UCLA Anderson School of Management
Daniel Andrei, Derivative Markets MGMTMFE 406, Winter 2018

Problem Set 4

These exercises do not need to be turned in for credit.

1 Exotic options

Consider a financial market with a riskless money market account which pays a constant interest rate r and a risky stock which pays a continuous porportional dividend at the constant rate δ. The price of the stock is reduced by the dividend payment and it follows that the stock price evolves according to

$$dS_t = S_t\left[\mu dt + \sigma dB_t\right] - \delta S_t dt \tag{1}$$

a. We call **paylater call** a European call option where the holder of the option pays no initial premium but pays an amount π at maturity only if the call ends up in the money. The payoff is therefore given by

$$S_T - K - \pi \quad \text{if} \quad S_T \geq K \tag{2}$$

$$0 \quad \text{if} \quad S_T < K \tag{3}$$

Let $c(S_0, 0; K, T)$ denote the price of a European vanilla call option at date 0 with maturity T and strike K. Compute the value of π as a function of $c(S_0, 0; K, T)$.

b. We call **power option** a European call option with payoff given by

$$\max\left[S_T^2 - K, 0\right] \tag{4}$$

Compute the dynamics of S_T^2 and then deduce a value for the power option.

1 a. We have

$$0 = \mathbb{E}^Q\left[e^{-rT}\max[S_T - K, 0]\right] - \mathbb{E}^Q\left[e^{-rT}\pi\mathbf{1}_{S_T>K}\right] \tag{5}$$

where $\mathbb{E}^Q[\cdot]$ is the expectation under the risk-neutral measure. Hence

$$\pi = \frac{C_0}{e^{-rT}\text{Prob}^Q[S_T \geq K]} = C_0 e^{rT}\Phi\left(\frac{\ln(S_0/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^{-1} \tag{6}$$

where $\Phi(\cdot)^-1$ represents the inverse normal cdf.

b. We know that the dynamics of the risky asset under the risk neutral measure has an r in the place of μ. Thus let's apply Ito's lemma on$f(S_t) = S_t^2$, where $dS_t = S_t[(r-\delta)dt + \sigma dB_t^Q]$.

$$\frac{dS_t^2}{S_t^2} = [2(r-\delta)dt + \sigma^2]dt + 2\sigma dB_t^Q \tag{7}$$

$$C = \mathbb{E}^Q\left[e^{-rT}\max[S_T^2 - K, 0]\right] \tag{8}$$
$$= \mathbb{E}^Q\left[e^{-rT}S_T^2\mathbf{1}_{S_T^2 \geq K}\right] - \mathbb{E}^Q\left[e^{-rT}K\mathbf{1}_{S_T^2 \geq K}\right] \tag{9}$$

The second term of (9) is

$$\mathbb{E}^Q\left[e^{-rT}K\mathbf{1}_{S_T^2 \geq K}\right] = e^{-rT}K\Phi\left(\frac{\ln(S_0^2/K) + (2(r-\delta) - \sigma^2)T}{2\sigma\sqrt{T}}\right) \tag{10}$$

The first term of (9) is

$$\mathbb{E}^Q\left[e^{-rT}S_T^2\mathbf{1}_{S_T^2 \geq K}\right] = \mathbb{E}^Q\left[e^{-rT}S_0^2 e^{(2(r-\delta)-\sigma^2)T + 2\sigma B_T^Q}\mathbf{1}_{S_T^2 \geq K}\right] \tag{11}$$
$$= S_0^2 e^{(r-2\delta+\sigma^2)T}\mathbb{E}^S\left[\mathbf{1}_{S_T^2 \geq K}\right] \tag{12}$$

where S is a new probability measure under which

$$\frac{dS_t^2}{S_t^2} = (2(r-\delta) + 5\sigma^2)dt + 2\sigma dW_t \tag{13}$$

and thus

$$\mathbb{E}^Q\left[e^{-rT}S_T^2\mathbf{1}_{S_T^2 \geq K}\right] = S_0^2 e^{(r-2\delta+\sigma^2)T}\Phi\left(\frac{\ln(S_0^2/K) + (2(r-\delta) + 3\sigma^2)T}{2\sigma\sqrt{T}}\right) \tag{14}$$

