

Sheet 2 solutions

April 27, 2018

Exercise 1: Linear Algebra

(a) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 0.25 & 0.1 \\ 0.2 & 0.5 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0.25 & -0.3 \\ -0.3 & 0.5 \end{pmatrix}.$$

Are they symmetric positive definite?

A symmetric matrix is a square matrix that is equal to its transpose: $\mathbf{M} = \mathbf{M}^T$. Its entries are symmetric with respect to the main diagonal, for example:

$$\mathbf{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

A symmetric $n \times n$ matrix \mathbf{M} is positive definite if the scalar $z^T \mathbf{M} z$ is positive for every non-zero column vector z of n real numbers. It is negative definite if $z^T \mathbf{M} z$ is negative, positive-semidefinite if $z^T \mathbf{M} z \geq 0$ and negative-semidefinite if $z^T \mathbf{M} z \leq 0$ for every non-zero vector z with appropriate dimension. A symmetric positive definite matrix has only positive eigenvalues. This is often easier to check than $z^T \mathbf{M} z$.

- Matrix \mathbf{A} is not symmetric!
- Matrix \mathbf{B} is symmetric. The eigenvalues are calculated as:

$$\begin{aligned} |\mathbf{B} - \lambda \mathbf{I}| &= 0 \\ \left| \begin{pmatrix} 0.25 & -0.3 \\ -0.3 & 0.5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| &= 0 \\ (0.25 - \lambda)(0.5 - \lambda) - 0.09 &= 0 \\ \lambda^2 - 0.75\lambda + 0.035 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{0.75}{2} \pm \sqrt{\left(\frac{0.75}{2}\right)^2 - 0.035} && \text{(pq-formula)} \\ \lambda_1 &= 0.7, \lambda_2 = 0.05 \end{aligned}$$

Both eigenvalues are positive, the matrix is symmetric and positive definite.

(b) For

$$\mathbf{C} = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix},$$

find the largest value for $\mu \in \mathbb{R}$ for which $C + \mu I$ is not symmetric positive definite.

We can check the eigenvalues for the largest value of μ for which $C + \mu I$ is not symmetric positive definite. The matrix

$$C + \mu I = \begin{pmatrix} -3 + \mu & 0 \\ 0 & 1 + \mu \end{pmatrix}$$

is in diagonal form, the eigenvalues are the entries of the diagonal. If at least one of the eigenvalues is smaller or equal to zero, the matrix is not symmetric positive definite:

$$\begin{aligned} (-3 + \mu) \leq 0 \quad \text{or} \quad (1 + \mu) \leq 0 \\ \mu \leq 3 \quad \text{or} \quad \mu \leq -1 \end{aligned}$$

The largest value for μ for which $C + \mu I$ is not symmetric positive definite is 3.

(c) Write a program in Python that determines whether a matrix is orthogonal.

A square matrix is orthogonal, if its columns and rows are orthogonal unit vectors, which is equivalent to:

$$\mathbf{M}^T \mathbf{M} = \mathbf{I}. \quad (1)$$

As an example, rotation matrices are always orthogonal. Please find the code listing below.

(d) Use this program to investigate whether

$$\mathbf{D} = \frac{1}{3} \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

is orthogonal.

Please find the code listing below. Matrix D is orthogonal.

```
import numpy as np

# c) program to verify matrix orthogonality

def check_orthogonal(M):
    # make sure the input is a matrix
    if len(np.shape(M)) != 2:
        print("error: input is not a matrix")
        return
    # make sure the input is a square matrix
```

```

dim = np.shape(M)[0]
if dim != np.shape(M)[1]:
    print("error: input is not a square matrix")
    return
A = np.dot(M, M.T)
if np.array_equal(A, np.identity(dim)):
    print("matrix is orthogonal")
else:
    print("matrix is not orthogonal")

# d) apply to given matrix

D = 1./3. * np.array(
    [[2, 2, -1],
     [2, -1, 2],
     [-1, 2, 2]])

check_orthogonal(D)

```

Exercise 2: 2D Transformations as Affine Matrices

The 2D pose of a robot w.r.t. a global coordinate frame is commonly written as $\mathbf{x} = (x, y, \theta)^T$, where (x, y) denotes its position in the xy -plane and θ its orientation. The homogeneous transformation matrix that represents a pose $\mathbf{x} = (x, y, \theta)^T$ w.r.t. to the origin $(0, 0, 0)^T$ of the global coordinate system is given by

$$X = \begin{pmatrix} \mathbf{R}(\theta) & \mathbf{t} \\ 0 & 1 \end{pmatrix}, \mathbf{R}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \mathbf{t} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- (a) While being at pose $\mathbf{x}_1 = (x_1, y_1, \theta_1)^T$, the robot senses a landmark l at position (l_x, l_y) w.r.t. to its local frame. Use the matrix X_1 to calculate the coordinates of l w.r.t. the global frame.

$$\text{Let } {}^g\mathbf{T}_{x_1} = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & x_1 \\ \sin \theta_1 & \cos \theta_1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } {}^{x_1}\mathbf{l} = \begin{bmatrix} l_x \\ l_y \\ 1 \end{bmatrix} \text{ then}$$

${}^g\mathbf{T}_{x_1}$ is the matrix expression in homogeneous form of pose \mathbf{x}_1 w.r.t. the global reference frame, while ${}^{x_1}\mathbf{l}$ is the vector expression in homogeneous form of the landmark w.r.t. the robot reference frame \mathbf{x}_1 .

