Selected Solutions to *Linear Algebra Done Wrong*

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Project Introduction

Linear Algebra Done Wrong by Sergei Treil is a well-known linear algebra reference book for students. While when I read this book and did the exercises, I found no solution manual was available online or through any other access. But the solution manual for the other famous book *Linear Algebra Done Right* can be easily found because *Dong Right* is more popular than *Dong Wrong* (without any doubt, *Dong Wrong* is an excellent book worth reading). Insights tend to hide behind mistakes and ambiguity. Without a good way to check the answer, we may miss them. The reference solution also helps to give the inspiration and save time when the hint given by Sergei is not sufficient.

Anyway, I scanned all and did most of the problems of the first 6 chapters (2014 version) and share those I think valuable to you (those are relatively hard or insightful from my perspective, I read this book for kind of reviewing and deeper mathematical understanding). The rest problems should be solvable even for new learners. Of course, there is no guarantee of the correctness of these solutions and they also may not be the best, either. One important thing is that the material is meant for reference to facilitate learning. Please avoid any copying and strictly obey the academic codes of your college. This really matters.

As I'm now a PhD student with limited time for this project, the update is sporadic and your advice and contributions are most welcomed. Please contact me at `huangjingonly@gmail.com` so that we can make this project better and really helpful for those in need.

Chapter 1. Basic Notations

1.4. Prove that a zero vector of a vector space V is unique.

Proof Suppose there exist 2 different zero vectors $\mathbf{0}_1$ and $\mathbf{0}_2$. So for any $\mathbf{v} \in V$, we have

$$\mathbf{v} + \mathbf{0}_1 = \mathbf{v}$$

$$\mathbf{v} + \mathbf{0}_2 = \mathbf{v}$$

Find the difference of the equations above, we get

$$\mathbf{0}_2 - \mathbf{0}_1 = \mathbf{v} - \mathbf{v} = \mathbf{0}_1/\mathbf{0}_2$$

then

$$\mathbf{0}_2 = \mathbf{0}_1 + \mathbf{0}_1/\mathbf{0}_2 = \mathbf{0}_1$$

So the zero vector is unique.

1.6. Prove that the additive inverse, defined in Axiom 4 of a vector space is unique.
Proof Assume there exist 2 different additive inverses $\mathbf{w}_1$ and $\mathbf{w}_2$ of vector $\boldsymbol{v} \in V$. Then

$$\mathbf{v} + \mathbf{w}_1 = \mathbf{0}$$

$$\mathbf{v} + \mathbf{w}_2 = \mathbf{0}$$

Obtain the difference of the two equations, we get

$$\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{0}$$

then

$$\mathbf{w}_1 = \mathbf{w}_2$$

So the additive inverse is unique.

1.7. Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v} \in V$.
Proof $0\mathbf{v} = (0\alpha)\mathbf{v} = \alpha(0\mathbf{v})$ for any scalar α, so $(1-\alpha)0\mathbf{v} = 0$ for all scalar $(1-\alpha)$. Then $0\mathbf{v} = \mathbf{0}$.

1.8. Prove that for any vector $\mathbf{v}$ its additive inverse $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.
Proof $\mathbf{v} + (-1)\mathbf{v} = (1-1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ and we know form **1.6** that the additive inverse is unique. So $-\mathbf{v} = (-1)\mathbf{v}$.

2.5. Let a system of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ be linearly independent but not generating. Show that it is possible to find a vector $\mathbf{v}_{r+1}$ such that the system $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r, \mathbf{v}_{r+1}$ is linear independent.
Proof Take $\mathbf{v}_{r+1}$ that can not be represented as $\sum_{k=1}^{r} \alpha_k \mathbf{v}_k$. It is possible because $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ is not generating. Now we need to show $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r, \mathbf{v}_{r+1}$ is linear independent. Suppose that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r, \mathbf{v}_{r+1}$ is linear dependent. i.e.

