Problem Set 5

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Question 1

1. Ito's lemma: $dM = \frac{\partial M}{\partial t} dt + \frac{\partial M}{\partial S_1} dS_1 + \frac{\partial M}{\partial S_2} dS_2 + \frac{1}{2}\frac{\partial^2 M}{\partial S_1^2}(dS_1)^2 + \frac{1}{2}\frac{\partial^2 M}{\partial S_2^2}(dS_2)^2 + \frac{1}{2}\frac{\partial^2 M}{\partial S_1 \partial S_2} dS_1 dS_2$

$\because$ only consider delta approach, we assume $\Gamma = \frac{\partial^2 M}{\partial S^2} = 0$. and assume $dt = 1$

$\therefore dM = \Delta_1 dS_1 + \Delta_2 dS_2 = \Delta_1 \mu_1 S_1 dt + \Delta_1 \sigma_1 S_1 dW_1 + \Delta_2 \mu_2 S_2 dt + \Delta_2 \sigma_2 S_2 dW_2$

$= (\Delta_1 \mu_1 S_1 + \Delta_2 \mu_2 S_2) dt + \Delta_1 \sigma_1 S_1 dW_1 + \Delta_2 \sigma_2 S_2 dW_2$

$E(dM) = \Delta_1 \mu_1 S_1 + \Delta_2 \mu_2 S_2$ $\quad Var(dM) = \Delta_1^2 \sigma_1^2 S_1^2 dt + \Delta_2^2 \sigma_2^2 S_2^2 dt + 2 \cdot \Delta_1 \Delta_2 \sigma_1 \sigma_2 S_1 S_2 \cdot \rho \, dt$

$= \Delta_1^2 \sigma_1^2 S_1^2 + \Delta_2^2 \sigma_2^2 S_2^2 + 2\Delta_1 \Delta_2 \sigma_1 \sigma_2 S_1 S_2 \rho$.

$P(M < M_0 - VaR) = 0.01$.

$P(dM < -VaR) = 0.01$.

$\mu - 2.326\sigma = -VaR$

$VaR = 2.326\sigma - \mu =$

$2.326\sqrt{(\Delta_1^2 \sigma_1^2 S_1^2 + \Delta_2^2 \sigma_2^2 S_2^2 + 2\Delta_1 \Delta_2 \sigma_1 \sigma_2 S_1 S_2 \rho)}$

$- \Delta_1 \mu_1 S_1 - \Delta_2 \mu_2 S_2$

2. $dM = \Delta_1 dS_1 + \Delta_2 dS_2 + \frac{1}{2}\Gamma_1 (dS_1)^2 + \frac{1}{2}\Gamma_2 (dS_2)^2 + \Gamma_{12} dS_1 dS_2$

$= \Delta_1 \mu_1 S_1 dt + \Delta_1 \sigma_1 S_1 dW_1 + \Delta_2 \mu_2 S_2 dt + \Delta_2 \sigma_2 S_2 dW_2 + \frac{1}{2}\Gamma_1 \sigma_1^2 S_1^2 dt + \frac{1}{2}\Gamma_2 \cdot \sigma_2^2 S_2^2 dt + \Gamma_{12} \sigma_1 \sigma_2 S_1 S_2 \rho \, dt$

$E(dM) = \Delta_1 \mu_1 S_1 + \Delta_2 \mu_2 S_2 + \frac{1}{2}\Gamma_1 \sigma_1^2 S_1^2 + \frac{1}{2}\Gamma_2 \sigma_2^2 S_2^2 + \Gamma_{12} \sigma_1 \sigma_2 S_1 S_2 \rho$.

$Var(dM) = \Delta_1^2 \sigma_1^2 S_1^2 + \Delta_2^2 \sigma_2^2 S_2^2 + 2\Delta_1 \Delta_2 \sigma_1 \sigma_2 S_1 S_2 \rho$

$VaR = 2.326 \cdot \sigma - \mu$

$= 2.326\sqrt{(\Delta_1^2 \sigma_1^2 S_1^2 + \Delta_2^2 \sigma_2^2 S_2^2 + 2\rho \Delta_1 \Delta_2 \sigma_1 \sigma_2 S_1 S_2)}$

$- \Delta_1 \mu_1 S_1 - \Delta_2 \mu_2 S_2 - \frac{1}{2}\Gamma_1 \sigma_1^2 S_1^2 - \frac{1}{2}\Gamma_2 \sigma_2^2 S_2^2 - \Gamma_{12} \sigma_1 \sigma_2 S_1 S_2 \rho$.

Question 1

In [1]:
```
from scipy.stats import norm, multivariate_normal
import numpy as np
```

3.

