Problem.1.

1. $P(W < W_0 - VaR) = 1-c.$

$cdf = 1 - e^{-\lambda x} = 1 - c.$

$x = -\frac{1}{\lambda} \cdot \log c.$

$\because x = W_0 - VaR.$

$\therefore VaR = W_0 + \frac{1}{\lambda} \cdot \log(c)$

$\because \lambda = \frac{1}{W_0}$

$\therefore VaR = W_0(1 + \log c).$

$W_0 = 200, \quad c = 99.9\%$

$VaR = 199.8$

2. $P(W > W_0 + VaR) = 1-c$

$cdf = 1 - e^{-\lambda x} = c.$

$x = -\frac{1}{\lambda} \log(1-c)$

$\because x = W_0 + VaR$

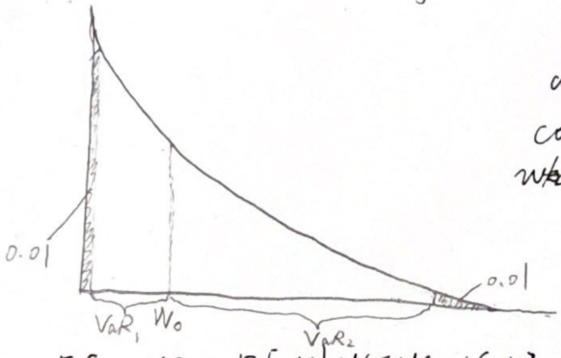
$\therefore VaR = -W_0 - W_0 \cdot \log(1-c)$

$= -W_0(1 + \log(1-c))$

$W_0 = 200, \quad c = 99.9\%$

$VaR = 1181.55$

3.



Because of the asymmetry of exponential distribution. when you short the asset, its value could increase a lot more than it could decrease when you buy it. Therefore: $VaR_2 = 1181.55 >> 199.8.$

4. $ES = W_0 - E[W \mid W \leq W_0 - VaR]$

$$= W_0 - \frac{\int_{-\infty}^{W_0 - VaR} W \cdot f(W)\, dW}{\int_{-\infty}^{W_0 - VaR} f(W) \cdot dW}$$

In which:

$$\int_{-\infty}^{W_0 - VaR} f(W) dW = 1 - c.$$

$\because$ For exponential distribution: $f(W) = 0$ for $W < 0$

$$\therefore \int_{-\infty}^{W_0 - VaR} W \cdot f(W) dW = \int_{0}^{W_0 - VaR} W \cdot f(W) \cdot dW.$$

$f(W) = \lambda \cdot e^{-\lambda W}.$

Denote $A = \int_0^{W_0 - VaR} W \cdot f(W) \cdot dW$

Integrating by parts:

$$A = -W \cdot e^{-\lambda W} \Big|_0^{W_0 - VaR} + \int_0^{W_0 - VaR} e^{-\lambda W} dW$$

$$= (VaR - W_0) \cdot e^{\lambda(VaR - W_0)} + (-\tfrac{1}{\lambda}) \cdot e^{-\lambda W} \Big|_0^{W_0 - VaR}$$

$$= (VaR - W_0) \cdot e^{\lambda(VaR - W_0)} + \left[(-\tfrac{1}{\lambda}) \cdot e^{\lambda(VaR - W_0)} + \tfrac{1}{\lambda}\right]$$

$$= (VaR - W_0) \cdot e^{(VaR - W_0)/W_0} + W_0 - W_0 \cdot e^{(VaR - W_0)/W_0}$$

4. $\therefore ES = W_0 - \frac{A}{1-c}$

in which $A = (VaR - 2W_0)e^{(VaR-W_0)/W_0} + W_0$

For question 1 $\therefore$ $ES_1 = 199.9$

~~For question 2 : $ES_2 =$~~

For question 2 : $ES = E[W \mid W \geq W_0 + VaR] - W_0$

$$ES = \frac{\int_{W_0+VaR}^{\infty} W \cdot f(W) \cdot dW}{1-c} - W_0$$

same as question 1 above.

