0 Lp

- Duality (weak duality)
- max-flow min-cut
- sssp (29.2-2, 29.2-3)
- strong duality.

(Linear-inequality Feasibility, Maximum Bipartite Matching) $\Rightarrow$ $\boxed{\text{ref. Section 8.7}}$ of "Combinatorial Optimization" - Algorithms and complexity. (Christos H. Papadimitriou & Kenneth Steiglitz)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0. \end{aligned}$$

①

Max-Flow Problem

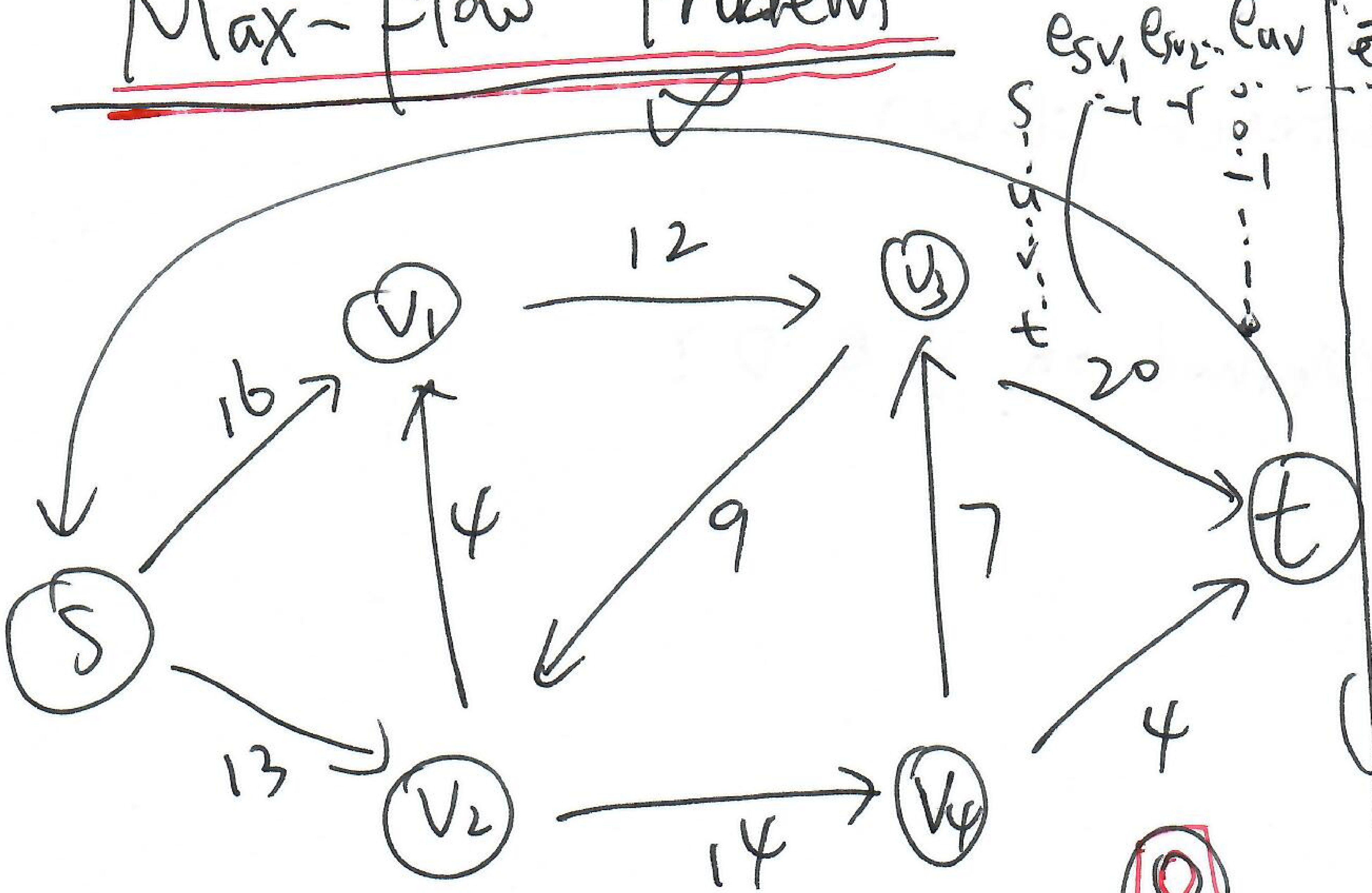


Fig. 26.1(a) of CLRS.(3rd edition). ⓪

min $c_E^T d_E$

s.t. $\begin{pmatrix}A\\I_E\end{pmatrix}^T\begin{pmatrix}p_V\\d_E\end{pmatrix} \geq e$

$(A^T\ \ I_E)\begin{pmatrix}p_V\\d_E\end{pmatrix} \geq e = \begin{pmatrix}0\\\vdots\\1\end{pmatrix}$ |E|

p_V is free |N|

$d_E \geq 0$ |E| ③

---

min $c_E^T d_E$

$(A^T\ \ I_E)\begin{pmatrix}p_V\\d_E\end{pmatrix} \geq e = \begin{pmatrix}0\\\vdots\\1\end{pmatrix}$ |E|

p_V is binary |N|

d_E is binary |E| ④

看边 $(uv)\in E$: $A^Tp + d \geq e$

③ min c^Td $\quad \begin{pmatrix}-1\\1\end{pmatrix}\cdot\begin{pmatrix}p_u\\p_v\end{pmatrix} + d_{uv} \geq e_{uv}$ $\quad u \to v$

