Chapter 9

Linear Fractional Transformation

This chapter introduces a new matrix function: linear fractional transformation (LFT). We show that many interesting control problems can be formulated in an LFT framework and thus can be treated using the same technique.

9.1 Linear Fractional Transformations

This section introduces the matrix linear fractional transformations. It is well known from the one-complex-variable function theory that a mapping $F : \mathbb{C} \mapsto \mathbb{C}$ of the form

$$F(s) = \frac{a + bs}{c + ds}$$

with a, b, c, and $d \in \mathbb{C}$ is called a *linear fractional transformation*. In particular, if $c \neq 0$ then $F(s)$ can also be written as

$$F(s) = \alpha + \beta s(1 - \gamma s)^{-1}$$

for some α, β and $\gamma \in \mathbb{C}$. The linear fractional transformation described above for scalars can be generalized to the matrix case.

Definition 9.1 Let M be a complex matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)},$$

and let $\Delta_\ell \in \mathbb{C}^{q_2 \times p_2}$ and $\Delta_u \in \mathbb{C}^{q_1 \times p_1}$ be two other complex matrices. Then we can formally define a *lower LFT* with respect to Δ_ℓ as the map

$$\mathcal{F}_\ell(M, \bullet) : \ \mathbb{C}^{q_2 \times p_2} \mapsto \mathbb{C}^{p_1 \times q_1}$$

with

$$\mathcal{F}_\ell(M, \Delta_\ell) := M_{11} + M_{12}\Delta_\ell(I - M_{22}\Delta_\ell)^{-1}M_{21}$$

provided that the inverse $(I - M_{22}\Delta_\ell)^{-1}$ exists. We can also define an *upper LFT* with respect to Δ_u as

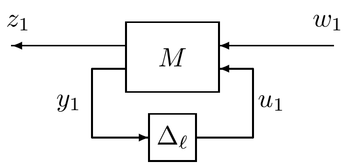
$$\mathcal{F}_u(M, \bullet) : \ \mathbb{C}^{q_1 \times p_1} \mapsto \mathbb{C}^{p_2 \times q_2}$$

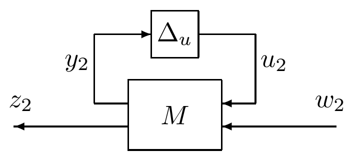
with

$$\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$

provided that the inverse $(I - M_{11}\Delta_u)^{-1}$ exists.

The matrix M in the preceding LFTs is called the *coefficient matrix*. The motivation for the terminologies of *lower* and *upper* LFTs should be clear from the following diagram representations of $\mathcal{F}_\ell(M, \Delta_\ell)$ and $\mathcal{F}_u(M, \Delta_u)$:





The diagram on the left represents the following set of equations:

$$\begin{aligned} \begin{bmatrix} z_1 \\ y_1 \end{bmatrix} &= M \begin{bmatrix} w_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ u_1 \end{bmatrix}, \\ u_1 &= \Delta_\ell \, y_1 \end{aligned}$$

while the diagram on the right represents

$$\begin{aligned} \begin{bmatrix} y_2 \\ z_2 \end{bmatrix} &= M \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u_2 \\ w_2 \end{bmatrix}, \\ u_2 &= \Delta_u \, y_2. \end{aligned}$$

It is easy to verify that the mapping defined on the left diagram is equal to $\mathcal{F}_\ell(M, \Delta_\ell)$ and the mapping defined on the right diagram is equal to $\mathcal{F}_u(M, \Delta_u)$. So from the above diagrams, $\mathcal{F}_\ell(M, \Delta_\ell)$ is a transformation obtained from closing the *lower* loop on the left diagram; similarly, $\mathcal{F}_u(M, \Delta_u)$ is a transformation obtained from closing the *upper* loop on the right diagram. In most cases, we shall use the general term *LFT* in referring to both upper and lower LFTs and assume that the context will distinguish the situations since one can use either of these notations to express a given object. Indeed, it is clear that $\mathcal{F}_u(N, \Delta) = \mathcal{F}_\ell(M, \Delta)$ with $N = \begin{bmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{bmatrix}$. It is usually not crucial which expression is used; however, it is often the case that one expression is more convenient than the other for a given problem. It should also be clear to the reader that in writing $\mathcal{F}_\ell(M, \Delta)$ [or $\mathcal{F}_u(M, \Delta)$] it is implied that Δ has compatible dimensions.

A useful interpretation of an LFT [e.g., $\mathcal{F}_\ell(M,\Delta)$] is that $\mathcal{F}_\ell(M,\Delta)$ has a nominal mapping, M_{11}, and is perturbed by Δ, while M_{12}, M_{21}, and M_{22} reflect a prior knowledge as to how the perturbation affects the nominal map, M_{11}. A similar interpretation can be applied to $\mathcal{F}_u(M,\Delta)$. This is why LFT is particularly useful in the study of perturbations, which is the focus of the next chapter.

