Chapter 7

Balanced Model Reduction

Simple linear models/controllers are normally preferred over complex ones in control system design for some obvious reasons: They are much easier to do analysis and synthesis with. Furthermore, simple controllers are easier to implement and are more reliable because there are fewer things to go wrong in the hardware or bugs to fix in the software. In the case when the system is infinite dimensional, the model/controller approximation becomes essential. In this chapter we consider the problem of reducing the order of a linear multivariable dynamical system. There are many ways to reduce the order of a dynamical system. However, we shall study only one of them: the balanced truncation method. The main advantage of this method is that it is simple and performs fairly well.

A model order-reduction problem can, in general, be stated as follows: Given a full-order model $G(s)$, find a lower-order model (say, an rth order model G_r), such that G and G_r are close in some sense. Of course, there are many ways to define the closeness of an approximation. For example, one may desire that the reduced model be such that

$$G = G_r + \Delta_a$$

and Δ_a is small in some norm. This model reduction is usually called an *additive* model reduction problem. We shall be only interested in $\mathcal{L}_\infty$ norm approximation in this book. Once the norm is chosen, the additive model reduction problem can be formulated as

$$\inf_{\deg(G_r)\leq r} \|G - G_r\|_\infty .$$

In general, a practical model reduction problem is inherently frequency-weighted (i.e., the requirement on the approximation accuracy at one frequency range can be drastically different from the requirement at another frequency range). These problems can, in general, be formulated as frequency-weighted model reduction problems:

$$\inf_{\deg(G_r)\leq r} \|W_o(G - G_r)W_i\|_\infty$$

with an appropriate choice of W_i and W_o. We shall see in this chapter how the balanced realization can give an effective approach to the aforementioned model reduction problems.

7.1 Lyapunov Equations

Testing stability, controllability, and observability of a system is very important in linear system analysis and synthesis. However, these tests often have to be done indirectly. In that respect, the Lyapunov theory is sometimes useful. Consider the following Lyapunov equation:

$$A^*Q + QA + H = 0 \tag{7.1}$$

with given real matrices A and H. It is well known that this equation has a unique solution iff $\lambda_i(A) + \bar{\lambda}_j(A) \neq 0, \forall i, j$. In this section, we shall study the relationships between the solution Q and the stability of A. The following results are standard.

Lemma 7.1 *Assume that A is stable, then the following statements hold:*

(i) $Q = \int_0^\infty e^{A^*t} H e^{At} dt$.

(ii) $Q > 0$ *if* $H > 0$ *and* $Q \geq 0$ *if* $H \geq 0$.

(iii) If $H \geq 0$, *then* (H, A) *is observable iff* $Q > 0$.

An immediate consequence of part (iii) is that, given a stable matrix A, a pair (C, A) is observable if and only if the solution to the following Lyapunov equation

$$A^*Q + QA + C^*C = 0$$

is positive definite, where Q is the *observability Gramian*. Similarly, a pair (A, B) is controllable if and only if the solution to

$$AP + PA^* + BB^* = 0$$

is positive definite, where P is the *controllability Gramian*.

In many applications, we are given the solution of the Lyapunov equation and need to conclude the stability of the matrix A.

Lemma 7.2 *Suppose Q is the solution of the Lyapunov equation (7.1), then*

(i) $\mathrm{Re}\lambda_i(A) \leq 0$ *if* $Q > 0$ *and* $H \geq 0$.

(ii) A *is stable if* $Q > 0$ *and* $H > 0$.

(iii) A *is stable if* $Q \geq 0$, $H \geq 0$, *and* (H, A) *is detectable.*

Proof. Let λ be an eigenvalue of A and $v \neq 0$ be a corresponding eigenvector, then $Av = \lambda v$. Premultiply equation (7.1) by v^* and postmultiply equation (7.1) by v to get

$$2\text{Re}\ \lambda(v^*Qv) + v^*Hv = 0.$$

Now if $Q > 0$, then $v^*Qv > 0$, and it is clear that $\text{Re}\lambda \leq 0$ if $H \geq 0$ and $\text{Re}\lambda < 0$ if $H > 0$. Hence (i) and (ii) hold. To see (iii), we assume $\text{Re}\lambda \geq 0$. Then we must have $v^*Hv = 0$ (i.e., $Hv = 0$). This implies that λ is an unstable and unobservable mode, which contradicts the assumption that (H, A) is detectable. □

7.2 Balanced Realizations

Although there are infinitely many different state-space realizations for a given transfer matrix, some particular realizations have proven to be very useful in control engineering and signal processing. Here we will only introduce one class of realizations for stable transfer matrices that are most useful in control applications. To motivate the class of realizations, we first consider some simple facts.

Lemma 7.3 *Let* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *be a state-space realization of a (not necessarily stable) transfer matrix* $G(s)$. *Suppose that there exists a symmetric matrix*

$$P = P^* = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with P_1 *nonsingular such that*

$$AP + PA^* + BB^* = 0.$$

Now partition the realization (A, B, C, D) *compatibly with* P *as*

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then $\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ *is also a realization of* G. *Moreover,* (A_{11}, B_1) *is controllable if* A_{11} *is stable.*

Proof. Use the partitioned P and (A, B, C) to get

$$0 = AP + PA^* + BB^* = \begin{bmatrix} A_{11}P_1 + P_1A_{11}^* + B_1B_1^* & P_1A_{21}^* + B_1B_2^* \\ A_{21}P_1 + B_2B_1^* & B_2B_2^* \end{bmatrix},$$

which gives $B_2 = 0$ and $A_{21} = 0$ since P_1 is nonsingular. Hence, part of the realization is not controllable:

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right].$$

Finally, it follows from Lemma 7.1 that (A_{11}, B_1) is controllable if A_{11} is stable. □

We also have the following:

Lemma 7.4 *Let* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *be a state-space realization of a (not necessarily stable) transfer matrix* $G(s)$*. Suppose that there exists a symmetric matrix*

$$Q = Q^* = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with Q_1 *nonsingular such that*

$$QA + A^*Q + C^*C = 0.$$

Now partition the realization (A, B, C, D) *compatibly with* Q *as*

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then $\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ *is also a realization of* G*. Moreover,* (C_1, A_{11}) *is observable if* A_{11} *is stable.*

The preceding two lemmas suggest that to obtain a minimal realization from a stable nonminimal realization, one only needs to eliminate all states corresponding to the zero block diagonal term of the controllability Gramian P and the observability Gramian Q. In the case where P is not block diagonal, the following procedure can be used to eliminate noncontrollable subsystems:

1. Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a stable realization.

