Lecture 5
ARMA Models

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Outline

Autoregressive Models

ARMA Models

- **parsimonious** description of (univariate) time series (mimicking autocorrelation etc.)
- very useful tools for forecasting (and commonly used in industry)
  - forecasting sales, earnings revenue growth at the firm level or at the industry level
  - forecasting GPD growth, inflation at the national level

Autoregressive process of order 1

- lagged returns might be useful in predicting returns.
- we consider a model that allows for this:

$$r_{t+1} = \phi_0 + \phi_1 r_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathrm{WN}(0, \sigma_\varepsilon^2)$$

  - $\{\varepsilon_t\}$ represents the 'news':

$$\varepsilon_t = r_t - E_{t-1}[r_t]$$

    ε_t is what you know about the process at t but not at $t-1$
  - Economists often call ε_t the 'shocks' or 'innovations'.
- this model is referred to as an **AR(1)**

Transition density

Definition

Given an information set $\mathcal{F}_t$, the **transition density** of a random variable r_{t+1} is the conditional distribution of r_{t+1} given by:

$$r_{t+1} \sim p(r_{t+1} | \mathcal{F}_t; \boldsymbol{\theta})$$

- The information set $\mathcal{F}_t$ is often (but not always) the history of the process $r_t, r_{t-1}, r_{t-2}, \ldots$
- In this case, the transition density is written:

$$r_{t+1} \sim p(r_{t+1} | r_t, r_{t-1}, \ldots, ; \boldsymbol{\theta})$$

- A transition density is **Markov** if it depends on its finite past.

AR(1) transition density

- Consider the AR(1) model with Gaussian shocks

$$r_{t+1} = \phi_0 + \phi_1 r_t + \varepsilon_{t+1}, \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2)$$

- The transition density is **Markov of order 1**.

$$r_{t+1} \sim p(r_{t+1} | r_t; \boldsymbol{\theta})$$

  the rest of the history $r_{t-2}, r_{t-3}, \ldots$ is irrelevant.

- With Gaussian shocks ε_t, the transition density is:

$$r_{t+1} \;\sim\; \mathrm{N}(\phi_0 + \phi_1 r_t, \sigma_\varepsilon^2)$$

- conditional mean and conditional variance:

$$\begin{aligned} E\left[r_{t+1} | r_t\right] &= \phi_0 + \phi_1 r_t, \\ V\left[r_{t+1} | r_t\right] &= V\left[\varepsilon_{t+1}\right] = \sigma_\varepsilon^2. \end{aligned}$$

Unconditional mean of AR(1)

- assume that the series is covariance-stationary
- compute the unconditional mean μ.
  - take unconditional expectations:

$$E\left[r_{t+1}\right] = \phi_0 + \phi_1 E\left[r_t\right].$$

  - use stationarity: $E\left[r_{t+1}\right] = E\left[r_t\right] = \mu$:

$$\mu = \phi_0 + \phi_1 \mu,$$

    and solving for the unconditional mean:

$$\mu = \frac{\phi_0}{1 - \phi_1}.$$

- mean exists if $\phi_1 \neq 1$ and is zero if $\phi_0 = 0$

Mean Reversion

- if $\phi_1 \neq 1$, we can rewrite the AR(1) process as:

$$r_{t+1} - \mu = \phi_1 \left(r_t - \mu \right) + \varepsilon_{t+1}.$$

- suppose $0 < \phi_1 < 1$
  - when $r_t > \mu$, the process is expected to get **closer** to the mean:

$$E_t[r_{t+1} - \mu] = \phi_1 \left(r_t - \mu \right) < \left(r_t - \mu \right).$$

  - when $r_t < \mu$, the process is expected to get **closer** to the mean:

$$E_t[r_{t+1} - \mu] = \phi_1 \left(r_t - \mu \right) > \left(r_t - \mu \right).$$

- the smaller ϕ_1, the higher the speed of mean reversion

Mean Reversion

- we can rewrite the AR(1) process as:

$$r_{t+2} - \mu = \phi_1^2 (r_t - \mu) + \phi_1 \varepsilon_{t+1} + \varepsilon_{t+2}.$$

