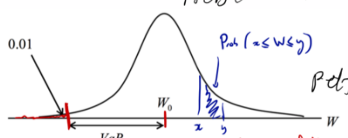
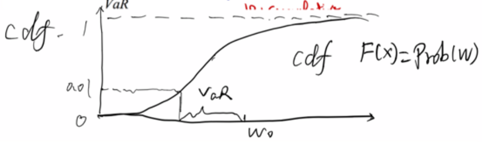
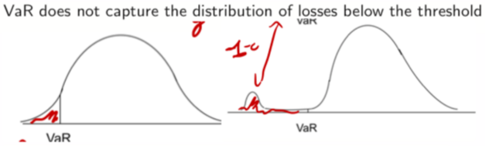
Modigliani-Miller theorem: in the absence of frictions, capital structure does not matter

Definition. Let W be a random variable. The *Value-at-Risk* at confidence level c relative to base level W_0 is the smallest non-negative number denoted by VaR such that $Prob(W < W_0 - VaR) \le 1-c$.





VaR does not capture the distribution of losses below the threshold



Expected shortfall: expected loss given loss larger than VaR

$$ES = W_0 - \mathbb{E}[W|W \le W_0 - VaR]$$

$ES \ge VaR$

$$= W_0 - \frac{\int_{-\infty}^{W_0-VaR} W f(W) dW}{\int_{-\infty}^{W_0-VaR} f(W) dW} = 1-c$$

$E(W) = \int_{-\infty}^{\infty} wf(w)dw$. $E(W|W \le a) = \frac{\int_{-\infty}^{a} wf(w)dw}{\int_{-\infty}^{a} f(w)dw}$

Also called C-VaR and Tail Loss conditional VaR

$W - W_0 \sim N(\mu, \sigma)$. $VaR = -(\mu + z(1-c)\cdot\sigma)$.

$ES = -\mu + \sigma \cdot \frac{e^{-z(c)^2/2}}{\sqrt{2\pi}(1-c)}$

Losses in successive days are independent, normally distributed, and have a mean of zero $\quad T\text{-day VaR} = \text{1-day VaR} \times \sqrt{T}$

$$T\text{-day ES} = \text{1-day ES} \times \sqrt{T}$$

Autocorrelation ρ between losses on successive days, replace $\sqrt{T}$ by

$$\sqrt{T + 2(T-1)\rho + 2(T-2)\rho^2 + 2(T-3)\rho^3 + \ldots + 2\rho^{T-1}}$$

$$VaR_{total} = \sqrt{\sum_i \sum_j VaR_i VaR_j \rho_{ij}} = \sqrt{\sum_i VaR_i^2 + 2\sum_{i<j} \rho_{ij} VaR_i VaR_j}$$

approx, but exact formula for normal.

Marginal VaR

$$DVaR_i = \frac{\partial VaR}{\partial x_i} \approx \frac{f(x_1, \ldots x_i + \Delta x_i \ldots x_n) - f(x_1, \ldots x_i, \ldots x_n)}{\Delta x_i}$$

increase by \$1.

Component VaR

$$CVaR_i = x_i \frac{\partial VaR}{\partial x_i}$$ how much VaR change if position ↑ 1%

Decomposition (Euler Theorem): $VaR = \sum_i CVaR_i$

$f(\lambda x_1, \ldots \lambda x_n) = \lambda f(x_1, \ldots x_n)$.

differenciate w.r.t. λ set $\lambda = 1$:

$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots x_n \frac{\partial f}{\partial x_n} = f$

Risk Adjusted Rate of Return on Capital (RAROC): profit per unit of necessary capital, i.e. profit per unit of VaR

$$RAROC = \frac{\text{Profit}}{\text{VaR}}$$

Define a coherent risk measure as the amount of cash that has to be added to a portfolio to make its risk acceptable

Properties of coherent risk measure

- If one portfolio always produces a worse outcome than another its risk measure should be greater
- If we add an amount of cash K to a portfolio its risk measure should go down by K
- Changing the size of a portfolio by a factor λ should result in the risk measure being multiplied by λ
- The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged

