MFE 409 Lecture 2a
Measuring Value-at-Risk

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Spring 2019

UCLA Anderson

Lecture Objectives

Measuring Value-at-Risk:

- How to judge validity of a VaR estimate?
- Historical approach
- Model-building approach
- How to get a measure for a given approach but also how to choose an appropriate approach

Outline

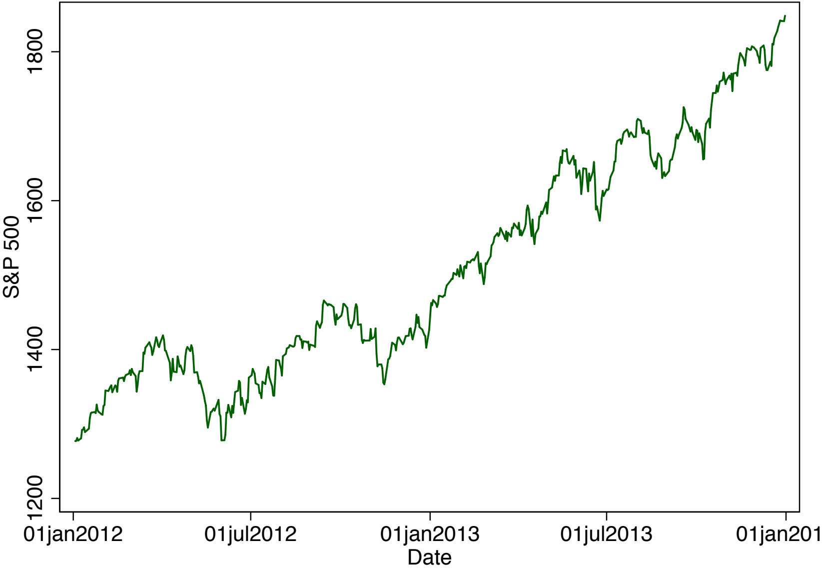
1. Back-testing

2. Historical Simulation

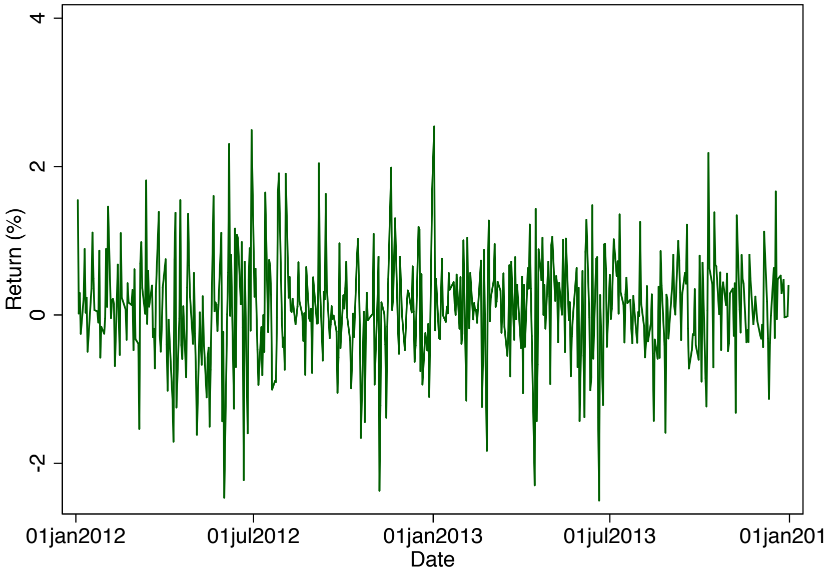
Back-Testing

- **Back-testing**: How well a current procedure would have performed if applied in the past
  - Investment strategy
  - Risk measure
- Our context: How would a method to compute Value-at-Risk would have performed in the past?

