for most practical problems the original system model corresponds to a causal physical system for which we can always enforce $E = I$, we prefer to use the generalized system representation in (7.13) in conjunction with computational techniques for descriptor systems to prevent possible loss of accuracy due to ill-conditioned inversion of E and, simultaneously, to preserve the most general problem formulation for arbitrary E. As it will be apparent in the next section, the computational aspects are practically the same for systems in both standard and descriptor forms.

For the intermediary (or even final) forms of the fault detection filter $Q(\lambda)$ (implementation form) and $R(\lambda)$ (internal form) it is possible to enforce at the end of each computational step state-space representations of the form

$$\left[\, Q(\lambda) \;\; R(\lambda) \,\right] = \left[\begin{array}{c|cc} A_Q - \lambda E_Q & B_Q & B_R \\ \hline C_Q & D_Q & D_R \end{array}\right],$$

where the descriptor pair $(A_Q - \lambda E_Q, C_Q)$ is observable. This is done either explicitly, via explicit updating formulas or, implicitly, by using minimal realization techniques to enforce all possible pole-zero cancellations.

In the following sections we describe the computational aspects of the main steps of the synthesis procedures presented in Chaps. 5 and 6. The emphasis of the presentation is to give an overview of the basic computations performed at each synthesis step, but without entering into algorithmic details. However, for the interested readers these details are partly explained in the Chap. 10 and, further, in the references provided in the Sects. 7.11 and 10.6. Some special algorithms for descriptor systems, which are instrumental for solving the computational problems for the synthesis of fault detection filters are described in Sect. 10.4. Available software implementations are mentioned in Sect. 10.5.

7.4 Nullspace-Based Reduction

In this section we discuss the numerical computations performed at Step 1) of all synthesis procedures presented in this book. In particular, this is the main step of **Procedure EFD**, where the resulting (stable) filter represents a solution of the EFDP, provided the fault detectability conditions are fulfilled. This step typically involves two main computations. The first one is the computation of $N_l(\lambda)$, a proper rational left nullspace basis of the $(p + m_u) \times (m_u + m_d)$ TFM

$$G(\lambda) = \begin{bmatrix} G_u(\lambda) & G_d(\lambda) \\ I_{m_u} & 0 \end{bmatrix}. \tag{7.14}$$

This serves to set the first factor of $Q(\lambda)$ to $Q_1(\lambda) := N_l(\lambda)$ and to initialize the filter synthesis by setting the TFM of the implementation form of the filter to $Q(\lambda) = Q_1(\lambda)$. The second computation is the determination of the nonzero TFMs of the reduced proper system (5.11)

$$\left[\, \overline{G}_w(\lambda)\ \overline{G}_f(\lambda) \,\right] := N_l(\lambda)F(\lambda)\,, \tag{7.15}$$

where

$$F(\lambda) := \left[\begin{array}{cc} G_w(\lambda) & G_f(\lambda) \\ 0 & 0 \end{array}\right]. \tag{7.16}$$

This serves to initialize the TFM of the internal form as $R(\lambda) = \left[\, \overline{G}_w(\lambda)\ \overline{G}_f(\lambda) \,\right]$. For both computations we rely on the numerically stable algorithm described in Sect. 10.3.2 to compute proper nullspace basis. Additionally, we discuss the checking of solvability conditions for several problems using the resulting state-space representation of the TFM $\overline{G}_f(\lambda)$.

In what follows, we assume that $p > r_d := \operatorname{rank} G_d(\lambda)$, which guarantees the existence of a nonempty left nullspace basis with $p - r_d$ rational basis vectors. Using the realization (7.13), state-space realizations of $G(\lambda)$ and $F(\lambda)$ are

$$G(\lambda) = \left[\begin{array}{c|cc} A - \lambda E & B_u & B_d \\ \hline C & D_u & D_d \\ 0 & I_{m_u} & 0 \end{array}\right], \qquad F(\lambda) = \left[\begin{array}{c|cc} A - \lambda E & B_w & B_f \\ \hline C & D_w & D_f \\ 0 & 0 & 0 \end{array}\right]. \tag{7.17}$$

