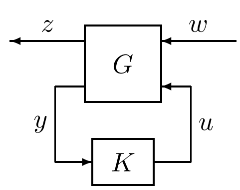
Chapter 14

$\mathcal{H}_\infty$ Control

In this chapter we consider $\mathcal{H}_\infty$ control theory. Specifically, we formulate the optimal and suboptimal $\mathcal{H}_\infty$ control problems in Section 14.1. However, we will focus on the suboptimal case in this book and discuss why we do so. In Section 14.2 a suboptimal controller is characterized together with an algebraic proof for a class of simplified problems while leaving the more general problems to a later section. The behavior of the $\mathcal{H}_\infty$ controller as a function of performance level γ is considered in Section 14.3. The optimal controllers are also briefly considered in this section. Some other interpretations of the $\mathcal{H}_\infty$ controllers are given in Section 14.4. Section 14.5 presents the formulas for an optimal $\mathcal{H}_\infty$ controller. Section 14.6 considers again the standard $\mathcal{H}_\infty$ control problem but with some assumptions in the previous sections relaxed. Since the proof techniques in Section 14.2 can, in principle, be applied to this general case except with some more involved algebra, the detailed proof for the general case will not be given; only the formulas are presented. We shall indicate how the assumptions in the general case can be relaxed further to accommodate other more complicated problems in Section 14.7. Section 14.8 considers the integral control in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theory and Section 14.9 considers how the general $\mathcal{H}_\infty$ solution can be used to solve the $\mathcal{H}_\infty$ filtering problem.

14.1 Problem Formulation

Consider the system described by the block diagram



where the plant G and controller K are assumed to be real rational and proper. It will be assumed that state-space models of G and K are available and that their realizations are assumed to be stabilizable and detectable. Recall again that a controller is said to be *admissible* if it internally stabilizes the system. Clearly, stability is the most basic requirement for a practical system to work. Hence any sensible controller has to be admissible.

> **Optimal $\mathcal{H}_\infty$ Control:** *Find all admissible controllers $K(s)$ such that $\|T_{zw}\|_\infty$ is minimized.*

It should be noted that the optimal $\mathcal{H}_\infty$ controllers as just defined are generally not unique for MIMO systems. Furthermore, finding an optimal $\mathcal{H}_\infty$ controller is often both numerically and theoretically complicated, as shown in Glover and Doyle [1989]. This is certainly in contrast with the standard $\mathcal{H}_2$ theory, in which the optimal controller is unique and can be obtained by solving two Riccati equations without iterations. Knowing the achievable optimal (minimum) $\mathcal{H}_\infty$ norm may be useful theoretically since it sets a limit on what we can achieve. However, in practice it is often not necessary and sometimes even undesirable to design an optimal controller, and it is usually much cheaper to obtain controllers that are very close in the norm sense to the optimal ones, which will be called *suboptimal controllers*. A suboptimal controller may also have other nice properties (e.g., lower bandwidth) over the optimal ones.

> **Suboptimal $\mathcal{H}_\infty$ Control:** *Given $\gamma > 0$, find all admissible controllers $K(s)$, if there are any, such that $\|T_{zw}\|_\infty < \gamma$.*

For the reasons mentioned above, we focus our attention in this book on suboptimal control. When appropriate, we briefly discuss what will happen when γ approaches the optimal value.

14.2 A Simplified $\mathcal{H}_\infty$ Control Problem

The realization of the transfer matrix G is taken to be of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

The following assumptions are made:

(i) (A, B_1) is controllable and (C_1, A) is observable;

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable;

(iii) $D_{12}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Two additional assumptions that are implicit in the assumed realization for $G(s)$ are that $D_{11} = 0$ and $D_{22} = 0$. As we mentioned in the last chapter, $D_{22} \neq 0$ does not pose any problem since it is easy to form an equivalent problem with $D_{22} = 0$ by a linear fractional transformation on the controller $K(s)$. However, relaxing the assumption $D_{11} = 0$ complicates the formulas substantially.

The $\mathcal{H}_\infty$ solution involves the following two Hamiltonian matrices:

$$H_\infty := \begin{bmatrix} A & \gamma^{-2}B_1B_1^* - B_2B_2^* \\ -C_1^*C_1 & -A^* \end{bmatrix}, \quad J_\infty := \begin{bmatrix} A^* & \gamma^{-2}C_1^*C_1 - C_2^*C_2 \\ -B_1B_1^* & -A \end{bmatrix}.$$

The important difference here from the $\mathcal{H}_2$ problem is that the (1,2)-blocks are not sign definite, so we cannot use Theorem 12.4 in Chapter 12 to guarantee that $H_\infty \in \mathrm{dom(Ric)}$ or $\mathrm{Ric}(H_\infty) \geq 0$. Indeed, these conditions are intimately related to the existence of $\mathcal{H}_\infty$ suboptimal controllers. Note that the (1,2)-blocks are a suggestive combination of expressions from the $\mathcal{H}_\infty$ norm characterization in Chapter 4 (or bounded real lemma in Chapter 12) and from the $\mathcal{H}_2$ synthesis of Chapter 13. It is also clear that if γ approaches infinity, then these two Hamiltonian matrices become the corresponding $\mathcal{H}_2$ control Hamiltonian matrices. The reasons for the form of these expressions should become clear through the discussions and proofs for the following theorem.

Theorem 14.1 *There exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ iff the following three conditions hold:*

(i) $H_\infty \in \mathrm{dom(Ric)}$ *and* $X_\infty := \mathrm{Ric}(H_\infty) > 0$*;*

(ii) $J_\infty \in \mathrm{dom(Ric)}$ *and* $Y_\infty := \mathrm{Ric}(J_\infty) > 0$*;*

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

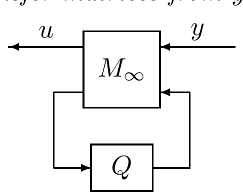
Moreover, when these conditions hold, one such controller is

$$K_{\mathrm{sub}}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty := A + \gamma^{-2}B_1B_1^*X_\infty + B_2F_\infty + Z_\infty L_\infty C_2$$
$$F_\infty := -B_2^*X_\infty, \quad L_\infty := -Y_\infty C_2^*, \quad Z_\infty := (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.$$

Furthermore, the set of all admissible controllers such that $\|T_{zw}\|_\infty < \gamma$ equals the set of all transfer matrices from y to u in



$$M_\infty(s) = \left[\begin{array}{c|cc} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$.

We shall only give a proof of the first part of the theorem; the proof for the all-controller parameterization needs much more work and is omitted (see Zhou, Doyle, and Glover [1996] for a comprehensive treatment of the related topics). We shall first show some preliminary results.

Lemma 14.2 *Suppose that $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, with $X = X^* > 0$, and $Y = Y^* > 0$. Let r be a positive integer. Then there exist matrices $X_{12} \in \mathbb{R}^{n\times r}$, $X_2 \in \mathbb{R}^{r\times r}$ such that $X_2 = X_2^*$*

$$\begin{bmatrix} X & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} > 0 \qquad \text{and} \qquad \begin{bmatrix} X & X_{12} \\ X_{12}^* & X_2 \end{bmatrix}^{-1} = \begin{bmatrix} Y & \star \\ \star & \star \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0 \qquad \text{and} \qquad \text{rank} \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + r.$$

Proof. ($\Leftarrow$) By the assumption, there is a matrix $X_{12} \in \mathbb{R}^{n\times r}$ such that $X - Y^{-1} = X_{12}X_{12}^*$. Defining $X_2 := I_r$ completes the construction.

($\Rightarrow$) Using Schur complements,

$$Y = X^{-1} + X^{-1}X_{12}(X_2 - X_{12}^*X^{-1}X_{12})^{-1}X_{12}^*X^{-1}.$$

Inverting, using the matrix inversion lemma, gives

$$Y^{-1} = X - X_{12}X_2^{-1}X_{12}^*.$$

Hence, $X - Y^{-1} = X_{12}X_2^{-1}X_{12}^* \geq 0$, and, indeed, $\text{rank}(X - Y^{-1}) = \text{rank}(X_{12}X_2^{-1}X_{12}^*) \leq r$. □

Lemma 14.3 *There exists an rth-order admissible controller such that $\|T_{zw}\|_\infty < \gamma$ only if the following three conditions hold:*

(i) There exists a $Y_1 > 0$ such that

$$AY_1 + Y_1A^* + Y_1C_1^*C_1Y_1/\gamma^2 + B_1B_1^* - \gamma^2 B_2B_2^* < 0. \tag{14.1}$$

(ii) There exists an $X_1 > 0$ such that

$$X_1A + A^*X_1 + X_1B_1B_1^*X_1/\gamma^2 + C_1^*C_1 - \gamma^2 C_2^*C_2 < 0. \tag{14.2}$$

(iii) $\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0 \qquad \text{rank} \begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \leq n + r.$

Proof. Suppose that there exists an rth-order controller $K(s)$ such that $\|T_{zw}\|_\infty < \gamma$. Let $K(s)$ have a state-space realization

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right].$$

Then

$$T_{zw} = \mathcal{F}_\ell(G, K) = \left[\begin{array}{cc|c} A + B_2\hat{D}C_2 & B_2\hat{C} & B_1 + B_2\hat{D}D_{21} \\ \hat{B}C_2 & \hat{A} & \hat{B}D_{21} \\ \hline C_1 + D_{12}\hat{D}C_2 & D_{12}\hat{C} & D_{12}\hat{D}D_{21} \end{array}\right] =: \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right].$$

Denote

$$R = \gamma^2 I - D_c^* D_c, \qquad \tilde{R} = \gamma^2 I - D_c D_c^*.$$

By Corollary 12.3, there exists an $\tilde{X} = \begin{bmatrix} X_1 & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} > 0$ such that

$$\tilde{X}(A_c + B_c R^{-1} D_c^* C_c) + (A_c + B_c R^{-1} D_c^* C_c)^* \tilde{X} + \tilde{X} B_c R^{-1} B_c^* \tilde{X} + C_c^* \tilde{R}^{-1} C_c < 0. \quad (14.3)$$

This gives, after much algebraic manipulation,

$$\begin{aligned} &X_1 A + A^* X_1 + X_1 B_1 B_1^* X_1/\gamma^2 + C_1^* C_1 - \gamma^2 C_2^* C_2 \\ &+ (X_1 B_1 \hat{D} + X_{12}\hat{B} + \gamma^2 C_2^*)(\gamma^2 I - \hat{D}^*\hat{D})^{-1}(X_1 B_1 \hat{D} + X_{12}\hat{B} + \gamma^2 C_2^*)^* < 0, \end{aligned}$$

which implies that

$$X_1 A + A^* X_1 + X_1 B_1 B_1^* X_1/\gamma^2 + C_1^* C_1 - \gamma^2 C_2^* C_2 < 0.$$

