Chapter 4

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Spaces

The most important objective of a control system is to achieve certain performance specifications in addition to providing internal stability. One way to describe the performance specifications of a control system is in terms of the size of certain signals of interest. For example, the performance of a tracking system could be measured by the size of the tracking error signal. In this chapter, we look at several ways of defining a signal's size (i.e., at several norms for signals). Of course, which norm is most appropriate depends on the situation at hand. For that purpose, we shall first introduce the Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$. Some state-space methods of computing real rational $\mathcal{H}_2$ and $\mathcal{H}_\infty$ transfer matrix norms are also presented.

4.1 Hilbert Spaces

Recall the inner product of vectors defined on a Euclidean space $\mathbb{C}^n$:

$$\langle x, y \rangle := x^* y = \sum_{i=1}^{n} \bar{x}_i y_i \quad \forall x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{C}^n.$$

Note that many important metric notions and geometrical properties, such as length, distance, angle, and the energy of physical systems, can be deduced from this inner product. For instance, the length of a vector $x \in \mathbb{C}^n$ is defined as

$$\|x\| := \sqrt{\langle x, x \rangle}$$

and the angle between two vectors $x, y \in \mathbb{C}^n$ can be computed from

$$\cos \angle(x, y) = \frac{\langle x, y \rangle}{\|x\|\,\|y\|}, \quad \angle(x, y) \in [0, \pi].$$

The two vectors are said to be *orthogonal* if $\angle(x, y) = \frac{\pi}{2}$.

We now consider a natural generalization of the inner product on $\mathbb{C}^n$ to more general (possibly infinite dimensional) vector spaces.

Definition 4.1 Let V be a vector space over $\mathbb{C}$. An *inner product*[1] on V is a complex-valued function,

$$\langle \cdot, \cdot \rangle : V \times V \longmapsto \mathbb{C}$$

such that for any $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(iii) $\langle x, x \rangle > 0$ if $x \neq 0$.

A vector space V with an inner product is called an *inner product space*.

It is clear that the inner product defined above induces a norm $\|x\| := \sqrt{\langle x, x \rangle}$, so that the norm conditions in Chapter 2 are satisfied. In particular, the distance between vectors x and y is $d(x, y) = \|x - y\|$.

Two vectors x and y in an inner product space V are said to be *orthogonal* if $\langle x, y \rangle = 0$, denoted $x \perp y$. More generally, a vector x is said to be orthogonal to a set $S \subset V$, denoted by $x \perp S$, if $x \perp y$ for all $y \in S$.

The inner product and the inner product induced norm have the following familiar properties.

Theorem 4.1 *Let V be an inner product space and let $x, y \in V$. Then*

(i) $|\langle x, y \rangle| \leq \|x\| \, \|y\|$ *(Cauchy-Schwarz inequality). Moreover, the equality holds if and only if $x = \alpha y$ for some constant α or $y = 0$.*

(ii) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ *(Parallelogram law).*

(iii) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ *if $x \perp y$.*

A *Hilbert space* is a complete inner product space with the norm induced by its inner product. For example, $\mathbb{C}^n$ with the usual inner product is a (finite dimensional) Hilbert space. More generally, it is straightforward to verify that $\mathbb{C}^{n \times m}$ with the inner product defined as

$$\langle A, B \rangle := \operatorname{trace} A^* B = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{a}_{ij} b_{ij} \quad \forall A, B \in \mathbb{C}^{n \times m}$$

is also a (finite dimensional) Hilbert space.

A well-know infinite dimensional Hilbert space is $\mathcal{L}_2[a, b]$, which consists of all square integrable and Lebesgue measurable functions defined on an interval $[a, b]$ with the inner product defined as

$$\langle f, g \rangle := \int_a^b f(t)^* g(t) dt$$

[1]The property (i) in the following list is the other way around to the usual mathematical convention since we want to have $\langle x, y \rangle = x^* y$ rather than $y^* x$ for $x, y \in \mathbb{C}^n$.

for $f, g \in \mathcal{L}_2[a, b]$. Similarly, if the functions are vector or matrix-valued, the inner product is defined correspondingly as

$$\langle f, g \rangle := \int_a^b \operatorname{trace}\left[f(t)^* g(t)\right] dt.$$

Some spaces used often in this book are $\mathcal{L}_2[0, \infty), \mathcal{L}_2(-\infty, 0], \mathcal{L}_2(-\infty, \infty)$. More precisely, they are defined as

$\mathcal{L}_2 = \mathcal{L}_2(-\infty, \infty)$: Hilbert space of matrix-valued functions on $\mathbb{R}$, with inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} \operatorname{trace}\left[f(t)^* g(t)\right] dt.$$

$\mathcal{L}_{2+} = \mathcal{L}_2[0, \infty)$: subspace of $\mathcal{L}_2(-\infty, \infty)$ with functions zero for $t < 0$.

$\mathcal{L}_{2-} = \mathcal{L}_2(-\infty, 0]$: subspace of $\mathcal{L}_2(-\infty, \infty)$ with functions zero for $t > 0$.

4.2 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Spaces

Let $S \subset \mathbb{C}$ be an open set, and let $f(s)$ be a complex-valued function defined on S:

$$f(s) : S \longmapsto \mathbb{C}.$$

Then $f(s)$ is said to be *analytic at a point* z_0 in S if it is differentiable at z_0 and also at each point in some neighborhood of z_0. It is a fact that if $f(s)$ is analytic at z_0 then f has continuous derivatives of all orders at z_0. Hence, a function analytic at z_0 has a power series representation at z_0. The converse is also true (i.e., if a function has a power series at z_0, then it is analytic at z_0). A function $f(s)$ is said to be *analytic in* S if it has a derivative or is analytic at each point of S. A matrix-valued function is analytic in S if every element of the matrix is analytic in S. For example, all real rational stable transfer matrices are analytic in the right-half plane and e^{-s} is analytic everywhere.

