# Questions about Fourier Transform

## 1. Can $y'$ be extracted out of the summation?

It is known as the *discrete Fourier transform (DFT)* of the $N$ samples $y_n$ and is commonly written as $c_k$ where $c_k = N\gamma_k$. While the integral approach we used in the derivation is approximate, it is relatively easy to show that the DFT is an exact result. Consider:

$$\sum_{n=0}^{N-1} c_k \exp\left(i\frac{2\pi kn}{N}\right) = \sum_{k=0}^{N-1}\sum_{n'=0}^{N-1} y_{n'} \exp\left(-i\frac{2\pi kn'}{N}\right) \exp\left(i\frac{2\pi kn}{N}\right)$$

$$= \sum_{k=0}^{N-1} y_{n'} \sum_{n'=0}^{N-1} \exp\left(i\frac{2\pi k(n-n')}{N}\right)$$

It should be this way:

$$= \sum_{k=0}^{N-1}\sum_{n'=0}^{N-1} y_{n'} \exp\left(i\frac{2\pi k(n-n')}{N}\right)$$

## 2. How does this equation come?

Making a change of variables such that $N-n \to n$ and using the identity $cos\theta = \frac{1}{2}(e^{-i\theta}+e^{i\theta})$ we get the *discrete cosine transform (DCT)*:

$$c_k = \sum_{n=0}^{\frac{1}{2}N} y_n \exp\left(-i\frac{2\pi kn}{N}\right) + \sum_{n=1}^{\frac{1}{2}N-1} y_n \exp\left(i\frac{2\pi kn}{N}\right)$$

$$\boxed{= y_0 + y_{\frac{N}{2}} cos\left(\frac{2\pi k(\frac{N}{2})}{N}\right) + 2\sum_{n=1}^{\frac{1}{2}N-1} y_n cos\left(\frac{2\pi kn}{N}\right)}$$

The inverse of this is very similar, differing only by the leading $\frac{1}{N}$ term:

$$y_n = \frac{1}{N}\left[c_0 + c_{\frac{N}{2}} cos\left(\frac{2\pi k(\frac{N}{2})}{N}\right) + 2\sum_{n=1}^{\frac{1}{2}N-1} c_k cos\left(\frac{2\pi kn}{N}\right)\right]$$

We have $c_k$ in this form:

$$c_k = \sum_{n=0}^{\frac{1}{2}N} y_n \exp\left(-i\frac{2\pi kn}{N}\right) + \sum_{n=1}^{\frac{1}{2}N-1} y_n \exp\left(i\frac{2\pi kn}{N}\right)$$

we can split the term $\sum_{n=0}^{\frac{1}{2}N} y_n \exp\left(-i\frac{2\pi kn}{N}\right)$ into three parts including the first element, last element, and elements in the middle. Then we have

$$\sum_{n=0}^{\frac{1}{2}N} y_n \exp\left(-i\frac{2\pi kn}{N}\right)$$

$$= y_0 + y_{\frac{N}{2}} \exp(-i\frac{2\pi k\frac{N}{2}}{N}) + \sum_{n=1}^{\frac{1}{2}N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right)$$

For the term $\exp(-i\frac{2\pi k\frac{N}{2}}{N})$ above, since the angle in the complex plane is $-\frac{2\pi k\frac{N}{2}}{N} = -k\pi$, the value of imaginary part is 0, leaving the real part only, that is, $\cos(-\frac{2\pi k\frac{N}{2}}{N}) = \cos(\frac{2\pi k\frac{N}{2}}{N})$. Then we have

$$\sum_{n=0}^{\frac{1}{2}N} y_n \exp\left(-i\frac{2\pi kn}{N}\right)$$

$$= y_0 + y_{\frac{N}{2}} \cos(\frac{2\pi k\frac{N}{2}}{N}) + \sum_{n=1}^{\frac{1}{2}N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right)$$

Therefore, $c_k$ can be rewritten as

$$
\begin{aligned}
c_k &= \sum_{n=0}^{\frac{1}{2}N} y_n \exp\left(-i\frac{2\pi kn}{N}\right) + \sum_{n=1}^{\frac{1}{2}N-1} y_n \exp\left(i\frac{2\pi kn}{N}\right) \\
&= y_0 + y_{\frac{N}{2}} \cos(\frac{2\pi k\frac{N}{2}}{N}) + \sum_{n=1}^{\frac{1}{2}N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right) + \sum_{n=1}^{\frac{1}{2}N-1} y_n \exp\left(i\frac{2\pi kn}{N}\right) \\
&= y_0 + y_{\frac{N}{2}} \cos(\frac{2\pi k\frac{N}{2}}{N}) + \sum_{n=1}^{\frac{1}{2}N-1} y_n \left[\exp\left(-i\frac{2\pi kn}{N}\right) + \exp\left(i\frac{2\pi kn}{N}\right)\right]
\end{aligned}
$$