2. Compute the controllability Gramian $P \geq 0$ from

$$AP + PA^* + BB^* = 0.$$

3. Diagonalize P to get $P = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}^*$ with $\Lambda_1 > 0$ and $\begin{bmatrix} U_1 & U_2 \end{bmatrix}$ unitary.

4. Then $G(s) = \left[\begin{array}{c|c} U_1^* A U_1 & U_1^* B \\ \hline C U_1 & D \end{array}\right]$ is a controllable realization.

A dual procedure can also be applied to eliminate nonobservable subsystems.

Now assume that $\Lambda_1 > 0$ is diagonal and is partitioned as $\Lambda_1 = \mathrm{diag}(\Lambda_{11}, \Lambda_{12})$ such that $\lambda_{\max}(\Lambda_{12}) \ll \lambda_{\min}(\Lambda_{11})$; then it is tempting to conclude that one can also discard those states corresponding to Λ_{12} without causing much error. However, this is not necessarily true, as shown in the following example.

Example 7.1 Consider a stable transfer function

$$G(s) = \frac{3s + 18}{s^2 + 3s + 18}.$$

Then $G(s)$ has a state-space realization given by

$$G(s) = \left[\begin{array}{cc|c} -1 & -4/\alpha & 1 \\ 4\alpha & -2 & 2\alpha \\ \hline -1 & 2/\alpha & 0 \end{array}\right]$$

where α is any nonzero number. It is easy to check that the controllability Gramian of the realization is given by

$$P = \begin{bmatrix} 0.5 & \\ & \alpha^2 \end{bmatrix}.$$

Since the last diagonal term of P can be made arbitrarily small by making α small, the controllability of the corresponding state can be made arbitrarily weak. If the state corresponding to the last diagonal term of P is removed, we get a transfer function

$$\hat{G} = \left[\begin{array}{c|c} -1 & 1 \\ \hline -1 & 0 \end{array}\right] = \frac{-1}{s+1},$$

which is not close to the original transfer function in any sense. The problem may be easily detected if one checks the observability Gramian Q, which is

$$Q = \begin{bmatrix} 0.5 & \\ & 1/\alpha^2 \end{bmatrix}.$$

Since $1/\alpha^2$ is very large if α is small, this shows that the state corresponding to the last diagonal term is strongly observable.

This example shows that the controllability (or observability) Gramian alone cannot give an accurate indication of the dominance of the system states in the input/output behavior. This motivates the introduction of a balanced realization that gives balanced Gramians for controllability and observability.

Suppose $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is stable (i.e., A is stable). Let P and Q denote the controllability Gramian and observability Gramian, respectively. Then by Lemma 7.1, P and Q satisfy the following Lyapunov equations:

$$AP + PA^* + BB^* = 0 \tag{7.2}$$

$$A^*Q + QA + C^*C = 0, \tag{7.3}$$

and $P \geq 0$, $Q \geq 0$. Furthermore, the pair (A, B) is controllable iff $P > 0$, and (C, A) is observable iff $Q > 0$.

Suppose the state is transformed by a nonsingular T to $\hat{x} = Tx$ to yield the realization

$$G = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right].$$

Then the Gramians are transformed to $\hat{P} = TPT^*$ and $\hat{Q} = (T^{-1})^*QT^{-1}$. Note that $\hat{P}\hat{Q} = TPQT^{-1}$, and therefore the eigenvalues of the product of the Gramians are invariant under state transformation.

Consider the similarity transformation T, which gives the eigenvector decomposition

$$PQ = T^{-1}\Lambda T, \quad \Lambda = \operatorname{diag}(\lambda_1 I_{s_1}, \ldots, \lambda_N I_{s_N}).$$

Then the columns of T^{-1} are eigenvectors of PQ corresponding to the eigenvalues $\{\lambda_i\}$. Later, it will be shown that PQ has a real diagonal Jordan form and that $\Lambda \geq 0$, which are consequences of $P \geq 0$ and $Q \geq 0$.

Although the eigenvectors are not unique, in the case of a minimal realization they can always be chosen such that

$$\hat{P} = TPT^* = \Sigma,$$

$$\hat{Q} = (T^{-1})^*QT^{-1} = \Sigma,$$

where $\Sigma = \operatorname{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_N I_{s_N})$ and $\Sigma^2 = \Lambda$. This new realization with controllability and observability Gramians $\hat{P} = \hat{Q} = \Sigma$ will be referred to as a *balanced realization* (also called internally balanced realization). The decreasingly ordered numbers, $\sigma_1 > \sigma_2 > \ldots > \sigma_N \geq 0$, are called the *Hankel singular values* of the system.

More generally, if a realization of a stable system is not minimal, then there is a transformation such that the controllability and observability Gramians for the transformed realization are diagonal and the controllable and observable subsystem is balanced. This is a consequence of the following matrix fact.

Theorem 7.5 *Let P and Q be two positive semidefinite matrices. Then there exists a nonsingular matrix T such that*

$$TPT^* = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad (T^{-1})^* Q T^{-1} = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

respectively, with $\Sigma_1,\ \Sigma_2,\ \Sigma_3$ diagonal and positive definite.

Proof. Since P is a positive semidefinite matrix, there exists a transformation T_1 such that

$$T_1 P T_1^* = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Now let

$$(T_1^*)^{-1} Q T_1^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}$$

and there exists a unitary matrix U_1 such that

$$U_1 Q_{11} U_1^* = \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_1 > 0$$

Let

$$(T_2^*)^{-1} = \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}$$

and then

$$(T_2^*)^{-1}(T_1^*)^{-1} Q T_1^{-1}(T_2)^{-1} = \begin{bmatrix} \Sigma_1^2 & 0 & \hat{Q}_{121} \\ 0 & 0 & \hat{Q}_{122} \\ \hat{Q}_{121}^* & \hat{Q}_{122}^* & Q_{22} \end{bmatrix}$$

But $Q \geq 0$ implies $\hat{Q}_{122} = 0$. So now let

$$(T_3^*)^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\hat{Q}_{121}^* \Sigma_1^{-2} & 0 & I \end{bmatrix}$$

giving

$$(T_3^*)^{-1}(T_2^*)^{-1}(T_1^*)^{-1} Q T_1^{-1}(T_2)^{-1}(T_3)^{-1} = \begin{bmatrix} \Sigma_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_{22} - \hat{Q}_{121}^* \Sigma_1^{-2} \hat{Q}_{121} \end{bmatrix}$$

