Gradient of Multi-class SVM Loss Function

Given:

- W is weight with a shape of D × C.
- β_{nj} is a binary matrix to indicate the sign of margins.
- $\sum_j \beta_{nj}$ is the count of positive margins for the jth sample.

$$f(x_n, W)_c = x_{nd} W_{dc}$$

$$L_n = \sum_{j \neq y_n} \max(0, x_{nd} W_{dj} - x_{nd} W_{d,y_n} + \Delta)$$

$$L = \frac{1}{N} \sum_{n}^{N} \sum_{j \neq y_n} \max(0, x_{nd} W_{dj} - x_{nd} W_{d,y_n} + \Delta) + \lambda \sum_{c=1}^{C} \sum_{d} W_{c,d}^2$$

The gradient of the loss function with respect to the weight matrix W is:

$$\beta_{nj} = \begin{cases} 1 & \text{if } (x_{nd} W_{dj} - x_{nd} W_{d,y_n} + \Delta) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$L_n = \sum_{j \neq y_n} \beta_{nj} (x_{nd} W_{dj} - x_{nd} W_{d,y_n} + \Delta)$$

$$\frac{\partial L_n}{\partial W_{d,y_n}} = - \sum_{j \neq y_n} \beta_{nj} x_{nd}$$

$$\frac{\partial L_n}{\partial W_{dc}} = \beta_{nc} x_{nd}, \text{ when } c \neq y_n$$

For all samples:

$$\frac{\partial L}{\partial W_{d,y_n}} = -\sum_{n=1}^{N} \sum_{j \neq y_n} \beta_{nj} x_{nd}$$
$$\frac{\partial L}{\partial W_{dc}} = \sum_{n=1}^{N} \beta_{nc} x_{nd}, \text{ when } c \neq y_n$$

The final gradient of the loss function with respect to W is:

$$\begin{aligned}
\frac{\partial L}{\partial W_{dc}} &= \frac{1}{N} \sum_{n=1}^{N} \left(\beta_{nc} + \delta_{c,y_n} \left(-\sum_{j \neq y_n} \beta_{nj} - \beta_{nc} \right) \right) x_{nd} + 2\lambda W_{dc} \\
&= \frac{1}{N} \sum_{n=1}^{N} \left(\beta_{nc} - \sum_{j} \beta_{nj} \right) x_{nd} + 2\lambda W_{dc}
\end{aligned}$$