- suppose $0 < \phi_1 < 1$
  - when $r_t > \mu$, the process is expected to get **closer** to the mean:

$$E_t[r_{t+2} - \mu] = \phi_1^2 (r_t - \mu) < (r_t - \mu).$$

  - when $r_t < \mu$, the process is expected to get **closer** to the mean:

$$E_t[r_{t+2} - \mu] = \phi_1^2 (r_t - \mu) > (r_t - \mu).$$

Half Life

- we can rewrite the AR(1) process as:

$$r_{t+h} - \mu = \phi^h (r_t - \mu) + \phi^{h-1} \varepsilon_{t+1} + \ldots + \varepsilon_{t+h}.$$

- suppose $0 < \phi_1 < 1$
  - at the **half-life**, the process is expected to cover **1/2** of the distance to the mean:

$$E_t[r_{t+h} - \mu] = \phi_1^h (r_t - \mu) = .5 (r_t - \mu).$$

- the half-life is defined by setting $\phi_1^h = 0.5$ and solving

$$h = \log(0.5) / \log(\phi_1)$$

Variance of AR(1)

Compute the unconditional variance:

- take the expectation of the square of :

$$r_{t+1} - \mu = \phi_1 (r_t - \mu) + \varepsilon_{t+1}.$$

- we obtain the following expression for the unconditional variance:

$$V[r_{t+1}] = \frac{\sigma_\varepsilon^2}{1 - \phi_1^2},$$

  provided that $\phi_1^2 < 1$ because the variance has to be positive and bounded

- covariance stationarity requires that

$$-1 < \phi_1 < 1.$$

- in addition, if $-1 < \phi_1 < 1$, we can show that the series is covariance stationary because the mean and variance are finite

Continuous-Time Model

Definition

In a continuous-time model, the log of stock prices, $p_t = \log P_t$, follows an **Ornstein-Uhlenbeck process** if:

$$dp_t = \kappa(\mu_p - p_t)dt + \sigma_p dB_t \tag{1}$$

Continuous-time version of a discrete-time, Gaussian AR(1) process.

Suppose we observe the process (1) at discrete intervals Δt, then this is equivalent to:

$$p_t = \mu + \phi_1(p_{t-1} - \mu) + \sigma\varepsilon_t \qquad \varepsilon_t \sim \mathrm{N}(0,1)$$

where

- $\phi_1 = \exp(-\kappa\Delta t)$
- $\mu = \mu_p$
- $\sigma^2 = (1 - \exp(-2\kappa\Delta t))\frac{\sigma_p^2}{2\kappa}$.

Dynamic Multipliers

- use the expression for the mean of the AR(1) to obtain:

$$r_{t+1} - \mu = \phi_1 \left(r_t - \mu \right) + \varepsilon_{t+1}.$$

- by repeated substitution, we get:

$$r_t - \mu = \sum_{i=0}^{t} \phi_1^i \varepsilon_{t-i} + \phi^{t+1} \left(r_{-1} - \mu \right).$$

- value of r_t at t is stated as a function of the **history of shocks** $\left\{ \varepsilon_\tau \right\}_{\tau=0}^{\tau=t}$ and its value at time $t = -1$
- effect of shocks die out over time provided that $-1 < \phi_1 < 1$.

Dynamic Multipliers

calculate the effect of a change ε_0 on r_t:

$$\frac{\partial[r_t - \mu]}{\partial \varepsilon_0} = \phi_1^t.$$

$$\frac{\partial[r_{t+j} - \mu]}{\partial \varepsilon_t} = \phi_1^j.$$

in a covariance stationary model, *dynamic multiplier* only depends on j, not on t

Again, note that we need $|\phi_1| < 1$ for a stationary (non-explosive) system where shocks die out: $\lim_{j\to\infty} \phi_1^j = 0$

MA(infinity) representation

- use the expression for the mean of the AR(1) to obtain:

$$r_{t+1} - \mu = \phi_1 \left(r_t - \mu \right) + \varepsilon_{t+1}.$$

- by repeated substitution:

$$r_t - \mu = \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}.$$

  - **linear** function of past innovations!
  - fits into class of linear time series

