Chapter 18

Miscellaneous Topics

This chapter considers two somewhat different problems. The first section gives a brief introduction into the problem of model validation and the second section considers the mixed real and complex μ analysis and synthesis.

18.1 Model Validation

A key to the success of the robust control theory developed in this book is to have appropriate descriptions of the uncertain system (whether it is an additive uncertainty model or a general linear fractional model). Then an important question is how one can decide if a model description is appropriate (i.e., how to validate a model).

For simplicity of presentation, we have chosen to present the discrete time model validation in this section with the understanding that a continuous time problem can be approximated by fast sampling. Suppose we have modeled a set of uncertain dynamical systems by

$$\mathbf{\Delta} := \{\Delta : \ \Delta \in \mathcal{H}_\infty, \quad \|\Delta\|_\infty \leq 1\}$$

where the $\mathcal{H}_\infty$ norm of a discrete time system $\Delta \in \mathcal{H}_\infty$ is defined as $\|\Delta(z)\|_\infty = \sup_{|z|>1} \overline{\sigma}(\Delta(z))$. In order to verify whether this model assumption is correct, some experimental data are collected. For example, let the input to the system be the sequence $u = (u_0, u_1, \ldots, u_{l-1})$ and the output $y = (y_0, y_1, \ldots, y_{l-1})$. A natural question is whether these data are consistent with our modeling assumption. In other words, does there exist a model $\Delta \in \mathbf{\Delta}$ such that the output of the Δ for the period of $t = 0, 1, \ldots, l-1$ is exactly $y = (y_0, y_1, \ldots, y_{l-1})$ with the input $u = (u_0, u_1, \ldots, u_{l-1})$? If there does not exist a such Δ, then the model is invalidated. If there exists a such Δ, however, it does not mean that the model is validated but it only means that the model is not validated by this set of data and it may be invalidated by another set of data in the future. Hence it is actually more accurate to say our validation procedure is model invalidation.

Let Δ be a stable, causal, linear, time-invariant system with transfer matrix

$$\Delta(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots$$

where $h_i, i = 0, 1, \ldots$ are the matrix Markov parameters. Suppose we have applied the input sequence $u = (u_0, u_1, \ldots, u_{l-1})$ to the system and collected the output for the period $t = 0, 1, \ldots, \ell - 1$, $y = (y_0, y_1, \ldots, y_{l-1})$. Then the input and output sequences are related by a Toeplitz matrix:

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{l-1} \end{bmatrix} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ h_{l-1} & h_{l-2} & \cdots & h_0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{l-1} \end{bmatrix}.$$

This equation shows that, for $u_0 \neq 0$ and SISO Δ, the inputs and outputs uniquely determine the first ℓ Markov parameters of the transfer function $\Delta(z)$. The model is validated (or more accurately not invalidated) if the remaining Markov parameters can be chosen so that $\Delta(z) \in \mathbf{\Delta}$. The existence of such a choice is the classical tangential Carathéodory-Fejér interpolation problem, for which a solution to the MIMO case can be found in Foias and Frazho [1990, page 195]. We shall state this result in the following theorem. But we shall define some notation first.

Let $(v_0, v_1, \ldots, v_{\ell-1}, v_\ell, v_{\ell+1}, \ldots)$ be a sequence and let π_ℓ denote the truncation operator such that

$$\pi_\ell(v_0, v_1, \ldots, v_{\ell-1}, v_\ell, v_{\ell+1}, \ldots) = (v_0, v_1, \ldots, v_{\ell-1}).$$

Let $v = (v_0, v_1, \ldots, v_{\ell-1})$ be a sequence of vectors and denote

$$T_v := \begin{bmatrix} v_0 & 0 & \cdots & 0 \\ v_1 & v_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ v_{l-1} & v_{l-2} & \cdots & v_0 \end{bmatrix}.$$

Theorem 18.1 *Given $u = (u_0, u_1, \ldots, u_{l-1})$ and $y = (y_0, y_1, \ldots, y_{l-1})$, there exists a $\Delta \in \mathcal{H}_\infty$, $\|\Delta\|_\infty \leq 1$ such that*

$$y = \pi_\ell \Delta u$$

if and only if $T_y^* T_y \leq T_u^* T_u$.

Note that the output of Δ after time $t = \ell - 1$ is irrelevant to the test. The condition $T_y^* T_y \leq T_u^* T_u$ is equivalent to

$$\sum_{j=1}^{i} \|y_j\|^2 \leq \sum_{j=1}^{i} \|u_j\|^2, i = 0, 1, \ldots, \ell - 1$$

or

$$\|\pi_i y\|_2 \le \|\pi_i u\|_2 \,, i = 0, 1, \ldots, \ell - 1,$$

which is obviously necessary. In fact, the last condition holds for stable, linear, time-varying operator Δ with $\sup_{u\neq 0} \frac{\|\Delta u\|_2}{\|u\|_2} \le 1$; see Poolla et al. [1994]. Note that if $u_0 \neq 0$, then T_u is of full column rank and the condition can also be written as $\overline{\sigma}\left(T_y(T_u^* T_u)^{-\frac{1}{2}}\right) \le 1$.