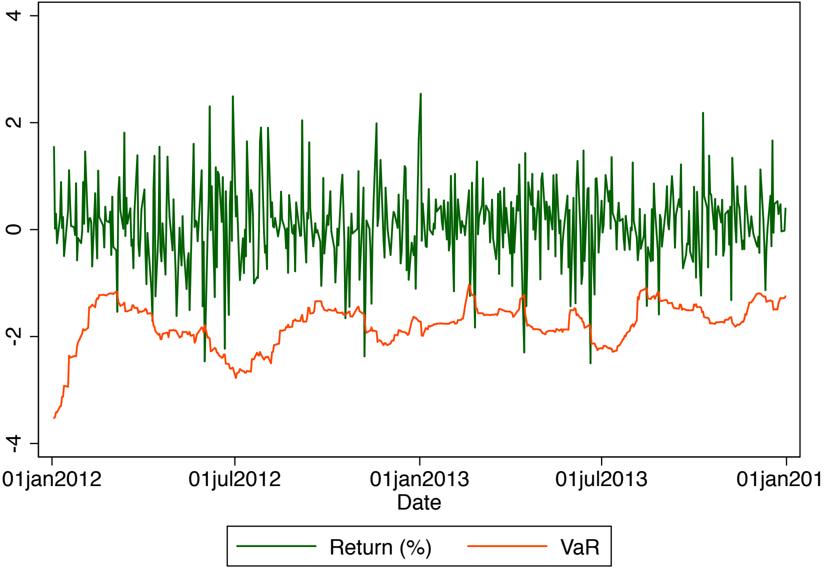
S&P500 Index, 2012-2013



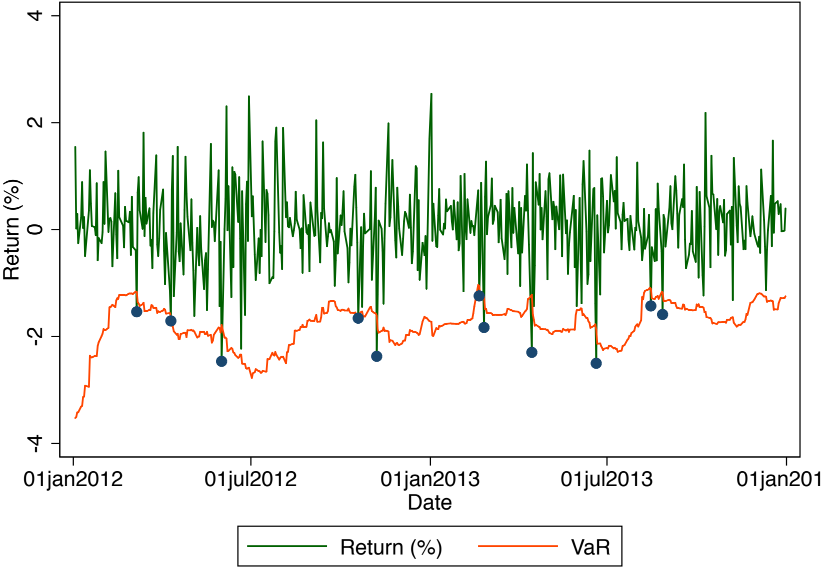
S&P500 Daily Returns, 2012-2013



A 99% VaR Measure

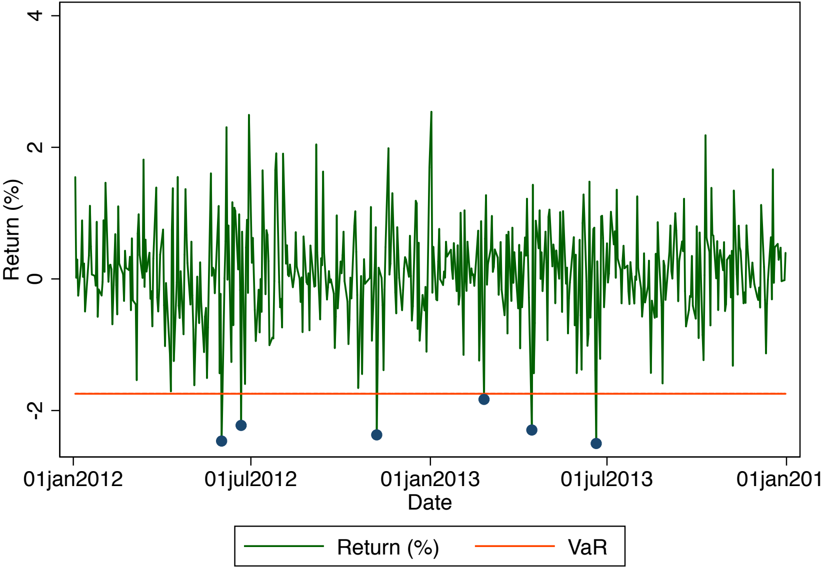


Exceptions



Return (%)
4
2
0
-2
-4
01jan2012
01jul2012
01jan2013
01jul2013
01jan201
Date
Return (%)
VaR

