Assignment 1

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Ex 1.

By Taylor expansion.

$$u(x_j+\Delta x) = u(x_j) + u^{(1)}(x_j)\Delta x + \frac{u^{(2)}(x_j)}{2!}(\Delta x)^2 + \frac{u^{(3)}(x_j)}{3!}\Delta x^3 + \frac{u^{(4)}(x_j)}{4!}\Delta x^4 + \frac{u^{(5)}(x_j)}{5!}\Delta x^5 + \frac{u^{(6)}(x_j)}{6!}\Delta x^6 + O(\Delta x^7) \quad ①$$

$$u(x_j-\Delta x) = u(x_j) - u^{(1)}(x_j)\Delta x + \frac{u^{(2)}(x_j)}{2!}(\Delta x^2) - \frac{u^{(3)}(x_j)}{3!}\Delta x^3 + \frac{u^{(4)}(x_j)}{4!}\Delta x^4 - \frac{u^{(5)}(x_j)}{5!}\Delta x^5 + \frac{u^{(6)}(x_j)}{6!}\Delta x^6 + O(\Delta x^7) \quad ②$$

$$① - ② : \quad u(x_j+\Delta x) - u(x_j-\Delta x) = 2\cdot u^{(1)}(x_j)\Delta x + 2\cdot\frac{u^{(3)}(x_j)}{3!}\Delta x^3 + 2\cdot\frac{u^{(5)}(x_j)}{5!}\Delta x^5 + O((\Delta x)^7) \quad ③$$

By subsitituting Δx with $2\Delta x$ and $3\Delta x$, we obtain:

$$u(x_j+2\Delta x) - u(x_j-2\Delta x) = 2\cdot 2\,u^{(1)}(x_j)\Delta x + 2^3\cdot 2\frac{u^{(3)}(x_j)}{3!}\Delta x^3 + 2^5\cdot 2\frac{u^{(5)}(x_j)}{5!}\Delta x^5 + O(\Delta x^7) \quad ④$$

$$u(x_j+3\Delta x) - u(x_j-3\Delta x) = 3\cdot 2\,u^{(1)}(x_j)\Delta x + 3^3\cdot 2\frac{u^{(3)}(x_j)}{3!}\Delta x^3 + 3^5\cdot 2\frac{u^{(5)}(x_j)}{5!}\Delta x^5 + O(\Delta x^7) \quad ⑤$$

By taking a appropriate linear combination of ③, ④ and ⑤, and keep only $u^{(1)}(x_j)$

We would like to Cancel out $u^{(3)}(x_j)$ and $u^{(5)}(x_j)$

$$\begin{cases} 2a + 2\cdot 2\cdot b + 3\cdot 2\cdot c = 1 \\ 2\cdot\frac{a}{3!} + 2^3\cdot 2\cdot\frac{b}{3!} + 3^3\cdot 2\cdot\frac{c}{3!} = 0 \\ 2\cdot\frac{a}{5!} + 2^5\cdot 2\cdot\frac{b}{5!} + 3^5\cdot 2\cdot\frac{c}{5!} = 0 \end{cases} \quad (*)$$

Solution to $(*)$ $\quad a = \frac{45}{60} \quad b = -\frac{9}{60} \quad c = \frac{1}{60}$

$a\cdot ③ + b\cdot ④ + c\,⑤$:

$$\frac{45}{60}\left(u(x_j+\Delta x) - u(x_j-\Delta x)\right) - \frac{9}{60}\left(u(x_j+2\Delta x) - u(x_j-2\Delta x)\right) + \frac{1}{60}\left(u(x_j+3\Delta x) - u(x_j-3\Delta x)\right)$$

$$= u^{(1)}(x_j)\Delta x + O((\Delta x)^7)$$

Finally, we get

$$\left.\frac{du}{dx}\right|_{x_j} = \frac{-u_{j-3} + 9u_{j-2} - 45u_{j-1} + 45u_{j+1} - 9u_{j+2} + u_{j+3}}{60\Delta x}$$

Ex2:

By 6-th order accurate central finite difference approximation.

$$-ik\,C_m(k)\,e^{ik(x-C_m(k)t)} = -C\,\frac{-u_{j-3}+9u_{j-2}-45u_{j-1}+45u_{j+1}-9u_{j+2}+u_{j+3}}{60\Delta x} \quad ①$$

$$\begin{cases} u_{j-3} = e^{ik(x-C_m(k)t)}\,e^{-3ik\Delta x} & u_{j+3} = e^{ik(x-C_m(k)t)}\,e^{3ik\Delta x} \\ u_{j-2} = e^{ik(x-C_m(k)t)}\,e^{-2ik\Delta x} & u_{j+2} = e^{ik(x-C_m(k)t)}\,e^{2ik\Delta x} \\ u_{j-1} = e^{ik(x-C_m(k)t)}\,e^{-ik\Delta x} & u_{j+1} = e^{ik(x-C_m(k)t)}\,e^{ik\Delta x}. \end{cases} \quad ②$$

