Chapter 13

$\mathcal{H}_2$ Optimal Control

In this chapter we treat the optimal control of linear time-invariant systems with a quadratic performance criterion.

13.1 Introduction to Regulator Problem

Consider the following dynamical system:

$$\dot{x} = Ax + B_2 u, \quad x(t_0) = x_0 \tag{13.1}$$

where x_0 is given but arbitrary. Our objective is to find a control function $u(t)$ defined on $[t_0, T]$, which can be a function of the state $x(t)$, such that the state $x(t)$ is driven to a (small) neighborhood of origin at time T. This is the so-called *regulator problem*. One might suggest that this regulator problem can be trivially solved for any $T > t_0$ if the system is controllable. This is indeed the case if the controller can provide arbitrarily large amount of energy since, by the definition of controllability, one can immediately construct a control function that will drive the state to zero in an arbitrarily short time. However, this is not practical since any physical system has energy limitation (i.e., the actuator will eventually saturate). Furthermore, large control action can easily drive the system out of the region, where the given linear model is valid. Hence certain limitations have to be imposed on the control in practical engineering implementation. The constraints on control u may be measured in many different ways; for example,

$$\int_{t_0}^{T} \|u\| \, dt, \quad \int_{t_0}^{T} \|u\|^2 \, dt, \quad \sup_{t \in [t_0, T]} \|u\| \, ;$$

That is, in terms of $\mathcal{L}_1$ norm, $\mathcal{L}_2$ norm, and $\mathcal{L}_\infty$ norm or, more generally, weighted $\mathcal{L}_1$ norm, $\mathcal{L}_2$ norm, and $\mathcal{L}_\infty$ norm

$$\int_{t_0}^{T} \|W_u u\| \, dt, \quad \int_{t_0}^{T} \|W_u u\|^2 \, dt, \quad \sup_{t \in [t_0, T]} \|W_u u\|$$

for some constant weighting matrix W_u.

Similarly, one might also want to impose some constraints on the transient response $x(t)$ in a similar fashion:

$$\int_{t_0}^{T} \|W_x x\|\, dt, \quad \int_{t_0}^{T} \|W_x x\|^2\, dt, \quad \sup_{t\in[t_0,T]} \|W_x x\|$$

for some weighting matrix W_x. Hence the regulator problem can be posed as an optimal control problem with certain combined performance index on u and x. In this chapter, we shall be concerned exclusively with the $\mathcal{L}_2$ performance problem or quadratic performance problem. Moreover, we shall focus on the infinite time regulator problem (i.e., $T \to \infty$) and, without loss of generality, we shall assume $t_0 = 0$. In this case, our problem is as follows: Find a control $u(t)$ defined on $[0,\infty)$ such that the state $x(t)$ is driven to the origin as $t \to \infty$ and the following performance index is minimized:

$$\min_u \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{13.2}$$

for some $Q = Q^*$, S, and $R = R^* > 0$. This problem is traditionally called a *linear quadratic regulator* problem or, simply, an LQR problem. Here we have assumed $R > 0$ to emphasize that the control energy has to be finite (i.e., $u(t) \in \mathcal{L}_2[0,\infty)$). So $\mathcal{L}_2[0,\infty)$ is the space over which the integral is minimized. Moreover, it is also generally assumed that

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0. \tag{13.3}$$

Since R is positive definite, it has a square root, $R^{1/2}$, which is also positive-definite. By the substitution

$$u \leftarrow R^{1/2} u,$$

we may as well assume at the start that $R = I$. In fact, we can even assume $S = 0$ by using a pre-state feedback $u = -S^* x + v$ provided some care is exercised; however, this will not be assumed in the sequel. Since the matrix in equation (13.3) is positive semi-definite with $R = I$, it can be factored as

$$\begin{bmatrix} Q & S \\ S^* & I \end{bmatrix} = \begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix} \begin{bmatrix} C_1 & D_{12} \end{bmatrix}.$$

Then equation (13.2) can be rewritten as

$$\min_{u\in\mathcal{L}_2[0,\infty)} \|C_1 x + D_{12} u\|_2^2 .$$

In fact, the LQR problem is posed traditionally as the minimization problem:

$$\min_{u \in \mathcal{L}_2[0,\infty)} \|C_1 x + D_{12} u\|_2^2 \tag{13.4}$$

$$\text{subject to: } \dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{13.5}$$

without explicitly mentioning the condition that the control should drive the state to the origin. Instead some assumptions are imposed on Q, S, and R (or, equivalently, on C_1 and D_{12}) to ensure that the optimal control law u has this property. To see what assumption one needs to make to ensure that the minimization problem formulated in equations (13.4) and (13.5) has a sensible solution, let us consider a simple example with $A = 1$, $B = 1, Q = 0, S = 0$, and $R = 1$:

$$\min_{u \in \mathcal{L}_2[0,\infty)} \int_0^\infty u^2 dt, \quad \dot{x} = x + u, \quad x(0) = x_0.$$