Another 99% VaR Measure



Number of Exceptions

- Say we measure the daily VaR with confidence c

- On a given day:
  - Probability of exception: $1 - c$
  - Probability of no exception: c

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Number of Exceptions

- Say we measure the daily VaR with confidence c
- On a given day:
  - Probability of exception: $1 - c$
  - Probability of no exception: c
- For 99% VaR, a 2-year sample should have on average 5 exceptions

Distribution of Number of Exceptions

- What if we see 6 exceptions? 11 exceptions?

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- Binomial distribution: $P(\#\text{ exceptions} \geq m) = 1 - F(m-1|n, 1-c)$
  - $F(.|n,p)$ c.d.f. of a binomial with n trials and success probability p

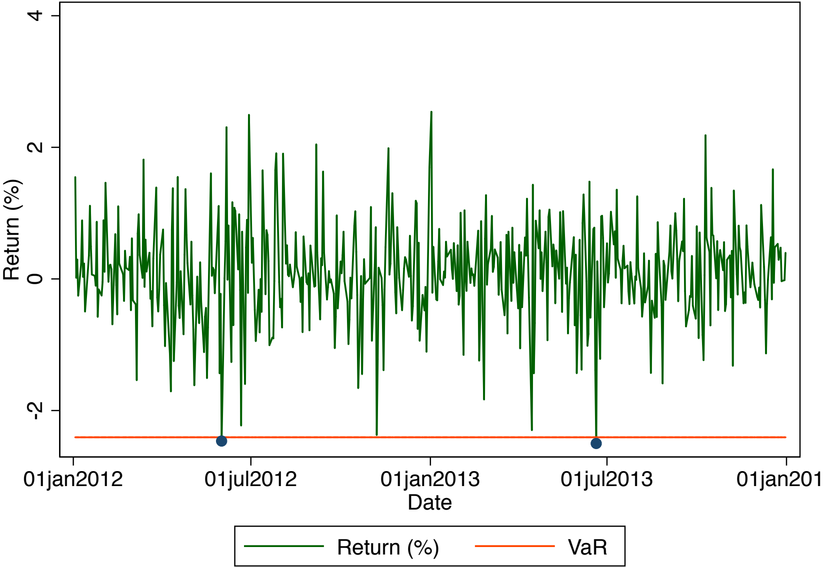
Application

- 99% - daily VaR
- 2 years: 502 daily returns
- Probability of 6 or more exceptions:
- Probability of 11 or more exceptions:

Application

- 99% - daily VaR
- 2 years: 502 daily returns
- Probability of 6 or more exceptions: 38.76%
- Probability of 11 or more exceptions: 1.3%

Another VaR Measure



Other Tests

- Probability of observing less (or equal) than m exceptions:

