ARMA model intuition

Consider the following system for excess stock return dynamics:

$$\begin{aligned} r_{t+1} &= x_t + \varepsilon_{t+1} \\ x_{t+1} &= \mu + \rho \left(x_t - \mu \right) + \eta_{t+1}, \end{aligned}$$

where $corr\left(\varepsilon, \eta\right) = c$, $|\rho| < 1$.

- In words, realized returns $\left(r_{t+1}\right)$ has unconditional mean μ and x_t captures time-variation in conditional expected returns (the risk premium)
- In particular: $E_t\left[r_{t+1}\right] = x_t$, and $E\left[x_t\right] = \mu$

Assume you *do **not** observe x_t* for any t

- Assume you *do* observe the history of returns, $\left\{r_{t-j}\right\}_{j=0}^{\infty}$
- This is realistic: we observe the history of stock returns, not of expected stock returns

Question: What is the best estimator of expected stock returns if we constrain our estimate to be linear in the data we observe?

- I.e., best linear estimator based on lagged returns?

ARMA model intuition: cont'd

A linear estimator based on stock returns only means:

$$\begin{aligned} E\left[x_t | r_t, r_{t-1}, r_{t-2}, ...\right] &= \phi_0 + \phi_1 r_t + \phi_2 r_{t-1} + \phi_3 r_{t-2} + ... \\ &= \phi_0 + \sum_{j=0}^{\infty} \phi_j r_{t-j}. \end{aligned}$$

- This estimator is not empirically estimable, as it has an infinite number of coefficients and as it requires an infinite history of returns
- But, let's leave that aside for this slide. Note that if we could estimate the coefficients by OLS, we would run a forecasting regression with r_{t+1} on the left hand side:

$$\begin{aligned} r_{t+1} &= x_t + \varepsilon_{t+1} \\ &= \phi_0 + \sum_{j=0}^{\infty} \phi_j r_{t-j} + \varepsilon_{t+1} \end{aligned}$$

This is a multiple regression. What matters for the OLS estimates?

- The variance and covariances of returns at all leads and lags
  - $Cov\left(r_{t+1}, r_{t-j}\right) \neq 0$ tells you that you can predict r_{t+1} using r_{t-j}
- The unconditional mean nails down the regression intercept, as usual

Autocorrelation function

To summarize: In order to find our predictor we need

- The unconditional mean of returns
- The unconditional variance of returns
- The autocorrelation function of returns
  - This function summarizes the amount of predictability of returns based on lagged returns in a linear estimator
  - This is why the autocorrelation function is so important

Capturing the autocorrelation function

Next, let's consider what estimator we need given the return dynamics we wrote down on the first slide

- Hopefully, we don't need to identify an infinite number of regression coefficients!

Note that r_{t+1} can be thought of as a *signal* of expected returns

- it is the expected return x_t plus noise (ε_{t+1})
- Thus, returns is an AR(1) plus noise
- We showed in the midterm that the univariate process for r in this case is an ARMA(1,1)
    - We showed this by showing an ARMA(1,1) captures the autocorrelation function

ARMA(1,1)

The idea is: If we have a model that matches the autocorrelation of r, we have the best linear predictor of r

- The best predictor of r_{t+1} is the best estimator of x_t since $E\left[r_{t+1}|x_t\right] = x_t$

So:

1. Derive the autocorrelation function of the return dynamics we wrote on the first slide
2. Note that this autocorrelation function can be matched by an ARMA(1,1) process
3. Estimate the ARMA(1,1) on returns data (or in our analytical case, derive the ARMA(1,1) coefficients

Since the ARMA(1,1) captures the autocorrelation function of returns, it is the best linear prediction model we can find using a linear function of lagged returns

- This is why ARMA models are so important. With the right ARMA specification, we have the best linear predictor!

Best predictor: discussion

ARMA(1,1) is the best predictor of future returns when using a linear function of lagged returns

In general:

- A nonlinear predictor may be better
  - though it is not in this particular case as I wrote down a linear model
- Introducing other data than just lagged returns may improve the prediction

For instance: assume $x_t = \delta_0 + \delta_1 dp_t$

- In this example, dp_t is the log dividend-price ratio
- Now, the best predictor is simply $\delta_0 + \delta_1 dp_t$ which can be identified by running a forecasting regression of future returns on the lagged dp

In sum: ARMA models are general benchmark models that give the best univariate linear predictions. That does not mean you cannot introduce other observable predictors and improve. But, it gives you a baseline prediction that is theoretically grounded (capturing the autocorrelation function)