Chapter 12

Algebraic Riccati Equations

We studied the Lyapunov equation in Chapter 7 and saw the roles it played in some applications. A more general equation than the Lyapunov equation in control theory is the so-called *algebraic Riccati equation* or ARE for short. Roughly speaking, Lyapunov equations are most useful in system analysis while AREs are most useful in control system synthesis; particularly, they play central roles in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control.

Let A, Q, and R be real $n \times n$ matrices with Q and R symmetric. Then an algebraic Riccati equation is the following matrix equation:

$$A^*X + XA + XRX + Q = 0. \tag{12.1}$$

Associated with this Riccati equation is a $2n \times 2n$ matrix:

$$H := \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}. \tag{12.2}$$

A matrix of this form is called a *Hamiltonian matrix*. The matrix H in equation (12.2) will be used to obtain the solutions to the equation (12.1). It is useful to note that the spectrum of H is symmetric about the imaginary axis. To see that, introduce the $2n \times 2n$ matrix:

$$J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

having the property $J^2 = -I$. Then

$$J^{-1}HJ = -JHJ = -H^*$$

so H and $-H^*$ are similar. Thus λ is an eigenvalue iff $-\bar{\lambda}$ is.

This chapter is devoted to the study of this algebraic Riccati equation.

12.1 Stabilizing Solution and Riccati Operator

Assume that H has no eigenvalues on the imaginary axis. Then it must have n eigenvalues in Re $s < 0$ and n in Re $s > 0$. Consider the n-dimensional invariant spectral subspace, $\mathcal{X}_-(H)$, corresponding to eigenvalues of H in Re $s < 0$. By finding a basis for $\mathcal{X}_-(H)$, stacking the basis vectors up to form a matrix, and partitioning the matrix, we get

$$\mathcal{X}_-(H) = \mathrm{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1, X_2 \in \mathbb{C}^{n\times n}$. ($X_1$ and X_2 can be chosen to be real matrices.) If X_1 is nonsingular or, equivalently, if the two subspaces

$$\mathcal{X}_-(H), \quad \mathrm{Im} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{12.3}$$

are complementary, we can set $X := X_2 X_1^{-1}$. Then X is uniquely determined by H (i.e., $H \longmapsto X$ is a function, which will be denoted Ric). We will take the domain of Ric, denoted dom(Ric), to consist of Hamiltonian matrices H with two properties: H has no eigenvalues on the imaginary axis and the two subspaces in equation(12.3) are complementary. For ease of reference, these will be called the *stability property* and the *complementarity property*, respectively. This solution will be called the *stabilizing solution*. Thus, $X = \mathrm{Ric}(H)$ and

$$\mathrm{Ric}: \ \mathrm{dom(Ric)} \subset \mathbb{R}^{2n\times 2n} \longmapsto \mathbb{R}^{n\times n}.$$

Remark 12.1 It is now clear that to obtain the stabilizing solution to the Riccati equation, it is necessary to construct bases for the stable invariant subspace of H. One way of constructing this invariant subspace is to use eigenvectors and generalized eigenvectors of H. Suppose λ_i is an eigenvalue of H with multiplicity k (then $\lambda_{i+j} = \lambda_i$ for all $j = 1, \ldots, k-1$), and let v_i be a corresponding eigenvector and $v_{i+1}, \ldots, v_{i+k-1}$ be the corresponding generalized eigenvectors associated with v_i and λ_i. Then v_j are related by

$$\begin{aligned}
(H - \lambda_i I)v_i &= 0 \\
(H - \lambda_i I)v_{i+1} &= v_i \\
&\vdots \\
(H - \lambda_i I)v_{i+k-1} &= v_{i+k-2},
\end{aligned}$$

and the span$\{v_j, j = i, \ldots, i+k-1\}$ is an invariant subspace of H. The sum of all invariant subspaces corresponding to stable eigenvalues is the stable invariant subspace $\mathcal{X}_-(H)$. $\diamond$

Example 12.1 Let

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of H are $1, 1, -1, -1$, and the corresponding eigenvectors and generalized eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -2 \end{bmatrix}, \; v_2 = \begin{bmatrix} -1 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} \; v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; v_4 = \begin{bmatrix} 1 \\ 3/2 \\ 0 \\ 0 \end{bmatrix}.$$

It is easy to check that $\{v_3, v_4\}$ form a basis for the stable invariant subspace $\mathcal{X}_-(H)$, $\{v_1, v_2\}$ form a basis for the antistable invariant subspace, and $\{v_1, v_3\}$ form a basis for another invariant subspace corresponding to eigenvalues $\{1, -1\}$ so

$$\overline{X} = 0, \quad \tilde{X} = \begin{bmatrix} -10 & 6 \\ 6 & -4 \end{bmatrix}, \quad \hat{X} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

are all solutions of the ARE with the property

$$\lambda(A + R\overline{X}) = \{-1, -1\}, \quad \lambda(A + R\tilde{X}) = \{1, 1\}, \quad \lambda(A + R\hat{X}) = \{1, -1\}.$$

Thus only $\overline{X}$ is the stabilizing solution. The stabilizing solution can be found using the following MATLAB command:

$\gg$ $[\mathbf{X_1}, \mathbf{X_2}] = \mathbf{ric\_schr}(\mathbf{H}),\ \mathbf{X} = \mathbf{X_2}/\mathbf{X_1}$

The following well-known results give some properties of X as well as verifiable conditions under which H belongs to dom(Ric).