Using the above theorem, we can derive solutions to some model validation problems easily. For example, consider a set of additive models shown in Figure 18.1.

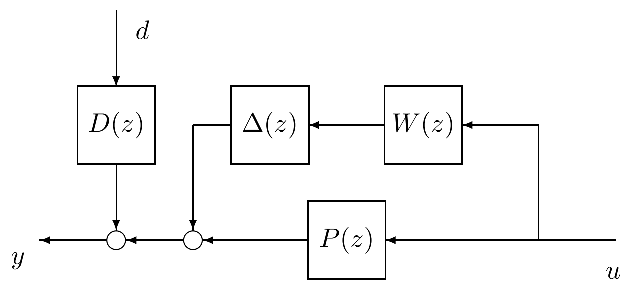

Figure 18.1: Model validation for additive uncertainty

In this case,

$$y = (P + \Delta W)u + Dd, \quad \|\Delta\|_\infty \le 1$$

where $P(z), W(z), D(z)$ and $\Delta(z)$ are causal, linear, time-invariant systems (but not necessarily stable). The disturbance d is assumed to come from a convex set, $d \in \mathcal{D}_{\text{convex}}$; for example, $\mathcal{D}_{\text{convex}} = \{d : \; d \in \ell_2[0,\infty), \|d\|_2 \le 1\}$. For simplicity, we shall also assume that $W(\infty)$ is of full column rank. Let

$$D(z) = D_0 + D_1 z^{-1} + D_2 z^{-2} + \cdots.$$

Theorem 18.2 *Given a set of input-output data* $u_{\text{expt}} = (u_0, u_1, \ldots, u_{\ell-1})$ *with* $u_0 \neq 0$, $y_{\text{expt}} = (y_0, y_1, \ldots, y_{\ell-1})$ *for the additive perturbed uncertainty system with an additive disturbance* $d \in \mathcal{D}_{\text{convex}}$, *where* $\mathcal{D}_{\text{convex}}$ *is a convex set, let*

$$\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{\ell-1}) = \pi_\ell(W u_{\text{expt}})$$

$$\hat{y} = (\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_{\ell-1}) = y_{\text{expt}} - \pi_\ell P u_{\text{expt}}.$$

Then there exists a $\Delta \in \mathcal{H}_\infty$, $\|\Delta\|_\infty \le 1$ *such that*

$$y_{\text{expt}} = \pi_\ell\left((P + \Delta W)u_{\text{expt}} + Dd\right)$$

for some $d \in \mathcal{D}_{\text{convex}}$ *if and only if there exists a* $d = (d_0, d_1, \ldots, d_{l-1}) \in \pi_\ell \mathcal{D}_{\text{convex}}$ *such that*

$$\overline{\sigma}\left[(T_{\hat{y}} - T_D T_d)(T_{\hat{u}}^* T_{\hat{u}})^{-1/2}\right] \le 1$$

where

$$T_D := \begin{bmatrix} D_0 & 0 & \cdots & 0 \\ D_1 & D_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ D_{l-1} & D_{l-2} & \cdots & D_0 \end{bmatrix}.$$

Proof. Note that the system input-output equation can be written as

$$(y - Pu) - Dd = \Delta(Wu).$$

Since P, W, D, and Δ are causal, linear, and time invariant, we have $\pi_\ell Dd = \pi_\ell D \pi_\ell d$, $\pi_\ell(y - Pu) = y_{\text{expt}} - \pi_\ell P \pi_\ell u = y_{\text{expt}} - \pi_\ell P u_{\text{expt}}$ and $\pi_\ell Wu = \pi_\ell W \pi_\ell u = \pi_\ell W u_{\text{expt}}$. Denote

$$\hat{d} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_{\ell-1}) = \pi_\ell(Dd).$$

Then it is easy to show that

$$\begin{bmatrix} \hat{d}_0 \\ \hat{d}_1 \\ \vdots \\ \hat{d}_{\ell-1} \end{bmatrix} = T_D \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{\ell-1} \end{bmatrix}$$

and $T_{\hat{d}} = T_D T_d$. Now note that

$$T_{\pi_\ell(y-Pu-Dd)} = T_{\pi_\ell(y-Pu)} - T_{\pi_\ell(Dd)} = T_{\hat{y}} - T_D T_d, \quad T_{\pi_\ell Wu} = T_{\hat{u}}$$

and $\pi_\ell \Delta Wu = \pi_\ell \Delta \pi_\ell (Wu)$ since Δ is causal. Applying Theorem 18.1, there exists a $\Delta \in \mathcal{H}_\infty$, $\|\Delta\|_\infty \leq 1$ such that

$$\pi_\ell\left[(y - Pu) - Dd\right] = \pi_\ell \Delta(Wu) = \pi_\ell \Delta \pi_\ell (Wu)$$

if and only if

$$(T_{\hat{y}} - T_D T_d)^*(T_{\hat{y}} - T_D T_d) \leq T_{\hat{u}}^* T_{\hat{u}}$$

which is equivalent to

$$\overline{\sigma}\left[(T_{\hat{y}} - T_D T_d)(T_{\hat{u}}^* T_{\hat{u}})^{-\frac{1}{2}}\right] \leq 1.$$

