Final Project

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Part 1

0.1 a

To prove wellposedness, it is enough to show operator $\mathcal{L}$ is semi-bounded, where the operator is as follows,

$$\mathcal{L}u = -U_0(x)\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}.$$

Then we could derive conditions of semi-boundness by computing the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$.

$$< \mathcal{L}u, \mu >= \int_0^{2\pi} u(\frac{\partial U_0}{\partial x}\bar{\mu} + U_0(x)\frac{\partial \bar{\mu}}{\partial x}) + \nu u\frac{\partial^2 \bar{\mu}}{\partial x^2} dx \tag{1}$$

From equation (1), we could obtain $\mathcal{L}^* = \frac{\partial U_0}{\partial x} + U_0(x)\frac{\partial}{\partial x} + \nu\frac{\partial^2}{\partial x^2}$.

$$\begin{aligned} < u, (\mathcal{L}+\mathcal{L}^*)u > &=< u, \frac{\partial U_0}{\partial x}u > +2\nu < u, \frac{\partial^2 u}{\partial x^2} > \\ &=< u, \frac{\partial U_0}{\partial x}u > -2\nu < \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} > \\ &\le \sup_x \frac{\partial U_0}{\partial x}(x)\|u\|^2_{L^2} - 2\nu\|\frac{\partial u}{\partial x}\|^2_{L^2} \end{aligned} \tag{2}$$

Therefore, one of sufficient conditions is $\frac{\partial U_0}{\partial x}(x)$ is bounded and ν is nonnegative.

0.2 b

$$\begin{aligned} \|\mathcal{P}_N\mathcal{L}(I-\mathcal{P}_N)u\| &= \|\mathcal{P}_N(-U_0\frac{\partial u(x)}{\partial x} + \mathcal{P}_N\nu\frac{\partial^2 u}{\partial x^2} + U_0\frac{\partial \mathcal{P}_N u}{\partial x} - \nu\frac{\partial^2 \mathcal{P}_N u}{\partial x^2})\| \\ &= \|\mathcal{P}_N(-U_0\frac{\partial u(x)}{\partial x} + U_0\frac{\partial \mathcal{P}_N u}{\partial x})\| \\ &= \|\mathcal{P}_N(U_0(\mathcal{P}_N\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}))\| \\ &\le C\|U_0\|\|\mathcal{P}_N\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x})\| \end{aligned} \tag{3}$$

because $\|\mathcal{P}_N\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\| \to 0$ and $\|\mathcal{P}_N u(0) - u(0)\| \to 0$, it is consistent. The error should decrease exponentially if we assume solution has infinitely sufficient smoothness.

0.3 c

In matrix form, the Fourier-collocation approximation to the PDE is

$$\frac{du_N(t)}{dt} = -U_0Du_N + \nu D^2u_N.$$

We continue by multiplying with u_N^T from the left,

$$\begin{aligned}
u_N^T \frac{du_N(t)}{dt} &= -U_0 u_N^T D u_N + \nu u_N^T D^2 u_N \\
&= -U_0 u_N^T D u_N + \nu (D^T u_N)^T (D u_N) \\
&= -U_0 u_N^T D u_N - \nu (D u_N)^T (D u_N) \\
&= -U_0 u_N^T D u_N - \nu \| D u_N \|_{l^2}^2 \\
&\leq -U_0 u_N^T D u_N = -\frac{U_0}{2} u_N^T (D + D^T) u_N = 0.
\end{aligned} \tag{4}$$

Therefore, we have $\frac{d\|u_N\|_{l^2}^2}{dt} \leq 0$. Finally, we conclude that the Fourier-collocation is stable.

Part 2 and Part 3

0.4 a

I wrote python codes which could implement Fourier collocation method and Fourier Galerkin method with odd number of nodes to approximate solution.

- Fourier collocation method:

$$\frac{du_N}{dt} = -u_N \otimes D u_N + \nu D^2 u_N$$

  where $\otimes$ stands for elementwise multiplication.

Listing 1: Fourier collocation method with time step controlled by CFL

```
def FourierCollocation_RK(u0, x, t, nu, c, h, cfl, N):
    #u0 is initial solution
    #x is grid points
    #t is time evaluated
    #nu is 0.1
    #c is 4.0
    #h is step size in space
    #cfl is conditional number
    #N is number of grids
    return u
```

- Fourier Galerkin method:

$$\frac{da_n(t)}{dt} = \begin{cases} -\nu n^2 a_n(t) - \sum_{-\frac{N}{2} \leq k \leq \frac{N}{2}} a_{n-k}(t)(ik)a_k(t) & \text{if } n = 0 \\ -\nu n^2 a_n(t) - \sum_{n-\frac{N}{2} \leq k \leq \frac{N}{2}} a_{n-k}(t)(ik)a_k(t) & \text{if } n > 0 \\ -\nu n^2 a_n(t) - \sum_{-\frac{N}{2} \leq k \leq \frac{N}{2}+n} a_{n-k}(t)(ik)a_k(t) & \text{if } n < 0 \end{cases} \tag{5}$$