A well-know property of the analytic functions is the so-called *maximum modulus theorem.*

Theorem 4.2 *If $f(s)$ is defined and continuous on a closed-bounded set S and analytic on the interior of S, then $|f(s)|$ cannot attain the maximum in the interior of S unless $f(s)$ is a constant.*

The theorem implies that $|f(s)|$ can only achieve its maximum on the boundary of S; that is,

$$\max_{s \in S} |f(s)| = \max_{s \in \partial S} |f(s)|$$

where ∂S denotes the boundary of S. Next we consider some frequently used complex (matrix) function spaces.

$\mathcal{L}_2(j\mathbb{R})$ Space

$\mathcal{L}_2(j\mathbb{R})$ or simply $\mathcal{L}_2$ is a Hilbert space of matrix-valued (or scalar-valued) functions on $j\mathbb{R}$ and consists of all complex matrix functions F such that the following integral is bounded:

$$\int_{-\infty}^{\infty} \operatorname{trace}\left[F^*(j\omega)F(j\omega)\right] d\omega < \infty.$$

The inner product for this Hilbert space is defined as

$$\langle F, G\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left[F^*(j\omega)G(j\omega)\right] d\omega$$

for $F, G \in \mathcal{L}_2$, and the inner product induced norm is given by

$$\|F\|_2 := \sqrt{\langle F, F\rangle}.$$

For example, all real rational strictly proper transfer matrices with no poles on the imaginary axis form a subspace (not closed) of $\mathcal{L}_2(j\mathbb{R})$ that is denoted by $\mathcal{RL}_2(j\mathbb{R})$ or simply $\mathcal{RL}_2$.

$\mathcal{H}_2$ Space[2]

$\mathcal{H}_2$ is a (closed) subspace of $\mathcal{L}_2(j\mathbb{R})$ with matrix functions $F(s)$ analytic in $\operatorname{Re}(s) > 0$ (open right-half plane). The corresponding norm is defined as

$$\|F\|_2^2 := \sup_{\sigma>0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left[F^*(\sigma+j\omega)F(\sigma+j\omega)\right] d\omega\right\}.$$

It can be shown[3] that

$$\|F\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left[F^*(j\omega)F(j\omega)\right] d\omega.$$

Hence, we can compute the norm for $\mathcal{H}_2$ just as we do for $\mathcal{L}_2$. The real rational subspace of $\mathcal{H}_2$, which consists of all strictly proper and real rational stable transfer matrices, is denoted by $\mathcal{RH}_2$.

$\mathcal{H}_2^\perp$ Space

$\mathcal{H}_2^\perp$ is the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$; that is, the (closed) subspace of functions in $\mathcal{L}_2$ that are analytic in the open left-half plane. The real rational subspace of $\mathcal{H}_2^\perp$, which consists of all strictly proper rational transfer matrices with all poles in the open right-half plane, will be denoted by $\mathcal{RH}_2^\perp$. It is easy to see that if G is a strictly proper, stable, and real rational transfer matrix, then $G \in \mathcal{H}_2$ and $G^\sim \in \mathcal{H}_2^\perp$. Most of our study in this book will be focused on the real rational case.

[2] The $\mathcal{H}_2$ space and $\mathcal{H}_\infty$ space defined in this subsection together with the $\mathcal{H}_p$ spaces, $p \geq 1$, which will not be introduced in this book, are usually called Hardy spaces and are named after the mathematician G. H. Hardy (hence the notation of $\mathcal{H}$).

[3] See Francis [1987].

The $\mathcal{L}_2$ spaces defined previously in the frequency domain can be related to the $\mathcal{L}_2$ spaces defined in the time domain. Recall the fact that a function in $\mathcal{L}_2$ space in the time domain admits a bilateral Laplace (or Fourier) transform. In fact, it can be shown that this bilateral Laplace transform yields an isometric isomorphism between the $\mathcal{L}_2$ spaces in the time domain and the $\mathcal{L}_2$ spaces in the frequency domain (this is what is called *Parseval's relations*):

$$\mathcal{L}_2(-\infty,\infty) \cong \mathcal{L}_2(j\mathbb{R})$$
$$\mathcal{L}_2[0,\infty) \cong \mathcal{H}_2$$
$$\mathcal{L}_2(-\infty,0] \cong \mathcal{H}_2^\perp .$$

As a result, if $g(t) \in \mathcal{L}_2(-\infty,\infty)$ and if its bilateral Laplace transform is $G(s) \in \mathcal{L}_2(j\mathbb{R})$, then

$$\|G\|_2 = \|g\|_2 .$$

Hence, whenever there is no confusion, the notation for functions in the time domain and in the frequency domain will be used interchangeably.

$$\begin{tikzcd}
\mathcal{L}_2[0,\infty) \arrow[r, shift left, "\text{Laplace Transform}"] & \mathcal{H}_2 \arrow[l, shift left, "\text{Inverse Transform}"] \\
\mathcal{L}_2(-\infty,\infty) \arrow[u, "P_+"] \arrow[d, "P_-"'] \arrow[r, shift left, "\text{Laplace Transform}"] & \mathcal{L}_2(j\mathbb{R}) \arrow[l, shift left, "\text{Inverse Transform}"] \arrow[u, "P_+"'] \arrow[d, "P_-"] \\
\mathcal{L}_2(-\infty,0] \arrow[r, shift left, "\text{Laplace Transform}"] & \mathcal{H}_2^\perp \arrow[l, shift left, "\text{Inverse Transform}"]
\end{tikzcd}$$