Note that $T_{\hat{u}}$ is of full column rank since $W(\infty)$ is of full column rank and $u_0 \neq 0$, which implies $\hat{u}_0 \neq 0$. □

Note that

$$\inf_{d \in \mathcal{D}_{\text{convex}}} \overline{\sigma}\left[(T_{\hat{y}} - T_D T_d)(T_{\hat{u}}^* T_{\hat{u}})^{-\frac{1}{2}}\right] \leq 1$$

is a convex problem and can be checked numerically.

Many other classes of model validation problems can be solved analogously. For example, consider a coprime factor model validation problem with

$$y = (M + \Delta_M W_M)^{-1}(N + \Delta_N W_N)u + d$$

where M, N, W_M, and W_N are causal, linear, time-invariant systems, and $\Delta_M, \Delta_N \in \mathcal{H}_\infty$, $\left\| \begin{bmatrix} \Delta_M & \Delta_N \end{bmatrix} \right\|_\infty \leq 1$. Then the problem can be solved by multiplying $M + \Delta_M W_M$ from the left of the system equation and rewriting the system equation as

$$(My - Nu - Md) = \begin{bmatrix} \Delta_M & \Delta_N \end{bmatrix} \begin{bmatrix} W_M(d-y) \\ W_N u \end{bmatrix}.$$

The model validation of a general LFT uncertainty system is considered in Davis [1995] and Chen and Wang [1996]. For continuous time model validation, see Rangan and Poolla [1996] and Smith and Dullerud [1996].

18.2 Mixed μ Analysis and Synthesis

In Chapter 10, we considered analysis and synthesis of systems with *complex* uncertainties. However, in practice, many systems involve parametric uncertainties that are real (for example, the uncertainty about a spring constant in a mechanical system). In this case, one has to cover this real parameter variation with a complex disk in order to use the complex μ analysis and synthesis tools, which usually results in a conservative solution. In this section, we shall consider briefly the analysis and synthesis problems with possibly both real parametric and complex uncertainties.

The mixed real and complex μ involves three types of blocks: *repeated real scalar, repeated complex scalar*, and *full* blocks. Three nonnegative integers, S_r, S_c, and F, represent the number of *repeated real scalar* blocks, the number of *repeated complex scalar* blocks, and the number of *full* blocks, and they satisfy

$$\sum_{i=1}^{S_r} k_i + \sum_{i=1}^{S_c} r_i + \sum_{j=1}^{F} m_j = n.$$

The ith repeated real scalar block is $k_i \times k_i$, the jth repeated complex scalar block is $r_j \times r_j$, and the ℓth full block is $m_\ell \times m_\ell$. The admissible set of uncertainties $\mathbf{\Delta} \subset \mathbb{C}^{n \times n}$ is defined as

$$\begin{aligned} \mathbf{\Delta} \; = & \left\{ \text{diag} \; \left[\phi_1 I_{k_1}, \ldots, \phi_{s_r} I_{k_{s_r}}, \delta_1 I_{r_1}, \ldots, \delta_{s_c} I_{r_{s_c}}, \right.\right. \\ & \left.\left. \Delta_1, \ldots, \Delta_F \right] : \phi_i \in \mathbb{R}, \; \delta_j \in \mathbb{C}, \; \Delta_\ell \in \mathbb{C}^{m_\ell \times m_\ell} \right\}. \end{aligned} \tag{18.1}$$

The mixed μ is defined in the same way as for the complex μ: Let $M \in \mathbb{C}^{n \times n}$; then

$$\mu_{\mathbf{\Delta}}(M) := \left(\min \left\{ \overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0 \right\} \right)^{-1} \tag{18.2}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) := 0$. Or, equivalently,

$$\frac{1}{\mu_{\mathbf{\Delta}}(M)} := \inf\left\{\alpha : \ \det(I - \alpha M\Delta) = 0, \ \overline{\sigma}(\Delta) \le 1, \ \Delta \in \mathbf{\Delta}\right\}.$$