Autocovariances of an AR(1)

- take the unconditional expectation:

$$(r_t - \mu)\left(r_{t-j} - \mu\right) = \phi_1 \left(r_{t-1} - \mu\right)\left(r_{t-j} - \mu\right) + \varepsilon_t \left(r_{t-j} - \mu\right).$$

- this yields:

$$E\left[(r_t - \mu)\left(r_{t-j} - \mu\right)\right] = \phi_1 E\left[\left(r_{t-1} - \mu\right)\left(r_{t-j} - \mu\right)\right] + E\left[\varepsilon_t \left(r_{t-j} - \mu\right)\right].$$

- or, using notation from Lecture 9:

$$\begin{aligned} \gamma_j &= \phi_1 \gamma_{j-1}, \qquad j > 0 \\ \gamma_0 &= \phi_1 \gamma_{-1} + \sigma_\varepsilon^2, \qquad j = 0 \end{aligned}$$

- note that $\gamma_{-j} = \gamma_j$

Autocorrelation Function

- it immediately implies that the ACF is:

$$\rho_j = \phi_1 \rho_{j-1}, \qquad j \geq 0$$

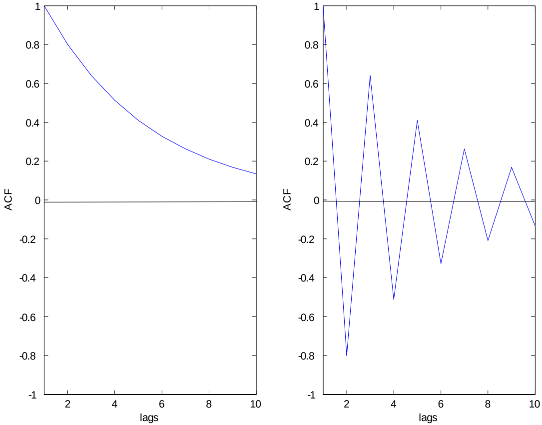
and $\rho_0 = 1$

- combing these two equations imply that:

$$\rho_j = \phi_1^j$$

  - exponential decay at a rate ϕ_1

Autocorrelation Function of an AR(1)



Autocorrelation Function for AR(1). The left panel considers $\phi_1 = 0.8$. The right panel considers $\phi_1 = -0.8$.

AR(p)

Definition

The $\mathbf{AR}(p)$ model is defined as:
$r_t = \phi_0 + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{WN}\left(0, \sigma_\varepsilon^2\right)$

- other lagged returns might be useful in predicting returns
- similar to multiple regression model with p lagged variables as explanatory variables
- the $\mathbf{AR}(p)$ is **Markov of order p**.

Conditional Moments

- conditional mean and conditional variance:

$$
\begin{aligned}
E\left[r_{t+1}|r_t,...,r_{t-p+1}\right] &= \phi_0+\phi_1 r_t+\ldots+\phi_p r_{t-p+1} \\
V\left[r_{t+1}|r_t,...,r_{t-p+1}\right] &= V\left[\varepsilon_{t+1}\right]=\sigma_\varepsilon^2
\end{aligned}
$$

- moments conditional on $r_t,\ldots,r_{t-p+1}$ are not correlated with $r_{t-i}, i\geq p$

AR(2)

- consider the model:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t \qquad \varepsilon_t \sim \text{WN}\left(0, \sigma_\varepsilon^2\right)$$

- take unconditional expectations to compute the mean

$$E\left[r_t\right] = \phi_0 + \phi_1 E\left[r_{t-1}\right] + \phi_2 E\left[r_{t-2}\right]$$

- Assuming stationarity and solving for the mean:

$$E\left[r_t\right] = \mu = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

  provided that $\phi_1 + \phi_2 \neq 1$.