The physical meaning of an LFT in control science is obvious if we take M as a proper transfer matrix. In that case, the LFTs defined previously are simply the closed-loop transfer matrices from $w_1 \mapsto z_1$ and $w_2 \mapsto z_2$, respectively; that is,

$$T_{zw1} = \mathcal{F}_\ell(M,\Delta_\ell), \qquad T_{zw2} = \mathcal{F}_u(M,\Delta_u)$$

where M may be the controlled plant and Δ may be either the system model uncertainties or the controllers.

Definition 9.2 An LFT, $\mathcal{F}_\ell(M,\Delta)$, is said to be *well-defined (or well-posed)* if $(I - M_{22}\Delta)$ is invertible.

Note that this definition is consistent with the well-posedness definition of the feedback system, which requires that the corresponding transfer matrix be invertible in $\mathcal{R}_p(s)$. It is clear that the study of an LFT that is not well-defined is meaningless; hence throughout this book, whenever an LFT is invoked, it will be assumed implicitly that it is well-defined. It is also clear from the definition that, for any M, $\mathcal{F}_\ell(M,0)$ is well-defined; hence any function that is not well-defined at the origin cannot be expressed as an LFT in terms of its variables. For example, $f(\delta) = 1/\delta$ is not an LFT of δ.

In some literature, LFT is used to refer to the following matrix functions:

$$(A+BQ)(C+DQ)^{-1} \qquad \text{or} \qquad (C+QD)^{-1}(A+QB)$$

where C is usually assumed to be invertible due to practical consideration. The following results follow from some simple algebra.

Lemma 9.1 *Suppose C is invertible. Then*

$$\begin{aligned} (A+BQ)(C+DQ)^{-1} &= \mathcal{F}_\ell(M,Q) \\ (C+QD)^{-1}(A+QB) &= \mathcal{F}_\ell(N,Q) \end{aligned}$$

with

$$M = \begin{bmatrix} AC^{-1} & B - AC^{-1}D \\ C^{-1} & -C^{-1}D \end{bmatrix}, \quad N = \begin{bmatrix} C^{-1}A & C^{-1} \\ B - DC^{-1}A & -DC^{-1} \end{bmatrix}.$$

The converse also holds if M satisfies certain conditions.

Lemma 9.2 *Let $\mathcal{F}_\ell(M,Q)$ be a given LFT with* $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$.

(a) If M_{12} is invertible, then

$$\mathcal{F}_\ell(M,Q) = (C+QD)^{-1}(A+QB)$$

with $A = M_{12}^{-1}M_{11}$, $B = M_{21} - M_{22}M_{12}^{-1}M_{11}$, $C = M_{12}^{-1}$, and $D = -M_{22}M_{12}^{-1}$; that is,

$$\begin{aligned} \begin{bmatrix} A & C \\ B & D \end{bmatrix} &= \mathcal{F}_\ell\left(\left[\begin{array}{cc:c} 0 & 0 & -I \\ M_{21} & 0 & M_{22} \\ \hdashline M_{11} & I & 0 \end{array}\right], -M_{12}^{-1} \right) \\ &= \mathcal{F}_\ell\left(\left[\begin{array}{cc:c} 0 & 0 & -I \\ M_{21} & 0 & M_{22} \\ \hdashline M_{11} & I & M_{12}+E \end{array}\right], E^{-1} \right) \end{aligned}$$

for any nonsingular matrix E.

(b) If M_{21} is invertible, then

$$\mathcal{F}_\ell(M,Q) = (A+BQ)(C+DQ)^{-1}$$

with $A = M_{11}M_{21}^{-1}$, $B = M_{12} - M_{11}M_{21}^{-1}M_{22}$, $C = M_{21}^{-1}$, and $D = -M_{21}^{-1}M_{22}$; that is,

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \mathcal{F}_\ell\left(\left[\begin{array}{cc:c} 0 & M_{12} & M_{11} \\ 0 & 0 & I \\ \hdashline -I & M_{22} & 0 \end{array}\right], -M_{21}^{-1} \right) \\ &= \mathcal{F}_\ell\left(\left[\begin{array}{cc:c} 0 & M_{12} & M_{11} \\ 0 & 0 & I \\ \hdashline -I & M_{22} & M_{21}+E \end{array}\right], E^{-1} \right) \end{aligned}$$

for any nonsingular matrix E.

However, for an arbitrary LFT $\mathcal{F}_\ell(M,Q)$, neither M_{21} nor M_{12} is necessarily square and invertible; therefore, the alternative fractional formula is more restrictive.