Let $\rho_R(M)$ be the real spectral radius (i.e., the largest magnitude of the real eigenvalues of M). For example, if a 4×4 matrix M has eigenvalues $1 \pm j3, -2, 1,$ then $\rho(M) = |1 + j3| = \sqrt{10}$ and $\rho_R(M) = |-2| = 2$. It is easy to see that

$$\mu_{\mathbf{\Delta}}(M) = \max_{\Delta \in \mathbf{B\Delta}} \rho_R(M\Delta)$$

where $\mathbf{B\Delta} := \{\Delta : \ \Delta \in \mathbf{\Delta}, \ \overline{\sigma}(\Delta) \le 1\}$. Note that $\max_{\Delta \in \mathbf{B\Delta}} \rho_R(M\Delta) = \max_{\Delta \in \mathbf{B\Delta}} \rho(M\Delta)$ if $s_r = 0$. [This should not be confused with the fact that, for a given matrix $\Delta \in \mathbf{B\Delta}$ and M, $\rho_R(M\Delta)$ may not be equal to $\rho(M\Delta)$. For example, $M = 2e^{j\frac{\pi}{4}}$ and $\Delta = 1$; then $\rho(M\Delta) = 2$ but $\rho_R(M\Delta) = 0$ since M has no real eigenvalues. However, one can choose another $\Delta_1 = e^{-j\frac{\pi}{4}}$ such that $\rho_R(M\Delta_1) = 2 = \rho(M\Delta)$.]

Define

$$\mathcal{Q} = \{\Delta \in \mathbf{\Delta} : \ \phi_i \in [-1, 1], \ |\delta_i| = 1, \ \Delta_i \Delta_i^* = I_{m_i}\}$$

$$\mathcal{D} = \left\{ \begin{array}{l} \operatorname{diag}\left[\tilde{D}_1, \ldots, \tilde{D}_{s_r}, D_1, \ldots, D_{s_c}, d_1 I_{m_1}, \ldots, d_{F-1} I_{m_{F-1}}, I_{m_F}\right] : \\ \tilde{D}_i \in \mathbb{C}^{k_i \times k_i}, \ \tilde{D}_i = \tilde{D}_i^* > 0, \ D_i \in \mathbb{C}^{r_i \times r_i}, \ D_i = D_i^* > 0, \ d_j \in \mathbb{R}, d_j > 0 \end{array} \right\}.$$

$$\mathcal{G} = \left\{\operatorname{diag}\left[G_1, \ldots, G_{s_r}, 0, \ldots, 0\right] : \quad G_i = G_i^* \in \mathbb{C}^{k_i \times k_i}\right\}.$$

It was shown in Young [1993] that

$$\mu_{\mathbf{\Delta}}(M) = \max_{Q \in \mathcal{Q}} \rho_R(QM).$$

Note that the above maximization is not necessarily achieved on the vertices for the real parameters; hence one must search over the entire interval for each real parameter. Again this maximization problem can have many local maximums and a power algorithm has been developed in Young [1993] to compute a lower bound.

It should also be noted that even though the complex μ (i.e., $s_r = 0$) is a continuous function of the data, the mixed μ (i.e., $s_r \neq 0$) may only be upper semicontinuous; see Packard and Pandey [1993]. It was also shown in Braatz et al. [1994] and Toker and Özbay [1995] that the computation of μ is a NP hard problem, which means that it may not be computable in a polynomial time. Of course, it should not be interpreted as every μ problem will not be solvable in a polynomial time; merely some might not.

Obviously, the upper bound for the complex μ can be applied for the mixed μ when the intervals of the real parameters are covered by complex disks. However, a better bound can be obtained for the mixed μ by exploiting the phase information of the real parameters. To motivate the improved bound for the mixed μ, we consider again the upper bound for the complex μ problem. It is known that

$$\mu_{\mathbf{\Delta}}(M) \le \inf_{D \in \mathcal{D}} \overline{\sigma}(DMD^{-1}).$$

This bound can be reformulated using linear matrix inequalities by noting the following:

$$\overline{\sigma}(DMD^{-1}) \le \beta \iff (DMD^{-1})^* DMD^{-1} \le \beta^2 I \iff M^* D^* DM - \beta^2 D^* D \le 0.$$

Since D is nonsingular and $D^* D \in \mathcal{D}$, we have

$$\mu_{\mathbf{\Delta}}(M) \le \inf_{D \in \mathcal{D}} \min_{\beta} \left\{ \beta : \; M^* D M - \beta^2 D \le 0 \right\}.$$

The following upper bound for the mixed μ was derived by Fan, Tits, and Doyle [1991] and reformulated in the current form by Young [1993].