Value-at-Risk ✗   Expected Shortfall ✓

Probability of observing k exceptions: $\quad \frac{n!}{k!(n-k)!}(1-c)^k c^{n-k}$

Two sided test (for large n): $\quad$ Chi-squared 5% threshold: 3.84

$$-2\ln\left[c^{n-m}(1-c)^m\right] + 2\ln\left[(1-m/n)^{n-m}(m/n)^m\right] \sim \chi^2(1)$$

Stressed VaR (or ES): VaR (or ES) for the worst consecutive 251-day period in the historical sample

(Stressed VaR) $\ge$ VaR ? Yes. VaR 8th worst, Pigeon's hole

Standard error of the estimate: $\quad \frac{1}{f(x)}\sqrt{\frac{c(1-c)}{n}}$

$f(x)$: p.d.f. at quantile c $\quad$ Need to know distribution

Normal pdf: $\quad f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

Observation i receives weight: $\quad \lambda^{n-i}\frac{1-\lambda}{1-\lambda^n}$ $\quad \lambda = 0.995$

Sort observations, VaR is the scenario just over $1-c$ cumulative weight

Power law: X follows a power law, with

$Prob(X > x) = Kx^{-1/\xi}$ $\quad$ Also called Pareto distribution

$\xi < 1$ controls thickness of tail: low ξ, thin tail

Regress $log[Prob(X > x)]$ on $log(x)$: slope $-1/\xi$

- In historical distribution: $Prob(X > x_i) = rank(x_i)/n$

Extreme value theory:

Key result: a wide range of probability distributions have common properties in the tail

Tail distribution: $\quad F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)}$ $\quad P(u \le X \le u+y)$

$P(X \le u+y | X \ge u)$ $\quad P(X \ge u)$

Result: as u becomes large, $F_u(y)$ converges to a generalized Pareto distribution: $\quad G_{\xi,\beta}(y) = 1 - \left[1 + \xi\frac{y}{\beta}\right]^{-1/\xi}$

Partial distribution function $\quad g_{\xi,\beta}(y) = \frac{1}{\beta}\left(1 + \frac{\xi y}{\beta}\right)^{-1/\xi - 1}$

Maximize log likelihood: $\quad \max_{\xi,\beta} \sum_{i \in tail} \ln\left[g_{\xi,\beta}(v_i - u)\right]$

Probability distribution:

$Prob(Loss > V) = \underbrace{[1 - F(u)]}_{n_u/n}[1 - G_{\xi,\beta}(V-u)] = P(Loss > u) \times P(Loss > V | Loss > u)$

V is VaR if this is $1-c$ $\quad VaR = u + \frac{\beta}{\xi}\left(\left[\frac{n}{n_u}(1-c)\right]^{-\xi} - 1\right)$.

$1 - c = \frac{n_u}{n} \times \left(1 + \frac{\xi(V-u)}{\beta}\right)^{-1/\xi}$ $\quad ES = \frac{VaR + \beta - \xi u}{1 - \xi}$

$\underbrace{\Delta P}_{\text{portfolio gain}} = \sum_i \alpha_i R_i \sim \mathcal{N}(0, \sigma_P^2)$ $\quad \sigma_P^2 = \sum_i \sum_j \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}$

$\underbrace{\Delta P}_{\text{portfolio gain}} = \alpha' \mathbf{R} \sim \mathcal{N}(0, \sigma_P^2)$ $\quad \sigma_P^2 = \alpha' \Sigma \alpha$

Want to hedge a position R_p using a hedging instrument R_h

Optimal hedging position: $\quad \alpha_{hedge} = -\rho\frac{\sigma_p}{\sigma_h}$

Variance of the hedged portfolio: $\quad$ Minimum variance $= \sigma_p^2(1-\rho^2)$

VaR of the hedged portfolio: $\quad$ Minimum VaR $= VaR_p\sqrt{1-\rho^2}$

Only depends of correlation ρ

Only count trading days: $\sigma_{yr} = \sigma_{day} \times \sqrt{252}$

mean: $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_{t-i}$ $\quad$ vol $\quad \sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(R_{t-i} - \bar{R})^2$