Theorem 12.1 *Suppose* $H \in \mathrm{dom(Ric)}$ *and* $X = \mathrm{Ric}(H)$. *Then*

(i) X *is real symmetric;*

(ii) X *satisfies the algebraic Riccati equation*

$$A^*X + XA + XRX + Q = 0;$$

(iii) $A + RX$ *is stable.*

Proof. (i) Let X_1, X_2 be as before. It is claimed that

$$X_1^* X_2 \text{ is Hermitian.} \tag{12.4}$$

To prove this, note that there exists a stable matrix H_- in $\mathbb{R}^{n\times n}$ such that

$$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_- \ .$$

(H_- is a matrix representation of $H|_{\mathcal{X}_-(H)}$.) Premultiply this equation by

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* J$$

to get

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* JH \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* J \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_- \ . \tag{12.5}$$

Since JH is symmetric, so is the left-hand side of equation (12.5) and so is the right-hand side:

$$\begin{aligned}(-X_1^*X_2 + X_2^*X_1)H_- &= H_-^*(-X_1^*X_2 + X_2^*X_1)^* \\ &= -H_-^*(-X_1^*X_2 + X_2^*X_1).\end{aligned}$$

This is a Lyapunov equation. Since H_- is stable, the unique solution is

$$-X_1^*X_2 + X_2^*X_1 = 0.$$

This proves equation (12.4). Hence $X := X_2X_1^{-1} = (X_1^{-1})^*(X_1^*X_2)X_1^{-1}$ is Hermitian. Since X_1 and X_2 can always be chosen to be real and X is unique, X is real symmetric.

(ii) Start with the equation

$$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_-$$

and postmultiply by X_1^{-1} to get

$$H\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} X_1 H_- X_1^{-1}. \tag{12.6}$$

Now pre-multiply by $[X \quad -I]$:

$$[X \quad -I]H\begin{bmatrix} I \\ X \end{bmatrix} = 0.$$

This is precisely the Riccati equation.

(iii) Premultiply equation (12.6) by $[I \quad 0]$ to get

$$A + RX = X_1 H_- X_1^{-1}.$$

Thus $A + RX$ is stable because H_- is. □

Now we are going to state one of the main theorems of this section; it gives the necessary and sufficient conditions for the existence of a unique stabilizing solution of equation (12.1) under certain restrictions on the matrix R.

Theorem 12.2 *Suppose H has no imaginary eigenvalues and R is either positive semidefinite or negative semidefinite. Then $H \in \mathrm{dom}(\mathrm{Ric})$ if and only if (A, R) is stabilizable.*

Proof. ($\Leftarrow$) To prove that $H \in \mathrm{dom}(\mathrm{Ric})$, we must show that

$$\mathcal{X}_-(H), \quad \mathrm{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

are complementary. This requires a preliminary step. As in the proof of Theorem 12.1 define X_1, X_2, H_- so that

$$\mathcal{X}_-(H) = \mathrm{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_- \quad . \tag{12.7}$$

We want to show that X_1 is nonsingular (i.e., Ker $X_1 = 0$). First, it is claimed that Ker X_1 is H_- invariant. To prove this, let $x \in \mathrm{Ker}\, X_1$. Premultiply equation (12.7) by $[I \quad 0]$ to get

$$AX_1 + RX_2 = X_1 H_- \quad . \tag{12.8}$$

Premultiply by $x^* X_2^*$, postmultiply by x, and use the fact that $X_2^* X_1$ is symmetric [see equation (12.4)] to get

$$x^* X_2^* R X_2 x = 0.$$

Since R is semidefinite, this implies that $RX_2 x = 0$. Now postmultiply equation (12.8) by x to get $X_1 H_- x = 0$ (i.e., $H_- x \in \mathrm{Ker}\, X_1$). This proves the claim.

Now to prove that X_1 is nonsingular, suppose, on the contrary, that Ker $X_1 \neq 0$. Then $H_-|_{\mathrm{Ker}\, X_1}$ has an eigenvalue, λ, and a corresponding eigenvector, x:

$$H_- x = \lambda x \tag{12.9}$$

$$\mathrm{Re}\, \lambda < 0, \quad 0 \neq x \in \mathrm{Ker}\, X_1.$$

Premultiply equation (12.7) by $[0 \quad I]$:

$$-QX_1 - A^*X_2 = X_2 H_- \ . \tag{12.10}$$

Postmultiply the above equation by x and use equation (12.9):

$$(A^* + \lambda I)X_2 x = 0.$$

Recall that $RX_2x = 0$; we have

$$x^* X_2^* [A + \overline{\lambda} I \quad R] = 0.$$

Then the stabilizability of (A, R) implies $X_2x = 0$. But if both $X_1x = 0$ and $X_2x = 0$, then $x = 0$ since $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ has full column rank, which is a contradiction.