  And,

$$u_N(x,t) = \sum_{|n| \leq \frac{N}{2}} a_n(t) e^{inx}.$$

Listing 2: Fourier Galerkin method with time step controlled by CFL

```
def FourierGalerkin_RK(a0, x, t, nu, c, cfl, N):
    #a0 is initial coefficients
    #x is grid points
    #t is time evaluated
    #nu is 0.1
    #c is 4.0
    #cfl is conditional number
    #N is number of grids
    return a
```

0.5 b

The definition of the time-step restriction is consistent with theories in the Finite Difference Methods. Because time step decreases linearly with respect to step size in the space for convection problem, and decreases quadratically for diffusion problem. From experiments, we could see the larger the number of grids points is, the smaller CFL is. In summar, the maximum CFL is 0.2 for collocation method, and the maximum CFL is 1.0 for galerkin method.

0.6 c

Listing 3: Compare approximated solutions by two methods with exact solutions

```
    python compare.py
```

Even though it seems error gets stuck when N is small, the convergence is almost exponential, which correspond to what I would expect.

0.7 d

Plot exact solutions and approximated solutions by collocation and galerkin methods. It seems collocation is better than galerkin method in term of speed. Besides collocation method is not problem dependent, while in galerkin method, for each problem we need to derive the equations for the expansion coefficients of the numerical solution.

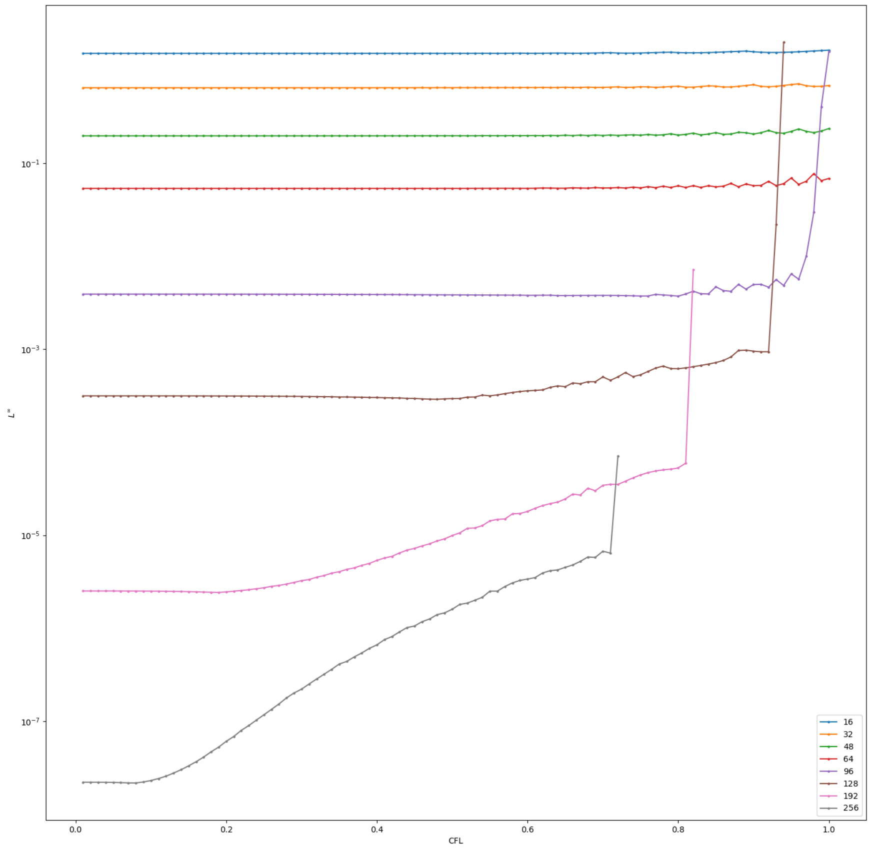

Figure 1: The maximum pointwise error of the numerical solutions, measured at $\frac{\pi}{8}$, as a function of CFL for the collocation method.

