Chapter 15

Controller Reduction

We have shown in the previous chapters that the $\mathcal{H}_\infty$ control theory and μ synthesis can be used to design robust performance controllers for highly complex uncertain systems. However, since a great many physical plants are modeled as high-order dynamical systems, the controllers designed with these methodologies typically have orders comparable to those of the plants. Simple linear controllers are normally preferred over complex linear controllers in control system designs for some obvious reasons: They are easier to understand and computationally less demanding; they are also easier to implement and have higher reliability since there are fewer things to go wrong in the hardware or bugs to fix in the software. Therefore, a lower-order controller should be sought whenever the resulting performance degradation is kept within an acceptable level. There are usually three ways to arrive at a lower-order controller. A seemingly obvious approach is to design lower-order controllers directly based on the high-order models. However, this is still largely an open research problem. Another approach is first to reduce the order of a high-order plant and, second, based on the reduced plant model, design a lower-order controller. A potential problem associated with this approach is that such a lower-order controller may not even stabilize the full-order plant since the error information between the full-order model and the reduced-order model is not considered in the design of the controller. On the other hand, one may seek to design first a high-order, high-performance controller and subsequently proceed with a reduction of the designed controller. This approach is usually referred to as controller reduction. A crucial consideration in controller order reduction is to take into account the closed loop so that closed-loop stability is guaranteed and the performance degradation is minimized with the reduced-order controllers. The purpose of this chapter is to introduce several controller reduction methods that can guarantee closed-loop stability and possibly closed-loop performance as well.

15.1 $\mathcal{H}_\infty$ Controller Reductions

In this section, we consider an $\mathcal{H}_\infty$ performance-preserving controller order reduction problem. We consider the feedback system shown in Figure 15.1 with a generalized plant realization given by

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

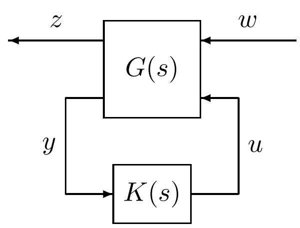


Figure 15.1: Closed-loop system diagram

The following assumptions are made:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A2) D_{12} has full column rank and D_{21} has full row rank;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

As stated in Chapter 14, all stabilizing controllers satisfying $\|T_{zw}\|_\infty < \gamma$ can be parameterized as

$$K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathcal{RH}_\infty, \quad \|Q\|_\infty < \gamma \tag{15.1}$$

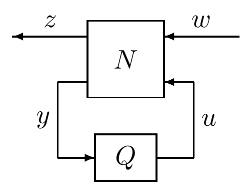
where M_∞ is of the form

$$M_\infty = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right]$$

such that $\hat{D}_{12}$ and $\hat{D}_{21}$ are invertible and $\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1$ and $\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$ are both stable (i.e., M_{12}^{-1} and M_{21}^{-1} are both stable).

The problem to be considered here is to find a controller $\hat{K}$ with a minimal possible order such that the $\mathcal{H}_\infty$ performance requirement $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ is satisfied. This is clearly equivalent to finding a Q so that it satisfies the preceding constraint and the order of $\hat{K}$ is minimized. Instead of choosing Q directly, we shall approach this problem from a different perspective. The following lemma is useful in the subsequent development and can be regarded as a special case of Theorem 10.6 (main loop theorem).

Lemma 15.1 *Consider a feedback system shown below*



where N is a suitably partitioned transfer matrix

$$N(s) = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}.$$

Then the closed-loop transfer matrix from w to z is given by

$$T_{zw} = \mathcal{F}_\ell(N, Q) = N_{11} + N_{12}Q(I - N_{22}Q)^{-1}N_{21}.$$

Assume that the feedback loop is well-posed [i.e., $\det(I - N_{22}(\infty)Q(\infty)) \neq 0$] and either $N_{21}(j\omega)$ has full row rank for all $\omega \in \mathbb{R} \cup \infty$ or $N_{12}(j\omega)$ has full column rank for all $\omega \in \mathbb{R} \cup \infty$ and $\|N\|_\infty \leq 1$; then $\|\mathcal{F}_\ell(N, Q)\|_\infty < 1$ if $\|Q\|_\infty < 1$.