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$$= F(m|n, 1-c)$$

Other Tests

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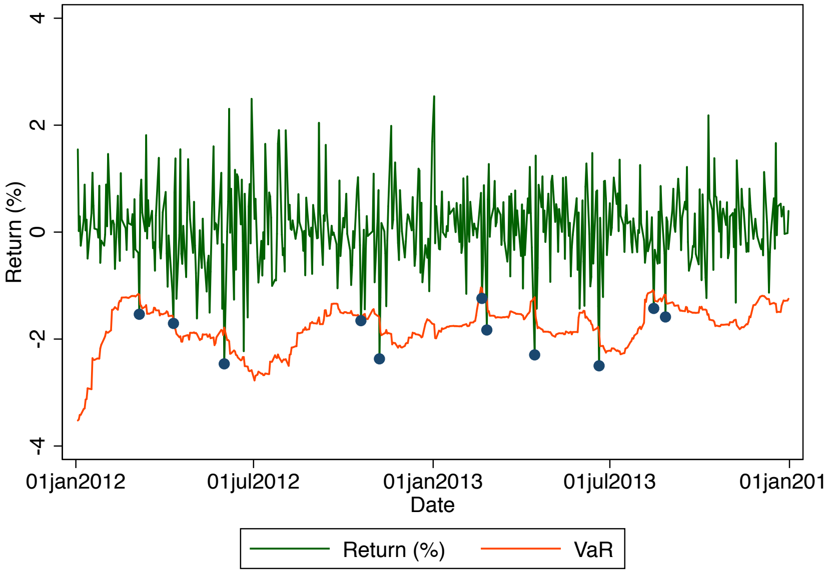
$$\sum_{k=0}^{m} \frac{n!}{k!(n-k)!}(1-c)^k c^{n-k}$$
$$= F(m|n, 1-c)$$

- Two sided test (for large n):

$$-2\ln\left[c^{n-m}(1-c)^m\right] + 2\ln\left[(1-m/n)^{n-m}(m/n)^m\right] \sim \chi^2(1)$$

  - Chi-squared 5% threshold: 3.84

Bunching



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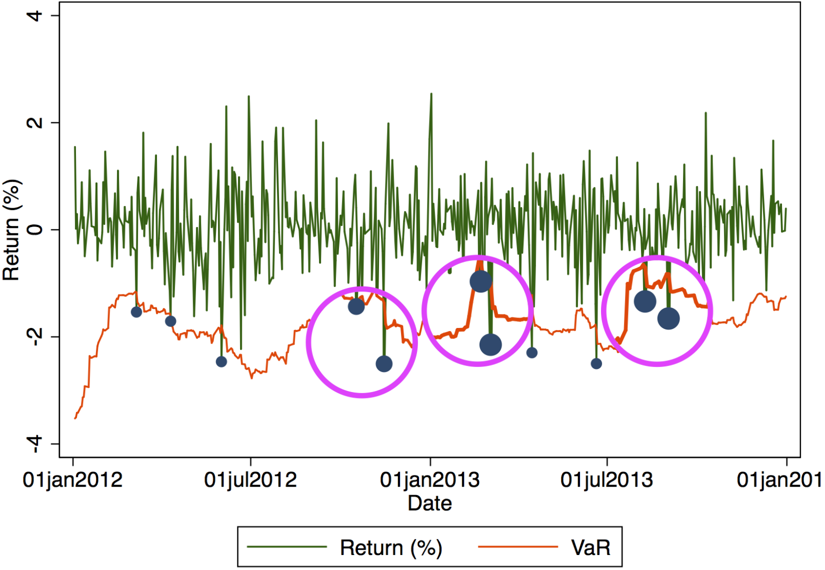

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- Assume the future will be drawn from the same distribution as the past
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  - You have n past observations of daily **returns**
  - Assume the next return will be any of these draws with probability $1/n$

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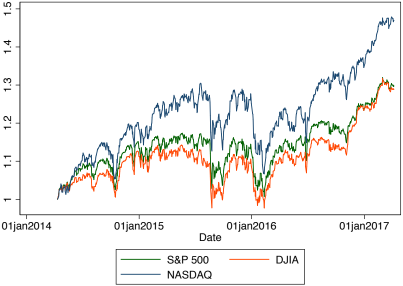
- Formally, to compute the daily value-at-risk at confidence level c:
  - You have n past observations of daily **returns**
  - Assume the next return will be any of these draws with probability $1/n$
  - The VaR corresponds to the loss in the $[(1-c) \times n]$-th worst past realization
    - ⋆ if not integer, round up

Example

- Assume we are 04/11/2017
- You have \$4m invested in S&P500, \$5m in NASDAQ Composite, \$1m in DJIA
- You know the value of the indices for the last 3 years (file *indices.xls*)

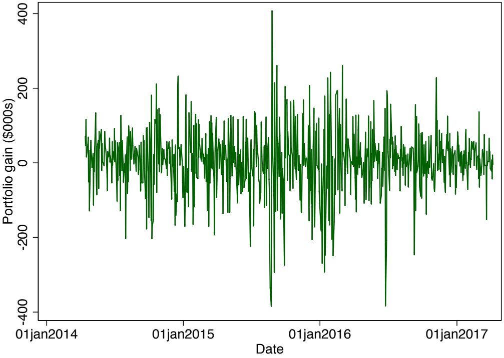


- What is your 1-day 99% VaR?

