MFE 409 Lecture 5
Credit Risk

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UCLA Anderson

Lecture Objectives

Credit Risk

- Understand specificities of credit risk
- Three methods to measure credit risk
- Ingredients for dynamic models of credit risk

What is Special about Credit Risk?

What is Special about Credit Risk?

- Limited upside, but large downside

- Default is rare
  - Difficulty of estimating default probabilities from past data

  - Greater reliance on models than for market risk

  - Difficulty of estimating default correlations from past data (but loss is very sensitive to correlations)

Estimating Default Probabilities

Three main methods:

Estimating Default Probabilities

Three main methods:

1. Use historical data and credit ratings

2. Use CDS spreads

3. Use structural model: Merton model (KMV)

Outline

1. Historical Method

2. Using Credit Default Swaps

3. Structural Method

Credit Ratings

- Rating agencies assess the creditworthiness of corporate bonds

S&P	Moody's	(Fitch same as S&P)
AAA	Aaa	
AA	Aa	
A	A	
BBB	Baa	↑ investment grade
BB	Ba	↓ non-investment grade
B	B	junk bonds
CCC	Caa	
⋮	⋮	
Default	*Default*	

- Aversion to reversals: "through the cycle" (hence slow moving)
- Most banks have their own internal ratings systems for borrowers.
- Use a mix of accounting and model-based information

Altman's Z-Score

- Use historical data to understand link between default and accounting ratios
- Focus on five measures
  - X_1=Working Capital/Total Assets
  - X_2=Retained Earnings/Total Assets
  - X_3=EBIT/Total Assets
  - X_4=Market Value of Equity/Book Value of Liabilities
  - X_5=Sales/Total Assets

$$Z = 1.2 \times X_1 + 1.4 \times X_2 + 3.3 \times X_3 + 0.6 \times X_4 + 0.99 \times X_5$$

- If the $Z > 3.0$ default is unlikely; if $2.7 < Z < 3.0$ we should be on alert. If $1.8 < Z < 2.7$ there is a moderate chance of default; if $Z < 1.8$ there is a high chance of default

Historical Default Probabilities

- Moody's, 1970-2013

	Time (years)						
	1	2	3	4	5	7	10
Aaa	0.000	0.013	0.013	0.037	0.104	0.241	0.489
Aa	0.022	0.068	0.136	0.260	0.410	0.682	1.017
A	0.062	0.199	0.434	0.679	0.958	1.615	2.759
Baa	0.174	0.504	0.906	1.373	1.862	2.872	4.623
Ba	1.110	3.071	5.371	7.839	10.065	13.911	19.323
B	3.904	9.274	14.723	19.509	23.869	31.774	40.560
Caa	15.894	27.003	35.800	42.796	48.828	56.878	66.212

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- High ratings have *steep slope*, but low ratings have shallow slope

Rating Transitions

Initial rating	Rating at year-end (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	*0.02*	*0.33*	*5.95*	*86.93*	*5.30*	*1.17*	*1.12*	*0.18*
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0	0.22	1.30	2.38	11.24	64.86	19.79

- Transition matrix $\mathbb{T} = \{p_{ij}\}$
- Markov Chain over credit ratings
- n-year transition: $\mathbb{T}^n$

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- Transition matrix $\mathbb{T} = \{p_{ij}\}$
- Markov Chain over credit ratings
- n-year transition: $\mathbb{T}^n$
- Default is an absorbing state

Conditional Default Probability

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- Historical default table gives *unconditional default probabilities*
- What is the probability that a Caa bond defaults in year 3 conditional on surviving until the end of year 2?

P(A | B) = P(A, B) / P(B) = 8.8 / 73% = 12%

P(default in year 3 ~~AND survive until the end of year 2~~) = 35.8 - 27

P(surviving until the end of year 2) = 100 - 27 = 73%

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- Historical default table gives *unconditional default probabilities*
- What is the probability that a Caa bond defaults in year 3 conditional on surviving until the end of year 2?
  - Probability of surviving until the end of year 2: $1 - 27.003\% = 72.997\%$
  - Conditional probability: $(35.8\% - 27.003\%)/72.997\% = 12.05\%$

Default Intensity

- Note $V(t)$: probability of surviving up to t, $Q(t) = 1 - V(t)$: probability of default by time t

(on survival)