On the other hand, let

$$\tilde{Y} = \gamma^2 \tilde{X}^{-1}$$

and partition $\tilde{Y}$ as $\tilde{Y} = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}^* & Y_2 \end{bmatrix} > 0$. Then

$$(A_c + B_c R^{-1} D_c^* C_c)\tilde{Y} + \tilde{Y}(A_c + B_c R^{-1} D_c^* C_c)^* + \tilde{Y} C_c^* \tilde{R}^{-1} C_c \tilde{Y} + B_c R^{-1} B_c^* < 0. \quad (14.4)$$

This gives

$$\begin{aligned} &A Y_1 + Y_1 A^* + B_1 B_1^* - \gamma^2 B_2 B_2^* + Y_1 C_1^* C_1 Y_1/\gamma^2 \\ &+ (Y_1 C_1^* \hat{D}^* + Y_{12}\hat{C}^* + \gamma^2 B_2)(\gamma^2 I - \hat{D}\hat{D}^*)^{-1}(Y_1 C_1^* \hat{D}^* + Y_{12}\hat{C}^* + \gamma^2 B_2)^* < 0, \end{aligned}$$

which implies that

$$A Y_1 + Y_1 A^* + B_1 B_1^* - \gamma^2 B_2 B_2^* + Y_1 C_1^* C_1 Y_1/\gamma^2 < 0.$$

By Lemma 14.2, given $X_1 > 0$ and $Y_1 > 0$, there exists X_{12} and X_2 such that $\tilde{Y} = \gamma^2 \tilde{X}^{-1}$ or $\tilde{Y}/\gamma = (\tilde{X}/\gamma)^{-1}$:

$$\begin{bmatrix} X_1/\gamma & X_{12}/\gamma \\ X_{12}^*/\gamma & X_2/\gamma \end{bmatrix}^{-1} = \begin{bmatrix} Y_1/\gamma & \star \\ \star & \star \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0 \qquad \operatorname{rank} \begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \leq n + r.$$

□

To show that the inequalities in the preceding lemma imply the existence of the stabilizing solutions to the Riccati equations of X_∞ and Y_∞, we need the following theorem.

Theorem 14.4 *Let $R \geq 0$ and suppose (A, R) is controllable and there is an $X = X^*$ such that*

$$\mathcal{Q}(X) := XA + A^*X + XRX + Q < 0. \tag{14.5}$$

Then there exists a solution $X_+ > X$ to the Riccati equation

$$X_+A + A^*X_+ + X_+RX_+ + Q = 0 \tag{14.6}$$

such that $A + RX_+$ is antistable.

Proof. Let $R = BB^*$ for some B. Note the fact that (A, R) is controllable iff (A, B) is. Let X be such that $\mathcal{Q}(X) < 0$. Since (A, B) is controllable, there is an F_0 such that

$$A_0 := A - BF_0$$

is antistable. Now let $X_0 = X_0^*$ be the unique solution to the Lyapunov equation

$$X_0A_0 + A_0^*X_0 - F_0^*F_0 + Q = 0.$$

Define

$$\hat{F}_0 := F_0 + B^*X,$$

and we have the following equation:

$$(X_0 - X)A_0 + A_0^*(X_0 - X) = \hat{F}_0^*\hat{F}_0 - \mathcal{Q}(X) > 0.$$

The antistability of A_0 implies that

$$X_0 > X.$$

Starting with X_0, we shall define a nonincreasing sequence of Hermitian matrices $\{X_i\}$. Associated with $\{X_i\}$, we shall also define a sequence of antistable matrices $\{A_i\}$ and a sequence of matrices $\{F_i\}$. Assume inductively that we have already defined matrices $\{X_i\}$, $\{A_i\}$, and $\{F_i\}$ for i up to $n-1$ such that X_i is Hermitian and

$$\begin{gathered} X_0 \geq X_1 \geq \cdots \geq X_{n-1} > X, \\ A_i = A - BF_i \text{ is antistable}, \ i = 0, \ldots, n-1; \\ F_i = -B^* X_{i-1}, \ i = 1, \ldots, n-1; \\ X_i A_i + A_i^* X_i = F_i^* F_i - Q, \ i = 0, 1, \ldots, n-1. \end{gathered} \tag{14.7}$$

Next, introduce

$$\begin{aligned} F_n &= -B^* X_{n-1}, \\ A_n &= A - BF_n. \end{aligned}$$

First we show that A_n is antistable. Using equation (14.7), with $i = n-1$, we get

$$X_{n-1} A_n + A_n^* X_{n-1} + Q - F_n^* F_n - (F_n - F_{n-1})^* (F_n - F_{n-1}) = 0. \tag{14.8}$$

Let

$$\hat{F}_n := F_n + B^* X;$$

then

$$(X_{n-1} - X) A_n + A_n^* (X_{n-1} - X) = -\mathcal{Q}(X) + \hat{F}_n^* \hat{F}_n + (F_n - F_{n-1})^* (F_n - F_{n-1}) > 0, \tag{14.9}$$

which implies that A_n is antistable by Lyapunov theorem since $X_{n-1} - X > 0$.

Now we introduce X_n as the unique solution of the Lyapunov equation:

$$X_n A_n + A_n^* X_n = F_n^* F_n - Q. \tag{14.10}$$

Then X_n is Hermitian. Next, we have

$$(X_n - X) A_n + A_n^* (X_n - X) = -\mathcal{Q}(X) + \hat{F}_n^* \hat{F}_n > 0,$$

and, by using equation (14.8),

$$(X_{n-1} - X_n) A_n + A_n^* (X_{n-1} - X_n) = (F_n - F_{n-1})^* (F_n - F_{n-1}) \geq 0.$$

Since A_n is antistable, we have

$$X_{n-1} \geq X_n > X.$$

We have a nonincreasing sequence $\{X_i\}$, and the sequence is bounded below by $X_i > X$. Hence the limit

$$X_+ := \lim_{n \to \infty} X_n$$

exists and is Hermitian, and we have $X_+ \geq X$. Passing the limit $n \to \infty$ in equation (14.10), we get $\mathcal{Q}(X_+) = 0$. So X_+ is a solution of equation (14.6).

Note that $X_+ - X \geq 0$ and

$$(X_+ - X) A_+ + A_+^* (X_+ - X) = -\mathcal{Q}(X) + (X_+ - X) R (X_+ - X) > 0. \tag{14.11}$$

Hence, $X_+ - X > 0$ and $A_+ = A + RX_+$ is antistable. □

Lemma 14.5 *There exists an admissible controller such that* $\|T_{zw}\|_\infty < \gamma$ *only if the following three conditions hold:*

(i) There exists a stabilizing solution $X_\infty > 0$ *to*

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0. \tag{14.12}$$

(ii) There exists a stabilizing solution $Y_\infty > 0$ *to*

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0. \tag{14.13}$$

(iii) $\begin{bmatrix} \gamma Y_\infty^{-1} & I_n \\ I_n & \gamma X_\infty^{-1} \end{bmatrix} > 0 \quad \text{or} \quad \rho(X_\infty Y_\infty) < \gamma^2.$

Proof. Applying Theorem 14.4 to part (i) of Lemma 14.3, we conclude that there exists a $Y > Y_1 > 0$ such that

$$AY + YA^* + YC_1^* C_1 Y/\gamma^2 + B_1 B_1^* - \gamma^2 B_2 B_2^* = 0$$

and $A + C_1^* C_1 Y/\gamma^2$ is antistable. Let $X_\infty := \gamma^2 Y^{-1}$; we have

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0 \tag{14.14}$$

and

$$A + (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty = -X_\infty^{-1} (A + C_1^* C_1 X_\infty^{-1}) X_\infty = -X_\infty^{-1} (A + C_1^* C_1 Y/\gamma^2) X_\infty$$

is stable.

Similarly, applying Theorem 14.4 to part (ii) of Lemma 14.3, we conclude that there exists an $X > X_1 > 0$ such that

$$XA + A^* X + XB_1 B_1^* X/\gamma^2 + C_1^* C_1 - \gamma^2 C_2^* C_2 = 0$$

and $A + B_1 B_1^* X/\gamma^2$ is antistable. Let $Y_\infty := \gamma^2 X^{-1}$, we have

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0 \tag{14.15}$$

and $A + (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty$ is stable.

Finally, note that the rank condition in part (iii) of Lemma 14.3 is automatically satisfied by $r \geq n$, and

$$\begin{bmatrix} \gamma Y_\infty^{-1} & I_n \\ I_n & \gamma X_\infty^{-1} \end{bmatrix} = \begin{bmatrix} X/\gamma & I_n \\ I_n & Y/\gamma \end{bmatrix} > \begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0$$

or $\rho(X_\infty Y_\infty) < \gamma^2$. □

Proof of Theorem 14.1: To complete the proof, we only need to show that the controller K_{sub} given in Theorem 14.1 renders $\|T_{zw}\|_\infty < \gamma$. Note that the closed-loop transfer function with K_{sub} is given by

$$T_{zw} = \left[\begin{array}{cc|c} A & B_2F_\infty & B_1 \\ -Z_\infty L_\infty C_2 & \hat{A}_\infty & -Z_\infty L_\infty D_{21} \\ \hline C_1 & D_{12}F_\infty & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right].$$

Define

$$P = \begin{bmatrix} \gamma^2 Y_\infty^{-1} & -\gamma^2 Y_\infty^{-1} Z_\infty^{-1} \\ -\gamma^2 (Z_\infty^*)^{-1} Y_\infty^{-1} & \gamma^2 Y_\infty^{-1} Z_\infty^{-1} \end{bmatrix}.$$

Then it is easy to show that $P > 0$ and

$$PA_c + A_c^* P + PB_c B_c^* P/\gamma^2 + C_c^* C_c = 0.$$

Moreover,

$$A_c + B_c B_c^* P/\gamma^2 = \begin{bmatrix} A + B_1 B_1^* Y_\infty^{-1} & B_2 F_\infty - B_1 B_1^* Y_\infty^{-1} Z_\infty^{-1} \\ 0 & A + B_1 B_1^* X_\infty/\gamma^2 + B_2 F_\infty \end{bmatrix}$$

has no eigenvalues on the imaginary axis since $A + B_1 B_1^* X_\infty/\gamma^2 + B_2 F_\infty$ is stable and $A + B_1 B_1^* Y_\infty^{-1}$ is antistable. Thus, by Corollary 12.3, $\|T_{zw}\|_\infty < \gamma$. □

