Gradient of Multi-class SVM Loss Function

Given:

- W is weight with a shape of C $\times$ D.
- β_{nj} is a binary matrix to indicate the sign of margins.

$$
\begin{aligned}
f(x_n, W)_c &= x_{nd} W_{dc} \\
L_n &= \sum_{j \neq y_n} \max(0, x_{nd} W_{dj} - x_{nd} W_{d,y_n} + \Delta)
\end{aligned}
$$

The gradient of the loss function with respect to the weight matrix W is:

$$
\beta_{nj} = \begin{cases} 1 & \text{if } x_{nd} W_{dj} - x_{nd} W_{d,y_n} + \Delta > 0 \\ 0 & \text{otherwise} \end{cases}
$$

Thus,

$$
\begin{aligned}
L_n &= \sum_{j \neq y_n} \beta_{nj} (x_{nd} W_{dj} - x_{nd} W_{d,y_n} + \Delta) \\
\frac{\partial L_n}{\partial W_{y_n}} &= - \sum_{j \neq y_n} \beta_{nj} x_{nd} \\
\frac{\partial L_n}{\partial W_{dc}} &= \beta_{nc} x_{nd}, \text{ when } c \neq y_n
\end{aligned}
$$

For all samples:

$$
\begin{aligned}
\frac{\partial L}{\partial W_{d,y_n}} &= - \sum_{n=1}^{N} \sum_{j \neq y_n} \beta_{nj} x_{nd} \\
\frac{\partial L}{\partial W_{dc}} &= \sum_{n=1}^{N} \beta_{nc} x_{nd}, \text{ when } c \neq y_n
\end{aligned}
$$

The final gradient of the loss function with respect to W is:

$$
\begin{aligned}
\frac{\partial L}{\partial W} &= \left(\frac{\partial L}{\partial W_1}, \frac{\partial L}{\partial W_2}, \ldots, \frac{\partial L}{\partial W_C} \right) \\
\frac{\partial L}{\partial W_{cd}} &= \sum_{n=1}^{N} \left(\beta_{nj} - \sum_{k \neq y_n} \beta_{nk} \right) x_{nd}
\end{aligned}
$$