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Image Registration 2:

Spring 2021

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Handouts & Lecture Notes

- **Report in Scientific American (June 2014):**

  "In each study, however, those who wrote out their notes by hand had a stronger conceptual understanding and were more successful in applying and integrating the material than those who used [sic] took notes with their laptops."

  The Pen Is Mightier Than the Keyboard

  P. A. Mueller, D. M. Oppenheimer, *Psychological Science,* Vol 25,Issue 6, pp. 1159 – 1168, April-23-2014.

- **Handouts are to aid note taking, not a total replacement for note taking**
- **Podcasts, slides, pdfs etc on BlackBoard**

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Image Warping & Regularisation

Warp Regularisation

- Arbitrary image warp $\phi : \underline{x} \mapsto \phi(\underline{x})$ is ill-posed problem
- Need to constrain/control image warps:

By construction:

e.g. AAM: Triangles & barycentric coordinates

- Parametric Image Warps:

  Warp defined by some small set of parameters

  Spline with some set of control/knot points

  Interpolation/extrapolation of displacement

By warping penalty:

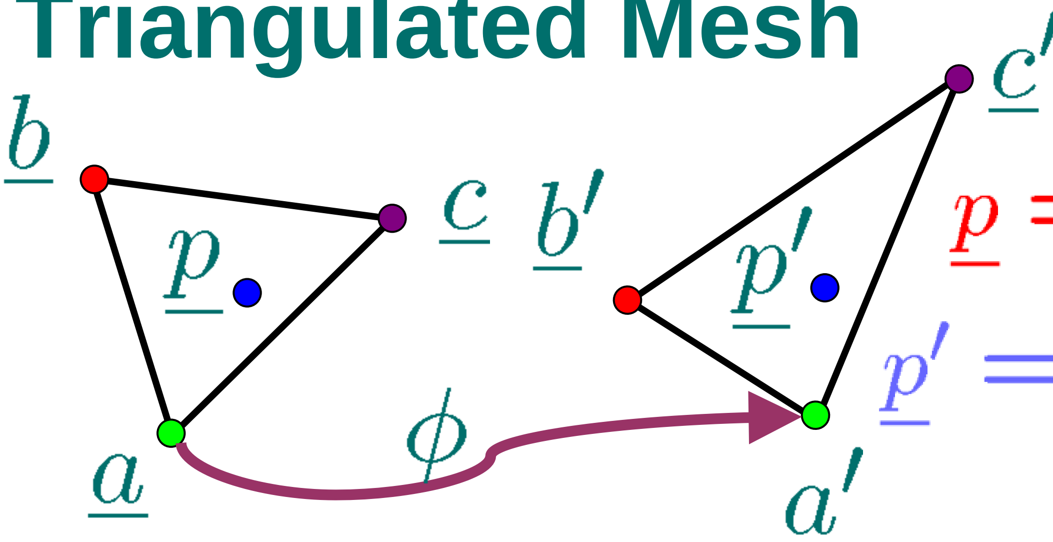
- Non-Parametric Image Warps:

  Trade-off between image match versus smoothness

  e.g. elastic and fluid registration

Parametric Image Warps

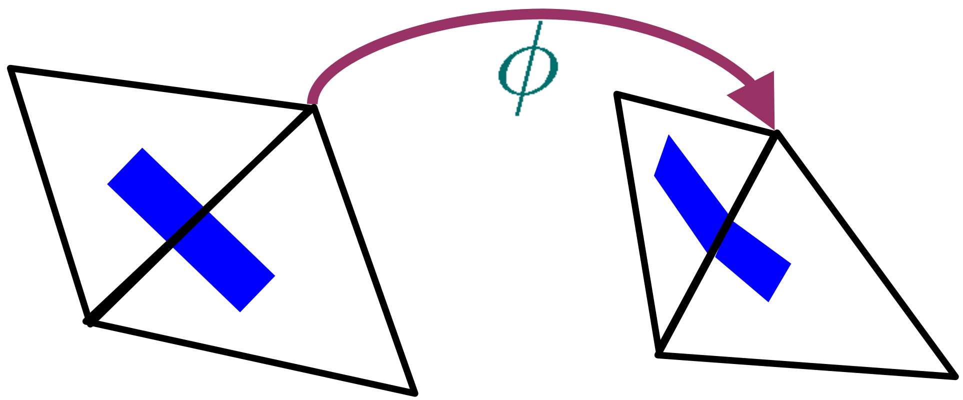
Triangulated Mesh



$$\underline{p} = \alpha\underline{a} + \beta\underline{b} + \gamma\underline{c}$$