The computational method of left nullspace bases exploits the simple fact that $N_l(\lambda)$ is a left nullspace basis of $G(\lambda)$ if and only if, for a suitable $M_l(\lambda)$,

$$Y_l(\lambda) := [\, M_l(\lambda)\ N_l(\lambda) \,] \tag{7.18}$$

is a left nullspace basis of the associated system matrix

$$S(\lambda) = \left[\begin{array}{ccc} A - \lambda E & B_u & B_d \\ C & D_u & D_d \\ 0 & I_{m_u} & 0 \end{array}\right]. \tag{7.19}$$

Thus, to compute a proper rational left nullspace basis $N_l(\lambda)$ of $G(\lambda)$ we can determine first a proper rational left nullspace basis $Y_l(\lambda)$ of $S(\lambda)$ and then, $N_l(\lambda)$ simply results as

$$N_l(\lambda) = Y_l(\lambda) \left[\begin{array}{c} 0 \\ I_{p+m_u} \end{array}\right]. \tag{7.20}$$

Since $Y_l(\lambda)$, of the form (7.18), is a left nullspace basis of $S(\lambda)$ in (7.19), it is easy to show that

$$Y_l(\lambda) \left[\begin{array}{c|c|c} A - \lambda E & B_w & B_f \\ C & D_w & D_f \\ 0 & 0 & 0 \end{array}\right] = \left[\, 0 \;\middle|\; \overline{G}_w(\lambda) \;\middle|\; \overline{G}_f(\lambda) \,\right],$$

and, therefore, $N_l(\lambda)F(\lambda)$ in (7.15) results as

$$\left[\; \overline{G}_w(\lambda) \;\; \overline{G}_f(\lambda) \;\right] = Y_l(\lambda) \begin{bmatrix} B_w & B_f \\ D_w & D_f \\ 0 & 0 \end{bmatrix}. \tag{7.21}$$

We recall shortly the computation of $Y_l(\lambda)$, using the approach presented in Sect. 10.3.2. Let U and V be orthogonal matrices such that the transformed pencil $\widetilde{S}(\lambda) := US(\lambda)V$ is in the Kronecker-like staircase form (see Sect. 10.1.6)

$$\widetilde{S}(\lambda) = \left[\begin{array}{cc} A_r - \lambda E_r & A_{r,l} - \lambda E_{r,l} \\ \hline 0 & A_l - \lambda E_l \\ \hline 0 & C_l \end{array}\right], \tag{7.22}$$

where the descriptor pair $(A_l - \lambda E_l, C_l)$ is observable, E_l is non-singular, and $A_r - \lambda E_r$ has full row rank excepting possibly a finite set of values of λ (i.e., the invariant zeros of $S(\lambda)$). The proper rational left nullspace basis $Y_l(\lambda)$ of $S(\lambda)$ can be determined as

$$Y_l(\lambda) = \left[\begin{array}{c|c|c} 0 & C_l(\lambda E_l - A_l)^{-1} & I_{p-r_d} \end{array}\right] U\,. \tag{7.23}$$

We compute now

$$U \left[\begin{array}{cc|c|c} 0 & 0 & B_w & B_f \\ I_p & 0 & D_w & D_f \\ 0 & I_{m_u} & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} * & * & * \\ \hline B_l & \overline{B}_w & \overline{B}_f \\ \hline D_l & \overline{D}_w & \overline{D}_f \end{array}\right], \tag{7.24}$$

where the row partitioning of the right hand side corresponds to the row partitioning of $\widetilde{S}(\lambda)$ in (7.22). With $Y_l(\lambda)$ in the form (7.23) and using (7.24), we obtain from (7.20) and (7.21)

$$[\, N_l(\lambda)\; \overline{G}_w(\lambda)\; \overline{G}_f(\lambda)\,] = \left[\begin{array}{c|ccc} A_l - \lambda E_l & B_l & \overline{B}_w & \overline{B}_f \\ \hline C_l & D_l & \overline{D}_w & \overline{D}_f \end{array}\right]. \tag{7.25}$$

The descriptor representation (7.25) has been obtained by performing exclusively orthogonal transformations on the system matrices. We can prove that all computed matrices are exact for a slightly perturbed system matrix pencil. It follows that the approach to compute the matrices of the realization (7.25) is *numerically backward stable*.

