Chapter 17

Gap Metric and ν-Gap Metric

In the previous chapters, we have seen that all of robust control design techniques assume that we have some description of the model uncertainties (i.e., we have a measure of the distance from the nominal plant to the set of uncertainty systems). This measure is usually chosen to be a metric or a norm. However, the operator norm can be a poor measure of the distance between systems in respect to feedback control system design. For example, consider

$$P_1(s) = \frac{1}{s}, \quad P_2(s) = \frac{1}{s+0.1}.$$

The closed-loop complementary sensitivity functions corresponding to P_1 and P_2 with unity feedback are relatively close and their difference is

$$\left\| P_1(I+P_1)^{-1} - P_2(I+P_2)^{-1} \right\|_\infty = 0.0909,$$

but the difference between P_1 and P_2 is

$$\|P_1 - P_2\|_\infty = \infty.$$

This shows that the closed-loop behavior of two systems can be very close even though the norm of the difference between the two open-loop systems can be arbitrarily large.

To deal with such problems, the gap metric and the ν-gap metric were introduced into the control literature by Zames and El-Sakkary [1980] (see also El-Sakkary [1985] and Vinnicombe [1993]) as being appropriate for the study of uncertainty in feedback systems. An alternative metric, the graph metric, was also introduced by Vidyasagar [1984] in terms of normalized coprime factorizations. All of these metrics are equivalent, and thus induce the same topology. This topology is the weakest in which feedback stability is a robust property. The metrics define notions of distance in the space of (possible) unstable systems that do not assume that the plants have the same number of poles in the right-half plane.

We shall briefly introduce the gap metric in Section 17.1 and study some of its applications in robust control. Our focus in this chapter is Sections 17.2–17.4, which study in some detail the ν-gap metric. In particular, we introduce the ν-gap metric in Section 17.2 and explore its frequency domain interpretation and applications in Section 17.3 and Section 17.4. Finally, we consider controller order reduction in the gap or ν-gap metric framework in Section 17.5.

17.1 Gap Metric

In this section we briefly introduce the gap metric and discuss some of its applications in controller design.

Let $P(s)$ be a $p \times m$ rational transfer matrix and let P have the following normalized right and left stable coprime factorizations:

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}.$$

That is,

$$M^{\sim}M + N^{\sim}N = I, \quad \tilde{M}\tilde{M}^{\sim} + \tilde{N}\tilde{N}^{\sim} = I.$$

The graph of the operator P is the subspace of $\mathcal{H}_2$ consisting of all pairs (u, y) such that $y = Pu$. This is given by

$$\begin{bmatrix} M \\ N \end{bmatrix} \mathcal{H}_2$$

and is a closed subspace of $\mathcal{H}_2$. The gap between two systems P_1 and P_2 is defined by

$$\delta_g(P_1, P_2) = \left\| \Pi_{\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}\mathcal{H}_2} - \Pi_{\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}\mathcal{H}_2} \right\|$$

where Π_K denotes the orthogonal projection onto K and $P_1 = N_1M_1^{-1}$ and $P_2 = N_2M_2^{-1}$ are normalized right coprime factorizations.

It is shown by Georgiou [1988] that the gap metric can be computed as follows:

Theorem 17.1 *Let $P_1 = N_1M_1^{-1}$ and $P_2 = N_2M_2^{-1}$ be normalized right coprime factorizations. Then*

$$\delta_g(P_1, P_2) = \max\left\{\vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1)\right\}$$

where $\vec{\delta}_g(P_1, P_2)$ is the directed gap and can be computed by

$$\vec{\delta}_g(P_1, P_2) = \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty .$$

The following procedures can be used in computing the directed gap $\vec{\delta}_g(P_1, P_2)$.

Computing $\vec{\delta}_g(P_1, P_2)$: Let

$$P_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \quad P_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right].$$

1. Let $P_i = N_i M_i^{-1}$, $i = 1, 2$ be normalized right coprime factorizations. Then

$$\left[\begin{array}{c} M_i \\ N_i \end{array}\right] = \left[\begin{array}{c|c} A_i + B_i F_i & B_i R_i^{-1/2} \\ \hline F_i & R_i^{-1/2} \\ C_i + D_i F_i & D_i R_i^{-1/2} \end{array}\right], \qquad \begin{array}{l} R_i = I + D_i^* D_i \\ \tilde{R}_i = I + D_i D_i^* \\ F_i = -R_i^{-1}(B_i^* X_i + D_i^* C_i) \end{array}$$

where

$$X_i = \operatorname{Ric}\left[\begin{array}{cc} A_i - B_i R_i^{-1} D_i^* C_i & -B_i R_i^{-1} B_i^* \\ -C_i^* \tilde{R}_i^{-1} C_i & -(A_i - B_i R_i^{-1} D_i^* C_i)^* \end{array}\right].$$

2. Define a generalized system

$$G(s) = \left[\begin{array}{cc} \left[\begin{array}{c} M_1 \\ N_1 \end{array}\right] & \left[\begin{array}{c} M_2 \\ N_2 \end{array}\right] \\ -I & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c:c} A_1 + B_1 F_1 & 0 & B_1 R_1^{-1/2} & 0 \\ 0 & A_2 + B_2 F_2 & 0 & B_2 R_2^{-1/2} \\ \hline F_1 & F_2 & R_1^{-1/2} & R_2^{-1/2} \\ C_1 + D_1 F_1 & C_2 + D_2 F_2 & D_1 R_1^{-1/2} & D_2 R_2^{-1/2} \\ \hdashline 0 & 0 & -I & 0 \end{array}\right].$$

3. Apply standard $\mathcal{H}_\infty$ algorithm to

$$\vec{\delta}_g(P_1, P_2) = \inf_{Q \in \mathcal{H}_\infty} \left\| \left[\begin{array}{c} M_1 \\ N_1 \end{array}\right] - \left[\begin{array}{c} M_2 \\ N_2 \end{array}\right] Q \right\|_\infty = \inf_{Q \in \mathcal{H}_\infty} \left\| \mathcal{F}_\ell(G, Q) \right\|_\infty .$$

Using the above procedure, it is easy to show that

$$\delta_g\left(\frac{1}{s}, \frac{1}{s + 0.1}\right) = 0.0995,$$

which confirms that the two systems given at the beginning of this chapter are indeed close in the gap metric. This example shows an important feature about the gap metric (similarly, the ν-gap metric defined in the next section): The distance between two plants, as measured by the gap metric δ_g (or the ν-gap metric δ_ν), has very little to do with any difference between their open-loop behavior (indeed, there is no reason why it should). This point will be further illustrated by an example in the next section.

A lower bound for the gap metric can also be obtained easily without actually solving the corresponding $\mathcal{H}_\infty$ optimization. Let

$$\Phi = \begin{bmatrix} M_2^\sim & N_2^\sim \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix}.$$

Then $\Phi^\sim \Phi = \Phi \Phi^\sim = I$ and

$$\begin{aligned} \vec{\delta}_g(P_1, P_2) &= \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} M_2^\sim & N_2^\sim \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix} \left\{ \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\} \right\|_\infty \\ &= \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} M_2^\sim M_1 + N_2^\sim N_1 - Q \\ -\tilde{N}_2 M_1 + \tilde{M}_2 N_1 \end{bmatrix} \right\|_\infty \geq \|\Psi(P_1, P_2)\|_\infty \end{aligned}$$

where

$$\Psi(P_1, P_2) := -\tilde{N}_2 M_1 + \tilde{M}_2 N_1 = \begin{bmatrix} \tilde{M}_2 & \tilde{N}_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}. \tag{17.1}$$

It will be seen in the next section that $\|\Psi(P_1, P_2)\|_\infty$ is related to the ν-gap metric. The above lower bound may actually be achieved. Consider, for example,

$$P_1 = \frac{k_1}{s+1}, \quad P_2 = \frac{k_2}{s+1}.$$

Then it is easy to verify that $P_i = N_i / M_i$, $i = 1, 2$, with

$$N_i = \frac{k_i}{s + \sqrt{1 + k_i^2}}, \quad M_i = \frac{s+1}{s + \sqrt{1 + k_i^2}},$$

are normalized coprime factorizations and it can be further shown, as in Georgiou and Smith [1990], that

$$\delta_g(P_1, P_2) = \|\Psi(P_1, P_2)\|_\infty = \begin{cases} \dfrac{|k_1 - k_2|}{|k_1| + |k_2|}, & \text{if } |k_1 k_2| > 1; \\ \\ \dfrac{|k_1 - k_2|}{\sqrt{(1 + k_1^2)(1 + k_2^2)}}, & \text{if } |k_1 k_2| \leq 1. \end{cases}$$

An immediate consequence of Theorem 17.1 is the connection between the uncertainties in the gap metric and the uncertainties characterized by the normalized coprime factors. The following corollary states that a ball of uncertainty in the directed gap is equivalent to a ball of uncertainty in the normalized coprime factors.