- using this expression for μ write the model in deviation from means:

$$r_t - \mu = \phi_1\left(r_{t-1} - \mu\right) + \phi_2\left(r_{t-2} - \mu\right) + \varepsilon_t$$

Autocorrelations of an AR(2)

- take the expectation of :

$$
\begin{aligned}
(r_t-\mu)(r_{t-j}-\mu) &= \phi_1(r_{t-1}-\mu)(r_{t-j}-\mu) \\
&\quad +\phi_2(r_{t-2}-\mu)(r_{t-j}-\mu)+\varepsilon_t(r_{t-j}-\mu)
\end{aligned}
$$

- this yields:

$$
\begin{aligned}
E\left[(r_t-\mu)(r_{t-j}-\mu)\right] &= \phi_1 E[(r_{t-1}-\mu)(r_{t-j}-\mu)] \\
&\quad + \phi_2 E[(r_{t-2}-\mu)(r_{t-j}-\mu)] \\
&\quad + E\left[\varepsilon_t(r_{t-j}-\mu)\right]
\end{aligned}
$$

- or, using different notation:

$$
\begin{aligned}
\gamma_j &= \phi_1\gamma_{j-1}+\phi_2\gamma_{j-2}, \qquad & j>0 \\
\gamma_0 &= \phi_1\gamma_{-1}+\phi_2\gamma_{-2}+\sigma_\varepsilon^2, \qquad & j=0
\end{aligned}
$$

Autocorrelations of an AR(2)

- the ACF:

$$\begin{aligned}
\rho_j &= \phi_1\rho_{j-1} + \phi_2\rho_{j-2}, & j \geq 2 \\
\rho_0 &= \phi_1\rho_{-1} + \phi_1\rho_{-2} + \sigma_\varepsilon^2/\gamma_0, & j = 0
\end{aligned}$$

  which implies that the ACF of an AR(2) satisfies a second-order difference equation:

$$\begin{aligned}
\rho_1 &= \phi_1\rho_0 + \phi_2\rho_1 \\
\rho_j &= \phi_1\rho_{j-1} + \phi_2\rho_{j-2}, & j \geq 2
\end{aligned}$$

Roots

Definition

The second-order difference equation for the ACF:

$$(1 - \phi_1 B - \phi_2 B^2)\rho_j = 0,$$

where B is the **back-shift operator**: $B\rho_j = \rho_{j-1}$

Note that we can write the above as:

$$(1 - \varpi_1 B)(1 - \varpi_2 B)\rho_j = 0$$

- A useful factorization
- Intuitively, the AR(2) is an "AR(1) on top of another AR(1)"
- From AR(1) math, we had that each AR(1) is stationary if its autocorrelation is less than one in absolute value.
- The 'roots' ϖ_j should satisfy similar property for AR(2) to be stationary

Finding the roots

A simple case:

$$\begin{aligned} 1-\phi_1 B-\phi_2 B^2 &= (1-\varpi_1 B)(1-\varpi_2 B) \\ &= 1-(\varpi_1+\varpi_2)B+\varpi_1\varpi_2 B^2 \end{aligned}$$

and so we solve using the relations:

$$\begin{aligned} \phi_1 &= \varpi_1+\varpi_2 \\ \phi_2 &= -\varpi_1\varpi_2 \end{aligned}$$

The solutions to this are the inverses to the solutions to the second order polynomial in the scalar-valued x:

$$(1-\phi_1 x-\phi_2 x^2)=0,$$

- the solutions to this equation are given by:

$$x_1, x_2 = \frac{\phi_1 \pm \sqrt{\phi_1^2+4\phi_2}}{-2\phi_2}$$

- the inverses are the **characteristic roots**: $\varpi_1 = x_1^{-1}$ and $\varpi_2 = x_2^{-1}$

Roots (real, distinct case)

- two characteristic roots: $\varpi_1 = x_1^{-1}$ and $\varpi_2 = x_2^{-1}$
- both characteristic roots are real-valued if the discriminant is greater than zero: $\phi_1^2 + 4\phi_2 > 0$
  - then we can factor the polynomial as:

$$(1 - \phi_1 B - \phi_2 B^2) = (1 - \varpi_1 B)(1 - \varpi_2 B)$$

  - *two AR(1) models on top of each other*
- The ACF will decay like an AR(1).