Theorem 18.3 *Let $M \in \mathbb{C}^{n \times n}$ and $\Delta \in \mathbf{\Delta}$. Then*

$$\mu_{\mathbf{\Delta}}(M) \le \inf_{D \in \mathcal{D}, G \in \mathcal{G}} \min_{\beta} \left\{ \beta : \; M^* D M + j(GM - M^* G) - \beta^2 D \le 0 \right\}.$$

Proof. Suppose we have a $Q \in \mathcal{Q}$ such that QM has a real eigenvalue $\lambda \in \mathbb{R}$. Then there is a vector $x \in \mathbb{C}^n$ such that

$$QMx = \lambda x.$$

Let $D \in \mathcal{D}$. Then $D^{\frac{1}{2}} \in \mathcal{D}$, $D^{\frac{1}{2}} Q = Q D^{\frac{1}{2}}$ and

$$D^{\frac{1}{2}} QMx = Q D^{\frac{1}{2}} Mx = \lambda D^{\frac{1}{2}} x.$$

Since $\overline{\sigma}(Q) \le 1$, it follows that

$$\lambda^2 \left\| D^{\frac{1}{2}} x \right\|^2 = \left\| Q D^{\frac{1}{2}} Mx \right\|^2 \le \left\| D^{\frac{1}{2}} Mx \right\|^2.$$

Hence

$$x^* (M^* D M - \lambda^2 D) x \ge 0.$$

Next, let $G \in \mathcal{G}$ and note that $Q^* G = QG = GQ$; then

$$x^* G M x = \left(\frac{1}{\lambda} QMx \right)^* GMx = \frac{1}{\lambda} x^* M^* Q^* G M x = \frac{1}{\lambda} x^* M^* Q G M x$$

$$= \frac{1}{\lambda} x^* M^* G Q M x = \frac{1}{\lambda} x^* M^* G (QMx) = x^* M^* G x.$$

That is,

$$x^* (GM - M^* G) x = 0.$$

Note that $j(GM - M^* G)$ is a Hermitian matrix, so it follows that for such x

$$x^* (M^* D M + j(GM - M^* G) - \lambda^2 D) x \ge 0.$$

It is now easy to see that if we have $D \in \mathcal{D}$, $G \in \mathcal{G}$ and $0 \le \beta \in \mathbb{R}$ such that

$$M^* D M + j(GM - M^* G) - \beta^2 D \le 0$$

then $|\lambda| \leq \beta$, and hence $\mu_{\mathbf{\Delta}}(M) \leq \beta$. □

This upper bound has an interesting interpretation: covering the uncertainties on the real axis using possibly off-axis disks. To illustrate, let $M \in \mathbb{C}$ be a scalar and $\Delta \in [-1, 1]$. We can cover this real interval using a disk as shown in Figure 18.2.

The off-axis disk can be expressed as

$$j\frac{G}{\beta} + \sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \tilde{\Delta}, \quad \tilde{\Delta} \in \mathbb{C},\ \left|\tilde{\Delta}\right| \leq 1.$$

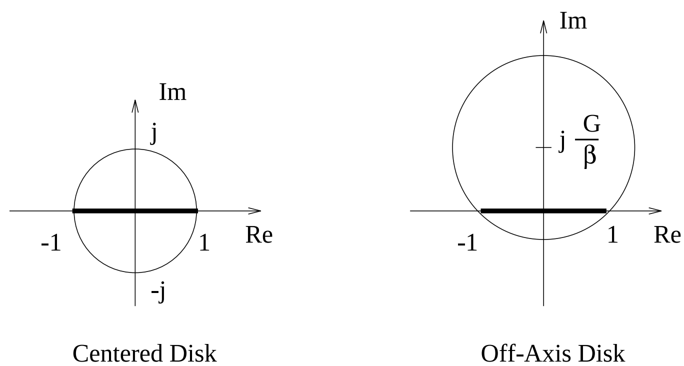

Figure 18.2: Covering real parameters with disks

Hence $1 - \Delta\frac{M}{\beta} \neq 0$ for all $\Delta \in [-1, 1]$ is guaranteed if

$$1 - \left(j\frac{G}{\beta} + \sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \tilde{\Delta}\right)\frac{M}{\beta} \neq 0, \quad \tilde{\Delta} \in \mathbb{C},\ \left|\tilde{\Delta}\right| \leq 1$$

$$\iff \quad 1 - \frac{\sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \frac{M}{\beta}}{1 - j\frac{G}{\beta}\frac{M}{\beta}}\ \tilde{\Delta} \neq 0, \quad \tilde{\Delta} \in \mathbb{C},\ \left|\tilde{\Delta}\right| \leq 1$$

$$\iff \quad \left(\frac{\sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \frac{M}{\beta}}{1 - j\frac{G}{\beta}\frac{M}{\beta}}\right)^* \left(\frac{\sqrt{1 + \left(\frac{G}{\beta}\right)^2}\ \frac{M}{\beta}}{1 - j\frac{G}{\beta}\frac{M}{\beta}}\right) \leq 1$$

$$\iff \quad \frac{M^*}{\beta}\frac{M}{\beta} + j\left(\frac{G}{\beta}\frac{M}{\beta} - \frac{M^*}{\beta}\frac{G}{\beta}\right) - 1 \leq 0$$

$$\iff \quad M^*M + j(GM - M^*G) - \beta^2 \leq 0.$$

The scaling G allows one to exploit the phase information about the real parameters so that a better upper bound can be obtained. We shall demonstrate this further using a simple example.

