UCLA Anderson School of Management

Solutions to Quizz #1

Problem 1. Solve the partial differential equation

$$\begin{aligned} \frac{\partial F}{\partial t} - \frac{\partial F}{\partial x}\mu x + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} &= 0 \\ F(T, x) &= x \end{aligned}$$

Hint: The solution to the SDE

$$dx_t = -\mu x dt + \sigma dW_t$$

is given by

$$x_T = x_t e^{-\mu(T-t)} + \sigma \int_t^T e^{-\mu(T-s)} dW_s$$

Solution:

$$\begin{aligned} F(t, x_t) &= E(x_T) = x_t e^{-\mu(T-t)} + \sigma E_t \left\{ \int_t^T e^{-\mu(T-s)} dW_s \right\} \\ &= x_t e^{-\mu(T-t)} \end{aligned}$$

Problem 2. Suppose that $X_t = [x_t, y_t]$ is a two-dimensional vector where both x_t and y_t follow (independent) Wiener processes.

Show that $z_t = x_t^2 + y_t^2 - 2t$ is a martingale. (Hint: Apply Ito's Lemma)

Solution: Applying Ito's Lemma leads to

$$\begin{aligned} dz_t &= 2x_t dx_t + 2y_t dy_t + 2dt - 2dt \\ &= 2x_t dx_t + 2y_t dy_t \end{aligned}$$

Since dx_t, dy_t are Wiener processes and there is no drift term in the above equation, z_t is a martingale.

Problem 3. Suppose that

$$dX_t = u_t dt + dW_t$$

and let M_t be given by

$$M_t = e^{-\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds}$$

Prove that

$$Y_t = X_t M_t$$

is a martingale. (Hint: Ito's Lemma implies that $dM_t = -u_t M_t dW_t$)

Solution: Applying Ito's Lemma gives

$$\begin{aligned} dY_t &= M_t dX_t + X_t dM_t + dM_t dX_t \\ &= M_t(u_t dt + dW_t) - X_t M_t u_t dW_t - u_t M_t dt \\ &= M_t(1 - X_t u_t) dW_t \end{aligned}$$

Since there is no "dt" term Y_t is a martingale.