Proof. We shall assume that N_{21} has full row rank. The case when N_{12} has full column rank can be shown in the same way.

To show that $\|T_{zw}\|_\infty < 1$, consider the closed-loop system at any frequency $s = j\omega$ with the signals fixed as complex constant vectors. Let $\|Q\|_\infty =: \epsilon < 1$ and note that $T_{wy} = N_{21}^{+}(I - N_{22}Q)$, where N_{21}^{+} is a right inverse of N_{21}. Also let $\kappa := \|T_{wy}\|_\infty$. Then $\|w\|_2 \leq \kappa\|y\|_2$, and $\|N\|_\infty \leq 1$ implies that $\|z\|_2^2 + \|y\|_2^2 \leq \|w\|_2^2 + \|u\|_2^2$. Therefore,

$$\|z\|_2^2 \leq \|w\|_2^2 + (\epsilon^2 - 1)\|y\|_2^2 \leq [1 - (1 - \epsilon^2)\kappa^{-2}]\|w\|_2^2,$$

which implies $\|T_{zw}\|_\infty < 1$. □

15.1.1 Additive Reduction

Consider the class of (reduced-order) controllers that can be represented in the form

$$\hat{K} = K_0 + W_2 \Delta W_1,$$

where K_0 may be interpreted as a nominal, higher-order controller, and Δ is a stable perturbation with stable, minimum phase and invertible weighting functions W_1 and W_2. Suppose that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$. A natural question is whether it is possible to obtain a reduced-order controller $\hat{K}$ in this class such that the $\mathcal{H}_\infty$ performance bound remains valid when $\hat{K}$ is in place of K_0. Note that this is somewhat a special case of the preceding general problem: The specific form of $\hat{K}$ means that $\hat{K}$ and K_0 must possess the same right-half plane poles, thus to a certain degree limiting the set of attainable reduced-order controllers.

Suppose $\hat{K}$ is a suboptimal $\mathcal{H}_\infty$ controller; that is, there is a $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$ such that $\hat{K} = \mathcal{F}_\ell(M_\infty, Q)$. It follows from simple algebra that

$$Q = \mathcal{F}_\ell(\bar{K}_a^{-1}, \hat{K})$$

where

$$\bar{K}_a^{-1} := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M_\infty^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Furthermore, it follows from straightforward manipulations that

$$\begin{aligned} \|Q\|_\infty < \gamma \quad & \Longleftrightarrow \quad \left\|\mathcal{F}_\ell(\bar{K}_a^{-1}, \hat{K})\right\|_\infty < \gamma \\ & \Longleftrightarrow \quad \left\|\mathcal{F}_\ell(\bar{K}_a^{-1}, K_0 + W_2 \Delta W_1)\right\|_\infty < \gamma \\ & \Longleftrightarrow \quad \left\|\mathcal{F}_\ell(\tilde{R}, \Delta)\right\|_\infty < 1 \end{aligned}$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_2 \end{bmatrix}$$

and R is given by the star product

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(\bar{K}_a^{-1}, \begin{bmatrix} K_o & I \\ I & 0 \end{bmatrix}).$$

It is easy to see that $\tilde{R}_{12}$ and $\tilde{R}_{21}$ are both minimum phase and invertible and hence have full column and full row rank, respectively, for all $\omega \in \mathbb{R} \cup \infty$. Consequently, by invoking Lemma 15.1, we conclude that if $\tilde{R}$ is a contraction and $\|\Delta\|_\infty < 1$, then $\left\|\mathcal{F}_\ell(\tilde{R}, \Delta)\right\|_\infty < 1$. This guarantees the existence of a Q such that $\|Q\|_\infty < \gamma$ or, equivalently, the existence of a $\hat{K}$ such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$. This observation leads to the following theorem.