It should be pointed out that some seemingly simple functions do not have simple LFT representations. For example,

$$(A+QB)(I+QD)^{-1}$$

cannot always be written in the form of $\mathcal{F}_\ell(M,Q)$ for some M; however, it can be written as

$$(A+QB)(I+QD)^{-1} = \mathcal{F}_\ell(N,\Delta)$$

with

$$N = \left[\begin{array}{c|cc} A & I & A \\ \hline -B & 0 & -B \\ D & 0 & D \end{array}\right], \qquad \Delta = \left[\begin{array}{cc} Q & \\ & Q \end{array}\right].$$

Note that the dimension of Δ is twice of Q.

The following lemma shows that the inverse of an *LFT* is still an LFT.

Lemma 9.3 *Let* $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ *and* M_{22} *is nonsingular. Then*

$$\left(\mathcal{F}_u(M, \Delta)\right)^{-1} = \mathcal{F}_u(N, \Delta)$$

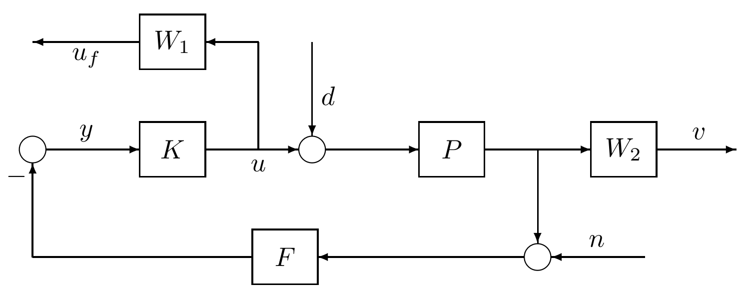
with N, *is given by*

$$N = \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1} \\ M_{22}^{-1}M_{21} & M_{22}^{-1} \end{bmatrix}.$$

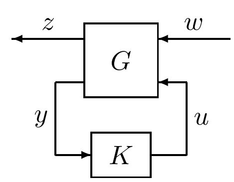
LFT is a very convenient tool to formulate many mathematical objects. We shall illustrate this by the following two examples.

Simple Block Diagrams

A feedback system with the following block diagram



can be rearranged as an LFT:



with

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, \quad z = \begin{pmatrix} v \\ u_f \end{pmatrix}, \quad G = \left[\begin{array}{cc|c} W_2P & 0 & W_2P \\ 0 & 0 & W_1 \\ \hline -FP & -F & -FP \end{array}\right].$$

A state-space realization for the generalized plant G can be obtained by directly realizing the transfer matrix G using any standard multivariable realization techniques (e.g., Gilbert realization). However, the direct realization approach is usually complicated. Here we shall show another way to obtain the realization for G based on the realizations of each component. To simplify the expression, we shall assume that the plant P is strictly proper and P, F, W_1, and W_2 have, respectively, the following state-space realizations:

$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array}\right], \quad F = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right], \quad W_1 = \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right], \quad W_2 = \left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right].$$

That is,

$$\dot{x}_p = A_p x_p + B_p(d+u), \quad y_p = C_p x_p,$$
$$\dot{x}_f = A_f x_f + B_f(y_p + n), \quad -y = C_f x_f + D_f(y_p + n),$$
$$\dot{x}_u = A_u x_u + B_u u, \quad u_f = C_u x_u + D_u u,$$
$$\dot{x}_v = A_v x_v + B_v y_p, \quad v = C_v x_v + D_v y_p.$$

Now define a new state vector

$$x = \begin{bmatrix} x_p \\ x_f \\ x_u \\ x_v \end{bmatrix}$$

and eliminate the variable y_p to get a realization of G as

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

with

$$A = \begin{bmatrix} A_p & 0 & 0 & 0 \\ B_f C_p & A_f & 0 & 0 \\ 0 & 0 & A_u & 0 \\ B_v C_p & 0 & 0 & A_v \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_p & 0 \\ 0 & B_f \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_p \\ 0 \\ B_u \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} D_v C_p & 0 & 0 & C_v \\ 0 & 0 & C_u & 0 \end{bmatrix}, \quad D_{11} = 0, \quad D_{12} = \begin{bmatrix} 0 \\ D_u \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -D_f C_p & -C_f & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -D_f \end{bmatrix}, \quad D_{22} = 0.$$

Parametric Uncertainty: A Mass/Spring/Damper System

One natural type of uncertainty is unknown coefficients in a state-space model. To motivate this type of uncertainty description, we shall begin with a familiar mechanical system, shown in Figure 9.1.

