Multivariate Laplace Method

$$\int_D h(x)\cdot e^{nf(x)}\,dx, \quad x=(x_1,\cdots x_d)$$

$$\approx\left(\frac{2\pi}{n}\right)^{\frac{d}{2}}\cdot h(x_0)\cdot e^{nf(x_0)}\cdot\left|\mathrm{Hess}_f(x_0)\right|^{-\frac{1}{2}}$$

$$I=\int_D h(x)\cdot e^{n\log(g(x))^{\frac{2}{n}}}\,dx.$$

$$\begin{cases}\dfrac{\partial\log(g(x))^{\frac{2}{n}}}{\partial x_i}=\dfrac{2}{n}\cdot\dfrac{1}{g}\cdot\dfrac{\partial g}{\partial x_i}\\[2ex] \dfrac{\partial^2\log(g(x))^{\frac{2}{n}}}{\partial x_i\partial x_j}=\dfrac{2}{n}\cdot\underbrace{\left(-\dfrac{1}{g^2}\cdot\dfrac{\partial g}{\partial x_i}\dfrac{\partial g}{\partial x_j}+\dfrac{1}{g}\cdot\dfrac{\partial^2 g}{\partial x_i\partial x_j}\right)}_{h_{ij}}\end{cases}$$

$$|H|^{-\frac{1}{2}}=\left(\frac{2}{n}\right)^{-\frac{d}{2}}\cdot|(h_{ij})|$$

$$\Rightarrow\begin{cases}I\approx\pi^{\frac{d}{2}}\cdot h(x_0)\cdot g^2(x_0)\cdot|(h_{ij})|\\[1ex] h_{ij}=-\dfrac{1}{g^2}\cdot\dfrac{\partial g}{\partial x_i}\cdot\dfrac{\partial g}{\partial x_j}+\dfrac{1}{g}\dfrac{\partial^2 g}{\partial x_i\partial x_j}\end{cases}$$

$$\begin{cases} g=k(x,x_{n+1})-k_n^T(x_{n+1})\cdot K_n^{-1}\cdot k_n(x_{n+1})\end{cases}$$

$$\frac{\partial g}{\partial x_i} = -\frac{2}{\theta}\cdot(x(i) - x_{n+1}(i)\cdot k(x, x_{n+1})$$

$$+\frac{2}{\theta}\cdot k_n^T(x_{n+1})\cdot K_n^{-1}\cdot \mathrm{diag}(x(i) - x_{1:n}(i))\cdot k_n(x)$$

$$\frac{\partial^2 g}{\partial x_i^2} = -\frac{2}{\theta}k(x, x_{n+1}) + \frac{4}{\theta^2}\cdot(x(i) - x_{n+1}(i))^2\cdot k(x, x_{n+1})$$

$$+\frac{2}{\theta}k_n^T(x_{n+1})\cdot K_n^{-1}\cdot k_n(x)$$

$$-\frac{4}{\theta^2}\cdot k_n^T(x_{n+1})\cdot K_n^{-1}\cdot \mathrm{diag}(x(i) - x_{1:n}(i))^2\cdot k_n(x)$$

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{4}{\theta^2}(x(i) - x_{n+1}(i))(x(j) - x_{n+1}(j))\cdot k(x, x_{n+1})$$

$$-\frac{4}{\theta^2}\cdot k_n^T(x_{n+1})\cdot K_n^{-1}\cdot \mathrm{diag}((x(i) - x_{1:n}(i))(x(j) - x_{1:n}(j)))\, k_n^T(x)$$