In practise, $\bar{R} = 0$ $\quad n-1 \to n$. $\quad \sigma^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} R_{t-i}^2$

Likelihood for one observation $\quad \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-R_i^2}{2\sigma^2}\right)$

Log likelihood $\quad \mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\left[-\log(\sigma^2) - \frac{R_{t-i}^2}{\sigma^2}\right]$

First-order condition w.r.t. σ^2 $\quad 0 = -\frac{n}{\sigma^2} + \frac{1}{\sigma^4}\sum_{i=1}^{n} R_{t-i}^2$

Weighting scheme + long-run variance

$$\sigma_t^2 = \gamma V_L + \sum_{i=1}^{n}\alpha_i R_{t-i}^2 \quad \text{with } 1 = \gamma + \sum_{i=1}^{n}\alpha_i$$

ARCH(m), autoregressive conditional heteroskedasticity

$\sigma_t^2 = \omega + \sum_{i=1}^{m}\alpha_i R_{t-i}^2$ $\quad$ if $\alpha_i = 1/m$, $\omega = 0$, rolling window estimate.

EWMA, exponentially weighted moving average $\quad \alpha_i = (1-\lambda)\lambda^i$

$\sigma_t^2 = \sum_{i=1}^{\infty}(1-\lambda)\lambda^{i-1} R_{t-i}^2 = (1-\lambda)R_{t-1}^2 + \lambda\underbrace{\sum_{i=2}^{\infty}(1-\lambda)\lambda^{i-2}R_{t-i}^2}_{\sigma_{t-1}^2}$

$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)R_t^2$ $\quad \lambda = 0.94$ till 2006 after 0.995

GARCH(1,1), generalized autoregressive conditional heteroskedasticity

$\sigma_t^2 = \underbrace{\gamma V_L}_{\omega} + \alpha R_{t-1}^2 + \beta\sigma_{t-1}^2$ $\quad$ EWMA + long-run average

If $\gamma = 0$, EWMA. For stability, $\alpha + \beta < 1$. $\alpha + \beta + \gamma = 1$

Initialize at $\sigma_0 = \sqrt{V_L} = \sqrt{\omega/(1-\alpha-\beta)}$

Ljung-Box Statistic $\quad n\sum_{k=1}^{K} w_k c_k^2 \quad w_k = \frac{n+2}{n-k}$

k-days in future, $\mathbb{E}_t\left[\sigma_{t+k}^2\right] = V_L + (\alpha+\beta)^k(\sigma_t^2 - V_L)$

VIX systematically higher than realized volatility

Risk adjustment implicit in options, that make VIX higher than future realized volatility

It does not mean that market expectations are systematically too high

Because higher volatility implies a higher price, if OTM options have higher implied volatility than ATM option $\to$ market expects negative skewness $\quad$ the difference in implied volatilities is time varying

high *negative* skewness imply high downside risk

Skew Index $= 100 - 10 \times$ Implied Expected Skewness

Higher positive index $\to$ higher downside risk

Model-building useful for $\quad$ Large portfolios $\quad$ Limited data

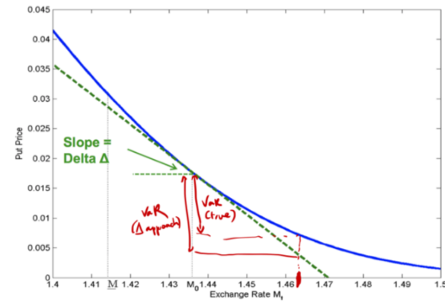
Taking account of nonlinearities

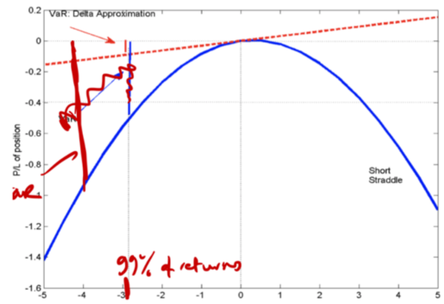
Historical simulations useful for: $\quad$ Non-normal situations