Figure 4.1: Relationships among function spaces

Define an orthogonal projection

$$P_+ : \mathcal{L}_2(-\infty,\infty) \longmapsto \mathcal{L}_2[0,\infty)$$

such that, for any function $f(t) \in \mathcal{L}_2(-\infty,\infty)$, we have $g(t) = P_+ f(t)$ with

$$g(t) := \begin{cases} f(t), & \text{for } t \ge 0; \\ 0, & \text{for } t < 0. \end{cases}$$

In this book, P_+ will also be used to denote the projection from $\mathcal{L}_2(j\mathbb{R})$ onto $\mathcal{H}_2$. Similarly, define P_- as another orthogonal projection from $\mathcal{L}_2(-\infty,\infty)$ onto $\mathcal{L}_2(-\infty,0]$ (or $\mathcal{L}_2(j\mathbb{R})$ onto $\mathcal{H}_2^\perp$). Then the relationships between $\mathcal{L}_2$ spaces and $\mathcal{H}_2$ spaces can be shown as in Figure 4.1.

Other classes of important complex matrix functions used in this book are those bounded on the imaginary axis.

$\mathcal{L}_\infty(j\mathbb{R})$ Space

$\mathcal{L}_\infty(j\mathbb{R})$ or simply $\mathcal{L}_\infty$ is a Banach space of matrix-valued (or scalar-valued) functions that are (essentially) bounded on $j\mathbb{R}$, with norm

$$\|F\|_\infty := \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

The rational subspace of $\mathcal{L}_\infty$, denoted by $\mathcal{RL}_\infty(j\mathbb{R})$ or simply $\mathcal{RL}_\infty$, consists of all proper and real rational transfer matrices with no poles on the imaginary axis.

$\mathcal{H}_\infty$ Space

$\mathcal{H}_\infty$ is a (closed) subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane. The $\mathcal{H}_\infty$ norm is defined as

$$\|F\|_\infty := \sup_{\mathrm{Re}(s)>0} \overline{\sigma}\left[F(s)\right] = \sup_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

The second equality can be regarded as a generalization of the maximum modulus theorem for matrix functions. (See Boyd and Desoer [1985] for a proof.) The real rational subspace of $\mathcal{H}_\infty$ is denoted by $\mathcal{RH}_\infty$, which consists of all proper and real rational stable transfer matrices.

$\mathcal{H}_\infty^-$ Space

$\mathcal{H}_\infty^-$ is a (closed) subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open left-half plane. The $\mathcal{H}_\infty^-$ norm is defined as

$$\|F\|_\infty := \sup_{\mathrm{Re}(s)<0} \overline{\sigma}\left[F(s)\right] = \sup_{\omega\in\mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

The real rational subspace of $\mathcal{H}_\infty^-$ is denoted by $\mathcal{RH}_\infty^-$, which consists of all proper, real rational, antistable transfer matrices (i.e., functions with all poles in the open right-half plane).

Let $G(s) \in \mathcal{L}_\infty$ be a $p \times q$ transfer matrix. Then a multiplication operator is defined as

$$M_G : \mathcal{L}_2 \longmapsto \mathcal{L}_2$$

$$M_G f := Gf.$$

In writing the preceding mapping, we have assumed that f has a compatible dimension. A more accurate description of the foregoing operator should be

$$M_G : \mathcal{L}_2^q \longmapsto \mathcal{L}_2^p.$$

That is, f is a q-dimensional vector function with each component in $\mathcal{L}_2$. However, we shall suppress all dimensions in this book and assume that all objects have compatible dimensions.

A useful fact about the multiplication operator is that the norm of a matrix G in $\mathcal{L}_\infty$ equals the norm of the corresponding multiplication operator.

Theorem 4.3 *Let $G \in \mathcal{L}_\infty$ be a $p \times q$ transfer matrix. Then $\|M_G\| = \|G\|_\infty$.*

Remark 4.1 It is also true that this operator norm equals the norm of the operator restricted to $\mathcal{H}_2$ (or $\mathcal{H}_2^\perp$); that is,

$$\|M_G\| = \|M_G|_{\mathcal{H}_2}\| := \sup\left\{\|Gf\|_2 : \ f \in \mathcal{H}_2, \ \|f\|_2 \le 1\right\}.$$

This will be clear in the proof where an $f \in \mathcal{H}_2$ is constructed. $\diamond$

Proof. By definition, we have

$$\|M_G\| = \sup\left\{\|Gf\|_2 : \ f \in \mathcal{L}_2, \ \|f\|_2 \le 1\right\}.$$

First we see that $\|G\|_\infty$ is an upper bound for the operator norm:

$$\begin{aligned}
\|Gf\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(j\omega)G^*(j\omega)G(j\omega)f(j\omega)\ d\omega \\
&\le \|G\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(j\omega)\|^2\ d\omega \\
&= \|G\|_\infty^2 \|f\|_2^2.
\end{aligned}$$

To show that $\|G\|_\infty$ is the least upper bound, first choose a frequency ω_0 where $\overline{\sigma}\left[G(j\omega)\right]$ is maximum; that is,

$$\overline{\sigma}\left[G(j\omega_0)\right] = \|G\|_\infty,$$

and denote the singular value decomposition of $G(j\omega_0)$ by

$$G(j\omega_0) = \overline{\sigma} u_1(j\omega_0)v_1^*(j\omega_0) + \sum_{i=2}^{r} \sigma_i u_i(j\omega_0)v_i^*(j\omega_0)$$

where r is the rank of $G(j\omega_0)$ and u_i, v_i have unit length.