Constructing the VaR

- Construct returns for the indices: if index is I_t, return is $r_t = (I_t/I_{t-1}) - 1$
- Construct hypothetical portfolio returns: (gains/losses)

$$r_t^{\text{Portfolio}} = \$4\text{m} \times r_t^{\text{S\&P500}} + \$5\text{m} \times r_t^{\text{NASDAQ}} + \$1\text{m} \times r_t^{\text{DJIA}}$$



Constructing the VaR

- Sort the 753 realizations from worse to best

stressed VaR

1.	24aug2015	-384.4229
2.	24jun2016	-383.3271
3.	21aug2015	-334.4092
4.	01sep2015	-293.692
5.	13jan2016	-292.5246
6.	28sep2015	-273.9006
7.	07jan2016	-269.3122
8.	05feb2016	-249.1592
9.	15jan2016	-247.4063
10.	09sep2016	-246.4139
11.	20aug2015	-246.061
12.	29jun2015	-223.0912
13.	27jun2016	-207.9397
14.	11dec2015	-206.0038

Average →

ES = $310,000

VaR = $249,000

Constructing the VaR

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- Value-at-risk corresponds to the $753 \times 1\% = 8$-th worst realization: $\$249,000$

Expected Shortfall

- Can use the same method to compute expected shortfall

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- Average of the $[(1-c)\times n]$ worst realizations
  - Still round up

Stressed VaR

- Stressed VaR (or ES): VaR (or ES) for the worst consecutive 251-day period in the historical sample

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- Stressed VaR (or ES): VaR (or ES) for the worst consecutive 251-day period in the historical sample

  = the one with the largest VaR

- Introduced by regulators to capture the idea that some periods are worse than others

  VaR = 8th worst

- (Stressed VaR) $\geq$ VaR ? YES

Accuracy of VaR

- If you backtest historical VaR, you find exactly **?** deviations

Accuracy of VaR

- If you backtest historical VaR, you find exactly $(1-c) \times n$ deviations
- But if you had the true VaR, you would sometimes find more, sometimes find less: historical VaR is not perfectly accurate
- Standard error of the estimate:

$$\frac{1}{f(x)}\sqrt{\frac{c(1-c)}{n}}$$

  - $f(x)$: p.d.f. at quantile c
  - Need to know distribution!

Example: Accuracy of VaR

- Back to portfolio example
- Historical VaR: $249,000

Example: Accuracy of VaR

- Back to portfolio example
- Historical VaR: $249,000
- Approximate by a normal (in $000s): mean 4, standard deviation 87

$$
\begin{aligned}
x &= \mu + \sigma\Phi^{-1}(0.01) && = -198.4 \\
f(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) && = 3.06 \times 10^{-4} \\
\text{StdDev(VaR)} &= \frac{1}{f(x)}\sqrt{\frac{0.99 \times 0.01}{753}} && = 12
\end{aligned}
$$

[VaR − 1.96 StdDev(VaR), VaR + 1.96 StdDev(VaR)]

- 95% confidence interval for the VaR is between $229,000 and $269,000 → not that precise

Bootstrap

- **Bootstrap**:

Bootstrap

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- Suppose there are 500 daily changes and you want to calculate a 95% confidence interval for VaR

  1. Sample 500,000 times with replacement from daily changes to obtain 1000 sets of changes over 500 days

  2. Calculate VaR for each set

  3. Calculate a confidence interval by taking the range between the 2.5% lowest and 97.5% largest value

How Much Historical Data?

- Portfolio example used 3 years of data
- How much data would you like to use?

