$d(AB) = AdB + BdA + dAdB$

Feynman Kac:

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} - r(x)F = -h(x)$$

$$F(T,x) = \Phi(x)$$

$dX = \mu(t,X)dt + \sigma(t,X)dW$

$$F(t,x) = E_t\left[\int_t^T e^{-\int_t^s r(X_u)du} h(X_s)ds + e^{-\int_t^T r(Xu)du}\Phi(X_T)\right]$$

Geometric Brownian Motion:

$dX_t = \alpha X_t dt + \sigma X_t dW_t$ $\quad$ X *is log normal*

$$logX_T \sim N\left\{logX_t + \left(\alpha - \frac{1}{2}\sigma^2\right)(T-t);\ \sigma^2(T-t)\right\}$$

if $T = t, t = 0$:

$$logX_t \sim N\left\{logX_0 + \left(\alpha - \frac{1}{2}\sigma^2\right)t;\ \sigma^2 t\right\}$$

$E_t(X_T) = X_t e^{\alpha(T-t)}$

$Var(X_T) = X_t^2\left[e^{(2\alpha-\sigma^2)(T-t)} - e^{2\alpha(T-t)}\right]$

$E_0(X_t) = X_0 e^{\alpha t}$

$Var(X_t) = X_0\left[e^{(2\alpha-\sigma^2)t} - e^{2\alpha t}\right]$

For a log normal distribution:

$X \sim N\{\mu; \sigma^2\}$ $\quad$ $E(X) = e^{\mu + \frac{1}{2}\sigma^2}$

$Var(X) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$ $\quad$ $E(X^2) = e^{2\mu+2\sigma^2}$

$E(X^n) = E^{n\mu + \frac{1}{2}n^2\sigma^2}$

$Var(X) = E(X^2) - \left(E(X)\right)^2$

Moment generating function of normal:

$M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ $\quad$ $E(X) = M_x'(0) = \mu$

$E(X^2) = M_x''(0) = \sigma^2 + \mu$ $\quad$ $E(X^n) = M_x^{(n)}(0)$

Risk free asset is that it has no diffusion dW term:

$dB(t) = rB(t)dt$

$dS(t) = \alpha Sd(t)t + \sigma S(t)d\overline{W}(t)$

No Arbitrage approach to Black Schole

$\Pi(t) = F(t,S)$

$d\Pi = \alpha_\pi \Pi dt + \sigma_\pi \Pi(t)d\overline{W}(t)$

$$\alpha_\pi(t) = \frac{F_t + \alpha SF_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{F}$$

$$\sigma_\pi(t) = \frac{\sigma SF_s}{F}$$

Consider a portfolio based on: 1. The stock, 2. The derivative asset:

$dV = V[u_s\alpha + u_\pi\alpha_\pi]dt + V[u_s\sigma + u_\pi\sigma_\pi]d\overline{W}$

$u_s + u_\pi = 1$ $\quad$ $u_s\alpha + u_\pi\alpha_\pi = r$ $\quad$ $u_s\sigma + u_\pi\sigma_\pi$

$$u_s = \frac{\sigma_\pi}{\sigma_\pi - \sigma} \qquad u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma}$$

$$u_s = \frac{SF_s}{SF_s - F} \qquad u_\pi = \frac{-F}{SF_s - F}$$

Black Scholes Equation:

$$F_t + rSF_s + \frac{1}{2}S^2\sigma^2 F_{ss} - rF = 0$$

$$F(T,s) = \Phi(s)$$

Under Q-Measurement, Risk Neutral Valuation:

$dS = rSdt + S\sigma dW$

$F(t,s) = e^{-r(T-t)}E_{t,s}^Q\left[\Phi\left(\mathrm{S(T)}\right)\right]$

By Ito's lemma:

$S(T) = s \cdot e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W(T)-W(t))}$

$$z = \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W(T) - W(t)\right)$$

$$z \sim N\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t);\ \sigma^2(T-t)\right\}$$

$$F(t,s) = e^{-r(T-t)}\int_{-\infty}^{\infty}\Phi(se^z)f(z)dz$$

$$E\left(g(x)\right) = \int_{-\infty}^{\infty}\mathrm{g}(x)f(x)dx \quad f(x) \text{ is the pdf of } x$$

pdf of normal: $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$E^Q[\max(se^z - K, 0)] = 0 \cdot Q(se^z < K) + \int_{\log\left(\frac{K}{s}\right)}^{\infty}(\mathrm{s}e^{\mathrm{z}} - \mathrm{K})f(z)dz$$