$(\Rightarrow)$ This is obvious since $H \in \mathrm{dom(Ric)}$ implies that X is a stabilizing solution and that $A + RX$ is asymptotically stable. It also implies that (A, R) must be stabilizable. □

The following result is the so-called *bounded real lemma*, which follows immediately from the preceding theorem.

Corollary 12.3 *Let $\gamma > 0$, $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$, and*

$$H := \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix}$$

where $R = \gamma^2 I - D^*D$. Then the following conditions are equivalent:

(i) $\|G\|_\infty < \gamma$.

(ii) $\bar{\sigma}(D) < \gamma$ *and H has no eigenvalues on the imaginary axis.*

(iii) $\bar{\sigma}(D) < \gamma$ *and* $H \in \mathrm{dom(Ric)}$.

(iv) $\bar{\sigma}(D) < \gamma$ *and* $H \in \mathrm{dom(Ric)}$ *and* $\mathrm{Ric}(H) \geq 0$ *(*$\mathrm{Ric}(H) > 0$ *if (C, A) is observable).*

(v) $\bar{\sigma}(D) < \gamma$ *and there exists an $X = X^* \geq 0$ such that*

$$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C = 0$$

and $A + BR^{-1}D^*C + BR^{-1}B^*X$ has no eigenvalues on the imaginary axis.

(vi) $\bar{\sigma}(D) < \gamma$ *and there exists an $X = X^* > 0$ such that*

$$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C < 0.$$

(vii) There exists an $X = X^* > 0$ *such that*

$$\begin{bmatrix} XA + A^*X & XB & C^* \\ B^*X & -\gamma I & D^* \\ C & D & -\gamma I \end{bmatrix} < 0.$$

Proof. The equivalence between (i) and (ii) has been shown in Chapter 4. The equivalence between (iii) and (iv) is obvious by noting the fact that $A + BR^{-1}D^*C$ is stable if $\|G\|_\infty < \gamma$ (See Problem 12.15). The equivalence between (ii) and (iii) follows from the preceding theorem. It is also obvious that (iv) implies (v). We shall now show that (v) implies (i). Thus suppose that there is an $X \geq 0$ such that

$$X(A + BR^{-1}D^*C) + (A + BR^{-1}D^*C)^*X + XBR^{-1}B^*X + C^*(I + DR^{-1}D^*)C = 0$$

and $A + BR^{-1}(B^*X + D^*C)$ has no eigenvalues on the imaginary axis. Then

$$W(s) := \left[\begin{array}{c|c} A & -B \\ \hline B^*X + D^*C & R \end{array}\right]$$

has no zeros on the imaginary axis since

$$W^{-1}(s) = \left[\begin{array}{c|c} A + BR^{-1}(B^*X + D^*C) & BR^{-1} \\ \hline R^{-1}(B^*X + D^*C) & R^{-1} \end{array}\right]$$

has no poles on the imaginary axis. Next, note that

$$\begin{aligned} -X(j\omega I - A) - (j\omega I - A)^*X + XBR^{-1}D^*C + C^*DR^{-1}B^*X \\ +XBR^{-1}B^*X + C^*(I + DR^{-1}D^*)C = 0. \end{aligned}$$

Multiplying $B^*\{(j\omega I - A)^*\}^{-1}$ on the left and $(j\omega I - A)^{-1}B$ on the right of the preceding equation and completing square, we have

$$G^*(j\omega)G(j\omega) = \gamma^2 I - W^*(j\omega)R^{-1}W(j\omega).$$

Since $W(s)$ has no zeros on the imaginary axis, we conclude that $\|G\|_\infty < \gamma$.

The equivalence between (vi) and (vii) follows from Schur complement. It is also easy to show that (vi) implies (i) by following the similar procedure as above. To show that (i) implies (vi), let

$$\hat{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ \epsilon I & 0 \end{array}\right].$$

Then there exists an $\epsilon > 0$ such that $\left\|\hat{G}\right\|_\infty < \gamma$. Now (vi) follows by applying part (v) to $\hat{G}$. □

Theorem 12.4 *Suppose H has the form*

$$H = \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix}.$$

Then $H \in \mathrm{dom(Ric)}$ iff (A, B) is stabilizable and (C, A) has no unobservable modes on the imaginary axis. Furthermore, $X = \mathrm{Ric}(H) \geq 0$ if $H \in \mathrm{dom(Ric)}$, and $\mathrm{Ker}(X) = \{0\}$ if and only if (C, A) has no stable unobservable modes.

Note that $\mathrm{Ker}(X) \subset \mathrm{Ker}(C)$, so that the equation $XM = C^*$ always has a solution for M, and a minimum F-norm solution is given by X^+C^*.