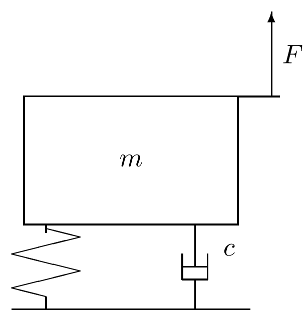


Figure 9.1: A mass/spring/damper system

The dynamical equation of the system motion can be described by

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}.$$

Suppose that the three physical parameters m, c, and k are not known exactly, but are believed to lie in known intervals. In particular, the actual mass m is within 10% of a nominal mass, $\bar{m}$, the actual damping value c is within 20% of a nominal value of $\bar{c}$, and the spring stiffness is within 30% of its nominal value of $\bar{k}$. Now introducing perturbations δ_m, δ_c, and δ_k, which are assumed to be unknown but lie in the interval $[-1, 1]$, the block diagram for the dynamical system is as shown in Figure 9.2.

It is easy to check that $\frac{1}{m}$ can be represented as an LFT in δ_m:

$$\frac{1}{m} = \frac{1}{\bar{m}(1+0.1\delta_m)} = \frac{1}{\bar{m}} - \frac{0.1}{\bar{m}}\delta_m(1+0.1\delta_m)^{-1} = \mathcal{F}_\ell\left(M_1, \delta_m\right)$$

with $M_1 = \begin{bmatrix} \frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 1 & -0.1 \end{bmatrix}$. Suppose that the input signals of the dynamical system are selected as $x_1 = x, x_2 = \dot{x}, F$, and the output signals are selected as $\dot{x}_1$ and $\dot{x}_2$. To represent the system model as an LFT of the natural uncertainty parameters δ_m, δ_c, and δ_k, we shall first isolate the uncertainty parameters and denote the inputs and outputs of δ_k, δ_c, and δ_m as y_k, y_c, y_m and u_k, u_c, u_m, respectively, as shown in Figure 9.3.

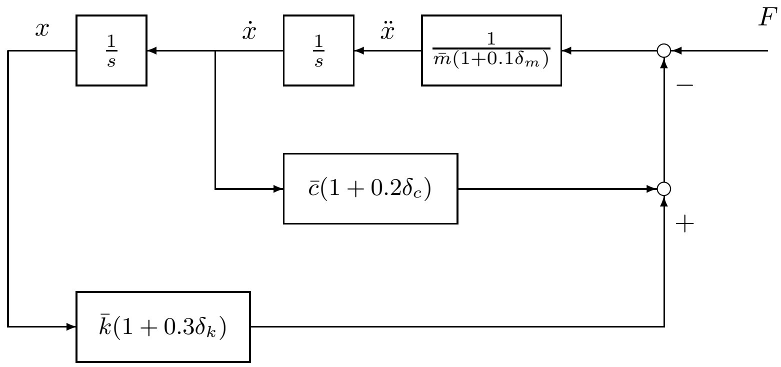


Figure 9.2: Block diagram of mass/spring/damper equation

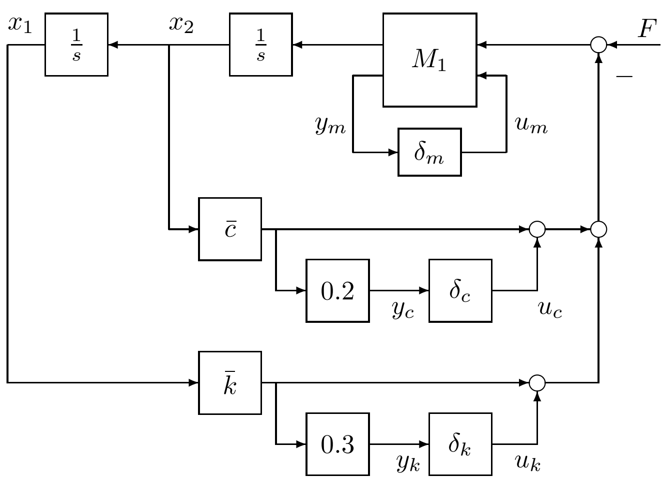


Figure 9.3: A block diagram for the mass/spring/damper system with uncertain parameters

Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y_k \\ y_c \\ y_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 0.3\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ F \\ u_k \\ u_c \\ u_m \end{bmatrix}, \quad \begin{bmatrix} u_k \\ u_c \\ u_m \end{bmatrix} = \Delta \begin{bmatrix} y_k \\ y_c \\ y_m \end{bmatrix}.$$

That is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathcal{F}_\ell(M, \Delta) \begin{bmatrix} x_1 \\ x_2 \\ F \end{bmatrix}$$

where

$$M = \left[\begin{array}{ccc:ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ \hdashline 0.3\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{array}\right], \quad \Delta = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{bmatrix}.$$

9.2 Basic Principle

We have studied two simple examples of the use of LFTs and, in particular, their role in modeling uncertainty. The basic principle at work here in writing a matrix LFT is often referred to as *"pulling out the Δ's"*. We will try to illustrate this with another picture. Consider a structure with four substructures interconnected in some known way, as shown in Figure 9.4. This diagram can be redrawn as a standard one via "pulling out the Δ's" in Figure 9.5.