- **Default intensity or hazard rate** $\lambda(t)$: conditional probability of defaulting between t and $t + \Delta t$ is $\lambda(t)\Delta t$

$$\lambda(t)\Delta t = \frac{V(t) - V(t+\Delta t)}{V(t)} = \frac{Q(t+\Delta t) - Q(t)}{1 - Q(t)}$$

$$-\lambda(t)V(t) = \frac{dV(t)}{dt}$$

$$\lambda(t) = -\frac{\frac{dV}{dt}}{V} = -\frac{d\log(V)}{dt}$$

decay rate

= minus growth rate

$$\int_{t_1}^{t_2} \lambda(t)\,dt = \log(V(t_1)) - \log(V(t_2))$$

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- **Default intensity or hazard rate** $\lambda(t)$: conditional probability of defaulting between t and $t + \Delta t$ is $\lambda(t)\Delta t$

$$\lambda(t)\Delta t = \frac{V(t) - V(t + \Delta t)}{V(t)}$$

$$-\lambda(t)V(t) = \frac{dV(t)}{dt}$$

if λ is constant

$V(t) = e^{-\lambda t}$

$$V(t) = e^{-\int_0^t \lambda(\tau)d\tau}$$

$$Q(t) = 1 - e^{-\int_0^t \lambda(\tau)d\tau}$$

Default Intensity

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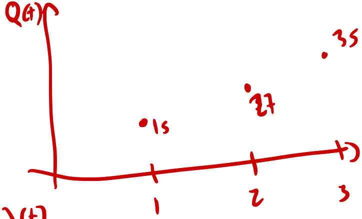
$$\lambda(t)\Delta t = \frac{V(t) - V(t + \Delta t)}{V(t)}$$

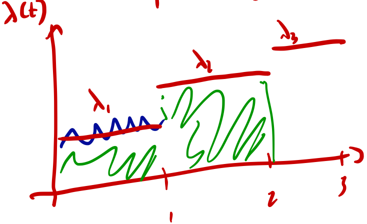
$$-\lambda(t)V(t) = \frac{dV(t)}{dt}$$

$$V(t) = e^{-\int_0^t \lambda(\tau)d\tau}$$

$$Q(t) = 1 - e^{-\int_0^t \lambda(\tau)d\tau}$$

- Construct $\lambda(t)$ for a Caa bond





$\forall x \in (0,1) : \lambda(x) = \lambda_1$

$$V(1) = e^{-\int_0^1 \lambda(t)\,dt}$$

$$= e^{-\int_0^1 \lambda_1\,dt}$$

$$= e^{-\lambda_1 \times 1}$$

$$= e$$

$$\lambda_1 = -\log(V_1)$$

$$V(2) = e^{-\int_0^2 \lambda(t)\,dt}$$

$$= e^{-\lambda_1 \times 1 - \lambda_2 \times 1}$$

$$= e$$

$$\lambda_2 = -\log(V(2)) + \log(V(1))$$

e.g.

$$V(1.5) - V(0.5) = e^{-\int_0^{1.5} \lambda(t)\,dt} - e^{-\int_0^{0.5} \lambda(t)\,dt}$$

$$= e^{-\lambda_1 - 0.5\lambda_2} - e^{-0.5\lambda_1}$$

Recovery Rate

- Bankruptcy process complicated
- Recovery rate: price at which bond trades about 30 days after default

Class	Mean(%)
Senior Secured	51.89
Senior Unsecured	36.69
Senior Subordinated	32.42
Subordinated	31.19
Junior Subordinated	23.95

distress funds

investors

Recovery Rate

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- Recovery rate negatively correlated with default rate

Outline

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2. Using Credit Default Swaps
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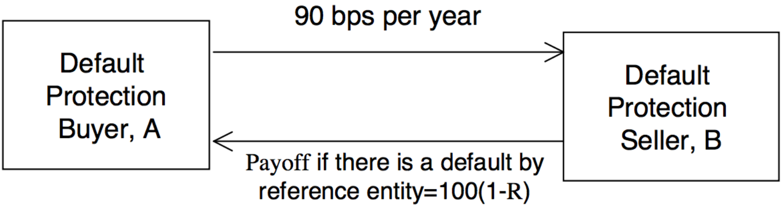
Credit Default Swaps



- R: recovery on the underlying bond

- 90bps: CDS spread

Credit Default Swaps

$$\text{CDS Spread} = \frac{\text{Total Amount Paid Per Year}}{\text{Notional Principal}}$$