It is clear that $u = 0$ is the optimal solution. However, the system with $u = 0$ is unstable and $x(t)$ diverges exponentially to infinity since $x(t) = e^t x_0$. The problem with this example is that this performance index does not "see" the unstable state x. Thus, to ensure that the minimization problem in equations (13.4) and (13.5) is sensible, we must assume that all unstable states can be "seen" from the performance index; that is, (C_1, A) must be detectable. An LQR problem with such an assumption will be called a *standard LQR problem*.

On the other hand, if the closed-loop stability is imposed on the preceding minimization, then it can be shown that $\min_{u \in \mathcal{L}_2[0,\infty)} \int_0^\infty u^2 dt = 2x_0^2$ and $u(t) = -2x(t)$ is the optimal control. This can also be generalized to a more general case where (C_1, A) is not necessarily detectable. Such a LQR problem will be referred to as an *Extended LQR problem*.

13.2 Standard LQR Problem

In this section, we shall consider the LQR problem as traditionally formulated.

Standard LQR Problem

Let a dynamical system be described by

$$\dot{x} = Ax + B_2 u, \quad x(0) = x_0 \text{ given but arbitrary} \tag{13.6}$$

$$z = C_1 x + D_{12} u \tag{13.7}$$

and suppose that the system parameter matrices satisfy the following assumptions:

(A1) (A, B_2) is stabilizable;

(A2) D_{12} has full column rank with $\begin{bmatrix} D_{12} & D_\perp \end{bmatrix}$ unitary;

(A3) (C_1, A) is detectable;

(A4) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

Find an optimal control law $u \in \mathcal{L}_2[0, \infty)$ such that the performance criterion $\|z\|_2^2$ is minimized.

Remark 13.1 Assumption (A1) is clearly necessary for the existence of a stabilizing control function u. The assumption (A2) is made for simplicity of notation and is actually a restatement that $R = D_{12}^* D_{12} = I$. Note also that $D_\perp$ drops out when D_{12} is square. It is interesting to point out that (A3) is not needed in the Extended LQR problem. The assumption (A3) enforces that the unconditional optimization problem will result in a stabilizing control law. In fact, the assumption (A3) together with (A1) guarantees that the input/output stability implies the internal stability; that is, $u \in \mathcal{L}_2$ and $z \in \mathcal{L}_2$ imply $x \in \mathcal{L}_2$, which will be shown in Lemma 13.1. Finally note that (A4) is equivalent to the condition that $(D_\perp^* C_1, A - B_2 D_{12}^* C_1)$ has no unobservable modes on the imaginary axis and is weaker than the popular assumption of detectability of $(D_\perp^* C_1, A - B_2 D_{12}^* C_1)$. (A4), together with the stabilizability of (A, B_2), guarantees by Corollary 12.7 that the following Hamiltonian matrix belongs to dom(Ric) and that $X = \mathrm{Ric}(H) \geq 0$:

$$\begin{aligned} H &= \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B_2 \\ -C_1^* D_{12} \end{bmatrix} \begin{bmatrix} D_{12}^* C_1 & B_2^* \end{bmatrix} \\ &= \begin{bmatrix} A - B_2 D_{12}^* C_1 & -B_2 B_2^* \\ -C_1^* D_\perp D_\perp^* C_1 & -(A - B_2 D_{12}^* C_1)^* \end{bmatrix}. \end{aligned} \tag{13.8}$$

Note also that if $D_{12}^* C_1 = 0$, then (A4) is implied by the detectability of (C_1, A). $\diamond$

Note that the Riccati equation corresponding to equation (13.8) is

$$(A - B_2 D_{12}^* C_1)^* X + X(A - B_2 D_{12}^* C_1) - X B_2 B_2^* X + C_1^* D_\perp D_\perp^* C_1 = 0. \tag{13.9}$$

Now let X be the corresponding stabilizing solution and define

$$F := -(B_2^* X + D_{12}^* C_1). \tag{13.10}$$

Then $A + B_2 F$ is stable. Denote

$$A_F := A + B_2 F, \quad C_F := C_1 + D_{12} F$$

and rearrange equation (13.9) to get

$$A_F^* X + X A_F + C_F^* C_F = 0. \tag{13.11}$$

Thus X is the observability Gramian of (C_F, A_F).

