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Solutions to Problem Set #4

Problem 1. Suppose that $\Pi(t, S_t)$ is the arbitrage-free price of a derivative security on S_t with some terminal payoff $\Pi(t, S_T) = \Phi(S_T)$ and r is the interest rate. Prove that under the probability measure Q we have that

$$d\Pi(t, S_t) = r\Pi(t, S_t)\, dt + \sigma_\Pi \Pi(t, S_t)\, dW_t$$

for an appropriate function $\sigma_\Pi(t, S_t)$.

Solution: Applying Ito's Lemma and noting that

$$dS_t = rS_t dt + \sigma(t, S_t)\, dW_t$$

implies that

$$\begin{aligned} d\Pi(t, S_t) &= \left[\frac{\partial \Pi}{\partial t} + r\frac{\partial \Pi}{\partial S_t}S_t + \frac{1}{2}\sigma^2(t, S_t)\, S_t^2 \frac{\partial^2 \Pi}{\partial S_t^2}\right] dt + \frac{\partial \Pi}{\partial S_t}\sigma(t, S_t)\, dW_t \\ &= r\Pi(t, S_t)\, dt + \frac{\partial \Pi}{\partial S_t}\sigma(t, S_t)\, dW_t \end{aligned}$$

where the second line follows from the fact that the price of $\Pi(t, S_t)$ needs to satisfy the Black-Scholes equation. Hence we can write

$$d\Pi(t, S_t) = r\Pi(t, S_t)\, dt + \sigma_\Pi \Pi(t, S_t)\, dW_t$$

where

$$\sigma_\Pi(t, S_t) = \frac{\frac{\partial \Pi(t, S_t)}{\partial S_t}\sigma(t, S_t)}{\Pi(t, S_t)}$$

problem

Problem 2. Suppose that the stock market follows the dynamics

$$dS_t = \mu S_t dt + \sigma S_t d\overline{W}_t$$

and the interest rate is constant and equal to r. Consider a "digital claim"

$$\Phi(S_T) = \begin{cases} 1 \text{ if } S_T > K \\ 0 \text{ if } S_T < K \end{cases}$$

where $K > 0$. Give a formula for the arbitrage-free price $\Pi(S_t, t)$ of the digital claim at time t when the stock price is S_t?

Solution: The risk-free dynamics under "Q" are given by

$$dS_t = rS_t dt + \sigma S_t dW_t$$

and accordingly

$$d\log(S_t) = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

Now

$$\begin{aligned}
\Pi(S_t, t) &= e^{-r(T-t)} E^Q\left[1_{\{S_T > K\}}\right] \\
&= e^{-r(T-t)} E^Q\left[1_{\{\log S_T > \log K\}}\right] \\
&= e^{-r(T-t)} \overset{Q}{\Pr}\left(\log S_T - \log K > 0\right) \\
&= e^{-r(T-t)} \overset{Q}{\Pr}\left(\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T - W_t\right) - \log K > 0\right) \\
&= e^{-r(T-t)} \overset{Q}{\Pr}\left(\left(\frac{W_T - W_t}{\sqrt{T-t}}\right) > \frac{\log\left(\frac{K}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= e^{-r(T-t)} N\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

problem

Problem 3. Keep the same dynamics for the stock price and the same assumption on the interest rate as above. Give a formula for the arbitrage-free price $\Pi(S_t, t)$ of a put option that has payoff

$$\Phi(S_T) = \max\{0, K - S_T\}$$

Solution: Solution:

$$\begin{aligned}
\Pi(S_t, t) &= e^{-r(T-t)} E^Q\left[1_{\{S_T < K\}}\left(K - S_T\right)\right] \\
&= e^{-r(T-t)}\left[K E^Q\left[1_{\{S_T < K\}}\right] - E^Q\left[1_{\{S_T < K\}} S_T\right]\right] \\
&= e^{-r(T-t)} K \overset{Q}{\Pr}\left(\log K - \log S_T > 0\right) \\
&\quad - \frac{1}{\sigma\sqrt{2\pi}\sqrt{(T-t)}} e^{-r(T-t)} \int_{-\infty}^{\log K} e^{\log(S_t) + x} e^{-\frac{1}{2}\frac{\left(x - \left(\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)\right)^2}{\sigma^2(T-t)}} \\
&= e^{-r(T-t)} K \overset{Q}{\Pr}\left(\log K - \log S_T > 0\right) \\
&\quad - S_t \frac{1}{\sigma\sqrt{2\pi}\sqrt{(T-t)}} e^{-r(T-t)} \int_{-\infty}^{\log K} e^{x} e^{-\frac{1}{2}\frac{\left(x - \left(\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)\right)^2}{\sigma^2(T-t)}}
\end{aligned}$$

Now we have

$$\begin{aligned}
\overset{Q}{\Pr}\left(\log K - \log S_T > 0\right) &= \Pr\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} < -\frac{W_T - W_t}{\sqrt{T-t}}\right) \\
&= \Pr\left(\frac{W_T - W_t}{\sqrt{T-t}} < -\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= N\left(-d_2\right)
\end{aligned}$$

where

$$d_2 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Also we have that

$$\begin{aligned} & e^x e^{-\frac{1}{2}\frac{\left(x-\left(\log S_t+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)\right)^2}{\sigma^2(T-t)}} \\ = \quad & e^{-\frac{1}{2}\frac{\left(x-\left(\log S_t+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)\right)}{\sigma^2(T-t)}} \end{aligned}$$

and hence

$$\frac{1}{\sigma\sqrt{2\pi}\sqrt{(T-t)}}\int_{-\infty}^{\log K} e^x e^{-\frac{1}{2}\frac{\left(x-\left(\log S_t+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)\right)^2}{\sigma^2(T-t)}} = N\left(-d_1\right)$$

where

$$d_1 = \log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)$$

Putting everything together we have

$$\Pi\left(S_t, t\right) = e^{-r(T-t)}\left[N\left(-d_2\right)K - S_t N\left(-d_1\right)\right]$$