By substituting ② into ①, we get

$$-ik\,C_m(k) = -C\,\frac{45(e^{ik\Delta x}-e^{-ik\Delta x})-9(e^{2ik\Delta x}-e^{-2ik\Delta x})+(e^{3ik\Delta x}-e^{-3ik\Delta x})}{60\Delta x} \quad ③$$

By Euler formula:

$$\begin{cases} e^{ik\Delta x} - e^{-ik\Delta x} = 2i\sin k\Delta x \\ e^{2ik\Delta x} - e^{-2ik\Delta x} = 2i\sin 2k\Delta x \\ e^{3ik\Delta x} - e^{-3ik\Delta x} = 2i\sin 3k\Delta x \end{cases} \quad ④$$

Combining ③ and ④, it holds.

$$ik\,C_m(k) = C\,\frac{i(45\sin k\Delta x) - 9\sin(2k\Delta x) + \sin(3k\Delta x))}{30\Delta x}$$

$$C_m(k) = \frac{C(45\sin(k\Delta x) - 9\sin(2k\Delta x) + \sin(3k\Delta x))}{30\Delta x k}$$

$$e_3(k,t) = kct \left| 1 - \frac{45\sin(k\Delta x) - 9\sin(2k\Delta x) + \sin(3k\Delta x)}{30k\Delta x} \right|$$

rewriting e_3 in terms of p and ν yields

$$e_3(p,\nu) = 2\pi\nu \left| 1 - \frac{45\sin(2\pi p^{-1}) - 9\sin(4\pi p^{-1}) + \sin(6\pi p^{-1})}{60\pi p^{-1}} \right| \qquad ①$$

where $p = \frac{2\pi}{k\Delta x}$ $\quad \nu = kct/2\pi$:

By Taylor expansion,

$$\left\{\begin{aligned}
\sin(2\pi p^{-1}) &= 2\pi p^{-1} - \frac{(2\pi p^{-1})^3}{3!} + \frac{(2\pi p^{-1})^5}{5!} - \frac{(2\pi p^{-1})^7}{7!} + O((2\pi p^{-1})^9) \\
\sin(4\pi p^{-1}) &= 4\pi p^{-1} - \frac{(4\pi p^{-1})^3}{3!} + \frac{(4\pi p^{-1})^5}{5!} - \frac{(4\pi p^{-1})^7}{7!} + O((4\pi p^{-1})^9) \\
\sin(6\pi p^{-1}) &= 6\pi p^{-1} - \frac{(6\pi p^{-1})^3}{3!} + \frac{(6\pi p^{-1})^5}{5!} - \frac{(6\pi p^{-1})^7}{7!} + O((6\pi p^{-1})^9)
\end{aligned}\right. \qquad ②$$

Combining ① and ② yields

$$e_3(p,\nu) = 2\pi\nu \left| \frac{45\times 2}{60\times 7!}(2\pi p^{-1})^6 - \frac{9\times 2^8}{60\times 7!}(2\pi p^{-1})^6 + \frac{3^7\times 2}{60\times 7!}(2\pi p^{-1})^6 \right|$$

$$= \frac{2\pi\nu}{70}\left(\frac{2\pi}{p}\right)^6$$

We introduce $P_3(\varepsilon_p, \nu)$ as a measure of the number of points per wavelength required to guarantee a phase error, $e_3 \le \varepsilon_p$.

$$e_3(p,\nu) \le \varepsilon_p$$

$$\Rightarrow \frac{2\pi\nu}{70}\left(\frac{2\pi}{p}\right)^6 \le \varepsilon_p$$

$$\Rightarrow \left(\frac{p}{2\pi}\right)^6 \ge \frac{2\pi\nu}{70\varepsilon_p}$$

$$\Rightarrow p \ge 2\pi \sqrt[6]{\frac{2\pi\nu}{70\varepsilon_p}}$$

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$P_3(0.1, v) = 5\sqrt[6]{v}$ $P_3(0.01, v) = 8\sqrt[6]{v}$

$P_2(0.1, v) = 7\sqrt[4]{v}$ $P_2(0.01, v) = 13\sqrt[4]{v}$

$P_1(0.1, v) = 20\sqrt{v}$ $P_1(0.01, v) = 64\sqrt{v}$

advantages of 6-th order method:
When low accuracy is required few efficiency is gained for short time integration. However, when high accuracy is required. 6-th order method is the optimal choice both for short and long time integration.

Ex 3.

When $k = 2$ $N = 20$ max_error $= 1.16 \times 10^{-6}$

When $k = 4$ $N = 28$ max_error $= 1.95 \times 10^{-6}$

When $k = 6$ $N = 34$ max_error $= 7.25 \times 10^{-6}$

When $k = 8$ $N = 42$ max_error $= 2.9 \times 10^{-6}$

When $k = 10$ $N = 48$ max_error $= 5.35 \times 10^{-6}$

When $k = 12$ $N = 54$ max_error $= 8.37 \times 10^{-6}$

please run ./run.sh on your terminal, then you can obtain plots and ~~not~~ numerical results.