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + ... + \alpha_r\mathbf{v}_r + \alpha_{r+1}\mathbf{v}_{r+1} = \mathbf{0}$$

and $\sum_{k=1}^{r+1} |\alpha_k| \neq 0$. If $\alpha_{r+1} = 0$, then

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + ... + \alpha_r\mathbf{v}_r = \mathbf{0}$$

and $\sum_{k=1}^{r} |\alpha_k| \neq 0$. This contradicts that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$ is linearly independent. So $\alpha_{r+1} \neq 0$. Thus $\mathbf{v}_{r+1}$ can be represented as

$$\mathbf{v}_{r+1} = -\frac{1}{\alpha_{r+1}} \sum_{k=1}^{r} \alpha_k \mathbf{v}_k$$

This contradicts the premise that $\mathbf{v}_{r+1}$ can not be represented by $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$. Thus, the system $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent.

3.2. Let a linear transformation in $\mathbb{R}^2$ be in the line $x_1 = x_2$. Find its matrix.
Solution 1. Reflection is a linear transformation. It is completely defined on the standard basis. And $\mathbf{e}_1 = [1\,0]^\mathsf{T} \overset{T}{\Rightarrow} \mathbf{r}_1 = [0\,1]^\mathsf{T}$, $\mathbf{e}_2 = [0\,1]^\mathsf{T} \overset{T}{\Rightarrow} \mathbf{r}_2 = [1\,0]^\mathsf{T}$. So the matrix is the combination of the two transformed standard basis as its first and second column. i.e.

$$T = [\mathbf{r}_1\,\mathbf{r}_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution 2. **(A more general method)** Let α be the angle between the x-axis and the line. The reflection can be achieved through following steps: first, rotate the line around the origin (z-axis in 3D space) $-\alpha$ so the line aligns with the x-axis (This line happen to pass through the origin, if not, translation is needed in advance to make the line pass through the origin and we need to use *homogeneous coordinates* since translation is not a linear transformation if represented in standard coordinates). Secondly, perform reflection about the x-axis. Lastly, we need to rotate the current frame back to its original location or perform other corresponding inverse transformation. So

$$T = Rotz(-\alpha) \cdot Ref \cdot Rotz(\alpha)$$

That is

$$\begin{aligned} T &= \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

3.7. Show that any linear transformation in $\mathbb{C}$ (treated as a complex vector space) is a multiplication by $\alpha \in \mathbb{C}$.
Proof Suppose a linear transformation $T : \mathbb{C} \Rightarrow \mathbb{C}$. $T(1) = a + ib$ and then

$T(-1) = -T(1) = -a - ib$. Note that $i^2 = -1$. Then $T(-1) = T(i^2) = iT(i)$. Thus

$$T(i) = \frac{-a - ib}{i} = i(a + ib)$$

So for any $\omega = x + iy \in \mathbb{C}$,

$$\begin{aligned} T(\omega) &= T(x + iy) = xT(1) + yT(i) \\ &= x(a + ib) + yi(a + ib) \\ &= (x + iy)(a + ib) \\ &= \omega T(1) \\ &= \omega\alpha \end{aligned}$$

and $\alpha = T(1)$.

5.4. Find the matrix of the orthogonal projection in $\mathbb{R}^2$ onto the line $x_1 = -2x_2$.

Solution

$$\begin{aligned} T &= R(\alpha) P R(-\alpha) \\ &= R(\alpha) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(-\alpha) \end{aligned}$$

and $\alpha = \tan^{-1}(-\frac{1}{2})$ so we can get the matrix is

$$T = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

5.7. Find the matrix of the reflection through the line $y = -2x/3$. Perform all the multiplications.

Solution *(Similar to 5.4 though not exactly the same.)*

$$\begin{aligned} T &= R(\alpha) Ref R(-\alpha) \\ &= R(\alpha) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R(-\alpha) \end{aligned}$$

and $\alpha = \tan^{-1}(-\frac{2}{3})$ so we can get the matrix is

$$T = \begin{bmatrix} \frac{5}{13} & -\frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{bmatrix}$$

6.1. Prove that if $A : V \to W$ is an isomorphism (i.e. an invertible linear transformation) and $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ is a basis in V, then $A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_n$ is

a basis in W.
Proof Any $w \in W$, $A^{-1}w = v \in V$ and

$$\begin{aligned} w = Av &= A[\mathbf{v}_1\ \mathbf{v}_2\ ...\ \mathbf{v}_n][v_1\ v_2\ ...\ v_n]^\mathsf{T} \\ &= [A\mathbf{v}_1\ A\mathbf{v}_2\ ...\ A\mathbf{v}_n][v_1\ v_2\ ...\ v_n]^\mathsf{T} \end{aligned}$$