Unknown structure of investment performance

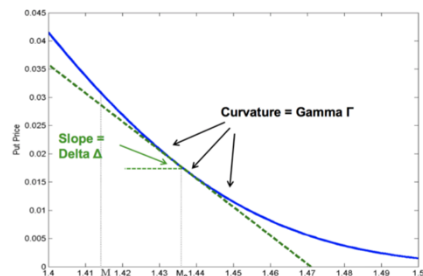
Key trade-off: making more assumptions vs. using a small part of data

Option $VaR = 2.32 \cdot \Delta S_t \cdot \sigma_S$.





$$P(S) \approx P(S_0) + \underbrace{P'(S_0)}_{\Delta}(S - S_0) + \frac{1}{2}\underbrace{P''(S_0)}_{\Gamma}(S - S_0)^2$$



Assume change in underlying price $S_{t+1} - S_t \sim \mathcal{N}(\mu_S, \sigma_S^2)$

Change in portfolio value: $P_{t+1} - P_t = \Delta \times (S_{t+1} - S_t) + \frac{1}{2}\Gamma \times (S_{t+1} - S_t)^2$

Can compute moments of $R_S = S_{t+1} - S_t$: $\quad \mathbb{E}[R_S] = \mu_S$

$\mathbb{E}[R_S^2] = \sigma_S^2 + \mu_S^2 \quad \mathbb{E}[R_S^3] = \mu_S^3 + 3\mu_S\sigma_S^2 \quad \mathbb{E}[R_S^4] = \mu_S^4 + 6\mu_S^2\sigma_S^2 + 3\sigma_S^4$

Obtain mean and variance of $P_{t+1} - P_t$:

$\mathbb{E}[P_{t+1} - P_t] = \Delta\mathbb{E}[R_S] + \frac{1}{2}\Gamma\mathbb{E}[R_S^2] = \Delta\mu_S + \frac{1}{2}\Gamma(\mu_S^2 + \sigma_S^2)$

$var[P_{t+1} - P_t] = \Delta^2 var[R_S] + \frac{1}{4}\Gamma^2 var[R_S^2] + \frac{1}{2}\Delta\Gamma cov[R_S, R_S^2]$

$= \Delta^2\sigma_S^2 + \frac{1}{2}\Gamma^2\sigma_S^2(2\mu_S^2 + \sigma_S^2) + 2\Delta\Gamma\mu_S\sigma_S^2$

$VaR(c) = -\mathbb{E}[P_{t+1} - P_t] + z(c) var[P_{t+1} - P_t]$

Cornish-Fisher expansion: asymptotic expansion for the quantile of a distribution $\quad$ Skewness: $\xi_P = \mathbb{E}[(R_P - \mu_P)^3]/\sigma_P^3$

Quantile $1-c$: $\quad \mu_P + \left(z(1-c) + \frac{1}{6}\left(z(1-c)^2 - 1\right)\xi_P\right)\sigma_P$

$P_{t+1} - P_t = \Delta \times (S_{t+1} - S_t) + \frac{1}{2}\Gamma \times (S_{t+1} - S_t)^2 + \nu(\sigma_{t+1} - \sigma_t) + \ldots$

Delta can be adjusted by trading the underlying

Gamma and Vega need trading of other options

TYPES OF RISK $\quad$ Market Risk $\quad$ Credit Risk

Liquidity Risk $\quad$ Operational Risk

Main goal of regulation: eliminate risk of bank failure

Create a stable economic environment where private individuals and businesses have confidence in the banking system

Basel I is focused on credit risk

$$\text{Cooke Ratio} = \frac{\text{Capital}}{\text{Risk-weighted Assets}} \quad \text{lower bound}$$

Capital: can be lost without the firm failing

Risk-weighted assets: quantity of assets that can default

Each asset receives a weight according to its risk: larger for more risky assets $\quad$ Risk weighted assets.