Next we assume that $G(s)$ has real coefficients and we shall construct a function $f(s) \in \mathcal{H}_2$ with real coefficients so that the norm is approximately achieved. [It will be clear in the following that the proof is much simpler if f is allowed to have complex coefficients, which is necessary when $G(s)$ has complex coefficients.]

If $\omega_0 < \infty$, write $v_1(j\omega_0)$ as

$$v_1(j\omega_0) = \begin{bmatrix} \alpha_1 e^{j\theta_1} \\ \alpha_2 e^{j\theta_2} \\ \vdots \\ \alpha_q e^{j\theta_q} \end{bmatrix}$$

where $\alpha_i \in \mathbb{R}$ is such that $\theta_i \in (-\pi, 0]$ and q is the column dimension of G. Now let $0 \le \beta_i \le \infty$ be such that

$$\theta_i = \angle\left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0}\right)$$

(with $\beta_i \to \infty$ if $\theta_i = 0$) and let f be given by

$$f(s) = \begin{bmatrix} \alpha_1 \frac{\beta_1 - s}{\beta_1 + s} \\ \alpha_2 \frac{\beta_2 - s}{\beta_2 + s} \\ \vdots \\ \alpha_q \frac{\beta_q - s}{\beta_q + s} \end{bmatrix} \hat{f}(s)$$

(with 1 replacing $\frac{\beta_i - s}{\beta_i + s}$ if $\theta_i = 0$), where a scalar function $\hat{f}$ is chosen so that

$$|\hat{f}(j\omega)| = \begin{cases} c & \text{if } |\omega - \omega_0| < \epsilon \text{ or } |\omega + \omega_0| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

where ϵ is a small positive number and c is chosen so that $\hat{f}$ has unit 2-norm (i.e., $c = \sqrt{\pi/2\epsilon}$). This, in turn, implies that f has unit 2-norm. Then

$$\begin{aligned} \|Gf\|_2^2 &\approx \frac{1}{2\pi}\left[\overline{\sigma}\left[G(-j\omega_0)\right]^2 \pi + \overline{\sigma}\left[G(j\omega_0)\right]^2 \pi\right] \\ &= \overline{\sigma}\left[G(j\omega_0)\right]^2 = \|G\|_\infty^2. \end{aligned}$$

Similarly, if $\omega_0 = \infty$, the conclusion follows by letting $\omega_0 \to \infty$ in the foregoing. □

Illustrative MATLAB Commands:

≫ **[sv, w]=sigma(A, B, C, D);** % frequency response of the singular values; or

≫ **w=logspace(l, h, n); sv=sigma(A, B, C, D, w);** % n points between 10^l and 10^h.

Related MATLAB Commands: semilogx, semilogy, bode, freqs, nichols, frsp, vsvd, vplot, pkvnorm

4.3 Computing $\mathcal{L}_2$ and $\mathcal{H}_2$ Norms

Let $G(s) \in \mathcal{L}_2$ and recall that the $\mathcal{L}_2$ norm of G is defined as

$$\begin{aligned} \|G\|_2 &:= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\{G^*(j\omega)G(j\omega)\}\, d\omega} \\ &= \|g\|_2 \\ &= \sqrt{\int_{-\infty}^{\infty} \operatorname{trace}\{g^*(t)g(t)\}\, dt} \end{aligned}$$

where $g(t)$ denotes the convolution kernel of G.

It is easy to see that the $\mathcal{L}_2$ norm defined previously is finite iff the transfer matrix G is strictly proper; that is, $G(\infty) = 0$. Hence, we will generally assume that the transfer matrix is strictly proper whenever we refer to the $\mathcal{L}_2$ norm of G (of course, this also applies to $\mathcal{H}_2$ norms). One straightforward way of computing the $\mathcal{L}_2$ norm is to use contour integral. Suppose G is strictly proper; then we have

$$\begin{aligned} \|G\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\{G^*(j\omega)G(j\omega)\}\, d\omega \\ &= \frac{1}{2\pi j}\oint \operatorname{trace}\{G^\sim(s)G(s)\}\, ds. \end{aligned}$$

The last integral is a contour integral along the imaginary axis and around an infinite semicircle in the left-half plane; the contribution to the integral from this semicircle equals zero because G is strictly proper. By the residue theorem, $\|G\|_2^2$ equals the sum of the residues of $\operatorname{trace}\{G^\sim(s)G(s)\}$ at its poles in the left-half plane.

Although $\|G\|_2$ can, in principle, be computed from its definition or from the method just suggested, it is useful in many applications to have alternative characterizations and to take advantage of the state-space representations of G. The computation of a $\mathcal{RH}_2$ transfer matrix norm is particularly simple.

Lemma 4.4 *Consider a transfer matrix*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

with A stable. Then we have

$$\|G\|_2^2 = \operatorname{trace}(B^*QB) = \operatorname{trace}(CPC^*) \tag{4.1}$$

where Q and P are observability and controllability Gramians that can be obtained from the following Lyapunov equations:

$$AP + PA^* + BB^* = 0 \qquad A^*Q + QA + C^*C = 0.$$

Proof. Since G is stable, we have

$$g(t) = \mathcal{L}^{-1}(G) = \begin{cases} Ce^{At}B, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and

$$\begin{aligned} \|G\|_2^2 &= \int_0^\infty \text{trace}\{g^*(t)g(t)\}\ dt = \int_0^\infty \text{trace}\{g(t)g(t)^*\}\ dt \\ &= \int_0^\infty \text{trace}\{B^*e^{A^*t}C^*Ce^{At}B\}\ dt = \int_0^\infty \text{trace}\{Ce^{At}BB^*e^{A^*t}C^*\}\ dt. \end{aligned}$$