Now the matrix M of the LFT can be obtained by computing the corresponding transfer matrix in the shadowed box.

We shall illustrate the preceding principle with an example. Consider an input/output relation

$$z = \frac{a + b\delta_2 + c\delta_1\delta_2^2}{1 + d\delta_1\delta_2 + e\delta_1^2} w =: Gw$$

where a, b, c, d, and e are given constants or transfer functions. We would like to write G as an LFT in terms of δ_1 and δ_2. We shall do this in three steps:

1. Draw a block diagram for the input/output relation with each δ separated as shown in Figure 9.6.

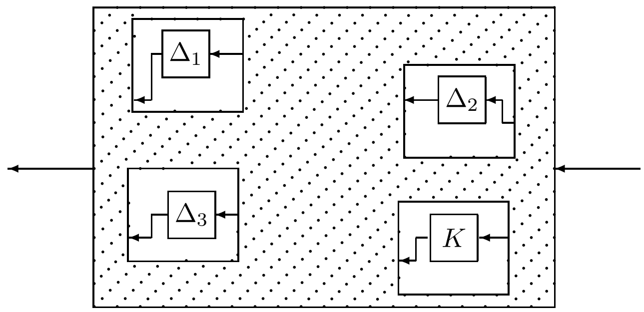


Figure 9.4: Multiple source of uncertain structure

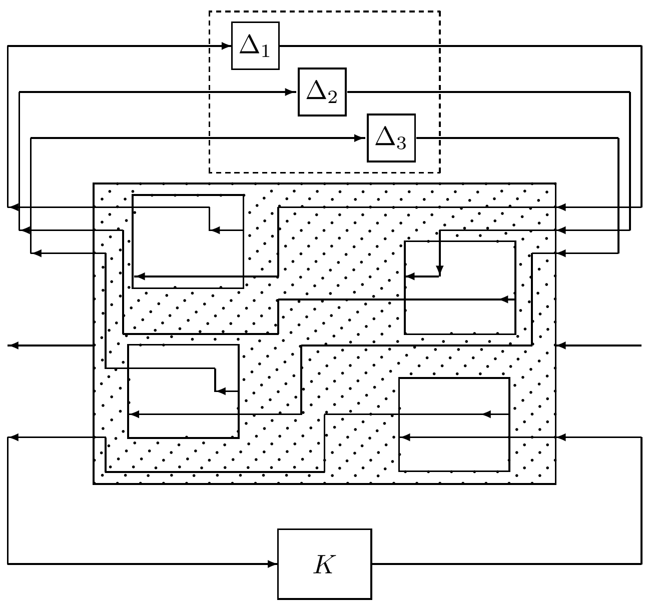


Figure 9.5: Pulling out the Δ's

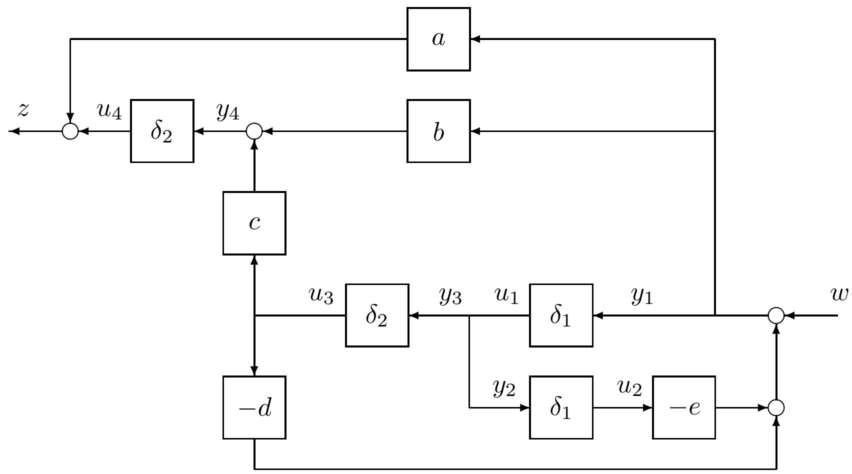


Figure 9.6: Block diagram for G

2. Mark the inputs and outputs of the δ's as y's and u's, respectively. (This is essentially *pulling out the* Δ*'s*.)

3. Write z and y's in terms of w and u's with all δ's taken out. (This step is equivalent to computing the transformation in the shadowed box in Figure 9.5.)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ z \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w \end{bmatrix}$$

where

$$M = \left[\begin{array}{cccc:c} 0 & -e & -d & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -be & -bd+c & 0 & b \\ \hdashline 0 & -ae & -ad & 1 & a \end{array}\right].$$

Then

$$z = \mathcal{F}_u(M, \Delta)w, \quad \Delta = \begin{bmatrix} \delta_1 I_2 & 0 \\ 0 & \delta_2 I_2 \end{bmatrix}.$$

All LFT examples in Section 9.1 can be obtained following the preceding steps.

