Empirical Methods in Finance
Homework 3: Solution

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Problem 1: Year-on-year quarterly data and ARMA dynamics

Assume the true quarterly log market earnings follow:

$$e_t = e_{t-1} + x_t,$$
$$x_t = \phi\, x_{t-1} + \varepsilon_t,$$

where $\mathbb{V}(\varepsilon_t) = \sigma_\varepsilon^2 = 1$ and ε_t is i.i.d. over time t.

The earnings data you are given is year-on-year earnings growth, with in logs is:

$$y_t \equiv e_t - e_{t-4}.$$

1. Assume $\phi = 0$. Derive autocovariances of order 0 through 5 for y_t.
2. Assume $\phi = 0$. Determine the number of AR lags and MA lags you need in the ARMA(p, q) process for y_t. Give the associated AR and MA coefficients.
3. Optional: Assume $0 < \phi < 1$. Repeat 1 and 2 under this assumption.

   Suggested Solution: In the general case of $0 < \phi < 1$, we have:

$$\mathbb{V}(x_t) = \phi^2 \mathbb{V}(x_t) + \sigma_\varepsilon^2$$
$$\mathbb{V}(x_t) = \frac{\sigma_\varepsilon^2}{1 - \phi^2}$$

and y_t can be written as

$$y_t = e_t - e_{t-4}$$

$$= x_{t-3} + x_{t-2} + x_{t-1} + x_t$$

Recalling that the autocovariances of the AR(1) process are given by

$$\mathbb{C}\left(x_t, x_{t-j}\right) = \frac{\phi^j}{1-\phi^2}\sigma_\varepsilon^2$$

We can now calculate the autocovariances:

$$\begin{aligned}
\mathbb{C}\left(y_t, y_t\right) &= \sum_{i=0}^{3}\sum_{\ell=0}^{3}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \left(4\frac{1}{1-\phi^2} + 6\frac{\phi}{1-\phi^2} + 4\frac{\phi^2}{1-\phi^2} + 2\frac{\phi^3}{1-\phi^2}\right)\sigma_\varepsilon^2 \\
&= 2 \cdot \frac{2+\phi+\phi^2}{1-\phi}\sigma_\varepsilon^2 \\
\mathbb{C}\left(y_t, y_{t-1}\right) &= \sum_{i=0}^{3}\sum_{\ell=1}^{4}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \frac{\left(1+\phi\right)\left(3+\phi^2\right)}{1-\phi}\sigma_\varepsilon^2 \\
\mathbb{C}\left(y_t, y_{t-2}\right) &= \sum_{i=0}^{3}\sum_{\ell=2}^{5}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \frac{2+2\phi+2\phi^2+\phi^3+\phi^4}{1-\phi}\sigma_\varepsilon^2 \\
\mathbb{C}\left(y_t, y_{t-3}\right) &= \sum_{i=0}^{3}\sum_{\ell=3}^{6}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \frac{\left(1+\phi\right)\left(1+\phi^2\right)^2}{1-\phi}\sigma_\varepsilon^2 \\
\mathbb{C}\left(y_t, y_{t-4}\right) &= \sum_{i=0}^{3}\sum_{\ell=4}^{7}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \frac{\phi\left(1+\phi\right)\left(1+\phi^2\right)^2}{1-\phi}\sigma_\varepsilon^2 \\
\mathbb{C}\left(y_t, y_{t-5}\right) &= \sum_{i=0}^{3}\sum_{\ell=5}^{8}\mathbb{C}\left(x_{t-i}, x_{t-\ell}\right) \\
&= \frac{\phi^2\left(1+\phi\right)\left(1+\phi^2\right)^2}{1-\phi}\sigma_\varepsilon^2
\end{aligned}$$

From the above, we see that for $j > 3$, the autocovariances behave like those of an

AR(1) process with parameter ϕ. Thus, we know that the ARMA representation is of the form

$$(1 - \phi\, B)\, y_t = \theta\,(B)\, \varepsilon_t$$

Expanding the left hand side, we have

$$\begin{aligned}(1 - \phi\, B)\, y_t &= x_t + (1 - \phi)\, x_{t-1} + (1 - \phi)\, x_{t-2} + (1 - \phi)\, x_{t-3} - \phi x_{t-4} \\ &= \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3}\end{aligned}$$

Thus, we see that the process is captured by an ARMA(1, 3) with coefficients

$$\begin{aligned}\phi\,(B) &= 1 - \phi\, B \\ \theta\,(B) &= 1 + B + B^2 + B^3\end{aligned}$$

Plugging in $\phi = 0$ and $\sigma_\varepsilon^2 = 1$ give the results in the simplified case.