Consider applying the control law $u = Fx$ to the system equations (13.6) and (13.7). The controlled system becomes

$$\begin{aligned} \dot{x} &= A_F x, \quad x(0) = x_0 \\ z &= C_F x \end{aligned}$$

or, equivalently,

$$\begin{aligned} \dot{x} &= A_F x + x_0 \delta(t), \quad x(0_-) = 0 \\ z &= C_F x. \end{aligned}$$

The associated transfer matrix is

$$G_c(s) = \left[\begin{array}{c|c} A_F & I \\ \hline C_F & 0 \end{array}\right]$$

and

$$\|G_c x_0\|_2^2 = x_0^* X x_0.$$

The proof of the following theorem requires a preliminary result about internal stability given input-output stability.

Lemma 13.1 *If $u, z \in \mathcal{L}_2[0,\infty)$ and (C_1, A) is detectable in the system described by equations (13.6) and (13.7), then $x \in \mathcal{L}_2[0,\infty)$. Furthermore, $x(t) \to 0$ as $t \to \infty$.*

Proof. Since (C_1, A) is detectable, there exists L such that $A + LC_1$ is stable. Let $\hat{x}$ be the state estimate of x given by

$$\dot{\hat{x}} = (A + LC_1)\hat{x} + (LD_{12} + B_2)u - Lz.$$

Then $\hat{x} \in \mathcal{L}_2[0,\infty)$ and $\hat{x} \to 0$ (see Problem 13.1) since z and u are in $\mathcal{L}_2[0,\infty)$. Now let $e = x - \hat{x}$; then

$$\dot{e} = (A + LC_1)e$$

and $e \in \mathcal{L}_2[0,\infty)$. Therefore, $x = e + \hat{x} \in \mathcal{L}_2[0,\infty)$. It is easy to see that $e(t) \to 0$ as $t \to \infty$ for any initial condition $e(0)$. Finally, $x(t) \to 0$ since $\hat{x} \to 0$. □

Theorem 13.2 *There exists a unique optimal control for the LQR problem, namely $u = Fx$. Moreover,*

$$\min_{u \in \mathcal{L}_2[0,\infty)} \|z\|_2 = \|G_c x_0\|_2 .$$

Note that the optimal control strategy is a constant gain state feedback, and this gain is independent of the initial condition x_0.

Proof. With the change of variable $v = u - Fx$, the system can be written as

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A_F & B_2 \\ C_F & D_{12} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}, \qquad x(0) = x_0. \tag{13.12}$$

Now if $v \in \mathcal{L}_2[0,\infty)$, then $x, z \in \mathcal{L}_2[0,\infty)$ and $x(\infty) = 0$ since A_F is stable. Hence $u = Fx + v \in \mathcal{L}_2[0,\infty)$. Conversely, if $u, z \in \mathcal{L}_2[0,\infty)$, then from Lemma 13.1 $x \in \mathcal{L}_2[0,\infty)$. So $v \in \mathcal{L}_2[0,\infty)$. Thus the mapping $v = u - Fx$ between $v \in \mathcal{L}_2[0,\infty)$ and those $u \in \mathcal{L}_2[0,\infty)$ that make $z \in \mathcal{L}_2[0,\infty)$ is one-to-one and onto. Therefore,

$$\min_{u \in \mathcal{L}_2[0,\infty)} \|z\|_2 = \min_{v \in \mathcal{L}_2[0,\infty)} \|z\|_2 .$$

By differentiating $x(t)^* X x(t)$ with respect to t along a solution of the differential equation (13.12) and by using equation (13.9) and the fact that $C_F^* D_{12} = -XB_2$, we see that

$$\begin{aligned} \frac{d}{dt} x^* X x &= \dot{x}^* X x + x^* X \dot{x} = x^*(A_F^* X + X A_F)x + 2x^* X B_2 v \\ &= -x^* C_F^* C_F x + 2x^* X B_2 v \\ &= -(C_F x + D_{12} v)^*(C_F x + D_{12} v) + 2x^* C_F^* D_{12} v + v^* v + 2x^* X B_2 v \\ &= -\|z\|^2 + \|v\|^2 . \end{aligned} \tag{13.13}$$

Now integrate equation (13.13) from 0 to ∞ to get

$$\|z\|_2^2 = x_0^* X x_0 + \|v\|_2^2 .$$

Clearly, the unique optimal control is $v = 0$, i.e., $u = Fx$. □

13.3 Extended LQR Problem

This section considers the extended LQR problem where no detectability assumption is made for (C_1, A).