Next find a unitary matrix U_2 such that

$$U_2 (Q_{22} - \hat{Q}_{121}^* \Sigma_1^{-2} \hat{Q}_{121}) U_2^* = \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_3 > 0$$

Define

$$(T_4^*)^{-1} = \begin{bmatrix} \Sigma_1^{-1/2} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U_2 \end{bmatrix}$$

and let

$$T = T_4 T_3 T_2 T_1$$

Then

$$TPT^* = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad (T^*)^{-1} Q T^{-1} = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

with $\Sigma_2 = I$. □

Corollary 7.6 *The product of two positive semidefinite matrices is similar to a positive semidefinite matrix.*

Proof. Let P and Q be any positive semidefinite matrices. Then it is easy to see that with the transformation given previously

$$TPQT^{-1} = \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

□

Corollary 7.7 *For any stable system* $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$*, there exists a nonsingular transformation* T *such that* $G = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$ *has controllability Gramian* P *and observability Gramian* Q *given by*

$$P = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

respectively, with $\Sigma_1, \ \Sigma_2, \ \Sigma_3$ *diagonal and positive definite.*

In the special case where $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a minimal realization, a balanced realization can be obtained through the following simplified procedure:

1. Compute the controllability and observability Gramians $P > 0, Q > 0$.
2. Find a matrix R such that $P = R^* R$.
3. Diagonalize RQR^* to get $RQR^* = U\Sigma^2 U^*$.
4. Let $T^{-1} = R^* U \Sigma^{-1/2}$. Then $TPT^* = (T^*)^{-1} Q T^{-1} = \Sigma$ and $\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$ is balanced.

Assume that the Hankel singular values of the system are decreasingly ordered so that $\Sigma = \mathrm{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_N I_{s_N})$ with $\sigma_1 > \sigma_2 > \ldots > \sigma_N$ and suppose $\sigma_r \gg \sigma_{r+1}$ for some r. Then the balanced realization implies that those states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_N$ are less controllable and less observable than those states corresponding to $\sigma_1, \ldots, \sigma_r$. Therefore, truncating those less controllable and less observable states will not lose much information about the system.

Two other closely related realizations are called *input normal realization* with $P = I$ and $Q = \Sigma^2$, and *output normal realization* with $P = \Sigma^2$ and $Q = I$. Both realizations can be obtained easily from the balanced realization by a suitable scaling on the states.

Next we shall derive some simple and useful bounds for the $\mathcal{H}_\infty$ norm and the $\mathcal{L}_1$ norm of a stable system.

Theorem 7.8 *Suppose*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \mathcal{RH}_\infty$$

is a balanced realization; that is, there exists

$$\Sigma = diag(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_N I_{s_N}) \geq 0$$

with $\sigma_1 > \sigma_2 > \ldots > \sigma_N \geq 0$, such that

$$A\Sigma + \Sigma A^* + BB^* = 0 \qquad A^*\Sigma + \Sigma A + C^*C = 0$$

Then

$$\sigma_1 \leq \|G\|_\infty \leq \int_0^\infty \|g(t)\|\, dt \leq 2\sum_{i=1}^{N} \sigma_i$$

where $g(t) = Ce^{At}B$.

Remark 7.1 It should be clear that the inequalities stated in the theorem do not depend on a particular state-space realization of $G(s)$. However, use of the balanced realization does make the proof simple. $\diamond$

Proof. Let $G(s)$ have the following state-space realization:

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx. \end{aligned} \tag{7.4}$$

Assume without loss of generality that (A, B) is controllable and (C, A) is observable. Then Σ is nonsingular. Next, differentiate $x(t)^*\Sigma^{-1}x(t)$ along the solution of equation (7.4) for any given input w as follows:

$$\frac{d}{dt}(x^*\Sigma^{-1}x) = \dot{x}^*\Sigma^{-1}x + x^*\Sigma^{-1}\dot{x} = x^*(A^*\Sigma^{-1} + \Sigma^{-1}A)x + 2\langle w, B^*\Sigma^{-1}x\rangle$$

Using the equation involving controllability Gramian to substitute for $A^*\Sigma^{-1} + \Sigma^{-1}A$ and completion of the squares gives

$$\frac{d}{dt}(x^*\Sigma^{-1}x) = \|w\|^2 - \|w - B^*\Sigma^{-1}x\|^2$$

Integration from $t = -\infty$ to $t = 0$ with $x(-\infty) = 0$ and $x(0) = x_0$ gives

$$x_0^*\Sigma^{-1}x_0 = \|w\|_2^2 - \|w - B^*\Sigma^{-1}x\|_2^2 \leq \|w\|_2^2$$

Let $w = B^*\Sigma^{-1}x$; then $\dot{x} = (A + BB^*\Sigma^{-1})x = -\Sigma A^*\Sigma^{-1}x \implies x \in \mathcal{L}_2(-\infty, 0] \implies w \in \mathcal{L}_2(-\infty, 0]$ and

$$\inf_{w\in\mathcal{L}_2(-\infty,0]} \left\{ \|w\|_2^2 \;\middle|\; x(0) = x_0 \right\} = x_0^*\Sigma^{-1}x_0.$$

Given $x(0) = x_0$ and $w = 0$ for $t \geq 0$, the norm of $z(t) = Ce^{At}x_0$ can be found from

$$\int_0^\infty \|z(t)\|^2\, dt = \int_0^\infty x_0^* e^{A^*t}C^*Ce^{At}x_0 dt = x_0^*\Sigma x_0$$

To show $\sigma_1 \leq \|G\|_\infty$, note that

$$\|G\|_\infty = \sup_{w\in\mathcal{L}_2(-\infty,\infty)} \frac{\|g * w\|_2}{\|w\|_2} = \sup_{w\in\mathcal{L}_2(-\infty,\infty)} \frac{\sqrt{\int_{-\infty}^\infty \|z(t)\|^2\, dt}}{\sqrt{\int_{-\infty}^\infty \|w(t)\|^2\, dt}}$$

$$\geq \sup_{w\in\mathcal{L}_2(-\infty,0]} \frac{\sqrt{\int_0^\infty \|z(t)\|^2\, dt}}{\sqrt{\int_{-\infty}^0 \|w(t)\|^2\, dt}} = \sup_{x_0\neq 0} \sqrt{\frac{x_0^*\Sigma x_0}{x_0^*\Sigma^{-1}x_0}} = \sigma_1$$