① incoming outgoing
$p_v - p_u + d_{uv} \geq 0$ $\quad u \neq t, v \neq s$ $\forall (u,v)\in E$.

② $p_s - p_t + d_{ts} \geq 1$ $\quad (ts)\in E$

~~d_{uv}~~

⑤ $p_u \in \{0,1\}$ $\forall u \in V$

$d_{uv} \in \{0,1\}$. $\forall (u,v)\in E$.

Assign f_{uv} to each $(uv)\in E$.

P: max $f_{ts} = \overbrace{(0,0,\dots,1)}^{e^T,\ |E|} f$

s.t.

$A_{V\times E} f_E = 0_E$ |N|

node-edge incidence matrix (关联矩阵)

$I_E f_E \leq C_E$ |E|

$f_E \geq 0_E$ |E|

①

~~max $(0,0,0,\dots 1)f$~~

~~s.t.~~

Goal: The Dual of **P**:

D:

|N| p. $Af = 0$ ②

|E| d. $If \leq c$

(1) $d_{ts} = 0$ (∵ ③ min $\sum c_E^T d_E$)

(2) $p_s = 1$, $p_t = 0$ (∵ ②)

(3) Def.:

$S \triangleq \{v \in V \mid p_v = 1\}$

$\bar{S} \triangleq \{v \in V \mid p_v = 0\}$.

$(S, \bar{S})$ is a cut.

(4) We should verify that

$\sum c_E^T d_E = Cap(S, \bar{S}) = \sum_{u\in S}\sum_{v\in\bar{S}} c(u,v)$.

That is, we should show that

$d_{uv} = 1 \iff u \in S, v \in \bar{S}$

$\iff p_u = 1, p_v = 0$.

(That is true ∵ ① ③) ⑥

②

SSSP (29.2-2, 29.2-3).