Extended LQR Problem

Let a dynamical system be given by

$$\begin{aligned} \dot{x} &= Ax + B_2 u, \quad x(0) = x_0 \ \text{ given but arbitrary} \\ z &= C_1 x + D_{12} u \end{aligned}$$

with the following assumptions:

(A1) (A, B_2) is stabilizable;

(A2) D_{12} has full column rank with $\begin{bmatrix} D_{12} & D_\perp \end{bmatrix}$ unitary;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

Find an optimal control law $u \in \mathcal{L}_2[0,\infty)$ such that the system is internally stable (i.e., $x \in \mathcal{L}_2[0,\infty)$) and the performance criterion $\|z\|_2^2$ is minimized.

Assume the same notation as in the last section, we have:

Theorem 13.3 *There exists a unique optimal control for the extended LQR problem, namely $u = Fx$. Moreover,*

$$\min_{u \in \mathcal{L}_2[0,\infty)} \|z\|_2 = \|G_c x_0\|_2 \,.$$

Proof. The proof of this theorem is very similar to the proof of the standard LQR problem except that, in this case, the input/output stability may not necessarily imply the internal stability. Instead, the internal stability is guaranteed by the way of choosing control law.

Suppose that $u \in \mathcal{L}_2[0,\infty)$ is such a control law that the system is stable, i.e., $x \in \mathcal{L}_2[0,\infty)$. Then $v = u - Fx \in \mathcal{L}_2[0,\infty)$. On the other hand, let $v \in \mathcal{L}_2[0,\infty)$ and consider

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A_F & B_2 \\ C_F & D_{12} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \quad , \qquad x(0) = x_0.$$

Then $x, z \in \mathcal{L}_2[0,\infty)$ and $x(\infty) = 0$ since A_F is stable. Hence $u = Fx + v \in \mathcal{L}_2[0,\infty)$. Again the mapping $v = u - Fx$ between $v \in \mathcal{L}_2[0,\infty)$ and those $u \in \mathcal{L}_2[0,\infty)$ that make $z \in \mathcal{L}_2[0,\infty)$ and $x \in \mathcal{L}_2[0,\infty)$ is one to one and onto. Therefore,

$$\min_{u \in \mathcal{L}_2[0,\infty)} \|z\|_2 = \min_{v \in \mathcal{L}_2[0,\infty)} \|z\|_2 \,.$$

Using the same technique as in the proof of the standard LQR problem, we have

$$\|z\|_2^2 = x_0^* X x_0 + \|v\|_2^2 \,.$$

Thus, the unique optimal control is $v = 0$, i.e., $u = Fx$. $\square$

13.4 Guaranteed Stability Margins of LQR

Now we will consider the system described by equation (13.6) with the LQR control law $u = Fx$. The closed-loop block diagram is as shown in Figure 13.1.

The following result is the key to stability margins of an LQR control law.

Lemma 13.4 *Let $F = -(B_2^* X + D_{12}^* C_1)$ and define $G_{12} = D_{12} + C_1(sI - A)^{-1}B_2$. Then*

$$\left(I - B_2^*(-sI - A^*)^{-1}F^*\right)\left(I - F(sI - A)^{-1}B_2\right) = G_{12}^\sim(s)G_{12}(s).$$

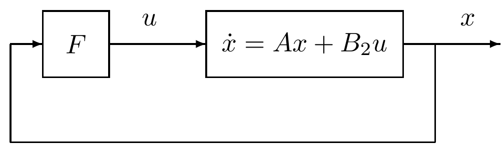


Figure 13.1: LQR closed-loop system

Proof. Note that the Riccati equation (13.9) can be written as

$$XA + A^*X - F^*F + C_1^*C_1 = 0.$$

Add and subtract sX to the above equation to get

$$-X(sI - A) - (-sI - A^*)X - F^*F + C_1^*C_1 = 0.$$

Now multiply the above equation from the left by $B_2^*(-sI - A^*)^{-1}$ and from the right by $(sI - A)^{-1}B_2$ to get

$$\begin{aligned}-B_2^*(-sI - A^*)^{-1}XB_2 - B_2^*X(sI - A)^{-1}B_2 - B_2^*(-sI - A^*)^{-1}F^*F(sI - A)^{-1}B_2 \\ +B_2^*(-sI - A^*)^{-1}C_1^*C_1(sI - A)^{-1}B_2 = 0.\end{aligned}$$

Using $-B_2^*X = F + D_{12}^*C_1$ in the above equation, we have

$$\begin{aligned}B_2^*(-sI - A^*)^{-1}F^* + F(sI - A)^{-1}B_2 - B_2^*(-sI - A^*)^{-1}F^*F(sI - A)^{-1}B_2 \\ +B_2^*(-sI - A^*)^{-1}C_1^*D_{12} + D_{12}^*C_1(sI - A)^{-1}B_2 \\ +B_2^*(-sI - A^*)^{-1}C_1^*C_1(sI - A)^{-1}B_2 = 0.\end{aligned}$$

Then the result follows from completing the square and from the fact that $D_{12}^*D_{12} = I$. □