Corollary 17.2 *Let P have a normalized coprime factorization $P = NM^{-1}$. Then for all $0 < b \leq 1$,*

$$\left\{P_1 : \ \vec{\delta}_g(P, P_1) < b\right\}$$

$$= \left\{P_1 : \quad P_1 = (N+\Delta_N)(M+\Delta_M)^{-1}, \ \Delta_N, \ \Delta_M \in \mathcal{H}_\infty, \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < b \right\}.$$

Proof. Suppose $\vec{\delta}_g(P, P_1) < b$ and let $P_1 = N_1 M_1^{-1}$ be a normalized right coprime factorization. Then there exists a $Q \in \mathcal{H}_\infty$ such that

$$\left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} Q \right\|_\infty < b.$$

Define

$$\begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} := \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} Q - \begin{bmatrix} M \\ N \end{bmatrix} \in \mathcal{H}_\infty.$$

Then $\left\| \begin{bmatrix} \Delta_M \\ \Delta_N \end{bmatrix} \right\|_\infty < b$ and $P_1 = (N_1 Q)(M_1 Q)^{-1} = (N+\Delta_N)(M+\Delta_M)^{-1}$.

To show the converse, note that $P_1 = (N+\Delta_N)(M+\Delta_M)^{-1}$ and there exists a $\tilde{Q}^{-1} \in \mathcal{H}_\infty$ such that $P_1 = \left\{(N+\Delta_N)\tilde{Q}\right\}\left\{(M+\Delta_M)\tilde{Q}\right\}^{-1}$ is a normalized right coprime factorization. Hence by definition, $\vec{\delta}_g(P, P_1)$ can be computed as

$$\vec{\delta}_g(P, P_1) = \inf_Q \left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M+\Delta_M \\ N+\Delta_N \end{bmatrix} \tilde{Q} Q \right\|_\infty \leq \left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M+\Delta_M \\ N+\Delta_N \end{bmatrix} \right\|_\infty < b$$

where the first inequality follows by taking $Q = \tilde{Q}^{-1} \in \mathcal{H}_\infty$. □

The following is a list of useful properties of the gap metric shown by Georgiou and Smith [1990].

- If $\delta_g(P_1, P_2) < 1$, then $\delta_g(P_1, P_2) = \vec{\delta}_g(P_1, P_2) = \vec{\delta}_g(P_2, P_1)$.
- If $b \leq \lambda(P) := \inf_{\mathrm{Re}\, s > 0} \underline{\sigma}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} \right)$, then

$$\left\{P_1 : \ \vec{\delta}(P, P_1) < b\right\} = \left\{P_1 : \ \delta(P, P_1) < b\right\}.$$

Recall that

$$\begin{aligned} b_{\mathrm{obt}}(P) &:= \left\{ \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty \right\}^{-1} \\ &= \sqrt{1-\lambda_{\max}(YQ)} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_H^2} \end{aligned}$$

and

$$b_{P,K} := \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty^{-1} = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I+KP)^{-1} \begin{bmatrix} I & K \end{bmatrix} \right\|_\infty^{-1}.$$

The following results were shown by Qiu and Davison [1992a].

Theorem 17.3 *Suppose the feedback system with the pair* (P_0, K_0) *is stable. Let* $\mathcal{P} := \{P : \ \delta_g(P, P_0) < r_1\}$ *and* $\mathcal{K} := \{K : \ \delta_g(K, K_0) < r_2\}$. *Then*

(a) The feedback system with the pair (P, K) *is also stable for all* $P \in \mathcal{P}$ *and* $K \in \mathcal{K}$ *if and only if*

$$\arcsin b_{P_0,K_0} \geq \arcsin r_1 + \arcsin r_2.$$

(b) The worst possible performance resulting from these sets of plants and controllers is given by

$$\inf_{P\in\mathcal{P},\ K\in\mathcal{K}} \arcsin b_{P,K} = \arcsin b_{P_0,K_0} - \arcsin r_1 - \arcsin r_2.$$

The sufficiency part of the theorem follows from Theorem 17.8 in the next section. Note that the theorem is still true if one of the uncertainty balls is taken as closed ball. In particular, one can take either $r_1 = 0$ or $r_2 = 0$.

Example 17.1 Consider

$$P_1 = \frac{s-1}{s+1}, \quad P_2 = \frac{2s-1}{s+1}.$$

Then $P_1 = N_1/M_1$ and $P_2 = N_2/M_2$ with

$$N_1 = \frac{1}{\sqrt{2}}\frac{s-1}{s+1}, \quad M_1 = \frac{1}{\sqrt{2}}, \quad N_2 = \frac{2s-1}{\sqrt{5}s+\sqrt{2}}, \quad M_2 = \frac{s+1}{\sqrt{5}s+\sqrt{2}}$$

are normalized coprime factorizations. It is easy to show that

$$\delta_g(P_1, P_2) = 1/3 > \|\Psi(P_1, P_2)\|_\infty = \sup_\omega \frac{|\omega|}{\sqrt{10\omega^2+4}} = \frac{1}{\sqrt{10}},$$

which implies that any controller K that stabilizes P_1 and achieves only $b_{P_1,K} > 1/3$ will actually stabilize P_2 by Theorem 17.3. The following MATLAB command can be used to compute the gap:

$$\gg \delta_{\mathbf{g}}(\mathbf{P_1}, \mathbf{P_2}) = \mathbf{gap}(\mathbf{P_1}, \mathbf{P_2}, \mathbf{tol})$$

Next, note that $b_{\text{obt}}(P_1) = 1/\sqrt{2}$ and the optimal controller achieving $b_{\text{obt}}(P_1)$ is $K_{\text{obt}} = 0$. There must be a plant P with $\delta_\nu(P_1, P) = b_{\text{obt}}(P_1) = 1/\sqrt{2}$ that can not be stabilized by $K_{\text{obt}} = 0$; that is, there must be an unstable plant P such that $\delta_\nu(P_1, P) = b_{\text{obt}}(P_1) = 1/\sqrt{2}$. A such P can be found using Corollary 17.2:

$$\{P : \ \delta_g(P_1, P) \le b_{\text{obt}}(P_1)\}$$

$$= \left\{ P : \quad P = \frac{N_1 + \Delta_N}{M_1 + \Delta_M}, \ \Delta_N, \ \Delta_M \in \mathcal{H}_\infty, \ \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty \le b_{\text{obt}}(P_1) \right\}.$$

that is, there must be $\Delta_N, \ \Delta_M \in \mathcal{H}_\infty$, $\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty = b_{\text{obt}}(P_1)$ such that

$$P = \frac{N_1 + \Delta_N}{M_1 + \Delta_M}$$

is unstable. Let

$$\Delta_N = 0, \quad \Delta_M = \frac{1}{\sqrt{2}} \frac{s-1}{s+1}.$$

Then

$$P = \frac{N_1 + \Delta_N}{M_1 + \Delta_M} = \frac{s-1}{2s}, \quad \delta_\nu(P_1, P) = b_{\text{obt}}(P_1) = 1/\sqrt{2}.$$

Example 17.2 We shall now consider the following question: Given an uncertain plant

$$P(s) = \frac{k}{s-1}, \quad k \in [k_1, k_2],$$

(a) Find the best nominal design model $P_0 = \dfrac{k_0}{s-1}$ in the sense

$$\inf_{k_0 \in [k_1, k_2]} \ \sup_{k \in [k_1, k_2]} \delta_g(P, P_0).$$

(b) Let k_1 be fixed and k_2 be variable. Find the k_0 so that the largest family of the plant P can be guaranteed to be stabilized a priori by any controller satisfying $b_{P_0,K} = b_{\text{obt}}(P_0)$.

For simplicity, suppose $k_1 \geq 1$. It can be shown that $\delta_g(P, P_0) = \frac{|k_0 - k|}{k_0 + k}$. Then the optimal k_0 for question (a) satisfies

$$\frac{k_0 - k_1}{k_0 + k_1} = \frac{k_2 - k_0}{k_2 + k_0};$$

that is, $k_0 = \sqrt{k_1 k_2}$ and

$$\inf_{k_0 \in [k_1, k_2]} \sup_{k \in [k_1, k_2]} \delta_g(P, P_0) = \frac{\sqrt{k_2} - \sqrt{k_1}}{\sqrt{k_2} + \sqrt{k_1}}.$$

To answer question (b), we note that by Theorem 17.3, a family of plants satisfying $\delta_g(P, P_0) \leq r$ with $P_0 = k_0/(s+1)$ is stabilizable a priori by any controller satisfying $b_{P_0,K} = b_{\mathrm{obt}}(P_0)$ if, and only if, $r < b_{P_0,K}$. Since $P_0 = N_0/M_0$ with

$$N_0 = \frac{k_0}{s + \sqrt{1 + k_0^2}}, \quad M_0 = \frac{s - 1}{s + \sqrt{1 + k_0^2}}$$

is a normalized coprime factorization, it is easy to show that

$$\left\| \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} \right\|_H = \frac{\sqrt{k_0^2 + (1 - \sqrt{1 + k_0^2})^2}}{2\sqrt{1 + k_0^2}}$$

and

$$b_{\mathrm{obt}}(P_0) = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + k_0^2}}\right)}.$$

Hence we need to find a k_0 such that

$$b_{\mathrm{obt}}(P_0) \geq \max\left\{\frac{k_0 - k_1}{k_0 + k_1}, \frac{k_2 - k_0}{k_2 + k_0}\right\};$$

that is,

$$\sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + k_0^2}}\right)} \geq \max\left\{\frac{k_0 - k_1}{k_0 + k_1}, \frac{k_2 - k_0}{k_2 + k_0}\right\}$$

for a largest possible k_2. The optimal k_0 is given by the solution of the equation:

$$\sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + k_0^2}}\right)} = \frac{k_0 - k_1}{k_0 + k_1}$$

and the largest $k_2 = k_0^2/k_1$. For example, if $k_1 = 1$, then $k_0 = 7.147$ and $k_2 = 51.0793$.