RISK-WEIGHTED CAPITAL FOR DERIVATIVES

Credit equivalent amount: $\quad \underbrace{\max(V, 0)}_{\text{current exposure}} + \underbrace{aL}_{\text{add-on}}$

L: principal $\quad a$: add-on factor. Varies by asset class and maturity

Tier 1 Capital: common equity, non-cumulative perpetual preferred shares

Tier 2 Capital: cumulative preferred stock, certain types of 99-year debentures, subordinated debt with an original life of more than 5 years

Capital requirement: Cooke ratio above 8%, half of it from Tier 1 capital

Netting: clause in the agreement that all transaction are considered as one in default

Current exposure: replace $\sum\max(V_i, 0)$ by $\max(\sum V_i, 0)$

Add-on: replace $\sum a_i L_i$ by $(0.4 + 0.6 \times NRR)\sum a_i L_i$

Net Replacement Ratio $\quad NRR = \frac{\max(\sum V_i, 0)}{\sum\max(V_i, 0)}$

1996 Amendment

Account for market risk, implemented in 1998

Assets of banks in two parts

- *Trading book*: marketable securities, derivatives. Marked to market.
- *Banking book*: assets expected to be held until maturity. Held at historical cost.

Under amendment:

- Credit risk charge for everything except positions in trading book in debt and equity traded securities, and commodities and foreign exchange
- Market risk charge for all asset in the trading book

Two ways to compute the market risk charge:

- Standardized approach: capital for each security class, not accounting for correlation
- Internal model-based approach: use model to compute VaR

Capital requirement for market risk: $\max(\text{VaR}_{t-1}, m_c \times \text{VaR}_{\text{avg}}) + SRC$

VaR: 10-day 99% Can use $\sqrt{10}\times$ 1-day VaR

Average over past 60 days

m_c depends of 1-year back-test performance: <5 exceptions, $m_c = 3$, grows gradually until $m_c = 4$ for 10 or more (with some discretion from regulator) SRC: specific risk charge, for risks with particular companies

Basel II Three pillars:

Minimum Capital Requirements: modifies credit risk, adds operational risk

Supervisory Review: communicate with supervisor, early intervention

Market Discipline: communicate with investors

Three methods:

Standardized Approach: Risk-weights depends of rating Adjustment for collateral

Foundation Internal Ratings Based (IRB) Approach

Expected loss from defaults: $\sum_i \text{EAD}_i \times \text{LGD}_i \times \text{PD}_i$

PD: probability that the counterparty will default within one year

EAD: exposure at default LGD: loss given default = 1 − recovery.

Approximation for 99.9% VaR: $\sum \text{EAD}_i \times \text{LGD}_i \times \text{WCDR}_i$

WCDR: worst-case default rate, default rate in the 99.9th percent worst aggregate outcome ρ (sometimes R): copula correlation

$$\text{WCDR}_i = \mathcal{N}\left[\frac{\mathcal{N}^{-1}(PD_i) + \sqrt{\rho}\,\mathcal{N}^{-1}(0.999)}{\sqrt{1-\rho}}\right]$$

Capital required: $\sum_i \text{EAD}_i \times \text{LGD}_i \times (\text{WCDR}_i - PD_i) \times \text{MA}_i$

Risk-weighted assets: $\times 12.5$

Basel III Six parts:

Capital Definition and Requirements Capital Conservation Buffer

Countercyclical Buffer Leverage Ratio

Liquidity Risk Counterparty Credit Risk

Tier 1 equity capital: 4.5% of risk-weighted assets

Total Tier 1 capital: 6% of risk-weighted assets

Total capital: 8% of risk-weighted assets

Capital conservation buffer

Need to accumulate additional 2.5% of risk-weighted assets in equity capital ahead of difficult times

Forced to retain earnings if under this threshold: 100% if <5.125%, ...