Corollary 13.5 *Suppose* $D_{12}^*C_1 = 0$. *Then*

$$\left(I - B_2^*(-sI - A^*)^{-1}F^*\right)\left(I - F(sI - A)^{-1}B_2\right) = I + B_2^*(-sI - A^*)^{-1}C_1^*C_1(sI - A)^{-1}B_2.$$

In particular,

$$\left(I - B_2^*(-j\omega I - A^*)^{-1}F^*\right)\left(I - F(j\omega I - A)^{-1}B_2\right) \geq I \qquad (13.14)$$

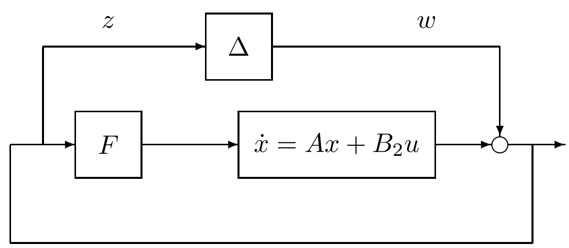
and

$$\left(I + B_2^*(-j\omega I - A^* - F^*B_2^*)^{-1}F^*\right)\left(I + F(j\omega I - A - B_2F)^{-1}B_2\right) \leq I. \qquad (13.15)$$

Note that the inequality (13.15) follows from taking the inverse of inequality (13.14).

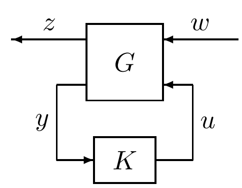
Define $G(s) = -F(sI - A)^{-1}B_2$ and assume for the moment that the system is single-input. Then the inequality (13.14) shows that the open-loop Nyquist diagram of the system $G(s)$ in Figure 13.1 never enters the unit disk centered at $(-1, 0)$ of the complex plane. Hence the system has at least a 6 dB ($= 20 \log 2$) gain margin and a 60^{o} phase margin in both directions. A similar interpretation may be generalized to multiple-input systems.

Next, it is noted that the inequality (13.15) can also be given some robustness interpretation. In fact, it implies that the closed-loop system in Figure 13.1 is stable even if the open-loop system $G(s)$ is perturbed additively by a $\Delta \in \mathcal{RH}_\infty$ as long as $\|\Delta\|_\infty < 1$. This can be seen from the following block diagram and the small gain theorem, where the transfer matrix from w to z is exactly $I + F(j\omega I - A - B_2F)^{-1}B_2$.



13.5 Standard $\mathcal{H}_2$ Problem

The system considered in this section is described by the following standard block diagram:



The realization of the transfer matrix G is taken to be of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

Notice the special off-diagonal structure of D: D_{22} is assumed to be zero so that G_{22} is strictly proper;[1] also, D_{11} is assumed to be zero in order to guarantee that the $\mathcal{H}_2$

[1]This assumption is made without loss of generality since a substitution of $K_D = K(I + D_{22}K)^{-1}$ would give the controller for $D_{22} \neq 0$.

problem is properly posed.[2]

The following additional assumptions are made for the output feedback $\mathcal{H}_2$ problem in this chapter:

(i) (A, B_2) is stabilizable and (C_2, A) is detectable;

(ii) $R_1 = D_{12}^* D_{12} > 0$ and $R_2 = D_{21} D_{21}^* > 0$;

(iii) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(iv) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

The first assumption is for the stabilizability of G by output feedback, and the third and the fourth assumptions together with the first guarantee that the two Hamiltonian matrices associated with the following $\mathcal{H}_2$ problem belong to dom(Ric). The assumptions in (ii) guarantee that the $\mathcal{H}_2$ optimal control problem is nonsingular.

$\mathcal{H}_2$ **Problem** *The $\mathcal{H}_2$ control problem is to find a proper, real rational controller K that stabilizes G internally and minimizes the $\mathcal{H}_2$ norm of the transfer matrix T_{zw} from w to z.*

In the following discussions we shall assume that we have state models of G and K. Recall that a controller is said to be admissible if it is internally stabilizing and proper. By Corollary 12.7 the two Hamiltonian matrices

$$H_2 := \begin{bmatrix} A - B_2 R_1^{-1} D_{12}^* C_1 & -B_2 R_1^{-1} B_2^* \\ -C_1^*(I - D_{12} R_1^{-1} D_{12}^*) C_1 & -(A - B_2 R_1^{-1} D_{12}^* C_1)^* \end{bmatrix}$$

$$J_2 := \begin{bmatrix} (A - B_1 D_{21}^* R_2^{-1} C_2)^* & -C_2^* R_2^{-1} C_2 \\ -B_1(I - D_{21}^* R_2^{-1} D_{21}) B_1^* & -(A - B_1 D_{21}^* R_2^{-1} C_2) \end{bmatrix}$$

