MFE 409 Lecture 2a
Measuring Value-at-Risk

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Lecture Objectives

Measuring Value-at-Risk:

- How to judge validity of a VaR estimate?
- Historical approach
- Model-building approach
- How to get a measure for a given approach but also how to choose an appropriate approach

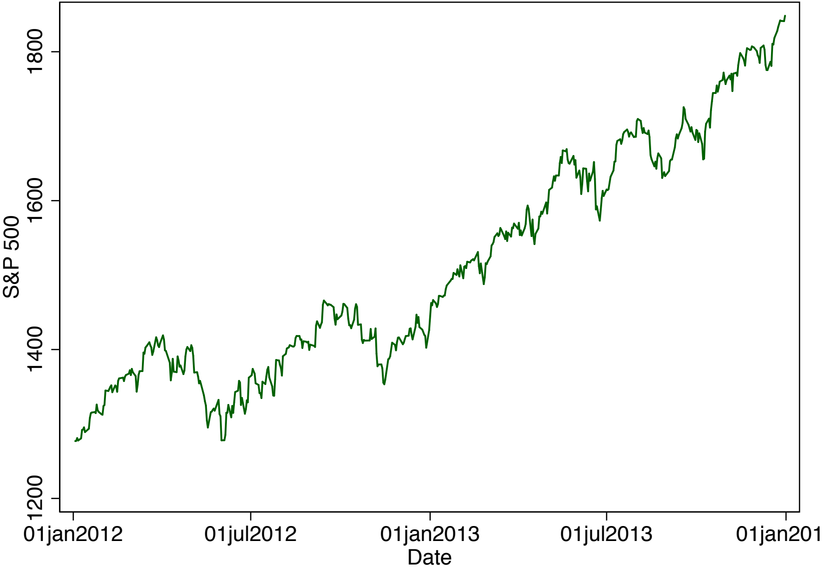
Outline

1. Back-testing
2. Historical Simulation

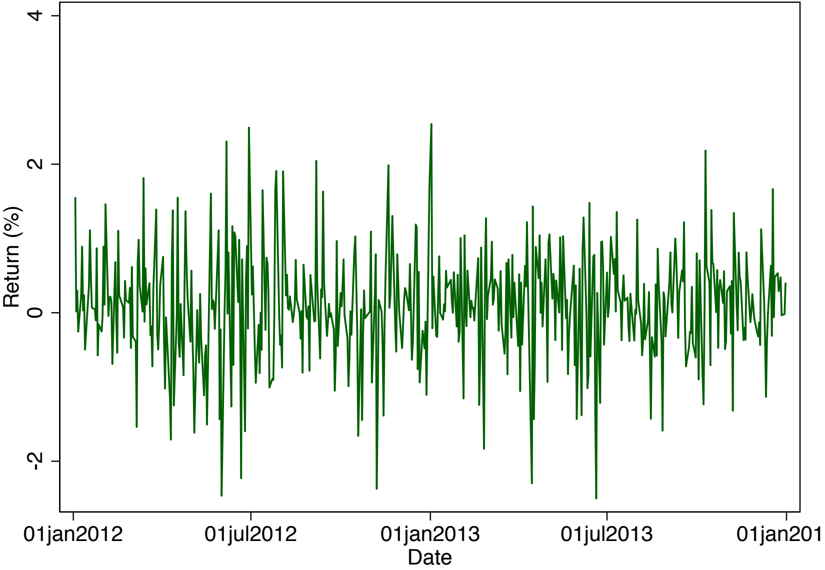
Back-Testing

- **Back-testing**: How well a current procedure would have performed if applied in the past
  - Investment strategy
  - Risk measure
- Our context: How would a method to compute Value-at-Risk would have performed in the past?