Countercyclical buffer (CCyB)

Same as capital conservation but left to discretion of national authorities Between 0% and 2.5% of of total risk-weighted assets

Leverage ratio: capital divided by exposure measure

- Capital: Tier 1 capital ▸ No risk-weighting
- Exposure: sum of on-balance-sheet exposures, derivatives exposures, securities financing transaction exposures, off-balance sheet items

Minimum leverage ratio of 3%

- Push to do more in the US, up to 5-6%
- UK: 4.05%, possibly up to 4.95%

Simple broad measure of credit risk, less subject to gaming

Liquidity Coverage Ratio (LCR) $\dfrac{\text{High-Quality Liquid Assets}}{\text{Net Cash Outflows in a 30-Day Period}}$

Bank's ability to survive a 30-day period of liquidity disruptions:

Must be greater than 100%

Net Stable Funding Ratio (NSFR) = $\dfrac{\text{Amount of Stable Funding}}{\text{Required Amount of Stable Funding}}$

Counterparty Credit Risk

Adjust profits for expected default of derivatives counterparties:

credit value adjustment (CVA)

Basel III requires CVA risk from changes in credit spreads to be included in calculation for market risk capital

Basel III explicitly account for those:

SIFI: systematically important financial institution

G-SIBs: global systematically important bank

Credit Risk

Historical Method

BBB | Baa ↑ investment grade

Altman's Z-Score

$Z = 1.2 \times X_1 + 1.4 \times X_2 + 3.3 \times X_3 + 0.6 \times X_4 + 0.99 \times X_5$

X_1=Working Capital/Total Assets

X_2=Retained Earnings/Total Assets X_3=EBIT/Total Assets

X_4=Market Value of Equity/Book Value of Liabilities

X_5=Sales/Total Assets

Historical default table gives *unconditional default probabilities*

Note $V(t)$: probability of surviving up to t, $Q(t) = 1 - V(t)$: probability of default by time t (no survival)

Default intensity or hazard rate $\lambda(t)$: conditional probability of defaulting between t and $t+\Delta t$ is $\lambda(t)\Delta t$

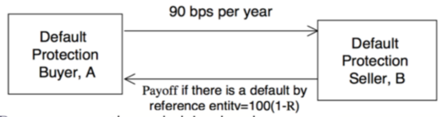
$$\lambda(t)\Delta t = \frac{V(t) - V(t+\Delta t)}{V(t)} = \frac{Q(t+\Delta t) - Q(t)}{1 - Q(t)} \quad -\lambda(t)V(t) = \frac{dV(t)}{dt}$$

$$\lambda(t) = -\frac{\frac{dv}{dt}}{v} = -\frac{d\log(v)}{dt}$$

$$\int_{t_1}^{t_2} \lambda(t)\,dt = \log(V(t_1)) - \log(V(t_2))$$

$$V(t) = e^{-\int_0^t \lambda(\tau)d\tau} \qquad Q(t) = 1 - e^{-\int_0^t \lambda(\tau)d\tau}$$

Credit Default Swaps



R: recovery on the underlying bond 90bps: CDS spread

$$\text{CDS Spread} = \frac{\text{Total Amount Paid Per Year}}{\text{Notional Principal}}$$

Estimating Default Probabilities from CDS

Payoff to selling protection:

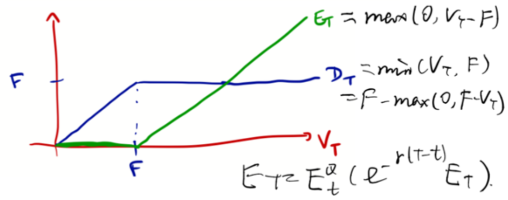
Π = Payment per period $- \bar{\lambda} \times (1 - R) \times$ Notional Principal

Assuming 0 profit: $\bar{\lambda} = \dfrac{\text{CDS Spread}}{1 - R}$

$$\text{CDS Spread}(T) = (1 - R)\frac{\int_0^T \lambda(\tau)e^{-\int_0^\tau r(u)+\lambda(u)du}d\tau}{\int_0^T e^{-\int_0^\tau r(u)+\lambda(u)du}d\tau}$$

Merton Model

Key idea: equity is a call option on asset value



$$E_T = E_t^{\mathbb{Q}}(e^{-r(T-t)}E_T)$$

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \qquad V_t: \text{asset value}.$$

$$V_t = V_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\,\underbrace{Z}_{\mathcal{N}(0,1)}\right]$$

Distance to Default

Firm defaults at date T on its debt when $V_T < F$

Probability of default (viewed from date 0) is:

$$\mathbb{P}[V_T < F] = \mathbb{P}\left[V_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z\right) < F\right]$$

$$= \mathbb{P}\left[Z < -\underbrace{\frac{\ln(V_0/F) + (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}_{d}\right]$$