Although not required, it is helpful to note the Wold Decomposition of this process is given by

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

$$\psi_j = \begin{cases} \sum_{\ell=0}^{j} \phi^j & j \leq 3 \\ \phi\, \psi_{j-1} & j > 3 \end{cases}$$

which allows us to verify the autocovariances using the relation

$$\gamma\,(h) = \sum_{j=0}^{\infty} \psi_{j+|h|} \psi_j$$

Problem 2: Market-timing and Sharpe ratios

Assume you have an estimate of expected annual excess market returns for each time t, called x_t. You estimate the regression

$$R_{t+1}^e = \alpha + \beta x_t + \epsilon_{t+1} \tag{1}$$

and obtain $\hat{\alpha} = 0$, $\hat{\beta} = 1$, and $\sigma\,(\hat{\epsilon}_{t+1}) = 15\%$. Further, the sample mean and standard deviation of x_t are both 5%.

1. Calculate the standard deviation of excess returns based on the information given.

Suggested Solution:

$$\begin{aligned} Var(R^e_{t+1}) &= \beta^2 Var(x_t) + Var(\epsilon_{t+1}) \\ &= 0.05^2 + 0.15^2 \\ &= 0.025 \end{aligned}$$

Thus, the standard deviation would be 15.81%.

2. Calculate the R^2 of the regression based on the information given.

   Suggested Solution:

$$R^2 = \frac{\beta^2\, Var(x_t)}{Var\left(R^e_{t+1}\right)} = 0.1$$

3. Calculate the sample Sharpe ratio of excess market returns based on the information given.

   Suggested Solution:

$$\begin{aligned} SR &= \frac{\mathbb{E}(R^e_{t+1})}{\sigma(R^e_{t+1})} \\ &= \frac{0.05}{0.1581} = 0.3163 \end{aligned}$$

4. Recall from investments that a myopic investors chooses a fraction of wealth

$$\alpha_t = \frac{\mathbb{E}_t(R^e_{t+1})}{\gamma \sigma^2_t(R^e_{t+1})}$$

   in the risky asset (the market) at each time t, where we assume the risk aversion coefficient, γ, equals 40/9. Further, assume that the residuals ϵ_{t+1} are i.i.d., so $\sigma_t\left(\epsilon_{t+1}\right) = 15\%$ for all t. Given this, calculate the weight the investor chooses to hold in the risky asset if $x_t = 0$ and if $x_t = 10\%$. What is conditional Sharpe ratio in each of these cases?

Suggested Solution: The weight the investor chooses to hold in the risky asset Thus

$$SR = \begin{cases} 0 & \text{if } x_t = 0 \\ \\ \dfrac{0.1}{0.15} = 0.667 & \text{if } x_t = 0.1 \end{cases},$$

and

$$\alpha_t = \begin{cases} 0 & \text{if } x_t = 0 \\ \\ \dfrac{0.1}{\frac{40}{9} \times 0.15^2} = 1 & \text{if } x_t = 0.1 \end{cases},$$

5. Assume T is large (i.e., $T \to \infty$) and that x_t is either 0% or 10% at each time t, with equal probability (0.5).

   (a) What is the unconditional average excess return for an investor that holds α_t each period?

   Suggested Solution:

$$\mathbb{E}(\alpha_t R^e_{t+1}) = 0.5 \times 10\% = 5\%$$

   (b) What is the unconditional standard deviation? The following may be helpful for calculating the unconditional variance. You could also simulate a very long series to check your math.