For SIMULINK users, it is much easier to do all the computations using SIMULINK block diagrams, as shown in the following example.

Example 9.1 Consider the HIMAT (highly maneuverable aircraft) control problem from the μ Analysis and Synthesis Toolbox (Balas et al. [1994]). The system diagram is shown in Figure 9.7 where

$$W_{\mathrm{del}} = \begin{bmatrix} \frac{50(s+100)}{s+10000} & 0 \\ 0 & \frac{50(s+100)}{s+10000} \end{bmatrix}, \quad W_p = \begin{bmatrix} \frac{0.5(s+3)}{s+0.03} & 0 \\ 0 & \frac{0.5(s+3)}{s+0.03} \end{bmatrix},$$

$$W_n = \begin{bmatrix} \frac{2(s+1.28)}{s+320} & 0 \\ 0 & \frac{2(s+1.28)}{s+320} \end{bmatrix},$$

$$P_0 = \left[\begin{array}{cccc|cc} -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 \\ 0 & -1.9 & 0.983 & 0 & -0.414 & 0 \\ 0.0123 & -11.7 & -2.63 & 0 & -77.8 & 22.4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 57.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.3 & 0 & 0 \end{array}\right]$$

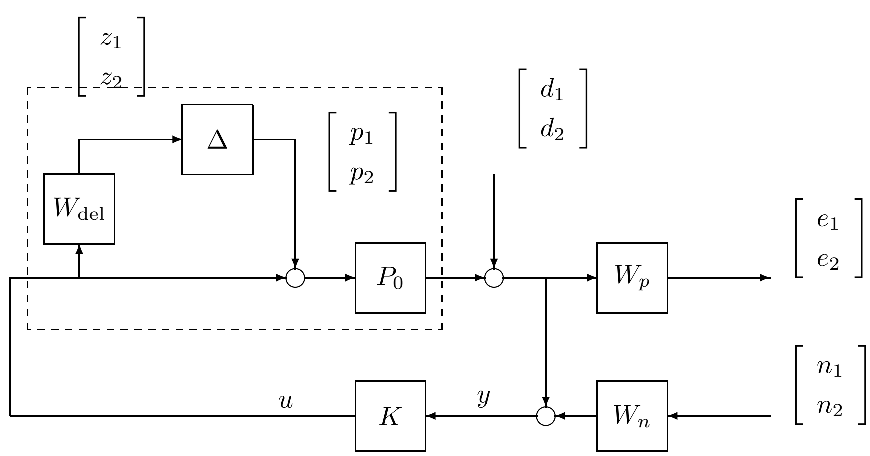


Figure 9.7: HIMAT closed-loop interconnection

The open-loop interconnection is

$$\begin{bmatrix} z_1 \\ z_2 \\ e_1 \\ e_2 \\ y_1 \\ y_2 \end{bmatrix} = \hat{G}(s) \begin{bmatrix} p_1 \\ p_2 \\ d_1 \\ d_2 \\ n_1 \\ n_2 \\ u_1 \\ u_2 \end{bmatrix}.$$

The SIMULINK block diagram of this open-loop interconnection is shown in Figure 9.8.

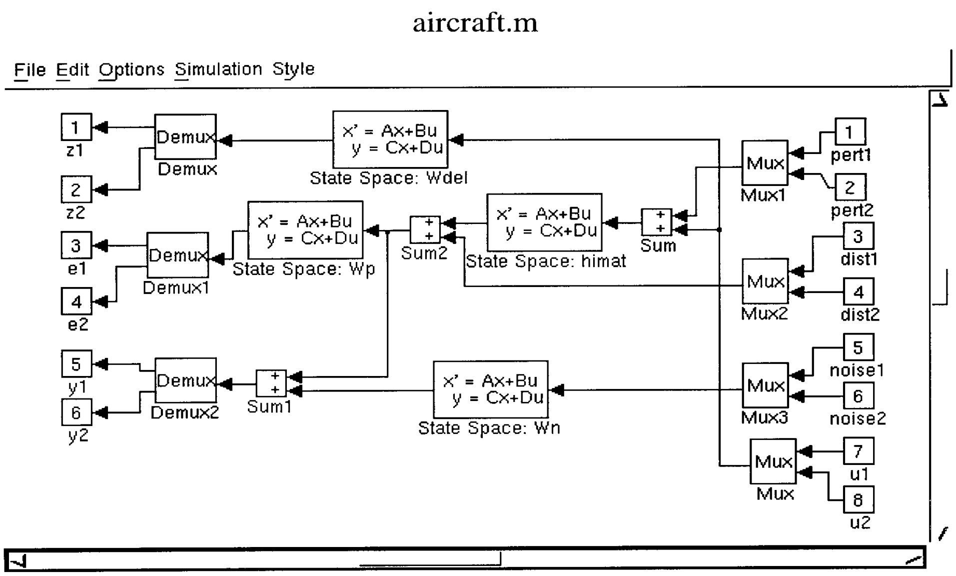


Figure 9.8: SIMULINK block diagram for HIMAT (aircraft.m)