$F(t,s) = sN\left(d_1(t,s)\right) - e^{-r(T-t)}KN\left(d_2(t,s)\right)$

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\log\left(\frac{s}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}$$

$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t}$

For a forward, $\Phi(S_T) = S_T - K$

$f(t) = e^{r(T-t)}S_t$ $\quad$ $K = S_0 e^{rT}$

Replication Method to Black Sholes:

$u^0 + u^* = 1$ $\quad$ u^0: *Bond weight*, u^*: *Stock weight*

$dV = V\{u^0 r + u^*\alpha\}dt + Vu^*\sigma d\overline{W}(t)$

$V(T) = \Phi\left(S(t)\right)$

$V(t) = F(t,S)$ $\quad$ *By Ito's Lemma*:

$$dV = \left\{F_t + \alpha SF_s + \frac{1}{2}\sigma^2 S^2 F_{ss}\right\}dt + \sigma SF_s d\overline{W}$$

$$dV = V\left\{\frac{F_t + \alpha SF_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{F}\right\}dt + V\frac{SF_s}{F}\sigma d\overline{W}$$

$$u^* = \frac{SF_s}{F} \qquad u^0 = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} = 1 - \frac{SF_s}{F}$$

$$h^0 = \frac{u^0 V}{B} = \frac{F - SF_s}{B} \qquad h^* = \frac{u^* V}{S} = F_s$$

The Greeks:

$$\Delta = \frac{\partial P}{\partial S} = N(d_1)$$

$$\Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}} \quad \varphi(d_1) \text{ is the pdf of } N(0,1)$$

$$\rho = \frac{\partial P}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2)$$

$$\Theta = \frac{\partial P}{\partial t} = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

$$v = \frac{\partial P}{\partial \sigma} = s\varphi(d_1)\sqrt{T-t}$$

Put-Call Parity:

$P(t,s) + s = Ke^{-r(T-t)} + c(t,s)$

Multidimensional Black Sholes

$$dS_i = \alpha_i S_i dt + S_i\sum_{j=1}^{n}\sigma_{ij}\,d\overline{W_i}(t)$$

σ *is* $n \times n$ *matrix* $\{\sigma_{ij}\}$

$dF = F \cdot \alpha_F dt + F \cdot \sigma_F d\overline{W}$

If 2 dimension:

$$dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S_1}\alpha_1 S_1 + \frac{\partial F}{\partial S_2}\alpha_2 S_2 + \frac{1}{2}\frac{\partial^2 F}{\partial S_1^2}S_1^2(\sigma_{11}^2 + \sigma_{12}^2) + \frac{1}{2}\frac{\partial^2 F}{\partial S_2^2}S_2^2(\sigma_{21}^2 + \sigma_{22}^2) + \frac{\partial^2 F}{\partial S_1 \partial S_2}S_1 S_2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})\right]dt + \left[\frac{\partial F}{\partial S_1}S_1\sigma_1 + \frac{\partial F}{\partial S_2}S_2\sigma_2\right]dW$$

$$\alpha_F = \frac{1}{F}\left[F_t + \sum_1^n \alpha_i S_i F_i + \frac{1}{2}tr\{\sigma^T D[S]F_{ss}D[S]\sigma\}\right]$$

$$\sigma_F = \frac{1}{F}\sum_1^n S_i F_i \sigma_i$$

$$F_{ss} = \left\{\frac{\partial^2 F}{\partial s_i \partial s_j}\right\}_{i,j=1}^n$$

$$u_B = 1 - \left(\sum_1^n u_i + u_F\right)$$

$$dV = V\left[\sum_1^n u_i\frac{dS_i}{S_i} + u_F\frac{dF}{F} + u_B\frac{dB}{B}\right]$$

$$dV = V\left[\sum_1^n u_i(\alpha_i - r) + u_F(\alpha_F - r) + r\right]dt + V\left[\sum_1^n u_i\sigma_i + u_F\sigma_F\right]d\overline{W}$$

$$\sum_1^n u_i\sigma_i + u_F\sigma_F = 0$$

$$\begin{bmatrix}\alpha_1 - r & \dots & \alpha_n - r & \alpha_F - r\\ \sigma_1^T & \dots & \sigma_n^T & \sigma_F^T\end{bmatrix}\begin{bmatrix}u_S\\ u_F\end{bmatrix} = \begin{bmatrix}\beta\\ 0\end{bmatrix}$$

σ_i^T *is column vector*

$$u_s = \begin{bmatrix}u_1\\ u_2\\ \dots\\ u_n\end{bmatrix} \qquad H = \begin{bmatrix}\alpha_1 - r & \dots & \alpha_n - r & \alpha_F - r\\ \sigma_1^T & \dots & \sigma_n^T & \sigma_F^T\end{bmatrix}$$