   Suggested Solution: Using the law of iterated expectation, we can write

$$\begin{aligned} Var(\alpha_t R^e_{t+1}) &= \mathbb{E}\left[\mathbb{E}_t\left[(\alpha_t R^e_{t+1})^2\right]\right] - \mathbb{E}\left[\mathbb{E}_t\left[\alpha_t R^e_{t+1}\right]\right]^2 \\ &= \mathbb{E}\left[\alpha_t^2 \mathbb{E}_t\left[(R^e_{t+1})^2\right]\right] - \mathbb{E}\left[\alpha_t \mathbb{E}_t\left[R^e_{t+1}\right]\right]^2 \\ &= \mathbb{E}\left[\alpha_t^2\left(x_t^2 + \sigma_t^2(\epsilon_{t+1})\right)\right] - \mathbb{E}\left[\alpha_t x_t\right]^2 \\ &= \frac{1}{2}\left(0.1^2 + 0.15^2\right) - \left(\frac{1}{2} \times 0.1\right)^2 \\ &= 0.01375 \end{aligned}$$

   (c) Finally, what is the unconditional Sharpe ratio of this strategy?

Suggested Solution:

$$SR = \frac{0.05}{\sqrt{0.01375}} = 0.4264$$

The unconditional Sharpe Ratio considering market timing is about 10% higher than the Sharpe Ratio of the simple buy and hold case.

(d) Now, assume the volatility of x_t is higher; x_t it can take the values -5% and $+15\%$ with equal probability.

i. What is the implied R^2 of a forecasting regression of future excess returns on x_t assuming again that $\hat{\alpha} = 0$ and $\hat{\beta} = 1$?

Suggested Solution: The unconditional mean of x_t is $\mathbb{E}[x_t] = \frac{1}{2}(-5\% + 15\%) = 5\%$. The unconditional variance of x_t is $Var(x_t) = \frac{1}{2}\left[(-0.1)^2 + 0.1^2\right] = 0.01$. So, the unconditional volatility of x_t is now 10%, two times higher than before.

$$\begin{aligned} Var(R^e_{t+1}) &= \beta^2 Var(x_t) + Var(\epsilon_{t+1}) \\ &= 0.01 + 0.15^2 \\ &= 0.0325 \end{aligned}$$

So the R^2 of the forecasting regression is three times higher:

$$R^2 = \frac{\beta^2 Var(x_t)}{Var\left(R^e_{t+1}\right)} = 0.3077$$

ii. What is the unconditional Sharpe ratio the investor that follows the risky asset share rule given above in (4)? Note that a higher R^2 implies a higher Sharpe ratio.

Suggested Solution: The weight the investor chooses to hold in the risky asset

$$\alpha_t = \begin{cases} \dfrac{-0.05}{\frac{40}{9} \times 0.15^2} = -0.5 & \text{if } x_t = -5\% \\ \\ \dfrac{0.15}{\frac{40}{9} \times 0.15^2} = 1.5 & \text{if } x_t = 15\% \end{cases}$$

The unconditional average excess return for an investor that holds α_t each period is

$$\mathbb{E}(\alpha_t R^e_{t+1}) = \frac{1}{2}(-0.5 \times -5\% + 1.5 \times 15\%) = 12.5\%$$

The unconditional standard deviation is

$$
\begin{aligned}
Var(\alpha_t R^e_{t+1}) &= \mathbb{E}\left[\mathbb{E}_t\left[(\alpha_t R^e_{t+1})^2\right]\right] - \mathbb{E}\left[\mathbb{E}_t\left[\alpha_t R^e_{t+1}\right]\right]^2 \\
&= \mathbb{E}\left[\alpha_t^2 \mathbb{E}_t\left[(R^e_{t+1})^2\right]\right] - \mathbb{E}\left[\alpha_t \mathbb{E}_t\left[R^e_{t+1}\right]\right]^2 \\
&= \mathbb{E}\left[\alpha_t^2\left(x_t^2 + \sigma_t^2(\epsilon_{t+1})\right)\right] - \mathbb{E}\left[\alpha_t x_t\right]^2 \\
&= \frac{1}{2}\left[(-0.5 \times -0.05)^2 + (1.5 \times 0.15)^2 + (-0.5^2 + 1.5^2) \times 0.15^2\right] \\
&- \left(\frac{1}{2}(-0.5 \times -0.05 + 1.5 \times 0.15)\right)^2 \\
&= 0.038125
\end{aligned}
$$

The unconditional Sharpe ratio of this strategy is

$$
SR = \frac{0.125}{\sqrt{0.038125}} = 0.6402
$$

We see the Sharpe ratio is now much higher with higher R^2.