According to Proposition 10.2, the realization $(A_l - \lambda E_l, B_l, C_l, D_l)$ of $N_l(\lambda)$ resulted in (7.25) represents a minimal proper rational left nullspace basis provided the realization (7.14) is controllable. In this case, according to Proposition 10.3, the realization of $N_l(\lambda)$ is also maximally controllable. Since, in general, $N_l(\lambda)$ has no infinite zeros, D_l has full row rank. However, in the case, when the realization (7.17) of $G(\lambda)$ is not controllable, then the descriptor realization $(A_l - \lambda E_l, B_l, C_l, D_l)$ does not represent, in general, a minimal proper left nullspace basis, because, it can be uncontrollable, or it can be controllable, but not maximally controllable. In both cases, a lower order basis exists.

Remark 7.1 The realization (7.13) can be always assumed minimal, in which case the realization (7.17) of $G(\lambda)$ is observable, but, in general, may be uncontrollable. For example, uncontrollable generalized eigenvalues of the pair (A, E) may exist, if some poles of $[\, G_w(\lambda) \; G_f(\lambda) \,]$, are not simultaneously poles of $[\, G_u(\lambda) \; G_d(\lambda) \,]$. Fortunately, the minimality of the realization of $G(\lambda)$ is typically fulfilled if only additive actuator and sensor faults are considered and $w \equiv 0$. In this case, B_f has partly the same columns as B_u (in the case of actuator faults) or zero columns (in the case of sensor faults). Controllability is also guaranteed if the noise input w accounts exclusively for the effects of parametric uncertainties and the nominal model is controllable (see Sect. 2.2.1). □

Remark 7.2 We can always determine a stable nullspace basis, using an output injection matrix K such that the pair $(A_l + KC_l, E_l)$ has stable generalized eigenvalues, or, alternatively, the spectrum $\Lambda(A_l + KC_l, E_l) = \Gamma$, where Γ is any symmetric set of n_l complex values in $\mathbb{C}_s$. Following the approach in Sect. 10.3.2, we perform an additional similarity transformation on $\widetilde{S}(\lambda)$ in (7.22), with the transformation matrix

$$\widehat{U} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & K \\ 0 & 0 & I \end{bmatrix}, \tag{7.26}$$

to obtain $\widehat{S}(\lambda) := \widehat{U}\widetilde{S}(\lambda)$ as

$$\widehat{S}(\lambda) = \left[\begin{array}{cc} A_r - \lambda E_r & A_{r,l} - \lambda E_{r,l} \\ \hline 0 & A_l + KC_l - \lambda E_l \\ \hline 0 & C_l \end{array}\right]. \tag{7.27}$$

After computing

$$\widehat{U} U \left[\begin{array}{cc|c|c} 0 & 0 & B_w & B_f \\ I_p & 0 & D_w & D_f \\ 0 & I_{m_u} & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} * & * & * \\ \hline B_l + KD_l & \overline{B}_w + K\overline{D}_w & \overline{B}_f + K\overline{D}_f \\ \hline D_l & \overline{D}_w & \overline{D}_f \end{array}\right], \tag{7.28}$$

we can form the realization for an alternative basis $\widetilde{N}_l(\lambda)$ and the corresponding reduced system $[\, \widetilde{G}_w(\lambda) \; \widetilde{G}_f(\lambda) \,]$ in the form

$$\left[\, \widetilde{N}_l(\lambda) \; \widetilde{G}_w(\lambda) \; \widetilde{G}_f(\lambda) \,\right] = \left[\begin{array}{c|ccc} A_l + KC_l - \lambda E_l & B_l + KD_l & \overline{B}_w + K\overline{D}_w & \overline{B}_f + K\overline{D}_f \\ \hline C_l & D_l & \overline{D}_w & \overline{D}_f \end{array}\right]. \tag{7.29}$$

It is easy to check that, with

$$\widetilde{M}_l(\lambda) = \left[\begin{array}{c|c} A_l + KC_l - \lambda E_l & K \\ \hline C_l & I \end{array}\right], \tag{7.30}$$

we implicitly determined the stable LCF

$$\left[\, N_l(\lambda)\ \overline{G}_w(\lambda)\ \overline{G}_f(\lambda) \,\right] = \widetilde{M}_l^{-1}(\lambda) \left[\, \widetilde{N}_l(\lambda)\ \widetilde{G}_w(\lambda)\ \widetilde{G}_f(\lambda) \,\right].$$

□

An important property of the resulting realizations of $N_l(\lambda)$, $\overline{G}_w(\lambda)$ and $\overline{G}_f(\lambda)$ in (7.25) is that they share the same observable pair $(A_l - \lambda E_l, C_l)$. However, in general, all three individual realizations may be uncontrollable (thus not minimal). However, we can easily show the following result.

