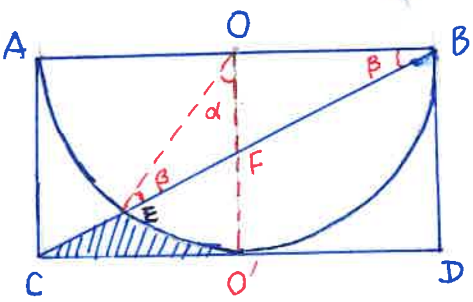
解法一（初等数学）



已知$|AB|=8$，$|AC|=4$，以AB为直径作半圆，圆心为O，$AC\perp AB$于A，$BD\perp AB$于B，连接BC交半圆圆弧于E。求阴影部分的面积。

解：设半圆与CD相切于O'，连接OO'交BC于F。

则F为BC的中点。故$|OF|=|FO'|=2$.

设$\angle OBE=\angle OEB=\beta$，$\angle EOO'=\alpha$.

$$S_{阴}=S_{\triangle CO'F}-S_{FEO'}=S_{\triangle CO'F}-(S_{扇\widehat{EOO'}}-S_{\triangle EOF})=S_{\triangle CO'F}-S_{扇EOO'}+S_{\triangle EOF}\cdots\text{①}$$

欲求$S_{扇EOO'}$，必求α的值。而由$\angle EOB+\angle OBE+\angle OEB=180^\circ$可得：

$$\alpha+90^\circ+\beta+\beta=180^\circ\Rightarrow\alpha+2\beta=\frac{\pi}{2}\Rightarrow\alpha=\frac{\pi}{2}-2\beta\quad\cdots\cdots\text{②}$$

欲求$\triangle EOF$的面积，则需求$\sin\alpha$的值。

$$S_{\triangle EOF}=\frac{1}{2}\cdot|OE|\cdot|OF|\sin\alpha=\frac{1}{2}\times4\times2\times\sin\alpha=4\sin\alpha\quad\cdots\cdots\text{③}$$

因此，我们需求出$\cos2\beta$的值，进而有必要分别求出$\sin\beta$和$\cos\beta$的值。

由勾股定理得：$|BC|^2=|AB|^2+|AC|^2=8^2+4^2=64+16=80\Rightarrow|BC|=\sqrt{80}=4\sqrt5\quad\cdots\cdots\text{④}$

$$\sin\beta=\frac{|AC|}{|BC|}=\frac{4}{4\sqrt5}=\frac{1}{\sqrt5}\quad\cdots\cdots\text{⑤}$$

$$\cos\beta=\frac{|AB|}{|BC|}=\frac{8}{4\sqrt5}=\frac{2}{\sqrt5}\quad\cdots\cdots\text{⑥}$$

由⑤和⑥可得：$\sin2\beta=2\sin\beta\cos\beta=2\times\frac{1}{\sqrt5}\times\frac{2}{\sqrt5}=\frac{4}{5}\quad\cdots\cdots\text{⑦}$

由⑦得$\cos2\beta=\sqrt{1-\sin^2 2\beta}=\sqrt{1-(\frac{4}{5})^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}\quad\cdots\cdots\text{⑧}$

由②得$\sin\alpha=\sin(\frac{\pi}{2}-2\beta)=\cos2\beta\quad\cdots\cdots\text{⑧}'$

代⑧入⑧'得　$\sin\alpha=\cos2\beta=\frac{3}{5}\quad\cdots\cdots\text{⑨}$

由⑨得：　$\alpha=\arcsin\frac{3}{5}\quad\cdots\cdots\text{⑩}$

代⑨入③：　$S_{\triangle EOF}=4\sin\alpha=4\times\frac{3}{5}=\frac{12}{5}\quad\cdots\cdots\text{⑪}$

$$S_{扇\widehat{EOO'}}=\frac{\alpha}{2\pi}\cdot\pi R^2=\frac{\alpha}{2\pi}\times\pi\times4^2=\frac{\alpha}{2}\times16=8\alpha\quad\cdots\cdots\text{⑫}$$

代⑩入⑫：

$$S_{扇\widehat{OEO'}}=8\alpha=8\times\arcsin\frac{3}{5}=8\arcsin\frac{3}{5}\quad\cdots\cdots\text{⑬}$$

$$S_{\triangle CO'F}=\frac{1}{2}|CO'|\cdot|O'F|=\frac{1}{2}\times4\times2=4\quad\cdots\cdots\text{⑭}$$