$$A = -W \cdot e^{-\lambda W}\Big|_{W_0+VaR}^{\infty} + (-\tfrac{1}{\lambda}) \cdot e^{-\lambda W}\Big|_{W_0+VaR}^{\infty}$$

$$= (2W_0 + VaR) \cdot e^{-(W_0+VaR)/W_0}$$

$\therefore ES_2 = \frac{A}{1-c} - W_0$

in which : $A = (2W_0 + VaR) \cdot e^{-(W_0+VaR)/W_0}$

$ES_2 = 1381.56$

Problem 2

1.

```
from scipy.stats import norm
import numpy as np
import matplotlib.pyplot as plt
import math

mu = 0.08
sigma1 = 0.12
sigma2 = 0.2
small = 0.00001
var = 2 * mu
c = 0.01
left = mu
right = 100
pi = 0.7
var_yours = mu - norm.ppf(0.01, loc = mu, scale = sigma2)
var_partner = mu - norm.ppf(0.01, loc = mu, scale = sigma1)
def cdf(var, p = pi):
    return p * norm.cdf(mu - var, loc = mu, scale = sigma1) + (1 - p) * norm.cdf(mu - var, loc = mu, sca


def value_at_risk(p, var = var, mu = mu):
    result = cdf(var)
    right = 100
    left = mu
    while (abs(result - c) > small):
        if result < c:
            right = var
        else:
            left = var
        var = (left + right) / 2
        result = cdf(var, p)
    return var

var_result = value_at_risk(pi)
pi_array = np.arange(0,1,0.01)
var_array = np.zeros(100)
for i,v in enumerate(pi_array):
    var_array[i] = value_at_risk(v)
print('VaR for your view is', var_yours)
```

```
## VaR for your view is 0.46526957480816816
```

```
print('VaR for partner view is', var_partner)
```

```
## VaR for partner view is 0.27916174488490086
```

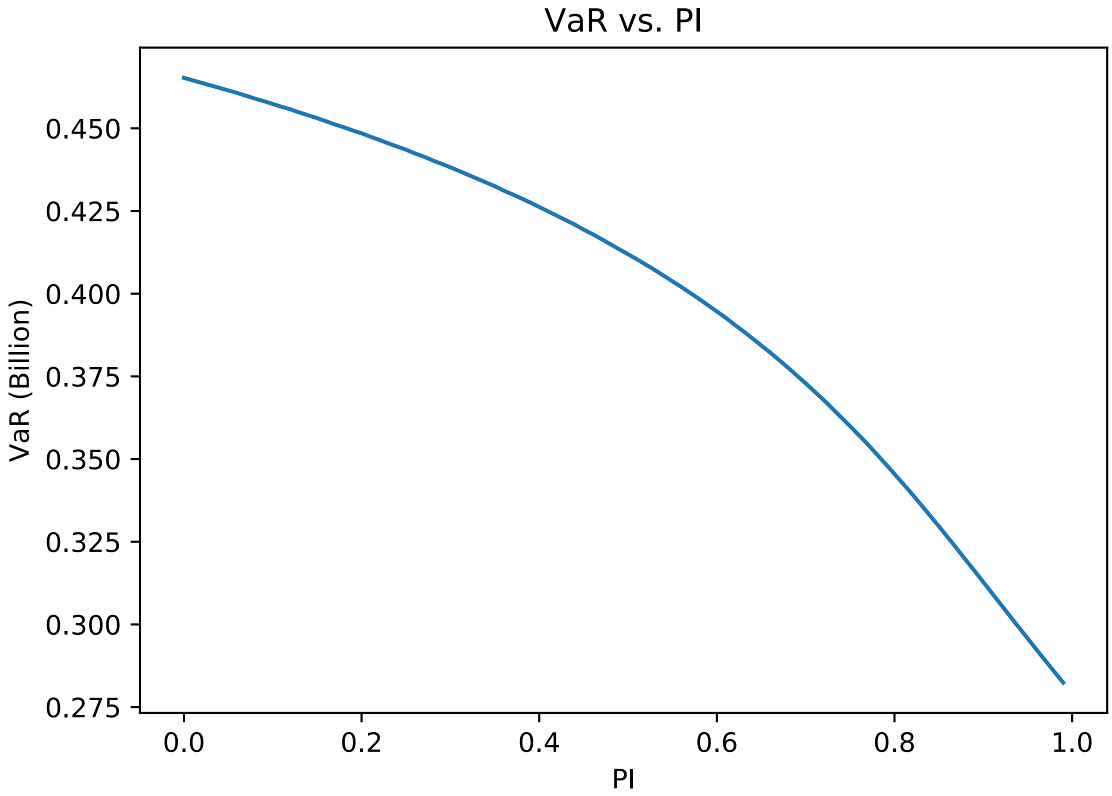
```
print('VaR for common view is', var_result)
```

```
## VaR for common view is 0.372900390625
```