Remark 14.1 It is appropriate to point out that the conditions stated in Lemma 14.3 are, in fact, necessary and sufficient; see Gahinet and Apkarian [1994] and Gahinet [1996] for a linear matrix inequality (LMI) approach to the $\mathcal{H}_\infty$ problem. But the necessity should be suitably interpreted. For example, if one finds an $X_1 > 0$ and a $Y_1 > 0$ satisfying conditions (i) and (ii) but not condition (iii), this does not imply that there is no admissible $\mathcal{H}_\infty$ controller since there might be other $X_1 > 0$ and $Y_1 > 0$ that satisfy all three conditions. For example, consider $\gamma = 1$ and

$$G(s) = \left[\begin{array}{c|cc} -1 & \begin{bmatrix} 1 & 0 \end{bmatrix} & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

It is easy to check that $X_1 = Y_1 = 0.5$ satisfy (i) and (ii) but not (iii). Nevertheless, we shall show in the next section that $\gamma_{\text{opt}} = 0.7321$ and thus a suboptimal controller exists for $\gamma = 1$. In fact, we can check that $1 < X_1 < 2,\ 1 < Y_1 < 2$ also satisfy (i), (ii) and (iii). ◇

Example 14.1 Consider the feedback system shown in Figure 6.3 with

$$P = \frac{50(s+1.4)}{(s+1)(s+2)}, \quad W_e = \frac{2}{s+0.2}, \quad W_u = \frac{s+1}{s+10}.$$

We shall design a controller so that the $\mathcal{H}_\infty$ norm from $w = \begin{bmatrix} d \\ d_i \end{bmatrix}$ to $z = \begin{bmatrix} e \\ \tilde{u} \end{bmatrix}$ is minimized. Note that

$$\begin{bmatrix} e \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} W_e(I+PK)^{-1} & W_e(I+PK)^{-1}P \\ -W_uK(I+PK)^{-1} & -W_uK(I+PK)^{-1}P \end{bmatrix} \begin{bmatrix} d \\ d_i \end{bmatrix} =: T_{zw} \begin{bmatrix} d \\ d_i \end{bmatrix}.$$

Then the problem can be set up in an LFT framework with

$$G(s) = \left[\begin{array}{cc:c} W_e & W_eP & -W_eP \\ 0 & 0 & -W_u \\ \hdashline I & P & -P \end{array}\right] = \left[\begin{array}{cccc|cc:c} -0.2 & 2 & 2 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 20 & -20 \\ 0 & 0 & -2 & 0 & 0 & 30 & -30 \\ 0 & 0 & 0 & -10 & 0 & 0 & -3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & -1 \\ \hdashline 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right].$$

A suboptimal $\mathcal{H}_\infty$ controller can be computed by using the following command:

$$\gg [\mathbf{K}, \mathbf{T_{zw}}, \gamma_{\mathbf{subopt}}] = \mathbf{hinfsyn}(\mathbf{G}, \mathbf{n_y}, \mathbf{n_u}, \gamma_{\min}, \gamma_{\max}, \mathbf{tol})$$

where n_y and n_u are the dimensions of y and u; $\gamma_{\min}$ and $\gamma_{\max}$ are, respectively a lower bound and an upper bound for γ_{opt}; and tol is a tolerance to the optimal value. Set $n_y = 1, n_u = 1$, $\gamma_{\min} = 0$, $\gamma_{\max} = 10$, tol $= 0.0001$; we get $\gamma_{\text{subopt}} = 0.7849$ and a suboptimal controller

$$K = \frac{12.82(s/10+1)(s/7.27+1)(s/1.4+1)}{(s/32449447.67+1)(s/22.19+1)(s/1.4+1)(s/0.2+1)}.$$

If we set tol $= 0.01$, we would get $\gamma_{\text{subopt}} = 0.7875$ and a suboptimal controller

$$\tilde{K} = \frac{12.78(s/10+1)(s/7.27+1)(s/1.4+1)}{(s/2335.59+1)(s/21.97+1)(s/1.4+1)(s/0.2+1)}.$$

The only significant difference between K and $\tilde{K}$ is the exact location of the far-away stable controller pole. Figure 14.1 shows the closed-loop frequency response of $\overline{\sigma}(T_{zw})$ and Figure 14.2 shows the frequency responses of S, T, KS, and SP.

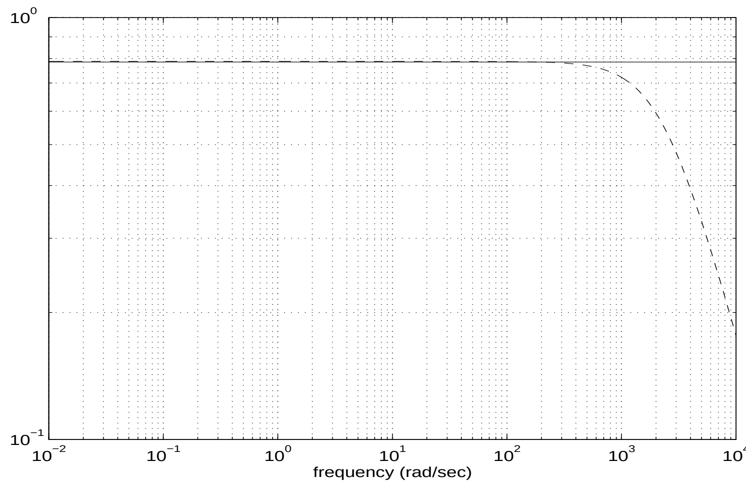


Figure 14.1: The closed-loop frequency responses of $\overline{\sigma}(T_{zw})$ with K (solid line) and $\tilde{K}$ (dashed line)

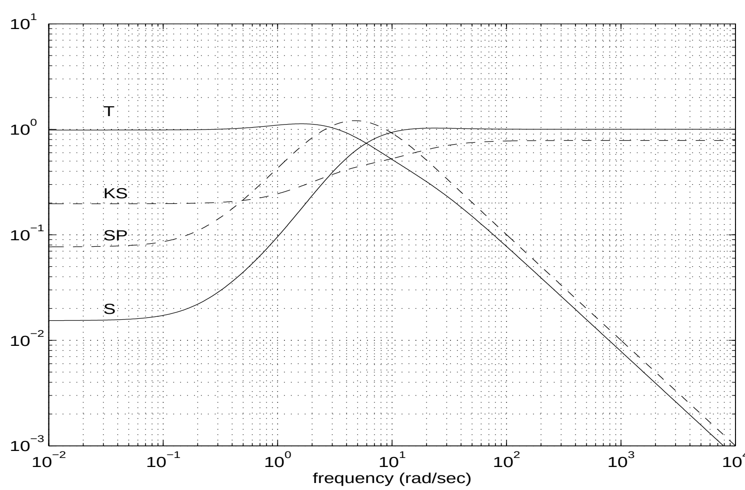


Figure 14.2: The frequency responses of S, T, KS, and SP with K

Example 14.2 Consider again the two-mass/spring/damper system shown in Figure 4.2. Assume that F_1 is the control force, F_2 is the disturbance force, and the measurements of x_1 and x_2 are corrupted by measurement noise:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + W_n \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \quad W_n = \begin{bmatrix} \dfrac{0.01(s+10)}{s+100} & 0 \\ 0 & \dfrac{0.01(s+10)}{s+100} \end{bmatrix}.$$

Our objective is to design a control law so that the effect of the disturbance force F_2 on the positions of the two masses, x_1 and x_2, are reduced in a frequency range $0 \le \omega \le 2$. The problem can be set up as shown in Figure 14.3, where $W_e = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$ is the performance weight and W_u is the control weight. In order to limit the control force, we shall choose

$$W_u = \frac{s+5}{s+50}.$$

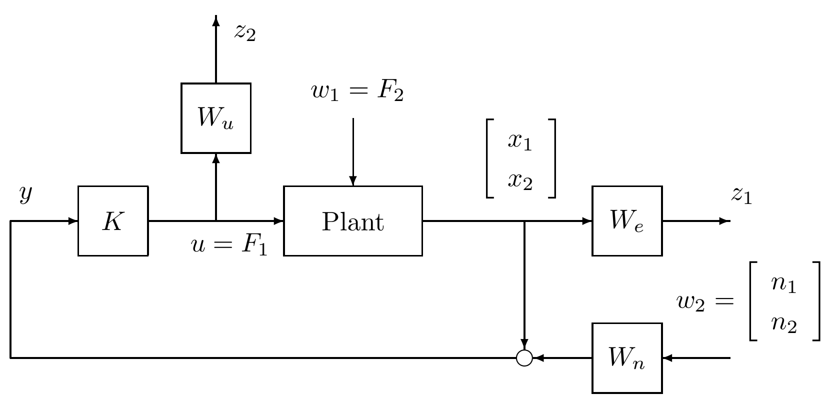


Figure 14.3: Rejecting the disturbance force F_2 by a feedback control

Now let $u = F_1$, $w = \begin{bmatrix} F_2 \\ n_1 \\ n_2 \end{bmatrix}$; then the problem can be formulated in an LFT form with

$$G(s) = \begin{bmatrix} \begin{bmatrix} W_e P_1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} W_e P_2 \\ W_u \end{bmatrix} \\ \begin{bmatrix} P_1 & W_n \end{bmatrix} & P_2 \end{bmatrix}$$

where P_1 and P_2 denote the transfer matrices from F_1 and F_2 to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, respectively. Let

$$W_1 = \frac{5}{s/2+1}, \quad W_2 = 0.$$

That is, we only want to reject the effect of the disturbance force F_2 on the position x_1. Then the optimal $\mathcal{H}_2$ performance is $\|\mathcal{F}_\ell(G, K_2)\|_2 = 2.6584$ and the $\mathcal{H}_\infty$ performance with the optimal $\mathcal{H}_2$ controller is $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 2.6079$ while the optimal $\mathcal{H}_\infty$ performance with an $\mathcal{H}_\infty$ controller is $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 1.6101$. This means that the effect of the disturbance force F_2 in the desired frequency rang $0 \le \omega \le 2$ will be effectively reduced with the $\mathcal{H}_\infty$ controller K_∞ by $5/1.6101 = 3.1054$ times at x_1. On the other hand, let

$$W_1 = 0, \quad W_2 = \frac{5}{s/2+1}.$$

That is, we only want to reject the effect of the disturbance force F_2 on the position x_2. Then the optimal $\mathcal{H}_2$ performance is $\|\mathcal{F}_\ell(G, K_2)\|_2 = 0.1659$ and the $\mathcal{H}_\infty$ performance with the optimal $\mathcal{H}_2$ controller is $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 0.5202$ while the optimal $\mathcal{H}_\infty$ performance with an $\mathcal{H}_\infty$ controller is $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 0.5189$. This means that the effect of the disturbance force F_2 in the desired frequency rang $0 \le \omega \le 2$ will be effectively reduced with the $\mathcal{H}_\infty$ controller K_∞ by $5/0.5189 = 9.6358$ times at x_2.