将⑪、⑬和⑭代入①：

$$S_{阴}=4-8\arcsin\frac{3}{5}+\frac{12}{5}=\frac{20}{5}+\frac{12}{5}-8\arcsin\frac{3}{5}=\frac{32}{5}-8\arcsin\frac{3}{5}$$

$$=8(\frac{4}{5}-\arcsin\frac{3}{5})$$

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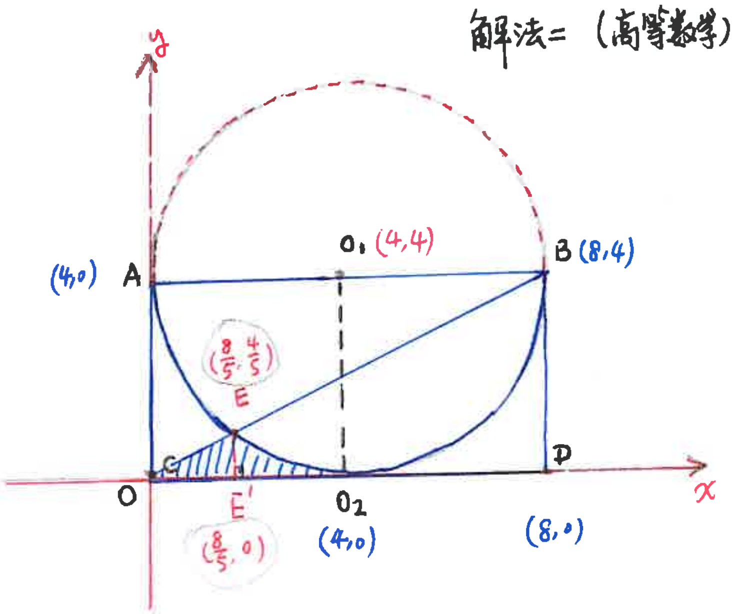
解法二 （高等数学）(1)



已知 $|AB|=8$, $|AC|=4$.
求阴影部分的面积。

解：以C点为坐标原心，OD为x轴，OA为y轴。以AB的中心O_1向CD作垂线交x轴于O_2.

则O_2的坐标为：$O_2(4,0)$；而A.B.C.D的坐标分别为：$A(4,0)$, $B(8,4)$, $C(0,0)$, $D(8,0)$.

设BC（即BO）与弧$\overset{\frown}{AO_2B}$相交于点E，由E向x轴作垂线交x轴于E'.

则 $S_阴 = S_{\triangle OE'E} + S_{◿EE'O_2}$ ······ ①

(1) 圆O_1的方程为：$(x-4)^2+(y-4)^2=4^2$ ······ ②

(2) 设过OB的直线方程为：$y=kx+b$

则直线OB过点(0,0)和B(8,4)，代点入方程得：

$$\begin{cases} 0 = k\times 0 + b \\ 4 = k\times 8 + b \end{cases} \Rightarrow \begin{cases} k=\frac{1}{2} \\ b=0 \end{cases}$$

故OB直线的方程为：$y=\frac{1}{2}x$ ······ ③

(3) 联立圆方程②和直线方程③求E的坐标。

$$\begin{cases} y=\frac{1}{2}x \\ (x-4)^2+(y-4)^2=4^2 \end{cases} \Rightarrow \begin{cases} x_1=8 \\ y_1=4 \end{cases} 和 \begin{cases} x_2=\frac{8}{5} \\ y_2=\frac{4}{5} \end{cases}$$

显然(x_1,y_1)是点B的坐标，故E的坐标为$(\frac{8}{5},\frac{4}{5})$ ······ ④

那么E'的坐标为$(\frac{8}{5},0)$ ······ ⑤

(4) 根据$E(\frac{8}{5},\frac{4}{5})$可求出$S_{\triangle OE'E}$.

$$S_{\triangle OE'E} = \int_0^{\frac{8}{5}} (\tfrac{1}{2}x)dx = \frac{1}{2}\int_0^{\frac{8}{5}} x\,dx = \frac{1}{2}\left[\tfrac{1}{2}x^2\right]\Big|_0^{\frac{8}{5}} = \frac{1}{4}\left[x^2\right]\Big|_0^{\frac{8}{5}} = \frac{1}{4}\times\frac{8}{5}\times\frac{8}{5} = \frac{16}{25}$$ ······ ⑥

当然，$S_{\triangle OE'E}$也可用几何方式求解，即 $S_{\triangle OE'E'} = \frac{1}{2}|OE'|\cdot|E'E| = \frac{1}{2}\times\frac{8}{5}\times\frac{4}{5} = \frac{16}{25}$

(5) 利用定积分求 $S_{◿EE'O_2}$.

由圆方程②得：$(y-4)^2 = 4^2-(x-4)^2 = 16-(x-4)^2$

$\Rightarrow |y-4| = \sqrt{16-(x-4)^2}$

由于在 $\overset{\frown}{EO_2}$ 上，$y<4$，故 $y-4<0$，因此 $4-y=\sqrt{16-(x-4)^2} \Rightarrow y = 4-\sqrt{16-(x-4)^2}$ ··· ⑦

解法二（高等数学）(2)

方程①在点$E'(\frac{8}{5},0)$和$O_2(4,0)$之间的定积分为：

$$A=\int_{\frac{8}{5}}^{4}(4-\sqrt{16-(4-x)^2})dx=\int_{\frac{8}{5}}^{4}4dx-\int_{\frac{8}{5}}^{4}\sqrt{16-(4-x)^2}\,dx$$

$$=4\int_{\frac{8}{5}}^{4}dx-\int_{\frac{8}{5}}^{4}\sqrt{16-(4-x)^2}\,dx=4\times(4-\frac{8}{5})-\int_{\frac{8}{5}}^{4}\sqrt{16-(4-x)^2}\,dx$$

$$=\frac{48}{5}-\int_{\frac{8}{5}}^{4}\sqrt{16-(4-x)^2}\,dx \quad \cdots\cdots ⑧$$

令 $B=\int_{\frac{8}{5}}^{4}\sqrt{16-(4-x)^2}\,dx$ ⑧′ 则 $A=\frac{48}{5}-B$ ··· ⑧″

现在对B求解，令 $4-x=4t$，则 $t=\frac{1}{4}(4-x)$，$x=4-4t$.