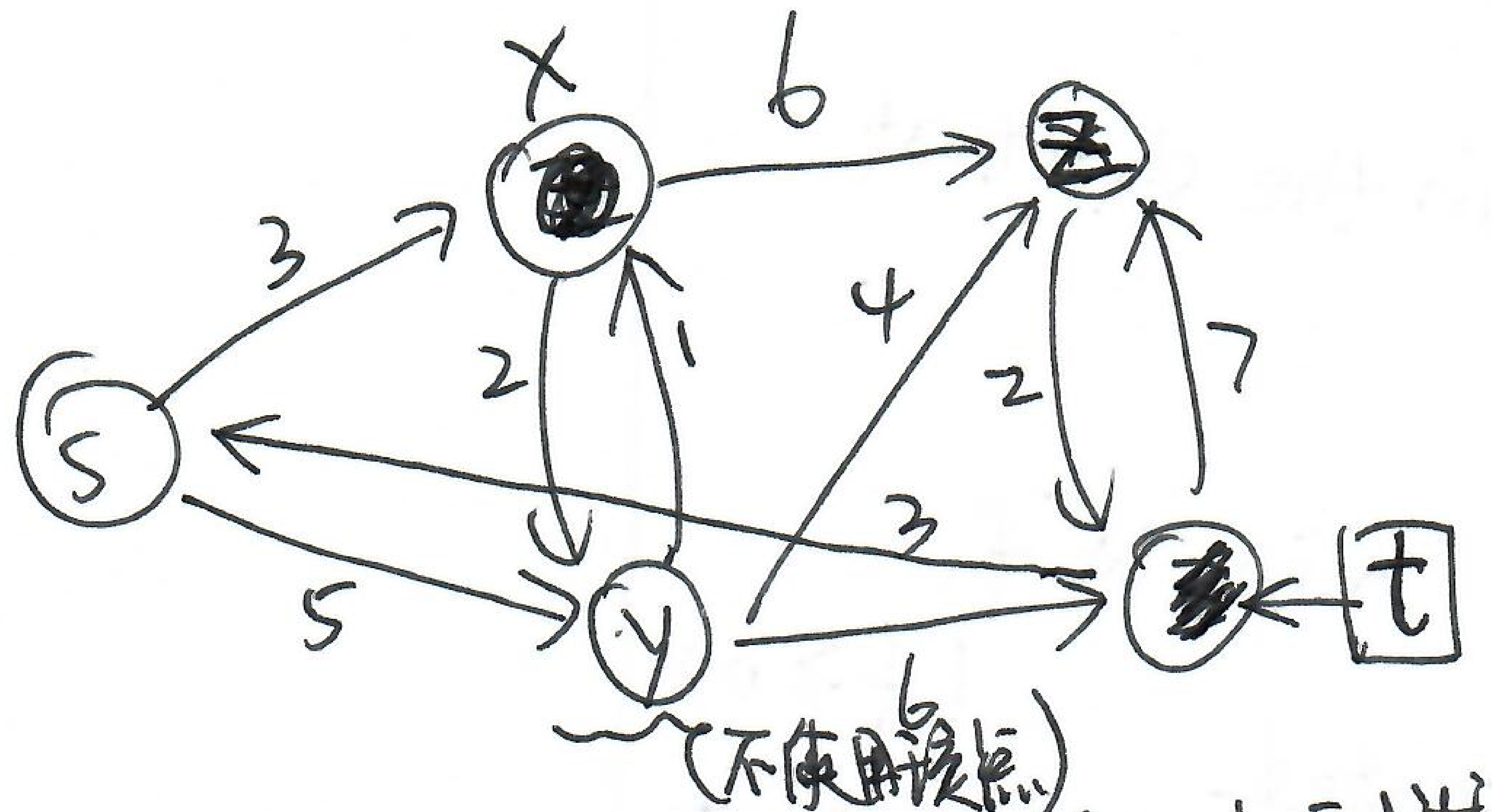


Fig 24.2 of CLRS (3rd Edition)

①

D: max d_t $s \rightsquigarrow t$.

s.t. $d_v \le d_u + w_{uv}$ $(u,v) \in E$. $(d_v - d_u \le w_{uv})$

$\boxed{d_s = 0}$ 这个条件有些"不合群".