Proof. It is clear from Theorem 12.2 that the stabilizability of (A, B) is necessary, and it is also sufficient if H has no eigenvalues on the imaginary axis. So we only need to show that, assuming (A, B) is stabilizable, H has no imaginary eigenvalues iff (C, A) has no unobservable modes on the imaginary axis. Suppose that $j\omega$ is an eigenvalue and $0 \neq \begin{bmatrix} x \\ z \end{bmatrix}$ is a corresponding eigenvector. Then

$$Ax - BB^*z = j\omega x$$

$$-C^*Cx - A^*z = j\omega z.$$

Rearrange:

$$(A - j\omega I)x = BB^*z \tag{12.11}$$

$$-(A - j\omega I)^*z = C^*Cx. \tag{12.12}$$

Thus

$$\langle z, (A - j\omega I)x \rangle = \langle z, BB^*z \rangle = \|B^*z\|^2$$

$$-\langle x, (A - j\omega I)^*z \rangle = \langle x, C^*Cx \rangle = \|Cx\|^2$$

so $\langle x, (A - j\omega I)^*z \rangle$ is real and

$$-\|Cx\|^2 = \langle (A - j\omega I)x, z \rangle = \overline{\langle z, (A - j\omega I)x \rangle} = \|B^*z\|^2.$$

Therefore, $B^*z = 0$ and $Cx = 0$. So from equations (12.11) and (12.12)

$$(A - j\omega I)x = 0$$

$$(A - j\omega I)^*z = 0.$$

Combine the last four equations to get

$$z^*[A - j\omega I \quad B] = 0$$

$$\begin{bmatrix} A - j\omega I \\ C \end{bmatrix} x = 0.$$

The stabilizability of (A, B) gives $z = 0$. Now it is clear that $j\omega$ is an eigenvalue of H iff $j\omega$ is an unobservable mode of (C, A).

Next, set $X := \mathrm{Ric}(H)$. We will show that $X \geq 0$. The Riccati equation is

$$A^*X + XA - XBB^*X + C^*C = 0$$

or, equivalently,

$$(A - BB^*X)^*X + X(A - BB^*X) + XBB^*X + C^*C = 0. \tag{12.13}$$

Noting that $A - BB^*X$ is stable (Theorem 12.1), we have

$$X = \int_0^\infty \mathrm{e}^{(A-BB^*X)^*t}(XBB^*X + C^*C)\mathrm{e}^{(A-BB^*X)t}dt. \tag{12.14}$$

Since $XBB^*X + C^*C$ is positive semidefinite, so is X.

Finally, we will show that Ker$X$ is nontrivial if and only if (C, A) has stable unobservable modes. Let $x \in \mathrm{Ker}X$, then $Xx = 0$. Premultiply equation (12.13) by x^* and postmultiply by x to get

$$Cx = 0.$$

Now postmultiply equation (12.13) again by x to get

$$XAx = 0.$$

We conclude that Ker(X) is an A-invariant subspace. Thus if Ker$(X) \neq \{0\}$, then there is a $0 \neq x \in \mathrm{Ker}(X)$ and a λ such that $\lambda x = Ax = (A - BB^*X)x$ and $Cx = 0$. Since $(A - BB^*X)$ is stable, Re$\lambda < 0$; thus λ is a stable unobservable mode. Conversely, suppose (C, A) has an unobservable stable mode λ (i.e., there is an x such that $Ax = \lambda x, Cx = 0$). By premultiplying the Riccati equation by x^* and postmultiplying by x, we get

$$2\mathrm{Re}\lambda x^*Xx - x^*XBB^*Xx = 0.$$

Hence $x^*Xx = 0$ (i.e., X is singular) since Re$\lambda < 0$. □

Example 12.2 This example shows that the observability of (C, A) is not necessary for the existence of a positive definite stabilizing solution. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Then (A, B) is stabilizable, but (C, A) is not detectable. However,

$$X = \begin{bmatrix} 18 & -24 \\ -24 & 36 \end{bmatrix} > 0$$

is the stabilizing solution.

Corollary 12.5 *Suppose that (A, B) is stabilizable and (C, A) is detectable. Then the Riccati equation*

$$A^*X + XA - XBB^*X + C^*C = 0$$

has a unique positive semidefinite solution. Moreover, the solution is stabilizing.

Proof. It is obvious from the preceding theorem that the Riccati equation has a unique stabilizing solution and that the solution is positive semidefinite. Hence we only need to show that any positive semidefinite solution $X \geq 0$ must also be stabilizing. Then by the uniqueness of the stabilizing solution, we can conclude that there is only one positive semidefinite solution. To achieve that goal, let us assume that $X \geq 0$ satisfies the Riccati equation but that it is not stabilizing. First rewrite the Riccati equation as

$$(A - BB^*X)^*X + X(A - BB^*X) + XBB^*X + C^*C = 0 \tag{12.15}$$

and let λ and x be an unstable eigenvalue and the corresponding eigenvector of $A - BB^*X$, respectively; that is,

$$(A - BB^*X)x = \lambda x.$$

Now premultiply and postmultiply equation (12.15) by x^* and x, respectively, and we have

$$(\bar{\lambda} + \lambda)x^*Xx + x^*(XBB^*X + C^*C)x = 0.$$

This implies

$$B^*Xx = 0, \quad Cx = 0$$

since $\mathrm{Re}(\lambda) \geq 0$ and $X \geq 0$. Finally, we arrive at

$$Ax = \lambda x, \quad Cx = 0.$$

That is, (C, A) is not detectable, which is a contradiction. Hence $\mathrm{Re}(\lambda) < 0$ (i.e., $X \geq 0$ is the stabilizing solution). □

Lemma 12.6 *Suppose D has full column rank and let $R = D^*D > 0$; then the following statements are equivalent:*