So we can see $[A\mathbf{v}_1\ A\mathbf{v}_2\ ...\ A\mathbf{v}_n]$ is in the form of a basis in W. Next we show that $A\mathbf{v}_1\ A\mathbf{v}_2\ ...\ A\mathbf{v}_n$ is linearly independent. If not, suppose $A\mathbf{v}_1$ can be expressed as a linear combination of $A\mathbf{v}_2\ A\mathbf{v}_3\ ...\ A\mathbf{v}_n$ without loss of generality. Multiplying them with A in the left side, it results in that $\mathbf{v}_1$ can be expressed by $\mathbf{v}_2\ ...\ \mathbf{v}_n$, which contradicts the fact that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ is a basis in V. So the proposition is proved.

7.4. Let $\mathbf{X}$ and $\mathbf{Y}$ be subspaces of a vector space $\mathbf{V}$. Using the previous exercise, show that $\mathbf{X} \cup \mathbf{Y}$ is a subspace if and only if $\mathbf{X} \subset \mathbf{Y}$ or $\mathbf{Y} \subset \mathbf{X}$.
Proof The sufficiency is obvious and easy to verify. For the necessity, suppose $\mathbf{X} \not\subset \mathbf{Y}$ nor $\mathbf{Y} \not\subset \mathbf{X}$ and $\mathbf{X} \cup \mathbf{Y}$ is a subspace of $\mathbf{V}$. Then there are vectors $\mathbf{x} \in \mathbf{X}$, $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{x} \notin \mathbf{Y}$, $\mathbf{y} \notin \mathbf{X}$. According to **7.3**, $\mathbf{x} + \mathbf{y} \notin \mathbf{X}$, $\mathbf{x} + \mathbf{y} \notin \mathbf{Y}$. So, $\mathbf{x} + \mathbf{y} \notin \mathbf{X} \cup \mathbf{Y}$. i.e.,
$\mathbf{x} \in \mathbf{X} \cup \mathbf{Y}$, $\mathbf{y} \in \mathbf{X} \cup \mathbf{Y}$, but $\mathbf{x} + \mathbf{y} \notin \mathbf{X} \cup \mathbf{Y}$, which contradicts $\mathbf{X} \cup \mathbf{Y}$ is a subspace. Thus, $\mathbf{X} \subset \mathbf{Y}$ or $\mathbf{Y} \subset \mathbf{X}$.

8.5. A transformation T in $\mathbb{R}^3$ is a rotation about the line $y = x + 3$ in the x-y plane through an angle γ. Write a 4×4 matrix corresponding to this transformation.
You can leave the result as a product of matrices.
Solution For a general spatial rotation around a given direction (suppose the direction is given by a vector) through an angle γ, the 3×3 rotation matrix can be given by:

$$R = R_x^{-1} R_y^{-1} R_z(\gamma) R_y R_x,$$

where the rotation by γ is assumed to be performed around z-axis. R_x and R_y are rotations used to align the direction with z-axis and can be determined by simple trigonometry.
For the problem given, the line $y = x + 3$ doesn't go through the origin, so extra step T_0 is needed to translate the line to make it pass the origin and homogeneous coordinates are applied:

$$R = T_0^{-1} R_x^{-1} R_y^{-1} R_z(\gamma) R_y R_x T_0.$$

According to the description,

$$\begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix} = R_x^{-1} R_y^{-1} R_z(\gamma) R_y R_x.$$

The corresponding matrix is then

$$R = T_0^{-1} \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix} T_0.$$

T_0 is not unique for the translation to make two parallel lines align.