Theorem 15.2 *Suppose W_1 and W_2 are stable, minimum phase and invertible transfer matrices such that $\tilde{R}$ is a contraction. Let K_0 be a stabilizing controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$. Then $\hat{K}$ is also a stabilizing controller such that $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$ if*

$$\|\Delta\|_\infty = \left\|W_2^{-1}(\hat{K} - K_0)W_1^{-1}\right\|_\infty < 1.$$

Since $\tilde{R}$ can always be made contractive for sufficiently small W_1 and W_2, there are infinite many W_1 and W_2 that satisfy the conditions in the theorem. It is obvious that to make $\left\|W_2^{-1}(\hat{K} - K_0)W_1^{-1}\right\|_\infty < 1$ for some $\hat{K}$, one would like to select the "largest" W_1 and W_2 such that $\tilde{R}$ is a contraction.

Lemma 15.3 *Assume that $\|R_{22}\|_\infty < \gamma$ and define*

$$L = \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix} = \mathcal{F}_\ell\left(\left[\begin{array}{cc:cc} 0 & -R_{11} & 0 & R_{12} \\ -R_{11}^\sim & 0 & R_{21}^\sim & 0 \\ \hdashline 0 & R_{21} & 0 & -R_{22} \\ R_{12}^\sim & 0 & -R_{22}^\sim & 0 \end{array}\right], \gamma^{-1}I\right).$$

Then $\tilde{R}$ is a contraction if W_1 and W_2 satisfy

$$\begin{bmatrix} (W_1^\sim W_1)^{-1} & 0 \\ 0 & (W_2 W_2^\sim)^{-1} \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix}.$$

Proof. See Goddard and Glover [1993]. □

An algorithm that maximizes $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$ has been developed by Goddard and Glover [1993]. The procedure below, devised directly from the preceding theorem, can be used to generate a required reduced-order controller that will preserve the closed-loop $\mathcal{H}_\infty$ performance bound $\left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma$.

1. Let K_0 be a full-order controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$;
2. Compute W_1 and W_2 so that $\tilde{R}$ is a contraction;
3. Use the weighted model reduction method in Chapter 7 or any other methods to find a $\hat{K}$ so that $\left\|W_2^{-1}(\hat{K} - K_0)W_1^{-1}\right\|_\infty < 1$.

Note that all controller reduction methods introduced in this book are only sufficient; that is, there may be desired reduced-order controllers that cannot be found from the proposed procedures.

15.1.2 Coprime Factor Reduction

The $\mathcal{H}_\infty$ controller reduction problem can also be considered in the coprime factor framework. For that purpose, we need the following alternative representation of all admissible $\mathcal{H}_\infty$ controllers:

Lemma 15.4 *The family of all admissible controllers such that $\|T_{zw}\|_\infty < \gamma$ can also be written as*

$$\begin{aligned} K(s) = \mathcal{F}_\ell(M_\infty, Q) &= (\Theta_{11}Q + \Theta_{12})(\Theta_{21}Q + \Theta_{22})^{-1} := UV^{-1} \\ &= (Q\tilde{\Theta}_{12} + \tilde{\Theta}_{22})^{-1}(Q\tilde{\Theta}_{11} + \tilde{\Theta}_{21}) := \tilde{V}^{-1}\tilde{U} \end{aligned}$$

where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$, and UV^{-1} and $\tilde{V}^{-1}\tilde{U}$ are, respectively, right and left coprime factorizations over $\mathcal{RH}_\infty$, and

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & \hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{B}_1\hat{D}_{21}^{-1} \\ \hline \hat{C}_1 - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{12} - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{D}_{11}\hat{D}_{21}^{-1} \\ -\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{D}_{21}^{-1} \end{array}\right]$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} \\ \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_1 - \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} & -\hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{C}_2 - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{21} - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{D}_{11} & -\hat{D}_{22}\hat{D}_{12}^{-1} \\ \hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{12}^{-1}\hat{D}_{11} & \hat{D}_{12}^{-1} \end{array}\right]$$

$$\Theta^{-1} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_2\hat{D}_{12}^{-1} & \hat{B}_1 - \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} \\ \hline -\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{12}^{-1} & -\hat{D}_{12}^{-1}\hat{D}_{11} \\ \hat{C}_2 - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{22}\hat{D}_{12}^{-1} & \hat{D}_{21} - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{D}_{11} \end{array}\right]$$

$$\tilde{\Theta}^{-1} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{B}_1\hat{D}_{21}^{-1} & \hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} \\ \hline \hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{21}^{-1} & \hat{D}_{21}^{-1}\hat{D}_{22} \\ \hat{C}_1 - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{D}_{11}\hat{D}_{21}^{-1} & \hat{D}_{12} - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{D}_{22} \end{array}\right].$$