The question asks to compute the landmark coordinate w.r.t. the global frame, i.e. ${}^g\mathbf{l}$:

$${}^g\mathbf{l} = {}^g\mathbf{T}_{x_1} \cdot {}^{x_1}\mathbf{l} \quad (2)$$

- (b) Now imagine that you are given the landmark's coordinates w.r.t. to the global frame. How can you calculate the coordinates that the robot will sense in his local frame?

We are given ${}^g\mathbf{l}$ and ${}^g\mathbf{T}_{x_1}$ and we want to compute ${}^{x_1}\mathbf{l}$. We can solve this either by taking (2) and solving with respect to ${}^{x_1}\mathbf{l}$ by multiplying to the left and right hand side $({}^g\mathbf{T}_{x_1})^{-1}$. Or we follow the same logic as the previous exercise, i.e.

$${}^{x_1}\mathbf{l} = {}^{x_1}\mathbf{T}_g \cdot {}^g\mathbf{l} = ({}^g\mathbf{T}_{x_1})^{-1} \cdot {}^g\mathbf{l}$$

- (c) The robot moves to a new pose $\mathbf{x}_2 = (x_2, y_2, \theta_2)^T$ w.r.t. the global frame. Find the transformation matrix T_{12} that represents the new pose w.r.t. to \mathbf{x}_1 . Hint: Write T_{12} as a product of homogeneous transformation matrices.

$$\text{Let } {}^g\mathbf{T}_{x_2} = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & x_2 \\ \sin \theta_2 & \cos \theta_2 & y_2 \\ 0 & 0 & 1 \end{bmatrix}$$

${}^g\mathbf{T}_{x_2}$ is the matrix expression in homogeneous form of pose \mathbf{x}_2 w.r.t. the global reference frame. This time we need to compute the homogeneous matrix form of the pose \mathbf{x}_2 expressed w.r.t. the reference frame of \mathbf{x}_1 , i.e. ${}^{x_1}\mathbf{T}_{x_2}$. Again, we follow the rules of transformation concatenation and we find:

$${}^{x_1}\mathbf{T}_{x_2} = {}^{x_1}\mathbf{T}_g \cdot {}^g\mathbf{T}_{x_2} = ({}^g\mathbf{T}_{x_1})^{-1} \cdot {}^g\mathbf{T}_{x_2} = T_{12}$$

- (d) The robot is at position \mathbf{x}_2 . Where is the landmark $\mathbf{l} = (l_x, l_y)$ w.r.t. the robot's local frame now?

Compute the landmark \mathbf{l} w.r.t. the reference frame of \mathbf{x}_2 , i.e. ${}^{x_2}\mathbf{l}$!

Since we computed ${}^{x_1}\mathbf{T}_{x_2}$ in the previous exercise we can just reuse it as follows:

$${}^{x_2}\mathbf{l} = {}^{x_2}\mathbf{T}_{x_1} \cdot {}^{x_1}\mathbf{l} = ({}^{x_1}\mathbf{T}_{x_2})^{-1} \cdot {}^{x_1}\mathbf{l}$$

In case we want to express it only in terms of the ${}^g\mathbf{T}_{x_1}$ and ${}^g\mathbf{T}_{x_2}$, we can apply the matrix inversion property $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and find:

$$\begin{aligned} ({}^{x_1}\mathbf{T}_{x_2})^{-1} &= (({}^g\mathbf{T}_{x_1})^{-1} \cdot {}^g\mathbf{T}_{x_2})^{-1} = ({}^g\mathbf{T}_{x_2})^{-1} \cdot {}^g\mathbf{T}_{x_1} \\ \implies {}^{x_2}\mathbf{l} &= ({}^g\mathbf{T}_{x_2})^{-1} \cdot {}^g\mathbf{T}_{x_1} \cdot {}^{x_1}\mathbf{l} \end{aligned}$$

We could have found the same result by directly applying the rules of transformation concatenation.