In [2]:
```
r = 0.00005
sigma1 = 0.02
sigma2 = 0.02
rho = 0.4
mu1 = 0.0003
mu2 = 0.0003
T = 126
s1 = 99
s2 = 101
K = 100
tau = T - 0
```

In [4]:
```
def parameters(s1,s2,gamma,sigma1,sigma2,sigma,tau,rho):
    alpha = gamma + sigma1 * np.sqrt(tau)
    beta = (np.log(s2/s1) - 0.5 * sigma * sigma * tau) / (sigma * np.sqrt(tau))
    theta = (rho * sigma2 - sigma1) / sigma
    return alpha, beta, theta
```

In [7]:
```
def option_price(s1,s2,mu1,mu2,sigma1,sigma2,tau,rho,K,r):
    gamma1 = (np.log(s1/K) + (r - 0.5 * sigma1 * sigma1) * tau) / (sigma1 * np.sqrt(tau))
    gamma2 = (np.log(s2/K) + (r - 0.5 * sigma2 * sigma2) * tau) / (sigma2 * np.sqrt(tau))
    sigma = np.sqrt(sigma1 * sigma1 + sigma2 * sigma2 - 2 * rho * sigma1 * sigma2)
    alpha1, beta1, theta1 = parameters(s1,s2,gamma1,sigma1,sigma2,sigma,tau,rho)
    alpha2, beta2, theta2 = parameters(s2,s1,gamma2,sigma2,sigma1,sigma,tau,rho)
    bivariate1 = multivariate_normal(mean=[0,0],cov=[[1,theta1],[theta1,1]])
    N1 = bivariate1.cdf(np.vstack([alpha1,beta1]).T)
    bivariate2 = multivariate_normal(mean=[0,0],cov=[[1,theta2],[theta2,1]])
    N2 = bivariate2.cdf(np.vstack([alpha2,beta2]).T)
    bivariate3 = multivariate_normal(mean=[0,0],cov=[[1,rho],[rho,1]])
    N3 = bivariate3.cdf(np.vstack([gamma1,gamma2]).T)
    price = s1 * N1 + s2 * N2 - K * np.exp(-r * tau) * N3
    return price
```

In [8]:
```
price = option_price(s1,s2,mu1,mu2,sigma1,sigma2,tau,rho,K,r)
print(price)
```

```
3.8434420467809467
```

The price of the option at date 0 is 3.84344.

4.

Delta approach

In [9]:
```
def option_delta(s1,s2,mu1,mu2,sigma1,sigma2,tau,rho,K,r,delta_s1=0,delt
a_s2=0):
    option_price_up = option_price(s1 + delta_s1,s2 + delta_s2,mu1,mu2,s
igma1,sigma2,tau,rho,K,r)
    option_price_down = option_price(s1 - delta_s1,s2 - delta_s2,mu1,mu2
,sigma1,sigma2,tau,rho,K,r)
    return (option_price_up - option_price_down) / (2 * delta_s1 + 2 * d
elta_s2)
```

In [10]:
```
delta_s = 0.01
delta1 = option_delta(s1,s2,mu1,mu2,sigma1,sigma2,tau,rho,K,r,delta_s1=d
elta_s)
delta2 = option_delta(s1,s2,mu1,mu2,sigma1,sigma2,tau,rho,K,r,delta_s2=d
elta_s)
VaR_delta = 2.326 * np.sqrt((delta1 * sigma1 * s1) ** 2 + (delta2 * sigm
a2 * s2) ** 2 + 2 * delta1 * delta2 * sigma1 * sigma2 * s1 * s2 * rho) -
delta1 * mu1 * s1 - delta2 * mu2 * s2
print(VaR_delta)
```

```
1.2318530065771665
```

VaR for delta approach is 1.231853.

Delta and Gamma approach

In [11]:
```
def option_gamma(s1,s2,mu1,mu2,sigma1,sigma2,tau,rho,K,r,delta_s1=0,delt
a_s2=0):
    delta_up = option_delta(s1 + delta_s1,s2 + delta_s2,mu1,mu2,sigma1,s
igma2,tau,rho,K,r,delta_s1=delta_s1,delta_s2=delta_s2)
    delta_down = option_delta(s1 - delta_s1,s2 - delta_s2,mu1,mu2,sigma1
,sigma2,tau,rho,K,r,delta_s1=delta_s1,delta_s2=delta_s2)
    return (delta_up - delta_down) / (2 * delta_s1 + 2 * delta_s2)
gamma1 = option_gamma(s1,s2,mu1,mu2,sigma1,sigma2,tau,rho,K,r,delta_s1=d
elta_s,delta_s2=0)
gamma2 = option_gamma(s1,s2,mu1,mu2,sigma1,sigma2,tau,rho,K,r,delta_s1=0
,delta_s2=delta_s)
delta_up1 = option_delta(s1,s2 + delta_s,mu1,mu2,sigma1,sigma2,tau,rho,K
,r,delta_s1=delta_s,delta_s2=0)
delta_down1 = option_delta(s1,s2 - delta_s,mu1,mu2,sigma1,sigma2,tau,rho
,K,r,delta_s1=delta_s,delta_s2=0)
gamma12 = (delta_up1 - delta_down1) / (2 * delta_s)
```