Example 18.1 Let

$$G(s) = \frac{s^2 + 2s + 1}{s^3 + s^2 + 2s + 1}.$$

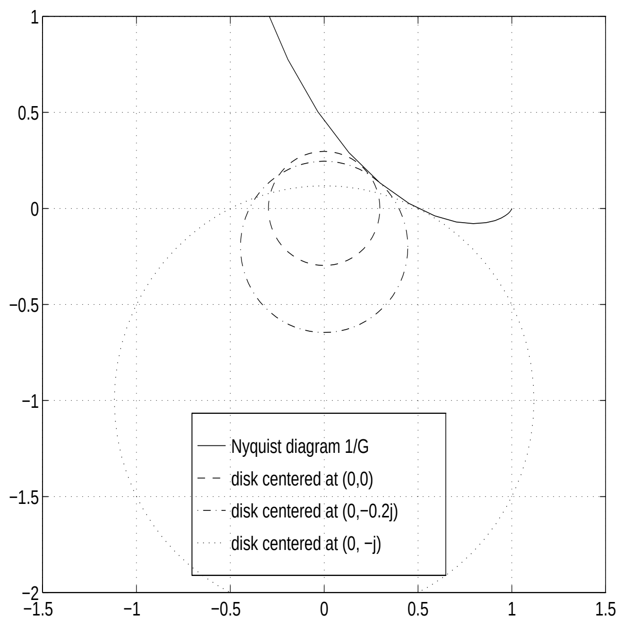

Figure 18.3: Computing the real stability margin by covering with disks

We are interested in finding the largest k such that $1 + \Delta G(s)$ has no zero in the right-half plane for all $\Delta \in [-k, k]$. Of course, the largest k can be found very easily by using well-known stability test, which gives

$$k_{\max} = \left(\sup_{\omega} \mu_{\mathbf{\Delta}}(G(j\omega))\right)^{-1} = \left(\sup_{\omega} \max_{\phi \in [-1,1]} \rho_R(\phi G(j\omega))\right)^{-1}$$

$$= \left(\sup_{\omega} \{|G(j\omega)| : \Im G(j\omega) = 0\}\right)^{-1} = \inf_{\omega} \left\{ \frac{1}{|G(j\omega)|} : \Im G(j\omega) = 0 \right\} = 0.5.$$

Now we use the complex covering idea to find the best possible k. Note that we only need to find the smallest $|\Delta|$ so that $1 + \Delta G(j\omega_0) = 0$ for some ω_0 or, equivalently, $\Delta + 1/G(j\omega_0) = 0$. The frequency response of $1/G$ and the disks covering an interval $[-k, k]$ are shown in Figure 18.3. It is clear that a centered disk would give $k = 1/\|G\|_\infty = 0.2970$ and an off-axis disk centered at $(0, -0.2j)$ would give $k = 0.3984$ while an off-axis disk centered at $(0, -j)$ would give the exactly value $k = 0.5$.

The following alternative characterization of the upper bound is useful in the mixed μ synthesis.

Theorem 18.4 *Given $\beta > 0$, there exist $D \in \mathcal{D}$ and $G \in \mathcal{G}$ such that*

$$M^*DM + j(GM - M^*G) - \beta^2 D \leq 0$$

if and only if there are $D_1 \in \mathcal{D}$ and $G_1 \in \mathcal{G}$ such that

$$\overline{\sigma}\left(\left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right)(I + G_1^2)^{-\frac{1}{2}}\right) \leq 1.$$

Proof. Let $D = D_1^2$ and $G = \beta D_1 G_1 D_1$. Then

$$M^*DM + j(GM - M^*G) - \beta^2 D \leq 0$$

$$\iff M^*D_1^2M + j(\beta D_1G_1D_1M - \beta M^*D_1G_1D_1) - \beta^2 D_1^2 \leq 0$$

$$\iff (D_1MD_1^{-1})^*(D_1MD_1^{-1}) + j(\beta G_1D_1MD_1^{-1} - \beta(D_1MD_1^{-1})^*G_1) - \beta^2 I \leq 0$$

$$\iff \left(\frac{D_1MD_1^{-1}}{\beta} - jG_1\right)^*\left(\frac{D_1MD_1^{-1}}{\beta} - jG_1\right) - (I + G_1^2) \leq 0$$

$$\iff \overline{\sigma}\left[\left(\frac{D_1MD_1^{-1}}{\beta} - jG_1\right)(I + G_1^2)^{-\frac{1}{2}}\right] \leq 1.$$

□

Similarly, the following corollary can be shown.