Chapter 2. Systems of Linear Equations

3.8. Show that if the equation $A\mathbf{x} = \mathbf{0}$ has unique solution (i.e. if echelon form of A has pivot in every column), then A is left invertible.
Proof $A\mathbf{x} = \mathbf{0}$ has unique solution, then the solution is trivial solution. The echelon form of A has pivot at every column. A is $m \times n$ matrix, then $m \geq n$. The row number is greater or equal to the column number. The reduced echelon form of A is denoted as

$$A_{re} = \begin{bmatrix} I_{n\times n} \\ \mathbf{0}_{(m-n)\times n} \end{bmatrix}.$$

And suppose A_{re} is obtained by a sequence of elementary row operation $E_1, E_2, ..., E_k$,

$$A_{re} = E_k \, ... \, E_2 E_1 A$$

E_i is $m \times m$. The left inverse of A is the first n rows of the product of E_i. i.e.

$$E_{left} = I_{n\times m} E_k \, ... \, E_2 E_1,$$

where

$$I_{n\times m} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots \end{bmatrix}_{n\times m},$$

is used to extract the $I_{n\times n}$ identity matrix in A_{re}. $E_{left}A = I$ so A is left invertible.

5.5. Let vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be a basis in V. Show that $\mathbf{u} + \mathbf{v} + \mathbf{w}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{w}$ is also a basis in V.
Solution For any vector $\mathbf{x} \in V$, suppose $\mathbf{x} = x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$. Then it is easy to figure out the $\mathbf{x} = x_1(\mathbf{u} + \mathbf{v} + \mathbf{w}) + (x_2 - x_1)(\mathbf{v} + \mathbf{w}) + (x_3 - x_2 - x_1)\mathbf{w}$. Thus $\mathbf{u} + \mathbf{v} + \mathbf{w}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{w}$ is also a basis in V.

7.4 Prove that is $A : X \to Y$ and V is a subspace of X then dim $AV \leq$ rank A. (AV here means the subspace V transformed by the transformation A, i.e., any vector in AV can be represented as $A\mathbf{v}, \mathbf{v} \in V$). Deduce from here that rank$(AB) \leq$ rank$A$.

Proof $\dim AV \leq \dim AX \leq \dim \mathrm{Ran} A = \mathrm{rank} A$
Suppose that the column vectors of A compose a basis of space V. Then $\mathrm{rank}(AB) \leq \mathrm{rank} A$.

7.5 Prove that if $A : X \to Y$ and V is a subspace of X then dim $AV \leq$ dim V. Deduce from here that rank$(AB) \leq$ rank B.
Proof Suppose dim $V = k$ and let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ be a basis of V. AV is defined by $A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_k$. dim $AV =$ rank $[A\mathbf{v}_1, A\mathbf{v}_2, ..., A\mathbf{v}_k] \leq k =$ dim V.

Similarly, assume rank $B = k$ and $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_k$ are linearly independent column vectors in B. Then rank $AB =$ rank $[A\mathbf{b}_1, A\mathbf{b}_2, ..., A\mathbf{b}_k] \leq k =$ rank B.

7.6 Prove that if the product AB of two $n \times n$ matrices is invertible, then both A and B are invertible. Do not use determinant for this problem.
Proof AB is invertible, rank$(AB) = n$. From Problem 7.5, we have rank$(AB) = n \leqslant$ rank$(A) \leqslant n$. Thus rank$(A) = n$. So is B. A, B have full rank and are invertible.

7.7 Prove that if $A\mathbf{x} = \mathbf{0}$ has unique solution, then the equation $A^\mathsf{T}\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$. (*Hint:* count pivots)
Proof Suppose $A \in \mathbb{R}^m \times \mathbb{R}^n$. Note that for $A\mathbf{x} = \mathbf{0}$, there is always a trivial solution $\mathbf{x} = \mathbf{0} \in \mathbb{R}^n$. And we know the trivial solution is unique, which also indicates that the echelon form of A has a pivot at every column. Accordingly, the echelon form of A^T has a pivot at every row (Think that the echelon form of A^T is completed by column reduction that corresponds to the row reduction of A). So $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$.

7.14 Is it possible for a real matrix A that Ran $A =$ Ker A^T? Is it possible for a complex A?
Solution Both are not possible. Suppose A is $m \times n$ and Ran $A =$ Ker A^T. Then Ran $A \subset$ Ker A^T, i.e., $A^\mathsf{T}A\mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in \mathbb{R}^n$. This holds only when $A^\mathsf{T}A = 0_{n\times n}$. Then $A = 0_{m\times n}$. (Use the row vectors of A^T and check the diagonal entries of $A^\mathsf{T}A$ equal to 0. It will lead to the conclusion that the row vectors are all zero vector.)