Roots (complex-valued case)

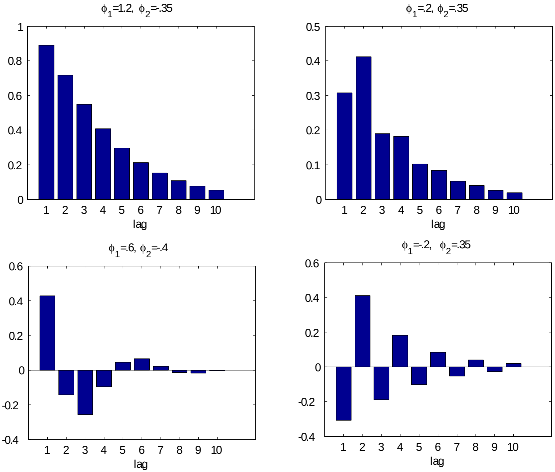
- two characteristic roots: $\varpi_1 = x_1^{-1}$ and $\varpi_2 = x_2^{-1}$

- both characteristic roots are complex-valued if the discriminant is negative: $\phi_1^2 + 4\phi_2 < 0$

- Then, $\varpi_1 = x_1^{-1}$ and $\varpi_2 = x_2^{-1}$ are complex numbers.

- The ACF will look like damped sine and cosine waves.

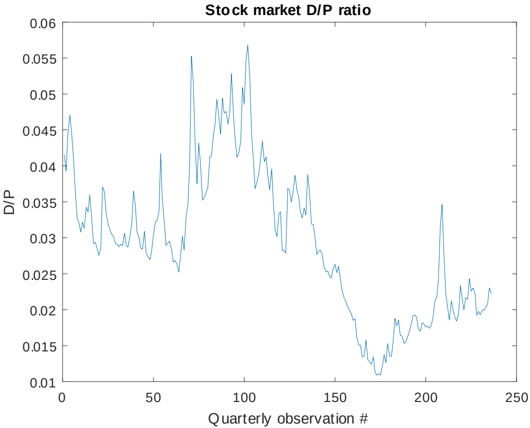
Autocorrelation for AR(2)



Autocorrelation Function for AR(2) processes.

AR(2) Example: The Dividend Price Ratio

- The stock market Dividend to Price ratio is:
  - Sum of last year's dividends to firms in the market divided by current market value
  - A "Valuation Ratio"
  - Very slow-moving (persistent); quarterly postWW2 data for U.S.:



Estimate AR(2) on this variable

ARIMA(2,0,0) Model:

Conditional Probability Distribution: Gaussian

Parameter	Value	Standard Error	t Statistic
Constant	0.00123254	0.00074679	1.65045
AR{1}	1.09319	0.0527929	20.7072
AR{2}	-0.137308	0.051282	-2.67752
Variance	7.84588e-06	1.92026e-07	40.8583

- Stationarity test:

$$1 - 1.09319x + 0.13731x^2 = 0$$

  - Roots greater than 1, so stationary despite $\phi_1 = 1.093 > 1$ as $\phi_2 = -0.137$.
  - Unconditional mean:

$$\mu = \frac{0.00123254}{1 - 1.09319 + 0.13731} = 0.0279$$

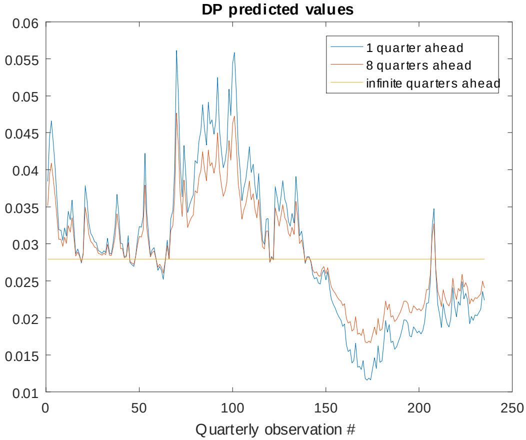
AR(2) DP prediction

Pred_DP1 = uncond_mean + phi1*(DP(2:end)-uncond_mean) + phi2*(DP(1:end-1)-uncond_mean);

Pred_DP2 = uncond_mean + phi1*(Pred_DP1-uncond_mean) + phi2*(DP(2:end)-uncond_mean);

Pred_DP3 = uncond_mean + phi1*(Pred_DP2-uncond_mean) + phi2*(Pred_DP1-uncond_mean);

etc.