We shall now show the other inequalities. Since

$$G(s) := \int_0^\infty g(t)e^{-st}dt,\ \mathrm{Re}(s) > 0,$$

by the definition of $\mathcal{H}_\infty$ norm, we have

$$\begin{aligned}
\|G\|_\infty &= \sup_{\mathrm{Re}(s)>0} \left\| \int_0^\infty g(t)e^{-st}dt \right\| \\
&\leq \sup_{\mathrm{Re}(s)>0} \int_0^\infty \left\| g(t)e^{-st} \right\| dt \\
&\leq \int_0^\infty \|g(t)\|\, dt.
\end{aligned}$$

To prove the last inequality, let e_i be the ith unit vector and define

$$E_1 = \begin{bmatrix} e_1 & \cdots & e_{s_1} \end{bmatrix}, \quad E_2 = \begin{bmatrix} e_{s_1+1} & \cdots & e_{s_1+s_2} \end{bmatrix}, \quad \ldots,$$
$$E_N = \begin{bmatrix} e_{s_1+\cdots+s_{N-1}+1} & \cdots & e_{s_1+\cdots+s_N} \end{bmatrix}.$$

Then $\sum_{i=1}^N E_i E_i^* = I$ and

$$\begin{aligned}
\int_0^\infty \|g(t)\|\, dt &= \int_0^\infty \left\| Ce^{At/2} \sum_{i=1}^N E_i E_i^* e^{At/2} B \right\| dt \\
&\leq \sum_{i=1}^N \int_0^\infty \left\| Ce^{At/2} E_i E_i^* e^{At/2} B \right\| dt \\
&\leq \sum_{i=1}^N \int_0^\infty \left\| Ce^{At/2} E_i \right\| \left\| E_i^* e^{At/2} B \right\| dt \\
&\leq \sum_{i=1}^N \sqrt{\int_0^\infty \left\| Ce^{At/2} E_i \right\|^2 dt} \sqrt{\int_0^\infty \left\| E_i^* e^{At/2} B \right\|^2 dt} \leq 2 \sum_{i=1}^N \sigma_i
\end{aligned}$$

where we have used Cauchy-Schwarz inequality and the following relations:

$$\int_0^\infty \left\| Ce^{At/2} E_i \right\|^2 dt = \int_0^\infty \lambda_{\max} \left(E_i^* e^{A^* t/2} C^* C e^{At/2} E_i \right) dt = 2\lambda_{\max} \left(E_i^* \Sigma E_i \right) = 2\sigma_i$$

$$\int_0^\infty \left\| E_i^* e^{At/2} B \right\|^2 dt = \int_0^\infty \lambda_{\max} \left(E_i^* e^{At/2} B B^* e^{A^* t/2} E_i \right) dt = 2\lambda_{\max} \left(E_i^* \Sigma E_i \right) = 2\sigma_i$$

$\square$

Example 7.2 Consider a system

$$G(s) = \left[\begin{array}{cc|c} -1 & -2 & 1 \\ 1 & 0 & 0 \\ \hline 2 & 3 & 0 \end{array}\right]$$

It is easy to show that the Hankel singular values of G are $\sigma_1 = 1.6061$ and $\sigma_2 = 0.8561$. The $\mathcal{H}_\infty$ norm of G is $\|G\|_\infty = 2.972$ and the $\mathcal{L}_1$ norm of $g(t)$ can be computed as

$$\int_0^\infty |g(t)|dt = h_1 + h_2 + h_3 + h_4 + \dots$$

where $h_i, i = 1, 2, \dots$ are the variations of the step response of G shown in Figure 7.1, which gives $\int_0^\infty |g(t)|dt \approx 3.5$. (See Problem 7.2.)

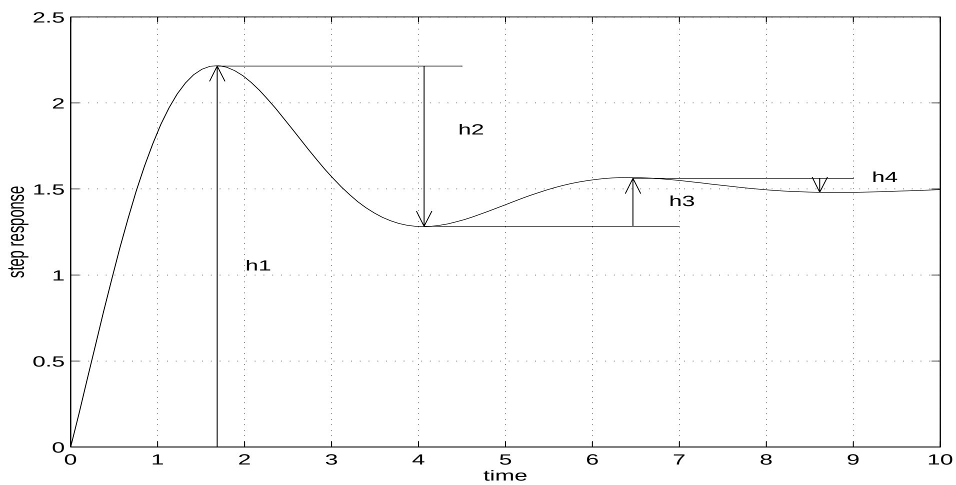

Figure 7.1: Estimating the $\mathcal{L}_1$ norm of $g(t)$

So we have

$$1.6061 = \sigma_1 \le \|G\|_\infty = 2.972 \le \int_0^\infty |g(t)|dt = 3.5 \le 2(\sigma_1 + \sigma_2) = 4.9244.$$

Illustrative MATLAB Commands:

$\gg$ **[Ab, Bb, Cb, sig, Tinv]=balreal(A, B, C);** % sig is a vector of Hankel singular values and Tinv $= T^{-1}$;

$\gg$ $[\mathbf{G_b}, \mathbf{sig}] = \mathbf{sysbal}(\mathbf{G})$;

Related MATLAB Commands: ssdelete, ssselect, modred, strunc

7.3 Model Reduction by Balanced Truncation

Consider a stable system $G \in \mathcal{RH}_\infty$ and suppose $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a balanced realization (i.e., its controllability and observability Gramians are equal and diagonal). Denote the balanced Gramians by Σ; then

$$A\Sigma + \Sigma A^* + BB^* = 0 \tag{7.5}$$

$$A^*\Sigma + \Sigma A + C^*C = 0. \tag{7.6}$$

Now partition the balanced Gramian as $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$ and partition the system accordingly as

$$G = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

The following theorem characterizes the properties of these subsystems.