Proposition 7.1 *If the realization (7.13) of* $\left[\, G_u(\lambda)\ G_d(\lambda)\ G_w(\lambda)\ G_f(\lambda) \,\right]$ *is irreducible, then the realization (7.25) is minimal.*

Proof The pair $(A_l - \lambda E_l, C_l)$ is observable by construction. To prove the controllability of the pair $(A_l - \lambda E_l, [\, B_l\ \overline{B}_w\ \overline{B}_f \,])$ we apply the same technique as in the proof of Proposition 10.2. Since the realization (7.13) of $\left[\, G_u(\lambda)\ G_d(\lambda)\ G_w(\lambda)\ G_f(\lambda) \,\right]$ is controllable, the realization

$$\left[\begin{array}{cccc} G_u(\lambda) & G_d(\lambda) & G_w(\lambda) & G_f(\lambda) \\ I_{m_u} & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cccc} A - \lambda E & B_u & B_d & B_w & B_f \\ \hline C & D_u & D_d & D_w & D_f \\ 0 & I_{m_u} & 0 & 0 & 0 \end{array}\right]$$

is controllable as well. Due to the controllability of the pair $(A - \lambda E, [\, B_u\ B_d\ B_w\ B_f \,])$, the pencil $[\, A - \lambda E\ \ B_u\ \ B_d\ \ B_w\ \ B_f \,]$ has full row rank, and thus the reduced pencil

$$U \left[\begin{array}{ccc|c|c|cc} A - \lambda E & B_u & B_d & B_w & B_f & 0 & 0 \\ C & D_u & D_d & D_w & D_f & I_p & 0 \\ 0 & I_{m_u} & 0 & 0 & 0 & 0 & I_{m_u} \end{array}\right] \left[\begin{array}{c|c} V & 0 \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{cc|c|c|c} A_r - \lambda E_r & A_{r,l} - \lambda E_{r,l} & * & * & * \\ 0 & A_l - \lambda E_l & \overline{B}_w & \overline{B}_f & B_l \\ 0 & C_l & \overline{D}_w & \overline{D}_f & D_l \end{array}\right]$$

has full row rank as well. It follows that $\left[\, A_l - \lambda E_l\ \ B_l\ \ \overline{B}_w\ \ \overline{B}_f \,\right]$ has full row rank and thus the pair $(A_l - \lambda E_l, \left[\, B_l\ \overline{B}_w\ \overline{B}_f \,\right])$ is controllable. The minimality is implied by irreducibility, because E_l is invertible. ■

Remark 7.3 If $m_u = m_d = 0$ and $[\, G_w(\lambda)\ G_f(\lambda) \,]$ is not proper, then realizations of the form (7.25) can be obtained by computing $[\, N_l(\lambda)\ \overline{G}_w(\lambda)\ \overline{G}_f(\lambda) \,]$, a proper left nullspace basis satisfying

$$[\, N_l(\lambda)\ \overline{G}_w(\lambda)\ \overline{G}_f(\lambda) \,] \left[\begin{array}{cc} G_w(\lambda) & G_f(\lambda) \\ -I_{m_w} & 0 \\ 0 & -I_{m_f} \end{array}\right] = 0\,.$$

The described nullspace computation approach leads to a state-space realization of the three TFMs $N_l(\lambda)$, $\overline{G}_f(\lambda)$ and $\overline{G}_w(\lambda)$ as in (7.25), which share the observable pair $(A_l - \lambda E_l, C_l)$. □

Proposition 7.1 and Remark 7.3 show that the initial synthesis problems formulated for the system $\big[\, G_u(\lambda)\; G_d(\lambda)\; G_w(\lambda)\; G_f(\lambda) \,\big]$ with minimal realization (7.13) have been reduced to simpler problems formulated for a reduced proper system $\big[\, 0\; 0\; \overline{G}_w(\lambda)\; \overline{G}_f(\lambda) \,\big]$, without control and disturbance inputs, such that the compound TFM

$$\big[\, Q(\lambda)\;\; R_w(\lambda)\;\; R_f(\lambda) \,\big] := \big[\, N_l(\lambda)\; \overline{G}_w(\lambda)\; \overline{G}_f(\lambda) \,\big] \,, \tag{7.31}$$