当 $x=4$ 时，$t=\frac{1}{4}(4-x)=\frac{1}{4}(4-4)=0$； ⑨

当 $x=\frac{8}{5}$ 时，$t=\frac{1}{4}(4-x)=\frac{1}{4}(4-\frac{8}{5})=1-\frac{2}{5}=\frac{3}{5}$. ⑩

$16-(4-x)^2=16-(4t)^2=16(1-t^2)$ ⑪

$dx=d(4-4t)=-4dt$. ⑫

代⑨—⑫入⑧′，得 $B=\int_{\frac{3}{5}}^{0}\sqrt{16(1-t^2)}\,d(4-4t)=-16\int_{\frac{3}{5}}^{0}\sqrt{1-t^2}\,dt$

$$=16\int_{0}^{\frac{3}{5}}\sqrt{1-t^2}\,dt \quad \cdots\cdots ⑬$$

接下来求$\int\sqrt{1-t^2}\,dt$. 设 $D=\int\sqrt{1-t^2}\,dt$. 令 $t=\sin x$.

则 $D=\int\sqrt{1-\sin^2x}\,d(\sin x)=\int\sqrt{\cos^2x}\cdot\cos x\,dx=\int\cos^2x\,dx$.

而 $\cos 2x=\cos^2x-\sin^2x=\cos^2x-(1-\sin^2x)=2\cos^2x-1$

则 $\cos^2x=\frac{1+\cos 2x}{2}$

所以 $D=\int\sqrt{1-t^2}\,dt=\int\cos^2x\,dx=\int\frac{1+\cos 2x}{2}dx=\int\frac{1}{2}dx+\int\frac{\cos 2x}{2}dx$

$=\frac{1}{2}x+\frac{1}{4}\int\cos 2x\,d(2x)=\frac{1}{2}x+\frac{1}{4}\sin 2x+C$. C为常数.

于是⑬变形为：

$$B=16\int_0^{\frac{3}{5}}\sqrt{1-t^2}\,dt\ \overset{t=\sin x}{=\!=\!=}\ 16\int_0^{\arcsin\frac{3}{5}}\sqrt{1-\sin^2x}\,d(\sin x)$$

$$=16\left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{\arcsin\frac{3}{5}}=16\left[\frac{1}{2}\arcsin\frac{3}{5}+\frac{1}{4}\sin 2(\arcsin\frac{3}{5})\right] \cdots ⑭$$

解法二（高等数学）(3)

$\sin 2x = 2\sin x\cos x$

当 $x=\arcsin\frac{3}{5}$ 时，$\sin x=\frac{3}{5}$，$\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-(\frac{3}{5})^2}=\frac{4}{5}$.

故 $\sin 2(\arcsin\frac{3}{5}) = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$ ······ ⑭'

代⑭'入⑭得：$B = 16\left[\frac{1}{2}\arcsin\frac{3}{5} + \frac{1}{4}\times\frac{24}{25}\right] = 8\arcsin\frac{3}{5} + \frac{96}{25}$ ··· ⑮

代⑮入⑧'得：$A = \frac{48}{5} - B = \frac{48}{5} - \left(8\arcsin\frac{3}{5} + \frac{96}{25}\right)$

$= \frac{48}{5} - \frac{96}{25} - 8\arcsin\frac{3}{5} = \frac{48\times5-48\times2}{25} - 8\arcsin\frac{3}{5}$

$= \frac{48\times3}{25} - 8\arcsin\frac{3}{5}$ ······ ⑯

而 $S_{EE'O_2} = A = \frac{16\times9}{25} - 8\arcsin\frac{3}{5}$ ············ ⑯'

将⑯'和⑥代入①，得：

$S_阴 = S_{\triangle OE'E} + S_{\triangle EE'O_2} = \frac{16}{25} + \left(\frac{16\times9}{25} - 8\arcsin\frac{3}{5}\right)$

$= \frac{16}{25} + \frac{16}{25}\times9 - 8\arcsin\frac{3}{5} = \frac{16}{25}\times10 - 8\arcsin\frac{3}{5}$

$= \frac{32\times5}{25} - 8\arcsin\frac{3}{5} = \frac{32}{5} - 8\arcsin\frac{3}{5}$

$= 8\left(\frac{4}{5} - \arcsin\frac{3}{5}\right)$

即 $S_阴 = 8\left(\frac{4}{5} - \arcsin\frac{3}{5}\right)$ #