Fixed Income Week 6 Notes*

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Homework 5

Hints for the homework:

- For this homework you will be implementing the two-factor Vasicek model. This is a variation of the N2 model that was presented, in which the two independent processes are Vasicek processes. To summarize the model you saw in class:
- Let X and Y be two factors (unobservable) that you believe drive the term-structure. You could say that X represents inflation and Y represents the short rate. In fact these could be any factors that you choose.
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- $dX = (\alpha_x - \beta_x X)dt + \sigma_x dZ_x$
- $dY = (\alpha_y - \beta_y Y)dt + \sigma_y dZ_y$
- $corr(dZ_x, dZ_y) = 0$
- $r_s = X_s + Y_s$
- $D(s,T) = \mathbb{E}[exp(-\int X_s ds)exp(-\int Y_s ds)]$
- $D(s,T) = A_X(T)A_Y(T)exp(-B_X(T)X_s - B_Y(T)Y_s)$
- $A_i(T) = exp\big[(\frac{\sigma_i^2}{2\beta_i^2} - \frac{\alpha_i}{\beta_i})T + (\frac{\alpha_i}{\beta_i^2} - \frac{\sigma_i^2}{\beta_i^3})(1 - exp(-\beta_i T)) + \frac{\sigma_i^2}{4\beta_i^3}(1 - exp(-2\beta_i T))\big]$
- $B_i(T) = \frac{1}{\beta_i}(1 - exp(\beta_i T))$

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- $i = X, Y$
- $ytm(s,T) = \frac{-ln(A_X(T))}{T} + \frac{B_X(T)}{T}X_s - \frac{-ln(A_Y(T))}{T} + \frac{B_Y(T)}{T}Y_s$
- Note that the yield to maturity computed above is for zero coupon bonds with maturity T and time t

Solution steps in words

- The data that we sent you contains both CMS data and CMT data. For this homework we will focus only on the CMT data.
- We will begin the process by assuming any starting values for the parameters α_X, β_X, σ_X, β_Y, σ_Y
- As suggested in the homework, set the value of $\alpha_Y = 0$
- In the first pass we will only use the values of $CMT(0.25)$ and $CMT(10)$ for each date.
- Given your assumed values of the parameters you can solve for $A_X(0.25)$,$B_X(0.25)$,$A_Y(0.25)$,$B_Y(0.$
- You can also solve for$A_X(10)$,$B_X(10)$,$A_Y(10)$,$B_Y(10)$
- Given these values of the constants, and the ytm on the bonds with maturity $T =$ 0.25and $T = 10$, you can back out the value of$X$and $Y$for each date
- But there is a problem, the ytm formula given above applies to only zero-coupon bonds where as the data for $T = 0.25$ and $T = 10$ is given for coupon paying bonds. Here you make a simplifying assumption that the ytm on the bonds with these two maturities are the same for the coupon paying and zero coupon bonds. Turns out that this assumption does not make too much of a difference in the solution of our problem
- Now given this information, you can solve for the values of X and Y for each date in the data-set by solving the following simultaneous equation: (I do this for the first line in the data-set
- $CMT(0.25) = 0.0675 = \frac{-ln(A_X(0.25))}{0.25} + \frac{B_X(0.25)}{0.25}X_s - \frac{-ln(A_Y(0.25))}{0.25} + \frac{B_Y(0.25)}{0.25}Y_s$
- $CMT(10) = 0.0897 = \frac{-ln(A_X(10))}{10} + \frac{B_X(10)}{10}X_s - \frac{-ln(A_Y(10))}{0.25} + \frac{B_Y(10)}{10}Y_s$

- You can do this now for each date in the data-set. At the end of this process you should have a time-series for X and Y for each date (just like the CMT rates for bonds with maturity $T = 2 - 7$ for each date)

- Now given the value of the X and$Y$and the parameters, you can solve for the $D(T)$ function for $T = 0.5, 1.0, 1.5...10$ for each date in the data- set. For a given date the value of $X$and $Y$is as you calculated above and the values of the constants $A_i(T)$, $B_i(T)$for $i = X, Y$ will change depending on T. At the end of this step you have the $D(T)$ function for maturities $0.5 - 10$ for each date in the data-set.

- Given the $D(T)$ function you can compute the par rates on bonds with maturities $T = 2, 3, 5, 7$. Thus, for each date you now have computed yields for bonds with these maturities

- As a final step compare the computed yields to the given yields. To do this compute the root-mean-squared-error. That is take the difference of computed and given yields for each maturity; square the error; add them up; take square-root; divide by 4.

- Once you have this value go back to step 1 and change the values of the selected parameters so that you can get a lower value of the RMSE. The objective is to pick the parameters that give you the lowest RMSE

- You can get any of the optimization functions in matlab, R etc. to do the looping described in the last step for you. To do this:

- First define a function that for a given value of the parameters computes the: constants for $T = 0.25$and $T = 10$; Backs out the value of X and Y for all dates; Uses these values to compute the $D(T)$ function for all dates for $T = 0.5, 1.0, 1.5...10$; Given these values of $D(T)$ computes the par rates; Computes the root mean squared error of the given and computed par rates. This function should return the value of the error

- Then use a MATLAB optimization function such as 'fminunc', 'lsqnon- lin', etc. A function like 'fminunc' will take in the initial parameters and the objective function described in the previous step. And will continue looping over it till it achieves or determines the parameters that provide it with the global minimum value of the RMSE error.

- At the end of all of this you should have solved for the parameters and the values of X and Y for each date such that the model prices and the actual prices are as close to each other as possible

Question 5

- At the end of the above process you have the time series of X and Y. Compute the mean and standard deviation of this time series and compare that to the mean and standard deviation using the model parameters. In order to compute the mean and standard deviation from model parameters assume that $T \to \infty$. Thus:
- $\mathbb{E}[X] = \frac{\alpha_X}{\beta_X}$
- $\mathbb{V}[X] = \frac{\sigma_X^2}{2\beta_X}$

Question 6

- Compute the difference between the given and computed par rates for each maturity. You can plot the deviations for each maturity and comment on the time-series properties. Alternatively, fit an AR model on the deviations and comment on the parameters and its values and what they imply.

Question 7

- For question 7, we want to compare hedging approaches:
- The value of your portfolio is given by:
- $V = P_T + N_1 P_2 + N_{10} P_{10}$
- In the first approach you want to compute N_2 and N_{10} such that the duration and convexity of your portfolio is zero:
- $P_T MD_T + N_1 P_2 MD_2 + N_{10} P_{10} MD_{10} = 0$
- $P_T C_T + N_1 P_2 C_2 + N_{10} P_{10} C_{10} = 0$
- In the second approach you want to compute N2 and N10 such that the derivative with respect to X and Y of your portfolio is zero:
- $\frac{d}{dX} P_T + N_1 \frac{d}{dX} P_2 + N_{10} \frac{d}{dX} P_{10} = 0$

- $\frac{d}{dY}P_T + N_1\frac{d}{dY}P_2 + N_{10}\frac{d}{dY}P_{10} = 0$
- You can compute the derivatives in two ways, either use:
- $\frac{d}{dX}P_T = \frac{C(T)}{2}\sum_{i=1}^{2T}\frac{d}{dX}D(\frac{i}{2}) + \frac{d}{dX}D(T)$
- $\frac{d}{dX}P_T = -\frac{C(T)}{2}\sum_{i=1}^{2T}D(\frac{i}{2})B_X(\frac{i}{2}) - D(T)B_X(T)$
- Or just compute a small change in the value of X and Y and compute the impact on the price (like the DV01 approach).