In general, given a family of plant P, it is not easy to see how to choose a best nominal model P_0 such that (a) or (b) is true. This is still a very important open question.

17.2 ν-Gap Metric

The shortfall of the gap metric is that it is not easily related to the frequency response of the system. On the other hand, the ν-gap metric to be introduced in this section has a clear frequency domain interpolation and can, in general, be computed from frequency response. The presentation given in this section, Sections 17.3, 17.4, and 17.5 are based on Vinnicombe [1993a, 1993b], to which readers are referred for further detailed discussions.

To define the new metric, we shall first review a basic concept from the complex analysis.

Definition 17.1 *Let $g(s)$ be a scalar transfer function and let Γ denote a Nyquist contour indented around the right of any imaginary axis poles of $g(s)$, as shown in Figure 17.1. Then the winding number of $g(s)$ with respect to this contour, denoted by* $\mathrm{wno}(g)$*, is the number of counterclockwise encirclements around the origin by $g(s)$ evaluated on the Nyquist contour Γ. (A clockwise encirclement counts as a negative encirclement.)*

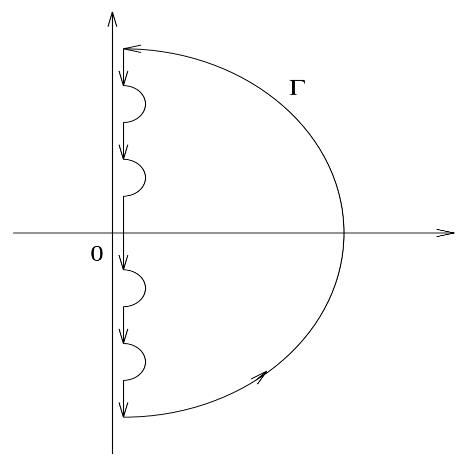

Figure 17.1: The Nyquist contour

The following argument principle is standard and can be found from any complex analysis book.

Lemma 17.4 (The Argument Principle) *Let Γ be a closed contour in the complex plane. Let $f(s)$ be a function analytic along the contour; that is, $f(s)$ has no poles on Γ. Assume $f(s)$ has Z zeros and P poles inside Γ. Then $f(s)$ evaluated along the contour Γ once in an anticlockwise direction will make $Z - P$ anticlockwise encirclements of the origin.*

Let $G(s)$ be a matrix (or scalar) transfer matrix. We shall denote $\eta(G)$ and $\eta_0(G)$, respectively, the number of open right-half plane and imaginary axis poles of $G(s)$.

The winding number has the following properties:

Lemma 17.5 *Let g and h be biproper rational scalar transfer functions and let F be a square transfer matrix. Then*

(a) $\text{wno}(gh) = \text{wno}(g) + \text{wno}(h)$;

(b) $\text{wno}(g) = \eta(g^{-1}) - \eta(g)$;

(c) $\text{wno}(g^{\sim}) = -\text{wno}(g) - \eta_0(g^{-1}) + \eta_0(g)$;

(d) $\text{wno}(1+g) = 0$ *if* $g \in \mathcal{RL}_\infty$ *and* $\|g\|_\infty < 1$;

(e) $\text{wno}\det(I+F) = 0$ *if* $F \in \mathcal{RL}_\infty$ *and* $\|F\|_\infty < 1$.

Proof. Part (a) is obvious by the argument principle. To show part (b), note that by the argument principle $\text{wno}(g)$ equals the excess of the number of open right-half plane zeros of g over the number of open right-half plane poles of g; that is, $\text{wno}(g) = \eta(g^{-1}) - \eta(g)$, since the number of right-half plane zeros of g is the number of right-half plane poles of g^{-1}. Next, suppose the order of g in part (c) is n. Then $\eta(g^{\sim}) = n - \eta(g) - \eta_0(g)$ and $\eta\left[(g^{\sim})^{-1}\right] = n - \eta(g^{-1}) - \eta_0(g^{-1})$, which gives $\text{wno}(g^{\sim}) = \eta\left[(g^{\sim})^{-1}\right] - \eta(g^{\sim}) = \eta(g) - \eta(g^{-1}) - \eta_0(g^{-1}) + \eta_0(g) = -\text{wno}(g) - \eta_0(g^{-1}) + \eta_0(g)$. Part (d) follows from the fact that $1 + \text{Re}\, g(j\omega) > 0,\ \forall\omega$ since $\|g\|_\infty < 1$. Finally, part (e) follows from part (d) and $\det(I+F) = \prod_{i=1}^m (1 + \lambda_i(F))$ with $|\lambda_i(F)| < 1$. $\square$

Example 17.3 Let

$$g_1 = \frac{1.2(s+3)}{s-5}, \quad g_2 = \frac{s-1}{s-2}, \quad g_3 = \frac{2(s-1)(s-2)}{(s+3)(s+4)}, \quad g_4 = \frac{(s-1)(s+3)}{(s-2)(s-4)}.$$

Figure 17.2 shows the functions, g_1, g_2, g_3, and g_4, evaluated on the Nyquist contour Γ. Clearly, we have

$$\text{wno}(g_1) = -1, \quad \text{wno}(g_2) = 0, \quad \text{wno}(g_3) = 2, \quad \text{wno}(g_4) = -1$$

and they are consistent with the results computed from using Lemma 17.5.

The ν-gap metric introduced in Vinnicombe [1993a, 1993b] is defined as follows:

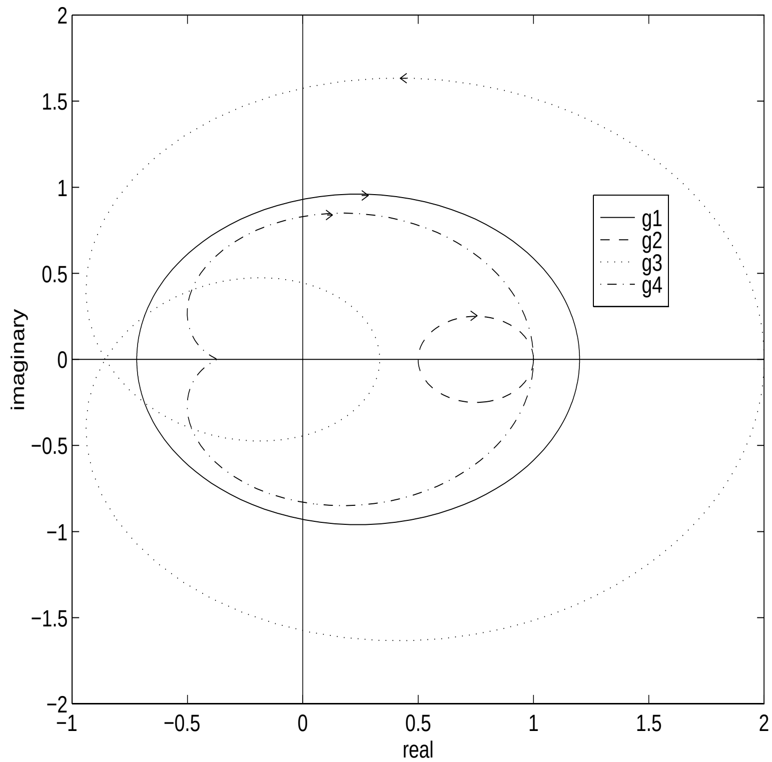

Figure 17.2: g_1, g_2, g_3, and g_4 evaluated on Γ

Definition 17.2 *The ν-gap metric is defined as*

$$\delta_\nu(P_1, P_2) = \begin{cases} \|\Psi(P_1, P_2)\|_\infty, & \text{if } \det\Theta(j\omega) \neq 0 \ \forall\omega \\ & \text{and } \operatorname{wno}\det\Theta(s) = 0, \\ & \\ 1, & \text{otherwise} \end{cases}$$

where $\Theta(s) := N_2^\sim N_1 + M_2^\sim M_1$ *and* $\Psi(P_1, P_2) := -\tilde{N}_2 M_1 + \tilde{M}_2 N_1$.

Note that it can be shown as in Vinnicombe [1993a] that

$$\delta_\nu(P_1, P_2) = \delta_\nu(P_2, P_1) = \delta_\nu(P_1^T, P_2^T)$$

and δ_ν is indeed a metric (a proof of this fact is quite complex).