The $\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ can be computed by

$$\gg \ [\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}] = \mathbf{linmod}('\mathbf{aircraft}')$$

which gives

$$A = \begin{bmatrix} -10000I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 & 0 \\ 0 & 0 & -1.9 & 0.983 & 0 & 0 & 0 & 0 \\ 0 & 0.0123 & -11.7 & -2.63 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -54.087 & 0 & 0 & -0.018 & 0 & 0 \\ 0 & 0 & 0 & 0 & -54.087 & 0 & -0.018 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -320I_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & -703.5624 & 0 \\ 0 & 0 & 0 & 0 & 0 & -703.5624 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4140 & 0 & 0 & 0 & -0.4140 & 0 \\ -77.8 & 22.4 & 0 & 0 & -77.8 & 22.4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9439I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -25.2476I_2 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 703.5624I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 28.65 & 0 & 0 & -0.9439 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 28.65 & 0 & -0.9439 & 0 & 0 \\ 0 & 0 & 57.3 & 0 & 0 & 0 & 0 & 25.2476 & 0 \\ 0 & 0 & 0 & 0 & 57.3 & 0 & 0 & 0 & 25.2476 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 50 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \end{bmatrix}.$$

9.3 Redheffer Star Products

The most important property of LFTs is that any interconnection of LFTs is again an LFT. This property is by far the most often used and is the heart of LFT machinery. Indeed, it is not hard to see that most of the interconnection structures discussed earlier (e.g., feedback and cascade) can be viewed as special cases of the so-called *star product*.

Suppose that P and K are compatibly partitioned matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

such that the matrix product $P_{22}K_{11}$ is well-defined and square, and assume further that $I - P_{22}K_{11}$ is invertible. Then the *star product of* P *and* K *with respect to this partition* is defined as

$$P \star K := \begin{bmatrix} \mathcal{F}_l(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1} K_{12} \\ K_{21}(I - P_{22}K_{11})^{-1} P_{21} & \mathcal{F}_u(K, P_{22}) \end{bmatrix}. \tag{9.1}$$

Note that this definition is dependent on the partitioning of the matrices P and K. In fact, this star product may be well-defined for one partition and not well-defined for another; however, we will not explicitly show this dependence because it is always clear from the context. In a block diagram, this dependence appears, as shown in Figure 9.9.

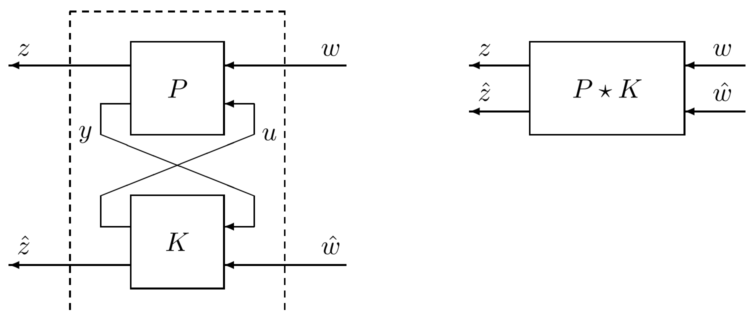


Figure 9.9: Interconnection of LFTs

Now suppose that P and K are transfer matrices with state-space representations:

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad K = \left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_{K1} & D_{K11} & D_{K12} \\ C_{K2} & D_{K21} & D_{K22} \end{array}\right].$$

Then the transfer matrix

$$P \star K : \begin{bmatrix} w \\ \hat{w} \end{bmatrix} \mapsto \begin{bmatrix} z \\ \hat{z} \end{bmatrix}$$

has a representation

$$P \star K = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

where

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A + B_2\tilde{R}^{-1}D_{K11}C_2 & B_2\tilde{R}^{-1}C_{K1} \\ B_{K1}R^{-1}C_2 & A_K + B_{K1}R^{-1}D_{22}C_{K1} \end{bmatrix} \\
\bar{B} &= \begin{bmatrix} B_1 + B_2\tilde{R}^{-1}D_{K11}D_{21} & B_2\tilde{R}^{-1}D_{K12} \\ B_{K1}R^{-1}D_{21} & B_{K2} + B_{K1}R^{-1}D_{22}D_{K12} \end{bmatrix} \\
\bar{C} &= \begin{bmatrix} C_1 + D_{12}D_{K11}R^{-1}C_2 & D_{12}\tilde{R}^{-1}C_{K1} \\ D_{K21}R^{-1}C_2 & C_{K2} + D_{K21}R^{-1}D_{22}C_{K1} \end{bmatrix} \\
\bar{D} &= \begin{bmatrix} D_{11} + D_{12}D_{K11}R^{-1}D_{21} & D_{12}\tilde{R}^{-1}D_{K12} \\ D_{K21}R^{-1}D_{21} & D_{K22} + D_{K21}R^{-1}D_{22}D_{K12} \end{bmatrix}
\end{aligned}$$