The lemma follows from the fact that the controllability Gramian of (A, B) and the observability Gramian of (C, A) can be represented as

$$Q = \int_0^\infty e^{A^*t}C^*Ce^{At}\ dt, \quad P = \int_0^\infty e^{At}BB^*e^{A^*t}\ dt,$$

which can also be obtained from

$$AP + PA^* + BB^* = 0 \qquad A^*Q + QA + C^*C = 0.$$

□

To compute the $\mathcal{L}_2$ norm of a rational transfer function, $G(s) \in \mathcal{RL}_2$, using the state-space approach, let $G(s) = [G(s)]_+ + [G(s)]_-$ with $G_+ \in \mathcal{RH}_2$ and $G_- \in \mathcal{RH}_2^\perp$; then

$$\|G\|_2^2 = \|[G(s)]_+\|_2^2 + \|[G(s)]_-\|_2^2$$

where $\|[G(s)]_+\|_2$ and $\|[G(s)]_-\|_2 = \|[G(-s)]_+\|_2 = \|([G(s)]_-)^\sim\|_2$ can be computed using the preceding lemma.

Still another useful characterization of the $\mathcal{H}_2$ norm of G is in terms of hypothetical input-output experiments. Let e_i denote the ith standard basis vector of $\mathbb{R}^m$, where m is the input dimension of the system. Apply the impulsive input $\delta(t)e_i$ [$\delta(t)$ is the unit impulse] and denote the output by $z_i(t)(= g(t)e_i)$. Assume $D = 0$; then $z_i \in \mathcal{L}_{2+}$ and

$$\|G\|_2^2 = \sum_{i=1}^m \|z_i\|_2^2.$$

Note that this characterization of the $\mathcal{H}_2$ norm can be appropriately generalized for nonlinear time-varying systems; see Chen and Francis [1992] for an application of this norm in sampled-data control.

Example 4.1 Consider a transfer matrix

$$G = \begin{bmatrix} \dfrac{3(s+3)}{(s-1)(s+2)} & \dfrac{2}{s-1} \\ \dfrac{s+1}{(s+2)(s+3)} & \dfrac{1}{s-4} \end{bmatrix} = G_s + G_u$$

with

$$G_s = \left[\begin{array}{cc|cc} -2 & 0 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad G_u = \left[\begin{array}{cc|cc} 1 & 0 & 4 & 2 \\ 0 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Then the command $\mathbf{h2norm(G_s)}$ gives $\|G_s\|_2 = 0.6055$ and $\mathbf{h2norm(cjt(G_u))}$ gives $\|G_u\|_2 = 3.182$. Hence $\|G\|_2 = \sqrt{\|G_s\|_2^2 + \|G_u\|_2^2} = 3.2393$.

Illustrative MATLAB Commands:

$\gg$ $\mathbf{P = gram(A, B)}$; $\mathbf{Q = gram(A', C')}$; **or** $\mathbf{P = lyap(A, B * B')}$;

$\gg$ $\mathbf{[G_s, G_u] = sdecomp(G)}$; % decompose into stable and antistable parts.

4.4 Computing $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ Norms

We shall first consider, as in the $\mathcal{L}_2$ case, how to compute the ∞ norm of an $\mathcal{RL}_\infty$ transfer matrix. Let $G(s) \in \mathcal{RL}_\infty$ and recall that the $\mathcal{L}_\infty$ norm of a matrix rational transfer function G is defined as

$$\|G\|_\infty := \sup_\omega \overline{\sigma}\{G(j\omega)\}.$$

The computation of the $\mathcal{L}_\infty$ norm of G is complicated and requires a search. A control engineering interpretation of the infinity norm of a scalar transfer function G is the distance in the complex plane from the origin to the farthest point on the Nyquist plot of G, and it also appears as the peak value on the Bode magnitude plot of $|G(j\omega)|$. Hence the ∞ norm of a transfer function can, in principle, be obtained graphically.

To get an estimate, set up a fine grid of frequency points:

$$\{\omega_1, \cdots, \omega_N\}.$$

Then an estimate for $\|G\|_\infty$ is

$$\max_{1 \le k \le N} \overline{\sigma}\{G(j\omega_k)\}.$$

This value is usually read directly from a Bode singular value plot. The $\mathcal{RL}_\infty$ norm can also be computed in state-space.

Lemma 4.5 *Let* $\gamma > 0$ *and*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RL}_\infty. \tag{4.2}$$

Then $\|G\|_\infty < \gamma$ *if and only if* $\overline{\sigma}(D) < \gamma$ *and the Hamiltonian matrix* H *has no eigenvalues on the imaginary axis where*

$$H := \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix} \tag{4.3}$$

and $R = \gamma^2 I - D^*D$.

Proof. Let $\Phi(s) = \gamma^2 I - G^\sim(s)G(s)$. Then it is clear that $\|G\|_\infty < \gamma$ if and only if $\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$. Since $\Phi(\infty) = R > 0$ and since $\Phi(j\omega)$ is a continuous function of ω, $\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$ if and only if $\Phi(j\omega)$ is nonsingular for all $\omega \in \mathbb{R} \cup \{\infty\}$; that is, $\Phi(s)$ has no imaginary axis zero. Equivalently, $\Phi^{-1}(s)$ has no imaginary axis pole. It is easy to compute by some simple algebra that

$$\Phi^{-1}(s) = \left[\begin{array}{c|c} H & \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix} & R^{-1} \end{array}\right].$$

Thus the conclusion follows if the above realization has neither uncontrollable modes nor unobservable modes on the imaginary axis. Assume that $j\omega_0$ is an eigenvalue of H but not a pole of $\Phi^{-1}(s)$. Then $j\omega_0$ must be either an unobservable mode of $(\begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix}, H)$ or an uncontrollable mode of $(H, \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix})$. Now suppose $j\omega_0$ is an unobservable mode of $(\begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix}, H)$. Then there exists an $x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$ such that

$$Hx_0 = j\omega_0 x_0, \ \begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix} x_0 = 0.$$

These equations can be simplified to

$$\begin{aligned} (j\omega_0 I - A)x_1 &= 0 \\ (j\omega_0 I + A^*)x_2 &= -C^*Cx_1 \\ D^*Cx_1 + B^*x_2 &= 0. \end{aligned}$$

Since A has no imaginary axis eigenvalues, we have $x_1 = 0$ and $x_2 = 0$. This contradicts our assumption, and hence the realization has no unobservable modes on the imaginary axis.