Theorem 7.9 *Assume that Σ_1 and Σ_2 have no diagonal entries in common. Then both subsystems (A_{ii}, B_i, C_i), $i = 1, 2$ are asymptotically stable.*

Proof. It is clearly sufficient to show that A_{11} is asymptotically stable. The proof for the stability of A_{22} is similar. Note that equations (7.5) and (7.6) can be written in terms of their partitioned matrices as

$$\begin{aligned}
A_{11}\Sigma_1 + \Sigma_1 A_{11}^* + B_1 B_1^* &= 0 && (7.7)\\
\Sigma_1 A_{11} + A_{11}^*\Sigma_1 + C_1^* C_1 &= 0 && (7.8)\\
A_{21}\Sigma_1 + \Sigma_2 A_{12}^* + B_2 B_1^* &= 0 && (7.9)\\
\Sigma_2 A_{21} + A_{12}^*\Sigma_1 + C_2^* C_1 &= 0 && (7.10)\\
A_{22}\Sigma_2 + \Sigma_2 A_{22}^* + B_2 B_2^* &= 0 && (7.11)\\
\Sigma_2 A_{22} + A_{22}^*\Sigma_2 + C_2^* C_2 &= 0. && (7.12)
\end{aligned}$$

By Lemma 7.3 or Lemma 7.4, Σ_1 can be assumed to be positive definite without loss of generality. Then it is obvious that $\mathrm{Re}\lambda_i(A_{11}) \le 0$ by Lemma 7.2. Assume that A_{11} is not asymptotically stable; then there exists an eigenvalue at $j\omega$ for some ω. Let V be a basis matrix for $\mathrm{Ker}(A_{11} - j\omega I)$. Then we have

$$(A_{11} - j\omega I)V = 0, \tag{7.13}$$

which gives

$$V^*(A_{11}^* + j\omega I) = 0.$$

Equations (7.7) and (7.8) can be rewritten as

$$(A_{11} - j\omega I)\Sigma_1 + \Sigma_1(A_{11}^* + j\omega I) + B_1B_1^* = 0 \tag{7.14}$$
$$\Sigma_1(A_{11} - j\omega I) + (A_{11}^* + j\omega I)\Sigma_1 + C_1^*C_1 = 0. \tag{7.15}$$

Multiplication of equation (7.15) from the right by V and from the left by V^* gives $V^*C_1^*C_1V = 0$, which is equivalent to

$$C_1V = 0.$$

Multiplication of equation (7.15) from the right by V now gives

$$(A_{11}^* + j\omega I)\Sigma_1 V = 0.$$

Analogously, first multiply equation (7.14) from the right by $\Sigma_1 V$ and from the left by $V^*\Sigma_1$ to obtain

$$B_1^*\Sigma_1 V = 0.$$

Then multiply equation (7.14) from the right by $\Sigma_1 V$ to get

$$(A_{11} - j\omega I)\Sigma_1^2 V = 0.$$

It follows that the columns of $\Sigma_1^2 V$ are in $\mathrm{Ker}(A_{11} - j\omega I)$. Therefore, there exists a matrix $\bar{\Sigma}_1$ such that

$$\Sigma_1^2 V = V\bar{\Sigma}_1^2.$$

Since $\bar{\Sigma}_1^2$ is the restriction of Σ_1^2 to the space spanned by V, it follows that it is possible to choose V such that $\bar{\Sigma}_1^2$ is diagonal. It is then also possible to choose $\bar{\Sigma}_1$ diagonal and such that the diagonal entries of $\bar{\Sigma}_1$ are a subset of the diagonal entries of Σ_1.

Multiply equation (7.9) from the right by $\Sigma_1 V$ and equation (7.10) by V to get

$$\begin{aligned} A_{21}\Sigma_1^2 V + \Sigma_2 A_{12}^*\Sigma_1 V &= 0 \\ \Sigma_2 A_{21} V + A_{12}^*\Sigma_1 V &= 0, \end{aligned}$$

which gives

$$(A_{21}V)\bar{\Sigma}_1^2 = \Sigma_2^2(A_{21}V).$$

This is a Sylvester equation in $(A_{21}V)$. Because $\bar{\Sigma}_1^2$ and Σ_2^2 have no diagonal entries in common, it follows that

$$A_{21}V = 0 \tag{7.16}$$

is the unique solution. Now equations (7.16) and (7.13) imply that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix} = j\omega \begin{bmatrix} V \\ 0 \end{bmatrix},$$

which means that the A-matrix of the original system has an eigenvalue at $j\omega$. This contradicts the fact that the original system is asymptotically stable. Therefore, A_{11} must be asymptotically stable. □

Corollary 7.10 *If Σ has distinct singular values, then every subsystem is asymptotically stable.*

The stability condition in Theorem 7.9 is only sufficient as shown in the following example.

Example 7.3 Note that

$$\frac{(s-1)(s-2)}{(s+1)(s+2)} = \left[\begin{array}{cc|c} -2 & -2.8284 & -2 \\ 0 & -1 & -1.4142 \\ \hline 2 & 1.4142 & 1 \end{array}\right]$$

is a balanced realization with $\Sigma = I$, and every subsystem of the realization is stable. On the other hand,

$$\frac{s^2-s+2}{s^2+s+2} = \left[\begin{array}{cc|c} -1 & 1.4142 & 1.4142 \\ -1.4142 & 0 & 0 \\ \hline -1.4142 & 0 & 1 \end{array}\right]$$

is also a balanced realization with $\Sigma = I$, but one of the subsystems is not stable.

Theorem 7.11 *Suppose $G(s) \in \mathcal{RH}_\infty$ and*

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

is a balanced realization with Gramian $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2)$

$$\begin{aligned} \Sigma_1 &= \mathrm{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_r I_{s_r}) \\ \Sigma_2 &= \mathrm{diag}(\sigma_{r+1} I_{s_{r+1}}, \sigma_{r+2} I_{s_{r+2}}, \ldots, \sigma_N I_{s_N}) \end{aligned}$$

and

$$\sigma_1 > \sigma_2 > \cdots > \sigma_r > \sigma_{r+1} > \sigma_{r+2} > \cdots > \sigma_N$$

where σ_i has multiplicity s_i, $i = 1, 2, \ldots, N$ and $s_1 + s_2 + \cdots + s_N = n$. Then the truncated system

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is balanced and asymptotically stable. Furthermore,

$$\|G(s) - G_r(s)\|_\infty \le 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_N)$$