Stationarity

- Recall: The modulus of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$. Thus, for real numbers the modulus is simply the absolute value.

Result:

- An AR(1) process is stationary if its characteristic root is less than one, i.e. if $1/x = \phi_1$ is less than one in modulus. This condition implies that $\rho_j = \phi_1^j$ converges to zero as $j \to \infty$.

- An AR(2) process is stationary if the two characteristic roots ϖ_1 and ϖ_2 (the inverses of the solutions to those two equations) are less than one in modulus.

Stationarity of AR(p)

- **An AR(p) process is stationary if all p characteristic roots of the below polymonial are less than one in modulus**

$$1 - \phi_1 x - \phi_2 x^2 - ... - \phi_p x^p = 0$$

- see chapter 2 in Hamilton (1994) for details.

Partial Autocorrelation Function

Definition

The PACF of a stationary series is defined as $\{\phi_{j,j}\}, j = 1, \ldots, n$

$$
\begin{aligned}
r_t &= \phi_{0,1} + \boldsymbol{\phi}_{1,1} r_{t-1} + v_{1t} \\
r_t &= \phi_{0,2} + \phi_{1,2} r_{t-1} + \boldsymbol{\phi}_{2,2} r_{t-2} + v_{2t} \\
r_t &= \phi_{0,3} + \phi_{1,3} r_{t-1} + \phi_{2,3} r_{t-2} + \boldsymbol{\phi}_{3,3} r_{t-3} + v_{3t} \\
& \ldots
\end{aligned}
$$

- These are simple multiple regressions that can be estimated with least squares.
- $\phi_{p,p}$ shows the incremental contribution of r_{t-p} to r_t over an $AR(p-1)$ model

PACF

Definition

The **sample partial autocorrelations (PACF)** of a time series are defined as $\widehat{\phi}_{1,1}, \widehat{\phi}_{2,2}, \ldots, \widehat{\phi}_{p,p}, \ldots,$

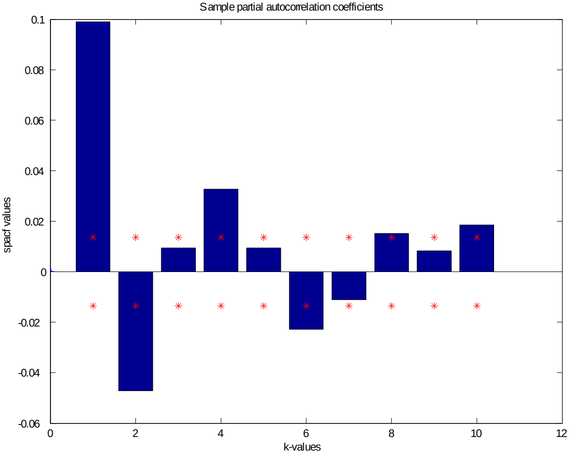
Partial Autocorrelation Function

The PACF of an $AR(p)$ satisfies:

1. $\hat{\phi}_{p,p} \to \phi_p$ as sample size increases
2. $\hat{\phi}_{j,j} \to 0$ for $j > p$

- for an AR(p) series, the sample PACF cuts off after lag p
- $\Rightarrow$ look at the sample PACF to determine an appropriate value of p

PACF of Daily Log Returns



PACF for Daily log Returns on VW-CRSP Index. Two standard error bands around zero. 1926-2007.