belong to dom(Ric), and, moreover, $X_2 := \mathrm{Ric}(H_2) \geq 0$ and $Y_2 := \mathrm{Ric}(J_2) \geq 0$. Define

$$F_2 := -R_1^{-1}(B_2^* X_2 + D_{12}^* C_1), \quad L_2 := -(Y_2 C_2^* + B_1 D_{21}^*) R_2^{-1}$$

and

$$A_{F_2} := A + B_2 F_2, \quad C_{1F_2} := C_1 + D_{12} F_2$$
$$A_{L_2} := A + L_2 C_2, \quad B_{1L_2} := B_1 + L_2 D_{21}$$
$$\hat{A}_2 := A + B_2 F_2 + L_2 C_2$$
$$G_c(s) := \left[\begin{array}{c|c} A_{F_2} & I \\ \hline C_{1F_2} & 0 \end{array}\right], \qquad G_f(s) := \left[\begin{array}{c|c} A_{L_2} & B_{1L_2} \\ \hline I & 0 \end{array}\right].$$

Before stating the main theorem, we note the following fact:

[2] Recall that a rational proper stable transfer function is an $\mathcal{RH}_2$ function iff it is strictly proper.

Lemma 13.6 *Let $U, V \in \mathcal{RH}_\infty$ be defined as*

$$U := \left[\begin{array}{c|c} A_{F_2} & B_2 R_1^{-1/2} \\ \hline C_{1F_2} & D_{12} R_1^{-1/2} \end{array}\right], \quad V := \left[\begin{array}{c|c} A_{L_2} & B_{1L_2} \\ \hline R_2^{-1/2} C_2 & R_2^{-1/2} D_{21} \end{array}\right].$$

Then U is an inner and V is a co-inner, $U^\sim G_c \in \mathcal{RH}_2^\perp$, and $G_f V^\sim \in \mathcal{RH}_2^\perp$.

Proof. The proof uses standard manipulations of state-space realizations. From U we get

$$U^\sim(s) = \left[\begin{array}{c|c} -A_{F_2}^* & -C_{1F_2}^* \\ \hline R_1^{-1/2} B_2^* & R_1^{-1/2} D_{12}^* \end{array}\right].$$

Then it is easy to compute

$$U^\sim U = \left[\begin{array}{cc|c} -A_{F_2}^* & -C_{1F_2}^* C_{1F_2} & -C_{1F_2}^* D_{12} R_1^{-1/2} \\ 0 & A_{F_2} & B_2 R_1^{-1/2} \\ \hline R_1^{-1/2} B_2^* & R_1^{-1/2} D_{12}^* C_{1F_2} & I \end{array}\right]$$

$$U^\sim G_c = \left[\begin{array}{cc|c} -A_{F_2}^* & -C_{1F_2}^* C_{1F_2} & 0 \\ 0 & A_{F_2} & I \\ \hline R_1^{-1/2} B_2^* & R_1^{-1/2} D_{12}^* C_{1F_2} & 0 \end{array}\right].$$

Now do the similarity transformation

$$\begin{bmatrix} I & -X_2 \\ 0 & I \end{bmatrix}$$

on the states of the preceding transfer matrices and note that

$$A_{F_2}^* X_2 + X_2 A_{F_2} + C_{1F_2}^* C_{1F_2} = 0.$$

We get

$$U^\sim U = \left[\begin{array}{cc|c} -A_{F_2}^* & 0 & 0 \\ 0 & A_{F_2} & B_2 R_1^{-1/2} \\ \hline R_1^{-1/2} B_2^* & 0 & I \end{array}\right] = I$$

$$U^\sim G_c = \left[\begin{array}{cc|c} -A_{F_2}^* & 0 & -X_2 \\ 0 & A_{F_2} & I \\ \hline R_1^{-1/2} B_2^* & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} -A_{F_2}^* & -X_2 \\ \hline R_1^{-1/2} B_2^* & 0 \end{array}\right] \in \mathcal{RH}_2^\perp.$$

It follows by duality that $G_f V^\sim \in \mathcal{RH}_2^\perp$ and V is a co-inner. □

Theorem 13.7 *There exists a unique optimal controller*

$$K_{\rm opt}(s) := \left[\begin{array}{c|c} \hat{A}_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right].$$