Computing $\boldsymbol{\delta_\nu(P_1, P_2)}$: Let

$$P_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \quad P_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right].$$

1. Let $P_i = N_i M_i^{-1}$, $i = 1, 2$ be normalized right coprime factorizations. Then

$$\begin{bmatrix} M_i \\ N_i \end{bmatrix} = \left[\begin{array}{c|c} A_i + B_i F_i & B_i R_i^{-1/2} \\ \hline F_i & R_i^{-1/2} \\ C_i + D_i F_i & D_i R_i^{-1/2} \end{array}\right], \qquad \begin{array}{l} R_i = I + D_i^* D_i \\ \tilde{R}_i = I + D_i D_i^* \\ F_i = -R_i^{-1}(B_i^* X_i + D_i^* C_i) \end{array}$$

   where

$$X_i = \operatorname{Ric}\begin{bmatrix} A_i - B_i R_i^{-1} D_i^* C_i & -B_i R_i^{-1} B_i^* \\ -C_i^* \tilde{R}_i^{-1} C_i & -(A_i - B_i R_i^{-1} D_i^* C_i)^* \end{bmatrix}.$$

2. Compute the zeros of

$$\Theta(s) := N_2^\sim N_1 + M_2^\sim M_1 = \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}^\sim \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}.$$

   Let n_0 = number of imaginary axis zeros of Θ, n_+ = number of open right-half plane zeros of Θ, and n = number of open right-half plane poles of Θ. Then wno $\det(N_2^\sim N_1 + M_2^\sim M_1) = n_+ - n$.

3. If either $n_0 \neq 0$ or $n_+ \neq n$, $\delta_\nu(P_1, P_2) = 1$. Otherwise, $\delta_\nu(P_1, P_2) = \|\Psi(P_1, P_2)\|_\infty$ with $\Psi(P_1, P_2) = -\tilde{N}_2 M_1 + \tilde{M}_2 N_1$:

$$\begin{bmatrix} \tilde{M}_i \\ \tilde{N}_i \end{bmatrix} = \left[\begin{array}{c|cc} A_i + L_i C_i & L_i & B_i + L_i D_i \\ \hline \tilde{R}_i^{-1/2} C_i & \tilde{R}_i^{-1/2} & \tilde{R}_i^{-1/2} D_i \end{array}\right]$$

$$L_i = -(B_i D_i^* + Y_i C_i^*)\tilde{R}_i^{-1}$$

   where

$$Y_i = \operatorname{Ric}\begin{bmatrix} (A_i - B_i D_i^* \tilde{R}_i^{-1} C_i)^* & -C_i^* \tilde{R}_i^{-1} C_i \\ -B_i R_i^{-1} B_i^* & -(A_i - B_i D_i^* \tilde{R}_i^{-1} C_i) \end{bmatrix}.$$

The MATLAB command **nugap** can be used to carry out the preceding computation:

$$\gg \ \delta_\nu(\mathbf{P_1}, \mathbf{P_2}) = \mathbf{nugap}(\mathbf{P_1}, \mathbf{P_2}, \mathbf{tol})$$

where tol is the computational tolerance.

Example 17.4 Consider, for example, $P_1 = 1$ and $P_2 = \frac{1}{s}$. Then

$$M_1 = N_1 = \frac{1}{\sqrt{2}}, \quad M_2 = \frac{s}{s+1}, \quad N_2 = \frac{1}{s+1}.$$

Hence

$$\Theta(s) = \frac{1}{\sqrt{2}}\frac{1-s}{1-s} = \frac{1}{\sqrt{2}}, \quad \Psi(P_1, P_2) = \frac{1}{\sqrt{2}}\frac{s-1}{s+1},$$

and $\delta_\nu(P_1, P_2) = \frac{1}{\sqrt{2}}$. (Note that Θ has no poles or zeros!)

The ν-gap metric can also be computed directly from the system transfer matrices without first finding the normalized coprime factorizations.

Theorem 17.6 *The ν-gap metric can be defined as*

$$\delta_\nu(P_1, P_2) = \begin{cases} \|\Psi(P_1, P_2)\|_\infty, & \text{if } \det(I + P_2^\sim P_1) \neq 0 \ \forall\omega \ \text{ and} \\ & \operatorname{wno}\det(I + P_2^\sim P_1) + \eta(P_1) - \eta(P_2) - \eta_0(P_2) = 0, \\ & \\ 1, & \text{otherwise} \end{cases}$$

where $\Psi(P_1, P_2)$ can be written as

$$\Psi(P_1, P_2) = (I + P_2 P_2^\sim)^{-1/2}(P_1 - P_2)(I + P_1^\sim P_1)^{-1/2}.$$

Proof. Since the number of unstable zeros of M_1 (M_2) is equal to the number of unstable poles of P_1 (P_2), and

$$N_2^\sim N_1 + M_2^\sim M_1 = M_2^\sim (I + P_2^\sim P_1) M_1,$$

we have

$$\begin{aligned} \operatorname{wno}\det(N_2^\sim N_1 + M_2^\sim M_1) &= \operatorname{wno}\det\{M_2^\sim (I + P_2^\sim P_1) M_1\} \\ &= \operatorname{wno}\det M_2^\sim + \operatorname{wno}\det(I + P_2^\sim P_1) + \operatorname{wno}\det M_1. \end{aligned}$$

Note that $\operatorname{wno}\det M_1 = \eta(P_1)$, $\operatorname{wno}\det M_2^\sim = -\operatorname{wno}\det M_2 - \eta_0(M_2^{-1}) = -\eta(P_2) - \eta_0(P_2)$, and

$$\operatorname{wno}\det(N_2^\sim N_1 + M_2^\sim M_1) = -\eta(P_2) - \eta_0(P_2) + \operatorname{wno}\det(I + P_2^\sim P_1) + \eta(P_1).$$

Furthermore,

$$\det(N_2^\sim N_1 + M_2^\sim M_1) \neq 0, \ \forall\omega \iff \det(I + P_2^\sim P_1) \neq 0, \ \forall\omega.$$

The theorem follows by noting that

$$\Psi(P_1, P_2) = (I + P_2 P_2^\sim)^{-1/2}(P_1 - P_2)(I + P_1^\sim P_1)^{-1/2}$$

since $\Psi(P_1, P_2) = -\tilde{N}_2 M_1 + \tilde{M}_2 N_1 = \tilde{M}_2(P_1 - P_2)M_1$ and

$$\tilde{M}_2^\sim \tilde{M}_2 = (I + P_2 P_2^\sim)^{-1}, \quad M_1 M_1^\sim = (I + P_1^\sim P_1)^{-1}.$$

□

This alternative formula is useful when doing the hand calculation or when computing from the frequency response of the plants since it does not need to compute the normalized coprime factorizations.

Example 17.5 Consider two plants $P_1 = 1$ and $P_2 = 1/s$. Then wno $\det(1 + P_2^\sim P_1) =$ wno$[(s-1)/s] = 1$, as shown in Figure 17.3(a), and wno $\det(1 + P_2^\sim P_1) + \eta(P_1) - \eta(P_2) - \eta_0(P_2) = 0$. On the other hand, wno $\det(1 + P_1^\sim P_2) + \eta(P_2) - \eta(P_1) =$ wno $(s+1)/s = 0$, as shown in Figure 17.3(b).

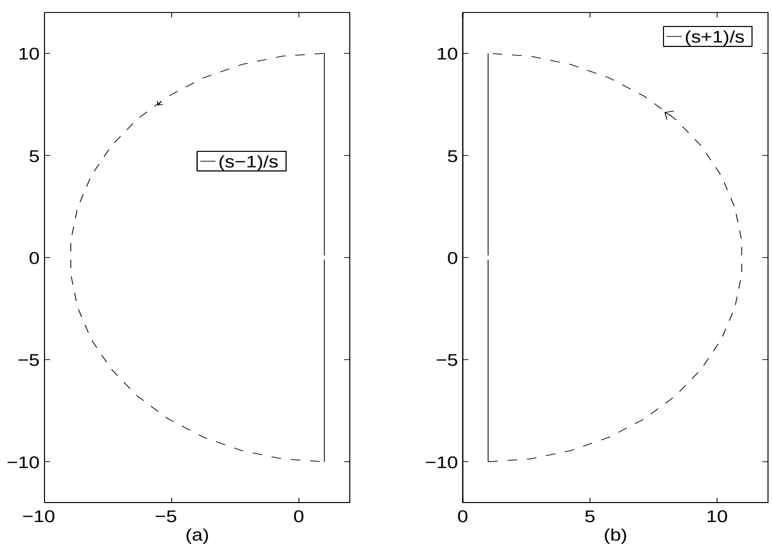

Figure 17.3: $\dfrac{s-1}{s}$ and $\dfrac{s+1}{s}$ evaluated on Γ

Similar to the gap metric, it is shown by Vinnicombe [1993a, 1993b] that the ν-gap metric can also be characterized as an optimization problem (however, we shall not use it for computation).

Theorem 17.7 *Let $P_1 = N_1 M_1^{-1}$ and $P_2 = N_2 M_2^{-1}$ be normalized right coprime factorizations. Then*

$$\delta_\nu(P_1, P_2) = \inf_{\substack{Q, Q^{-1} \in \mathcal{L}_\infty \\ \text{wno } \det(Q) = 0}} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty .$$

Moreover, $\delta_g(P_1, P_2) b_{\text{obt}}(P_1) \leq \delta_\nu(P_1, P_2) \leq \delta_g(P_1, P_2)$.