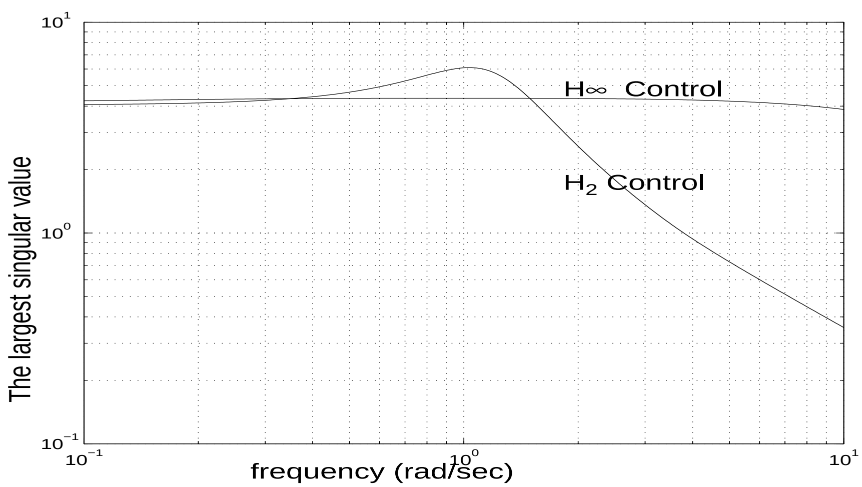


Figure 14.4: The largest singular value plot of the closed-loop system T_{zw} with an $\mathcal{H}_2$ controller and an $\mathcal{H}_\infty$ controller

Finally, set

$$W_1 = W_2 = \frac{5}{s/2+1}.$$

That is, we want to reject the effect of the disturbance force F_2 on both x_1 and x_2. Then the optimal $\mathcal{H}_2$ performance is $\|\mathcal{F}_\ell(G, K_2)\|_2 = 4.087$ and the $\mathcal{H}_\infty$ performance with the optimal $\mathcal{H}_2$ controller is $\|\mathcal{F}_\ell(G, K_2)\|_\infty = 6.0921$ while the optimal $\mathcal{H}_\infty$ performance with an $\mathcal{H}_\infty$ controller is $\|\mathcal{F}_\ell(G, K_\infty)\|_\infty = 4.3611$. This means that the effect of the disturbance force F_2 in the desired frequency rang $0 \le \omega \le 2$ will only be effectively reduced with the $\mathcal{H}_\infty$ controller K_∞ by $5/4.3611 = 1.1465$ times at both x_1 and x_2. This result shows clearly that it is very hard to reject the disturbance effect on both positions at the same time. The largest singular value Bode plots of the closed-loop system are shown in Figure 14.4. We note that the $\mathcal{H}_\infty$ controller typically gives a relatively flat frequency response since it tries to minimize the peak of the frequency response. On the other hand, the $\mathcal{H}_2$ controller would typically produce a frequency response that rolls off fast in the high-frequency range but with a large peak in the low-frequency range.

14.3 Optimality and Limiting Behavior

In this section, we will discuss, without proof, the behavior of the $\mathcal{H}_\infty$ suboptimal solution as γ varies, especially as γ approaches the infima achievable norm, denoted by γ_{opt}. Since Theorem 14.1 gives necessary and sufficient conditions for the existence of an admissible controller such that $\|T_{zw}\|_\infty < \gamma$, γ_{opt} is the infimum over all γ such that conditions (i)–(iii) are satisfied. Theorem 14.1 does not give an explicit formula for γ_{opt}, but, just as for the $\mathcal{H}_\infty$ norm calculation, it can be computed as closely as desired by a search technique.

Although we have not focused on the problem of $\mathcal{H}_\infty$ *optimal* controllers, the assumptions in this book make them relatively easy to obtain in most cases. In addition to describing the qualitative behavior of suboptimal solutions as γ varies, we will indicate why the descriptor version of the controller formulas below can usually provide formulas for the optimal controller when $\gamma = \gamma_{\mathrm{opt}}$.

As $\gamma \to \infty$, $H_\infty \to H_2$, $X_\infty \to X_2$, etc., and $K_{\mathrm{sub}} \to K_2$. This fact is the result of the particular choice of the suboptimal controller. While it could be argued that K_{sub} is a natural choice, this connection with $\mathcal{H}_2$ actually hints at deeper interpretations. In fact, K_{sub} is the minimum entropy solution (see Section 14.4) as well as the minimax controller for $\|z\|_2^2 - \gamma^2\|w\|_2^2$.

If $\gamma_2 \ge \gamma_1 > \gamma_{\mathrm{opt}}$, then $X_\infty(\gamma_1) \ge X_\infty(\gamma_2)$ and $Y_\infty(\gamma_1) \ge Y_\infty(\gamma_2)$. Thus X_∞ and Y_∞ are decreasing functions of γ, as is $\rho(X_\infty Y_\infty)$. At $\gamma = \gamma_{\mathrm{opt}}$, any one of the three conditions in Theorem 14.1 can fail. If only condition (iii) fails, then it is relatively straightforward to show that the descriptor formulas below for $\gamma = \gamma_{\mathrm{opt}}$ are optimal; that is, the optimal controller is given by

$$(I - \gamma_{\mathrm{opt}}^{-2} Y_\infty X_\infty)\dot{\hat{x}} = A_s \hat{x} - L_\infty y \tag{14.16}$$

$$u = F_\infty \hat{x} \tag{14.17}$$

where $A_s := A + B_2F_\infty + L_\infty C_2 + \gamma_{\rm opt}^{-2}Y_\infty A^*X_\infty + \gamma_{\rm opt}^{-2}B_1B_1^*X_\infty + \gamma_{\rm opt}^{-2}Y_\infty C_1^*C_1$. (See Example 14.3.)

The formulas in Theorem 14.1 are not well-defined in the optimal case because the term $(I - \gamma_{\rm opt}^{-2}X_\infty Y_\infty)$ is not invertible. It is possible but far less likely that conditions (i) or (ii) would fail before (iii). To see this, consider (i) and let γ_1 be the largest γ for which H_∞ fails to be in dom(Ric) because the H_∞ matrix fails to have either the stability property or the complementarity property. The same remarks will apply to (ii) by duality.

If complementarity fails at $\gamma = \gamma_1$, then $\rho(X_\infty) \to \infty$ as $\gamma \to \gamma_1$. For $\gamma < \gamma_1$, H_∞ may again be in dom(Ric), but X_∞ will be indefinite. For such γ, the controller $u = -B_2^*X_\infty x$ would make $\|T_{zw}\|_\infty < \gamma$ but would not be stabilizing. (See part 1 of Example 14.3.) If the stability property fails at $\gamma = \gamma_1$, then $H_\infty \notin$ dom(Ric) but Ric can be extended to obtain X_∞ so that a controller can be obtained to make $\|T_{zw}\|_\infty = \gamma_1$. The stability property will also not hold for any $\gamma \le \gamma_1$, and no controller whatsoever exists that makes $\|T_{zw}\|_\infty < \gamma_1$. In other words, if stability breaks down first, then the infimum over stabilizing controllers equals the infimum over all controllers, stabilizing or otherwise. (See part 2 of Example 14.3.) In view of this, we would typically expect that complementarity would fail first.

Complementarity failing at $\gamma = \gamma_1$ means $\rho(X_\infty) \to \infty$ as $\gamma \to \gamma_1$, so condition (iii) would fail at even larger values of γ, unless the eigenvectors associated with $\rho(X_\infty)$ as $\gamma \to \gamma_1$ are in the null space of Y_∞. Thus condition (iii) is the most likely of all to fail first. If condition (i) or (ii) fails first because the stability property fails, the formulas in Theorem 14.1 as well as their descriptor versions are optimal at $\gamma = \gamma_{\rm opt}$. This is illustrated in Example 14.3. If the complementarity condition fails first, [but (iii) does not fail], then obtaining formulas for the optimal controllers is a more subtle problem.

Example 14.3 Let an interconnected dynamical system realization be given by

$$G(s) = \left[\begin{array}{c|cc} a & \begin{bmatrix} 1 & 0 \end{bmatrix} & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

Then all assumptions for output feedback problem are satisfied and

$$H_\infty = \begin{bmatrix} a & \frac{1-\gamma^2}{\gamma^2} \\ -1 & -a \end{bmatrix}, \quad J_\infty = \begin{bmatrix} a & \frac{1-\gamma^2}{\gamma^2} \\ -1 & -a \end{bmatrix}.$$

The eigenvalues of H_∞ and J_∞ are given by

$$\left\{ \pm\frac{\sqrt{(a^2+1)\gamma^2 - 1}}{\gamma} \right\}.$$

If $\gamma > \dfrac{1}{\sqrt{a^2+1}}$, then $\mathcal{X}_-(H_\infty)$ and $\mathcal{X}_-(J_\infty)$ exist and

$$\mathcal{X}_-(H_\infty) = \operatorname{Im}\begin{bmatrix} \frac{\sqrt{(a^2+1)\gamma^2-1}-a\gamma}{\gamma} \\ 1 \end{bmatrix}$$

$$\mathcal{X}_-(J_\infty) = \operatorname{Im}\begin{bmatrix} \frac{\sqrt{(a^2+1)\gamma^2-1}-a\gamma}{\gamma} \\ 1 \end{bmatrix}.$$

We shall consider two cases:

1) $a > 0$: In this case, the complementary property of dom(Ric) will fail before the stability property fails since

$$\sqrt{(a^2+1)\gamma^2-1} - a\gamma = 0$$

when $\gamma = 1$.

Nevertheless, if $\gamma > \dfrac{1}{\sqrt{a^2+1}}$ and $\gamma \neq 1$, then $H_\infty \in \text{dom(Ric)}$ and

$$X_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2-1} - a\gamma} = \begin{cases} > 0; & \text{if } \gamma > 1 \\ < 0; & \text{if } \frac{1}{\sqrt{a^2+1}} < \gamma < 1. \end{cases}$$

Note that if $\gamma > 1$, then $H_\infty \in \text{dom(Ric)}$, $J_\infty \in \text{dom(Ric)}$, and

$$X_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2-1} - a\gamma} > 0$$

$$Y_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2-1} - a\gamma} > 0.$$

Hence conditions (i) and (ii) in Theorem 14.1 are satisfied, and we need to check condition (iii). Since

$$\rho(X_\infty Y_\infty) = \frac{\gamma^2}{(\sqrt{(a^2+1)\gamma^2-1} - a\gamma)^2},$$

it is clear that $\rho(X_\infty Y_\infty) \to \infty$ when $\gamma \to 1$. So condition (iii) will fail before condition (i) or (ii) fails.

2) $a < 0$: In this case, the complementary property is always satisfied, and, furthermore, $H_\infty \in \text{dom(Ric)}$, $J_\infty \in \text{dom(Ric)}$, and

$$X_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2-1} - a\gamma} > 0$$

$$Y_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2 - 1} - a\gamma} > 0$$

for $\gamma > \dfrac{1}{\sqrt{a^2+1}}$.