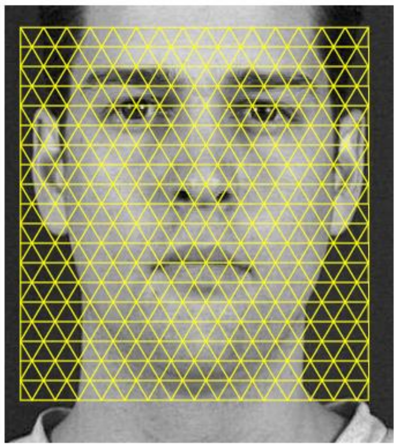
$$\underline{p}' = \alpha\underline{a}' + \beta\underline{b}' + \gamma\underline{c}'$$

- **Triangulated mesh and barycentric coordinates**
- **Tends to bend objects at edges**

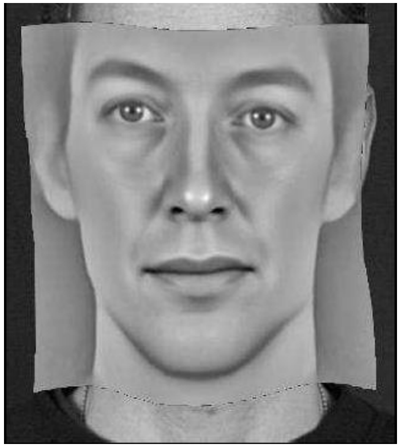


Continuous but not differentiable at edges

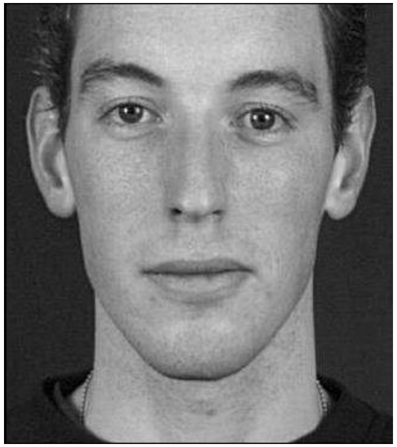
Image Registration: Triangulated Meshes



example mesh

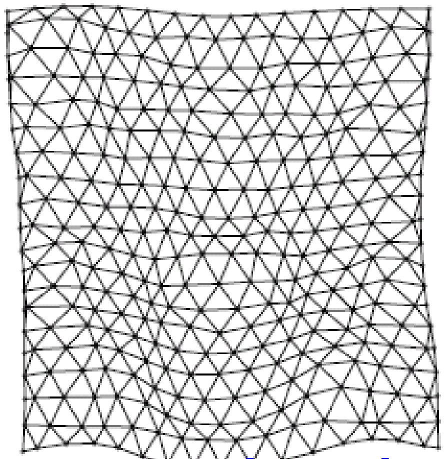


warped reference

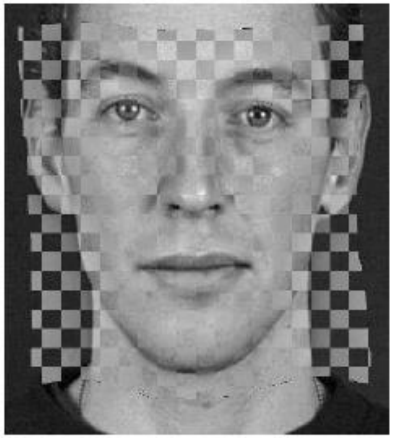


Target

Note:
Uses group of images, reference is a mean face

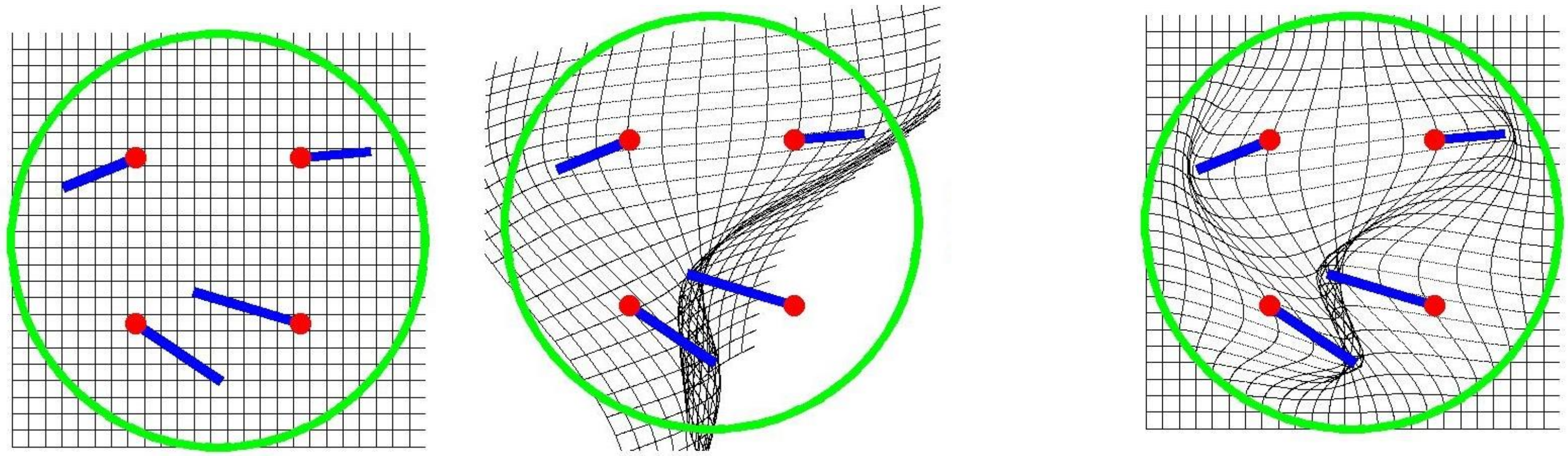


warped mesh



comparison

Thin-Plate & Clamped-Plate Splines



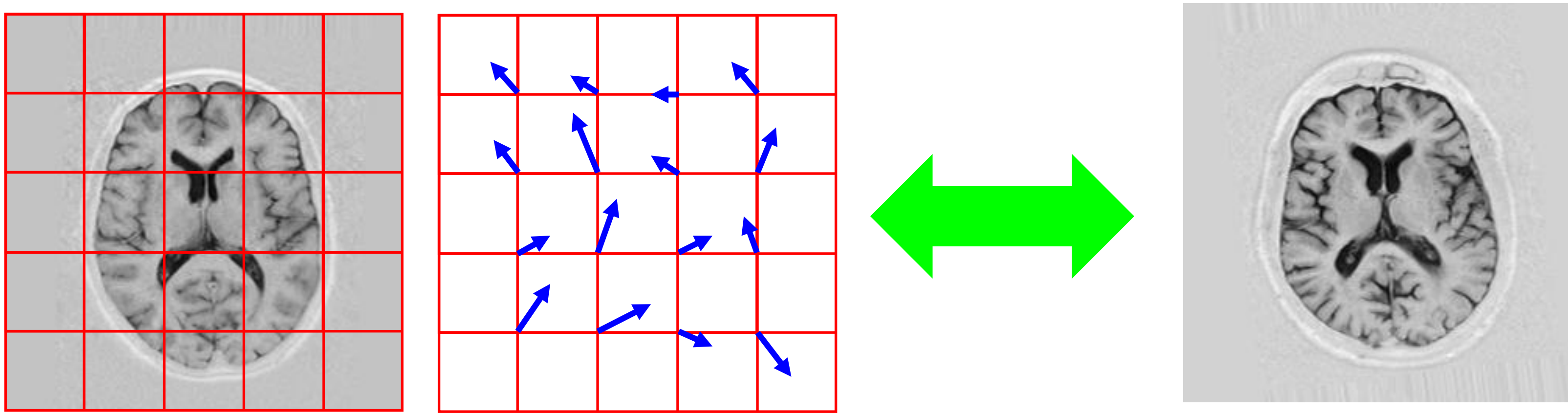
Initial knotpoint positions and displacements and unit circle.

Biharmonic Thin-Plate Spline (TPS) asymptotically flat. (Duchon 1976)

Biharmonic Clamped-Plate (CPS) zero outside unit circle. (Marsland & Twining 2002)

- **Displacement field: analytic function of knotpoints**
- **Can fold if displacements too large**
- **Other splines: B-splines (Rueckert), cubic splines etc**

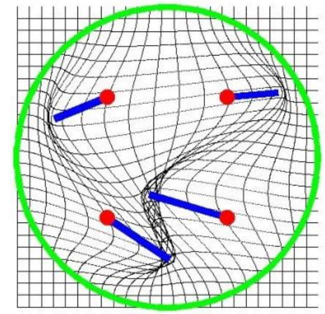
Parametric Registration:



- **Initialise:** **Rigid (translation & rotation), affine (+scale)**
- **Define initial knotpoint positions**
- **Define knotpoint displacements**
- **Extrapolate/interpolate to find warp** $\phi : \underline{x} \mapsto \phi(\underline{x})$
- **Create warped image and do comparison**
- **Optimise displacements (& initial positions)**
- **Repeat at finer scale**

Non-Parametric Image Warps

Warp Regularisation:



Parametric Warps:

- **Limited number of degrees of freedom**
- **Controls warps & easier to optimise**
- **Still have to check for folding**
- **Limited in terms of flexibility**

Dense, non-parametric Warps:

- **Pixel-by-pixel deformations**

Varying notation:
where it goes to
OR
where it came from

$$\phi : \underline{x} \mapsto \phi(\underline{x}) \qquad \underline{u} : \underline{x} \mapsto \underline{x} \boxed{+} \underline{u}(\underline{x})$$

deformation field **displacement field**

- **Need to add regularisation term to control warps**
- **Physics-inspired algorithms:**

Elastic solid or visco-elastic fluid

Framework: Non-Parametric Registration

- Two **images**: $\mathcal{I}_{\mathrm{s}}(\underline{x})$ & $\mathcal{I}_{\mathrm{t}}(\underline{x})$ & **warp**: $\phi(\underline{x})$
- **Image-matching/difference term**: $\mathcal{D}(\mathcal{I}_{\mathrm{s}}, \mathcal{I}_{\mathrm{t}}; \phi)$

  Warp & resample, push-forward or pull-back mapping
  SAD, SSD, mutual information etc
- **Warp regularisation term**: $\mathcal{R}(\phi)$
- **Total objective function/energy:**

$$\mathcal{E} = \underbrace{\boxed{\mathcal{D}(\mathcal{I}_{\mathrm{s}}, \mathcal{I}_{\mathrm{t}}; \phi)}}_{\text{image matching term}} + \underbrace{\boxed{\lambda}}_{\text{trade-off parameter}} \underbrace{\boxed{\mathcal{R}(\phi)}}_{\text{warp roughness term}}$$

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Energies & Forces

Energy minimisation:

$$\mathcal{E} = \mathcal{D}(\mathcal{I}_{\mathrm{s}}, \mathcal{I}_{\mathrm{t}}; \phi) + \lambda \mathcal{R}(\phi)$$

- **Less-smooth warps vs better match/smaller difference**

Forces:

- **Image forces: tries to align structures**
- **Warp forces: resists deformation**
- **Forces = 'gradient' of relevant energy term**

Solution:

- **Minimize energy OR zero net force**

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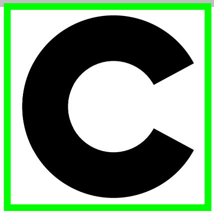
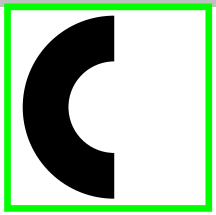
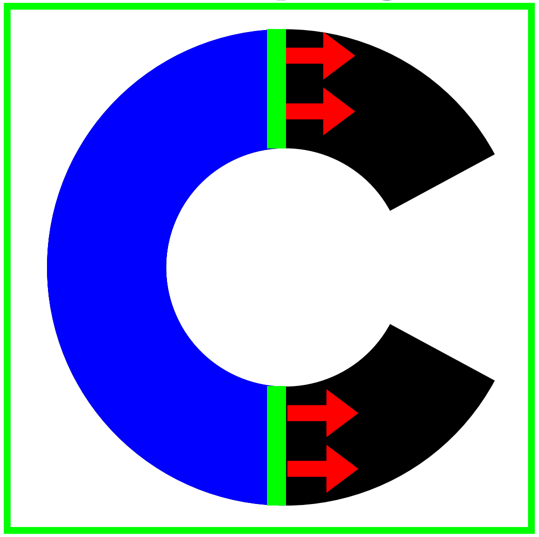
Image Forces:



