Selected Solutions to *Linear Algebra Done Wrong*

HUANG Jing

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Introduction of This Project

Linear Algebra Done Wrong by Sergei Treil is a well-known and popular linear algebra reference book for students. While when I read this book and did the exercises, I found no solution manual was available online or through any other access. So there was no good way to check my solution. But insights tends to hide behind mistakes or ambiguity. Moreover, there would be occasions where the hint given by Sergei is not sufficient. Solution could give the clue and save time. Anyway, I scanned all and did most of the problems of the first 6 chapters (2014 version) and share those I think valuable to you. Of course, there is no guarantee of the correctness of these solutions and they also may not be the best, either. As I'm now a PhD student with limited time for this project, the update is sporadic and your advice and contributions are most expected. Please contact me at *huangjingonly@gmail.com* and we can make this project better and really helpful for those in need.

Chapter 1. Basic Notations

1.4. Prove that a zero vector of a vector space V is unique.
Proof Suppose there exist 2 different zero vectors $\mathbf{0}_1$ and $\mathbf{0}_2$. So for any $\mathbf{v} \in V$, we have

$$\mathbf{v} + \mathbf{0}_1 = \mathbf{v}$$

$$\mathbf{v} + \mathbf{0}_2 = \mathbf{v}$$

Find the difference of the equations above, we get

$$\mathbf{0}_2 - \mathbf{0}_1 = \mathbf{v} - \mathbf{v} = \mathbf{0}_1/\mathbf{0}_2$$

then

$$\mathbf{0}_2 = \mathbf{0}_1 + \mathbf{0}_1/\mathbf{0}_2 = \mathbf{0}_1$$

So the zero vector is unique.

1.6. Prove that the additive inverse, defined in Axiom 4 of a vector space is unique.

Proof Assume there exist 2 different additive inverses $\mathbf{w}_1$ and $\mathbf{w}_2$ of vector $\boldsymbol{v} \in V$. Then

$$\mathbf{v} + \mathbf{w}_1 = \mathbf{0}$$

$$\mathbf{v} + \mathbf{w}_2 = \mathbf{0}$$

Obtain the difference of the two equations, we get

$$\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{0}$$

then

$$\mathbf{w}_1 = \mathbf{w}_2$$

So the additive inverse is unique.

1.7. Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v} \in V$.

Proof $0\mathbf{v} = (0\alpha)\mathbf{v} = \alpha(0\mathbf{v})$ for any scalar α, so $(1 - \alpha)0\mathbf{v} = 0$ foe all scalar $(1 - \alpha)$. Then $0\mathbf{v} = \mathbf{0}$.

1.8. Prove that for any vector $\mathbf{v}$ its additive inverse $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.

Proof $\mathbf{v} + (-1)\mathbf{v} = (1 - 1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ and we know form **1.6** that the additive inverse is unique. So $-\mathbf{v} = (-1)\mathbf{v}$.

2.5. Let a system of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ be linearly independent but not generating. Show that it is possible to find a vector $\mathbf{v}_{r+1}$ such that the system $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r, \mathbf{v}_{r+1}$ is linear independent.

Proof Take $\mathbf{v}_{r+1}$ that can not be represented as $\sum_{k=1}^{r} \alpha_k \mathbf{v}_k$. It's possible because $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ is not generating. Now we need to show $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r, \mathbf{v}_{r+1}$ is linear independent. Suppose that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r, \mathbf{v}_{r+1}$ is linear dependent. i.e.

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + ... + \alpha_r \mathbf{v}_r + \alpha_{r+1} \mathbf{v}_{r+1} = \mathbf{0}$$

and $\sum_{k=1}^{r+1} |\alpha_k| \neq 0$. If $\alpha_{r+1} = 0$, then

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + ... + \alpha_r \mathbf{v}_r = \mathbf{0}$$

and $\sum_{k=1}^{r} |\alpha_k| \neq 0$. This contradicts that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ is linearly independent. So $\alpha_{r+1} \neq 0$. Thus $\mathbf{v}_{r+1}$ can be represented as

$$\mathbf{v}_{r+1} = -\frac{1}{\alpha_{r+1}} \sum_{k=1}^{r} \alpha_k \mathbf{v}_k$$

This contradicts the premise that $\mathbf{v}_{r+1}$ can not be represented by $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$. Thus, the system $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent.

3.2. Let a linear transformation in $\mathbb{R}^2$ be in the line $x_1 = x_2$. Find its matrix.

Solution 1. Reflection is linear transformation. It is completely defined on the standard basis. And $\mathbf{e}_1 = [1\ 0]^{\mathbf{T}} \overset{T}{\Rightarrow} \mathbf{r}_1 = [0\ 1]^{\mathbf{T}}$, $\mathbf{e}_2 = [0\ 1]^{\mathbf{T}} \overset{T}{\Rightarrow} \mathbf{r}_2 = [1\ 0]^{\mathbf{T}}$. So the matrix is the combination of the two transformed standard basis as its first and second column. i.e.

$$T = [\mathbf{r}_1\ \mathbf{r}_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution 2. (A more general method) Let α be the angle between the $x-$axis and the line. The reflection can be achieved through following steps: first, rotate the line around the origin ($z-$axis in 3D space) $-\alpha$ so the line aligns with the $x-$axis (This line happen to pass through the origin, if not, translation is needed in advance to make the line pass through the origin and we need to use *homogeneous coordinates* since translation is not a linear transformation if represented in standard coordinates). Secondly, perform reflection about the $x-$axis. Lastly, we need to rotate the current frame back to its original location or perform other corresponding inverse transformation. So

$$T = Rotz(-\alpha) \cdot Ref \cdot Rotz(\alpha)$$

That is

$$
\begin{aligned}
T &= \begin{bmatrix} cos(-\alpha) & -sin(-\alpha) \\ sin(-\alpha) & cos(-\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} cos(-\frac{\pi}{4}) & -sin(-\frac{\pi}{4}) \\ sin(-\frac{\pi}{4}) & cos(-\frac{\pi}{4}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} cos(\frac{\pi}{4}) & -sin(\frac{\pi}{4}) \\ sin(\frac{\pi}{4}) & cos(\frac{\pi}{4}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{aligned}
$$

3.7. Show that any linear transformation in $\mathbb{C}$ (treated as a complex vector space) is a multiplication by $\alpha \in \mathbb{C}$.

Proof Suppose a linear transformation $T : \mathbb{C} \Rightarrow \mathbb{C}$. $T(1) = a + ib$ and then $T(-1) = -T(1) = -a - ib$. Note that $i^2 = -1$. Then $T(-1) = T(i^2) = iT(i)$. Thus

$$T(i) = \frac{-a - ib}{i} = i(a + ib)$$

So for any $\omega = x + iy \in \mathbb{C}$,

$$
\begin{aligned}
T(\omega) = T(x + iy) &= xT(1) + yT(i) \\
&= x(a + ib) + yi(a + ib) \\
&= (x + iy)(a + ib) \\
&= \omega T(1) \\
&= \omega \alpha
\end{aligned}
$$

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and $\alpha = T(1)$.

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Chapter 2. Systems of linear equations