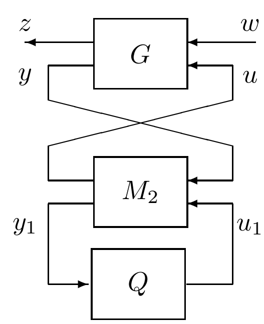
Moreover,

$$\min \|T_{zw}\|_2^2 = \|G_c B_1\|_2^2 + \|R_1^{1/2} F_2 G_f\|_2^2 = \operatorname{trace}(B_1^* X_2 B_1) + \operatorname{trace}(R_1 F_2 Y_2 F_2^*).$$

Proof. Consider the all-stabilizing controller parameterization $K(s) = \mathcal{F}_\ell(M_2, Q)$, $Q \in \mathcal{RH}_\infty$ with

$$M_2(s) = \left[\begin{array}{c|cc} \hat{A}_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

and consider the following system diagram:



Then $T_{zw} = \mathcal{F}_\ell(N, Q)$ with

$$N = \left[\begin{array}{cc|cc} A_{F_2} & -B_2 F_2 & B_1 & B_2 \\ 0 & A_{L_2} & B_{1L_2} & 0 \\ \hline C_{1F_2} & -D_{12} F_2 & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right]$$

and

$$T_{zw} = G_c B_1 - U R_1^{1/2} F_2 G_f + U R_1^{1/2} Q R_2^{1/2} V.$$

It follows from Lemma 13.6 that $G_c B_1$ and U are orthogonal. Thus

$$\begin{aligned} \|T_{zw}\|_2^2 &= \|G_c B_1\|_2^2 + \left\| U R_1^{1/2} F_2 G_f - U R_1^{1/2} Q R_2^{1/2} V \right\|_2^2 \\ &= \|G_c B_1\|_2^2 + \left\| R_1^{1/2} F_2 G_f - R_1^{1/2} Q R_2^{1/2} V \right\|_2^2. \end{aligned}$$

Since G_f and V are also orthogonal by Lemma 13.6, we have

$$\begin{aligned} \|T_{zw}\|_2^2 &= \|G_cB_1\|_2^2 + \left\| R_1^{1/2}F_2G_f - R_1^{1/2}QR_2^{1/2}V \right\|_2^2 \\ &= \|G_cB_1\|_2^2 + \left\| R_1^{1/2}F_2G_f \right\|_2^2 + \left\| R_1^{1/2}QR_2^{1/2} \right\|_2^2. \end{aligned}$$

This shows clearly that $Q = 0$ gives the unique optimal control, so $K = \mathcal{F}_\ell(M_2, 0)$ is the unique optimal controller. □

The optimal $\mathcal{H}_2$ controller, K_{opt}, and the closed-loop transfer matrix, T_{zw}, can be obtained by the following MATLAB program:

$\gg$ $[\mathbf{K}, \mathbf{T_{zw}}] = \mathbf{h2syn}(\mathbf{G}, \mathbf{n_y}, \mathbf{n_u})$

where n_y and n_u are the dimensions of y and u, respectively.

Related MATLAB Commands: lqg, lqr, lqr2, lqry, reg, lqe

13.6 Stability Margins of $\mathcal{H}_2$ Controllers

It is well-known that a system with LQR controller has at least 60° phase margin and 6 dB gain margin. However, it is not clear whether these stability margins will be preserved if the states are not available and the output feedback $\mathcal{H}_2$ (or LQG) controller has to be used. The answer is provided here through a counterexample from Doyle [1978]: There are no guaranteed stability margins for a $\mathcal{H}_2$ controller.

Consider a single-input and single-output two-state generalized dynamical system:

$$G(s) = \left[\begin{array}{c|cc} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \sqrt{\sigma} & 0 \\ \sqrt{\sigma} & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} \sqrt{q} & \sqrt{q} \\ 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

It can be shown analytically that

$$X_2 = \begin{bmatrix} 2\alpha & \alpha \\ \alpha & \alpha \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 2\beta & \beta \\ \beta & \beta \end{bmatrix}$$

and

$$F_2 = -\alpha \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad L_2 = -\beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$\alpha = 2 + \sqrt{4+q}\ , \quad \beta = 2 + \sqrt{4+\sigma}.$$

Then the optimal output $\mathcal{H}_2$ controller is given by

$$K_{\text{opt}} = \left[\begin{array}{cc|c} 1-\beta & 1 & \beta \\ -(\alpha+\beta) & 1-\alpha & \beta \\ \hline -\alpha & -\alpha & 0 \end{array}\right].$$

Suppose that the resulting closed-loop controller (or plant G_{22}) has a scalar gain k with a nominal value $k = 1$. Then the controller implemented in the system is actually

$$K = kK_{\text{opt}},$$

and the closed-loop system A matrix becomes

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -k\alpha & -k\alpha \\ \beta & 0 & 1-\beta & 1 \\ \beta & 0 & -\alpha-\beta & 1-\alpha \end{bmatrix}.$$