With higher volatility, the VaR will be higher.
With PI between 0 and 1, common view will be ranging between your view and partner's view.

2.

```
pi_array = np.arange(0,1,0.01)
var_array = np.zeros(100)
for i,v in enumerate(pi_array):
    var_array[i] = value_at_risk(v)
plt.plot(pi_array, var_array)
plt.xlabel('PI')
plt.ylabel('VaR (Billion)')
plt.title('VaR vs. PI')
plt.show()
```



The VaR of common view is ranging from that of your view 0.4653 to that of partner view 0.2792. The higher the probability PI, the VaR of common view will be more towards that of partner view.

3. We can use simulation to compute the VaR of portfolio. With millions of draws from the gamma distribution of σ, we can compute the VaR for each σ. After that, we take the average of the VaRs. And the average VaR is the VaR of the portfolio.

Problem 3.

1. let $X_t = \log S_t$.

By Ito's lemma: $dX_t = \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial S} dS + \frac{1}{2}\frac{\partial^2 X}{\partial S^2} (dS)^2$

$$dX_t = \cancel{\mu dt} (\mu - \tfrac{1}{2}\sigma^2) dt + \sigma dW_t.$$

$\log S_T = X_T \sim N(X_0 + (\mu - \frac{1}{2}\sigma^2)T, \sigma^2 \cdot T.)$.

If X is a random variable with quartile c equal to x_0. then the quartile c of $g(X)$ is $g(x_0)$ if g is a monotone function.

$$P(S_T < S_0 - VaR) = 1 - c. \Longleftrightarrow$$

$$P(\log S_T < \log(S_0 - VaR)\} = 1 - c = 0.01$$

$$\frac{\log(S_0 - VaR) - \{X_0 + (\mu - \frac{1}{2}\sigma^2)T\}}{\sigma \cdot \sqrt{T}} = -2.327. \quad T = \frac{10}{252}, \quad S_0 = 50$$

$$VaR = 50 \cdot (1 - e^{\frac{5}{126}(\mu - \frac{1}{2}\sigma^2) - 0.4635 \cdot \sigma}).$$

Input $\mu = 0.07$ $\sigma = 0.16$. $\Rightarrow$ $VaR = 3.468$

2. Let x denote the value you want to borrow by bonds.

$$\frac{100 - x \times (e^{0.02 \times \frac{10}{252}} - 1)}{VaR} \times 50 = x + 100 \qquad \Rightarrow x \approx ~~1426.57~~ \ 1326.57 \text{ million}$$

Therefore, in your portfolio: Long 1426.57 million stock and
Short 1326.57 million bonds.

Problem 3

3.

```
import math
def eur_call(S,T,r,sigma,K):
    d1 = (math.log(S/K) + (r + sigma * sigma / 2) * math.sqrt(T)) / (sigma * math.sqrt(T))
    d2 = d1 - sigma * math.sqrt(T)
    return S * norm.cdf(d1) - K * math.exp(-r * T) * norm.cdf(d2)

S = 50
T = 0.25
K = 50
r = 0.02
sigma = 0.16
c0 = eur_call(S,T,r,sigma,K)
print('c0 =', c0)
```

```
## c0 = 1.7108771746126656
```

