Empirical Methods in Finance

The I-GARCH Model*

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In the TA session yesterday, a few people were confused about the I-GARCH(1,1) model and in particular regarding Question 21 in both Finals 2017 and 2018:

> *When we use the I-GARCH(1,1) process with zero intercept in the vol specification for forecasting the variance at long horizons, which one of the following is correct.*

In this short note, I try to give a more detailed answer to these two questions and hope to clarify this point.

1 The GARCH Model

Recall from Lecture 8 notes the GARCH model representation: ε_t follows a GARCH(m, s) model if

$$\varepsilon_t = \sigma_t \eta_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2, \tag{1}$$

where η_t is i.i.d. with mean zero and unit variance, $\alpha_0 > 0$, $\alpha_i \geq 0$ for $i > 0$, $\beta_j \geq 0$ for $j > 0$, and

$$\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1.$$

The unconditional variance is

$$\mathbb{E}\left[\varepsilon_t^2\right] = \frac{\alpha_0}{1 - \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i)} \tag{2}$$

*Please see Chapter 3 of Tsay's textbook for a detailed treatment of conditional heteroscedastic models.

In particular, as we learned in class, the multi-step volatility forecast for a GARCH$(1,1)$ model is:

$$\sigma_t^2(h) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2(h-1), \tag{3}$$

where h is the forecast horizon.

2 The I-GARCH Model

If the AR polynomial of the GARCH representation has a unit root, i.e.

$$\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) = 1,$$

then we have an Integrated GARCH (I-GARCH) model. So, from equation (2), the unconditional variance in an I-GARCH model is *not* defined. This seems hard to justify for an excess return series.

In particular, for an I-GARCH$(1,1)$ model we have $\alpha_1 = 1-\beta_1$. Thus, ε_t follows an I-GARCH$(1,1)$ model if

$$\varepsilon_t = \sigma_t\eta_t, \quad \sigma_t^2 = \alpha_0 + (1-\beta_1)\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2, \tag{4}$$

with $0 < \beta_1 < 1$.

Plugging in $\alpha_1 + \beta_1 = 1$ in equation (3), the volatility forecast for long horizon will be:

$$\sigma_t^2(h) = \alpha_0 + \sigma_t^2(h-1) \tag{5}$$

$$= (h-1)\alpha_0 + \sigma_t^2(1), \tag{6}$$

where equation (6) is derived by repeated substitution of equation (5). Consequently, the effect of $\sigma_t^2(1)$ on future volatilities is persistent, and the volatility forecasts form a straight line with slope α_0.

2.1 I-GARCH(1,1) Process with Zero Intercept

The case of zero intercept ($\alpha_0 = 0$) is of particular interest in studying the I-GARCH(1,1) model. In this case, from equation (6), the volatility forecasts are fixed and simply equal to $\sigma_t^2(1)$

for all forecast horizons.[1]

So, in the **2017 Final exam**, the correct answer is **(b)**:

> *When we use the I-GARCH(1,1) process with zero intercept in the vol specification for forecasting the variance at long horizons,*
>
> ***(b) the forecast is fixed at the conditional variance.***

The model is also an *exponential smoothing* model for the $\{\varepsilon_t^2\}$ series. To see this, rewrite the model in equation (4) as

$$\begin{aligned}
\sigma_t^2 &= (1-\beta_1)\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2 \\
&= (1-\beta_1)\varepsilon_{t-1}^2 + \beta_1\left[(1-\beta_1)\varepsilon_{t-2}^2 + \beta_1\sigma_{t-2}^2\right] \\
&= (1-\beta_1)\varepsilon_{t-1}^2 + \beta_1(1-\beta_1)\varepsilon_{t-2}^2 + \beta_1^2\sigma_{t-2}^2
\end{aligned}$$

By repeated substitution, we have

$$\sigma_t^2 = (1-\beta_1)\left[\varepsilon_{t-1}^2 + \beta_1\varepsilon_{t-2}^2 + \beta_1^2\varepsilon_{t-3}^2 + \cdots\right], \tag{7}$$

which is the well-known exponential smoothing formation with $\beta_1 < 1$ being the discounting factor. Thus, in I-GARCH(1,1) with zero intercept, the forecast is an exponentially-weighted moving average of lagged squared residuals $\{\varepsilon_{t-i}^2\}$.

So, in **the 2018 Final exam**, the correct answer is **(b)**:

> *When we use the I-GARCH(1,1) process with zero intercept in the vol specification for forecasting the variance at long horizons,*
>
> ***(b) the forecast is an exponentially weighted moving average of past squared residuals.***

[1] This special I-GARCH(1,1) model is the volatility model used in RiskMetrics, which is an approach for calculating value at risk.