- Five year maturity most common, often 1, 3, 5, 7, 10
- Payments are usually made quarterly in arrears (pay if no default occurred during the quarter, and...)
- In the event of default there is a final accrual payment by the buyer (payment made in the period in which default occurred)
- Settlement dates: March 20, June 20, Sept 20, Dec 20
- Often trade with fixed coupon upfront payment

Credit Default Swaps

- Settlement can be specified as
  - delivery of the bonds (physical settlement)
  - cash settlement (avoids scamble for underlying bond)
- Often use *cheapest-to-deliver bond*, price determined in an auction

Credit Default Swaps

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  - delivery of the bonds (physical settlement)
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- Often use *cheapest-to-deliver bond*, price determined in an auction
- CDS outstanding is many times cash bonds outstanding
  - Anyone can “issue” the bond of reference entity by selling protection

Estimating Default Probabilities from CDS

Approximate approach:

- Payoff to selling protection:

$$\Pi = \text{Payment per period} - \bar{\lambda} \times (1 - R) \times \text{Notional Principal}$$

- Assuming 0 profit:

$$\bar{\lambda} = \frac{\text{CDS Spread}}{1 - R}$$

Bootstrapping Default Probabilities from CDS

$r(t)$: instantaneous forward rate

- Term structure of CDS Spreads allows to recover default intensity using bootstrap method
- Use a more exact formula for CDS Spread:

$$\text{CDS Spread}(T) = (1-R)\frac{\int_0^T \lambda(\tau)e^{-\int_0^\tau r(u)+\lambda(u)du}d\tau}{\int_0^T e^{-\int_0^\tau r(u)+\lambda(u)du}d\tau}$$

- Use piecewise constant hazard rate $\lambda(t)$ to fit the various spreads

$$(1-R)\times \text{Not.}\times\int_0^T \lambda(\tau)\, e^{-\int_0^\tau \lambda(u)du}\, e^{-\int_0^\tau r(u)du}\, d\tau$$

$$= \text{CDS Spread}\times \text{Not.}\times\int_0^T e^{-\int_0^\tau \lambda(u)du}\, e^{-\int_0^\tau r(u)du}\, d\tau$$

Bootstrapping Default Probabilities from CDS

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- Use piecewise constant hazard rate $\lambda(t)$ to fit the various spreads
- Reconstitute hazard rate for 60% recovery, $r=0$, and:

Maturity	3yr	5yr	10yr
CDS Spread (bp)	50	60	100

$$CDS(3) = (1-R)\,\frac{\int_0^3 \lambda(\tau)\, e^{-\int_0^\tau \lambda(u)\,du}\, d\tau}{\int_0^3 e^{-\int_0^\tau \lambda(u)\,du}\, d\tau}$$

$$= (1-R)\,\frac{\int_0^3 \lambda_1\, e^{-\lambda_1 \tau}\, d\tau}{\int_0^3 e^{-\lambda_1 \tau}\, d\tau}$$

$$CDS(3) = (1-R)\,\lambda_1$$

$$\text{"TOP"} = \int_0^3 \lambda(\tau)\, e^{-\int_0^\tau \lambda(u)\,du}\, d\tau + \int_3^5 \lambda(\tau)\, e^{-\int_0^\tau \lambda(u)\,du}\, d\tau$$

$$= \int_0^3 \lambda_1 e^{-\lambda_1 \tau}\, d\tau + \int_3^5 \lambda_2\, e^{-\lambda_1 \times 3 - \lambda_2(\tau - 3)}\, d\tau$$

$$= \lambda_1 \int_0^3 e^{-\lambda_1 \tau} + \lambda_2\, e^{-3\lambda_1} \int_3^5 e^{-\lambda_2(\tau-3)}\, d\tau$$

Comparing Hazard Rates

- Historical 7-year hazard rate from Moody's data (1970-2013)
- Implied 7-year default intensities from bond prices, Merrill Lynch data (1996-2007)

Rating	Historical Hazard Rate (% per annum)	Hazard Rate from bonds (% per annum)	Ratio	Difference
Aaa	0.034	0.596	17.3	0.561
Aa	0.098	0.728	7.4	0.630
A	0.233	1.145	5.8	0.912
Baa	0.416	2.126	5.1	1.709
Ba	2.140	4.671	2.2	2.531
B	5.462	8.017	1.5	2.555
Caa	12.016	18.395	1.5	6.379