(i) $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ *has full column rank for all ω.*

(ii) $\big((I - DR^{-1}D^*)C,\ A - BR^{-1}D^*C\big)$ *has no unobservable modes on the $j\omega$ axis.*

Proof. Suppose $j\omega$ is an unobservable mode of $\big((I - DR^{-1}D^*)C,\, A - BR^{-1}D^*C\big)$; then there is an $x \neq 0$ such that

$$(A - BR^{-1}D^*C)x = j\omega x, \quad (I - DR^{-1}D^*)Cx = 0;$$

that is,

$$\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -R^{-1}D^*C & I \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = 0.$$

But this implies that

$$\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} \tag{12.16}$$

does not have full-column rank. Conversely, suppose equation (12.16) does not have full-column rank for some ω; then there exists $\begin{bmatrix} u \\ v \end{bmatrix} \neq 0$ such that

$$\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Now let

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I & 0 \\ -R^{-1}D^*C & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ R^{-1}D^*C & I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \neq 0$$

and

$$(A - BR^{-1}D^*C - j\omega I)x + By = 0 \tag{12.17}$$

$$(I - DR^{-1}D^*)Cx + Dy = 0. \tag{12.18}$$

Premultiply equation (12.18) by D^* to get $y = 0$. Then we have

$$(A - BR^{-1}D^*C)x = j\omega x, \quad (I - DR^{-1}D^*)Cx = 0;$$

that is, $j\omega$ is an unobservable mode of $\big((I - DR^{-1}D^*)C,\, A - BR^{-1}D^*C\big)$. □

Remark 12.2 If D is not square, then there is a $D_\perp$ such that $\begin{bmatrix} D_\perp & DR^{-1/2} \end{bmatrix}$ is unitary and that $D_\perp D_\perp^* = I - DR^{-1}D^*$. Hence, in some cases we will write the condition (ii) in the preceding lemma as $(D_\perp^* C, A - BR^{-1}D^*C)$ having no imaginary unobservable modes. Of course, if D is square, the condition is simplified to $A - BR^{-1}D^*C$ having no imaginary eigenvalues. Note also that if $D^*C = 0$, condition (ii) becomes (C, A) having no imaginary unobservable modes. ◇

Corollary 12.7 *Suppose D has full column rank and denote $R = D^*D > 0$. Let H have the form*

$$\begin{aligned} H &= \begin{bmatrix} A & 0 \\ -C^*C & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C^*D \end{bmatrix} R^{-1} \begin{bmatrix} D^*C & B^* \end{bmatrix} \\ &= \begin{bmatrix} A - BR^{-1}D^*C & -BR^{-1}B^* \\ -C^*(I - DR^{-1}D^*)C & -(A - BR^{-1}D^*C)^* \end{bmatrix}. \end{aligned}$$

Then $H \in \operatorname{dom}(\mathrm{Ric})$ iff (A, B) is stabilizable and $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full-column rank for all ω. Furthermore, $X = \mathrm{Ric}(H) \geq 0$ if $H \in \operatorname{dom}(\mathrm{Ric})$, and $\operatorname{Ker}(X) = \{0\}$ if and only if $(D_\perp^* C, A - BR^{-1}D^*C)$ has no stable unobservable modes.

Proof. This is the consequence of Lemma 12.6 and Theorem 12.4. □

Remark 12.3 It is easy to see that the detectability (observability) of $(D_\perp^* C, A - BR^{-1}D^*C)$ implies the detectability (observability) of (C, A); however, the converse is, in general, not true. Hence the existence of a stabilizing solution to the Riccati equation in the preceding corollary is not guaranteed by the stabilizability of (A, B) and detectability of (C, A). Furthermore, even if a stabilizing solution exists, the positive definiteness of the solution is not guaranteed by the observability of (C, A) unless $D^*C = 0$. As an example, consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then (C, A) is observable, (A, B) is controllable, and

$$A - BD^*C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D_\perp^* C = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

A Riccati equation with the preceding data has a nonnegative definite stabilizing solution since $(D_\perp^* C, A - BR^{-1}D^*C)$ has no unobservable modes on the imaginary axis. However, the solution is not positive definite since $(D_\perp^* C, A - BR^{-1}D^*C)$ has a stable unobservable mode. On the other hand, if the B matrix is changed to

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then the corresponding Riccati equation has no stabilizing solution since, in this case, $(A - BD^*C)$ has eigenvalues on the imaginary axis although (A, B) is controllable and (C, A) is observable. ◇

Related MATLAB Commands: ric_eig, are

12.2 Inner Functions

A transfer function N is called *inner* if $N \in \mathcal{RH}_\infty$ and $N^\sim N = I$ and *co-inner* if $N \in \mathcal{RH}_\infty$ and $NN^\sim = I$. Note that N need not be square. Inner and co-inner are dual notions (i.e., N is an inner iff N^T is a co-inner). A matrix function $N \in \mathcal{RL}_\infty$ is called *all-pass* if N is square and $N^\sim N = I$; clearly a square inner function is all-pass. We will focus on the characterizations of inner functions here, and the properties of co-inner functions follow by duality.