2 Security design

Value the M&I stock purchase contract (pages 458-460 in McDonald 3rd edition) assuming that the 3-year interest rate is 3%, the dividend yield on the stock is 2%, and the M&I volatility is 15%. *Note: you are asked to value the* ***second*** *component of the M&I issue. Please ignore the first component (interest in the trust).*

2 The payoff of the stock purchase contract consists in:

a. Purchase forward $0.6699 \times S_{MI}$ at price \$25

b. Sell 0.6699 call options with strike \$37.32

c. Buy 0.5402 call options with strike \$46.28

d. Receive annual coupon payments of $0.6699 \times \$37.32 \times 2.6\% = \$25 \times 2.6\%$

Focus on the first 3 items (a-c):

$$\text{Payoff} = 0.6699 \times (S_{MI} - \max[S_{MI} - 37.32, 0]) + 0.5402 \times \max[S_{MI} - 46.28, 0] - 25 \tag{15}$$

The six inputs necessary to value the options are: $S_0 = 37.32$, $K_1 = 37.32$, $K_2 = 46.28$, $\sigma = 0.15$, $r = 0.03$, $\delta = 0.02$, and $T = 3$. We have for the two option prices:

$$\text{Call}(37.32) = 4.112 \tag{16}$$
$$\text{Call}(46.28) = 1.385 \tag{17}$$

Then, the present value of $0.6699 \times S_{MI} - 25$ is equal to

$$PV(F_{0,T} - 25) = PV\left(0.6699 \times 37.32 \times e^{(0.03-0.02)\times 3} - 25\right) = 0.69646 \tag{18}$$

Turn now to the item (d) above. The annual coupon payment we receive is valued at:

$$25 \times 0.026 \times \left(e^{-0.03} + e^{-0.06} + e^{-0.09}\right) = 1.83699 \tag{19}$$

Now, we can sum up all payments to get an initial price for the stock purchase agreement of

$$\text{Price} = 0.69646 - 0.6699 \times 4.122 + 0.5402 \times 1.394 + 1.83704 \tag{20}$$
$$= \$0.527 \tag{21}$$

3 An equity-linked structured note

Consider a structured investment product that is available from Credit Suisse for an initial cost of 1$ and whose payoff in 3 years is given by

$$1 + \max\left[\alpha, c\left(\frac{S_3}{S_0} - 1\right)\right] \tag{22}$$

where α, c are positive constants and S_t denotes the level of the S&P500 at time t. The riskless interest rate is 7%, the volatility of the S&P500 is equal to 20% per year and the dividend yield on the index is $\delta = 2\%$ per year.

a. What does this product do?

b. Show that the terminal payoff of the strucutred product can be written as $\beta + (c/S_0)\max(S_3 - K, 0)$ for some constants β and K to be determined.

c. Find an equation that relates the value of α to the maximum value of c that Credit Suisse can offer while being sure to not loose any money on the trade. Compute the maximal value of c when $\alpha = 2\%$.

d. Assume that Credit Suisse offers you to invest in a contract with $c = 0.8$ and $\alpha = 2\%$. What should you do? What if the volatility was equal to 35%?

3 a. This product insures a constant return worth α. If the S&P does not increase significantly the return is α and if it does the return is $c\left(\frac{S_3}{S_0} - 1\right)$.

b. The payoff of the option (see Figure 1) satisfies

$$v_T = 1 + \alpha + \max\left(c\left(\frac{S_3}{S_0} - 1\right) - \alpha, 0\right) \tag{23}$$

$$= 1 + \alpha + \frac{c}{S_0}\max\left(S_3 - \left(S_0 + \frac{\alpha S_0}{c}\right), 0\right) \equiv \beta + \frac{c}{S_0}\max\left(S_3 - K(\alpha, c), 0\right) \tag{24}$$