Corollary 18.5 $\mu_{\mathbf{\Delta}}(M) \leq r\beta$ *if there are $D_1 \in \mathcal{D}$ and $G_1 \in \mathcal{G}$ such that*

$$\overline{\sigma}\left(\left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right)(I + G_1^2)^{-\frac{1}{2}}\right) \leq r \leq 1.$$

Proof. This follows by noting that

$$\overline{\sigma}\left(\left(\frac{D_1 M D_1^{-1}}{\beta} - jG_1\right)(I+G_1^2)^{-\frac{1}{2}}\right) \le r \le 1$$

$$\Longrightarrow \left(\frac{D_1 M D_1^{-1}}{r\beta} - j\frac{G_1}{r}\right)^* \left(\frac{D_1 M D_1^{-1}}{r\beta} - j\frac{G_1}{r}\right) \le I + G_1^2 \le I + \left(\frac{G_1}{r}\right)^2.$$

Let $G_2 = \dfrac{G_1}{r} \in \mathcal{G}$. Then

$$\left(\frac{D_1 M D_1^{-1}}{r\beta} - jG_2\right)^* \left(\frac{D_1 M D_1^{-1}}{r\beta} - jG_2\right) \le I + G_2^2$$

$$\Longrightarrow \overline{\sigma}\left(\left(\frac{D_1 M D_1^{-1}}{r\beta} - jG_2\right)(I+G_2^2)^{-\frac{1}{2}}\right) \le 1$$

$$\Longrightarrow \mu_{\mathbf{\Delta}}(M) \le r\beta.$$

□

Note that this corollary is not necessarily true if $r > 1$. It is fairly easy to check that the well-posedness condition, main loop theorem, robust stability, and robust performance theorems for the mixed μ setup are exactly the same as the ones for complex μ problems.

We are now in the position to consider the synthesis problem with mixed uncertainties. Consider again the general system diagram in Figure 18.4. By the robust performance condition, we need to find a stabilizing controller K so that

$$\min_K \sup_\omega \mu_{\mathbf{\Delta}}\left(\mathcal{F}_\ell(P, K)\right) \le \beta.$$

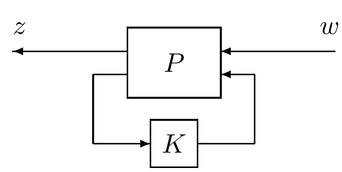

Figure 18.4: Synthesis framework

By Theorems 18.3 and 18.4, $\mu_{\mathbf{\Delta}}\left(\mathcal{F}_\ell\left(P(j\omega), K(j\omega)\right)\right) \le \beta,\ \forall\omega$ if there are frequency-dependent scaling matrices $D_\omega \in \mathcal{D}$ and $G_\omega \in \mathcal{G}$ such that

$$\sup_\omega \overline{\sigma}\left[\left(\frac{D_\omega\left(\mathcal{F}_\ell\left(P(j\omega), K(j\omega)\right)\right) D_\omega^{-1}}{\beta} - jG_\omega\right)(I+G_\omega^2)^{-\frac{1}{2}}\right] \le 1, \quad \forall\omega.$$

Similar to the complex μ synthesis, we can now describe a mixed μ synthesis procedure that involves $D, G - K$ iterations.

$D, G - K$ Iteration:

(1) Let K be a stabilizing controller. Find initial estimates of the scaling matrices $D_\omega \in \mathcal{D}$, $G_\omega \in \mathcal{G}$ and a scalar $\beta_1 > 0$ such that

$$\sup_\omega \overline{\sigma}\left[\left(\frac{D_\omega\left(\mathcal{F}_\ell\left(P(j\omega), K(j\omega)\right)\right) D_\omega^{-1}}{\beta_1} - jG_\omega\right)(I + G_\omega^2)^{-\frac{1}{2}}\right] \le 1, \quad \forall \omega.$$

Obviously, one may start with $D_\omega = I$, $G_\omega = 0$, and a large $\beta_1 > 0$.

(2) Fit the frequency response matrices D_ω and jG_ω with $D(s)$ and $G(s)$ so that

$$D(j\omega) \approx D_\omega, \quad G(j\omega) \approx jG_\omega, \quad \forall\ \omega.$$

Then for $s = j\omega$

$$\sup_\omega \overline{\sigma}\left(\left(\frac{D_\omega\left(\mathcal{F}_\ell\left(P(j\omega), K(j\omega)\right)\right) D_\omega^{-1}}{\beta_1} - jG_\omega\right)(I + G_\omega^2)^{-\frac{1}{2}}\right)$$

$$\approx \sup_\omega \overline{\sigma}\left[\left(\frac{D(s)\left(\mathcal{F}_\ell\left(P(s), K(s)\right)\right) D^{-1}(s)}{\beta_1} - G(s)\right)(I + G^\sim(s)G(s))^{-\frac{1}{2}}\right].$$