representing the initial synthesis at Step 1) of the synthesis procedures, has a minimal realization given by (7.25). Moreover, the stability of the initial synthesis can be enforced, as discussed in Remark 7.2, by using a suitable output injection matrix K such that the pair $(A_l + KC_l, E_l)$ has only stable generalized eigenvalues. Accordingly, the alternative initial synthesis

$$\big[\, Q(\lambda)\;\; R_w(\lambda)\;\; R_f(\lambda) \,\big] := \big[\, \widetilde{N}_l(\lambda)\; \widetilde{G}_w(\lambda)\; \widetilde{G}_f(\lambda) \,\big] \tag{7.32}$$

can be chosen, whose minimal realization is given in (7.29). An important aspect to mention is that, similarly to the original realization (7.13), both realizations (7.25) and (7.29), share the same observable pairs $(A_l - \lambda E_l, C_l)$ and $(A_l + KC_l - \lambda E_l, C_l)$, respectively. Any further updating of the initial synthesis can be done by preserving these properties.

Updating of the initial synthesis takes place in **Procedure EFDI** to solve the EFDIP, in **Procedure AFDI** to solve the AFDIP, and in **Procedure GENSPEC** to compute the maximally achievable fault specifications. The updating techniques employed in these procedures, can be conveniently described in terms of two rational matrices: $G(\lambda)$, the rational matrix for which a left nullspace basis has to be determined, and $F(\lambda)$, the rational matrix which has to be multiplied from left with the computed basis. The key property of the state-space realizations of $G(\lambda)$ and $F(\lambda)$, which is instrumental to perform the necessary updating is that they share the same state, descriptor, and output matrices. Note that, for the initial synthesis, the choices in (7.14) for $G(\lambda)$ and (7.16) for $F(\lambda)$, with the corresponding state-space realizations in (7.17), have been used.

The resulting realizations of $R_f(\lambda)$ (i.e., either $\overline{G}_f(\lambda)$ in (7.25) or $\widetilde{G}_f(\lambda)$ in (7.29)) allow to check various solvability conditions. The following result is the state-space version of Corollary 5.2 to characterize the solvability of the EFDP.

Corollary 7.1 *For the system (7.13) with $w \equiv 0$ the EFDP is solvable if and only if*

$$\begin{bmatrix} \overline{B}_{f_j} \\ \overline{D}_{f_j} \end{bmatrix} \neq 0, \quad j = 1, \ldots m_f \,, \tag{7.33}$$

where $\overline{B}_{f_j}$ and $\overline{D}_{f_j}$ are, respectively, the j-th columns of $\overline{B}_f$ and $\overline{D}_f$.

Proof Since the pair $(A_l - \lambda E_l, C_l)$ is observable, the fault input observability conditions $\overline{G}_{f_j}(\lambda) \neq 0$, for $j = 1, \ldots, m_f$, of Corollary 5.2 are equivalent to the conditions (7.33). ■

For the solvability of the EFDP, by imposing the strong detectability condition with respect to a set of frequencies Ω, we have the following state-space version of Corollary 5.3.

Corollary 7.2 *Let Ω be the set of frequencies which characterize the persistent fault signals and assume that the resulting descriptor realization in (7.25) is such that $\Lambda(A_l, E_l) \cap \Omega = \emptyset$. For the system (7.13) with $w \equiv 0$ the EFDP is solvable with the strong detectability condition with respect to Ω if and only if for all $\lambda_z \in \Omega$*

$$\operatorname{rank} \begin{bmatrix} A_l - \lambda_z E_l & \overline{B}_{f_j} \\ C_l & \overline{D}_{f_j} \end{bmatrix} > n_l, \quad j = 1, \ldots, m_f . \tag{7.34}$$

Proof The conditions (7.34) are equivalent with the strong detectability requirement of Corollary 5.3 for $\overline{G}_{f_j}(\lambda)$ to have no zero in Ω, for $j = 1, \ldots, m_f$. ■

These corollaries can be extended in a straightforward way to cover the fault isolability conditions for the solvability of EFDIP. For the solvability of the EFDIP with strong isolability condition we have the following state-space version of Corollary 5.6.

