定是一个最小生成树？

课外作业

- TC Ex.24.1: 2, 3, 4
- TC Ex.24.2: 2
- TC Ex.24.3: 2, 4, 7
- TC Ex.24.5: 2, 5
- TC Prob.24: 2, 3