Information Criteria

- information criteria help determine the **optimal lag length**
- the Akaike (1973) information criterion:

$$AIC = -2\ln(likelihood) + 2(number\ of\ parameters)$$

- the Bayesian information criterion of Schwarz (1978):

$$BIC = -2\ln(likelihood) + \ln T(number\ of\ parameters)$$

  - the BIC penalty depends on the sample size T
- for different values of p, compute $AIC(p)$ and/or $BIC(p)$ pick the lag length with the minimum AIC/BIC

Manufacturing White Noise

- to check the performance of the AR model you've selected: **check the residuals**!!
- residuals should look like **white noise**
  - look at the ACF of the residuals
  - perform Ljung-Box test on residuals
  - $Q(m) \sim \chi^2(m-p)$ where p is the lag length of the $AR(p)$ model

Forecasting

- suppose we have an AR(p) model
- we want to forecast r_{t+h} using all the info $\mathcal{F}_t$ available at t
- assume we choose the forecast to minimize the **mean square error**:

$$E\left[\left(y - y_{prediction}\right)^2\right]$$

- The conditional mean minimizes the mean squared forecast error.
- we will come back to **optimal forecasting** later

1-step ahead forecast error

- the $AR(p)$ model is given by:

$$r_{t+1} = \phi_0 + \phi_1 r_t + \ldots + \phi_p r_{t-p+1} + \varepsilon_{t+1}$$

- take the conditional expectation:

$$E_t\left[r_{t+1}\right] = \phi_0 + \phi_1 r_t + \ldots + \phi_p r_{t-p+1}$$

- the one-step ahead forecast error:

$$v_t(1) = r_{t+1} - \phi_0 - \sum_{i=1}^{p} \phi_i r_{t-i+1} = \varepsilon_{t+1}$$

- the variance of the one-step ahead forecast error:

$$V\left[v_t\left(1\right)\right] = \sigma_\varepsilon^2$$

  - if ε_t is normally distributed, then the 95 % confidence interval:

$$\pm 1.96\sigma_\varepsilon$$

2-step ahead forecast error

- the $AR(p)$ model is given by:

$$r_{t+2} = \phi_0 + \phi_1 r_{t+1} + \ldots + \phi_p r_{t-p+2} + \varepsilon_{t+2}$$

- we just take the conditional expectation:

$$E_t\left[r_{t+2}\right] = \phi_0 + \phi_1 \widehat{r}_t(1) + \ldots + \phi_p r_{t-p+2}$$

- the two-step ahead forecast error:

$$v_t(2) = \phi_1 v_t(1) + \varepsilon_{t+2} = \phi_1 \varepsilon_{t+1} + \varepsilon_{t+2}$$

- the variance of the two-step ahead forecast error:

$$V\left[v_t\left(2\right)\right] = \sigma_\varepsilon^2(1 + \phi_1^2)$$

  - the variance of the two-step ahead forecast error is larger than the variance of the one-step ahead forecast error

Multi-step ahead forecast error

Result:

The *h*-step ahead forecast is given by:

$$\hat{r}_t\left(h\right) = \phi_0 + \sum_{i=1}^{p} \phi_i \hat{r}_t\left(h-i\right)$$

where $\hat{r}_t\left(j\right) = r_{t+j}$ if $j < 0$.

- the *h*-step ahead forecast converges to the unconditional expectation $E(r_t)$ as $h \longrightarrow \infty$
- this is referred to as **mean reversion**

Estimation: conditional least squares

- assume we observe or can condition on the first p observations.
- AR(p) model is then a linear regression model:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + \varepsilon_t, \qquad t = p+1, \ldots, T$$

- using least squares, the fitted model is

$$\widehat{r}_t = \widehat{\phi}_0 + \widehat{\phi}_1 r_{t-1} + \ldots + \widehat{\phi}_p r_{t-p}$$

  and the residual is $v_t = r_t - \widehat{r}_t$

- the estimated variance of the residuals is:

$$\widehat{\sigma}_\varepsilon^2 = \frac{\sum_{t=p+1}^{T} v_t^2}{T - 2p - 1}$$

ML Estimation

- alternatively, we could use maximum likelihood.
- the log-likelihood function is:

$$\ln p(r_1, r_2, \ldots, r_T; \boldsymbol{\theta}) = \sum_{t=2}^{T} \ln p(r_t | r_{t-1}, \ldots, r_1; \boldsymbol{\theta}) + \ln p(r_1; \boldsymbol{\theta})$$

- for example, assume Gaussian shocks ε_t then $p(r_t | r_{t-1}, \ldots, r_{t-p}; \boldsymbol{\theta})$ is normal
- the difference between least squares and ML estimation of $(\phi_0, \phi_1, \ldots, \phi_p)$ are the initial distributions $p(r_1; \boldsymbol{\theta}), p(r_2 | r_1; \boldsymbol{\theta}) \ldots$.
- Conditional least squares of an AR(p) drops the first p terms in the likelihood.