It can be shown that the characteristic polynomial has the form

$$\det(sI - \tilde{A}) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

with

$$a_1 = \alpha + \beta - 4 + 2(k-1)\alpha\beta, \quad a_0 = 1 + (1-k)\alpha\beta.$$

Note that for closed-loop stability it is necessary to have $a_0 > 0$ and $a_1 > 0$. Note also that $a_0 \approx (1-k)\alpha\beta$ and $a_1 \approx 2(k-1)\alpha\beta$ for sufficiently large α and β if $k \neq 1$. It is easy to see that for sufficiently large α and β (or q and σ), the system is unstable for arbitrarily small perturbations in k in either direction. Thus, by choice of q and σ, the gain margins may be made arbitrarily small.

It is interesting to note that the margins deteriorate as control weight $(1/q)$ gets small (large q) and/or system driving noise gets large (large σ). In modern control folklore, these have often been considered ad hoc means of improving sensitivity.

It is also important to recognize that vanishing margins are not only associated with open-loop unstable systems. It is easy to construct minimum phase, open-loop stable counterexamples for which the margins are arbitrarily small.

The point of this example is that $\mathcal{H}_2$ (LQG) solutions, unlike LQR solutions, provide no global system-independent guaranteed robustness properties. Like their more classical colleagues, modern LQG designers are obliged to test their margins for each specific design.

It may, however, be possible to improve the robustness of a given design by relaxing the optimality of the filter with respect to error properties. A successful approach in this direction is the so called LQG loop transfer recovery (LQG/LTR) design technique. The idea is to design a filtering gain, L_2, in such way so that the LQG (or $\mathcal{H}_2$) control law will approximate the loop properties of the regular LQR control. This will not be explored further here; interested readers may consult related references.

13.7 Notes and References

Detailed treatment of $\mathcal{H}_2$ related theory, LQ optimal control, Kalman filtering, etc., can be found in Anderson and Moore [1989] or Kwakernaak and Sivan [1972]. The LQG/LTR control design was first introduced by Doyle and Stein [1981], and much work has been reported in this area since then. Additional results on the LQR stability margins can be found in Zhang and Fu [1996].

13.8 Problems

Problem 13.1 Let $v(t) \in L_2[0, \infty)$. Let $y(t)$ be the output of the system $G(s) = \frac{1}{s+1}$ with input v. Prove that $\lim_{t\to\infty} y(t) = 0$.

Problem 13.2 Parameterize all stabilizing controllers satisfying $\|T_{zw}\|_2 \leq \gamma$ for a given $\gamma > 0$.

Problem 13.3 Consider the feedback system in Figure 6.3 and suppose

$$P = \frac{s-10}{(s+1)(s+10)}, \quad W_e = \frac{1}{s+0.001}, \quad W_u = \frac{s+2}{s+10}.$$

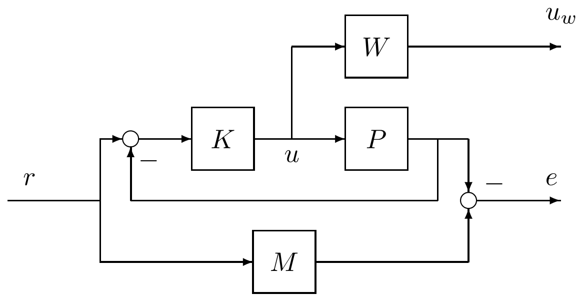
Design a controller that minimizes

$$\left\| \begin{bmatrix} W_e S_o \\ W_u K S_o \end{bmatrix} \right\|_2.$$

Simulate the time response of the system when r is a step.

Problem 13.4 Repeat Problem 13.3 when $W_e = 1/s$. (Note that the solution given in this chapter cannot be applied directly.)

Problem 13.5 Consider the model matching (or reference) control problem shown here:



Let $M(s) \in \mathcal{H}_\infty$ be a strictly proper transfer matrix and $W(s), W^{-1}(s) \in \mathcal{RH}_\infty$. Formulate an $\mathcal{H}_2$ control problem that minimizes u_w and the error e through minimizing the $\mathcal{H}_2$ norm of the transfer matrix from r to (e, u_w). Apply your formula to

$$M(s) = \frac{4}{s^2 + 2s + 4}, \quad P(s) = \frac{10(s+2)}{(s+1)^3}, \quad W(s) = \frac{0.1(s+1)}{s+10}.$$

Problem 13.6 Repeat Problem 13.5 with $W = \epsilon$ for $\epsilon = 0.01$ and 0.0001. Study the behavior of the controller when $\epsilon \to 0$.

Problem 13.7 Repeat Problem 13.5 and Problem 13.6 with

$$P = \frac{10(2-s)}{(s+1)^3}.$$