On the other hand, if Ran $A =$ Ker A^T, Ker $A^\mathsf{T} \subset$ Ran A. i.e., if $A^\mathsf{T}\mathbf{b} = \mathbf{0}$, then the function $A\mathbf{x} = \mathbf{b}$ has a solution. But we have $A = 0_{m\times n}$, then for arbitrary $b \in \mathbb{R}^m$, $A^\mathsf{T}\mathbf{b} = \mathbf{0}$ holds. But for $\mathbf{b} \neq \mathbf{0}$, $A\mathbf{x} = \mathbf{b}$ does not have a solution. This is contradictory. So it is not possible for the real or complex matrix A that Ran $A =$ Ker A^T.

8.5 Prove that if A and B are similar matrices then trace $A =$ trace B. (*Hint:* recall how trace(XY) and trace(YX) are related.)
Proof trace$(A) =$ trace$(Q^{-1}BQ) =$ trace$(Q^{-1}QB) =$ trace(B). (Note that

trace(AB) = trace(BA) as long as AB, BA can be performed.)

Chapter 3. Determinants

3.4 A square matrix ($n \times n$) is called skew-symmetric (or antisymmetric) if $A^{\mathsf{T}} = A$. Prove that if A is skew-symmetric and n is odd, then $\det A = 0$. Is thsi true for even n?
Proof $\det A = \det A^{\mathsf{T}} = \det -A = (-1)^n \det A$ by using the the properties of determinant and skew-symmetric matrices. If n is odd, $(-1)^n = -1$, we have $\det A = -\det A$, thus $\det A = 0$.

If n is even, we just have $\det A = \det A$ so this conclusion generally is not true.

3.5 A square matrix is called *nilpotent* if $A^k = \mathbf{0}$ for some positive integer k. Show that for a nilpotent matrix A, $\det A = 0$.
Proof $\det A^k = (\det A)^k = \det \mathbf{0} = 0$, thus $\det A = 0$.

3.6 Prove that if A and B are similar, then $\det A = \det B$.
proof A and B are similar, then $A = Q^{-1}BQ$ for an invertible matrix Q.Then

$$\begin{aligned}
\det A &= \det Q^{-1}BQ \\
&= (\det Q^{-1})(\det B)(\det Q) \\
&= (\det Q^{-1})(\det Q)(\det B) \\
&= (\det Q^{-1}Q)(\det B) \\
&= (\det I)(\det B) \\
&= \det B.
\end{aligned}$$

3.7 A real square matrix Q is called orthogonal if $Q^{\mathsf{T}}Q = I$. Prove that if Q is an orthogonal matrix then $\det Q = \pm 1$.
Proof $\det Q^{\mathsf{T}}Q = (\det Q^{\mathsf{T}})(\det Q) = (\det Q)^2 = \det I = 1$. Thus $\det Q = \pm 1$.

3.9 Let points A, B and C in the plane $\mathbb{R}^2$ have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. Show that the area of triangle ABC is the absolute value of

$$\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$

Hint: use row operation and geometric interpretation of 2×2 determinants (area).

Proof The area of triangle ABC is half of the parallelogram defined by neighbouring sides AB, AC. which also can be computed by

$$\begin{aligned} S_{\triangle ABC} &= \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\ &= \frac{1}{2}|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|, \end{aligned}$$

In the same time, if we use row reduction to check the determinant

$$\begin{aligned} \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} &= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\ &= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & 0 & y_3 - y_1 - (y_2 - y_1)\frac{x_3 - x_1}{x_2 - x_1} \end{vmatrix} . \\ &= \frac{1}{2}((x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)), \end{aligned}$$

We assume that $x_2 - x_1 \neq 0$ and it can be verified if $x_2 - x_1 = 0$, the result still holds. Thus we can see the conclusion holds.

3.10 Let A be a square matrix. Show that block triangular matrices

$$\begin{bmatrix} I & * \\ \mathbf{0} & A \end{bmatrix} \quad \begin{bmatrix} A & * \\ \mathbf{0} & I \end{bmatrix} \quad \begin{bmatrix} I & \mathbf{0} \\ * & A \end{bmatrix} \quad \begin{bmatrix} A & \mathbf{0} \\ * & I \end{bmatrix}$$

all have determinant equal to $\det A$. Here $*$ can be anything.
Proof Considering performing row reduction to make A be triangular, the whole matrix will also be triangular and the rest part on the diagonal is just I. Thus the determinant of the block matrix equals to $\det A$.
(Problem 3.11 and 3.12 are just applications of the conclusion of Problem 3.10. The hint just tells the answer.)