Note that N inner implies that N has at least as many rows as columns. For N inner and any $q \in \mathbb{C}^m$, $v \in \mathcal{L}_2$, $\|N(j\omega)q\| = \|q\|$, $\forall\omega$ and $\|Nv\|_2 = \|v\|_2$ since $N(j\omega)^*N(j\omega) = I$ for all ω. Because of these norm preserving properties, inner matrices will play an important role in the control synthesis theory in this book. In this section, we present a state-space characterization of inner transfer functions.

Lemma 12.8 *Suppose* $N = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ *and* $X = X^* \geq 0$ *satisfies*

$$A^*X + XA + C^*C = 0. \tag{12.19}$$

Then

(a) $D^*C + B^*X = 0$ *implies* $N^\sim N = D^*D$.

(b) (A, B) *is controllable, and* $N^\sim N = D^*D$ *implies that* $D^*C + B^*X = 0$.

Proof. Conjugating the states of

$$N^\sim N = \left[\begin{array}{cc|c} A & 0 & B \\ -C^*C & -A^* & -C^*D \\ \hline D^*C & B^* & D^*D \end{array}\right]$$

by $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ on the left and $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ on the right yields

$$\begin{aligned} N^\sim N &= \left[\begin{array}{cc|c} A & 0 & B \\ -(A^*X + XA + C^*C) & -A^* & -(XB + C^*D) \\ \hline B^*X + D^*C & B^* & D^*D \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -A^* & -(XB + C^*D) \\ \hline B^*X + D^*C & B^* & D^*D \end{array}\right]. \end{aligned}$$

Then (a) and (b) follow easily. $\square$

This lemma immediately leads to one characterization of inner matrices in terms of their state-space representations. Simply add the condition that $D^*D = I$ to Lemma 12.8 to get $N^\sim N = I$.

Corollary 12.9 *Suppose* $N = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is stable and minimal, and* X *is the observability Gramian. Then* N *is an inner if and only if*

(a) $D^*C + B^*X = 0$

(b) $D^*D = I$.

A transfer matrix $N_\perp$ is called a *complementary inner factor (CIF)* of N if $[N\, N_\perp]$ is square and is an inner. The dual notion of the complementary co-inner factor is defined in the obvious way. Given an inner N, the following lemma gives a construction of its CIF. The proof of this lemma follows from straightforward calculation and from the fact that $CX^+X = C$ since $\mathrm{Im}(I - X^+X) \subset \mathrm{Ker}(X) \subset \mathrm{Ker}(C)$.

Lemma 12.10 *Let* $N = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *be an inner and* X *be the observability Gramian. Then a CIF* $N_\perp$ *is given by*

$$N_\perp = \left[\begin{array}{c|c} A & -X^+C^*D_\perp \\ \hline C & D_\perp \end{array}\right]$$

where $D_\perp$ *is an orthogonal complement of* D *such that* $[D \;\; D_\perp]$ *is square and orthogonal.*

12.3 Notes and References

The general solutions of a Riccati equation are given by Martensson [1971]. The paper by Willems [1971] contains a comprehensive treatment of ARE and the related optimization problems. Some matrix factorization results are given in Doyle [1984]. Numerical methods for solving ARE can be found in Arnold and Laub [1984], Van Dooren [1981], and references therein. See Zhou, Doyle, and Glover [1996] and Lancaster and Rodman [1995] for a more extensive treatment of this subject.

12.4 Problems

Problem 12.1 Assume that $G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RL}_\infty$ is a stabilizable and detectable realization and $\gamma > \|G(s)\|_\infty$. Show that there exists a transfer matrix $M \in$

$\mathcal{RL}_\infty$ such that $M^\sim M = \gamma^2 I - G^\sim G$ and $M^{-1} \in \mathcal{RH}_\infty$. A particular realization of M is

$$M(s) = \left[\begin{array}{c|c} A & B \\ \hline -R^{1/2}F & R^{1/2} \end{array}\right]$$

where

$$\begin{aligned} R &= \gamma^2 I - D^* D \\ F &= R^{-1}(B^* X + D^* C) \\ X &= \operatorname{Ric}\begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix} \end{aligned}$$

and $X \geq 0$ if A is stable.

Problem 12.2 Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a stabilizable and detectable realization. Suppose $G^\sim(j\omega)G(j\omega) > 0$ for all ω or $\begin{bmatrix} A - j\omega & B \\ C & D \end{bmatrix}$ has full-column rank for all ω. Let

$$X = \operatorname{Ric}\begin{bmatrix} A - BR^{-1}D^*C & -BR^{-1}B^* \\ -C^*(I - DR^{-1}D^*)C & -(A - BR^{-1}D^*C)^* \end{bmatrix}$$

with $R := D^* D > 0$. Show

$$W^\sim W = G^\sim G$$

where $W^{-1} \in \mathcal{RH}_\infty$ and

$$W = \left[\begin{array}{c|c} A & B \\ \hline R^{-1/2}(D^*C + B^*X) & R^{1/2} \end{array}\right].$$