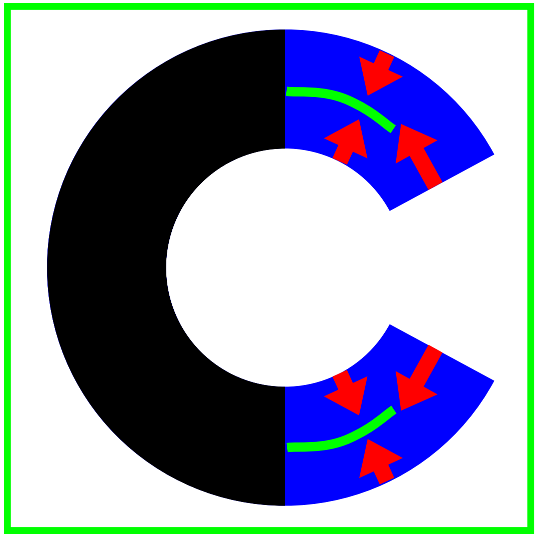


Two choices for moving image

Arch warping to C



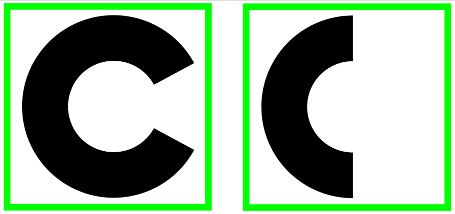
C warping to arch



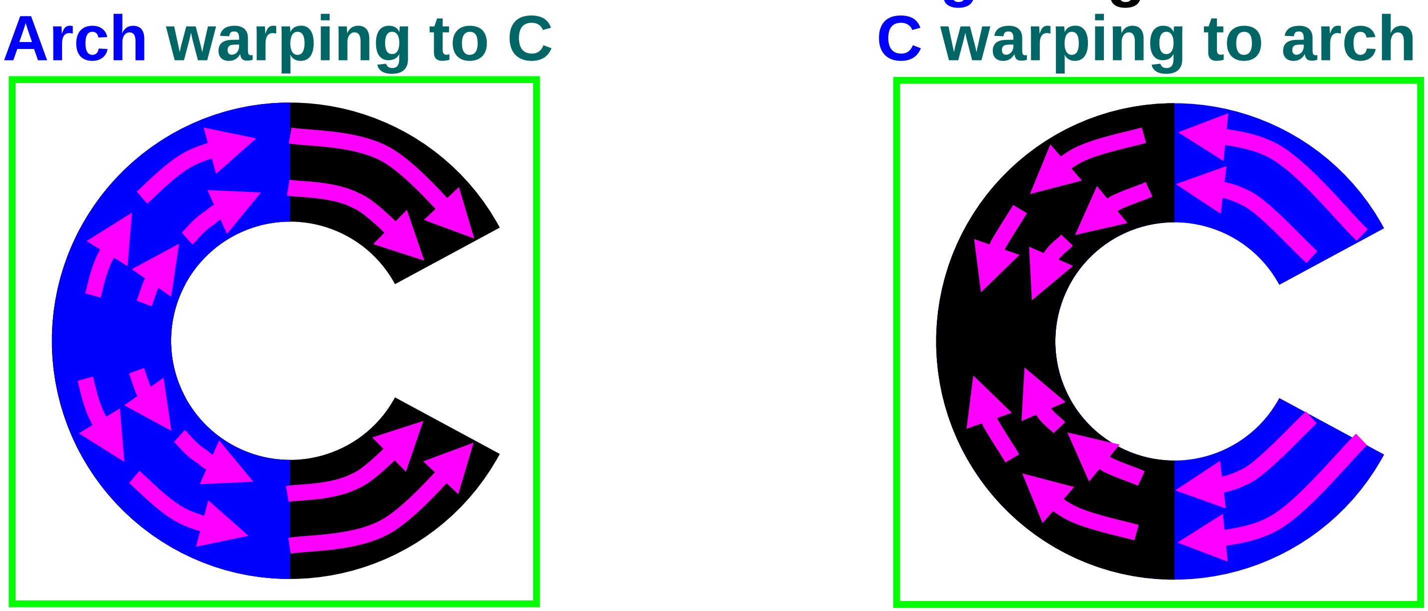
- Moving leading edge of arch improves image match
- Without warp regulariser, only these pixels move

- Image forces try to shrink non-overlap region
- Only shrink whole shape if warp regulariser acts

Preferred Solution:



Two choices for moving image



- Edge of arch moves to edge of C
- Rest of arch stretches as well

- Edge of C contracts to edge of arch
- Rest of C contracts as well

Only movement of object so far. What about background?

Suggestion: Elasticity

Make object elastic:

- Agrees with our intuition
- Allows more natural correspondence between warped and target object

Problems:

- Can’t warp just object - if we knew how to separate object & background, wouldn’t need registration!
- Real images not this simple

Solution:

Treat whole moving image as if printed on elastic sheet

Can both stretch and compress

- Need deformation energy for warped elastic sheet

Mathematical Aside:

Mathematics in Image Registration

- **Maths as a language:**
  - **Concise and precise statements**
  - **Own special symbols and own syntax**
  - **Need a logical approach**
- **If you can program, already have required skills**
- **Most cases:**
  - **Translation rather than manipulation**

Vector Calculus in 3D: Notation

$$\vec{\nabla} = \left(\frac{\partial}{\partial \mathrm{x}}, \frac{\partial}{\partial \mathrm{y}}, \frac{\partial}{\partial \mathrm{z}}\right)$$

first derivative, vector-like

Note: Will find $\mathbf{X}$ as cartesian coordinate and $\underline{x}$ as position vector

$$\text{div(ergence):}\, \vec{\nabla} \cdot \underline{n}(\boxed{\underline{x}}) = \frac{\partial \mathrm{n_x}}{\partial \boxed{\mathrm{x}}} + \frac{\partial \mathrm{n_y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{n_z}}{\partial \mathrm{z}}$$

first derivative acting on vector gives scalar

del squared:

$$\vec{\nabla}^2 \underline{n} = \left(\vec{\nabla}^2 \mathrm{n_x}, \vec{\nabla}^2 \mathrm{n_y}, \vec{\nabla}^2 \mathrm{n_z}\right)$$

$$\vec{\nabla}^2 \mathrm{n_x} = \frac{\partial^2 \mathrm{n_x}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{n_x}}{\partial \mathrm{y}^2} + \frac{\partial^2 \mathrm{n_x}}{\partial \mathrm{z}^2}$$

second derivative acting on vector gives another vector

- See Maths Primer for more details

Mathematics in Image Registration

vector field

vector field

$$\boxed{\underline{v}}(\boxed{\underline{x}},\boxed{\mathrm{t}}) = \frac{\partial \boxed{\underline{u}}(\underline{x},\mathrm{t})}{\boxed{\partial \mathrm{t}}} + (\boxed{\underline{v}}(\underline{x},\mathrm{t}) \cdot \boxed{\vec{\nabla}})\boxed{\underline{u}}(\underline{x},\mathrm{t})$$

position time

first derivatives

- **Relation between a pair of vector fields that depend on time and on position**
- **Related via first-derivatives acting on u**
- **If you know u, can compute Eulerian velocity v**

Mathematics in Image Registration

$$\boxed{\underline{F}^{\mathrm{visc}}(\underline{x},\mathrm{t})} = \boxed{\mu}\,\boxed{\vec{\nabla}^2}\,\boxed{\underline{v}}(\underline{x},\mathrm{t}) + \left(\boxed{\lambda+\mu}\right)\boxed{\vec{\nabla}\left(\vec{\nabla}\right.}\bullet\boxed{\underline{v}}(\underline{x},\mathrm{t})\Big)$$

constants

Eulerian velocity

result

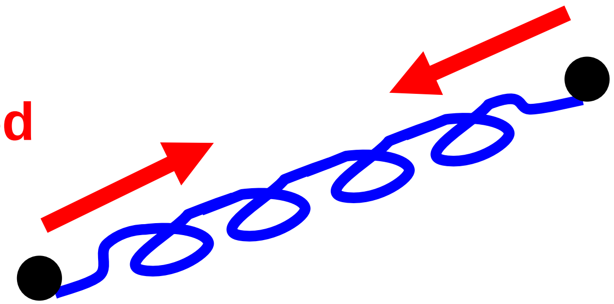
second-order spatial derivatives

- Output is a vector field, function of space & time
- Depends on second-order spatial derivatives of Eulerian velocity (previous slide)
- Two constants that need to be assigned values
- Viscous forces in a fluid, vary according to flow

Elastic Registration

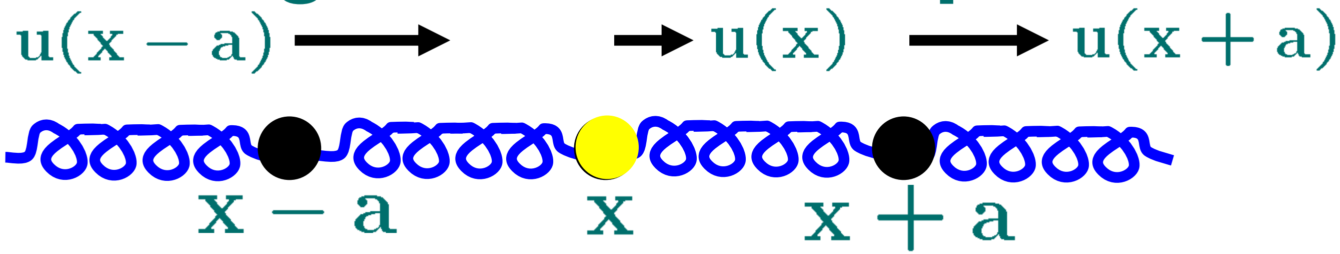
Elastic Registration: Thought Expt

- Single moving point, spring connected to its point of origin



- Where does the force act?
- Push-forward & take image value with me:
- Spring force acts at $\underline{x} + \underline{u}(\underline{x})$ but depends on $\underline{u}(\underline{x})$
- Pull-back: bring value at $\underline{x} + \underline{u}(\underline{x})$ back with me
- Image difference computed at $\underline{x}$ and spring force at $\underline{x}$
- Different coordinate systems can be confusing!