It is now easy to see that

$$\{P: \ \delta_\nu(P_0, P) < r\}$$
$$\supset \left\{ P = (N_0 + \Delta_N)(M_0 + \Delta_M)^{-1} : \ \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in \mathcal{H}_\infty, \ \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < r \right\}.$$

Define

$$\frac{1}{b_{P,K}(\omega)} := \overline{\sigma} \left(\begin{bmatrix} I \\ K(j\omega) \end{bmatrix} (I + P(j\omega)K(j\omega))^{-1} \begin{bmatrix} I & P(j\omega) \end{bmatrix} \right)$$

and

$$\psi(P_1(j\omega), P_2(j\omega)) = \overline{\sigma} \left(\Psi(P_1(j\omega), P_2(j\omega)) \right).$$

The following theorem states that robust stability can be checked using the frequency-by-frequency test.

Theorem 17.8 *Suppose (P_0, K) is stable and $\delta_\nu(P_0, P_1) < 1$. Then (P_1, K) is stable if*

$$b_{P_0,K}(\omega) > \psi(P_0(j\omega), P_1(j\omega)), \quad \forall \omega.$$

Moreover,

$$\arcsin b_{P_1,K}(\omega) \geq \arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega)), \quad \forall \omega$$

and

$$\arcsin b_{P_1,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P_1).$$

Proof. Let $P_1 = \tilde{M}_1^{-1} \tilde{N}_1$, $P_0 = N_0 M_0^{-1} = \tilde{M}_0^{-1} \tilde{N}_0$ and $K = UV^{-1}$ be normalized coprime factorizations, respectively. Then

$$\frac{1}{b_{P_1,K}(\omega)} = \overline{\sigma} \left(\begin{bmatrix} V \\ U \end{bmatrix} (\tilde{M}_1 V + \tilde{N}_1 U)^{-1} \begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix} \right) = \overline{\sigma} \left((\tilde{M}_1 V + \tilde{N}_1 U)^{-1} \right).$$

That is,

$$b_{P_1,K}(\omega) = \underline{\sigma}(\tilde{M}_1 V + \tilde{N}_1 U) = \underline{\sigma} \left(\begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix} \right).$$

Similarly,

$$b_{P_0,K}(\omega) = \underline{\sigma}(\tilde{M}_0 V + \tilde{N}_0 U) = \underline{\sigma}\left(\begin{bmatrix} \tilde{M}_0 & \tilde{N}_0 \end{bmatrix}\begin{bmatrix} V \\ U \end{bmatrix}\right).$$

Note that

$$\psi(P_0(j\omega), P_1(j\omega)) = \overline{\sigma}\left(\begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix}\begin{bmatrix} N_0 \\ -M_0 \end{bmatrix}\right)$$

$$\begin{bmatrix} N_0 & \tilde{M}_0^{\sim} \\ -M_0 & \tilde{N}_0^{\sim} \end{bmatrix}^{\sim}\begin{bmatrix} N_0 & \tilde{M}_0^{\sim} \\ -M_0 & \tilde{N}_0^{\sim} \end{bmatrix} = I.$$

To simplify the derivation, define

$$G_0 = \begin{bmatrix} N_0 \\ -M_0 \end{bmatrix}, \quad \tilde{G}_0 = \begin{bmatrix} \tilde{M}_0 & \tilde{N}_0 \end{bmatrix}, \quad \tilde{G}_1 = \begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix}, \quad F = \begin{bmatrix} V \\ U \end{bmatrix}.$$

Then

$$\psi(P_0, P_1) = \overline{\sigma}(\tilde{G}_1 G_0), \quad b_{P_0,K}(\omega) = \underline{\sigma}(\tilde{G}_0 F), \quad b_{P_1,K}(\omega) = \underline{\sigma}(\tilde{G}_1 F)$$

and

$$\begin{bmatrix} G_0 & \tilde{G}_0^{\sim} \end{bmatrix}^{\sim}\begin{bmatrix} G_0 & \tilde{G}_0^{\sim} \end{bmatrix} = I \Longrightarrow \begin{bmatrix} G_0 & \tilde{G}_0^{\sim} \end{bmatrix}\begin{bmatrix} G_0 & \tilde{G}_0^{\sim} \end{bmatrix}^{\sim} = I.$$

That is,

$$G_0 G_0^{\sim} + \tilde{G}_0^{\sim}\tilde{G}_0 = I.$$

Note that

$$I = \tilde{G}_1\tilde{G}_1^{\sim} = \tilde{G}_1(G_0 G_0^{\sim} + \tilde{G}_0^{\sim}\tilde{G}_0)\tilde{G}_1^{\sim} = (\tilde{G}_1 G_0)(\tilde{G}_1 G_0)^{\sim} + (\tilde{G}_1\tilde{G}_0^{\sim})(\tilde{G}_1\tilde{G}_0^{\sim})^{\sim}.$$

Hence

$$\underline{\sigma}^2(\tilde{G}_1\tilde{G}_0^{\sim}) = 1 - \overline{\sigma}^2(\tilde{G}_1 G_0).$$

Similarly,

$$I = F^{\sim}F = F^{\sim}(G_0 G_0^{\sim} + \tilde{G}_0^{\sim}\tilde{G}_0)F = (G_0^{\sim}F)^{\sim}(G_0^{\sim}F) + (\tilde{G}_0 F)^{\sim}(\tilde{G}_0 F)$$

$$\Longrightarrow \overline{\sigma}^2(G_0^{\sim}F) = 1 - \underline{\sigma}^2(\tilde{G}_0 F).$$

By the assumption, $\psi(P_0, P_1) < b_{P_0,K}(\omega)$; that is,

$$\overline{\sigma}(\tilde{G}_1 G_0) < \underline{\sigma}(\tilde{G}_0 F), \quad \forall\omega$$

and

$$\overline{\sigma}(G_0^{\sim}F) = \sqrt{1 - \underline{\sigma}^2(\tilde{G}_0 F)} < \sqrt{1 - \overline{\sigma}^2(\tilde{G}_1 G_0)} = \underline{\sigma}(\tilde{G}_1\tilde{G}_0^{\sim}).$$

Hence

$$\overline{\sigma}(\tilde{G}_1 G_0)\overline{\sigma}(G_0^{\sim}F) < \underline{\sigma}(\tilde{G}_1\tilde{G}_0^{\sim})\underline{\sigma}(\tilde{G}_0 F);$$

that is,

$$\overline{\sigma}(\tilde{G}_1 G_0 G_0^{\sim} F) < \underline{\sigma}(\tilde{G}_1 \tilde{G}_0^{\sim} \tilde{G}_0 F), \quad \forall\ \omega$$
$$\Longrightarrow \left\| (\tilde{G}_1 \tilde{G}_0^{\sim} G_0 F)^{-1} (\tilde{G}_1 G_0 G_0^{\sim} F) \right\|_\infty < 1.$$

Now

$$\begin{aligned}\tilde{G}_1 F &= \tilde{G}_1(\tilde{G}_0^{\sim} \tilde{G}_0 + G_0 G_0^{\sim})F = (\tilde{G}_1 \tilde{G}_0^{\sim} \tilde{G}_0 F) + (\tilde{G}_1 G_0 G_0^{\sim} F) \\ &= (\tilde{G}_1 \tilde{G}_0^{\sim} \tilde{G}_0 F)\left(I + (\tilde{G}_1 \tilde{G}_0^{\sim} \tilde{G}_0 F)^{-1}(\tilde{G}_1 G_0 G_0^{\sim} F)\right).\end{aligned}$$

By Lemma 17.5,

$$\text{wno } \det(\tilde{G}_1 F) = \text{wno } \det(\tilde{G}_1 \tilde{G}_0^{\sim} \tilde{G}_0 F) = \text{wno } \det(\tilde{G}_1 \tilde{G}_0^{\sim}) + \text{wno } \det(\tilde{G}_0 F).$$

Since (P_0, K) is stable $\Longrightarrow (\tilde{G}_0 F)^{-1} \in \mathcal{H}_\infty \Longrightarrow \eta((\tilde{G}_0 F)^{-1}) = 0$

$$\Longrightarrow \text{wno } \det(\tilde{G}_0 F) := \eta((\tilde{G}_0 F)^{-1}) - \eta(\tilde{G}_0 F) = 0.$$

Next, note that

$$P_0^T = (\tilde{N}_0^T)(\tilde{M}_0^T)^{-1}, \quad P_1^T = (\tilde{N}_1^T)(\tilde{M}_1^T)^{-1}$$

and $\delta_\nu(P_0^T, P_1^T) = \delta_\nu(P_0, P_1) < 1$; then, by definition of $\delta_\nu(P_0^T, P_1^T)$,

$$\text{wno } \det((\tilde{N}_0^T)^{\sim}(\tilde{N}_1^T) + (\tilde{M}_0^T)^{\sim}(\tilde{M}_1^T)) = \text{wno } \det(\tilde{G}_1 \tilde{G}_0^{\sim})^T = \text{wno } \det(\tilde{G}_1 \tilde{G}_0^{\sim}) = 0.$$

Hence wno $\det(\tilde{G}_1 F) = 0$, but wno $\det(\tilde{G}_1 F) := \eta((\tilde{G}_1 F)^{-1}) - \eta(\tilde{G}_1 F) = \eta((\tilde{G}_1 F)^{-1})$ since $\eta(\tilde{G}_1 F) = 0$, so $\eta((\tilde{G}_1 F)^{-1}) = 0$; that is, (P_1, K) is stable.