Pf: 见 ⑥

An optimal solution to the shortest-paths problem sets each $\bar{d}_v = \min_{u:(u,v)\in E} \{\bar{d}_u + w_{uv}\}$, so that $\bar{d}_v$ is the largest value that is $\le$ all of the values in the set $\{\bar{d}_u + w_{uv}\}$.

②

It is Bellman-Ford Alg.

What is the dual of the LP for 29.2-3? What does it mean? (ProblemOverflow)

min $w^T f$

s.t. $Af \ge \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{matrix} \leftarrow s \\ \\ \leftarrow t \end{matrix}$ 或者 $(1-n) \leftarrow s$. $\sum_v (d_v - d_s)$

$f \ge 0$.

⑤

In Matrix Form.

max d_t

s.t.

$A^T d \le w$

$\begin{cases} d_s = 0 \\ d_u \ge 0. \ (u \ne s) \end{cases}$

③

Q: Run Simplex Method on D:

④

29.2-3 ⑤ single source s to all vertices $\in V$.

max $\sum_{v \in V} d_v$

s.t.

$d_v \le d_u + w_{uv}$ $\forall (u,v) \in E$.

$d_s = 0$

Pf Goal: $\sum d_v = \sum d_v^*$ and "=" is achieved iff $d_v = d_v^*$ for all v.

Pf. Any optimal sol. ~~must satisfy~~

$\begin{cases} d_v = \min_{u:(u,v)\in E} \{d_u + w_{uv}\}, \\ d_s = 0. \end{cases}$

$d_v \le d_v^*$ (the distance from s to v).

$\boxed{\sum d_v \le \sum d_v^*}$.

$d_v = d_v^*$ (for all v) is feasible.

$\boxed{\sum d_v \ge \sum d_v^*}$.

~~and it is~~

So: $\boxed{\sum d_v = \sum d_v^*}$

and "=" is achieved only when $d_v = d_v^*$ for all v. ($\because d_v \le d_v^*$)

Pf. LP for $s \overset{SP}{\leadsto} t$ problem.

Goal:

~~$d_t = d_t^*$~~

$$\boxed{d_t^* = d_t}$$

optimal sol to LP $\qquad$ the real ~~distance~~ s-t distance.

Pf. Take a shortest path P: $\underset{s}{\overset{v_0}{\circ}} \to \overset{v_1}{\circ} \to \overset{v_2}{\circ} \cdots \circ \to \underset{t}{\overset{v_n}{\circ}}$ (s⇝t)

① We prove that $\forall x \in s \overset{SP}{\leadsto} t$: $\boxed{d_x^* \le d_x} \Rightarrow d_t^* \le d_t$.

By induction on the (length of) the shortest path P.

B.S: Basis: $d_s^* = 0 = d_t$.

I.H: $\forall i \ge 0$: $d_{v_i}^* \le d_{v_i}$.

I.S: To prove that: $d_{v_{i+1}}^* \le d_{v_{i+1}}$.

$v_i \circ \longrightarrow \circ\, v_{i+1}$

$d_{v_{i+1}}^* \le d_{v_{i+1}}$

v_{i+1} is next to v_i. (On the shortest path.)

$d_{v_{i+1}} = d_{v_i} + w(v_i v_{i+1})$

$d_{v_{i+1}} \ge d_{v_i}^* + w(v_i v_{i+1})$. (I.H.)

$d_{v_{i+1}}^* \le d_{v_i}^* + w(v_i v_{i+1}) \le d_{v_{i+1}}$. (constraints of LP)

~~$d_{v_{i+1}}^* \ge d_{v_{i+1}}$~~

In P:

$d_{v_{i+1}}^* = \min_{(u, v_{i+1})} \{ d_u^* + w_{(u, v_{i+1})} \}$

② $\forall v \in V$: d_v is feasible to LP.

$\because$ max d_t in LP

$d_t^* \ge d_t$.

Therefore: $d_t^* = d_t$.

O'

How about Running Simplex Method on $\underline{P}$?

⑤

SSSP.

~~To minimize the~~

Variables: f_{uv} for each edge $(u,v) \in E$.