Corollary 7.3 *For the system (7.13) with $w \equiv 0$ the EFDIP with strong isolability is solvable if and only if*

$$\operatorname{rank} \begin{bmatrix} A_l - \lambda E_l & \overline{B}_f \\ C_l & \overline{D}_f \end{bmatrix} = n_l + m_f . \tag{7.35}$$

Proof The strong isolability condition for the solvability of the EFDIP is equivalent with the left invertibility condition (3.21) for the reduced model $\overline{G}_f(\lambda)$. However, this is equivalent with the normal rank condition (7.35). ■

This corollary serves also to check the solvability conditions for the AMMP.

Example 7.3 Consider the continuous-time system with the TFMs

$$G_u(s) = \begin{bmatrix} \dfrac{s+1}{s+2} \\[2ex] \dfrac{s+2}{s+3} \end{bmatrix}, \quad G_d(s) = \begin{bmatrix} \dfrac{s-1}{s+1} \\[2ex] 0 \end{bmatrix}, \quad G_w(s) = 0, \quad G_f(s) = \begin{bmatrix} \dfrac{s+1}{s-2} & 0 \\[2ex] \dfrac{s+2}{s-3} & 1 \end{bmatrix} .$$

The compound TFM $[\, G_u(s) \; G_d(s) \; G_f(s) \,]$ has the standard state-space realization with matrices $E = I_5$ and

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad \left[\begin{array}{c|c|c} B_u & B_d & B_f \end{array}\right] = \left[\begin{array}{c|c|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right],$$

$$C = \left[\begin{array}{ccccc} -1 & 0 & -1 & \frac{3}{2} & 0 \\ 0 & -1 & 0 & 0 & \frac{5}{2} \end{array}\right], \quad \left[\begin{array}{c|c|c} D_u & D_d & D_f \end{array}\right] = \left[\begin{array}{c|c|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right].$$

It is easy to observe that the pair $(A, [\, B_u \;\; B_d \,])$ is not controllable, and therefore the realization of $G(s)$ in (7.17) is not controllable as well.

Using the nullspace computation approach, we obtain for $N_l(s)$ and $\overline{G}_f(s)$ the matrices of the descriptor realizations

$$A_l - sE_l = \left[\begin{array}{cc} 1.5522 & -1.0552 \\ -2.0011 & -2.5848 \end{array}\right] - s\left[\begin{array}{cc} -0.8442 & -0.1362 \\ 0 & -0.9672 \end{array}\right],$$

$$\left[\begin{array}{c|c} B_l & \overline{B}_f \end{array}\right] = \left[\begin{array}{ccc|cc} 0 & 0.3666 & 0.3666 & -0.5133 & 0.3666 \\ 0 & -0.1796 & -0.1796 & -1.9757 & -0.1796 \end{array}\right],$$

$$C_l = \left[\begin{array}{cc} 0 & 1.904 \end{array}\right],$$

$$\left[\begin{array}{c|c} D_l & \overline{D}_f \end{array}\right] = \left[\begin{array}{ccc|cc} 0 & 0.7071 & -0.7071 & 0.7071 & 0.7071 \end{array}\right].$$

The generalized eigenvalues of the pair (A_l, E_l) are -2.5 and 3. The pair $(A_l - sE_l, B_l)$ is not controllable, and even not stabilizable, because the unstable eigenvalue 3 is not controllable. The corresponding TFMs, scaled with $\sqrt{2}$, are

$$\sqrt{2}N_l(s) = \left[\begin{array}{ccc} 0 & \dfrac{s+3}{s+2.5} & -\dfrac{s+2}{s+2.5} \end{array}\right], \quad \sqrt{2}\overline{G}_f(s) = \left[\begin{array}{cc} \dfrac{(s+3)(s+2)}{(s-3)(s+2.5)} & \dfrac{s+3}{s+2.5} \end{array}\right].$$