$= \mathcal{N}(-d)$ d is the *distance do default*

Number of standard deviations away from the default point

Market value at date 0: price of a call option with strike F

$$E_0 = V_0\mathcal{N}(d_1) - Fe^{-rT}\mathcal{N}(d_2) \qquad d_1 = \frac{\ln(V_0/F) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(V_0/F) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad (1)$$

Variations in equity reveal variations in underlying

$$\sigma_E E_0 = \underbrace{\frac{\partial E}{\partial V}}_{\Delta}\sigma V_0 = \mathcal{N}(d_1)\sigma V_0 \quad (2)$$

Issues

Equity volatility may be affected by short term factors in equity market rather than fundamentals Liquidity, microstructure effects

Default is a complex phenomenon

Complex capital structure Coordination problems

Bankruptcy choice Bargaining between creditors and management

Merton model gives reasonable *ordinal ranking* of default risk, but the simple version of model does poor job of matching *cardinal default risk* - the actual probabilities of default

Liquidity

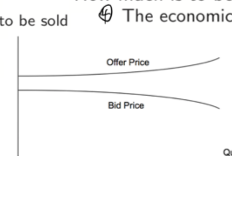
(1) **Market liquidity**, or trading liquidity
(2) **Funding liquidity**

Market liquidity: Ability to sell an asset on short notice

Price received for an asset depends on:

(1) Mid-market price (2) How much is to be sold
(3) How quickly it is to be sold (4) The economic environment

Role of *size*



Proportional bid-ask spread: $s = \dfrac{\text{Ask Price} - \text{Bid Price}}{\text{Mid-market Price}}$ $\dfrac{\text{Bid} + \text{Ask}}{2}$

Cost of liquidating a portfolio with positions α_i right now:

$$\sum_{i=1}^{n} \frac{1}{2}|\alpha_i| s_i(\alpha_i)$$

Stressed conditions: replace s_i by extreme historical value, e.g. 1% largest

Liquidity-adjusted VaR: If portfolio is likely liquidated in extreme bad performance, add liquidation cost to VaR calculation

Optimal Execution

bid-ask difference is $p(q)$ sell S shares over the next n days

q is the quantity sold on that day

daily standard deviation of returns is σ

$$\min_{\{q_t\}} \lambda \sqrt{\sum_{t=1}^{n} \sigma^2 x_t^2} + \sum_{t=1}^{n} \frac{1}{2}|q_t| p(q_t) \qquad \sum_{t=1}^{n} q_t = S$$

$$x_1 = S \qquad x_t = x_{t-1} - q_{t-1}$$

Funding liquidity: Ability to maintain sources of funding for running the firm's activities

Sources of funding liquidity:

(1) Cash and Treasury holdings (2) Ability to borrow
(3) Retail and wholesale deposits (4) Central bank borrowing

Some useful steps: (1) Plan for the lifetime of the strategy

(2) Analyze correlation of strategy performance and funding conditions

(3) Understand behavior of other participants in the markets: if everybody does the same thing, everybody will fall at the same time

Fatalist view: when everything goes bad, there is nothing to do

Why is bunching of exceptions the sign of an issue with a VaR measure?

SOLUTION: VaR measure fails to capture time variation in risk or extreme tail-risk events happened. A useful VaR measure has very few exceptions and randomly distributed along the time series.

When using exponential weighting to compute the VaR using the historical method, we choose typically a parameter $\lambda = 0.995$. When using exponential weighting to estimate volatility we choose typically a parameter $\lambda = 0.94$. Explain why such different values.

SOLUTION: Different λ helps to adjust the weight of historical data. Estimating 1% quantile needs a lot of data (it only uses outliers) whereas estimating volatility is easier (it uses all the data), therefore we use a smaller λ.

What are the advantages and limitations of the VIX as a measure of future volatility?