Similarly, a contradiction will also be arrived at if $j\omega_0$ is assumed to be an uncontrollable mode of $(H, \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix})$. □

Bisection Algorithm

Lemma 4.5 suggests the following bisection algorithm to compute $\mathcal{RL}_\infty$ norm:

(a) Select an upper bound γ_u and a lower bound γ_l such that $\gamma_l \leq \|G\|_\infty \leq \gamma_u$;

(b) If $(\gamma_u - \gamma_l)/\gamma_l \leq$specified level, stop; $\|G\| \approx (\gamma_u + \gamma_l)/2$. Otherwise go to the next step;

(c) Set $\gamma = (\gamma_l + \gamma_u)/2$;

(d) Test if $\|G\|_\infty < \gamma$ by calculating the eigenvalues of H for the given γ;

(e) If H has an eigenvalue on $j\mathbb{R}$, set $\gamma_l = \gamma$; otherwise set $\gamma_u = \gamma$; go back to step (b).

Of course, the above algorithm applies to $\mathcal{H}_\infty$ norm computation as well. Thus $\mathcal{L}_\infty$ norm computation requires a search, over either γ or ω, in contrast to $\mathcal{L}_2$ $(\mathcal{H}_2)$ norm computation, which does not. A somewhat analogous situation occurs for constant matrices with the norms $\|M\|_2^2 = \text{trace}(M^*M)$ and $\|M\|_\infty = \overline{\sigma}[M]$. In principle, $\|M\|_2^2$ can be computed exactly with a finite number of operations, as can the test for whether $\overline{\sigma}(M) < \gamma$ (e.g., $\gamma^2 I - M^*M > 0$), but the value of $\overline{\sigma}(M)$ cannot. To compute $\overline{\sigma}(M)$, we must use some type of iterative algorithm.

Remark 4.2 It is clear that $\|G\|_\infty < \gamma$ iff $\left\|\gamma^{-1}G\right\|_\infty < 1$. Hence, there is no loss of generality in assuming $\gamma = 1$. This assumption will often be made in the remainder of this book. It is also noted that there are other fast algorithms to carry out the preceding norm computation; nevertheless, this bisection algorithm is the simplest. $\diamond$

Additional interpretations can be given for the $\mathcal{H}_\infty$ norm of a stable matrix transfer function. When $G(s)$ is a single-input and single-output system, the $\mathcal{H}_\infty$ norm of the $G(s)$ can be regarded as the largest possible amplification factor of the system's steady-state response to sinusoidal excitations. For example, the steady-state response of the system with respect to a sinusoidal input $u(t) = U\sin(\omega_0 t + \phi)$ is

$$y(t) = U|G(j\omega_0)|\sin(\omega_0 t + \phi + \angle G(j\omega_0))$$

and thus the maximum possible amplification factor is $\sup_{\omega_0} |G(j\omega_0)|$, which is precisely the $\mathcal{H}_\infty$ norm of the transfer function.

In the multiple-input and multiple-output case, the $\mathcal{H}_\infty$ norm of a transfer matrix $G \in \mathcal{RH}_\infty$ can also be regarded as the largest possible amplification factor of the system's steady-state response to sinusoidal excitations in the following sense: Let the sinusoidal inputs be

$$u(t) = \begin{bmatrix} u_1\sin(\omega_0 t + \phi_1) \\ u_2\sin(\omega_0 t + \phi_2) \\ \vdots \\ u_q\sin(\omega_0 t + \phi_q) \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}.$$

Then the steady-state response of the system can be written as

$$y(t) = \begin{bmatrix} y_1 \sin(\omega_0 t + \theta_1) \\ y_2 \sin(\omega_0 t + \theta_2) \\ \vdots \\ y_p \sin(\omega_0 t + \theta_p) \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

for some $y_i,\ \theta_i,\ i = 1, 2, \ldots, p$, and furthermore,

$$\|G\|_\infty = \sup_{\phi_i, \omega_o, \hat{u}} \frac{\|\hat{y}\|}{\|\hat{u}\|}$$

where $\|\cdot\|$ is the Euclidean norm. The details are left as an exercise.

Example 4.2 Consider a mass/spring/damper system as shown in Figure 4.2.

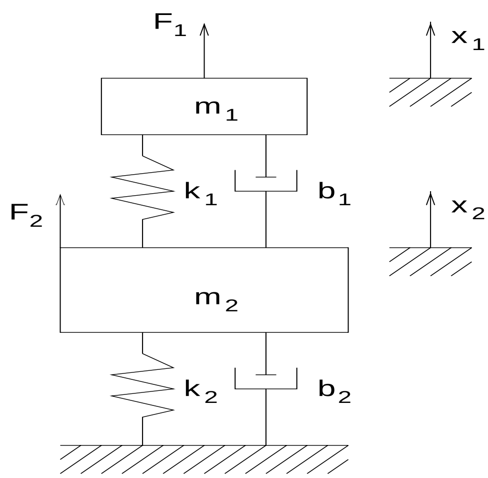

Figure 4.2: A two-mass/spring/damper system

The dynamical system can be described by the following differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + B \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