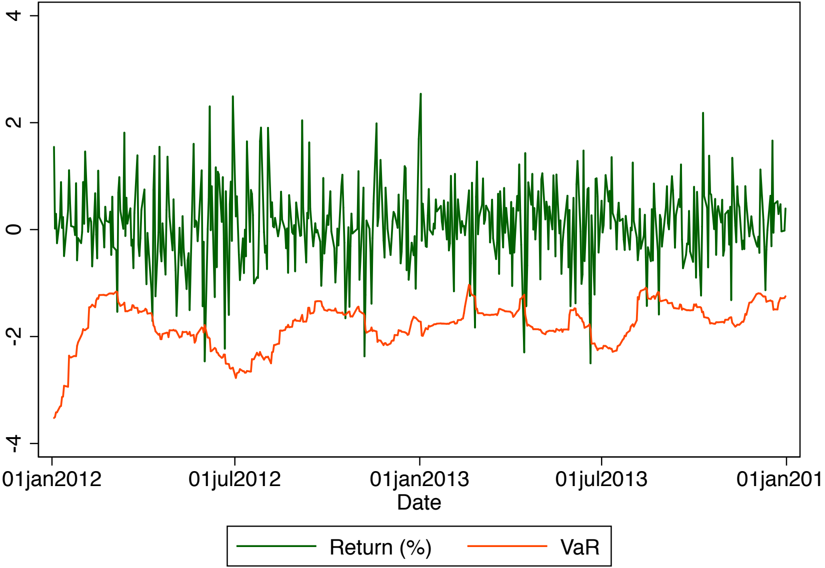
S&P500 Index, 2012-2013



S&P500 Daily Returns, 2012-2013

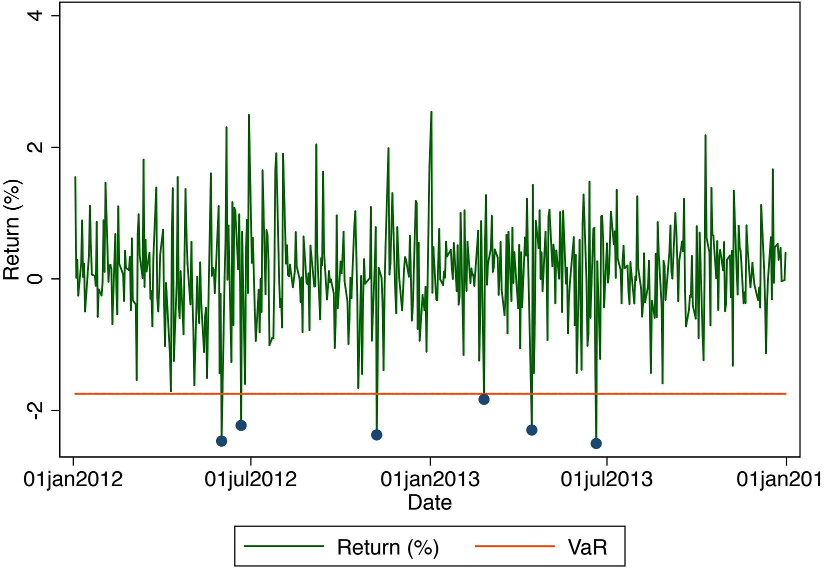


A 99% VaR Measure



Exceptions

Another 99% VaR Measure



Number of Exceptions

- Say we measure the daily VaR with confidence c

- On a given day:
    - Probability of exception:
    - Probability of no exception:

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- Binomial distribution: $P(\#\text{ exceptions} \geq m) = 1 - F(m-1|n, 1-c)$
  - $F(.|n,p)$ c.d.f. of a binomial with n trials and success probability p

Application

- 99% - daily VaR
- 2 years: 502 daily returns
- Probability of 6 or more exceptions:
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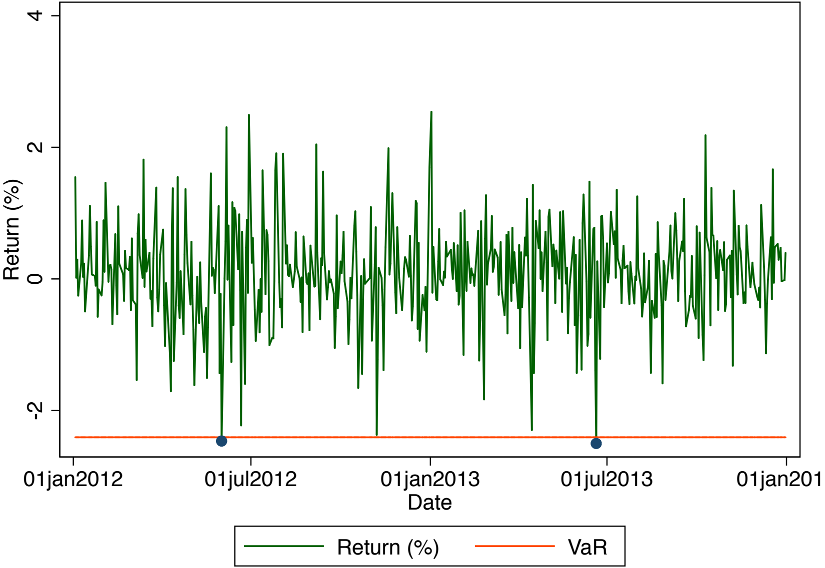
Another VaR Measure



