What can be sampled locally?

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Qian Luowen
Undergraduate Seminar

Preliminaries

- The LOCAL model
- Markov random fields (MRFs) and local CSP
- Local sampling problem
- Mixing rate

The LOCAL model

- $G(V, E)$ represents the processor network
- Δ is the maximum degree
- $n = |V|$

Markov random field

- Given a graph G(V, E) and a finite domain [q] = {1, 2, …, q},
  The probability measure of each configuration σ is proportional to the weight:

$$w(\sigma) := \prod_{e=uv\in E} A_e(\sigma_u,\sigma_v)\prod_{v\in V} b_v(\sigma_v),$$

Markov random field

- Given a graph G(V, E) and a finite domain [q] = {1, 2, …, q},
  The probability measure of each configuration σ is proportional to the weight:

$$w(\sigma) = \prod_{c=(f_c, S_c)\in\mathcal{C}} f_c(\sigma|_{S_c}),$$

Local sampling

- Measured by total variation distance:

$$d_{\mathrm{TV}}(\mu, \nu) = \sum_{\sigma \in \Omega} \frac{1}{2} |\mu(\sigma) - \nu(\sigma)| = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|.$$

Mixing rate

- Ergodicity: Irreducible + Aperiodic
- Mixing rate is defined as

$$\tau(\epsilon) = \max_{\sigma \in \Omega} \min \left\{ t : d_{\mathrm{TV}} \left(\pi_\sigma^{(t)}, \pi \right) \leq \epsilon \right\},$$

Luby's algorithm for maximal independent set (MIS)

Initialize I to an empty set.

While V is not empty:

1. Sample an independent set S in V.
2. Add the set S to I.
3. Remove from V the set S and all the neighbours of nodes in S.

Return I.

LubyGlauber algorithm

Algorithm 1: Pseudocode for vertex $v \in V$ in *LubyGlauber* algorithm

Input: Vertex $v \in V$ receives $\{A_{uv}\}_{u \in \Gamma(v)}$ and b_v as input.

initialize X_v to an arbitrary value in $[q]$;
for *$t = 1$ through T* **do**
    sample a real $\beta_v \in [0, 1]$ uniformly and independently;
    if $\beta_v > \max\{\beta_u \mid u \in \Gamma(v)\}$ **then**
        resample X_v according to marginal distribution $\mu_v(\cdot \mid X_{\Gamma(v)})$;
return X_v;

$$\forall c \in [q], \quad \mu_v(c \mid X_{\Gamma(v)}) = \frac{b_v(c) \prod_{u \in \Gamma(v)} A_{uv}(c, X_u)}{\sum_{a \in [q]} b_v(a) \prod_{u \in \Gamma(v)} A_{uv}(a, X_u)}.$$

LubyGlauber algorithm

- Assumption: Marginal distribution is well defined, chain is irreducible among all feasible configurations

Proposition 3.1. *The Markov chain* LubyGlauber *is reversible and has stationary distribution* μ*. Furthermore, under the above assumption,* $d_{\mathrm{TV}}(\mu_{\mathsf{LG}}, \mu)$ *converges to 0 as* $T \to \infty$*.*

Theorem 3.2. *Under the same assumption as Proposition 3.1, if the total influence* $\boldsymbol{\alpha} < 1$*, then the mixing rate of the* LubyGlauber *chain is* $\tau(\epsilon) = O\left(\frac{\Delta}{1-\boldsymbol{\alpha}} \log\left(\frac{n}{\epsilon}\right)\right)$.

Theorem 1.1. *If* $q \geq \alpha\Delta$ *for an arbitrary constant* $\alpha > 2$*, there is an algorithm which samples a uniform proper* q*-coloring within total variation distance* $\epsilon > 0$ *in* $O\left(\Delta \log\left(\frac{n}{\epsilon}\right)\right)$ *rounds of communications on any graph* $G(V, E)$ *with* $n = |V|$ *vertices and maximum degree* $\Delta = \Delta(n)$.

LocalMetropolis algorithm

Algorithm 2: Pseudocode for the *LocalMetropolis* algorithm

Input: Each vertex $v \in V$ receives $\{A_{uv}\}_{u \in \Gamma(v)}$ and b_v as input.

each $v \in V$ initializes X_v to an arbitrary value in $[q]$;
for $t = 1$ *through* T **do**
    foreach $v \in V$ **do**
        propose a random $\sigma_v \in [q]$ with probability $b_v(\sigma_v) / \sum_{c \in [q]} b_v(c)$;
    foreach $e = (u, v) \in E$ **do**
        pass the check independently with probability
         $A_e(\sigma_u, \sigma_v) A_e(X_u, \sigma_v) A_e(\sigma_u, X_v) / \left(\max_{i,j \in [q]} A_e(i, j)\right)^3$;
    foreach $v \in V$ **do**
        if *all edges e incident with v pass the checks* **then**
            $X_v \leftarrow \sigma_v$;
each $v \in V$ **returns** X_v;

LocalMetropolis algorithm

Theorem 4.1. *The Markov chain* LocalMetropolis *is reversible and has stationary distribution* μ. *Furthermore, under above assumptions,* $d_{\mathrm{TV}}(\mu_{\mathsf{LM}}, \mu)$ *converges to* 0 *as* $T \to \infty$.

Theorem 4.2. *If* $q \geq \alpha\Delta$ *for a constant* $\alpha > 2 + \sqrt{2}$, *the mixing rate of the* LocalMetropolis *chain for proper* q*-coloring on graphs with maximum degree at most* $\Delta = \Delta(n) \geq 9$ *is* $\tau(\epsilon) = O(\log(\frac{n}{\epsilon}))$, *where the constant factor in* $O(\cdot)$ *depends only on* α *but not on the maximum degree* Δ.

Lower bounds

Theorem 1.3. *For $\Delta \geq 6$, there exist infinitely many graphs $G(V, E)$ with maximum degree Δ and diameter $\mathrm{diam}(G) = \widetilde{\Omega}(\sqrt{|V|})$ such that any algorithm that samples uniform independent set in G within sufficiently small constant total variation distance ϵ requires $\Omega(\mathrm{diam}(G))$ rounds of communications, even assuming the vertices $v \in V$ to be aware of G.*

Thanks!