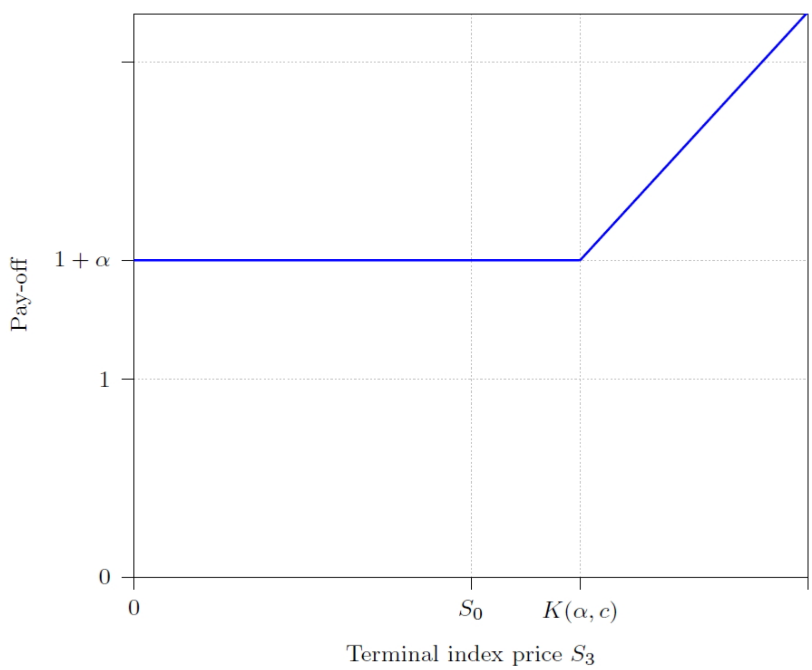


Figure 1: Payoff of the structured note

c. The condition for the bank to make money is

$$e^{-3r}(1 + \alpha) + \frac{c}{S_0}C_0(K(\alpha, c)) \leq 1 \tag{25}$$

where $C_t(k)$ is the arbitrage price at time t of a European call option with strike k and maturity $T = 3$. Thus, the maximum c that the bank can offer solves

$$e^{-3r}(1 + \alpha) + \frac{c}{S_0}C_0(K(\alpha, c)) = 1 \tag{26}$$

Solving the above equation shows that when $\alpha = 2\%$ the maximal value of c that the bank can offer is $c^* = 0.9513$.

d. if $c = 0.8$ then

$$e^{-3r}(1+\alpha) + \frac{0.8}{S_0} C_0(K(\alpha, 0.8)) = 0.9710 < 1 \tag{27}$$

so I shouldn't invest in the structured product because the price is to high relative to what it can pay me off. On the other hand, if $\sigma = 0.35$, then$c^* = 0.6447$ and

$$e^{-3r}(1+\alpha) + \frac{0.8}{S_0} C_0(K(\alpha, 0.8)) = 1.0413 > 1 \tag{28}$$

which shows that the bank is actually selling the product below its no-arbitrage value and thus allows me to extract some riskless profit. To do so I should follow this strategy:

- Buy the contract
- Borrow $(1+\alpha)e^{-3r}$ at the riskless rate
- Sell c/S_0 European calls with strike $K = K(\alpha, 0.8)$ and maturity $T = 3$. If this call is not available then I can replicate the desired position by using a delta hedging strategy:
  - Short $\frac{c}{S_0}\Delta_t$ units of the stock at time t
  - Long $\pi_{0t} = e^{-rt}\frac{c}{S_0}(\Delta_t - C_t(K))$ units of the riskless asset.

  At time 0, we have

  $$e^{-3r}(1+\alpha) + \frac{0.8}{S_0} C_0(K(\alpha, 0.8)) - 1 > 0 \tag{29}$$

  At time 3 the value of option equals what we have to reimurse. Thus, this is an arbitrage. Interpretation: if $c < c^*$, then nobody buys the option and the bank is unable to raise funds. If $c > c^*$ everybody profits from the arbitrage opportunity and the bank goes bankrupt.

4 Portfolio insurance

A fund holds a well-diversified portfolio that tracks the performance of the S&P500 and is currently worth 360,000,000. The current value of the index is 1,200 and the fund manager would like to buy insurance against a reduction of more than 5% in the index level at a six month horizon.