Since $\overline{G}_f(s)$ is unstable, we can choose the output injection matrix

$$K = \left[\begin{array}{c} 0.8503 \\ 3.0480 \end{array}\right],$$

such that the generalized eigenvalues of the pair $(A_l + KC_l, E_l)$ become -2.5 and -3. With $\widetilde{M}_l(s)$ defined in (7.30), we obtain the updated basis $\widetilde{N}_l(s) = \widetilde{M}_l(s)N_l(s)$ and the corresponding $\widetilde{G}_f(s) = \widetilde{M}_l(s)\overline{G}_f(s)$, whose realizations are given in (7.29). The resulting realization of $\widetilde{N}_l(s)$ is now controllable (due to output injection) and $Q(s) := \sqrt{2}\widetilde{N}_l(s)$ is a least McMillan degree solution, of order two, of the EFDP. The TFM $Q(s)$ of the implementation form and the corresponding TFM of the internal form $R_f(s)$ are

$$Q(s) = \left[\begin{array}{ccc} 0 & \dfrac{s-3}{s+2.5} & -\dfrac{(s+2)(s-3)}{(s+3)(s+2.5)} \end{array}\right], \quad R_f(s) = \left[\begin{array}{cc} \dfrac{s+2}{s+2.5} & \dfrac{s-3}{s+2.5} \end{array}\right].$$

The denominator factor $\widetilde{M}_l(s)$ employed for the stabilization of $\overline{G}_f(s)$ is

$$\widetilde{M}_l(s) = \frac{s-3}{s+3}.$$

Note that this factor needs not be computed explicitly.

It is worth to mention that the resulting filter $Q(s)$ is a proper nullspace basis of $G(s)$, but is not a minimal proper nullspace basis (it contains the non-constant factor $\widetilde{M}_l(s)$). Thus, this example illustrates the case when the peculiarities of the fault dynamics of $G_f(s)$ (e.g., unstable poles which are not poles of the underlying system) impose the use of a higher order fault detection filter than the order of a minimal nullspace basis.

The script **Ex7_3** in Listing 7.1 computes the results obtained in this example. ◊

Listing 7.1 Script **Ex7_3** to compute the results of Example 7.3

```
% Uses the Control Toolbox and the Descriptor System Tools

% define the state-space realizations of Gu, Gd and Gf
A = diag([ -2 -3 -1 2 3]);
Bu = [1 1 0 0 0]'; Bd = [0 0 2 0 0]';
Bf = [0 0 0 2 2;0 0 0 0 0]';
C = [-1 0 -1 1.5 0; 0 -1 0 0 2.5];
Du = [ 1 1]'; Dd = [1 0]'; Df = [1 0; 1 1];
p = 2; mu = 1; md = 1; mf = 2;        % enter dimensions
sys = ss(A,[Bu Bd Bf],C,[Du Dd Df]); % define system

% compute initial synthesis [Nl Gf_bar] in Q_Rf, where Nl is a
% left nullspace basis of [Gu Gd; I 0] and Gf_bar = Nl[Gf; 0];
Q_Rf = glnull([sys;eye(mu,mu+md+mf)],struct('m2',mf));

% compute a stable left coprime factorization [Nl~ Gf~] = Ml~[Nl Gf_bar]
% using explicitly computed output injection matrix K
[al,b,cl,d,el] = dssdata(Q_Rf);
k = gsfstab(al',el',cl',-3,-2).';      % assign one pole at -3
M = dss(al+k*cl,k,cl,1,el);            % Ml~
Q_Rf = dss(al+k*cl,b+k*d,cl,d,el);     % Ml~[Nl Gf_bar]

% alternative computation (comment out next line)
% [Q_Rf,M] = glcf(Q_Rf,struct('sdeg',-3,'smarg',-2));

% compute Q and Rf; scale to match example
Q = sqrt(2)*Q_Rf(:,1:p+mu); Rf = sqrt(2)*Q_Rf(:,p+mu+1:end);

% display results
minreal(zpk(Q)), minreal(zpk(Rf)), minreal(zpk(M))
```

Remark 7.4 We can easily extend the nullspace method to systems with parametric faults, described by equivalent linear models with additive faults of the form

$$\begin{aligned} E\lambda x(t) &= Ax(t) + B_u u(t) + B_d d(t) + B_f(t) f(t)\,, \\ y(t) &= Cx(t) + D_u u(t) + D_d d(t) + D_f(t) f(t)\,, \end{aligned} \tag{7.36}$$

where the fault input channel contains time-varying matrices with special structures (see (2.17) in Sect. 2.2.2). If we denote $\widetilde f_1(t) := B_f(t) f(t) \in \mathbb{R}^n$, $\widetilde f_2(t) := D_f(t) f(t) \in \mathbb{R}^p$, and $\widetilde f(t) = [\, \widetilde f_1^T(t)\;\; \widetilde f_2^T(t)\,]^T$, then the time-varying system (7.36) can be equivalently expressed in the form

$$\begin{aligned} E\lambda x(t) &= Ax(t) + B_u u(t) + B_d d(t) + [\, I_n\;\; 0\,]\widetilde f(t)\,, \\ y(t) &= Cx(t) + D_u u(t) + D_d d(t) + [\, 0\;\; I_p\,]\widetilde f(t)\,. \end{aligned} \tag{7.37}$$