Problem 12.3 A square $(m \times m)$ matrix function $G(s) \in \mathcal{RH}_\infty$ is said to be *positive real (PR)* if $G(j\omega) + G^*(j\omega) \geq 0$ for all finite ω; and $G(s)$ is said to be *strictly positive real (SPR)* if $G(j\omega) + G^*(j\omega) > 0$ for all $\omega \in \mathbb{R}$. Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state-space realization of $G(s)$ with A stable (not necessarily a minimal realization). Suppose there exist $X \geq 0$, Q, and W such that

$$XA + A^*X = -Q^*Q \tag{12.20}$$
$$B^*X + W^*Q = C \tag{12.21}$$
$$D + D^* = W^*W, \tag{12.22}$$

Show that $G(s)$ is positive real and

$$G(s) + G^\sim(s) = M^\sim(s)M(s)$$

with $M(s) = \left[\begin{array}{c|c} A & B \\ \hline Q & W \end{array}\right]$. Furthermore, if $M(j\omega)$ has full-column rank for all $\omega \in \mathbb{R}$, then $G(s)$ is strictly positive real.

Problem 12.4 Suppose (A, B, C, D) is a minimal realization of $G(s)$ with A stable and $G(s)$ positive real. Show that there exist $X \geq 0$, Q, and W such that

$$\begin{aligned} XA + A^*X &= -Q^*Q \\ B^*X + W^*Q &= C \\ D + D^* &= W^*W \end{aligned}$$

and

$$G(s) + G^{\sim}(s) = M^{\sim}(s)M(s)$$

with $M(s) = \left[\begin{array}{c|c} A & B \\ \hline Q & W \end{array}\right]$. Furthermore, if $G(s)$ is strictly positive real, then $M(j\omega)$ has full-column rank for all $\omega \in \mathbb{R}$.

Problem 12.5 Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state-space realization of $G(s) \in \mathcal{RH}_\infty$ with A stable and $R := D + D^* > 0$. Show that $G(s)$ is strictly positive real if and only if there exists a stabilizing solution to the following Riccati equation:

$$X(A - BR^{-1}C) + (A - BR^{-1}C)^*X + XBR^{-1}B^*X + C^*R^{-1}C = 0.$$

Moreover, $M(s) = \left[\begin{array}{c|c} A & B \\ \hline R^{-\frac{1}{2}}(C - B^*X) & R^{\frac{1}{2}} \end{array}\right]$ is minimal phase and

$$G(s) + G^{\sim}(s) = M^{\sim}(s)M(s).$$

Problem 12.6 Assume $p \geq m$. Show that there exists an *rcf* $G = NM^{-1}$ such that N is an *inner* if and only if $G^{\sim}G > 0$ on the $j\omega$ axis, including at ∞. This factorization is unique up to a constant unitary multiple. Furthermore, assume that the realization of $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is stabilizable and that $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$. Then a particular realization of the desired coprime factorization is

$$\begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A + BF & BR^{-1/2} \\ \hline F & R^{-1/2} \\ C + DF & DR^{-1/2} \end{array}\right] \in \mathcal{RH}_\infty$$

where

$$R = D^*D > 0$$

$$F = -R^{-1}(B^*X + D^*C)$$

and

$$X = \mathrm{Ric}\begin{bmatrix} A - BR^{-1}D^*C & -BR^{-1}B^* \\ -C^*(I - DR^{-1}D^*)C & -(A - BR^{-1}D^*C)^* \end{bmatrix} \geq 0.$$

Moreover, a complementary inner factor can be obtained as

$$N_\perp = \left[\begin{array}{c|c} A + BF & -X^\dagger C^* D_\perp \\ \hline C + DF & D_\perp \end{array}\right]$$

if $p > m$.

Problem 12.7 Assume that $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{R}_p(s)$ and (A, B) is stabilizable. Show that there exists a right coprime factorization $G = NM^{-1}$ such that $M \in \mathcal{RH}_\infty$ is an inner if and only if G has no poles on the $j\omega$ axis. A particular realization is

$$\begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A + BF & B \\ \hline F & I \\ C + DF & D \end{array}\right] \in \mathcal{RH}_\infty$$

where

$$F = -B^*X$$

$$X = \mathrm{Ric}\begin{bmatrix} A & -BB^* \\ 0 & -A^* \end{bmatrix} \geq 0.$$

Problem 12.8 A right coprime factorization of $G = NM^{-1}$ with $N, M \in \mathcal{RH}_\infty$ is called a *normalized right coprime factorization* if $M^\sim M + N^\sim N = I$; that is, if $\begin{bmatrix} M \\ N \end{bmatrix}$ is an inner. Similarly, an *lcf* $G = \tilde{M}^{-1}\tilde{N}$ is called a *normalized left coprime factorization* if $\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}$ is a co-inner. Let a realization of G be given by $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and define $R = I + D^*D > 0$ and $\tilde{R} = I + DD^* > 0$.