However, for $\gamma \le \dfrac{1}{\sqrt{a^2+1}}$, $H_\infty \notin \mathrm{dom(Ric)}$ since stability property fails. Nevertheless, in this case, if $\gamma_0 = \dfrac{1}{\sqrt{a^2+1}}$, we can extend the dom(Ric) to include those matrices H_∞ with imaginary axis eigenvalues as

$$\overline{\mathcal{X}_-}(H_\infty) = \mathrm{Im}\begin{bmatrix} -a \\ 1 \end{bmatrix}$$

such that $X_\infty = -\dfrac{1}{a}$ is a solution to the Riccati equation

$$A^* X_\infty + X_\infty A + C_1^* C_1 + \gamma_0^{-2} X_\infty B_1 B_1^* X_\infty - X_\infty B_2 B_2^* X_\infty = 0$$

and $A + \gamma_0^{-2} B_1 B_1^* X_\infty - B_2 B_2^* X_\infty = 0$. It can be shown that

$$\rho(X_\infty Y_\infty) = \frac{\gamma^2}{(\sqrt{(a^2+1)\gamma^2 - 1} - a\gamma)^2} < \gamma^2$$

is satisfied if and only if

$$\gamma > \sqrt{a^2+2} + a \quad \left(> \frac{1}{\sqrt{a^2+1}}\right).$$

So condition (iii) of Theorem 14.1 will fail before either (i) or (ii) fails.

In both $a > 0$ and $a < 0$ cases, the optimal γ for the output feedback is given by

$$\gamma_{\mathrm{opt}} = \sqrt{a^2+2} + a$$

and the optimal controller given by the descriptor formula in equations (14.16) and (14.17) is a constant. In fact,

$$u_{\mathrm{opt}} = -\frac{\gamma_{\mathrm{opt}}}{\sqrt{(a^2+1)\gamma_{\mathrm{opt}}^2 - 1} - a\gamma_{\mathrm{opt}}}\; y.$$

For instance, let $a = -1$ then $\gamma_{\mathrm{opt}} = \sqrt{3} - 1 = 0.7321$ and $u_{\mathrm{opt}} = -0.7321\, y$. Further,

$$T_{zw} = \left[\begin{array}{c|cc} -1.7321 & 1 & -0.7321 \\ \hline 1 & 0 & 0 \\ -0.7321 & 0 & -0.7321 \end{array}\right].$$

It is easy to check that $\|T_{zw}\|_\infty = 0.7321$.

14.4 Minimum Entropy Controller

Let T be a transfer matrix with $\|T\|_\infty < \gamma$. Then the entropy of $T(s)$ is defined by

$$I(T,\gamma) = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln\left|\det\left(I - \gamma^{-2}T^*(j\omega)T(j\omega)\right)\right| d\omega.$$

It is easy to see that

$$I(T,\gamma) = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \sum_i \ln\left|1 - \gamma^{-2}\sigma_i^2\left(T(j\omega)\right)\right| d\omega$$

and $I(T,\gamma) \geq 0$, where $\sigma_i\left(T(j\omega)\right)$ is the ith singular value of $T(j\omega)$. It is also easy to show that

$$\lim_{\gamma\to\infty} I(T,\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i \sigma_i^2\left(T(j\omega)\right) d\omega = \|T\|_2^2\,.$$

Thus the entropy $I(T,\gamma)$ is, in fact , a performance index measuring the tradeoff between the $\mathcal{H}_\infty$ optimality ($\gamma \to \|T\|_\infty$) and the $\mathcal{H}_2$ optimality ($\gamma \to \infty$).

It has been shown in Glover and Mustafa [1989] that the suboptimal controller given in Theorem 14.1 is actually the controller that satisfies the norm condition $\|T_{zw}\|_\infty < \gamma$ and minimizes the following entropy:

$$-\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln\left|\det\left(I - \gamma^{-2}T_{zw}^*(j\omega)T_{zw}(j\omega)\right)\right| d\omega.$$

Therefore, the given suboptimal controller is also called the minimum entropy controller [maximum entropy controller if the entropy is defined as $\tilde{I}(T,\gamma) = -I(T,\gamma)$].

Related MATLAB Commands: hinfsyne, hinffi

14.5 An Optimal Controller

To offer a general idea about the appearance of an optimal controller, we shall give in the following (without proof) the conditions under which an optimal controller exists and an explicit formula for an optimal controller.

Theorem 14.6 *There exists an admissible controller such that $\|T_{zw}\|_\infty \leq \gamma$ iff the following three conditions hold:*

(i) There exists a full column rank matrix

$$\begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} \in \mathbb{R}^{2n\times n}$$

such that

$$H_\infty \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} = \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} T_X, \quad \mathrm{Re}\ \lambda_i(T_X) \le 0\ \forall i$$

and

$$X_{\infty 1}^* X_{\infty 2} = X_{\infty 2}^* X_{\infty 1};$$

(ii) There exists a full column rank matrix

$$\begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} \in \mathbb{R}^{2n\times n}$$

such that

$$J_\infty \begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} = \begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} T_Y, \quad \mathrm{Re}\ \lambda_i(T_Y) \le 0\ \forall i$$

and

$$Y_{\infty 1}^* Y_{\infty 2} = Y_{\infty 2}^* Y_{\infty 1};$$

(iii)

$$\begin{bmatrix} X_{\infty 2}^* X_{\infty 1} & \gamma^{-1} X_{\infty 2}^* Y_{\infty 2} \\ \gamma^{-1} Y_{\infty 2}^* X_{\infty 2} & Y_{\infty 2}^* Y_{\infty 1} \end{bmatrix} \ge 0.$$

Moreover, when these conditions hold, one such controller is

$$K_{\mathrm{opt}}(s) := C_K (sE_K - A_K)^+ B_K$$

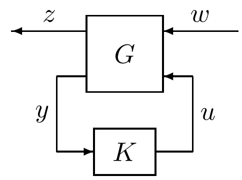
where

$$\begin{aligned} E_K &:= Y_{\infty 1}^* X_{\infty 1} - \gamma^{-2} Y_{\infty 2}^* X_{\infty 2} \\ B_K &:= Y_{\infty 2}^* C_2^* \\ C_K &:= -B_2^* X_{\infty 2} \\ A_K &:= E_K T_X - B_K C_2 X_{\infty 1} = T_Y^* E_K + Y_{\infty 1}^* B_2 C_K. \end{aligned}$$

Remark 14.2 It is simple to show that if $X_{\infty 1}$ and $Y_{\infty 1}$ are nonsingular and if $X_\infty = X_{\infty 2} X_{\infty 1}^{-1}$ and $Y_\infty = Y_{\infty 2} Y_{\infty 1}^{-1}$, then condition (iii) in the preceding theorem is equivalent to $X_\infty \ge 0$, $Y_\infty \ge 0$, and $\rho(Y_\infty X_\infty) \le \gamma^2$. So, in this case, the conditions for the existence of an optimal controller can be obtained from "taking the limit" of the corresponding conditions in Theorem 14.1. Moreover, the controller given above is reduced to the descriptor form given in equations (14.16) and (14.17). $\diamond$

14.6 General $\mathcal{H}_\infty$ Solutions

Consider the system described by the block diagram



where, as usual, G and K are assumed to be real rational and proper with K constrained to provide internal stability. The controller is said to be admissible if it is real rational, proper, and stabilizing. Although we are taking everything to be real, the results presented here are still true for the complex case with some obvious modifications. We will again only be interested in characterizing all suboptimal $\mathcal{H}_\infty$ controllers.

The realization of the transfer matrix G is taken to be of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

which is compatible with the dimensions of $z(t) \in \mathbb{R}^{p_1}$, $y(t) \in \mathbb{R}^{p_2}$, $w(t) \in \mathbb{R}^{m_1}$, $u(t) \in \mathbb{R}^{m_2}$, and the state $x(t) \in \mathbb{R}^n$. The following assumptions are made:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A2) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

Assumption (A1) is necessary for the existence of stabilizing controllers. The assumptions in (A2) mean that the penalty on $z = C_1x + D_{12}u$ includes a nonsingular, normalized penalty on the control u, that the exogenous signal w includes both plant disturbance and sensor noise, and that the sensor noise weighting is normalized and nonsingular. Relaxation of (A2) leads to singular control problems; see Stroorvogel [1992]. For those problems that have D_{12} full column rank and D_{21} full row rank but do not satisfy assumption (A2), a normalizing procedure is given in the next section so that an equivalent new system will satisfy this assumption.

Assumptions (A3) and (A4) are made for a technical reason: Together with (A1) they guarantee that the two Hamiltonian matrices in the corresponding $\mathcal{H}_2$ problem belong to dom(Ric), as we have seen in Chapter 13. Dropping (A3) and (A4) would make the solution very complicated. A further discussion of the assumptions and their possible relaxation will be provided in Section 14.7.

The main result is now stated in terms of the solutions of the X_∞ and Y_∞ Riccati equations together with the "state feedback" and "output injection" matrices F and L.

$$
\begin{aligned}
R &:= D_{1\bullet}^* D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{1\bullet} := [D_{11} \;\; D_{12}] \\
\tilde{R} &:= D_{\bullet 1} D_{\bullet 1}^* - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \\
H_\infty &:= \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^* C_1 & B^* \end{bmatrix} \\
J_\infty &:= \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C^* \\ -B_1 D_{\bullet 1}^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^* & C \end{bmatrix}
\end{aligned}
$$

$$
X_\infty := \mathrm{Ric}(H_\infty) \qquad Y_\infty := \mathrm{Ric}(J_\infty)
$$

$$
\begin{aligned}
F &:= \begin{bmatrix} F_{1\infty} \\ F_{2\infty} \end{bmatrix} := -R^{-1} \left[D_{1\bullet}^* C_1 + B^* X_\infty \right] \\
L &:= \begin{bmatrix} L_{1\infty} & L_{2\infty} \end{bmatrix} := -[B_1 D_{\bullet 1}^* + Y_\infty C^*] \tilde{R}^{-1}
\end{aligned}
$$

Partition D, $F_{1\infty}$, and $L_{1\infty}$ are as follows:

$$
\left[\begin{array}{c|c} & F' \\ \hline L' & D \end{array} \right] = \left[\begin{array}{c|ccc} & F_{11\infty}^* & F_{12\infty}^* & F_{2\infty}^* \\ \hline L_{11\infty}^* & D_{1111} & D_{1112} & 0 \\ L_{12\infty}^* & D_{1121} & D_{1122} & I \\ L_{2\infty}^* & 0 & I & 0 \end{array} \right].
$$

Remark 14.3 In the above matrix partitioning, some matrices may not exist depending on whether D_{12} or D_{21} is square. This issue will be discussed further later. For the time being, we shall assume that all matrices in the partition exist. $\diamond$

Theorem 14.7 *Suppose G satisfies the assumptions (A1)–(A4).*

(a) There exists an admissible controller $K(s)$ such that $||\mathcal{F}_\ell(G,K)||_\infty < \gamma$ (i.e., $\|T_{zw}\|_\infty < \gamma$) if and only if

(i) $\gamma > \max(\overline{\sigma}[D_{1111}, D_{1112},], \overline{\sigma}[D_{1111}^*, D_{1121}^*])$;

(ii) $H_\infty \in \mathrm{dom(Ric)}$ *with* $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$;

(iii) $J_\infty \in \mathrm{dom(Ric)}$ *with* $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$;

(iv) $\rho(X_\infty Y_\infty) < \gamma^2$.