Through the identity $\cos\theta = \frac{1}{2}(e^{-i\theta} + e^{i\theta})$, we have

$$
c_k = y_0 + y_{\frac{N}{2}} \cos(\frac{2\pi k\frac{N}{2}}{N}) + 2\sum_{n=1}^{\frac{1}{2}N-1} y_n \cos\left(\frac{2\pi kn}{N}\right)
$$

## 3. How does this term come?

Note that it is common to see DCT formulations where the sampling is taken at the *midpoint* of the sample intervals, known at a "Type-II" DCT, which is effectively the same, but implies that $y_n = y_{N-1-n}$ and for an even $N$:

$$
\begin{aligned}
c_k &= \sum_{n=0}^{\frac{1}{2}N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right) + \sum_{n=\frac{1}{2}N}^{N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right) \\
&= \exp\left(i\frac{\pi k}{N}\right)\left[\sum_{n=0}^{\frac{1}{2}N-1} y_n \exp\left(-i\frac{2\pi k(n+\frac{1}{2})}{N}\right) + \sum_{n=\frac{1}{2}N}^{N-1} y_{N-1-n} \exp\left(i\frac{2\pi k(N-\frac{1}{2}-n)}{N}\right)\right] \\
&= \exp\left(i\frac{\pi k}{N}\right)\left[\sum_{n=0}^{\frac{1}{2}N-1} y_n \exp\left(-i\frac{2\pi k(n+\frac{1}{2})}{N}\right) + \boxed{\sum_{n=0}^{\frac{1}{2}N-1} y_n \exp\left(i\frac{2\pi k(n+\frac{1}{2})}{N}\right)}\right] \\
&= 2\exp\left(i\frac{\pi k}{N}\right) \sum_{n=0}^{\frac{1}{2}N-1} y_n cos\left(\frac{2\pi k(n+\frac{1}{2})}{N}\right)
\end{aligned}
$$

Here, we want to make the second summation has the same format of the previous one, so we are doing the variable substitution. We want $y_{N-1-n}$ like $y_n$, so we let a new variable $n' = N-1-n$. Then $y_{N-1-n} = y_{n'}$ and $N - \frac{1}{2} - n = n' + \frac{1}{2}$. As for the summary boundary, when $n = \frac{1}{2}N$, $n' = N - 1 - \frac{1}{2}N = \frac{N}{2} - 1$, when $n = N-1$, $n' = N - 1 - (N-1) = 0$. Therefore, we have

$$
\begin{aligned}
&\sum_{n=\frac{1}{2}N}^{N-1} y_{N-1-n} \exp\left(i\frac{2\pi k(N-\frac{1}{2}-n)}{N}\right) \\
&= \sum_{n'=0}^{\frac{N}{2}-1} y_{n'} \exp\left(i\frac{2\pi k(n'+\frac{1}{2})}{N}\right) \\
&= \sum_{n=0}^{\frac{N}{2}-1} y_n \exp\left(i\frac{2\pi k(n+\frac{1}{2})}{N}\right)
\end{aligned}
$$

## 4. Is this a typo?

## Fast Fourier Transforms

We introduced the DFT in the following form:

$$c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right)$$

In the simple approach we discussed for solving this, we must sum over $N$ terms and repeat this $\frac{1}{2}N + 1$ times, so it scales with $N^2$ and becomes prohibitively expensive for large datasets (just try it). The numpy routines introduced in the image compression example hint that this is not the most efficient way of solving the problem. A cleverer approach was introduced by Gauss (in 1805) and it is simpler to understand if we assume the number of samples is a power of two, such that $N = 2_m$ where $m$ is an integer. If we split our sum into two equally sized samples, of odd and even $n$, then the sum of the even terms can be written as:

$$E_k = \sum_{r=0}^{\frac{1}{2}N-1} y_{2r} \exp\left(-i\frac{2\pi k(2r)}{N}\right) = \sum_{r=0}^{\frac{1}{2}N-1} y_{2r} \exp\left(-i\frac{2\pi kr}{\frac{1}{2}N}\right)$$

But this is just another Fourier transform with $\frac{1}{2}N$ samples instead of $N$. Similarly, for the odd terms we get:

$$\begin{aligned}\sum_{r=0}^{\frac{1}{2}N-1} y_{2r+1} \exp\left(-i\frac{2\pi k(2r+1)}{N}\right) &= \exp\left(-i\frac{2\pi k}{N}\right) \sum_{r=0}^{\frac{1}{2}N-1} y_{2r+1} \exp\left(-i\frac{2\pi kr}{\frac{1}{2}N}\right) \\ &= \exp\left(-i\frac{2\pi k}{N}\right) O_k\end{aligned}$$

where $O_k$ is another Fourier transform with $\frac{1}{2}N$ samples. Then the complete Fourier coefficient is the sum of the odd and even terms:

$$c_k = E_k + \boxed{\exp\left(-i\frac{2\pi k}{N} c O_k\right)}$$

`I believe so. It should be in this way:`

$$c_k = E_k + \exp\left(-i\frac{2\pi k}{N}\right) O_k$$