Other Tests

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$$= F(m|n, 1-c)$$

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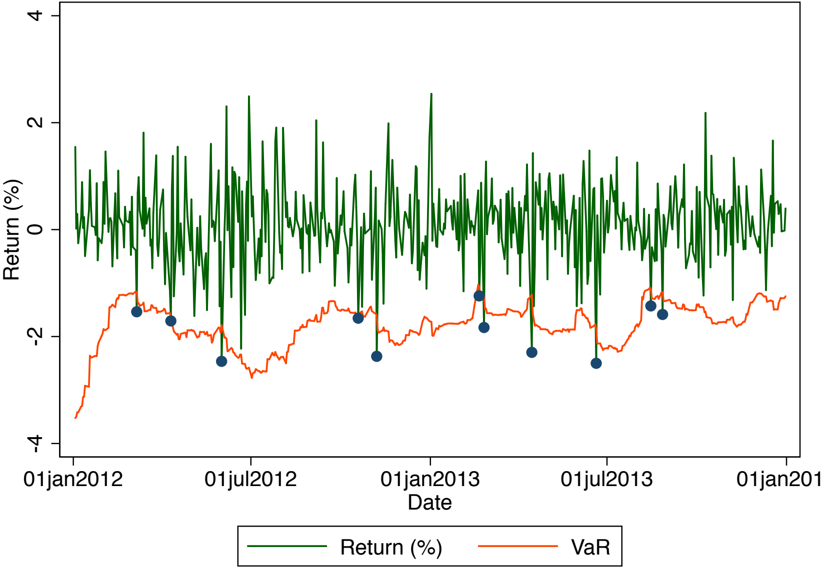
$$= F(m|n, 1-c)$$

- Two sided test (for large n):

$$-2\ln\left[c^{n-m}(1-c)^m\right] + 2\ln\left[(1-m/n)^{n-m}(m/n)^m\right] \sim \chi^2(1)$$

  - Chi-squared 5% threshold: 3.84

Bunching



4
2
0
-2
-4
Return (%)
01jan2012
01jul2012
01jan2013
01jul2013
01jan201
Date
Return (%)
VaR

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- Assume the future will be drawn from the same distribution as the past
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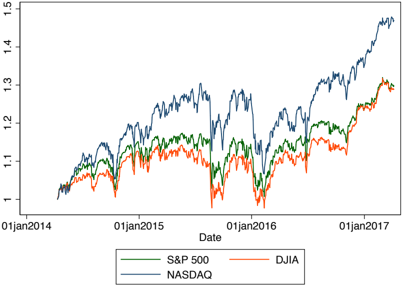
- Formally, to compute the daily value-at-risk at confidence level c:
  - You have n past observations of daily **returns**
  - Assume the next return will be any of these draws with probability $1/n$
  - The VaR corresponds to the loss in the $[(1 - c) \times n]$-th worst past realization
    - ⋆ if not integer, round up

Example

- Assume we are 04/11/2017
- You have \$4m invested in S&P500, \$5m in NASDAQ Composite, \$1m in DJIA
- You know the value of the indices for the last 3 years (file *indices.xls*)



- What is your 1-day 99% VaR?

Constructing the VaR

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Expected Shortfall

- Can use the same method to compute expected shortfall

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- Average of the $[(1-c)\times n]$ worst realizations
  - Still round up

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- Stressed VaR (or ES): VaR (or ES) for the worst consecutive 251-day period in the historical sample
- Introduced by regulators to capture the idea that some periods are worse than others
- (Stressed VaR) $\geq$ VaR ?

Accuracy of VaR

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- If you backtest historical VaR, you find exactly deviations
- But if you had the true VaR, you would sometimes find more, sometimes find less: historical VaR is not perfectly accurate
- Standard error of the estimate:

$$\frac{1}{f(x)}\sqrt{\frac{c(1-c)}{n}}$$

  - $f(x)$: p.d.f. at quantile c
  - Need to know distribution!