(b) Given that the conditions of part (a) are satisfied, then all rational internally stabilizing controllers $K(s)$ *satisfying* $||\mathcal{F}_\ell(G, K)||_\infty < \gamma$ *are given by*

$$K = \mathcal{F}_\ell(M_\infty, Q) \qquad \textit{for arbitrary } Q \in \mathcal{RH}_\infty \qquad \textit{such that} \quad \|Q\|_\infty < \gamma$$

where

$$M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]$$

$$\hat{D}_{11} = -D_{1121}D_{1111}^*(\gamma^2 I - D_{1111}D_{1111}^*)^{-1}D_{1112} - D_{1122}$$

$\hat{D}_{12} \in \mathbb{R}^{m_2 \times m_2}$ *and* $\hat{D}_{21} \in \mathbb{R}^{p_2 \times p_2}$ *are any matrices (e.g., Cholesky factors) satisfying*

$$\begin{aligned} \hat{D}_{12}\hat{D}_{12}^* &= I - D_{1121}(\gamma^2 I - D_{1111}^* D_{1111})^{-1} D_{1121}^*, \\ \hat{D}_{21}^*\hat{D}_{21} &= I - D_{1112}^*(\gamma^2 I - D_{1111}D_{1111}^*)^{-1}D_{1112}, \end{aligned}$$

and

$$\begin{aligned} \hat{B}_2 &= Z_\infty(B_2 + L_{12\infty})\hat{D}_{12}, \\ \hat{C}_2 &= -\hat{D}_{21}(C_2 + F_{12\infty}), \\ \hat{B}_1 &= -Z_\infty L_{2\infty} + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11}, \\ \hat{C}_1 &= F_{2\infty} + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2, \\ \hat{A} &= A + BF + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 \end{aligned}$$

where

$$Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.$$

(Note that if $D_{11} = 0$ then the formulas are considerably simplified.)

Some Special Cases:

Case 1: $D_{12} = I$

In this case

1. In part (a), (i) becomes $\gamma > \overline{\sigma}(D_{1121})$.

2. In part (b)

$$\begin{aligned}\hat{D}_{11} &= -D_{1122}\\ \hat{D}_{12}\hat{D}_{12}^* &= I - \gamma^{-2}D_{1121}D_{1121}^*\\ \hat{D}_{21}^*\hat{D}_{21} &= I.\end{aligned}$$

Case 2: $D_{21} = I$

In this case

1. In part (a), (i) becomes $\gamma > \overline{\sigma}(D_{1112})$.
2. In part (b)

$$\begin{aligned}\hat{D}_{11} &= -D_{1122}\\ \hat{D}_{12}\hat{D}_{12}^* &= I\\ \hat{D}_{21}^*\hat{D}_{21} &= I - \gamma^{-2}D_{1112}^*D_{1112}.\end{aligned}$$

Case 3: $D_{12} = I$ & $D_{21} = I$

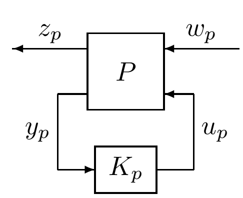
In this case

1. In part (a), (i) drops out.
2. In part (b)

$$\begin{aligned}\hat{D}_{11} &= -D_{1122}\\ \hat{D}_{12}\hat{D}_{12}^* &= I\\ \hat{D}_{21}^*\hat{D}_{21} &= I.\end{aligned}$$

14.7 Relaxing Assumptions

In this section, we indicate how the results of Section 14.6 can be extended to more general cases. Let a given problem have the following diagram, where $z_p(t) \in \mathbb{R}^{p_1}$, $y_p(t) \in \mathbb{R}^{p_2}$, $w_p(t) \in \mathbb{R}^{m_1}$, and $u_p(t) \in \mathbb{R}^{m_2}$:



The plant P has the following state-space realization with D_{p12} *full column rank and* D_{p21} *full row rank:*

$$P(s) = \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{array}\right].$$

The objective is to find all rational proper controllers $K_p(s)$ that stabilize P and $||\mathcal{F}_\ell(P, K_p)||_\infty < \gamma$. To solve this problem, we first transform it to the standard one treated in the last section. *Note that the following procedure can also be applied to the* $\mathcal{H}_2$ *problem (except the procedure for the case* $D_{11} \neq 0$*).*

Normalize D_{12} and D_{21}

Perform singular value decompositions to obtain the matrix factorizations

$$D_{p12} = U_p \begin{bmatrix} 0 \\ I \end{bmatrix} R_p, \quad D_{p21} = \tilde{R}_p \begin{bmatrix} 0 & I \end{bmatrix} \tilde{U}_p$$

such that U_p and $\tilde{U}_p$ are square and unitary and R_p and $\tilde{R}_p$ are square and invertible. Now let

$$z_p = U_p z, \quad w_p = \tilde{U}_p^* w, \quad y_p = \tilde{R}_p y, \quad u_p = R_p^{-1} u$$

and

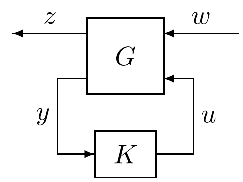
$$K(s) = R_p K_p(s) \tilde{R}_p$$

$$\begin{aligned} G(s) &= \begin{bmatrix} U_p^* & 0 \\ 0 & \tilde{R}_p^{-1} \end{bmatrix} P(s) \begin{bmatrix} \tilde{U}_p^* & 0 \\ 0 & R_p^{-1} \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A_p & B_{p1}\tilde{U}_p^* & B_{p2}R_p^{-1} \\ \hline U_p^* C_{p1} & U_p^* D_{p11} \tilde{U}_p^* & U_p^* D_{p12} R_p^{-1} \\ \tilde{R}_p^{-1} C_{p2} & \tilde{R}_p^{-1} D_{p21} \tilde{U}_p^* & \tilde{R}_p^{-1} D_{p22} R_p^{-1} \end{array}\right] \\ &=: \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \end{aligned}$$

Then

$$D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad D_{21} = \begin{bmatrix} 0 & I \end{bmatrix},$$

and the new system is as follows:



Furthermore, $||\mathcal{F}_\ell(P, K_p)||_\alpha = ||U_p \mathcal{F}_\ell(G, K)\tilde{U}_p||_\alpha = \|\mathcal{F}_\ell(G, K)\|_\alpha$ for $\alpha = 2$ or ∞ since U_p and $\tilde{U}_p$ are unitary. Moreover, Assumptions (A1), (A3), and (A4) remain unaffected.

Remove the Assumption $D_{22} = 0$

Suppose $K(s)$ is a controller for G with D_{22} set to zero. Then the controller for $D_{22} \neq 0$ is $K(I + D_{22}K)^{-1}$. Hence there is no loss of generality in assuming $D_{22} = 0$.

Relax (A3) and (A4)

Suppose that

$$G = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

which violates both (A3) and (A4) and corresponds to the robust stabilization of an integrator. If the controller $u = -\epsilon x$, where $\epsilon > 0$ is used, then

$$T_{zw} = \frac{-\epsilon s}{s + \epsilon}, \quad \text{with } \|T_{zw}\|_\infty = \epsilon.$$

Hence the norm can be made arbitrarily small as $\epsilon \to 0$, but $\epsilon = 0$ is not admissible since it is not stabilizing. This may be thought of as a case where the $\mathcal{H}_\infty$ optimum is not achieved on the set of admissible controllers. Of course, for this system, $\mathcal{H}_\infty$ *optimal* control is a silly problem, although the suboptimal case is not obviously so.

Relax (A1)

If assumption (A1) is violated, then it is obvious that no admissible controllers exist. Suppose (A1) is relaxed to allow unstabilizable and/or undetectable modes on the $j\omega$ axis and internal stability is also relaxed to also allow closed-loop $j\omega$ axis poles, but (A2)–(A4) is still satisfied. It can be easily shown that under these conditions the closed-loop $\mathcal{H}_\infty$ norm cannot be made finite and, in particular, that the unstabilizable and/or undetectable modes on the $j\omega$ axis must show up as poles in the closed-loop system.

Violate (A1) and either or both (A3) and (A4)

Sensible control problems can be posed that violate (A1) *and* either or both (A3) and (A4). For example, cases when A has modes at $s = 0$ that are unstabilizable through B_2 and/or undetectable through C_2 arise when an integrator is included in a weight on a disturbance input or an error term. In these cases, either (A3) or (A4) are also violated, or the closed-loop $\mathcal{H}_\infty$ norm cannot be made finite. In many applications, such problems can be reformulated so that the integrator occurs inside the loop (essentially using the internal model principle) and is hence detectable and stabilizable. We will show this process in the next section.

An alternative is to introduce ϵ perturbations so that (A1), (A3), and (A4) are satisfied. Roughly speaking, this would produce sensible answers for sensible problems, but the behavior as $\epsilon \to 0$ could be problematic.

Relax (A2)

In the cases that either D_{12} is not full column rank or that D_{21} is not full row rank, improper controllers can give a bounded $\mathcal{H}_\infty$ norm for T_{zw}, although the controllers will not be admissible by our definition. Such singular filtering and control problems have been well-studied in $\mathcal{H}_2$ theory and many of the same techniques go over to the $\mathcal{H}_\infty$ case (e.g., Willems [1981] and Willems, Kitapci, and Silverman [1986]). A complete solution to the singular problem can be found in Stroorvogel [1992].

14.8 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Integral Control

It is interesting to note that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design frameworks do not, in general, produce integral control. In this section we show how to introduce integral control into the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design framework through a simple disturbance rejection problem. We consider a feedback system shown in Figure 14.5. We shall assume that the frequency contents of the disturbance w are effectively modeled by the weighting $W_d \in \mathcal{RH}_\infty$ and the constraints on control signal are limited by an appropriate choice of $W_u \in \mathcal{RH}_\infty$. In order to let the output y track the reference signal r, we require K to contain an integrator [i.e., $K(s)$ has a pole at $s = 0$]. (In general, K is required to have poles on the imaginary axis.)