For this LTI system, we can compute, similarly as in (7.24),

$$U\left[\begin{array}{cc|cc} 0 & 0 & I_n & 0 \\ I_p & 0 & 0 & I_p \\ 0 & I_{m_u} & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} * & * \\ \hline B_l & \overline{B}_f \\ \hline D_l & \overline{D}_f \end{array}\right], \tag{7.38}$$

from which we obtain the descriptor system realizations of the proper left nullspace basis $(A_l - \lambda E_l, B_l, C_l, D_l)$ and of the reduced proper system $(A_l - \lambda E_l, \overline{B}_f, C_l, \overline{D}_f)$, with outputs $\overline{y}(t)$, as defined in (5.11), and inputs $\widetilde{f}(t)$. According to Remark 7.2, we can arbitrarily assign the dynamics of both the nullspace basis as well as of the reduced system, using an additional LTI prefilter of the form (7.30). Recall that any stable nullspace basis can serve as a fault detection filter, with outputs $r(t)$ and inputs $y(t)$ and $u(t)$, provided the fault detectability conditions for the reduced system are fulfilled (jointly with the stability requirement). This comes down to check (e.g., via simulations or by exploiting the special structures of matrices $B_f(t)$ and $D_f(t)$, see Sect. 2.2.2), that the residual signal $r(t)$ is sensitive to each parametric fault $f_i(t)$, for $i = 1, \ldots, k$. It follows that the nullspace method allows to address fault detection problems for parametric faults in a simple way, involving only multiplications of (structured) time-varying matrices with constant matrices. □

7.5 Least-order Synthesis

The synthesis of fault detection filters of least McMillan degree underlies an important computational paradigm, typically employed at Step 2) of several of the presented synthesis procedures. This paradigm concerns with the updating of the proper left nullspace basis $Q(\lambda) = N_l(\lambda)$, computed at Step 1), by determining a factor $Q_2(\lambda)$ such that the product $Q_2(\lambda)N_l(\lambda)$ has the least possible McMillan degree under the constraint that certain *admissibility conditions* are simultaneously fulfilled. A basic admissibility condition is the (problem dependent) solvability condition, which must be always fulfilled by the updated reduced system with the TFMs $Q_2(\lambda)\overline{G}_w(\lambda)$ and $Q_2(\lambda)\overline{G}_f(\lambda)$. For example, for the solvability of the EFDP and AFDP, all columns of $Q_2(\lambda)\overline{G}_f(\lambda)$ must be nonzero (see Corollaries 5.2 and 5.4), while for the solvability of the EMMP and AMMP with enforced strong isolability, $Q_2(\lambda)\overline{G}_f(\lambda)$ must be left invertible (i.e., must have full column rank) (see Corollaries 5.10 and 5.11). Certain regularization conditions are additionally imposed, as—for example, $Q_2(\lambda)\overline{G}_w(\lambda)$ to have full row rank when solving the AFDP, $Q_2(\lambda)\overline{G}_f(\lambda)$ to be invertible when solving the EMMP with strong isolability, or $[\, Q_2(\lambda)\overline{G}_w(\lambda) \;\; Q_2(\lambda)\overline{G}_f(\lambda)\,]$ to have full row rank when solving the AMMP. These additional conditions are enforced to ease the solution of some synthesis problems, however, they are not necessary for the solvability of the respective problems.

We assume that the initial nullspace-based synthesis computed at Step 1) of all synthesis procedures is $[\, Q(\lambda) \;\; R_w(\lambda) \;\; R_f(\lambda)\,] := [\, N_l(\lambda) \;\; \overline{G}_w(\lambda) \;\; \overline{G}_f(\lambda)\,]$ in (7.31) and has the state-space realization (7.25), with D_l of full row rank. To simplify the presentation, we denote $W(\lambda) := [\, Q(\lambda) \;\; R_w(\lambda) \;\; R_f(\lambda)\,]$ the $(p - r_d) \times (p + m_u +$