Proof. The stability of G_r follows from Theorem 7.9. We shall first show the one step model reduction. Hence we shall assume $\Sigma_2 = \sigma_N I_{s_N}$. Define the approximation error

$$
\begin{aligned}
E_{11} &:= \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] - \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] \\
&= \left[\begin{array}{ccc|c} A_{11} & 0 & 0 & B_1 \\ 0 & A_{11} & A_{12} & B_1 \\ 0 & A_{21} & A_{22} & B_2 \\ \hline -C_1 & C_1 & C_2 & 0 \end{array}\right]
\end{aligned}
$$

Apply a similarity transformation T to the preceding state-space realization with

$$
T = \begin{bmatrix} I/2 & I/2 & 0 \\ I/2 & -I/2 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I & I & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}
$$

to get

$$
E_{11} = \left[\begin{array}{ccc|c} A_{11} & 0 & A_{12}/2 & B_1 \\ 0 & A_{11} & -A_{12}/2 & 0 \\ A_{21} & -A_{21} & A_{22} & B_2 \\ \hline 0 & -2C_1 & C_2 & 0 \end{array}\right]
$$

Consider a dilation of $E_{11}(s)$:

$$
\begin{aligned}
E(s) &= \begin{bmatrix} E_{11}(s) & E_{12}(s) \\ E_{21}(s) & E_{22}(s) \end{bmatrix} \\
&= \left[\begin{array}{ccc|cc} A_{11} & 0 & A_{12}/2 & B_1 & 0 \\ 0 & A_{11} & -A_{12}/2 & 0 & \sigma_N \Sigma_1^{-1} C_1^* \\ A_{21} & -A_{21} & A_{22} & B_2 & -C_2^* \\ \hline 0 & -2C_1 & C_2 & 0 & 2\sigma_N I \\ -2\sigma_N B_1^* \Sigma_1^{-1} & 0 & -B_2^* & 2\sigma_N I & 0 \end{array}\right] \\
&=: \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]
\end{aligned}
$$

Then it is easy to verify that

$$
\tilde{P} = \begin{bmatrix} \Sigma_1 & 0 & \\ 0 & \sigma_N^2 \Sigma_1^{-1} & 0 \\ 0 & 0 & 2\sigma_N I_{s_N} \end{bmatrix}
$$

satisfies

$$
\begin{aligned}
\tilde{A}\tilde{P} + \tilde{P}\tilde{A}^* + \tilde{B}\tilde{B}^* &= 0 \\
\tilde{P}\tilde{C}^* + \tilde{B}\tilde{D}^* &= 0
\end{aligned}
$$

Using these two equations, we have

$$
\begin{aligned}
E(s)E^{\sim}(s) &= \left[\begin{array}{cc|c} \tilde{A} & -\tilde{B}\tilde{B}^* & \tilde{B}\tilde{D}^* \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & -\tilde{D}\tilde{B}^* & \tilde{D}\tilde{D}^* \end{array}\right] \\
&= \left[\begin{array}{cc|c} \tilde{A} & -\tilde{A}\tilde{P} - \tilde{P}\tilde{A}^* - \tilde{B}\tilde{B}^* & \tilde{P}\tilde{C}^* + \tilde{B}\tilde{D}^* \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & -\tilde{C}\tilde{P} - \tilde{D}\tilde{B}^* & \tilde{D}\tilde{D}^* \end{array}\right] \\
&= \left[\begin{array}{cc|c} \tilde{A} & 0 & 0 \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & 0 & \tilde{D}\tilde{D}^* \end{array}\right] \\
&= \tilde{D}\tilde{D}^* = 4\sigma_N^2 I
\end{aligned}
$$

where the second equality is obtained by applying a similarity transformation

$$T = \begin{bmatrix} I & \tilde{P} \\ 0 & I \end{bmatrix}$$

Hence $\|E_{11}\|_\infty \le \|E\|_\infty = 2\sigma_N$, which is the desired result.

The remainder of the proof is achieved by using the order reduction by one-step results and by noting that $G_k(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ obtained by the "kth" order partitioning is internally balanced with balanced Gramian given by

$$\Sigma_1 = \operatorname{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_k I_{s_k})$$

Let $E_k(s) = G_{k+1}(s) - G_k(s)$ for $k = 1, 2, \ldots, N-1$ and let $G_N(s) = G(s)$. Then

$$\overline{\sigma}\left[E_k(j\omega)\right] \le 2\sigma_{k+1}$$

since $G_k(s)$ is a reduced-order model obtained from the internally balanced realization of $G_{k+1}(s)$ and the bound for one-step order reduction holds.

Noting that

$$G(s) - G_r(s) = \sum_{k=r}^{N-1} E_k(s)$$

by the definition of $E_k(s)$, we have

$$\overline{\sigma}\left[G(j\omega) - G_r(j\omega)\right] \le \sum_{k=r}^{N-1} \overline{\sigma}\left[E_k(j\omega)\right] \le 2 \sum_{k=r}^{N-1} \sigma_{k+1}$$

This is the desired upper bound. □

A useful consequence of the preceding theorem is the following corollary.

Corollary 7.12 *Let $\sigma_i, i = 1, \ldots, N$ be the Hankel singular values of $G(s) \in \mathcal{RH}_\infty$. Then*

$$\|G(s) - G(\infty)\|_\infty \leq 2(\sigma_1 + \ldots + \sigma_N)$$

The above bound can be tight for some systems.

Example 7.4 Consider an nth-order transfer function

$$G(s) = \sum_{i=1}^{n} \frac{b_i}{s + a_i},$$

with $a_i > 0$ and $b_i > 0$. Then $\|G(s)\|_\infty = G(0) = \sum_{i=1}^{n} b_i/a_i$ and $G(s)$ has the following state-space realization:

$$G = \left[\begin{array}{cccc|c} -a_1 & & & & \sqrt{b_1} \\ & -a_2 & & & \sqrt{b_2} \\ & & \ddots & & \vdots \\ & & & -a_n & \sqrt{b_n} \\ \hline \sqrt{b_1} & \sqrt{b_2} & \cdots & \sqrt{b_n} & 0 \end{array}\right]$$

and the controllability and observability Gramians of the realization are given by

$$P = Q = \left[\frac{\sqrt{b_i b_j}}{a_i + a_j}\right]$$

It is easy to see that $\sigma_i = \lambda_i(P) = \lambda_i(Q)$ and

$$\sum_{i=1}^{n} \sigma_i = \sum_{i=1}^{n} \lambda_i(P) = \operatorname{trace}(P) = \sum_{i=1}^{n} \frac{b_i}{2a_i} = \frac{1}{2} G(0) = \frac{1}{2} \|G\|_\infty$$

In particular, let $a_i = b_i = \alpha^{2i}$; then $P = Q \to \frac{1}{2} I_n$ (i.e., $\sigma_j \to \frac{1}{2}$ as $\alpha \to \infty$). This example also shows that even when the Hankel singular values are extremely close, they may not be regarded as repeated singular values.