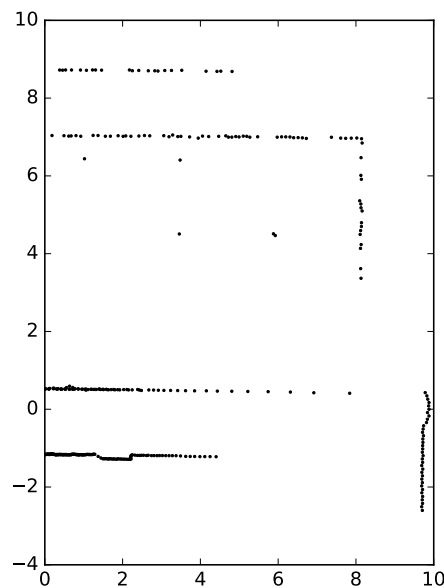
Exercise 3: Sensing

A robot is located at $x = 1.0\text{m}$, $y = 0.5\text{m}$, $\theta = \frac{\pi}{4}$. Its laser range finder is mounted on the robot at $x = 0.2\text{m}$, $y = 0.0\text{m}$, $\theta = \pi$ (with respect to the robot's frame of reference).

The distance measurements of one laser scan can be found in the file `laser_scan.dat`, which is provided on the website of this lecture. The first distance measurement is taken in the angle $\alpha = -\frac{\pi}{2}$ (in the frame of reference of the laser range finder), the last distance measurement has $\alpha = \frac{\pi}{2}$ (i.e., the field of view of the sensor is π), and all neighboring measurements are in equal angular distance (all angles in radians).

- (a) Use Python to plot all laser end-points in the frame of reference of the laser range finder.

Please find the code listing below.

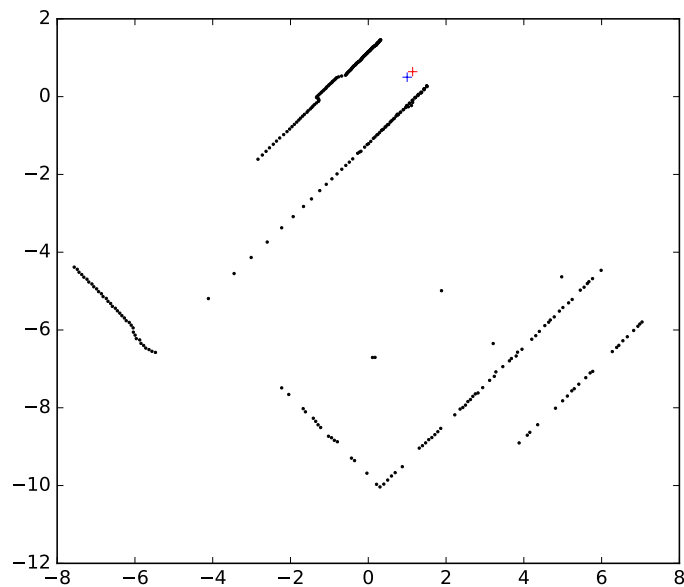


- (b) The provided scan exhibits an unexpected property. Identify it and suggest an explanation.

It appears as the laser can, at times, see through the walls, which shouldn't be possible. This can indeed happen if the "wall" is actually a semi transparent obstacle, such as a grid, a fence, or a glass.

- (c) Use homogeneous transformation matrices in Octave to compute and plot the center of the robot, the center of the laser range finder, and all laser end-points in world coordinates.

Please find the code listing below.



```
import numpy as np
import matplotlib.pyplot as plt
import math
pi = math.pi

# a) Load laserscan and plot in scanner frame

scan = np.loadtxt('laserscan.dat')
angle = np.linspace(-pi/2, pi/2, np.shape(scan)[0], endpoint='true')

x = scan * np.cos(angle);
y = scan * np.sin(angle);

plt.plot(x, y, '.k', markersize=3)

# Set the same scale on both axes
plt.gca().set_aspect('equal')
plt.savefig('scan1.pdf')

# c) Transform to global frame

# Define the transformation matrices
T_global_robot = np.array(
    [[np.cos(pi/4), -np.sin(pi/4), 1],
     [np.sin(pi/4),  np.cos(pi/4), 0.5],
     [0, 0, 1]])
```

```

T_robot_laser = np.array(
    [[np.cos(pi), -np.sin(pi), 0.2],
     [np.sin(pi),  np.cos(pi), 0.0],
     [0, 0, 1]])

# Compute the laser frame w.r.t. the global frame
T_global_laser = np.dot(T_global_robot, T_robot_laser)

# Apply the transformation to the scan points
w = np.ones((1, np.shape(x)[0]))[0]
scan_laser = np.array([x, y, w])
scan_global = np.dot(T_global_laser, scan_laser)

# Plot the laser points
plt.figure()
plt.plot(scan_global[0,:], scan_global[1,:], '.k', markersize=3)

# Plot robot pose in blue
plt.plot(T_global_robot[0,2], T_global_robot[1,2], '+b');

# Plot laser pose in red
plt.plot(T_global_laser[0,2], T_global_laser[1,2], '+r');

# Set the same scale on both axes
plt.gca().set_aspect('equal')
plt.savefig('scan2.pdf')

```