Outline

1. Historical Method
2. Using Credit Default Swaps
3. Structural Method

Merton Model

- **Key idea**: equity is a call option on asset value

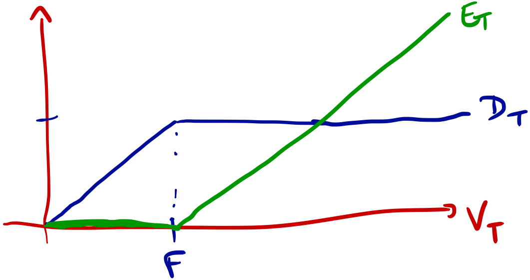
Assets	Liabilities
Assets	Debt
	Equity

Merton Model

- **Key idea**: equity is a call option on asset value

$$E_T = \max(0, V_T - F)$$

$$D_T = \min(V_T, F) = F - \max(0, F - V_T)$$



$$E_t = E_t^Q\left(e^{-r(T-t)} E_T\right)$$

- Single zero-coupon debt instrument with face value F, maturing at date T
  - Default only possible at date T
- D_t: market value of debt at date t
- V_t: market value of assets
- E_t: market value of equity

Model of Asset Value

- Assume market value of assets V_t follows a *geometric Brownian motion*

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t$$

- Log normal distribution for asset value:

$$V_t = V_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\sqrt{t} \underbrace{Z}_{\mathcal{N}(0,1)}\right]$$

Distance to Default

- Firm defaults at date T on its debt when $V_T < F$
- Probability of default (viewed from date 0) is:

$$\mathbb{P}\left(V_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\, Z \right) < F \right)$$

$$= \mathbb{P}\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\, Z < \log(F/V_0) \right)$$

$$= \mathbb{P}\left(Z < \frac{\log(F/V_0) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right)$$

$$= \mathcal{N}\left(\frac{\log(F/V_0) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right)$$

Distance to Default

- Firm defaults at date T on its debt when $V_T < F$
- Probability of default (viewed from date 0) is:

$$
\begin{aligned}
\mathbb{P}[V_T < F] &= \mathbb{P}\left[V_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z\right) < F\right] \\
&= \mathbb{P}\left[Z < -\underbrace{\frac{\ln(V_0/F) + (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}_{d}\right] \\
&= \mathcal{N}(-d)
\end{aligned}
$$

- d is the *distance do default*
  - Number of standard deviations away from the default point

Empirical Implementation

- Two main parameters: V_0, σ

- If debt and equity are both traded:

$V_0 = D_0 + E_0$

$\sigma =$. construct $V_t = D_t + E_t$

. compute its volatility

$\hookrightarrow \text{std}\left(\frac{V_{t+1} - V_t}{V_t}\right)$

Empirical Implementation

- Two main parameters: V_0, σ

- If debt and equity are both traded:
  - V_0 is sum of market value of debt and market value of equity
  - σ is volatility of V_t, can use historical data

- If only equity is available ...

Linking Equity and Asset Values

- Equity at date T: pays off only if debt is covered

$$E_T = \max(V_T - F, 0)$$

- Market value at date 0: price of a call option with strike F

$$E_0 = V_0 \mathcal{N}(d_1) - F e^{-rT} \mathcal{N}(d_2)$$

$$d_1 = \frac{\ln(V_0/F) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(V_0/F) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Obtaining Asset Volatility from Equity Volatility

$$\sigma_E = \cdots \cdots \sigma$$

$$E_t = C_{BS}(V_t)$$

$$dE_t = \cdots dt + \frac{\partial C_{BS}}{\partial V_t} dV_t$$

$$= \cdots dt + \frac{\partial C_{BS}}{\partial V_t} \sigma V_t dW_t$$

$$\frac{dE_t}{E_t} = \cdots dt + \frac{\partial C_{BS}}{\partial V_t} \sigma \frac{V_t}{E_t} dW_t$$

$$\sigma_E = \overbrace{\frac{\partial C_{BS}}{\partial V_t}}^{\Delta} \sigma \frac{V_t}{E_t} = \Delta \sigma \frac{V_t}{E_t} = N(d_1) \sigma \frac{V_t}{E_t}$$