Due to the property that if X is a random variable with quantile c equal to x0, then the quantile c of g(X) is g(x0) if g is a monotone function.
Because the price of call is a monotone function of stock price, $P(C_T < C_0 - VaR_c) = 1 - c$ is equivalent to $P(C(S_T) < C(S_0 - VaR_s) = 1 - c$.
Therefore, $C_0 - VaR_c = C(S_0 - VaR_s)$

```
r = 0.02
c_at_var = eur_call(S - 3.468, T - 10/252, r, sigma, K)
#print(c_at_var)
VaR_c = c0 - c_at_var
print('VaR of call is', VaR_c)
```

```
## VaR of call is 1.376634175835873
```

$$\frac{100 - x \times (e^{2\% \times \frac{10}{252}} - 1)}{1.3766} \times 1.7109 = x + 100$$

$$x = 24.26 million$$

In your portfolio, long 124.26 million ATM call options and short 24.26 million bonds.

4.

```
r = 0.02
def eur_put(S,T,r,sigma,K):
    d1 = (math.log(S/K) + (r + sigma * sigma / 2) * math.sqrt(T)) / (sigma * math.sqrt(T))
    d2 = d1 - sigma * math.sqrt(T)
    return K * math.exp(-r * T) * norm.cdf(-d2) - S * norm.cdf(-d1)
p0 = eur_put(S,T,r,sigma,K)
print('p0 =', p0)
```

```
## p0 = 1.461501134246781
```

Because stock has the mean of return larger than 0, so the expectation of stock price will be higher than S0 in 10 days. The VaR of put will be lower if we short put instead of long it. So that we can invest more on put within the same VaR constrain.
Same as call option, $P(P_T > P_0 + VaR_p) = 1 - c$ is equivalent to $P(P(S_T) > P(S_0 - VaR_s)) = 1 - c$.
Therefore, $P_0 + VaR_p = P(S_0 - VaR_s)$

```
r = 0.02
p_at_var = eur_put(S - 3.468, T - 10/252, r, sigma, K)
#print(p_at_var)
VaR_p = p_at_var - p0
print('VaR of put option is', VaR_p)
```

```
## VaR of put option is 2.130866119001105
```

$$\frac{100 + x \times (e^{2\% \times \frac{10}{252}} - 1)}{2.1309} \times 1.4615 = x - 100$$

$$x = 168.68 million$$

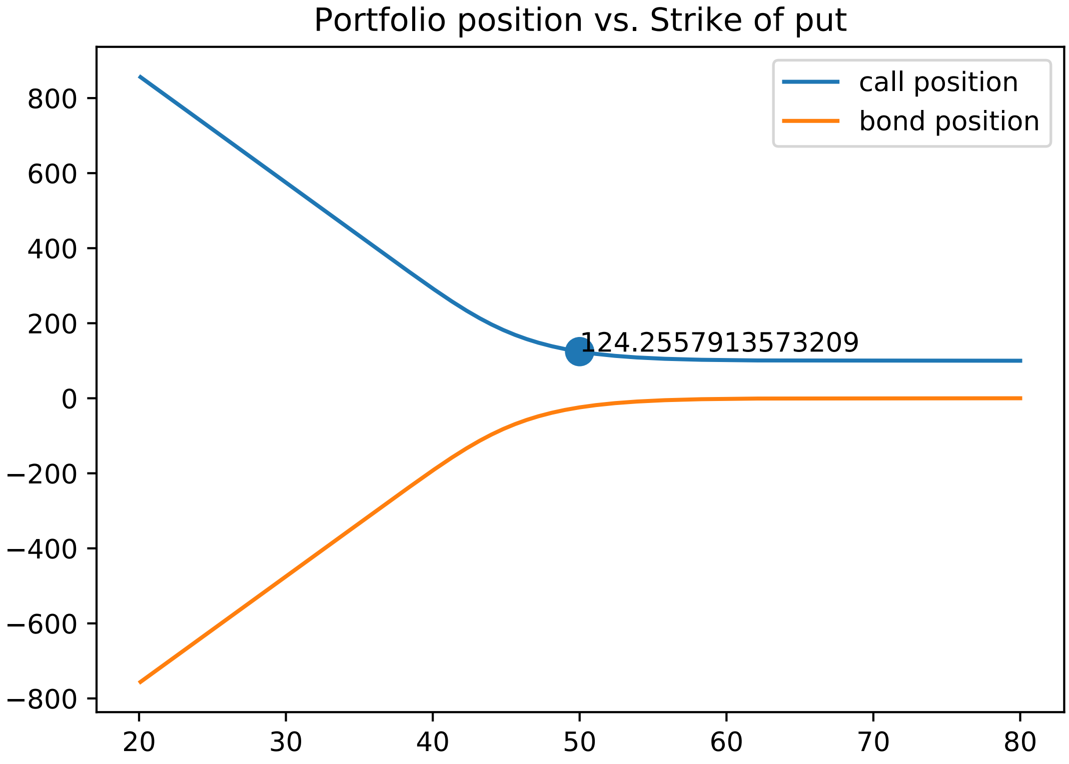
In your portfolio, short 68.68 million ATM put options and long 168.68 million bonds.