4.2 Let P be a *permutation matrix*, i.e., an $n \times n$ matrix consisting of zeros and ones and such that there is exactly one 1 in every row and every column.

a) Can you describe the corresponding linear transformation? That will explain the name.

b) Show that P is invertible. Can you describe P^{-1}?

c) Show that for some $N > 0$

$$P^N := \underbrace{PP \dots P}_{N \text{ times}} = I.$$

Use the fact that there are only finitely many permutations.

Solution a) Consider the linear transformation $\mathbf{y} = P\mathbf{x}$ and each row of P. There is only one 1 in each row of P. Suppose in the first row of P, $P_{1,j} = 1$, then $y_1 = \mathbf{p_1}\mathbf{x} = x_j$, where $\mathbf{p_1}$ is the first row of P. Namely x_j is moved to the 1st place after the linear transformation. Similarly, for the 2nd row of P, suppose $P_{2,k} = 1$, then $y_2 = x_k$, x_k is moved to the 2nd place, so on and so forth. There is also only 1 for each column, then we know after multiplying by the permutation matrix P, the elements in $\mathbf{x}$ change their order.

b) Suppose P is invertible, by multiplying P^{-1}, $\mathbf{x} = P^{-1}\mathbf{y}$. But we know $y_1 = x_j$, then we have $P^{-1}_{j,1} = 1$ so that x_j can return to its original position. Similarly, $y_2 = x_k$, then $P^{-1}_{k,2} = 1$. Following this we can see that $P^{-1}_{i,j} = P_{j,i}$ if $P_{j,i} = 1$ and the rest are all 0. So we can see P is invertible and $P^{-1} = P^{\mathsf{T}}$.

c) Note that $P\mathbf{x}, P^2\mathbf{x}, P^3\mathbf{x} \ldots P^N\mathbf{x}$ are all permutations of $(x_1, x_2, ..., x_n)$. If P^N can never equal to I, $P\mathbf{x}, P^2\mathbf{x}, P^3\mathbf{x} \ldots P^N\mathbf{x}$ will be different permutations. And N can be infinitely big, so there will be infinitely many permutations of $(x_1, x_2, ..., x_n)$, which is impossible. Thus there must be some $N > 0$, $P^N = I$.

Exercises Prat 5 and Part 7 in this chapter are normal. So I try to give some ideas and answers for reference:

- Problem 5.3, we can use the last column expansion and the left matrix $(A+tI)_{i,j}$ is a triangular matrix. The final expression is $\det(A+tI) = a_0 + a_1t + a_2t^2 + ... + a_{n-1}t^{n-1}$. The order of -1 in each term is even.

- Problem 5.7, $n!$ multiplications is needed. We can use induction to prove it.

- Problem 7.4 and Problem 7.5, consider $\det RA = (\det R)(\det A) = \det A$, where R is the rotation matrix with its determinant equal to 1. For proof of the parallelogram area, we can also use parameter angle, i.e., $\mathbf{v_1} = [x_1, y_1]^{\mathsf{T}} = [v_1 \cos\alpha,\ v_1 \sin\alpha]^{\mathsf{T}}$, $\mathbf{v_2} = [x_2, y_2]^{\mathsf{T}} = [v_2 \cos\beta,\ v_2 \sin\beta]^{\mathsf{T}}$. v_1, v_2 are the lengths of $\mathbf{v_1}, \mathbf{v_2}$, respectively. α, β represents the angle between the vector and x-axis positive direction. Then

$$\begin{aligned}\det A = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} &= x_1y_2 - x_2y_1 \\ &= v_1v_2(\cos\alpha\sin\beta - \cos\beta\sin\alpha) \\ &= v_1v_2\sin(\beta-\alpha).\end{aligned}$$

  $\beta - \alpha$ is the angle from $\mathbf{v_1}$ to $\mathbf{v_2}$.

Chapter 4. Introduction to Spectral Theory (Eigenvalues and Eigenvectors)

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