Call

$$dE_t = \Delta dV_t$$

$$std(dE_t) = \Delta\, std(dV_t)$$

$$std\left(\frac{dE_t}{E_t}\right) \times E_t$$

$$= \Delta\, std\left(\frac{dV_t}{V_t}\right) V_t$$

$$\sigma_E E_t = \Delta \sigma V_t$$

Obtaining Asset Volatility from Equity Volatility

- Variations in equity reveal variations in underlying

$$\sigma_E E_0 = \underbrace{\frac{\partial E}{\partial V}}_{\Delta} \sigma V_0$$

$$= \mathcal{N}(d_1)\sigma V_0$$

- 2 equations in 2 unknown to solve for V_0 and σ

Example

- A company's equity is \$3 million and the volatility of the equity is 80%
- The face value of debt is \$10 million and time to debt maturity is 1 year
- The risk-free rate is 5%
- What is the distance to default, the probability of default?

- Estimate σ and V_0
  - $\sigma, V_0 \xrightarrow{f} C_{BS}(\sigma, V_0)$
    $\longrightarrow \sigma N(d_1) \frac{V}{E}$
  - solve $f(\sigma, V_0) = (E_0, \sigma_E)$
- Compute d and $N(-d)$

Example

- A company's equity is \$3 million and the volatility of the equity is 80%
- The face value of debt is \$10 million and time to debt maturity is 1 year
- The risk-free rate is 5%
- What is the distance to default, the probability of default?
- $V_0 = 12.40$, $\sigma = 21.23\%$
- $d_2 = 1.1408$, probability of default is $\mathcal{N}(d_2) = 12.7\%$

Example (Continued)

- What is the expected recovery rate on the debt?

Example (Continued)

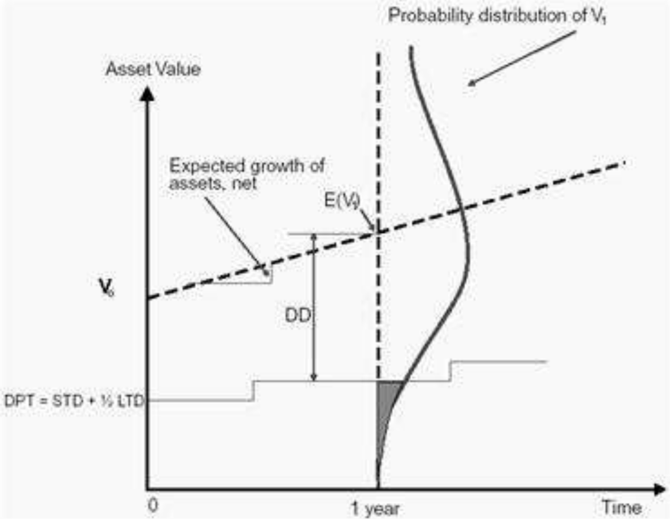
- What is the expected recovery rate on the debt?
- Present value of face value: 9.51
- Market value of debt: 9.40
- Expected loss: 1.2%
- Expected loss = probability of defaul $\times$ (1 - recovery rate)
- Recovery rate = 1 - 1.2% / 12.7% = 91%

Issues

- Equity volatility may be affected by short term factors in equity market rather than fundamentals
  - Liquidity, microstructure effects
- Default is a complex phenomenon
  - Complex capital structure
  - Coordination problems
  - Bankruptcy choice
  - Bargaining between creditors and management
- Merton model gives reasonable *ordinal ranking* of default risk, but the simple version of model does poor job of matching *cardinal default risk* - the actual probabilities of default

KMV Approach

- KMV (now part of Moody's) modify Merton model to bring historical default frequencies closer to the model



KMV Approach

- The default value of assets is not F, but

$$\hat{F} = \mathsf{STD} + \frac{1}{2} \times \mathsf{LTD}$$

  - STD is short-term debt, LTD is long-term debt
- The distance to default DD is approximated by

$$\mathsf{DD} = \frac{\mathbb{E}(V) - \hat{F}}{\sigma}$$

- The probability of default is not $\mathcal{N}(-\mathsf{DD})$
- Instead, estimate empirical relationship between DD and expected frequency of default (EFD)
- KMV provides "Credit Monitor" of estimated EFDs

Takeaways

- Difficult to use directly historical data for credit
- Use historical data for other firms + credit ratings, characteristics
- Use prices to know what the market thinks: CDS, bond prices, ...
- Model the default process, estimate parameters using properties of observable prices
- Issue: Difficult to go from risk-neutral to actual probabilities