$$R = I - D_{22}D_{K11}, \quad \tilde{R} = I - D_{K11}D_{22}.$$

In fact, it is easy to show that

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix} \star \begin{bmatrix} D_{K11} & C_{K1} \\ B_{K1} & A_K \end{bmatrix}, \\
\bar{B} &= \begin{bmatrix} B_1 & B_2 \\ D_{21} & D_{22} \end{bmatrix} \star \begin{bmatrix} D_{K11} & D_{K12} \\ B_{K1} & B_{K2} \end{bmatrix}, \\
\bar{C} &= \begin{bmatrix} C_1 & D_{12} \\ C_2 & D_{22} \end{bmatrix} \star \begin{bmatrix} D_{K11} & C_{K1} \\ D_{K21} & C_{K2} \end{bmatrix}, \\
\bar{D} &= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \star \begin{bmatrix} D_{K11} & D_{K12} \\ D_{K21} & D_{K22} \end{bmatrix}.
\end{aligned}$$

The MATLAB command **starp** can be used to compute the star product:

$$\gg \ \mathbf{P} \star \mathbf{K} = \mathbf{starp}(\mathbf{P}, \mathbf{K}, \mathbf{dimy}, \mathbf{dimu})$$

where dimy and dimu are the dimensions of y and u, respectively. In the particular case when $\dim(\hat{z}) = 0$ and $\dim(\hat{w}) = 0$, we have

$$\gg \ \mathcal{F}_\ell(\mathbf{P}, \mathbf{K}) = \mathbf{starp}(\mathbf{P}, \mathbf{K})$$

9.4 Notes and References

This chapter is based on the lecture notes by Packard [1991] and the paper by Doyle, Packard, and Zhou [1991].

9.5 Problems

Problem 9.1 Find M and N matrices such that $\tilde{\Delta} = M\Delta N$, where Δ is block diagonal.

1. $\tilde{\Delta} = \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix}$

2. $\tilde{\Delta} = \begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & 0 & \Delta_2 \end{pmatrix}$

3. $\tilde{\Delta} = \begin{pmatrix} \Delta_1 & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Delta_2 \\ 0 \end{bmatrix} & \Delta_3 \end{pmatrix}$

4. $\tilde{\Delta} = \begin{pmatrix} \Delta_1 & 0 & 0 \\ \Delta_2 & \Delta_3 & 0 \\ 0 & 0 & \Delta_4 \end{pmatrix}$

5. $\tilde{\Delta} = \begin{pmatrix} \Delta_1 & \Delta_3 & 0 \\ \Delta_2 & \Delta_4 & \Delta_6 \\ 0 & \Delta_5 & 0 \end{pmatrix}$

Problem 9.2 Let $G = (I - P(s)\Delta)^{-1}$. Find a matrix $M(s)$ such that $G = \mathcal{F}_u(M, \Delta)$.

Problem 9.3 Consider the unity feedback system with $G(s)$ of size 2×2. Suppose $G(s)$ has an uncertainty model of the form

$$G(s) = \begin{bmatrix} [1 + \Delta_{11}(s)]g_{11}(s) & [1 + \Delta_{12}(s)]g_{12}(s) \\ [1 + \Delta_{21}(s)]g_{21}(s) & [1 + \Delta_{22}(s)]g_{22}(s) \end{bmatrix}.$$

Suppose also that we wish to study robust stability of the feedback system. Pull out the Δ's and draw the appropriate block diagram in the form of a structured perturbation of a nominal system.

Problem 9.4 Let

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \qquad K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right].$$

Find state-space realizations for $\mathcal{F}_\ell(P, K)$ and $\mathcal{F}_\ell(P, \hat{D})$.

Problem 9.5 Suppose D_{21} is nonsingular and

$$M(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

Find a state-space realization for

$$\hat{M}(s) = \begin{bmatrix} \begin{bmatrix} 0 & M_{12} \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} M_{11} \\ I \end{bmatrix} \\ \begin{bmatrix} -I & M_{22} \end{bmatrix} & M_{21} + E \end{bmatrix}$$

where E is a constant matrix. Find the state-space realization for $\mathcal{F}_\ell(\hat{M}, E^{-1})$ when $E = I$.