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- How much data would you like to use?
- More data, more precise estimates
- But "future same as past" less likely to be true

Weighting of observations

- Use as much data as possible, but put more weight on recent data
- Weight observations with an exponential decay as you go back in time. = probability $\lambda < 1$
- Observation i receives weight:

$$\lambda^{n-i} \frac{1-\lambda}{1-\lambda^n}$$

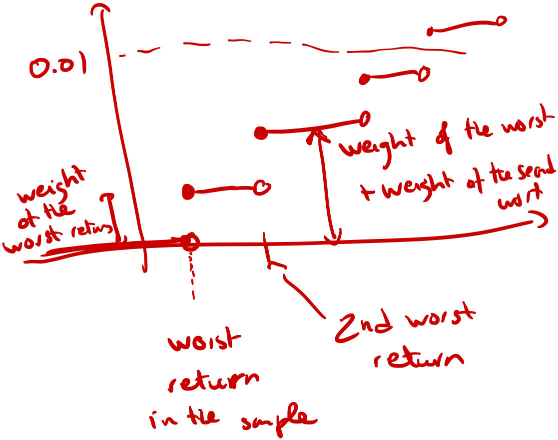
- Sort observations, VaR is the scenario just over $1-c$ cumulative weight

$\lambda = 0.995$

	Return		Weights
Old	x	λ^{n-1}	Small
	x		
		λ^2	
		λ	
Recent	x	1	Big

Sum and divide

c.d.f.



Portfolio Example with Weighting

- $\lambda = 0.995$

	Date	Return	Weight	Cumulative weight
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6.	28sep2015	-273.9006	.0007428	.0057085
7.	07jan2016	-269.3122	.001055	.0067636
8.	05feb2016	-249.1592	.0011663	.0079299
9.	15jan2016	-247.4063	.0010873	.0090171
10.	09sep2016	-246.4139	.0024737	.0114909
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Estimating the Tail

- Extreme tail estimated imprecisely with historical method: 99.9% would need multiple thousands of observations
- To get more precise estimates, make assumptions about the shape of the distribution
- Model the whole distribution, e.g. normal distribution
  - VaR depends of σ
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- Model the left tail of the distribution, e.g. using a power law
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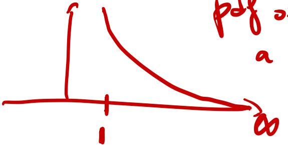
- **Extreme value theory**: this approach is valid for many distributions

Power Law



pdf of a power law

- **Power law**: X follows a power law, with

$$\text{Prob}(X > x) = Kx^{-1/\xi}$$

  - Also called Pareto distribution
  - $\xi < 1$ controls thickness of tail: low ξ, thin tail

Fat tail distribution

Power Law

- **Power law**: X follows a power law, with

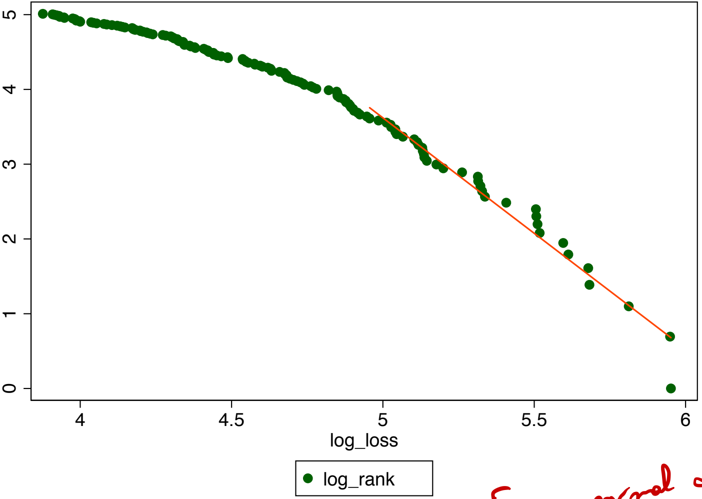
$$\text{Prob}(X > x) = Kx^{-1/\xi}$$

$$\log(\underline{\qquad}) = -\frac{1}{\xi}\log(x) + \log(K)$$

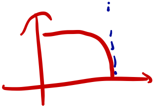
  - Also called Pareto distribution
  - $\xi < 1$ controls thickness of tail: low ξ, thin tail