The model reduction bound can also be loose for systems with Hankel singular values close to each other.

Example 7.5 Consider the balanced realization of a fourth-order system:

$$G(s) = \frac{(s-0.99)(s-2)(s-3)(s-4)}{(s+1)(s+2)(s+3)(s+4)}$$

$$= \left[\begin{array}{cccc|c} -9.2e+00 & -5.7e+00 & -2.7e+00 & 1.3e+00 & -4.3e+00 \\ 5.7e+00 & -8.1e-07 & -6.4e-01 & 1.5e-06 & 1.3e-03 \\ -2.7e+00 & 6.4e-01 & -7.9e-01 & 7.1e-01 & -1.3e+00 \\ -1.3e+00 & 1.5e-06 & -7.1e-01 & -2.7e-06 & -2.3e-03 \\ \hline 4.3e+00 & 1.3e-03 & 1.3e+00 & -2.3e-03 & 1.0e+00 \end{array}\right]$$

with Hankel singular values given by

$$\sigma_1 = 0.9998, \quad \sigma_2 = 0.9988, \quad \sigma_3 = 0.9963, \sigma_4 = 0.9923.$$

The approximation errors and the estimated bounds are listed in the following table. The table shows that the actual error for an rth-order approximation is almost the same as $2\sigma_{r+1}$, which would be the estimated bound if we regard $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_4$. In general, it is not hard to construct an nth-order system so that the rth-order balanced model reduction error is approximately $2\sigma_{r+1}$ but the error bound is arbitrarily close to $2(n-r)\sigma_{r+1}$. One method to construct such a system is as follows: Let $G(s)$ be a stable all-pass function, that is, $G^{\sim}(s)G(s) = I$. Then there is a balanced realization for G so that the controllability and observability Gramians are $P = Q = I$. Next, make a very small perturbation to the balanced realization, then the perturbed system has a balanced realization with distinct singular values and $P = Q \approx I$. This perturbed system will have the desired properties.

r	0	1	2	3
$\|G - G_r\|_\infty$	1.9997	1.9983	1.9933	1.9845
Bounds: $2\sum_{i=r+1}^{4} \sigma_i$	7.9744	5.9748	3.9772	1.9845
$2\sigma_{r+1}$	1.9996	1.9976	1.9926	1.9845

The balanced realization and truncation can be done using the following MATLAB commands:

$\gg$ $[\mathbf{G_b}, \mathbf{sig}] = \mathbf{sysbal}(\mathbf{G})$; % find a balanced realization G_b and the Hankel singular values sig.

$\gg$ $\mathbf{G_r} = \mathbf{strunc}(\mathbf{G_b}, \mathbf{2})$; % truncate to the second-order.

Related MATLAB Commands: reordsys, resid, Hankmr

7.4 Frequency-Weighted Balanced Model Reduction

This section considers the extension of the balanced truncation method to the frequency-weighted case. Given the original full-order model $G \in \mathcal{RH}_\infty$, the input weighting matrix $W_i \in \mathcal{RH}_\infty$, and the output weighting matrix $W_o \in \mathcal{RH}_\infty$, our objective is to find a lower-order model G_r such that

$$\|W_o(G - G_r)W_i\|_\infty$$

is made as small as possible. Assume that G, W_i, and W_o have the following state-space realizations:

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \quad W_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad W_o = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right]$$

with $A \in \mathbb{R}^{n\times n}$. Note that there is no loss of generality in assuming $D = G(\infty) = 0$ since otherwise it can be eliminated by replacing G_r with $D + G_r$.

Now the state-space realization for the weighted transfer matrix is given by

$$W_oGW_i = \left[\begin{array}{ccc|c} A & 0 & BC_i & BD_i \\ B_oC & A_o & 0 & 0 \\ 0 & 0 & A_i & B_i \\ \hline D_oC & C_o & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array}\right]$$

Let $\bar{P}$ and $\bar{Q}$ be the solutions to the following Lyapunov equations:

$$\bar{A}\bar{P} + \bar{P}\bar{A}^* + \bar{B}\bar{B}^* = 0 \tag{7.17}$$
$$\bar{Q}\bar{A} + \bar{A}^*\bar{Q} + \bar{C}^*\bar{C} = 0 \tag{7.18}$$

Then the input weighted Gramian P and the output weighted Gramian Q are defined by

$$P := \begin{bmatrix} I_n & 0 \end{bmatrix} \bar{P} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad Q := \begin{bmatrix} I_n & 0 \end{bmatrix} \bar{Q} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

It can be shown easily that P and Q satisfy the following lower-order equations:

$$\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}^* + \begin{bmatrix} BD_i \\ B_i \end{bmatrix} \begin{bmatrix} BD_i \\ B_i \end{bmatrix}^* = 0 \tag{7.19}$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix} + \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix}^* \begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} C^*D_o^* \\ C_o^* \end{bmatrix} \begin{bmatrix} C^*D_o^* \\ C_o^* \end{bmatrix}^* = 0 \tag{7.20}$$

The computation can be further reduced if $W_i = I$ or $W_o = I$. In the case of $W_i = I$, P can be obtained from

$$PA^* + AP + BB^* = 0 \tag{7.21}$$

while in the case of $W_o = I$, Q can be obtained from

$$QA + A^*Q + C^*C = 0 \tag{7.22}$$

Now let T be a nonsingular matrix such that

$$TPT^* = (T^{-1})^*QT^{-1} = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

(i.e., balanced) with $\Sigma_1 = \text{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r})$ and $\Sigma_2 = \text{diag}(\sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N})$ and partition the system accordingly as

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right]$$

Then a reduced-order model G_r is obtained as

$$G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right]$$

Unfortunately, there is generally no known a priori error bound for the approximation error and the reduced-order model G_r is not guaranteed to be stable either.