There are several ways to achieve the integral design. One approach is to introduce an integral in the performance weight W_e. Then the transfer function between w and z_1 is given by

$$z_1 = W_e(I + PK)^{-1}W_d w.$$

Now if the resulting controller K stabilizes the plant P and makes the norm (2-norm or ∞-norm) between w and z_1 finite, then K must have a pole at $s = 0$ that is the zero of the sensitivity function (assuming W_d has no zeros at $s = 0$). (This follows from the well-known internal model principle.) The problem with this approach is that the $\mathcal{H}_\infty$ (or $\mathcal{H}_2$) control theory presented in this chapter and in the previous chapters cannot be

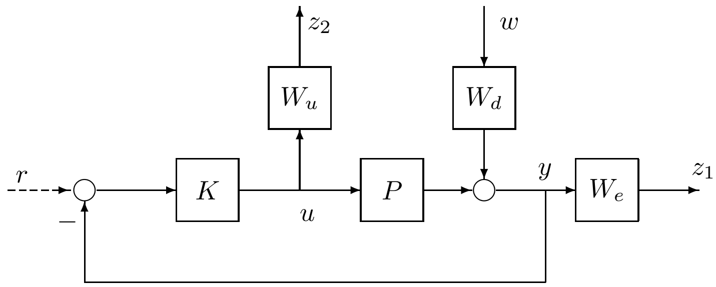


Figure 14.5: A simple disturbance rejection problem

applied to this problem formulation directly because the pole $s = 0$ of W_e becomes an uncontrollable pole of the feedback system and the very first assumption for the $\mathcal{H}_\infty$ (or $\mathcal{H}_2$) theory is violated.

However, the obstacles can be overcome by appropriately reformulating the problem. Suppose W_e can be factorized as follows:

$$W_e = \tilde{W}_e(s)M(s)$$

where $M(s)$ is proper, containing all the imaginary axis poles of W_e, and $M^{-1}(s) \in \mathcal{RH}_\infty$, $\tilde{W}_e(s)$ is stable and minimum phase. Now suppose there exists a controller $K(s)$ that contains the same imaginary axis poles that achieves the performance specifications. Then, without loss of generality, K can be factorized as

$$K(s) = -\hat{K}(s)M(s)$$

such that there is no unstable pole/zero cancellation in forming the product $\hat{K}(s)M(s)$. Now the problem can be reformulated as in Figure 14.6. Figure 14.6 can be put in the general LFT framework as in Figure 14.7 with

$$G(s) = \begin{bmatrix} \begin{bmatrix} \tilde{W}_e M W_d \\ 0 \end{bmatrix} & \begin{bmatrix} \tilde{W}_e M P \\ W_u \end{bmatrix} \\ M W_d & M P \end{bmatrix}.$$

We shall illustrate this design through a simple numerical example. Let

$$P = \frac{s-2}{(s+1)(s-3)} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 3 & 2 & 1 \\ \hline -2 & 1 & 0 \end{array}\right], \quad W_d = 1,$$

$$W_u = \frac{s+10}{s+100} = \left[\begin{array}{c|c} -100 & -90 \\ \hline 1 & 1 \end{array}\right], \quad W_e = \frac{1}{s}.$$

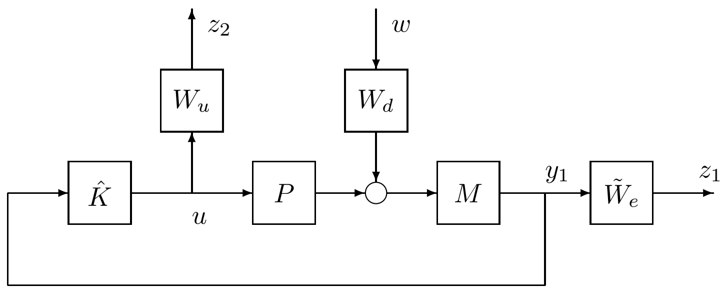


Figure 14.6: Disturbance rejection with imaginary axis poles

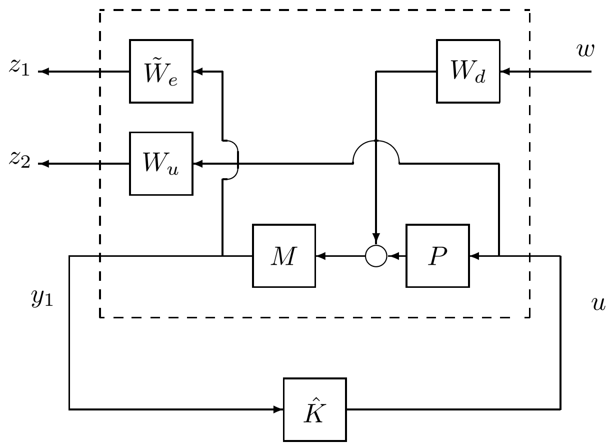


Figure 14.7: LFT framework for the disturbance rejection problem

Then we can choose without loss of generality that

$$M = \frac{s+\alpha}{s}, \quad \tilde{W}_e = \frac{1}{s+\alpha}, \quad \alpha > 0.$$

This gives the following generalized system:

$$G(s) = \left[\begin{array}{ccccc|c:c} -\alpha & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & -100 & 0 & 0 & 0 & 0 & -90 \\ 0 & 0 & 0 & -2\alpha & \alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 1 & -2 & 1 & 1 & 0 \end{array}\right].$$

The suboptimal $\mathcal{H}_\infty$ controller $\hat{K}_\infty$ for each α can be computed easily as

$$\hat{K}_\infty = \frac{-2060381.4(s+1)(s+\alpha)(s+100)(s-0.1557)}{(s+\alpha)^2(s+32.17)(s+262343)(s-19.89)},$$

which gives the closed-loop $\mathcal{H}_\infty$ norm 7.854. Hence the controller $K_\infty = -\hat{K}_\infty(s)M(s)$ is given by

$$K_\infty(s) = \frac{2060381.4(s+1)(s+100)(s-0.1557)}{s(s+32.17)(s+262343)(s-19.89)} \approx \frac{7.85(s+1)(s+100)(s-0.1557)}{s(s+32.17)(s-19.89)},$$

which is independent of α as expected. Similarly, we can solve an optimal $\mathcal{H}_2$ controller

$$\hat{K}_2 = \frac{-43.487(s+1)(s+\alpha)(s+100)(s-0.069)}{(s+\alpha)^2(s^2+30.94s+411.81)(s-7.964)}$$

and

$$K_2(s) = -\hat{K}_2(s)M(s) = \frac{43.487(s+1)(s+100)(s-0.069)}{s(s^2+30.94s+411.81)(s-7.964)}.$$

An approximate integral control can also be achieved without going through the preceding process by letting

$$W_e = \tilde{W}_e = \frac{1}{s+\epsilon}, \quad M(s) = 1$$

for a sufficiently small $\epsilon > 0$. For example, a controller for $\epsilon = 0.001$ is given by

$$K_\infty = \frac{316880(s+1)(s+100)(s-0.1545)}{(s+0.001)(s+32)(s+40370)(s-20)} \approx \frac{7.85(s+1)(s+100)(s-0.1545)}{s(s+32)(s-20)},$$

which gives the closed-loop $\mathcal{H}_\infty$ norm of 7.85. Similarly, an approximate $\mathcal{H}_2$ integral controller is obtained as

$$K_2 = \frac{43.47(s+1)(s+100)(s-0.0679)}{(s+0.001)(s^2+30.93s+411.7)(s-7.972)} \approx \frac{43.47(s+1)(s+100)(s-0.0679)}{s(s^2+30.93s+411.7)(s-7.972)}.$$

14.9 $\mathcal{H}_\infty$ Filtering

In this section we show how the filtering problem can be solved using the $\mathcal{H}_\infty$ theory developed earlier. Suppose a dynamic system is described by the following equations:

$$\dot{x} = Ax + B_1 w(t), \quad x(0) = 0 \tag{14.18}$$
$$y = C_2 x + D_{21} w(t) \tag{14.19}$$
$$z = C_1 x + D_{11} w(t) \tag{14.20}$$

The filtering problem is to find an estimate $\hat{z}$ of z in some sense using the measurement of y. The restriction on the filtering problem is that the filter has to be causal so that it can be realized (i.e., $\hat{z}$ has to be generated by a causal system acting on the measurements). We will further restrict our filter to be *unbiased*; that is, given $T > 0$ the estimate $\hat{z}(t) = 0 \ \forall t \in [0, T]$ if $y(t) = 0, \ \forall t \in [0, T]$. Now we can state our $\mathcal{H}_\infty$ filtering problem.

> $\mathcal{H}_\infty$ **Filtering**: Given a $\gamma > 0$, find a causal filter $F(s) \in \mathcal{RH}_\infty$ if it exists such that
> $$J := \sup_{w \in \mathcal{L}_2[0,\infty)} \frac{\|z - \hat{z}\|_2^2}{\|w\|_2^2} < \gamma^2$$
> with $\hat{z} = F(s)y$.

A diagram for the filtering problem is shown in Figure 14.8.

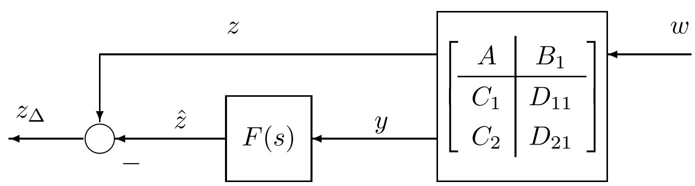
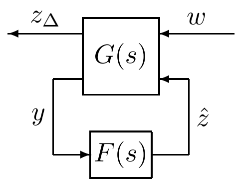


Figure 14.8: Filtering problem formulation

The preceding filtering problem can also be formulated in an LFT framework: Given a system shown below



$$G(s) = \left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & -I \\ C_2 & D_{21} & 0 \end{array}\right]$$

find a filter $F(s) \in \mathcal{RH}_\infty$ such that

$$\sup_{w \in \mathcal{L}_2} \frac{\|z_\Delta\|_2^2}{\|w\|_2^2} < \gamma^2. \tag{14.21}$$

Hence the filtering problem can be regarded as a special $\mathcal{H}_\infty$ problem. However, compared with control problems, there is no internal stability requirement in the filtering problem. Hence the solution to the above filtering problem can be obtained from the $\mathcal{H}_\infty$ solution in the previous sections by setting $B_2 = 0$ and dropping the internal stability requirement.