H must be singular

$$\alpha_i - r = \sum_{j=1}^n \sigma_{ij}\lambda_j, \quad i = 1, \dots, n,$$

$$\alpha_F - r = \textstyle\sum_{j=1}^n \sigma_{Fj}\,\lambda_j, \qquad \lambda = \begin{bmatrix}\lambda_1\\ \dots\\ \lambda_n\end{bmatrix}$$

$\alpha - r1_n = \sigma\lambda$ $\quad$ $\alpha_F - r = \sigma_F\lambda$

$$F_t + \sum_{i=1}^n r\,S_i F_i + \frac{1}{2}tr\{\sigma^T D[S]F_{ss}D[S]\sigma\} - rF = 0$$

$F = e^{-r(T-t)}E^Q\left[\Phi\left(S(T)\right)\right]$

Reducing the State Space

$$F(t, s_1, \dots, s_n) = s_n G\left(t, \frac{s_1}{s_n}, \dots, \frac{s_{n-1}}{s_n}\right)$$

$$z = \left(\frac{s_1}{s_n}, \dots, \frac{s_{n-1}}{s_n}\right)$$

$F_t(t,s) = s_n G_t(t,z)$

$F_i(t,s) = G_i(t,z), i = 1, \dots, n-1$

$$F_n(t,s) = G(t,z) - \sum_{j=1}^{n-1}\frac{s_j}{s_n}G_j(t,z)$$

$$F_{ij}(t,s) = \frac{1}{s_n}G_{ij}(t,z), \quad i,j = 1, \dots, n-1$$

$$F_{in}(t,s) = F_{ni}(t,s) = -\sum_{j=1}^{n-1} \frac{s_j}{S_n^2} G_{ij}(t,z),$$
$i = 1, \dots, n-1$
$$F_{nn} = -\sum_{i,j=1}^{n-1} \frac{s_i s_j}{S_n^3} G_{ij}(t,z)$$

For two dimension:

$F_1 = G_z \qquad F_2 = G - zG_z$

$F_{11} = \frac{1}{S_2} G_{zz} \quad F_{22} = \frac{S_1^2}{S_2^3} G_{zz}$

$G_t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) z^2 G_{zz} = 0$

$F(t, s_1, s_2) = s_1 N\big(d_1(t,z)\big) - s_2 N\big(d_2(t,z)\big)$

$$d_1(t,z) = -\frac{1}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}} \Big\{ logz + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T-t) \Big\}$$

$$d_2(t,z) = d_1(t,z) - \sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}$$

Dividends

$\delta = \delta(S_{t-})$ *is a function of* S_{t-}

$S_t = S_{t-} - \delta(S_{t-})$

$F^0(T, S_T) = \Phi(S_T)$

$F^1(T_1^-, S_{T^-}) = F^0\big(T_1,\ S_{T^-} - \delta(S_{T^-})\big)$

$F(t, S_t) = e^{-r(T-t)} E^Q[\Phi(S_T)]$

$dS_t = rS_t dt + \sigma S_t dW_t$

if $\delta(S_{t^-}) = \delta S_{t^-}, \delta$ *is a constant*:

$F_\delta(t, S_t) = F(t, (1-\delta)^n S_t)$

n is the number of dividend points in the interval (t, T]

Continuous dividends:

Q-measure:

$dS = (r - \delta) S dt + \sigma S dW$

$F(t, S_t) = e^{-r(T-t)} E^Q[\Phi(S_T)]$

$F_\delta(t, s) = F_0(t, se^{-\delta(T-t)})$

Forward with dividend:

$K = S_0 e^{(r-\delta)T}$

Foreign Exchange:

P measure:

$dX_t = X_t \alpha dt + X_t \sigma_X d\overline{W}_t$

Q measure:

$dX_t = X_t\big(r^d - r^f\big)dt + X_t \sigma_X dW_t$

This is similar to assuming that X_t is dividend paying asset.

Forward: $K = X_0 e^{(r^d - r^f)T}$

$F_\delta(t, S_t) = F\big(t, e^{-\delta(T-t)} S_t\big)$

$F(t,x) = xe^{-r^f(T-t)} N[d_1] - e^{-r^d(T-t)} K N[d_2]$

$$d_1(t,x) = \frac{1}{\sigma_X \sqrt{T-t}} \left\{ \log\left(\frac{x}{K}\right) + \left(r^d - r^f + \frac{1}{2}\sigma_X^2\right)(T-t) \right\}$$

$d_2(t,x) = d_1 - \sigma_X \sqrt{T-t}$