SOLUTION: Advantages: it is a real time and forward-looking measure. Limitation: risk-neutral (encodes variance risk premium, typically is above actual realized volatility)

ignore means (they are zero)

$$VaR = -\frac{1}{2}\sqrt{(x_1\sigma_1)^2 + (x_2\sigma_2)^2 + 2\rho\sigma_1\sigma_2x_1x_2}\,\Phi^{-1}(0.01)$$

Marginal VaR

$$MVaR(x_1) = -\frac{1}{2}\frac{2x_1\sigma_1^2 + 2\rho\sigma_1\sigma_2x_2}{\sqrt{(x_1\sigma_1)^2 + (x_2\sigma_2)^2 + 2\rho\sigma_1\sigma_2x_1x_2}}\Phi^{-1}(0.01)$$

Cvar

$$CVaR(x_1) = -\frac{1}{2}\frac{2x_1^2\sigma_1^2 + 2\rho\sigma_1\sigma_2x_2x_1}{\sqrt{(x_1\sigma_1)^2 + (x_2\sigma_2)^2 + 2\rho\sigma_1\sigma_2x_1x_2}}\Phi^{-1}(0.01)$$

$$\text{VaR} = CVaR_1 + CVaR_2 = x_1\frac{\partial \text{VaR}}{\partial x_1} + x_2\frac{\partial \text{VaR}}{\partial x_2} = x_1DVaR_1 + x_2DVaR_2.$$

Now for changes Δx_1 and Δx_2 we get $\tilde{\text{VaR}}$

$$\tilde{\text{VaR}} \approx (x_1 + \Delta x_1)DVaR_1 + (x_2 + \Delta x_2)DVaR_2 = \text{VaR} + \Delta x_1 DVaR_1 + \Delta x_2 DVaR_2.$$

Then the conditional likelihood is the joint conditional density is

$$f(r_n, r_{n-1}, ..., r_1|r_0) = \frac{1}{\sigma_n}\varphi\left(\frac{r_n}{\sigma_n}\right) \times ... \times \frac{1}{\sigma_1}\varphi\left(\frac{r_1}{\sigma_1}\right)$$

$$= \frac{1}{(2\pi)^{n/2}\sigma_n...\sigma_1}\exp\left(-0.5\sum_{t=1}^{n}\frac{r_t^2}{\sigma_t^2}\right)$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} 1 & \rho \\ & 1 \end{bmatrix}\right)$$ MLE estimation of correlation

Then, for any given pair of observation

$$f(X_1, X_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{X_1^2 + X_2^2 - 2\rho X_1 X_2}{2(1-\rho^2)}\right)$$

The joint likelihood

$$F = \Pi_{i=1}^{i=N} f(\{X_{1,i}, X_{2,i}\}; \rho) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^N \exp\left(-\frac{1}{2(1-\rho^2)}\sum_{i=1}^{N} X_{1,i}^2 + X_{2,i}^2 - 2\rho X_{1,i}X_{2,i}\right)$$

Take logs

$$-N\log\left(2\pi\sqrt{1-\rho^2}\right) - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{N} X_{1,i}^2 + X_{2,i}^2 - 2\rho X_{1,i}X_{2,i}$$

$$-N\log(2\pi) - \frac{N}{2}\log(1-\rho^2) - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{N} X_{1,i}^2 + X_{2,i}^2 - 2\rho X_{1,i}X_{2,i}$$

First order condition

$$\frac{-N\hat{\rho}}{1-\hat{\rho}^2} - \left(\frac{2\sum_{i=1}^{N} X_{1,i}X_{2,i} - 8\hat{\rho}\sum_{i=1}^{N} X_{1,i}X_{2,i}}{4(1-\hat{\rho}^2)^2}\right) = 0$$

Then the solution is a third order polynomial

$$-4N\hat{\rho}^3 + \left(8\sum_{i=1}^{N} X_{1,i}X_{2,i} - 4N\right)\hat{\rho} + 2\sum_{i=1}^{N} X_{1,i}X_{2,i} = 0$$

Where the "standard" estimator

$$\hat{\rho} = \frac{\sum_{i=1}^{N} X_{1,i}X_{2,i}}{\sqrt{\sum_{i=1}^{N} X_{1,i}^2 \sum_{i=1}^{N} X_{2,i}^2}}$$