$f_{uv} \in \{0,1\}$. $(f_{uv} \geq 0)$

indicates that whether $(u,v) \in E$ is in the shortest path from ~~s to~~ s to t.

Goal: $\min \sum_{uv \in E} w_{uv} f_{uv}$ — $\min w^T f$. $(f \geq 0)$

s.t. f_{uv} should enforce a directed path from s to t.

① $out(u) = in(u) = 1$

Outdegree(s) = 1 $\quad in(t) = 1$

$out(s) = \sum_{v \in V} f_{sv} = 1$

$in(t) = \sum_{v \in V} f_{vt} = 1$

BUT. we don't know which vertices are in the ~~sp~~ shortest path from s to t.

② What about other vertices?

$u \notin \{s,t\}$. u: $\begin{cases} \notin SP & \text{(u)}: in(u) = out(u) = 0 \\ \text{OR} & \\ \in SP & \to u \to: in(u) = out(u) = 1 \end{cases}$

$u \notin \{s,t\}$.

u: $in(u) = out(u)$.

~~f_{uv}~~ SP from s to t: a unit flow ~~of unit~~ from s to t.

③ $\boxed{P:}$ $\min \sum_{uv \in E} w_{uv} f_{uv}$ $(\min w^T f)$

incidence matrix

$$A f = \begin{pmatrix} -1 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix} \begin{matrix} \leftarrow s \\ \\ \leftarrow \text{other } u \notin \{s,t\} \\ \\ \leftarrow t. \end{matrix}$$

$f \geq 0$.

④ $\boxed{D:}$ $(-1, 0, \cdots, 1) \cdot d$.

$\max\ d_t - d_s$.

s.t.

$|E|$: $A^T d \leq w$

$d \geq 0$.

$\boxed{Q:}$ 没有 $d_s = 0$?

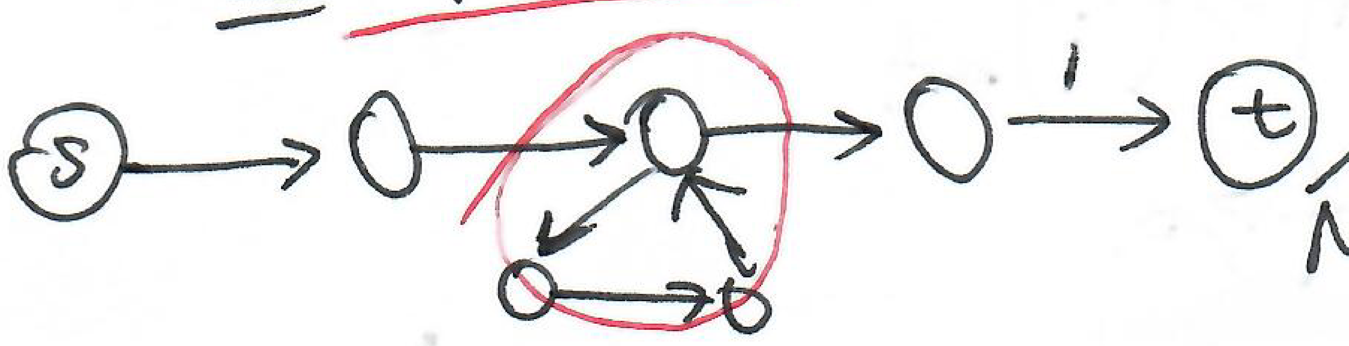
$\boxed{d_v - d_u \leq w_{uv}}$

$$\begin{array}{c|c} & \begin{matrix} u & v \end{matrix} \\ uv & \begin{pmatrix} -1 & 1 \end{pmatrix} \end{array} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_u \\ \vdots \\ d_v \end{pmatrix}$$

Thm. It has optimal solutions with $f_{uv} \in \{0,1\}$.

Q: path from s to t?

Pf. No.

A: 

cycle problem / Negative

③

Thm 29.10 (Strong LP Duality).