Proof. The results follow immediately from Lemma 9.2. □

Theorem 15.5 *Let $K_0 = \Theta_{12}\Theta_{22}^{-1}$ be the central $\mathcal{H}_\infty$ controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$ and let $\hat{U}, \hat{V} \in \mathcal{RH}_\infty$ with $\det \hat{V}(\infty) \neq 0$ be such that*

$$\left\| \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1/\sqrt{2}. \qquad (15.2)$$

Then $\hat{K} = \hat{U}\hat{V}^{-1}$ is also a stabilizing controller such that $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$.

Proof. Note that by Lemma 15.4, K is an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ if and only if there exists a $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$ such that

$$\begin{bmatrix} U \\ V \end{bmatrix} := \begin{bmatrix} \Theta_{11}Q + \Theta_{12} \\ \Theta_{21}Q + \Theta_{22} \end{bmatrix} = \Theta \begin{bmatrix} Q \\ I \end{bmatrix} \tag{15.3}$$

and

$$K = UV^{-1}.$$

Hence, to show that $\hat{K} = \hat{U}\hat{V}^{-1}$ with $\hat{U}$ and $\hat{V}$ satisfying equation (15.2) is also a stabilizing controller such that $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$, we need to show that there is another coprime factorization for $\hat{K} = UV^{-1}$ and a $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$ such that equation (15.3) is satisfied.

Define

$$\Delta := \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right)$$

and partition Δ as

$$\Delta := \begin{bmatrix} \Delta_U \\ \Delta_V \end{bmatrix}.$$

Then

$$\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} = \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \Theta \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix} \Delta = \Theta \begin{bmatrix} -\gamma\Delta_U \\ I - \Delta_V \end{bmatrix}$$

and

$$\begin{bmatrix} \hat{U}(I-\Delta_V)^{-1} \\ \hat{V}(I-\Delta_V)^{-1} \end{bmatrix} = \Theta \begin{bmatrix} -\gamma\Delta_U(I-\Delta_V)^{-1} \\ I \end{bmatrix}.$$

Define $U := \hat{U}(I-\Delta_V)^{-1}$, $V := \hat{V}(I-\Delta_V)^{-1}$, and $Q := -\gamma\Delta_U(I-\Delta_V)^{-1}$. Then UV^{-1} is another coprime factorization for $\hat{K}$. To show that $\hat{K} = UV^{-1} = \hat{U}\hat{V}^{-1}$ is a stabilizing controller such that $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$, we need to show that $\left\|\gamma\Delta_U(I-\Delta_V)^{-1}\right\|_\infty < \gamma$ or, equivalently, $\left\|\Delta_U(I-\Delta_V)^{-1}\right\|_\infty < 1$. Now

$$\begin{aligned} \Delta_U(I-\Delta_V)^{-1} &= \begin{bmatrix} I & 0 \end{bmatrix} \Delta \left(I - \begin{bmatrix} 0 & I \end{bmatrix} \Delta\right)^{-1} \\ &= \mathcal{F}_\ell\left(\begin{bmatrix} 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ I/\sqrt{2} & \begin{bmatrix} 0 & I/\sqrt{2} \end{bmatrix} \end{bmatrix}, \sqrt{2}\Delta \right) \end{aligned}$$

and, by Lemma 15.1, $\left\|\Delta_U(I-\Delta_V)^{-1}\right\|_\infty < 1$ since

$$\begin{bmatrix} 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ I/\sqrt{2} & \begin{bmatrix} 0 & I/\sqrt{2} \end{bmatrix} \end{bmatrix}$$

is a contraction and $\left\|\sqrt{2}\Delta\right\|_\infty < 1$. □

Similarly, we have the following theorem:

Theorem 15.6 *Let $K_0 = \tilde{\Theta}_{22}^{-1}\tilde{\Theta}_{21}$ be the central $\mathcal{H}_\infty$ controller such that $\|\mathcal{F}_\ell(G, K_0)\|_\infty < \gamma$ and let $\hat{\tilde{U}}, \hat{\tilde{V}} \in \mathcal{RH}_\infty$ with $\det \hat{\tilde{V}}(\infty) \neq 0$ be such that*

$$\left\| \left(\begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right) \tilde{\Theta}^{-1} \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1/\sqrt{2}.$$

Then $\hat{K} = \hat{\tilde{V}}^{-1}\hat{\tilde{U}}$ is also a stabilizing controller such that $\|\mathcal{F}_\ell(G, \hat{K})\|_\infty < \gamma$.

The preceding two theorems show that the sufficient conditions for $\mathcal{H}_\infty$ controller reduction problem are equivalent to frequency-weighted $\mathcal{H}_\infty$ model reduction problems.

$\mathcal{H}_\infty$ Controller Reduction Procedures

(i) Let $K_0 = \Theta_{12}\Theta_{22}^{-1} (= \tilde{\Theta}_{22}^{-1}\tilde{\Theta}_{21})$ be a suboptimal $\mathcal{H}_\infty$ central controller ($Q = 0$) such that $\|T_{zw}\|_\infty < \gamma$.

(ii) Find a reduced-order controller $\hat{K} = \hat{U}\hat{V}^{-1}$ (or $\hat{\tilde{V}}^{-1}\hat{\tilde{U}}$) such that

$$\left\| \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1/\sqrt{2}$$

or

$$\left\| \left(\begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix} \right) \tilde{\Theta}^{-1} \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1/\sqrt{2}.$$

Then the closed-loop system with the reduced-order controller $\hat{K}$ is stable and the performance is maintained with the reduced-order controller; that is,

$$\|T_{zw}\|_\infty = \left\|\mathcal{F}_\ell(G, \hat{K})\right\|_\infty < \gamma.$$

15.2 Notes and References

The main results presented in this chapter are based on the work of Goddard and Glover [1993, 1994]. Other controller reduction methods include the stability-oriented controller reduction criterion proposed by Enns [1984a, 1984b]; the weighted and unweighted coprime factor controller reduction methods studied by Liu and Anderson [1986, 1990]; Liu, Anderson, and Ly [1990]; Anderson and Liu [1989]; and Anderson [1993]; the normalized $\mathcal{H}_\infty$ controller reduction studied by Mustafa and Glover [1991]; the normalized coprime factor method studied by McFarlane and Glover [1990] in the $\mathcal{H}_\infty$ loop-shaping setup; and the controller reduction in the ν-gap metric setup studied by Vinnicombe [1993b]. Lenz, Khargonekar, and Doyle [1987] have also proposed another $\mathcal{H}_\infty$ controller reduction method with guaranteed performance for a class of $\mathcal{H}_\infty$ problems.

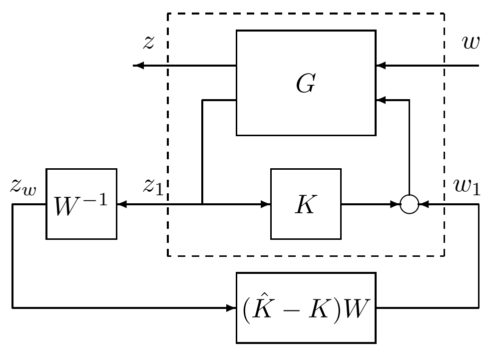
15.3 Problems

Problem 15.1 Find a lower-order controller for the system in Example 10.4 when $\gamma = 2$.

Problem 15.2 Find a lower-order controller for Problem 14.3 when $\gamma = 1.1\gamma_{\mathrm{opt}}$ where γ_{opt} is the optimal norm.

Problem 15.3 Find a lower-order controller for the HIMAT control problem in Problem 14.12 when $\gamma = 1.1\gamma_{\mathrm{opt}}$ where γ_{opt} is the optimal norm. Compare the controller reduction methods presented in this chapter with other available methods.