Elastic Registration: simple 1D Model



- **Displace** $\mathrm{x} \mapsto \phi(\mathrm{x}), \mathrm{x} \mapsto \mathrm{x} + \mathrm{u(x)}$
- **Spring: force proportional to extension**
- **Uniform translation: no net force**
- **Scaling: no net force**

$$\left.\begin{array}{l}\text{Uniform translation: no net force}\\ \text{Scaling: no net force}\end{array}\right\} \begin{array}{l}\phi(\mathrm{x}) = \mathrm{sx} + \mathrm{t},\\ \mathrm{F}(x) = 0\end{array}$$

- **Net force depends on second-derivatives of** $\mathrm{u(x)}, \phi(\mathrm{x})$

Navier-Lame/Navier-Cauchy Equation:

Second spatial derivatives, rotationally invariant, vector-valued:

$$\underline{F}_{\mathrm{elas}}(\underline{x}) = \mu \vec{\nabla}^2 \underline{u}(\underline{x}) + (\lambda + \mu) \vec{\nabla}(\vec{\nabla} \cdot \underline{u}(\underline{x}))$$

Two free parameters

Elastic Registration:

- **Forces:**

$$\underline{F}_{\mathbf{elas}}(\mathbf{x}) = \mu \vec{\nabla}^{\mathbf{2}} \underline{u}(\underline{x}) + (\lambda + \mu)\vec{\nabla}(\vec{\nabla} \cdot \underline{u}(\underline{x}))$$

- **Energies:**

$$\mathcal{E}_{\mathbf{elas}} = \int \mathbf{d}\underline{x} \left[\frac{\mu}{4} \sum_{\mathbf{i},\mathbf{j}} \left(\partial_{\mathbf{i}} \mathbf{u}_{\mathbf{j}} + \partial_{\mathbf{j}} \mathbf{u}_{\mathbf{i}} \right)^{\mathbf{2}} + \frac{\lambda}{\mathbf{2}} \left(\vec{\nabla} \cdot \underline{u} \right)^{\mathbf{2}} \right]$$

shorthand notation $\partial_{\mathbf{x}} \equiv \frac{\partial}{\partial \mathbf{x}}$ **functional derivative** $\underline{F}_{\mathbf{elas}}(\underline{x}) = -\frac{\delta \mathcal{E}_{\mathbf{elas}}}{\delta \underline{u}(\underline{x})}$

- **Elastic registration:**

general expression: $\mathcal{E} = \mathcal{D}(\mathcal{I}_{\mathbf{s}}, \mathcal{I}_{\mathbf{t}}; \phi) + \lambda \mathcal{R}(\phi)$

$$\phi(\underline{x}) = \underline{x} + \underline{u}(\underline{x})$$

warp & displacement

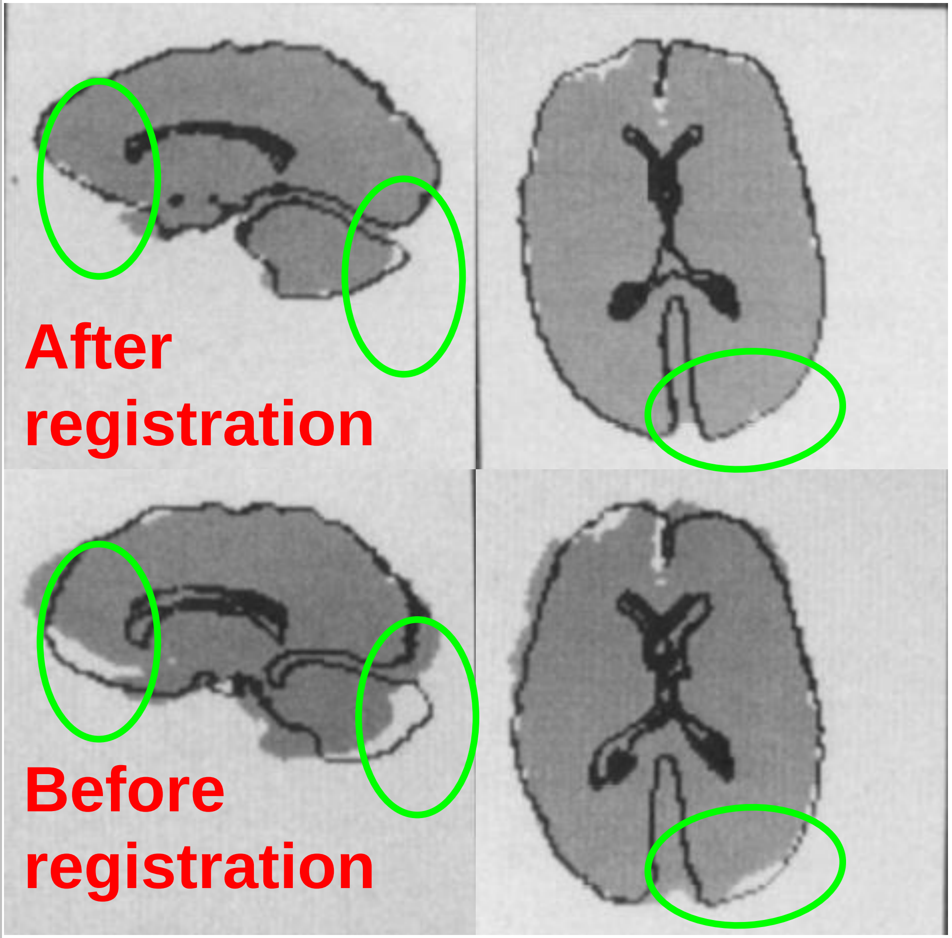
$$\boxed{\mathcal{R}(\phi) = \mathcal{E}_{\mathbf{elas}}(\underline{u})}$$

elastic regulariser

Elastic Registration:

Bajcsy & Kovacic, *Multiresolution elastic matching*,
Computer Vision, Graphics, and Image Processing, Volume 46, (1989)

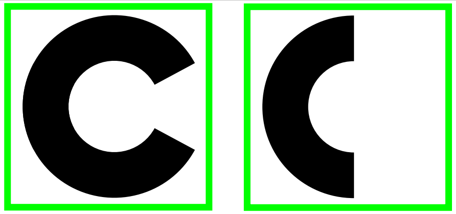
Atlas to Brain in 3D



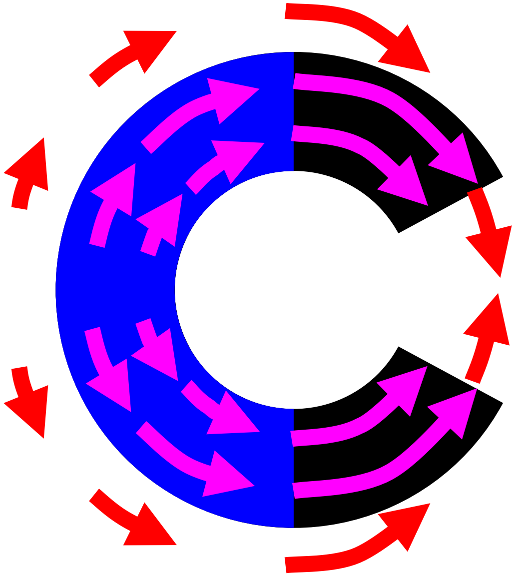
Rather old images, but pure elastic:

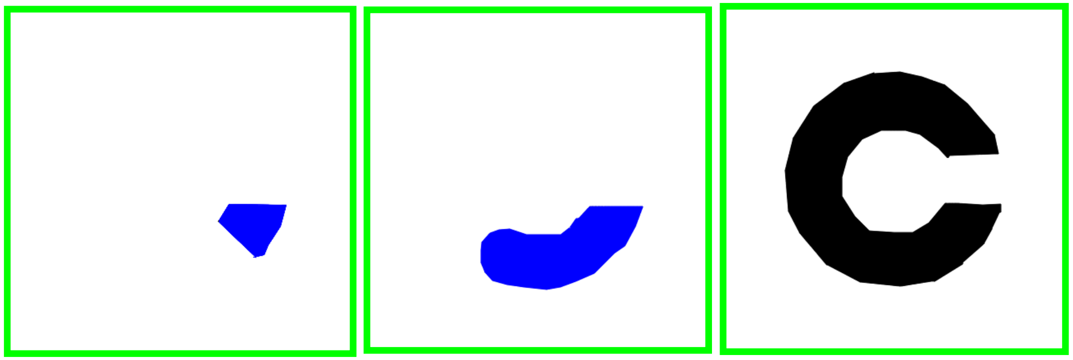
More recent papers use more complicated techniques

Elastic Registration



Direction of motion





moving image | result | target

- Background pulled along as well & compressed
- Elastic force increases as deformation increases
- Will it go all the way?
- Christensen et al: Elastic fails for large deformations

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Summary: Elastic Registration

- Potentially greater flexibility than parametric case
- Parametric case, choice of spline basis rather artificial
- Whereas elastic deformation accords with physical properties of actual objects
- Grossly simplified, elasticity same across whole image
- Elastic doesn’t fold or tear provided small deformations
- Works well for some types of images
- But limited to small deformations

  Can’t handle extreme cases as arch to C shows
- Has problems when deformations only localized

  Elasticity pulls rest of image along with it

Summary:

This Lecture:

- Warp regularisation
- Parametric warps

    triangulated meshes, splines

- Non-Parametric Warps:

    general framework,

    trade-off between match and warp smoothness

    Elastic registration: formulation & problems

Next Lecture:

- Fluids and flows of warps, groupwise registration

Additional Information:

Bernd Fischer and Jan Modersitzki,
Ill-posed medicine—an introduction to image registration,
Inverse Problems 24 (2008)

Jan Modersitzki,
Numerical Methods for Image Registration,
Oxford Science Publications