5.

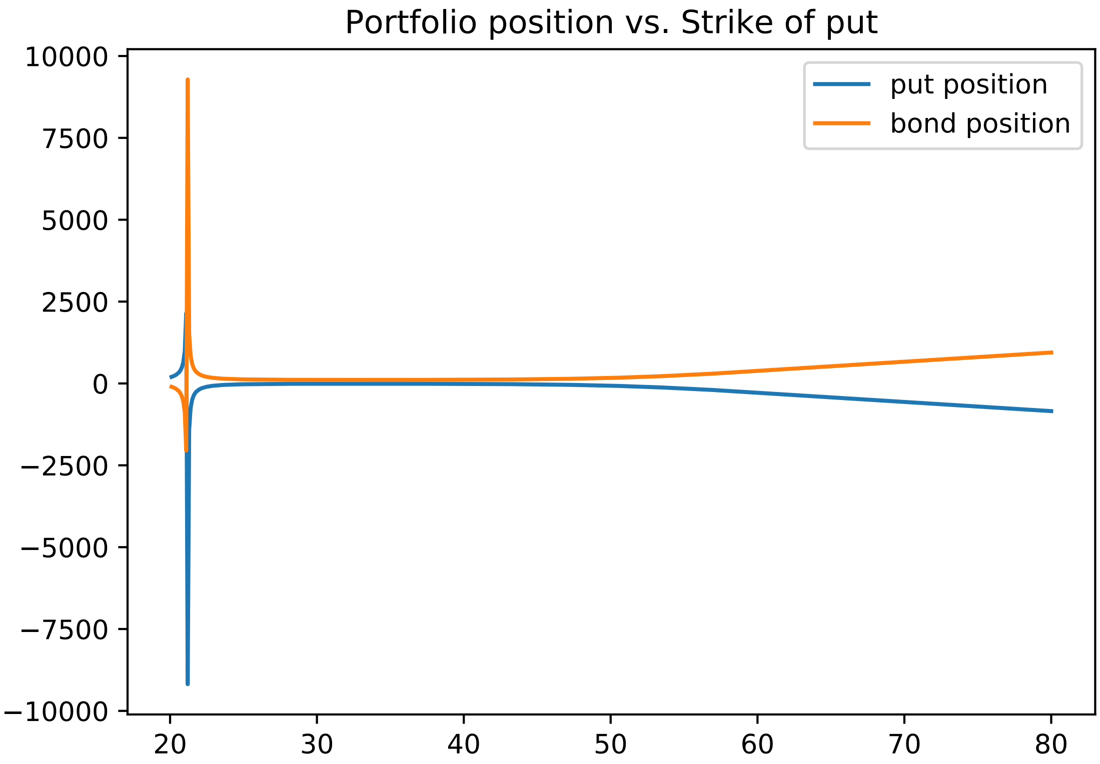
```
r = 0.02
def call_position(S,T,r,sigma,K):
    c0 = eur_call(S,T,r,sigma,K)
    c_at_var = eur_call(S - 3.468, T - 10/252, r, sigma, K)
    var = c0 - c_at_var
    return 100 * (1 - var / c0) / (var / c0 + math.exp(0.02 * 10 / 252) - 1) + 100

def put_position(S,T,r,sigma,K):
    p0 = eur_put(S,T,r,sigma,K)
    p_at_var = eur_put(S - 3.468, T - 10/252, r, sigma, K)
    var = p_at_var - p0
    return 100 * (var / p0 + 1) / (var / p0 - math.exp(0.02 * 10 / 252) + 1) - 100

put_returns = list()
call_returns = list()
stike = 0
S_exp = S * math.exp((mu - 0.5 * sigma * sigma) * 10 / 252)
put_positions = list()
call_positions = list()
max_return = 0
k_list = np.arange(80,20,-0.1)
for k in k_list:
    call_unit = call_position(S, T, r, sigma, k)
    call_positions.append(call_unit)
    put_unit = put_position(S, T, r, sigma, k)
    put_positions.append(-put_unit)
    c_0 = eur_call(S,T,r,sigma,k)
    p_0 = eur_put(S,T,r,sigma,k)
    c_end = eur_call(S_exp,T-10/252,r,sigma,k)
    p_end = eur_put(S_exp,T-10/252,r,sigma,k)
    retn_c = (c_end - p_0) * call_unit / c_0 - (call_unit - 100) * (math.exp(0.02 * 10/252) - 1)
    retn_p = (p_0 - p_end) * put_unit / p_0 + (put_unit + 100) * (math.exp(0.02 * 10/252) - 1)
    call_returns.append(retn_c)
    put_returns.append(retn_p)
    if retn_p > max_return:
        stike = k
        max_return = retn_p
print('Strike price of put with highest return is', stike)
```

```
## Strike price of put with highest return is 21.200000000003342
```

