UCLA Anderson **MGMTMFE403**

Fall 2017

Intro to Stochastic Calculus Quiz 2 Solutions

Problem 1. Exchange rates (30 Points)
The US-Euro exchange rate X_t follows the dynamics

$$dX_t = \mu_X X_t dt + \sigma_X X_t d\overline{W}_t$$

where μ_X, σ_X are constants and $\overline{W}_t$ is a standard Brownian motion. The dollar interest rate is constant and equal to r^d and the Euro interest rate is constant and equal to r^f. Suppose that we wish to obtain the arbitrage-free price of a derivative with payoff

$$\Phi(X_T) = \log(X_T)$$

at time T. Compute the price of such a derivative security at time 0 as a function of X_0.

Solutions to question 1

Solution: Under the risk neutral measure, the exchange rate follows the dynamics

$$dX_t = \left(r^d - r^f\right) X_t dt + \sigma_X X_t dW_t$$

Applying Ito's Lemma gives

$$d(\log(X_t)) = \left[r^d - r^f - \frac{\sigma_X^2}{2}\right] dt + \sigma_X dW_t$$

or

$$\log(X_T) = \log(X_0) + \left[r^d - r^f - \frac{\sigma_X^2}{2}\right] T + \sigma_X W_T.$$

Taking expectations on both sides leads to

$$E_0^Q \log(X_T) = \log(X_0) + \left[r^d - r^f - \frac{\sigma_X^2}{2}\right] T$$

Therefore the price of the derivative is

$$\Pi(X_0, 0) = e^{-rT}\left\{\log(X_0) + \left[r^d - r^f - \frac{\sigma_X^2}{2}\right] T\right\}.$$

Problem 2. Delta Hedeging (30 Points)

Suppose that the price of a stock at time t is equal to $S_t = 100$, and the price of a European Call option is equal to $C_t = 10$. The Delta of the option is $N(d_1) = 0.2$.

a) How may shares of stocks should you be holding in a portfolio aimed to replicate the payoff of the option at time T? (This could be a fraction.)

b) How many dollars should you be borrowing in the replicating portfolio?

c) Suppose that

$$\frac{dS_t}{S_t} = 0.1dt + 0.25d\overline{W}_t.$$

Moreover, suppose that $r = 0.02$. Fill in the missing information in the dots bellow

$$\frac{dC_t}{C_t} = ...dt + ...d\overline{W}_t.$$

(Hint: The Call option's replicating portfolio consists of the stock and the bond. Compute the fraction of that portfolio that is invested in stocks and the fraction invested in bonds.)

Solutions to question 2

Solution: a) You should be holding $N(d_1) = 0.2$ units of the stock.

b) The dollar amount invested in stocks is given by the value of the call option (10) minus the dollar value invested in stocks ($0.2 \times 100 = 20$) in the replicating portfolio. Therefore one would have to borrow 10 dollars.

c) To answer this question, we start by computing the fraction of the replicating portfolio invested in stocks $u^S = \frac{20}{15} = 2$, which means that the fraction invested in bonds is $u^0 = -1$. Therefore the dynamics of the replicating portfolio is given by

$$\begin{aligned}
\frac{dC_t}{C_t} &= \frac{dV_t^h}{V_t^h} = \left(u^0 r + u^S 0.1\right) dt + u^S 0.25 d\overline{W}_t \\
&= (-0.02 + 0.2)\, dt + 0.5 d\overline{W}_t \\
&= 0.18 dt + 0.5 d\overline{W}_t.
\end{aligned}$$

Problem 3. A "ratio" forward. (30 Points)

Suppose that the interest rate r is constant. Take two stocks $S_t^{(1)}$ and $S_t^{(2)}$ with dynamics

$$\begin{aligned}
dS_t^{(1)} &= \mu_1 S_t^{(1)} dt + \sigma_1 S_t^{(1)} d\overline{W}_t^{(1)} \\
dS_t^{(2)} &= \mu_2 S_t^{(2)} dt + \sigma_2 S_t^{(2)} d\overline{W}_t^{(1)}
\end{aligned}$$

Note that both stocks are affected by the same brownian motion.

a) Define:

$$z_t = \frac{S_t^{(1)}}{S_t^{(2)}}$$

Use Ito's Lemma to derive the dynamics of z_t under the risk neutral measure Q, i.e. fill the gaps in the expression below

$$dz_t = ...dt + ...dW_t^{(1)}.$$

Careful: First find the dynamics of $S_t^{(1)}$ and $S_t^{(2)}$ under the risk neutral measure Q and then use Ito's Lemma to find the dynamics of z_t under Q.

b) Compute the futures price for z_t, i.e., determine K so that

$$E_0^Q (z_T - K) = 0$$

Solutions to question 3 Solutions to question 3

Solution: a) Ito's Lemma implies that

$$\begin{aligned}
dz_t &= d\left(\frac{S_t^{(1)}}{S_t^{(2)}}\right) = \frac{dS_t^{(1)}}{S_t^{(2)}} + S_t^{(1)} d\left(\frac{1}{S_t^{(2)}}\right) + dS_t^{(1)} d\left(\frac{1}{S_t^{(2)}}\right) \\
&= r\frac{S_t^{(1)}}{S_t^{(2)}} dt + \sigma_1 \frac{S_t^{(1)}}{S_t^{(2)}} d\overline{W}_t^{(1)} \\
&\quad + S_t^{(1)} \left\{ -\frac{1}{\left(S_t^{(2)}\right)^2} \left[rS_t^{(2)} dt + \sigma_2 S_t^{(2)} d\overline{W}_t^{(1)} \right] + \frac{1}{\left(S_t^{(2)}\right)^3} \sigma_2^2 \left(S_t^{(2)}\right)^2 dt \right\} \\
&\quad - \sigma_1 \sigma_2 \frac{S_t^{(1)} S_t^{(2)}}{\left(S_t^{(2)}\right)^2} dt \\
&= z_t \left[\sigma_2^2 - \sigma_1 \sigma_2\right] dt + z_t \left(\sigma_1 - \sigma_2\right) d\overline{W}_t^{(1)}.
\end{aligned}$$

b) Since z_t is log-normal, we know that

$$E^Q\left(z_T\right) = z_0 e^{\left(\sigma_2^2 - \sigma_1 \sigma_2\right) T}.$$

Therefore

$$K = z_0 e^{\left(\sigma_2^2 - \sigma_1 \sigma_2\right) T}.$$