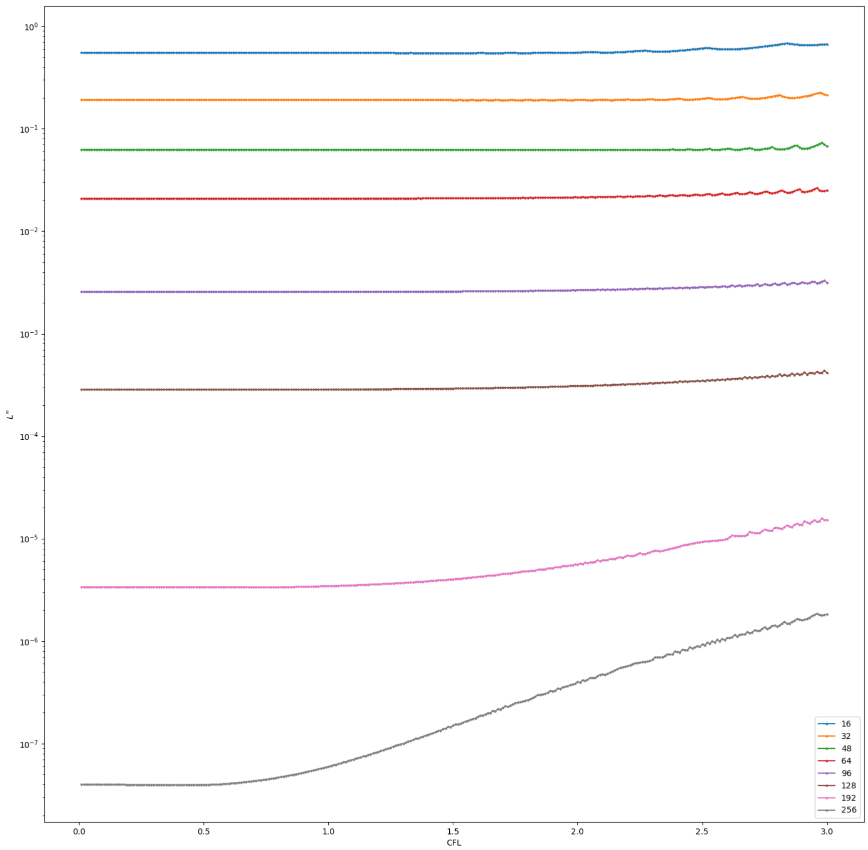

Figure 2: The maximum pointwise error of the numerical solutions, measured at $\frac{\pi}{8}$, as a function of CFL for the galerkin method.

	collocation	galerkin
16	0.7670206801740171	0.11461195516450884
32	0.27643113945753317	0.06078347206061485
48	0.06592869490839881	0.018836715863480347
64	0.014373073266606085	0.005065915301337043
96	0.0006118836067017241	0.0003239833199462083
128	2.442786149270404e-05	1.9572552794944187e-05
192	4.4927528808358375e-08	6.774765237693714e-08
256	1.3962520029053849e-10	2.293654155973286e-10

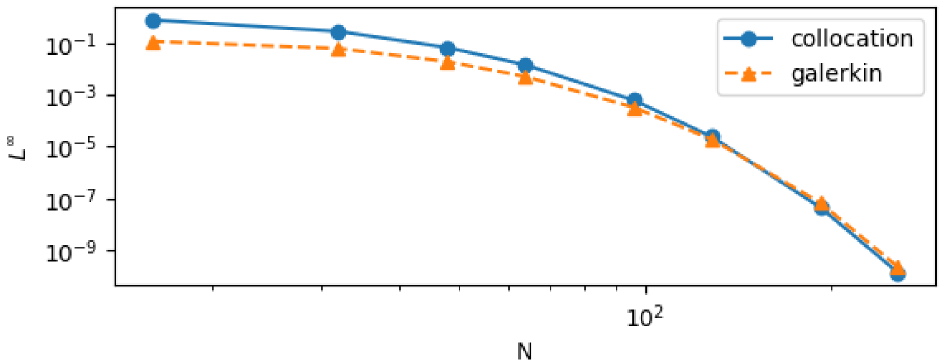

Figure 3: The maximum pointwise error of the numerical solutions, measured at $\frac{\pi}{4}$, as a function of N

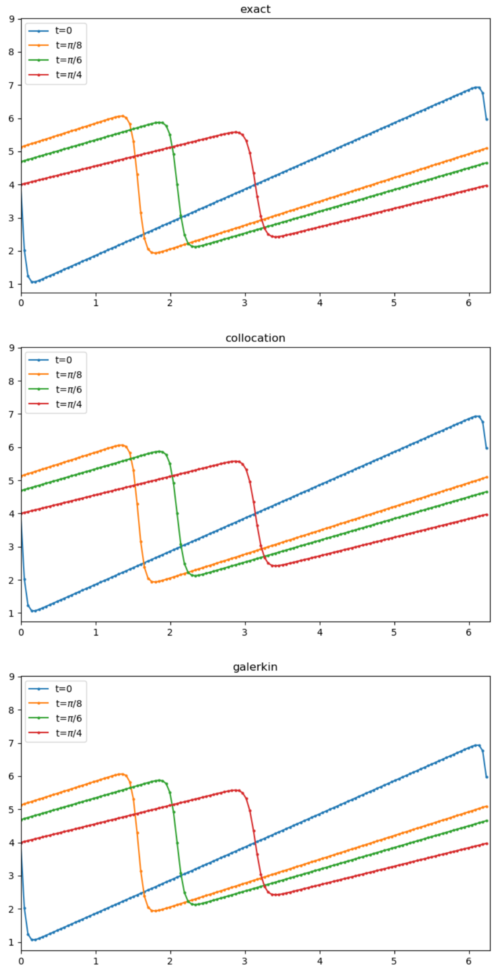

Figure 4: The solution for N = 128 at t = 0, $\pi/8$, $\pi/6$, $\pi/4$.