Example: Accuracy of VaR

- Back to portfolio example
- Historical VaR: $249,000

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- Historical VaR: \$249,000
- Approximate by a normal (in \$000s): mean 4, standard deviation 87

$$
\begin{aligned}
x &= \mu - \sigma\Phi^{-1}(0.01) && = -198.4 \\
f(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) && = 3.06 \times 10^{-4} \\
\text{StdDev(VaR)} &= \frac{1}{f(x)}\sqrt{\frac{0.99 \times 0.01}{753}} && = 12
\end{aligned}
$$

- 95% confidence interval for the VaR is between \$229,000 and \$269,000 $\rightarrow$ not that precise

Bootstrap

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- Suppose there are 500 daily changes and you want to calculate a 95% confidence interval for VaR
  1. Sample 500,000 times with replacement from daily changes to obtain 1000 sets of changes over 500 days
  2. Calculate VaR for each set
  3. Calculate a confidence interval by taking the range between the 2.5% lowest and 97.5% largest value

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- How much data would you like to use?
- More data, more precise estimates
- But "future same as past" less likely to be true

Weighting of observations

- Use as much data as possible, but put more weight on recent data
- Weight observations with an exponential decay as you go back in time.
- Observation i receives weight:

$$\lambda^{n-i}\frac{1-\lambda}{1-\lambda^n}$$

- Sort observations, VaR is the scenario just over $1-c$ cumulative weight

Portfolio Example with Weighting

- $\lambda = 0.995$

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Estimating the Tail

- Extreme tail estimated imprecisely with historical method: 99.9% would need multiple thousands of observations
- To get more precise estimates, make assumptions about the shape of the distribution
- Model the whole distribution, e.g. normal distribution
  - VaR depends of σ
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  - VaR depends of the shape of the left tail of the distribution
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- **Extreme value theory**: this approach is valid for many distributions

Power Law

- **Power law**: X follows a power law, with

$$\text{Prob}(X > x) = Kx^{-1/\xi}$$

  - Also called Pareto distribution
  - $\xi < 1$ controls thickness of tail: low ξ, thin tail

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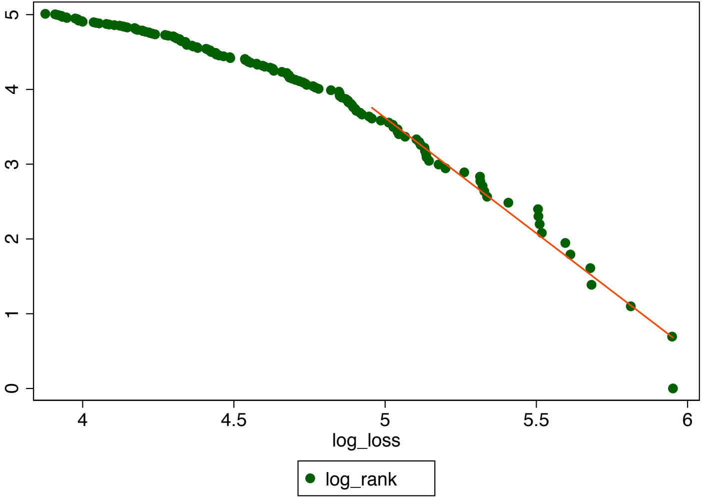
- Regress $log[\text{Prob}(X > x)]$ on $log(x)$: slope $-1/\xi$
  - In historical distribution: $\text{Prob}(X > x_i) = \text{rank}(x_i)/n$

Log-log Plot for Portfolio Loss



- Slope: -3, $\xi = 1/3$

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$$F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)}$$

- Result: as u becomes large, $F_u(y)$ converges to a generalized Pareto distribution:

$$G_{\xi,\beta}(y) = 1 - \left[1 + \xi\frac{y}{\beta}\right]^{-1/\xi}$$

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- Model of right tail! Remember to find the c-th quantile of losses

Estimating the Power Law

- Partial distribution function:
$$g_{\xi,\beta}(y) = \frac{1}{\beta}\left(1 + \frac{\xi y}{\beta}\right)^{-1/\xi - 1}$$

- Choose u: typically 95th percentile of historical distribution

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- Maximize log likelihood:

$$\max_{\xi,\beta} \sum_{i \in tail} \ln\left[g_{\xi,\beta}(v_i - u)\right]$$

VaR and ES for a Power Law

- Probability distribution:

$$\text{Prob}(\text{Loss} > \text{V}) = \underbrace{[1 - F(u)]}_{n_u/n}[1 - G_{\xi,\beta}(\text{V} - u)]$$

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- Can also obtain ES:

$$\text{ES} = \frac{\text{VaR} + \beta - \xi u}{1 - \xi}$$