Theorem 14.8 *Suppose (C_2, A) is detectable and*

$$\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

has full row rank for all ω. Let D_{21} be normalized and D_{11} partitioned conformably as

$$\begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} = \left[\begin{array}{cc} D_{111} & D_{112} \\ \hdashline 0 & I \end{array}\right].$$

Then there exists a causal $F(s) \in \mathcal{RH}_\infty$ such that $J < \gamma^2$ if and only if $\overline{\sigma}(D_{111}) < \gamma$ and $J_\infty \in \mathrm{dom(Ric)}$ with $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$, where

$$\begin{aligned} \tilde{R} &:= \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}^* - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix} \\ J_\infty &:= \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C_1^* & C_2^* \\ -B_1 D_{11}^* & -B_1 D_{21}^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{11} B_1^* & C_1 \\ D_{21} B_1^* & C_2 \end{bmatrix}. \end{aligned}$$

Moreover, if the above conditions are satisfied, then a rational causal filter $F(s)$ satisfying $J < \gamma^2$ is given by

$$\hat{z} = F(s) y = \left[\begin{array}{c|c} A + L_{2\infty} C_2 + L_{1\infty} D_{112} C_2 & -L_{2\infty} - L_{1\infty} D_{112} \\ \hline C_1 - D_{112} C_2 & D_{112} \end{array}\right] y$$

where

$$\begin{bmatrix} L_{1\infty} & L_{2\infty} \end{bmatrix} := -\begin{bmatrix} B_1 D_{11}^* + Y_\infty C_1^* & B_1 D_{21}^* + Y_\infty C_2^* \end{bmatrix} \tilde{R}^{-1}.$$

In the case where $D_{11} = 0$ and $B_1 D_{21}^* = 0$ the filter becomes much simpler:

$$\hat{z} = \left[\begin{array}{c|c} A - Y_\infty C_2^* C_2 & Y_\infty C_2^* \\ \hline C_1 & 0 \end{array}\right] y$$

where Y_∞ is the stabilizing solution to

$$Y_\infty A^* + A Y_\infty + Y_\infty (\gamma^{-2} C_1^* C_1 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0.$$

14.10 Notes and References

The first part of this chapter is based on Sampei, Mita, and Nakamichi [1990], Packard [1994], Doyle, Glover, Khargonekar, and Francis [1989], and Chen and Zhou [1996]. The proof of Theorem 14.4 comes from Ran and Vreugdenhil [1988]. The detailed derivation of the $\mathcal{H}_\infty$ solution for the general case is treated in Glover and Doyle [1988, 1989]. A fairly complete solution to the singular $\mathcal{H}_\infty$ problem is obtained in Stoorvogel [1992]. The $\mathcal{H}_\infty$ filtering and smoothing problems are considered in detail in Nagpal and Khargonekar [1991]. There is a rich body of literature on the LMI approach to $\mathcal{H}_\infty$ control and related problems. In particular, readers are referred to the monograph by Boyd, Ghaoui, Feron, and Balakrishnan [1994] for a comprehensive treatment of LMIs and their applications in control. See also the paper by Chilali and Gahinet [1996] for an application of LMIs in $\mathcal{H}_\infty$ control with closed-loop pole constraints.

14.11 Problems

Problem 14.1 Figure 14.9 shows a single-loop analog feedback system. The plant is

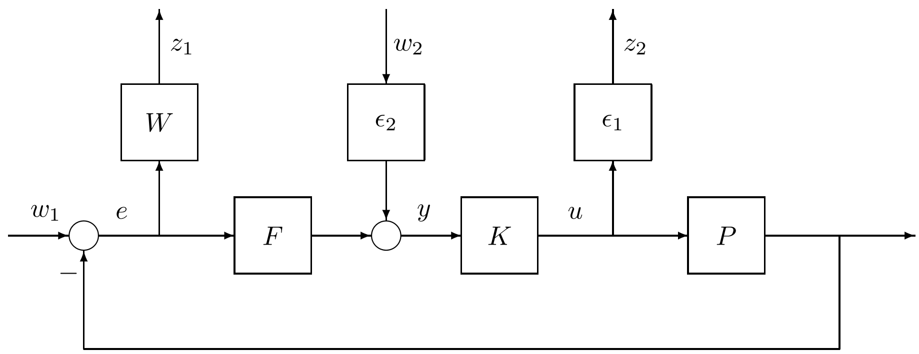


Figure 14.9: Analog feedback system

P and the controller K; F is an antialiasing filter for future digital implementation of the controller (it is a good idea to include F at the start of the analog design so that there are no surprises later due to additional phase lag). The basic control specification is to get good tracking over a certain frequency range, say $[0, \omega_1]$; that is, to make the magnitude of the transfer function from w_1 to e small over this frequency range. The weighted tracking error is z_1 in the figure, where the weight W is selected to be a low-pass filter with bandwidth ω_1. We could attempt to minimize the $\mathcal{H}_\infty$ norm from w_1 to z_1, but this problem is not regular. To regularize it, another input, w_2, is added and another signal, z_2, is penalized. The two weights ϵ_1 and ϵ_2 are small positive scalars. The design problem is to minimize the $\mathcal{H}_\infty$ norm

$$\text{from } w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \text{ to } z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Figure 14.9 can then be converted to the standard setup by stacking the states of P, F, and W to form the state of G.

The plant transfer function is taken to be

$$P(s) = \frac{20 - s}{(s + 0.01)(s + 20)}.$$

With a view toward subsequent digital control with $h = 0.5$, the filter F is taken to have bandwidth $\pi/0.5$, and Nyquist frequency ω_N:

$$F(s) = \frac{1}{(0.5/\pi)s + 1}.$$

The weight W is then taken to have bandwidth one-fifth the Nyquist frequency:

$$W(s) = \left[\frac{1}{(2.5/\pi)s+1}\right]^2.$$

Finally, ϵ_1 and ϵ_2 are both set to 0.01.

Run *hinfsyn* and show your plots of the closed-loop frequency responses.

Problem 14.2 Make the same assumptions as in Chapter 13 for $\mathcal{H}_2$ control and derive the $\mathcal{H}_\infty$ controller parameterization by using the normalization procedure and Theorem 14.7.

Problem 14.3 Consider the feedback system in Figure 6.3 and suppose

$$P = \frac{s-10}{(s+1)(s+10)}, \quad W_e = \frac{1}{s+0.001}, \quad W_u = \frac{s+2}{s+10}.$$

Design a controller that minimizes

$$\left\| \begin{bmatrix} W_e S_o \\ W_u K S_o \end{bmatrix} \right\|_\infty.$$

Simulate the time response of the system when r is a step.

Problem 14.4 Repeat Problem 14.3 when $W_e = 1/s$.

Problem 14.5 Consider again Problem 13.5 and design a controller that minimizes the $\mathcal{H}_\infty$ norm of the transfer matrix from r to (e, u_w).

Problem 14.6 Repeat Problem 14.5 with $W = \epsilon$ for $\epsilon = 0.01$ and 0.0001. Study the behavior of the controller when $\epsilon \to 0$.

Problem 14.7 Repeat Problem 14.5 and Problem 14.6 with

$$P = \frac{10(2-s)}{(s+1)^3}.$$

Problem 14.8 Let $N \in \mathcal{RH}_\infty^-$. The Nehari problem is the following approximation problem:

$$\inf_{Q \in \mathcal{RH}_\infty} \|N - Q\|_\infty.$$

Formulate the Nehari problem as a standard $\mathcal{H}_\infty$ control problem.

Problem 14.9 Consider a generalized plant

$$P = \left[\begin{array}{cc|cc} -4 & 25 & 0.8 & -1 \\ -10 & 29 & 0.9 & -1 \\ \hline 10 & -25 & 0 & 1 \\ 13 & 25 & 1 & 0 \end{array}\right]$$

with an SISO controller K. Find the optimal $\mathcal{H}_\infty$ performance γ_{opt}. Calculate the central controller for each $\gamma \in (\gamma_{\mathrm{opt}}, 2)$ and the corresponding $\mathcal{H}_\infty$ performance, γ_∞. Plot γ_∞ versus γ. Is γ_∞ monotonic with respect to γ? (See Ushida and Kimura [1996] for a detailed discussion.)

Problem 14.10 Let a satellite model be given by $P_o(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1.539^2 & -2 \times 0.003 \times 1.539 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1.7319 \times 10^{-5} \\ 0 \\ 3.7859 \times 10^{-4} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \quad D = 0.$$

Suppose the true system is described by

$$P = (N + \Delta_N)(M + \Delta_M)^{-1}$$

where $P_o = NM^{-1}$ is a normalized coprime factorization. Design a controller so that the controller stabilizes the largest

$$\Delta = \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}.$$

Problem 14.11 Consider the feedback system shown below and let

$$P = \frac{0.5(1-s)}{(s+2)(s+0.5)}, \quad W_1 = 50\frac{s/1.245+1}{s/0.007+1}, \quad W_2 = 0.1256\frac{s/0.502+1}{s/2+1}.$$

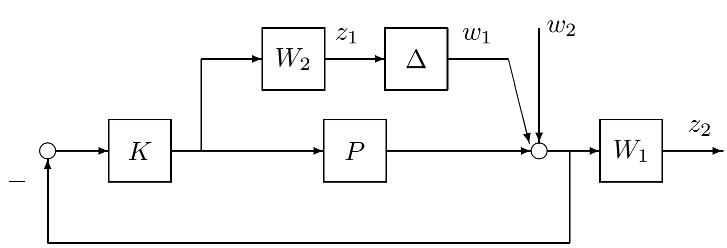


Figure 14.10: System with additive uncertainty

Then

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -W_2KS & -W_2KS \\ W_1S & W_1S \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = M \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where $S = (I + PK)^{-1}$.

(a) Design a controller K such that

$$\inf_{K \text{ stabilizing}} \|M\|_\infty .$$

(b) Design a controller K so that

$$\inf_{K \text{ stabilizing}} \sup_\omega \mu_\Delta(M), \qquad \Delta = \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix}.$$

Note that $\mu_\Delta(M) = |W_1 S| + |W_2 K S|$.

Problem 14.12 Design a μ-synthesis controller for the HIMAT control problem in Example 9.1.

Problem 14.13 Let $G(s) \in \mathcal{H}_\infty$ be a square transfer matrix and $\alpha > 0$. Show that G is (extended) strictly positive real (i.e., $G^*(j\omega) + G(j\omega) > 0,\ \forall\ \omega \in \mathbb{R} \cup \{\infty\}$) if and only if $\left\|(\alpha I - G)(\alpha I + G)^{-1}\right\|_\infty < 1$.

Problem 14.14 Consider a generalized system

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

and suppose we want to find a controller K such that $\mathcal{F}_\ell(G, K)$ is (extended) strictly postive real. Show that the problem can be converted to a standard $\mathcal{H}_\infty$ control problem by using the transformation in the last problem.

Problem 14.15 (Synthesis Using Popov Criterion) A stability criterion by Popov involves finding a controller and a multiplier matrix N such that

$$Q + (I + sN)\mathcal{F}_\ell(G, K)$$

is strictly positive real where Q is a constant matrix. Assume $D_{11} = 0$ and $D_{12} = 0$ in the realization of G. Find the $\tilde{G}$ so that

$$Q + (I + sN)\mathcal{F}_\ell(G, K) = \mathcal{F}_\ell(\tilde{G}, K)$$

Hence the results in the last problem can be applied.