Finally, note that

$$\tilde{G}_1 F = \tilde{G}_1(\tilde{G}_0^{\sim} \tilde{G}_0 + G_0 G_0^{\sim})F = (\tilde{G}_1 \tilde{G}_0^{\sim})(\tilde{G}_0 F) + (\tilde{G}_1 G_0)(G_0^{\sim} F)$$

and

$$\begin{aligned}\underline{\sigma}(\tilde{G}_1 F) &\geq \underline{\sigma}(\tilde{G}_1 \tilde{G}_0^{\sim})\underline{\sigma}(\tilde{G}_0 F) - \overline{\sigma}(\tilde{G}_1 G_0)\overline{\sigma}(G_0^{\sim} F) \\ &= \sqrt{1 - \overline{\sigma}^2(\tilde{G}_1 G_0)}\,\underline{\sigma}(\tilde{G}_0 F) - \overline{\sigma}(\tilde{G}_1 G_0)\sqrt{1 - \underline{\sigma}^2(\tilde{G}_0 F)} \\ &= \sin(\arcsin \underline{\sigma}(\tilde{G}_0 F) - \arcsin \overline{\sigma}(\tilde{G}_1 G_0)) \\ &= \sin(\arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega)))\end{aligned}$$

and, consequently,

$$\arcsin b_{P_1,K}(\omega) \geq \arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega))$$

and

$$\inf_\omega \arcsin b_{P_1,K}(\omega) \geq \inf_\omega \arcsin b_{P_0,K}(\omega) - \sup_\omega \arcsin \psi(P_0(j\omega), P_1(j\omega)).$$

That is, $\arcsin b_{P_1,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P_1)$. □

The significance of the preceding theorem can be illustrated using Figure 17.4. It is clear from the figure that $\delta_\nu(P_0, P_1) > b_{P_0,K}$. Thus a frequency-independent stability test cannot conclude that a stabilizing controller K for P_0 will stabilize P_1. However, the frequency-dependent test in the preceding theorem shows that K stabilizes both P_0 and P_1 since $b_{P_0,K}(\omega) > \psi(P_0(j\omega), P_1(j\omega))$ for all ω. Furthermore,

$$b_{P_1,K} \geq \inf_\omega \sin\left(\arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0, P_1)\right) > 0.$$

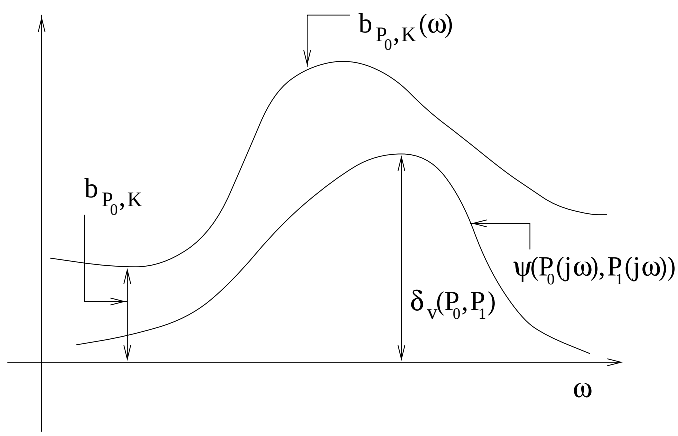

Figure 17.4: K stabilizes both P_0 and P_1 since $b_{P_0,K}(\omega) > \psi(P_0, P_1)$ for all ω

The following theorem is one of the main results on the ν-gap metric.

Theorem 17.9 *Let P_0 be a nominal plant and $\beta \leq \alpha < b_{\rm obt}(P_0)$.*

(i) For a given controller K,

$$\arcsin b_{P,K} > \arcsin \alpha - \arcsin \beta$$

for all P satisfying $\delta_\nu(P_0, P) \leq \beta$ if and only if $b_{P_0,K} > \alpha$.

(ii) For a given plant P,

$$\arcsin b_{P,K} > \arcsin \alpha - \arcsin \beta$$

for all K satisfying $b_{P_0,K} > \alpha$ if and only if $\delta_\nu(P_0, P) \leq \beta$.

Proof. The sufficiency follows essentially from Theorem 17.8. The necessity proof is harder, see Vinnicombe [1993a, 1993b] for details. □

The preceding theorem shows that any plant at a distance less than β from the nominal will be stabilized by any controller stabilizing the nominal with a stability margin of β. Furthermore, any plant at a distance greater than β from the nominal will be *destabilized* by *some* controller that stabilizes the nominal with a stability margin of at least β.

Similarly, one can consider the system robust performance with simultaneous perturbations on the plant and controller.

Theorem 17.10 *Suppose the feedback system with the pair (P_0, K_0) is stable. Then*

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K_0} - \arcsin \delta_\nu(P_0, P) - \arcsin \delta_\nu(K_0, K)$$

for any P and K.

Proof. Use the fact that $b_{P,K} = b_{K,P}$ and apply Theorem 17.8 to get

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P).$$

Dually, we have

$$\arcsin b_{P_0,K} \geq \arcsin b_{P_0,K_0} - \arcsin \delta_\nu(K_0, K).$$

Hence the result follows. □

Example 17.6 Consider again the following example, studied in Vinnicombe [1993b], with

$$P_1 = \frac{s-1}{s+1}, \quad P_2 = \frac{2s-1}{s+1}$$

and note that

$$1 + P_2^\sim P_1 = 1 + \frac{-2s-1}{-s+1}\frac{s-1}{s+1} = \frac{3s+2}{s+1}.$$

Then

$$1 + P_2^\sim(j\omega)P_1(j\omega) \neq 0, \quad \forall\omega, \quad \text{wno}\ \det(I + P_2^\sim P_1) + \eta(P_1) - \eta(P_2) = 0$$

and

$$\delta_\nu(P_1, P_2) = \|\Psi(P_1, P_2)\|_\infty = \sup_\omega \frac{|P_1 - P_2|}{\sqrt{1+|P_1|^2}\,\sqrt{1+|P_2|^2}} = \sup_\omega \frac{|\omega|}{\sqrt{10\omega^2+4}} = \frac{1}{\sqrt{10}}.$$

This implies that any controller K that stabilizes P_1 and achieves only $b_{P_1,K} > 1/\sqrt{10}$ will actually stabilize P_2. This result is clearly less conservative than that of using the gap metric. Furthermore, there exists a controller such that $b_{P_1,K} = 1/\sqrt{10}$ that *destabilizes* P_2. Such a controller is $K = -1/2$, which results in a closed-loop system with P_2 illposed.

Example 17.7 Consider the following example taken from Vinnicombe [1993b]:

$$P_1 = \frac{100}{2s+1}, \quad P_2 = \frac{100}{2s-1}, \quad P_3 = \frac{100}{(s+1)^2}.$$

P_1 and P_2 have very different open-loop characteristics—one is stable, the other unstable. However, it is easy to show that

$$\delta_\nu(P_1, P_2) = \delta_g(P_1, P_2) = 0.02, \quad \delta_\nu(P_1, P_3) = \delta_g(P_1, P_3) = 0.8988,$$

$$\delta_\nu(P_2, P_3) = \delta_g(P_2, P_3) = 0.8941,$$

which show that P_1 and P_2 are very close while P_1 and P_3 (or P_2 and P_3) are quite far away. It is not surprising that any reasonable controller for P_1 will do well for P_2 but not necessarily for P_3. The closed-loop step responses under unity feedback,

$$K_1 = 1,$$

are shown in Figure 17.5.

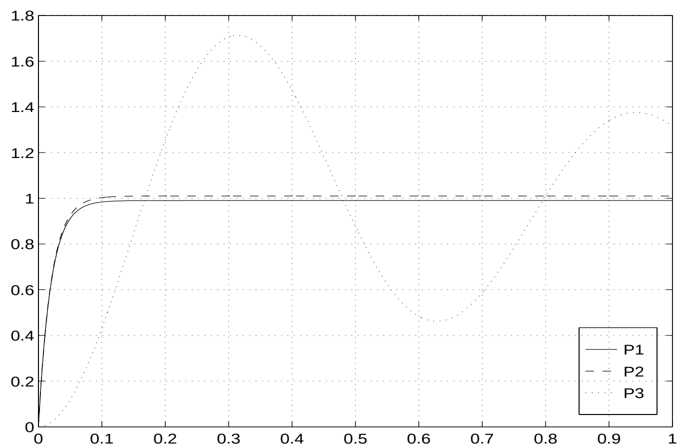

Figure 17.5: Closed-loop step responses with $K_1 = 1$

The corresponding stability margins for the closed-loop systems with P_1 and P_2 are

$$b_{P_1,K_1} = 0.7071, \quad \text{and} \quad b_{P_2,K_1} = 0.7,$$

respectively, which are very close to their maximally possible margins,

$$b_{\mathrm{obt}}(P_1) = 0.7106, \quad \text{and} \quad b_{\mathrm{obt}}(P_2) = 0.7036$$

(in fact, the optimal controllers for P_1 and P_2 are $K = 0.99$ and $K = 1.01$, respectively). While the stability margin for the closed-loop system with P_3 is

$$b_{P_3,K_1} = 0.0995,$$

which is far away from its optimal value, $b_{\text{obt}}(P_3) = 0.4307$, and results in poor performance of the closed loop. In fact, it is not hard to find a controller that will perform well for both P_1 and P_2 but will destabilize P_3.

