Ex1.

Wei Huang

$$\frac{\partial u}{\partial t} = -\sin(x)\frac{\partial u}{\partial x}$$

We seek a trigonometric polynomial,

$$u_N(x,t) = \sum_{|n|\le\frac{N}{2}} a_n(t)\, e^{inx}$$

and require that the residual

$$R_N(x,t) = \frac{\partial u_N(x,t)}{\partial t} + \sin(x)\frac{\partial u_N(x,t)}{\partial x}$$

is orthogonal to $\hat{B}_N$.

$$R_N(x,t) = \sum_{|n|\le\frac{N}{2}} \left(\frac{da_n(t)}{dt} + \frac{e^{ix}-e^{-ix}}{2i}(in)\, a_n(t) \right) e^{inx}$$

$$R_N(x,t) = \sum_{|n|\le\frac{N}{2}} \frac{da_n(t)}{dt} e^{inx} + \frac{1}{2}\sum_{|n|\le\frac{N}{2}} n e^{i(n+1)x} a_n(t) - \frac{1}{2}\sum_{|n|\le\frac{N}{2}} n e^{i(n-1)x} a_n(t)$$

$$= \sum_{|n|\le\frac{N}{2}} \frac{da_n(t)}{dt} e^{inx} + \frac{1}{2}\sum_{|n|\le\frac{N}{2}} (n-1) e^{inx} a_{n-1}(t) - \frac{1}{2}\sum_{|n|\le\frac{N}{2}} (n+1) e^{inx} a_{n+1}(t)$$

$$+ \frac{N}{4}\left(e^{i\frac{N+2}{2}x} a_{\frac{N}{2}}(t) + e^{-i\frac{N+2}{2}x} a_{-\frac{N}{2}}(t) \right)$$

where $a_{-(\frac{N}{2}+1)}(t) = a_{\frac{N}{2}+1}(t) = 0$.

Finally we get, for $|n| \le \frac{N}{2}$

$$\frac{da_n(t)}{dt} + \frac{n-1}{2} a_{n-1}(t) - \frac{n+1}{2} a_{n+1}(t) = 0$$

with $a_{-(\frac{N}{2}+1)}(t) = a_{\frac{N}{2}+1}(t) = 0$.

We notice that

$$P_N \mathcal{L} \neq \mathcal{L} P_N ,$$

and so the truncation error

$$P_N \mathcal{L}(I - P_N) u \neq 0 .$$

In this case $a_n(t)$ are not equal to Fourier coefficients $\hat{u}_n$

So $P_N u \neq u_N(x,t)$.

Ex 2. We could assume test functions are defined as

$$\psi_n(x) = \frac{2}{\pi} \sin(nx) .$$

$$\psi_n(0) = \psi_n(\pi) = 0 \quad \text{for all } n.$$

We seek a approximation

$$u_N(x,t) = \sum_{n=0}^{N} \hat{u}_n(t) \sin(nx)$$

and require that the residual

$$R_N(x,t) = \frac{\partial u_N(x,t)}{\partial t} + \sin(x) \frac{\partial u_N(x,t)}{\partial x}$$

is orthogonal to the test space.

$$R_N(x,t) = \sum_{n=0}^{N} \frac{\partial \hat{u}_N(t)}{\partial t} \sin(nx) + \sin(x) \sum_{n=0}^{N} n \hat{u}_n(t) \cos(nx)$$

$$\cancel{R_N(x,t)} = \sum_{n=0}^{N} \frac{\partial \hat{u}_N(t)}{\partial t} \sin(nx) + \sum_{n=0}^{N} n \hat{u}_n(t) \frac{\sin((n+1)x) - \sin((n-1)x)}{2} .$$

By defining $\hat{u}_{-1}(t) = \hat{u}_{N+1}(t) = 0$, we could get

$$R_N = \sum_{n=0}^{N} \frac{\partial \hat{u}_n(t)}{\partial t}\sin(nx) + \sum_{n=0}^{N}(n-1)\hat{u}_{n-1}(t)\frac{\sin(nx)}{2} + \frac{\hat{u}_N(t)}{2} N \sin((N+1)x)$$

$$- \sum_{n=0}^{N}(n+1)\hat{u}_{n+1}\frac{\sin(nx)}{2} - \frac{\hat{u}_0(t)}{2}\cdot 0 \sin(-x).$$

$$= \sum_{n=0}^{N}\left(\frac{\partial \hat{u}_n(t)}{\partial t} + \frac{n-1}{2}\hat{u}_{n-1}(t) - \frac{n+1}{2}\hat{u}_{n+1}(t)\right)\sin(nx) + \text{trancation error}.$$

Finally we obtain, for $0 \le n \le N$

$$\frac{\partial \hat{u}_n(t)}{\partial t} + \frac{n-1}{2}\hat{u}_{n-1}(t) - \frac{n+1}{2}\hat{u}_{n+1}(t) = 0$$

with $\hat{u}_{-1}(t) = \hat{u}_{N+1}(t) = 0$.

Ex3.

① Assume $\psi_n = \frac{2}{\pi}\cos(nx)$

② Assume

$$u_N(x,t) = \sum_{n=0}^{N+2}\hat{u}_n(t)\cos(nx)$$

③ $(R_N, \psi_n) = 0$ for $0 \le n \le N+2$

$$R_N = \sum_{n=0}^{N+2}\left(\frac{\partial \hat{u}_n(t)}{\partial t} - \frac{n+1}{2}\hat{u}_{n+1}(t) + \frac{n-1}{2}\hat{u}_{n-1}(t)\right)\cos(nx) + \text{trancation error}$$

with $\hat{u}_{-1}(t) = \hat{u}_{N+3}(t) = 0$

$$\Rightarrow \frac{\partial \hat{u}_n(t)}{\partial t} - \frac{n+1}{2}\hat{u}_{n+1}(t) + \frac{n-1}{2}\hat{u}_{n-1}(t) = 0$$

④
$$\begin{cases} \sum_{n=0}^{N+2}\hat{u}_n(t) = 0 \\ \sum_{n=0}^{N+2}(-1)^n\hat{u}_n(t) = 0 \end{cases}$$

Ex4.

Burgers equation could be rewritten as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \varepsilon \frac{\partial^2 u}{\partial x^2} = 0$$

We seek solution of the form.

$$u_N(x,t) = \sum_{|n|\le\frac{N}{2}} a_n(t) e^{inx} = \sum_{j=0}^{N-1} u_N(x_j,t) g_j(x)$$

$$R_N(x,t) = \frac{\partial u_N(x,t)}{\partial t} + u_N(x,t)\frac{\partial u_N(x,t)}{\partial x} - \varepsilon \frac{\partial^2 u_N(x,t)}{\partial x^2}$$

Vanishes at a specified set of grids y_j, in this case.

We assume $y_j = x_j$

this results in.

$$\frac{d u_N(x_j,t)}{\partial t} + \sum_{k=0}^{N-1} \left(u_N(x_j,t) D^{(1)}_{jk} \cancel{u_N(x_k,t)} - \varepsilon D^{(2)}_{jk} \right) u_N(x_k,t)$$

$D^{(1)}$ and $D^{(2)}$ are the differentiation matrices.