A very special frequency-weighted model reduction problem is the relative error model reduction problem where the objective is to find a reduced-order model G_r so that

$$G_r = G(I + \Delta_{\text{rel}})$$

and $\|\Delta_{\text{rel}}\|_\infty$ is made as small as possible. Δ_{rel} is usually called the *relative error*. In the case where G is square and invertible, this problem can be simply formulated as

$$\min_{\deg G_r \le r} \left\| G^{-1}(G - G_r) \right\|_\infty .$$

Of course, the dual approximation problem

$$G_r = (I + \Delta_{\text{rel}})G$$

can be obtained by taking the transpose of G. It turns out that the approximation G_r obtained below also serves as a multiplicative approximation:

$$G = G_r(I + \Delta_{\text{mul}})$$

where Δ_{mul} is usually called the *multiplicative error*.

Error bounds can be derived if the frequency-weighted balanced truncation method is applied to the relative and multiplicative approximations.

Theorem 7.13 *Let $G, G^{-1} \in \mathcal{RH}_\infty$ be an nth-order square transfer matrix with a state-space realization*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Let P and Q be the solutions to

$$PA^* + AP + BB^* = 0 \tag{7.23}$$

$$Q(A - BD^{-1}C) + (A - BD^{-1}C)^*Q + C^*(D^{-1})^*D^{-1}C = 0 \tag{7.24}$$

Suppose

$$P = Q = \operatorname{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r}, \sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N}) = \operatorname{diag}(\Sigma_1, \Sigma_2)$$

with $\sigma_1 > \sigma_2 > \ldots > \sigma_N \geq 0$, and let the realization of G be partitioned compatibly with Σ_1 and Σ_2 as

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

Then

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is stable and minimum phase. Furthermore,

$$\|\Delta_{\text{rel}}\|_\infty \leq \prod_{i=r+1}^{N} \left(1 + 2\sigma_i(\sqrt{1+\sigma_i^2} + \sigma_i)\right) - 1$$

$$\|\Delta_{\text{mul}}\|_\infty \leq \prod_{i=r+1}^{N} \left(1 + 2\sigma_i(\sqrt{1+\sigma_i^2} + \sigma_i)\right) - 1$$

Related MATLAB Commands: srelbal, sfrwtbal

7.5 Notes and References

Balanced realization was first introduced by Mullis and Roberts [1976] to study roundoff noise in digital filters. Moore [1981] proposed the balanced truncation method for model-reduction. The stability properties of the reduced-order model were shown by Pernebo and Silverman [1982]. The error bound for the balanced model reduction was shown by Enns [1984a, 1984b], and Glover [1984] subsequently gave an independent proof. The frequency-weighted balanced model-reduction method was also introduced by Enns [1984a, 1984b]. The error bounds for the relative error are derived in Zhou [1995]. Other related results are shown in Green [1988]. Other weighted model-reduction methods can be found in Al-Saggaf and Franklin [1988], Glover [1986b], Glover, Limebeer and Hung [1992], Green [1988], Hung and Glover [1986], Zhou [1995], and references therein. Discrete-time balance model-reduction results can be found in Al-Saggaf and Franklin [1987], Hinrichsen and Pritchard [1990], and references therein.

7.6 Problems

Problem 7.1 Use the following relation

$$\frac{d}{dt}\left(e^{At}Qe^{A^*t}\right) = Ae^{At}Qe^{A^*t} + e^{At}Qe^{A^*t}A$$

to show that $P = \int_0^\infty e^{At}Qe^{A^*t}dt$ solves

$$AP + PA^* + Q = 0$$

if A is stable.

Problem 7.2 Let $G(s) \in \mathcal{H}_\infty$ and let $g(t)$ be the inverse Laplace transform of $G(s)$. Let $h_i, i = 1, 2, \ldots$ be the variations of the step response of G. Show that

$$\int_0^\infty |g(t)|dt = h_1 + h_2 + h_3 + h_4 + \ldots$$

Problem 7.3 Let $Q \geq 0$ be the solution to

$$QA + A^*Q + C^*C = 0$$

Suppose Q has m zero eigenvalues. Show that there is a nonsingular matrix T such that

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & 0 & D \end{array}\right], \quad A_{22} \in \mathbb{R}^{m\times m}.$$

Apply the above result to the following state-space model:

$$A = \begin{bmatrix} -4 & -7 & -2 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = 0$$

Problem 7.4 Let

$$G(s) = \sum_{i=1}^{5} \frac{\alpha^{2i}}{s + \alpha^{2i}}$$

Find a balanced realization for each of the following α:

$$\alpha = 2,\ 4,\ 20,\ 100.$$

Discuss the behavior of the Hankel singular values as $\alpha \to \infty$.

Problem 7.5 Find a transformation so that $TPT^* = \Sigma^2, (T^*)^{-1}QT^{-1} = I$. (This realization is called output normalized realization.)

Problem 7.6 Consider the model reduction error:

$$E_{11} = \left[\begin{array}{ccc|c} A_{11} & 0 & A_{12}/2 & B_1 \\ 0 & A_{11} & -A_{12}/2 & 0 \\ A_{21} & -A_{21} & A_{22} & B_2 \\ \hline 0 & -2C_1 & C_2 & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & 0 \end{array}\right].$$

Show that

$$\tilde{P} = \begin{bmatrix} \Sigma_1 & 0 & \\ 0 & \sigma_N^2 \Sigma_1^{-1} & 0 \\ 0 & 0 & 2\sigma_N I_{s_N} \end{bmatrix}$$

satisfies

$$A_e \tilde{P} + \tilde{P} A_e^* + B_e B_e^* + \frac{1}{4\sigma_N^2} \tilde{P} C_e^* C_e \tilde{P} = 0.$$

Problem 7.7 Suppose P and Q are the controllability and observability Gramians of $G(s) = C(sI - A)^{-1}B \in \mathcal{RH}_\infty$. Let $G_d(z) = G(s)|_{s=\frac{z+1}{z-1}} = C_d(zI - A_d)^{-1}B_d + D_d$. Compute the controllability and observability Gramians P_d and Q_d and compare PQ and $P_d Q_d$.

Problem 7.8 Note that a delay can be approximated as

$$e^{-\tau s} \approx \left(\frac{1 - \frac{\tau}{2n} s}{1 + \frac{\tau}{2n} s} \right)^n$$

for a sufficiently large n. Let a process model $\dfrac{e^{-s}}{1 + Ts}$ be approximated by

$$G(s) = \left(\frac{1 - 0.05s}{1 + 0.05s} \right)^{10} \frac{1}{1 + sT}$$

For each $T = 0, 0.01, 0.1, 1, 10$, find a reduced-order model, if possible, using balanced truncation such that the approximation error is no greater than 0.1.