Of course, this does not necessarily mean that all controllers performing reasonably well with P_1 and P_2 will do badly with P_3, merely that some do — the unit feedback being an example. It may be harder to find a controller that will perform reasonably well with all three plants; the maximally stabilizing controller of P_3,

$$K_3 = \frac{2.0954s + 10.8184}{s + 23.2649},$$

is a such controller, which gives

$$b_{P_1,K_3} = 0.4307, \quad b_{P_2,K_3} = 0.4126, \quad \text{and} \quad b_{P_3,K_3} = 0.4307.$$

The step responses under this control law are shown in Figure 17.6.

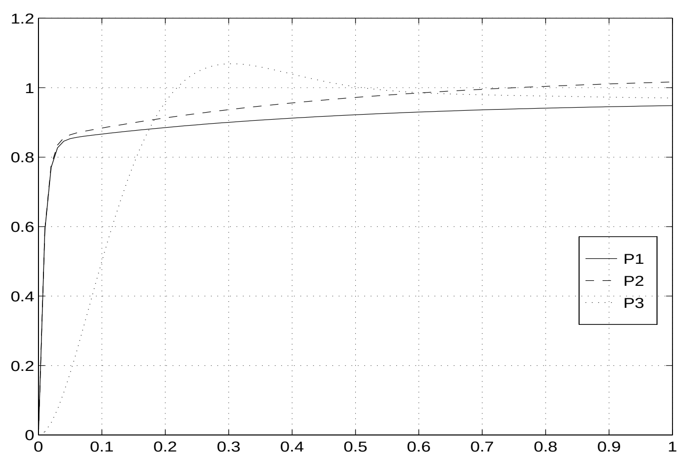

Figure 17.6: Closed-loop step responses with $K_3 = \dfrac{2.0954s + 10.8184}{s + 23.2649}$

17.3 Geometric Interpretation of ν-Gap Metric

The most salient feature of the ν-gap metric is that it can be computed pointwise in frequency domain:

$$\delta_\nu(P_1, P_2) = \sup_\omega \psi(P_1(j\omega),\ P_2(j\omega))$$

provided the winding number condition is satisfied. (For a more extensive coverage of material presented in this section and the next two sections, readers are encouraged to consult the original references by Vinnicombe [1992a, 1992b, 1993a, 1993b].)

In particular, for a single-input single-output system,

$$\psi(P_1(j\omega),\ P_2(j\omega)) = \frac{|P_1(j\omega) - P_2(j\omega)|}{\sqrt{1+|P_1(j\omega)|^2}\ \sqrt{1+|P_2(j\omega)|^2}}. \tag{17.2}$$

This function has the interpretation of being the chordal distance between $P_1(j\omega)$ and $P_2(j\omega)$. To illustrate this, consider the Riemann sphere, which is a unit sphere tangent at its "south pole" to the complex plant at its origin shown in Figure 17.7.

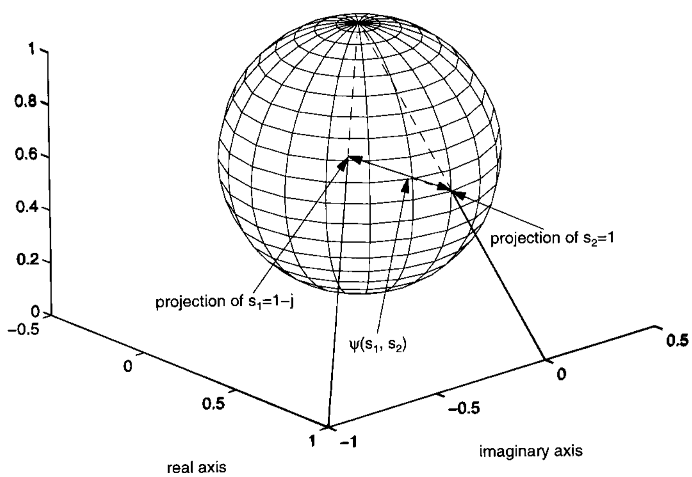

Figure 17.7: Projection onto the Riemann sphere

A point s_1 (e.g., $s_1 = 1-j$) in the complex plane is stereographically projected on the Riemann sphere by connecting the "north pole" to s_1 and determining the intersection of this straight line with the Riemann sphere, resulting in the projection, q_1, of s_1. The coordinates of q_1 are

$$x_1 = \frac{\mathrm{Re}s_1}{1+|s_1|^2}, \quad y_1 = \frac{\Im s_1}{1+|s_1|^2}, \quad z_1 = \frac{|s_1|^2}{1+|s_1|^2}.$$

Thus, the north pole represents the point at infinity and the unit circle is projected onto the "equator." The chordal distance between two points, s_1 and s_2, is the Euclidean distance between their stereographical projections, q_1 and q_2:

$$d(s_1, s_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2} = \frac{|s_1 - s_2|}{\sqrt{1+|s_1|^2}\,\sqrt{1+|s_2|^2}}.$$

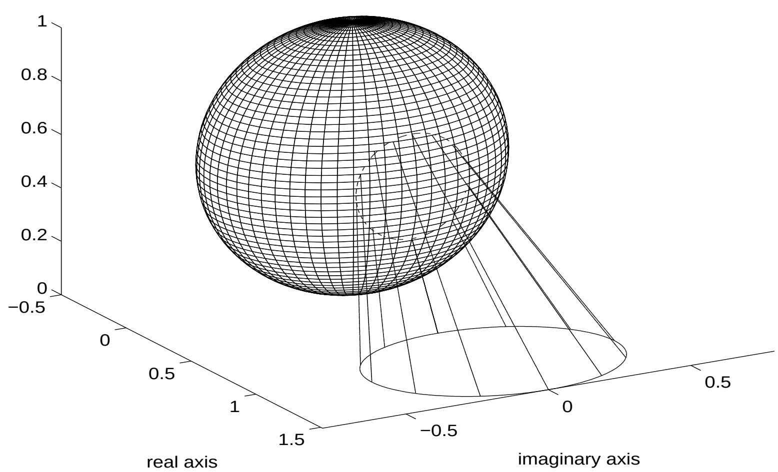

Figure 17.8: Projection of a disk on the Nyquist diagram onto the Riemann sphere

Now consider a circle of chordal radius r centered at $P_0(j\omega_0)$ on the Riemann sphere for some frequency ω_0; that is,

$$\frac{|P(j\omega_0) - P_0(j\omega_0)|}{\sqrt{1+|P(j\omega_0)|^2}\,\sqrt{1+|P_0(j\omega_0)|^2}} = r.$$

Let $P(j\omega_0) = R + jI$ and $P_0(j\omega_0) = R_0 + jI_0$. Then it is easy to show that

$$\left(R - \frac{R_0}{1-\alpha}\right)^2 + \left(I - \frac{I_0}{1-\alpha}\right)^2 = \frac{\alpha(1+|P_0|^2-\alpha)}{(1-\alpha)^2}, \quad \text{if } \alpha \neq 1$$

where $\alpha = r^2(1+|P_0|^2)$. This means that a ball of uncertainty on the ν-gap metric corresponds to a (large) ball of uncertainty on the Nyquist diagram. Figure 17.8 shows a circle of chordal radius 0.2 centered at the stereographical projection of $P_0(j\omega_0) = 1$ and the corresponding circle on the Nyquist diagram.

Figure 17.9 and Figure 17.10 illustrate the uncertainty on the Nyquist diagram corresponding to the balls of uncertainty on the Riemann sphere centered at p_0 with chordal radius 0.2. For example, an uncertainty of 0.2 at $|p_0(j\omega_0)| = 1$ for some ω_0 (i.e., $\delta_\nu(p_0, p) \le 0.2$) implies that $0.661 \le |p(j\omega_0)| \le 1.513$ and the phase difference between p_0 and p is no more than 23.0739^{o} at ω_0.

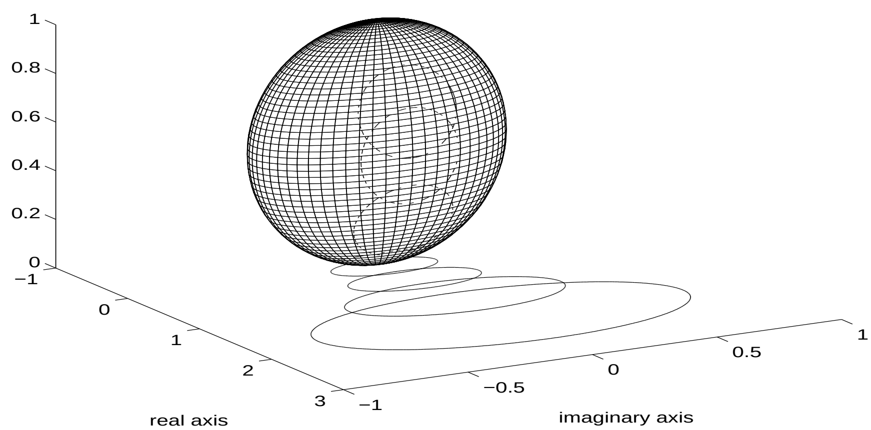

Figure 17.9: Uncertainty on the Riemann sphere and the corresponding uncertainty on the Nyquist diagram

Note that $\|\Psi(P_1, P_2)\|_\infty$ on its own without the winding number condition is useless for the study of feedback systems. This is illustrated through the following example.