Assume that the riskless rate is 6%, that the dividend yield on the index is 3% and that the annualized volatility of the index is 30%. **For this exercise, assume that options are written on 1 unit of the index.**

a. If the fund manager decides to obtain his insurance by investing in put options, how much would it cost?

b. Show that if the fund manager decides to use calls instead of puts the initial cost would be the same.

c. Assume that instead of buying options, the fund manager decides to synthetically create the option he needs by trading in the index and the riskless asset. How much should he initially invest in the riskless asset?

4 a. First, the portfolio represents

$$\pi_t = \frac{360,000,000}{1200} = 300,000 \tag{30}$$

units of the index. The strike of the put option written on 1 unit of the stock is $K = 0.95 \times 1200 = 1140$ and the investor needs 300,000 units of the put to insure himself. Figure 2 illustrates the value of the portfolio including the put option.

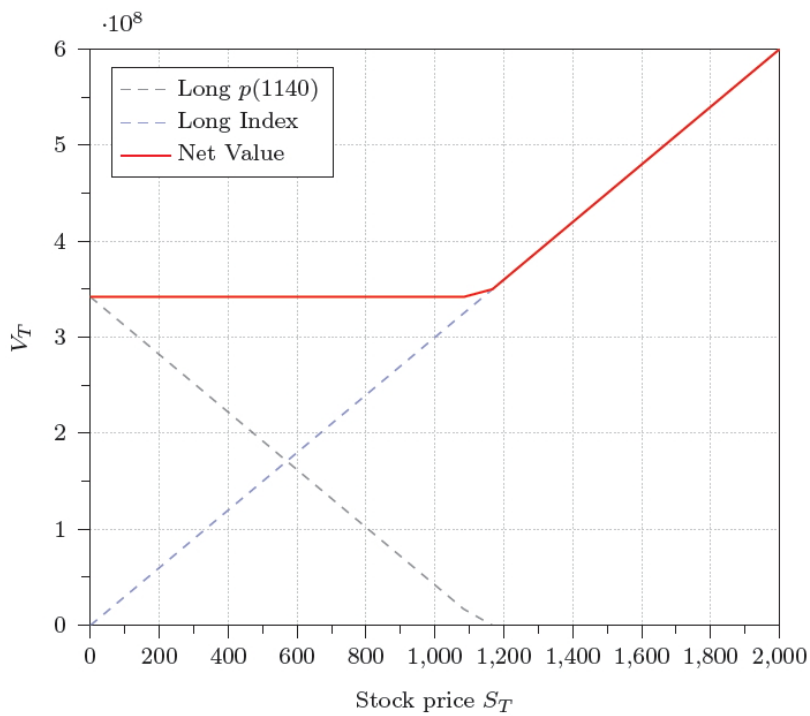


Figure 2: Payoff diagram for Question 4

Using the Black-Scholes formula, the price of a put option $p(1140)$ written on 1 unit of the stock is

$$p(1140) = 63.176 \tag{31}$$

Thus, the cost is worth $300,000 \times 63.176 = 18,952,687$.

b. We simply have to replicate the payoff of 300,000 put options with strike $K = 1140$ using calls, the stock, and the riskless asset. By no-arbitrage and by the put-call parity, we need to go long 300,000 units of call, short $e^{-\delta\tau}300,000$ units of the stock, and lend $Ke^{-r\tau}300,000$. In other words, the cost has to be the same.

c. The amount α invested in the riskless asset can directly be inferred from the Black-Scholes formula

$$300,000 p_t = 300,000 \left(Ke^{-r\tau}\Phi(-d_2) - S_t e^{-\delta\tau}\Phi(-d_1)\right) \tag{32}$$
$$\equiv \alpha_t + \pi_t S_t \tag{33}$$

Hence

$$\alpha_t = 300,000 Ke^{-r\tau}\Phi(-d_2) = 300,000 \times 432.3792 = 129,713,773 \tag{34}$$

This amount must be monitored frequently. As the value of the original portfolio declines, the delta of the put option becomes more negative and the proportion in the riskless asset must be increased. As the value of the original portfolio increases, the delta of the put becomes less negative and the proportion in the riskless asset must be decreased. Please refer to Section 18.13 in Hull (8th edition) for an additional example.