- Regress $log[\text{Prob}(X > x)]$ on $log(x)$: slope $-1/\xi$
  - In historical distribution: $\text{Prob}(X > x_i) = \text{rank}(x_i)/n$

Log-log Plot for Portfolio Loss



For a normal distribution:



- Slope: -3, $\xi = 1/3$

Extreme Value Theory

- Key result: a wide range of probability distributions have common properties in the tail

Pickands - Balkema - de Haan theorem
Second Theorem of EVT

Extreme Value Theory

- Key result: a wide range of probability distributions have common properties in the tail
- Tail distribution:

$\mathbb{P}(u \le X \le u+y)$

$\mathbb{P}(X \le u+y \mid X \geq u)$

$$F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)}$$

$\mathbb{P}(u \le X)$

- Result: as u becomes large, $F_u(y)$ converges to a generalized Pareto distribution:

$$G_{\xi,\beta}(y) = 1 - \left[1 + \xi \frac{y}{\beta}\right]^{-1/\xi}$$

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- Model of right tail! Remember to find the c-th quantile of losses

Estimating the Power Law

- Partial distribution function:

$$g_{\xi,\beta}(y) = \frac{1}{\beta}\left(1 + \frac{\xi y}{\beta}\right)^{-1/\xi-1}$$

- Choose u: typically 95th percentile of historical distribution

Estimating the Power Law

- Partial distribution function:

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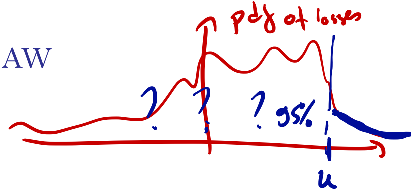
- Choose u: typically 95th percentile of historical distribution

- Maximize log likelihood:

$$\max_{\xi,\beta} \sum_{i \in tail} \ln\left[g_{\xi,\beta}(v_i - u)\right]$$

VaR and ES for a Power Law



- Probability distribution:

$$\text{Prob}(\text{Loss} > \text{V}) = \underbrace{[1 - F(u)]}_{n_u/n}[1 - G_{\xi,\beta}(\text{V} - u)]$$

$$= \mathbb{P}(\text{Loss} > u) \times \mathbb{P}(\text{Loss} > \text{V} \mid \text{Loss} > u)$$

VaR and ES for a Power Law

- Probability distribution:

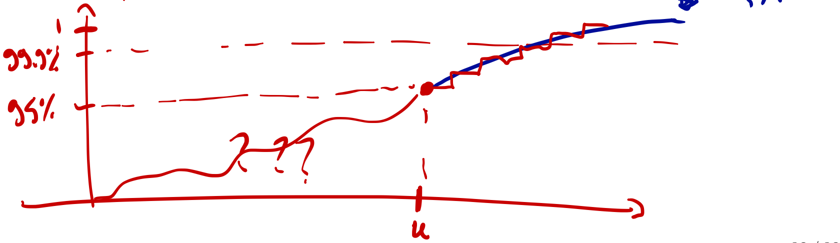
$$\text{Prob(Loss} > \text{V)} = \underbrace{[1 - F(u)]}_{n_u/n}[1 - G_{\xi,\beta}(\text{V} - u)]$$

$$1-c = \frac{n_u}{n} \times \left[1+\xi\frac{(V-u)}{\beta}\right]^{-1/\xi}$$

- V is VaR if this is $1 - c$

$(1-c)$

$$\text{VaR} = u + \frac{\beta}{\xi}\left(\left[\frac{n}{n_u}\right]^{-\xi} - 1\right)$$



VaR and ES for a Power Law

- Probability distribution:

$$\text{Prob}(\text{Loss} > \text{V}) = \underbrace{[1 - F(u)]}_{n_u/n}[1 - G_{\xi,\beta}(\text{V} - u)]$$

- V is VaR if this is $1 - c$

$$\text{VaR} = u + \frac{\beta}{\xi}\left(\left[\frac{n}{n_u}c\right]^{-\xi} - 1\right)$$

- Can also obtain ES:

$$\text{ES} = \frac{\text{VaR} + \beta - \xi u}{1 - \xi}$$