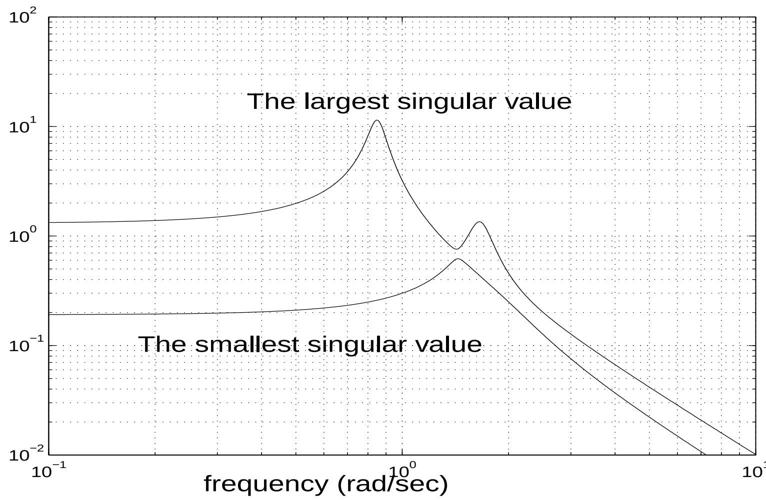

Figure 4.3: $\|G\|_\infty$ is the peak of the largest singular value of $G(j\omega)$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}.$$

Suppose that $G(s)$ is the transfer matrix from (F_1, F_2) to (x_1, x_2); that is,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = 0,$$

and suppose $k_1 = 1$, $k_2 = 4$, $b_1 = 0.2$, $b_2 = 0.1$, $m_1 = 1$, and $m_2 = 2$ with appropriate units. The following MATLAB commands generate the singular value Bode plot of the above system as shown in Figure 4.3.

$\gg$ **G=pck(A,B,C,D);**

$\gg$ **hinfnorm(G,0.0001)** or **linfnorm(G,0.0001)** % relative error ≤ 0.0001

$\gg$ **w=logspace(-1,1,200);** % **200** points between $1 = 10^{\mathbf{-1}}$ and $10 = 10^{\mathbf{1}}$;

$\gg$ **Gf=frsp(G,w);** % computing frequency response;

$\gg$ **[u,s,v]=vsvd(Gf);** % SVD at each frequency;

$\gg$ **vplot('liv, lm', s),** **grid** % plot both singular values and grid.

Then the $\mathcal{H}_\infty$ norm of this transfer matrix is $\|G(s)\|_\infty = 11.47$, which is shown as the peak of the largest singular value Bode plot in Figure 4.3. Since the peak is achieved at $\omega_{\max} = 0.8483$, exciting the system using the following sinusoidal input

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0.9614 \sin(0.8483t) \\ 0.2753 \sin(0.8483t - 0.12) \end{bmatrix}$$

gives the steady-state response of the system as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11.47 \times 0.9614 \sin(0.8483t - 1.5483) \\ 11.47 \times 0.2753 \sin(0.8483t - 1.4283) \end{bmatrix}.$$

This shows that the system response will be amplified 11.47 times for an input signal at the frequency $\omega_{\max}$, which could be undesirable if F_1 and F_2 are disturbance force and x_1 and x_2 are the positions to be kept steady.

Example 4.3 Consider a two-by-two transfer matrix

$$G(s) = \begin{bmatrix} \dfrac{10(s+1)}{s^2 + 0.2s + 100} & \dfrac{1}{s+1} \\ \dfrac{s+2}{s^2 + 0.1s + 10} & \dfrac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}.$$

A state-space realization of G can be obtained using the following MATLAB commands:

$\gg$ **G11=nd2sys([10,10],[1,0.2,100]);**

$\gg$ **G12=nd2sys(1,[1,1]);**

$\gg$ **G21=nd2sys([1,2],[1,0.1,10]);**

$\gg$ **G22=nd2sys([5,5],[1,5,6]);**

$\gg$ **G=sbs(abv(G11,G21),abv(G12,G22));**

Next, we set up a frequency grid to compute the frequency response of G and the singular values of $G(j\omega)$ over a suitable range of frequency.

$\gg$ **w=logspace(0,2,200);** % **200** points between $1 = 10^{\mathbf{0}}$ and $100 = 10^{\mathbf{2}}$;

$\gg$ **Gf=frsp(G,w);** % computing frequency response;

$\gg$ **[u,s,v]=vsvd(Gf);** % SVD at each frequency;

$\gg$ **vplot('liv, lm', s)**, **grid** % plot both singular values and grid;

$\gg$ **pkvnorm(s)** % find the norm from the frequency response of the singular values.

The singular values of $G(j\omega)$ are plotted in Figure 4.4, which gives an estimate of $\|G\|_\infty \approx 32.861$. The state-space bisection algorithm described previously leads to $\|G\|_\infty = 50.25 \pm 0.01$ and the corresponding MATLAB command is

$\gg$ **hinfnorm(G,0.0001)** or **linfnorm(G,0.0001)** % relative error ≤ 0.0001.

