Laplace Method

$f''(x)$ exists, $M \to +\infty$, $a, b < \infty$

$$\begin{cases} x_0 = \arg\max_x f(x) \\ f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \end{cases}$$

$$\int_a^b h(x) e^{Mf(x)} dx \approx h(x_0) e^{Mf(x_0)} \cdot \sqrt{\frac{2\pi}{n |f''(x_0)|}}$$

$$I = \int_0^1 \left(k(x, x_{n+1}) - k_n^T(x_{n+1}) \cdot K_n^{-1} \cdot k_n(x) \right)^2 dP(x)$$

$$= \int_0^1 h(x) \cdot g^2(x) dx$$

$$= \int_0^1 h(x) \cdot e^{n \cdot \log (g(x))^{\frac{2}{n}}} dx$$

$$\approx h(x_0) \cdot g^2(x_0) \cdot \sqrt{\frac{2\pi}{n}} \cdot \left| \frac{d^2 \log (g(x))^{\frac{2}{n}}}{dx^2} \Big|_{x=x_0} \right|^{-\frac{1}{2}}$$

$$\left(\log g(x)^{\frac{2}{n}} \right)' = (g(x))^{-\frac{2}{n}} \cdot \frac{2}{n} (g(x))^{\frac{2}{n} - 1} \cdot g'(x)$$

$$= \frac{2}{n} \cdot g'(x) / g(x)$$

$$\left(\frac{2}{n} g'(x)/g(x)\right)' = \frac{2}{n} \cdot \frac{g'' \cdot g - (g')^2}{g^2}$$

$$\Rightarrow \hat{I} \approx \sqrt{\pi}\, h(x_0) \cdot \frac{|g(x_0)|^3}{\sqrt{|g''(x_0)\, g(x_0) - (g'(x_0))^2|}}$$

if $x_0 = \arg\max_x h(x) \cdot g^2(x)$

$$I = \int_0^1 e^{n \log (h \cdot g^2)^{\frac{1}{n}}} dx$$

$$\approx h(x_0) \cdot g^2(x_0) \sqrt{\frac{2\pi}{n}} \cdot \left| \frac{d^2 \log (h \cdot g^2)^{\frac{1}{n}}}{dx^2} \right|_{x=x_0}^{-\frac{1}{2}}$$

$$r(x) = \sqrt{h(x)} \cdot g(x)$$

$$\Rightarrow \hat{I} = \sqrt{\pi} \cdot \frac{|r(x_0)|^3}{\sqrt{|r''(x_0) \cdot r(x_0) - (r'(x_0))^2|}}$$

$$\begin{cases} r = h^{\frac{1}{2}} \cdot g \\ r' = \frac{1}{2} h^{-\frac{1}{2}} \cdot h' \cdot g + h^{\frac{1}{2}} \cdot g' \\ r'' = -\frac{1}{4} h^{-\frac{3}{2}} \cdot (h')^2 \cdot g + \frac{1}{2} h^{-\frac{1}{2}} \cdot h'' \cdot g + h^{-\frac{1}{2}} \cdot h' \cdot g' \end{cases}$$

$$+ h^{\frac{1}{2}} \cdot g''$$

$$r'' \cdot r - (r')^2 = \frac{1}{2}\left(h'' - h^{-1} \cdot (h')^2\right) \cdot g^2 + h\left(gg'' - (g')^2\right)$$

$$\Rightarrow \hat{I} = \sqrt{\pi} \cdot h^{\frac{3}{2}} \frac{|g|^3}{\sqrt{\left| \frac{1}{2}\left(h'' - h^{-1} \cdot (h')^2\right) g^2 + h\left(gg'' - (g')^2\right) \right|}}$$

if $x_0 = \underset{x}{\arg\max}\, h(x)$

$$I = \int_0^1 g^2(x) \cdot e^{n \cdot \log (h(x))^{\frac{1}{n}}} dx$$

$$\simeq g^2(x_0) \cdot h(x_0) \cdot \sqrt{\frac{2\pi}{n}} \cdot \left| \frac{d^2 \log (h(x))^{\frac{1}{n}}}{dx^2} \right|_{x=x_0}^{-\frac{1}{2}}$$

$$\hat{I} = h^2 \cdot \sqrt{\frac{2\pi}{|h'' \cdot h - (h')^2|}} \cdot g^2$$