In [12]:
```
VaR_gamma = 2.326 * np.sqrt((delta1 * sigma1 * s1) ** 2 + (delta2 * sigm
a2 * s2) ** 2 + 2 * delta1 * delta2 * sigma1 * sigma2 * s1 * s2 * rho) -
delta1 * mu1 * s1 - delta2 * mu2 * s2 - 0.5 * gamma1 * sigma1 * sigma1 *
s1 * s1 - 0.5 * gamma2 * gamma2 * s2 * s2 - gamma12 * sigma1 * sigma2 *
s1 * s2 * rho
print(VaR_gamma)
```

```
1.2156959395119131
```

VaR for delta-gamma approach is 1.2156959.

For delta approach, we assume gamma is 0, and with delta-gamma approach, gamma is positive.
From Q2, we can see that VaR is smaller with positive gamma.

5.

In [15]:
```
simulation_count = 100000
normals = np.random.normal(0,1,simulation_count)
normals1 = np.random.normal(0,1,simulation_count)
s1_t1 = s1 + mu1 * s1 + s1 * sigma1 * normals
s2_t1 = s2 + mu2 * s2 + s2 * sigma2 * (rho * normals + np.sqrt(1 - rho *
rho) * normals1)
price_t1 = option_price(s1_t1,s2_t1,mu1,mu2,sigma1,sigma2,tau - 1,rho,K,
r)
VaR_sim = price - np.sort(price_t1)[int(simulation_count * 0.01) - 1]
print(VaR_sim)
```

```
1.1329095576438206
```

Simulated VaR is 1.13291.

1. Delta and Gamma are only the first and sencond term of the Taylor series. However, there are still higher degree terms are not captured by delta-gamma approach.
2. There is theta effect which is not captured by delta-gamma approach.

6.

Other types of risk are including Credit or Default Risk, Foreign-Exchange Risk, Interest Rate Risk and so on. If you had to worry about just one more risk, it should be interest rate risk.
I should consider some interest rate models to simulate and try to hedge interest rate risk. Some interest rate models include Merton model, Vesicek model, CIR model and so on.

3. Interview questions

1. The option with lower gamma has higher VaR

$dp = \Delta \cdot ds + \frac{1}{2}\Gamma \cdot (ds)^2$, with larger gamma, the second term will be larger, so that dp is larger. $P(dp < -VaR) = 0.01$ $P(-dp > VaR) = 0.01$

so that VaR is smaller with lager gamma.

2. $\frac{dr_1}{r_1} = u\,dt + \sigma\,dW$ $\quad r_2 = \frac{1}{r_1}$

Ito's lemma. $dr_2 = \frac{\partial r_2}{\partial t} \cdot dt + \frac{\partial r_2}{\partial r_1} dr_1 + \frac{1}{2}\frac{\partial^2 r_2}{\partial r_1^2} (dr_1)^2$

$= 0 + -\frac{1}{r_1^2}(u\,dt \cdot r_1 + \sigma r_1 dW) + \frac{1}{2} \times 2 \cdot \frac{1}{r_1^3} \cdot \sigma^2 r_1^2 dt$

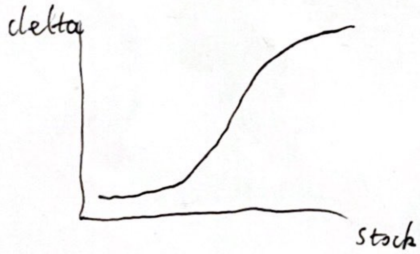
$= -\frac{1}{r_1} \cdot u\,dt + \frac{1}{r_1} \cdot \sigma^2 dt - \frac{1}{r_1} \cdot \sigma\,dW$

$= (\sigma^2 - u) \cdot r_2\,dt - r_2 \cdot \sigma\,dW$

$\frac{dr_2}{r_2} = (\sigma^2 - u) dt - \sigma\,dW$

drift $= \sigma^2 - u$ $\quad$ volatility $= \sigma$.

3. You should buy stock and borrow.



At first delta neutral.

Total delta = 0

Stock price decreases, and option delta will decrease as well.

Total delta will be negative.

In order to adjust the hedge to delta = 0 again. You should buy stock, whose delta is 1.

Therefore, you should buy stock and borrow.