(a) Suppose (A, B) is stabilizable and (C, A) has no unobservable modes on the imaginary axis. Show that there is a normalized right coprime factorization $G = NM^{-1}$

$$\begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A + BF & BR^{-1/2} \\ \hline F & R^{-1/2} \\ C + DF & DR^{-1/2} \end{array}\right] \in \mathcal{RH}_\infty$$

where
$$F = -R^{-1}(B^*X + D^*C)$$
and
$$X = \mathrm{Ric}\begin{bmatrix} A - BR^{-1}D^*C & -BR^{-1}B^* \\ -C^*\tilde{R}^{-1}C & -(A - BR^{-1}D^*C)^* \end{bmatrix} \geq 0.$$

(b) Suppose (C, A) is detectable and (A, B) has no uncontrollable modes on the imaginary axis. Show that there is a normalized left coprime factorization $G = \tilde{M}^{-1}\tilde{N}$
$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} := \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline \tilde{R}^{-1/2}C & \tilde{R}^{-1/2} & \tilde{R}^{-1/2}D \end{array}\right]$$
where
$$L = -(BD^* + YC^*)\tilde{R}^{-1}$$
and
$$Y = \mathrm{Ric}\begin{bmatrix} (A - BD^*\tilde{R}^{-1}C)^* & -C^*\tilde{R}^{-1}C \\ -BR^{-1}B^* & -(A - BD^*\tilde{R}^{-1}C) \end{bmatrix} \geq 0.$$

(c) Show that the controllability Gramian P and the observability Gramian Q of $\begin{bmatrix} M \\ N \end{bmatrix}$ are given by
$$P = (I + YX)^{-1}Y, \quad Q = X$$
while the controllability Gramian $\tilde{P}$ and observability Gramian $\tilde{Q}$ of $\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}$ are given by
$$\tilde{P} = Y, \quad \tilde{Q} = (I + XY)^{-1}X.$$

Problem 12.9 Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Find M_1 and M_2 such that $M_1^{-1}, M_2^{-1} \in \mathcal{RH}_\infty$ and
$$M_1 M_1^\sim = I + GG^\sim, \quad M_2^\sim M_2 = I + G^\sim G.$$

Problem 12.10 Let $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$, and consider the Sylvester equation
$$AX + XB = C$$
for an unknown matrix $X \in \mathbb{R}^{m\times n}$. Let
$$M = \begin{bmatrix} B & 0 \\ C & -A \end{bmatrix}, \quad N = \begin{bmatrix} B & 0 \\ 0 & -A \end{bmatrix}.$$

1. Let the columns of $\begin{bmatrix} U \\ V \end{bmatrix} \in \mathbb{C}^{n+m\times n}$ be the eigenvectors of M associated with the eigenvalues of B and suppose U is nonsingular. Show that
$$X = VU^{-1}$$
solves the Sylvester equation. Moreover, every solution of the Sylvester equation can be written in the above form.

2. Show that the Sylvester equation has a solution if and only if M and N are similar. (See Lancaster and Tismenetsky [1985, page 423].)

Problem 12.11 Let $A \in \mathbb{R}^{n\times n}$. Show that
$$P(t) = \int_0^t e^{A^*\tau} Q e^{A\tau} d\tau$$
satisfies
$$\dot{P}(t) = A^*P(t) + P(t)A + Q, \quad P(0) = 0.$$

Problem 12.12 A more general case of the above problem is when the given matrices are time varying and the initial condition is not zero. Let $A(t), Q(t), P_0 \in \mathbb{R}^{n\times n}$. Show that
$$P(t) = \Phi^T(t,t_0)P_0\Phi(t,t_0) + \int_{t_0}^t \Phi^T(t,\tau)Q(\tau)\Phi(t,\tau)d\tau$$
satisfies
$$\dot{P}(t) = A^*P(t) + P(t)A + Q(t), \quad P(t_0) = P_0$$
where $\Phi(t,\tau)$ is the state transition matrix for the system $\dot{x} = A(t)x$.

Problem 12.13 Let $A \in \mathbb{R}^{n\times n}$, $R = R^*$, $Q = Q^*$. Define
$$H = \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}.$$
Let
$$\Theta(t) = \begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ \Theta_{21}(t) & \Theta_{22}(t) \end{bmatrix} = e^{Ht}.$$
Show that
$$P(t) = (\Theta_{21}(t) + \Theta_{22}P_0)(\Theta_{11}(t) + \Theta_{12}(t)P_0)^{-1}$$
is the solution to the following differential Riccati equation:
$$-\dot{P}(t) = A^*P(t) + P(t)A + PRP + Q, \quad P(0) = P_0.$$

Problem 12.14 Let $A \in \mathbb{R}^{n\times n}$, $R = R^*$, $Q = Q^*$. Define

$$H = \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}.$$

Let

$$\Theta(t) = \begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ \Theta_{21}(t) & \Theta_{22}(t) \end{bmatrix} = e^{H(t-T)}.$$

Show that

$$P(t) = \Theta_{21}(t)\Theta_{11}^{-1}(t)$$

is the solution to the following differential Riccati equation:

$$-\dot{P}(t) = A^*P(t) + P(t)A + PRP + Q, \quad P(T) = 0.$$

Problem 12.15 Suppose $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ and $\|G\|_\infty < \gamma$. Show that $A + BR^{-1}D^*C$ with $R = \gamma^2 I - D^*D$ is stable. (Hint: Show $A + B(I - \Delta D/\gamma)^{-1}\Delta C/\gamma$ is stable for all Δ with $\|\Delta\| \le 1$.)