Example 17.8 Consider

$$P_1 = 1, \quad P_2 = \frac{s - 1 - \epsilon}{s - 1}.$$

It is clear that P_2 becomes increasingly difficult to stabilize as $\epsilon \to 0$ due to the near unstable pole/zero cancellation. In fact, any stabilizing controller for P_1 will destabilize all P_2 for ϵ sufficiently small. This is confirmed by noting that $b_{\mathrm{obt}}(P_1) = 1,\ b_{\mathrm{obt}}(P_2) \approx \epsilon/2$, and

$$\delta_g(P_1, P_2) = \delta_\nu(P_1, P_2) = 1, \quad \epsilon \ge -2.$$

However, $\|\Psi(P_1, P_2)\|_\infty = \frac{|\epsilon|}{\sqrt{4+4\epsilon+2\epsilon^2}} \approx \frac{\epsilon}{2}$ in itself fails to indicate the difficulty of the problem.

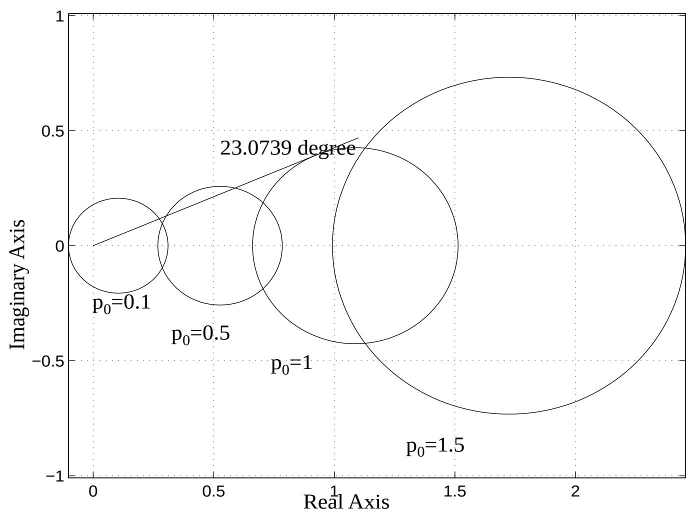

Figure 17.10: Uncertainty on the Nyquist diagram corresponding to the balls of uncertainty on the Riemann sphere centered at p_0 with chordal radius 0.2

17.4 Extended Loop-Shaping Design

Let $\mathcal{P}$ be a family of parametric uncertainty systems and let $P_0 \in \mathcal{P}$ be a nominal design model. We are interested in finding a controller so that we have the largest possible robust stability margin; that is,

$$\sup_K \inf_{P\in\mathcal{P}} b_{P,K}.$$

Note that by Theorem 17.8, for any $P_1 \in \mathcal{P}$, we have

$$\arcsin b_{P_1,K}(\omega) \geq \arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega)), \quad \forall \omega.$$

Now suppose we need $\inf_{P\in\mathcal{P}} b_{P,K} > \alpha$. Then it is sufficient to have

$$\arcsin b_{P_0,K}(\omega) - \arcsin \psi(P_0(j\omega), P_1(j\omega)) > \arcsin \alpha, \quad \forall \omega, \quad P_1 \in \mathcal{P};$$

that is,

$$b_{P_0,K}(\omega) > \sin\left(\arcsin \psi(P_0(j\omega), P_1(j\omega)) + \arcsin \alpha\right), \quad \forall \omega, \quad P_1 \in \mathcal{P}.$$

Let $W(s) \in \mathcal{H}_\infty$ be such that

$$|W(j\omega)| \geq \sin\left(\arcsin \psi(P_0(j\omega), P_1(j\omega)) + \arcsin \alpha\right), \quad \forall \omega, \quad P_1 \in \mathcal{P}.$$

Then it is sufficient to guarantee

$$\frac{|W(j\omega)|}{b_{P_0,K}(\omega)} < 1.$$

Let $P_0 = \tilde{M}_0^{-1}\tilde{N}_0$ be a normalized left coprime factorization and note that

$$\frac{1}{b_{P_0,K}(\omega)} := \overline{\sigma}\left(\begin{bmatrix} I \\ K(j\omega) \end{bmatrix}(I + P_0(j\omega)K(j\omega))^{-1}\tilde{M}_0^{-1}(j\omega)\right).$$

Then it is sufficient to find a controller so that

$$\left\|\begin{bmatrix} I \\ K \end{bmatrix}(I + P_0K)^{-1}\tilde{M}_0^{-1}W\right\|_\infty < 1.$$

The process can be iterated to find the largest possible α.

Combining the preceding robust stabilization idea and the $\mathcal{H}_\infty$ loop-shaping in Chapter 16, one can devise an extended loop-shaping design procedure as follows. (A more advanced loop-shaping procedure can be found in Vinnicombe [1993b].)

Design Procedure:

Let $\mathcal{P}$ be a family of parametric uncertain systems and let P_0 be a nominal model.

(a) Loop-Shaping: The singular values of the nominal plant are shaped, using a precompensator W_1 and/or a postcompensator W_2, to give a desired open-loop shape. The nominal plant P_0 and the shaping functions W_1, W_2 are combined to form the shaped plant, P_s, where $P_s = W_2P_0W_1$. We assume that W_1 and W_2 are such that P_s contains no hidden modes.

(b) Compute *frequency-by-frequency*:

$$f(\omega) = \sup_{P\in\mathcal{P}} \psi(P_s(j\omega), W_2(j\omega)P(j\omega)W_1(j\omega)).$$

Set $\alpha = 0$.

(b) Fit a stable and minimum phase rational transfer function $W(s)$ so that

$$|W(j\omega)| \geq \sin(\arcsin f(\omega) + \arcsin\alpha) \quad \forall\omega.$$

(c) Find a K_∞ such that

$$\beta := \inf_{K_\infty}\left\|\begin{bmatrix} I \\ K_\infty \end{bmatrix}(I + P_0K_\infty)^{-1}\tilde{M}_0^{-1}W\right\|_\infty.$$

(d) If $\beta \approx 1$, stop and the final controller is $K = W_1K_\infty W_2$. If $\beta \ll 1$, increase α and go back to (b). If $\beta \gg 1$, decrease α and go back to (b).

17.5 Controller Order Reduction

The controller order-reduction procedure described in Chapter 15 can, of course, be applied to the loop-shaping controller design in Chapter 16 and the gap or ν-gap metric optimization here. However, the controller order-reduction in the loop-shaping controller design, or gap metric, or ν-gap metric optimization is especially simple. The following theorem follows immediately from Theorems 17.7 and 17.10.

Theorem 17.11 *Let P_0 be a nominal plant and K_0 be a stabilizing controller such that $b_{P_0,K_0} \leq b_{\text{obt}}(P_0)$. Let $K_0 = UV^{-1}$ be a normalized coprime factorization and let $\hat{U}, \hat{V} \in \mathcal{RH}_\infty$ be such that*

$$\left\| \begin{bmatrix} U \\ V \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right\|_\infty \leq \varepsilon.$$

Then $K := \hat{U}\hat{V}^{-1}$ stabilizes P_0 if $\varepsilon < b_{P_0,K_0}$. Furthermore,

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K_0} - \arcsin \varepsilon - \arcsin \beta$$

for all $\{P : \delta_\nu(P, P_0) \leq \beta\}$.

Hence to reduce the controller order one only needs to approximate the normalized coprime factors of the controller. An algorithm for finding the best approximation is also presented in Vinnicombe [1993a, 1993b].

17.6 Notes and References

This chapter is based on Georgiou and Smith [1990] and Vinnicombe [1993a, 1993b]. Early studies of the gap metric can be found in Zames and El-Sakkary [1980] and El-Sakkary [1985]. The pointwise gap metric was introduced by Qiu and Davision [1992b].

17.7 Problems

Problem 17.1 *Calculate the gap $\delta_g(P_i, P_j)$ with*

$$P_1 = \frac{1}{s+1}, \quad P_2 = \frac{1}{s-1}, \quad P_3 = \frac{s+2}{(s+1)^2}, \quad P_4 = \frac{s-2}{(s+1)^2}, \quad P_5 = \frac{1}{(s+1)^2}.$$

Problem 17.2 *Let $P = \frac{10}{\tau s+1}, \tau \in [1, 3]$ and let $P_0 = \frac{10}{\tau_0 s+1}$. Find the optimal $\tau_0 \in [1, 3]$ minimizing*

$$\min_{\tau_o} \max_{\tau} \delta_g(P, P_0).$$

Problem 17.3 *Repeat Problems 17.1–17.2 for the ν-gap metric.*