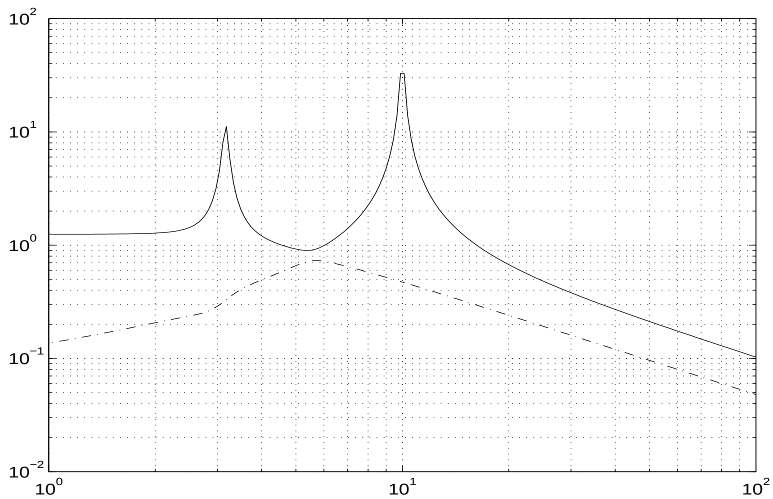

Figure 4.4: The largest and the smallest singular values of $G(j\omega)$

The preceding computational results show clearly that the graphical method can lead to a wrong answer for a lightly damped system if the frequency grid is not sufficiently dense. Indeed, we would get $\|G\|_\infty \approx 43.525, 48.286$ and 49.737 from the graphical method if $400, 800$, and 1600 frequency points are used, respectively.

Related MATLAB Commands: linfnorm, vnorm, getiv, scliv, var2con, xtract, xtracti

4.5 Notes and References

The basic concept of function spaces presented in this chapter can be found in any standard functional analysis textbook, for instance, Naylor and Sell [1982] and Gohberg and Goldberg [1981]. The system theoretical interpretations of the norms and function spaces can be found in Desoer and Vidyasagar [1975]. The bisection $\mathcal{L}_\infty$ norm computational algorithm was first developed in Boyd, Balakrishnan, and Kabamba [1989]. A more efficient $\mathcal{L}_\infty$ norm computational algorithm is presented in Bruinsma and Steinbuch [1990].

4.6 Problems

Problem 4.1 Let $G(s)$ be a matrix in $\mathcal{RH}_\infty$. Prove that

$$\left\| \begin{bmatrix} G \\ I \end{bmatrix} \right\|_\infty^2 = \|G\|_\infty^2 + 1.$$

Problem 4.2 (Parseval relation) Let $f(t), g(t) \in \mathcal{L}_2$, $F(j\omega) = \mathcal{F}\{f(t)\}$, and $G(j\omega) = \mathcal{F}\{g(t)\}$. Show that

$$\int_{-\infty}^{\infty} f(t)g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)G^*(j\omega)d\omega$$

and

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega.$$

Note that

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \quad f(t) = \mathcal{F}^{-1}(F(j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega.$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Problem 4.3 Suppose A is stable. Show

$$\int_{-\infty}^{\infty} (j\omega I - A)^{-1} d\omega = \pi I.$$

Suppose $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \mathcal{RH}_\infty$ and let $Q = Q^*$ be the observability Gramian. Use the above formula to show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} G^\sim(j\omega)G(j\omega)d\omega = B^*QB.$$

[Hint: Use the fact that $G^\sim(s)G(s) = F^\sim(s) + F(s)$ and $F(s) = B^*Q(sI - A)^{-1}B$.]

Problem 4.4 Compute the 2-norm and ∞-norm of the following systems:

$$G_1(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{s+3}{(s+1)(s-2)} \\ \dfrac{10}{s-2} & \dfrac{5}{s+3} \end{bmatrix}, \quad G_2(s) = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 3 & 1 \\ \hline 1 & 2 & 0 \end{array}\right]$$

$$G_3(s) = \left[\begin{array}{cc|c} -1 & -2 & 1 \\ 1 & 0 & 0 \\ \hline 2 & 3 & 0 \end{array}\right], \quad G_4(s) = \left[\begin{array}{ccc|cc} -1 & -2 & -3 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{array}\right].$$

Problem 4.5 Let $r(t) = \sin \omega t$ be the input signal to a plant

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

with $0 < \xi < 1/\sqrt{2}$. Find the steady-state response of the system $y(t)$. Also find the frequency ω that gives the largest magnitude steady-state response of $y(t)$.

Problem 4.6 Let $G(s) \in \mathcal{RH}_\infty$ be a $p \times q$ transfer matrix and $y = G(s)u$. Suppose

$$u(t) = \begin{bmatrix} u_1 \sin(\omega_0 t + \phi_1) \\ u_2 \sin(\omega_0 t + \phi_2) \\ \vdots \\ u_q \sin(\omega_0 t + \phi_q) \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}.$$

Show that the steady-state response of the system is given by

$$y(t) = \begin{bmatrix} y_1 \sin(\omega_0 t + \theta_1) \\ y_2 \sin(\omega_0 t + \theta_2) \\ \vdots \\ y_p \sin(\omega_0 t + \theta_p) \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

for some y_i and θ_i, $i = 1, 2, \ldots, p$. Show that $\sup_{\phi_i, \omega_o, \|\hat{u}\|_2 \le 1} \|\hat{y}\|_2 = \|G\|_\infty$.

Problem 4.7 Write a MATLAB program to plot, versus γ, the distance from the imaginary axis to the nearest eigenvalue of the Hamiltonian matrix for a given state-space model with stable A. Try it on

$$\begin{bmatrix} \dfrac{s+1}{(s+2)(s+3)} & \dfrac{s}{s+1} \\[2ex] \dfrac{s^2-2}{(s+3)(s+4)} & \dfrac{s+4}{(s+1)(s+2)} \end{bmatrix}.$$

Read off the value of the $\mathcal{H}_\infty$-norm. Compare with the MATLAB function *hinfnorm*.

Problem 4.8 Let $G(s) = \dfrac{1}{(s^2 + 2\xi s + 1)(s+1)}$. Compute $\|G(s)\|_\infty$ using the Bode plot and state-space algorithm, respectively for $\xi = 1, 0.1, 0.01, 0.001$ and compare the results.