The minimization problem is generically defined as **[ABT: place-holder; needs constraints etc.]**

$$\underset{\text{admissible } \omega}{\operatorname{argmin}} \; J(\omega), \tag{1}$$

where J is some objective function on the micro structure. Letting S denote the compliance tensor, we choose

$$J(\omega) = \frac{1}{2}\|S^H(\omega) - S^*\|_F^2, \tag{2}$$

for a microstructure with shape ω. There are several other possible choices for the objective functional J, such as deviation of elasticity tensor or error in displacement. In our setting, we are interested in particular values of Poisson ratios and shear moduli and the compliance.**[ABT: Justification for use of compliance]**

The microstructure boundary $\partial\omega$ is parameterized by a vector $\mathbf{p}$, consisting of, for instance, wire mesh node offsets and thicknesses. With proper assumptions, the derivative of $\partial\omega$ with respect to $\mathbf{p}$ is given by

$$\mathbf{v}_{p_\alpha}(\mathbf{y}, \mathbf{p}) := \frac{\partial \mathbf{y}}{\partial p_\alpha} \quad \text{for } \mathbf{y} \in \partial\omega, \tag{3}$$

and defines perturbation velocity fields over the boundary. Using $\mathbf{p}$ the minimization problem can be written as

$$\underset{\text{admissible } \mathbf{p}}{\operatorname{argmin}} \; J(\mathbf{p}) \text{ where } J(\mathbf{p}) = \frac{1}{2}\|S^H(\mathbf{p}) - S^*\|_F^2. \tag{4}$$

The derivative of the objective function is then

$$\frac{\partial J}{\partial p_\alpha} = [S^H - S^*] : \frac{\partial S^H}{\partial p_\alpha} = [S^H - S^*] : \mathrm{d}S^H[\mathbf{v}_{p_\alpha}], \tag{5}$$

where $\mathrm{d}S^H[\mathbf{v}_{p_\alpha}]$ is the shape derivative of S^H with perturbation $\mathbf{v}_{p_\alpha}$.

Shape derivative of elasticity tensor. The shape derivative of the homogenized elasticity tensor for microstructure with shape ω and perturbation $\mathbf{v}$ is defined as the Gâteaux derivative [1]

$$\mathrm{d}C^H[\mathbf{v}] := \lim_{t \downarrow 0} \frac{C^H(\omega(t, \mathbf{v})) - C^H(\omega)}{t}, \tag{6}$$

where $\omega(t, \mathbf{v}) := \{\mathbf{x} + t\mathbf{v} : \mathbf{x} \in \omega\}$. The homogenized elasticity tensor **[ABT: give ref]**, with a bit of manipulation, can be rewritten in the energy form as

$$C^H_{ijkl} = \frac{1}{|Y|}\int_\omega (\mathbf{e}^{ij} + \varepsilon(\mathbf{w}^{ij})) : C^{\text{base}} : (\mathbf{e}^{kl} + \varepsilon(\mathbf{w}^{kl}))\,\mathrm{d}\mathbf{y}. \tag{7}$$

Using this form, taking its variation with respect to a *permissible* boundary perturbation, integrating by parts, using the divergence theorem, and the fact that $\mathbf{w}^{ij}$ satisfies the cell problem, the shape derivative of C^H can be written as

$$\mathrm{d}C^H_{ijkl}[\mathbf{v}] = \frac{1}{|Y|}\int_{\partial\omega} [(\mathbf{e}^{ij} + \varepsilon(\mathbf{w}^{ij})) : C^{\text{base}} : (\mathbf{e}^{kl} + \varepsilon(\mathbf{w}^{kl}))](\mathbf{v} \cdot \hat{\mathbf{n}})\,\mathrm{d}A(\mathbf{y}). \tag{8}$$

Shape derivative of compliance tensor. The compliance tensor is the inverse of elasticity tensor, i.e. $S : C = I$. Using direct differentiation

$$\mathrm{d}S^H[\mathbf{v}] = -S^H : \mathrm{d}C^H[\mathbf{v}] : S^H. \tag{9}$$

Combining the results from Eq. (5), Eq. (8), and Eq. (9), one can compute $\frac{\partial J}{\partial p_\alpha}$.

Numerical computation. The integrand in Eq. (7) is cubic over each boundary element ($\mathbf{e}^{ij} + \varepsilon(\mathbf{w}^{ij})$ and $\mathbf{v} \cdot \hat{\mathbf{n}}$ are linear) and we use quadrature that evaluate the cubic function exactly.

References

[1] JEAN-P Zolésio and MC Delfour. Shapes and geometries: Analysis, differential calculus and optimization. *SIAM, Philadelphia*, 2001.