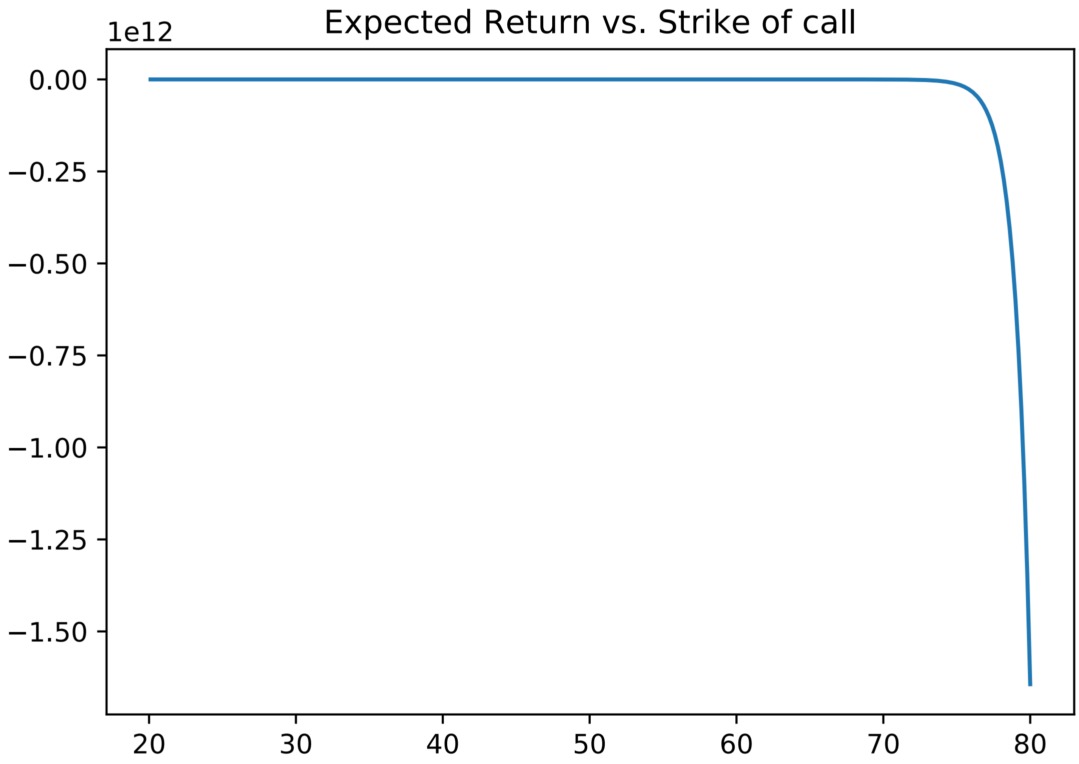
```
plt.plot(k_list, call_positions, label = 'call position')
plt.plot(k_list, [-x + 100 for x in call_positions], label = 'bond position')
plt.scatter(50, call_positions[300], s = 100)
plt.annotate(str(call_positions[300]),xy=(50,call_positions[300]))
plt.legend()
plt.title('Portfolio position vs. Strike of put')
plt.show()
```