5 Collars in acquisitions

Firm A has a stock price of \$38 and has made an offer for firm B where A promises to pay \$60/share for B, as long as A's stock price remains between \$35 and \$45. If the price of A is below \$35, A will pay 1.714 shares, and if the price of A is above \$45, A will pay 1.333 shares. The deal is expected to close in 9 months. Assume (for A's stock): $\sigma = 40\%$, $r = 6\%$, and $\delta = 0$.

a. How are values 1.714 and 1.333 arrived at?

b. Plot the value of A's offer for one B share as a function of A's stock price in 9 months. Clearly indicate on the diagram the slope of all the lines that you draw.

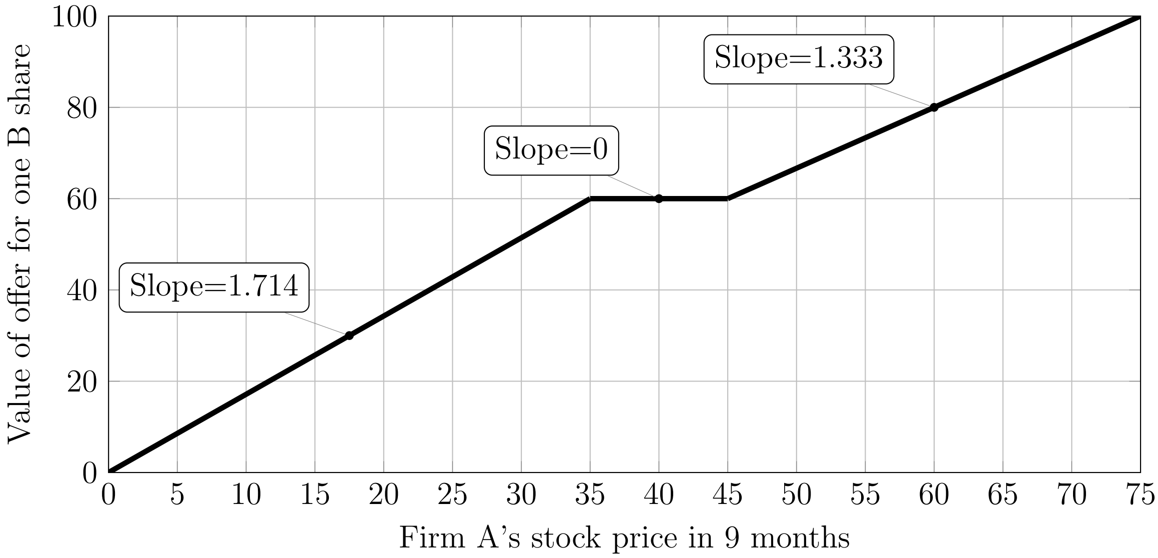
c. What is the value of the offer?

d. How sensitive is the value of the offer to the volatility of A's stock? More precisely, what is the value of the offer if the volatility of A's stock goes up to 45%? Here is some information that you might find useful:

Vega of a \$35-strike call	0.1134
Vega of a \$35-strike put	0.1134
Vega of a \$45-strike call	0.1291
Vega of a \$45-strike put	0.1291

5 a. This makes the amount paid (as a function of A's stock price) continuous at the collar points; i.e., $1.714 \times 35 = 60$ and $1.333 \times 45 = 60$.

b.



c. The deal can be looked at as giving Company B's shareholders 1.333 shares of A, a long position of 1.333 of 45-strike puts, and a short position on 1.714 of 35-strike puts.

   The 45-strike put is worth \$8.43. The 35-strike put is worth \$2.97.

   The value of this offer is

$$1.333 \times \$38 + 1.333 \times \$8.43 - 1.714 \times \$2.97 = \$56.82 \tag{35}$$

   Alternatively, one can value the deal as 1.714 shares, short 1.714 35-strike calls, and long 1.333 45-strike calls.

d. The Vega of the offer is

$$1.333 \times 0.1291 - 1.714 \times 0.1134 = -0.0224 \tag{36}$$

   If the volatility goes up to 45%, the value of the offer goes down to \$56.71.