(3) Let $D(s)$ be factorized as

$$D(s) = D_{ap}(s)D_{\min}(s), \quad D_{ap}^\sim(s)D_{ap}(s) = I, \quad D_{\min}(s),\ D_{\min}^{-1}(s) \in \mathcal{H}_\infty.$$

That is, D_{ap} is an all-pass and $D_{\min}$ is a stable and minimum phase transfer matrix. Find a normalized right coprime factorization

$$D_{ap}^\sim(s)G(s)D_{ap}(s) = G_N G_M^{-1}, \quad G_N, \quad G_M \in \mathcal{H}_\infty$$

such that

$$G_M^\sim G_M + G_N^\sim G_N = I.$$

Then

$$G_M^{-1} D_{ap}^\sim (I + G^\sim G)^{-1} D_{ap} (G_M^{-1})^\sim = I$$

and, for each frequency $s = j\omega$, we have

$$\overline{\sigma}\left[\left(\frac{D(s)\left(\mathcal{F}_\ell\left(P(s), K(s)\right)\right) D^{-1}(s)}{\beta_1} - G(s)\right)(I + G^\sim(s)G(s))^{-\frac{1}{2}}\right]$$

$$= \overline{\sigma}\left[\left(\frac{D_{\min}\left(\mathcal{F}_\ell\left(P, K\right)\right) D_{\min}^{-1}}{\beta_1} - D_{ap}^\sim G D_{ap}\right) D_{ap}^\sim (I + G^\sim G)^{-\frac{1}{2}}\right]$$

$$= \overline{\sigma}\left[\left(\frac{D_{\min}\left(\mathcal{F}_\ell\left(P,K\right)\right)D_{\min}^{-1}}{\beta_1} - G_N G_M^{-1}\right)D_{ap}^{\sim}(I+G^{\sim}G)^{-\frac{1}{2}}\right]$$

$$= \overline{\sigma}\left[\left(\frac{D_{\min}\left(\mathcal{F}_\ell\left(P,K\right)\right)D_{\min}^{-1}G_M}{\beta_1} - G_N\right)G_M^{-1}D_{ap}^{\sim}(I+G^{\sim}G)^{-\frac{1}{2}}\right]$$

$$= \overline{\sigma}\left[\frac{D_{\min}\left(\mathcal{F}_\ell\left(P,K\right)\right)D_{\min}^{-1}G_M}{\beta_1} - G_N\right].$$

(4) Define

$$P_a = \begin{bmatrix} D_{\min}(s) & \\ & I \end{bmatrix} P(s) \begin{bmatrix} D_{\min}^{-1}(s)G_M(s) & \\ & I \end{bmatrix} - \beta_1 \begin{bmatrix} G_N & \\ & 0 \end{bmatrix}$$

and find a controller K_{new} minimizing $\|\mathcal{F}_\ell(P_a,K)\|_\infty$.

(5) Compute a new β_1 as

$$\beta_1 = \sup_\omega \inf_{\tilde{D}_\omega\in\mathcal{D},\tilde{G}_\omega\in\mathcal{G}} \{\beta(\omega): \quad \Gamma \le 1\}$$

where

$$\Gamma := \overline{\sigma}\left[\left(\frac{\tilde{D}_\omega\mathcal{F}_\ell(P,K_{\text{new}})\tilde{D}_\omega^{-1}}{\beta(\omega)} - j\tilde{G}_\omega\right)(I+\tilde{G}_\omega^2)^{-\frac{1}{2}}\right].$$

(6) Find $\hat{D}_\omega$ and $\hat{G}_\omega$ such that

$$\inf_{\hat{D}_\omega\in\mathcal{D},\hat{G}_\omega\in\mathcal{G}} \overline{\sigma}\left[\left(\frac{\hat{D}_\omega\mathcal{F}_\ell(P,K_{\text{new}})\hat{D}_\omega^{-1}}{\beta_1} - j\hat{G}_\omega\right)(I+\hat{G}_\omega^2)^{-\frac{1}{2}}\right].$$

(7) Compare the new scaling matrices $\hat{D}_\omega$ and $\hat{G}_\omega$ with the previous estimates D_ω and G_ω. Stop if they are close, else replace D_ω, G_ω and K with $\hat{D}_\omega$, $\hat{G}_\omega$ and K_{new}, respectively, and go back to step (2).

18.3 Notes and References

The model validation problems are discussed in Smith and Doyle [1992] in the frequency domain; in Poolla, Khargonekar, Tikku, Krause, Nagpal [1994 in the discrete time domain (on which Section 18.1 is based); and in Rangan and Poolla [1996] and Smith and Dullerud [1996] in the continuous time domain. See also Davis [1995] and Chen and Wang [1996]. The mixed μ problems are discussed in detail in Young [1993] (on which Section 18.2 is based), Fan, Tits, and Doyle [1991], Packard and Pandey [1993], and references therein.

18.4 Problems

Problem 18.1 Write a MATLAB program for the additive model validation problem and try it on a simple experiment in your laboratory.