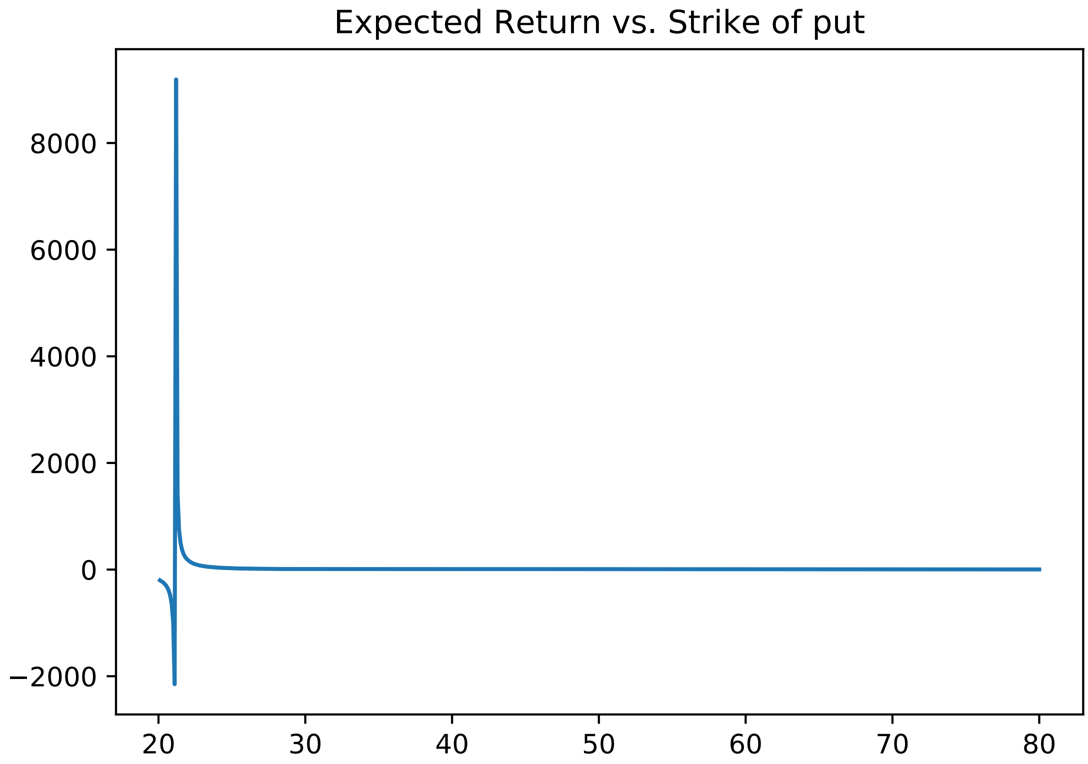
```
plt.plot(k_list, put_positions, label = 'put position')
plt.plot(k_list, [-x + 100 for x in put_positions], label = 'bond position')
plt.legend()
plt.title('Portfolio position vs. Strike of put')
plt.show()
```



```
plt.plot(k_list, call_returns)
plt.title('Expected Return vs. Strike of call')
plt.show()
```



```
plt.plot(k_list, put_returns)
plt.title('Expected Return vs. Strike of put')
plt.show()
```



According to the return plots, we should short put option with stike = 21.2.
In your portfolio, short put option 9185.47 million dollars and long 9285.47 million dollars. The expected return of this portfolio is 9192.76 million dollars.

6. With respect tp a constraint on expected shortfall, all the portfolio will have a low position on risky assets. And the return of all portfolio will be lower.