If a linear program P has a (bound) optimal solution x^*, then its dual D has a (bounded) optimal solution y^*, and $c^T x^* = b^T y^*$.

Pf. By construction.

Run simplex Method on P. ~~The~~ The final slack form is.

$$z = \nu' + \sum_{j \in N} c_j' x_j$$

$$x_i = b_i' - \sum_{j \in N} a_{ij}' x_j \quad \text{for } i \in B.$$

Then, the optimal ~~dual~~ solution to D is:

$$y_i^* = \begin{cases} -c_{n+i}' & \text{if } (n+i) \in N. \\ 0 & \text{o.w.} \end{cases}$$

$\boxtimes$.

(1) c_{n+i}' 对应于 x_{n+i}，对应于 y_i.(Lagrange Multiplier)

(2) $(n+i) \in N \Rightarrow x_{n+i}$ 处于"最上行".

$\Rightarrow y_i^*$ 处于"最上行".

Approach I:

Key Observation: The dual dictionary is the Negative Transpose of the primal dictionary.

{[illegible] Question (ProblemOverflow) ~~Approach I:~~

Use SageMath Here.

$z = 3x_1 + x_2 + 2x_3$

$x_4 = 3x_1 - x_2 -$

$x_5 =$

$x_6 =$

Approach II:

4

$$z = 3x_1 + x_2 + 2x_3 + 0 \cdot x_4 + 0x_5 + 0x_6 + 0$$

$$x_1 + x_2 + 3x_3 + x_4 = 30$$

$$2x_1 + 2x_2 + 5x_3 + x_5 = 24$$

$$4x_1 + x_2 + 2x_3 + x_6 = 36$$

z	x_1	x_2	x_3	x_4	x_5	x_6	
	3	1	2	0	0	0	0
	1	1	3	1	0	0	30
	2	2	5	0	1	0	24
	4	1	2	0	0	1	36.

$$\begin{array}{c|ccc} & n & m & 1 \\ \hline 1 & c & 0 & 0 \\ m & A & I & b \end{array}$$

$$\Downarrow$$

$$\left[\begin{array}{c|c|c} C_N & C_B & 0 \\ \hline N & B & b \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{c|c|c} C_N & \boxed{C_B} & 0 \\ \hline B^{-1}N & I & B^{-1}b \end{array}\right]$$

$$\Downarrow$$

stopping condition. optimal value.

$$r = \left[\begin{array}{c|c|c} C_N - C_B B^{-1} N \geq 0 & 0 & -C_B B^{-1} b \\ \hline B^{-1}N & I & \underbrace{B^{-1}b}_{\text{final } b.} \end{array}\right]$$

In the final slack form: $x_1, x_2, x_4 \in B$; $x_3, x_5, x_6 \in N$.

z	x_3	x_5	x_6	x_1	x_2	x_4	
	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$	0	0	0	28
							8
					I		4
							18

(Problem Overflow.)

Q2: Where is $-C_B B^{-1}$? That is what we want.

Q1: Optimal? $cx^* = -C_B B^{-1} b = y^* b$

$$\boxed{\underline{\text{Set}} \quad y^* = -C_B B^{-1}}.$$

$$C_N - \boxed{C_B B^{-1}} N = \overset{x_3 \; x_5 \; x_6}{(2, \boxed{0, 0})} - \overset{x_1 \; x_2 \; x_4}{(3, 1, \boxed{0})} \begin{array}{c} x_1 \; x_2 \; x_4 \\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 4 & 1 & 0 \end{pmatrix}^{-1} \end{array} \begin{array}{c} x_3 \; x_5 \; x_6 \\ \begin{pmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \end{array}$$

对应于 y_2 y_3

$$(3, 1, \boxed{0}) \cdot \begin{pmatrix} 0 & -\frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{1}{2} & 0 \end{pmatrix}$$

$$(\boxed{0}, \frac{1}{6}, \frac{2}{3}).$$