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Advanced Discretization Methods
Spring Semester 2020
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Exam

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Part 1

Consider the advection diffusion equation given as

$$\frac{\partial u(x,t)}{\partial t} + U_0(x)\frac{\partial u(x,t)}{\partial x} = \nu \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{1}$$

where $U_0(x)$ is periodic and bounded and ν is assumed to be constant. Also $u(x,t)$ is assumed to be smooth and periodic as is the initial condition.

(a) State sufficient conditions on $U_0(x)$ and ν that ensures Eq. 1 to be well-posed.

(b) Assume that Eq. 1 is approximated using Fourier Collocation method. Is the approximation consistent and what is the expected convergence rate when increasing N, the number of modes used in the approximation.

(c) Assume now that $U_0(x)$ is constant and Eq. 1 is approximated using a Fourier Collocation method with odd number of modes. Prove that the semi-discrete approximation, i.e. continuous time and approximated space, is stable.

Part 2

Consider now Burger's equation given as

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \nu \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{2}$$

where $u(x,t)$ is assumed periodic. Eq. 2 has an analytic solution given as

$$u(x,t) = c - 2\nu \frac{\frac{\partial \phi(x-ct,t+1)}{\partial x}}{\phi(x-ct,t+1)}, \qquad \phi(a,b) = \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(a-(2k+1)\pi)^2}{4\nu b}\right).$$

This solution represents a sawtooth-like traveling wave which propagates at the velocity c, while slowly decaying due to dissipation. In all subsequent tests use $c = 4.0$ and $\nu = 0.1$ and consider the problem in the standard interval of $x \in [0, 2\pi]$.

(a) Construct a program that solves Eq. 2 using Fourier Collocation method with an odd number of grid points, i.e. the grid points are given as

$$x_j = \frac{2\pi}{N+1} j = j\Delta x, \qquad j \in [0, N],$$

and use the exact solution at $t = 0$ as the initial condition.

To advance the equation in time, use 4th order Runge-Kutta method obtained by first defining the function, $F(u^n)$, as

$$\frac{du^n}{dt} = F(u^n) = -u^n \frac{\partial u^n}{\partial x} + \nu \frac{\partial^2 u^n}{\partial x^2},$$

where u^n represents the solution at $t = n\Delta t$ and Δt is the time step. The 4th order Runge-Kutta method for advancing from u^n to u^{n+1} is then defined as

$$u_1 = u^n + \frac{\Delta t}{2} F(u^n)$$

$$u_2 = u^n + \frac{\Delta t}{2} F(u_1)$$

$$u_3 = u^n + \Delta t F(u_2)$$

$$u^{n+1} = \frac{1}{3}\left(-u^n + u_1 + 2u_2 + u_3 + \frac{\Delta t}{2} F(u_3)\right),$$

where u_1, u_2 and u_3 are introduced for convenience as help-functions/arrays.

The time step, Δt, is given as

$$\Delta t \leq \mathrm{CFL} \times \left[\max_{x_j}\left(\frac{|u(x_j)|}{\Delta x} + \frac{\nu}{(\Delta x)^2}\right)\right]^{-1}. \tag{3}$$

(b) Determine by experiment the maximum value of the number CFL in Eq. 3 that results in a stable scheme for $N = 16, 32, 48, 64, 96, 128, 192, 256$. Does the definition of the time-step restriction as given in Eq. 3 seem reasonable (think of the CFL-condition known from Finite Difference Methods)? Use these values of CFL in what remains.

(c) Measure the L^∞-error between the computed solution at $t = \pi/4$ and the exact solution for $N = 16, 32, 48, 64, 96, 128, 192, 256$. What is the convergence rate observed and does it correspond to what you would expect?

(d) Plot the solution for $N = 128$ at $t = 0, \pi/8, \pi/6, \pi/4$.

Part 3

Consider the same problem, parameters, etc as in Part 2.

(a) Construct a program that solves Eq. 2 using Fourier Galerkin method.

Use the quadrature formula to compute the expansion coefficients of the initial conditions and use the 4th order Runge-Kutta method described in Part 2 to advance the Fourier-Galerkin equations in time.

The time step, Δt, is given as

$$\Delta t \leq \mathrm{CFL} \times \left[\max_{x_j}\left(|u(x_j)|k_{\max} + \nu(k_{\max})^2\right)\right]^{-1}, \tag{4}$$

where $k_{\max} = N/2$.

(b) Determine by experiment the maximum value of the number CFL in Eq. 4 that results in a stable scheme for $N = 16, 32, 48, 64, 96, 128, 192, 256$. Use these values of CFL in what remains.

(c) Measure the L^∞-error between the computed solution at $t = \pi/4$ and the exact solution for $N = 16, 32, 48, 64, 96, 128, 192, 256$. What is the convergence rate observed and does it correspond to what you would expect?

(d) Compare the results with those in Part 2. Do you see any significant difference between the two solutions? Any reason to use the Galerkin formulation?