Problem 15.4 Let G be a generalized plant and K be a stabilizing controller. Let $\Delta = \mathrm{diag}(\Delta_p, \Delta_k)$ be a suitably dimensioned perturbation and let $T_{\hat{z}\hat{w}}$ be the transfer matrix from $\hat{w} = \begin{bmatrix} w \\ w_1 \end{bmatrix}$ to $\hat{z} = \begin{bmatrix} z \\ z_1 \end{bmatrix}$ in the following diagram:



Let $W, W^{-1} \in \mathcal{H}_\infty$ be a given transfer matrix. Show that the following statements are equivalent:

1. $\mu_\Delta\left(\begin{bmatrix} I & 0 \\ 0 & W^{-1} \end{bmatrix} T_{\hat{z}\hat{w}}\right) < 1$;
2. $\|\mathcal{F}_\ell(G,K)\|_\infty < 1$ and $\left\|W^{-1}\mathcal{F}_u(T_{\hat{z}\hat{w}}, \Delta_p)\right\|_\infty < 1$ for all $\overline{\sigma}(\Delta_p) \le 1$;
3. $\left\|W^{-1}T_{z_1w_1}\right\|_\infty < 1$ and $\left\|\mathcal{F}_\ell\left(\begin{bmatrix} I & 0 \\ 0 & W^{-1} \end{bmatrix} T_{\hat{z}\hat{w}}, \Delta_k\right)\right\|_\infty < 1$ for all $\overline{\sigma}(\Delta_k) \le 1$.

Problem 15.5 In the part 3 of Problem 15.4, if we let $\Delta_k = (\hat{K} - K)W$, then $T_{z_1w_1} = G_{22}(I - KG_{22})^{-1}$ and $\mathcal{F}_\ell\left(\begin{bmatrix} I & 0 \\ 0 & W^{-1} \end{bmatrix} T_{\hat{z}\hat{w}}, \Delta_k\right) = \mathcal{F}_\ell(G, \hat{K})$. Thus $\hat{K}$ stabilizes the system and satisfies $\left\|\mathcal{F}_\ell(G,\hat{K})\right\|_\infty < 1$ if $\|\Delta_k\|_\infty = \left\|(\hat{K} - K)W\right\|_\infty \le 1$ and part 2 of

Problem 15.4 is satisfied by a controller K. Hence to reduce the order of the controller K, it is sufficient to solve a frequency-weighted model reduction problem if W can be calculated. In the single-input and single-output case, a "smallest" weighting function $W(s)$ can be calculated using part 2 of Problem 15.4 as follows:

$$|W(j\omega)| \geq \sup_{\overline{\sigma}(\Delta_p)\leq 1} |\mathcal{F}_u(T_{\hat{z}\hat{w}}(j\omega), \Delta_p)|.$$

Repeat Problems 15.1 and 15.2 using the foregoing method. (Hint: W can be computed frequency by frequency using μ software and then fitted by a stable and minimum phase transfer function.)

Problem 15.6 One way to generalize the method in Problem 15.5 to the MIMO case is to take a diagonal W

$$W = \operatorname{diag}(W_1, W_2, \ldots, W_m)$$

and let $\hat{W}_i$ be computed from

$$|\hat{W}_i(j\omega)| \geq \sup_{\overline{\sigma}(\Delta_p)\leq 1} \left\| e_i^T \mathcal{F}_u(T_{\hat{z}\hat{w}}(j\omega), \Delta_p) \right\|$$

where e_i is the ith unit vector. Next let $\alpha(s)$ be computed from

$$|\alpha(j\omega)| \geq \sup_{\overline{\sigma}(\Delta_p)\leq 1} \left\| \hat{W}^{-1} \mathcal{F}_u(T_{\hat{z}\hat{w}}(j\omega), \Delta_p) \right\|$$

where $\hat{W} = \operatorname{diag}(\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_m)$. Then a suitable W can be taken as

$$W = \alpha \hat{W}.$$

Apply this method to Problem 15.3.

Problem 15.7 Generalize the procedures in Problems 15.5 and 15.6 to problems with additional structured uncertainty cases. (A more general case can be found in Yang and Packard [1995].)