Example: ML Estimation of AR(1)

- assume the initial value r_1 comes from the stationary dist.
- unconditional moments:

$$E[r_1] = \frac{\phi_0}{1-\phi_1}, \qquad V[r_1] = \frac{\sigma_\varepsilon^2}{1-\phi_1^2},$$

- hence, the density $p(r_1;\theta)$ of the first observation r_1 is normal with the above (unconditional) mean and variance
- for $t > 1$, the conditional moments:

$$E[r_t|r_{t-1}] \; = \; \phi_0 + \phi_1 r_{t-1}, \qquad V[r_t|r_{t-1}] \; = \; \sigma_\varepsilon^2$$

- hence, the conditional density $p(r_t|r_{t-1};\boldsymbol{\theta})$ is normal with the above (conditional) mean and (conditional) variance

Example: ML Estimation of AR(1)

- the log-likelihood function is:

$$
\begin{aligned}
\ln p(r_1, r_2, \ldots, r_T; \boldsymbol{\theta}) &= \sum_{t=2}^{T} \ln p(r_t | r_{t-1}, \ldots, r_1; \boldsymbol{\theta}) + \ln p(r_1; \boldsymbol{\theta}) \\
&= -\frac{1}{2} \sum_{t=2}^{T} \left(\ln(2\pi) + \ln(\sigma_\varepsilon^2) + \frac{(r_t - \phi_0 - \phi_1 r_{t-1})^2}{\sigma_\varepsilon^2} \right) \\
&\quad + \ln p(r_1; \boldsymbol{\theta})
\end{aligned}
$$

- choose parameters $\boldsymbol{\theta} = (\phi_0, \phi_1, \sigma_\varepsilon^2)$ to maximize the log-likelihood function

$p(r_1; \boldsymbol{\theta})$ is typically chosen to be the stationary distribution

$$
r_1 \sim \mathrm{N}\left(\frac{\phi_0}{1-\phi_1}, \frac{\sigma_\varepsilon^2}{1-\phi_1^2}\right)
$$

Exact vs. Conditional ML

- the conditional ML estimator drops the initial condition
- exact log-likelihood function:

$$\ln p(r_1, r_2, \ldots, r_T; \boldsymbol{\theta}) = \sum_{t=2}^{T} \ln p(r_t | r_{t-1}, \ldots, r_1; \boldsymbol{\theta}) + \ln p(r_1 | \boldsymbol{\theta})$$

- conditional log-likelihood function:

$$\ln p(r_{p+1}, \ldots, r_T; \boldsymbol{\theta}) = \sum_{t=p+1}^{T} \ln p(r_t | r_{t-1}, \ldots, r_1; \boldsymbol{\theta})$$

- the conditional log-likelihood 'conditions' on the first data point and drops the first p terms.
- Conditional ML is the same as least squares. The solution can be calculated analytically.

Summary: AR(p) models

- *dynamic model*, e.g. AR(p):

$$r_t = \phi_0 + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + \varepsilon_t$$

  - constant determines mean through: $\mu = \frac{\phi_0}{1-\phi_1-\phi_2-\ldots-\phi_p}$
  - coefficients $(\phi_1, \phi_2, \ldots, \phi_p)$ must satisfy stationarity restrictions for a well-specified model:
  - objective: parsimonious model of dynamics of r_t

- For AR(p) models, you can maximize the conditional MLE in closed-form...conditional least squares....but there is no guarantee that it will satisfy the stationarity restrictions.

- Calculating the full MLE requires numerical optimization.

Application:

Bond Pricing

(Optional Material)

Bond Notation

- an n-period zero coupon bond pays one dollar n periods from now
- notation:
  - $P_t^{(n)}$ denotes the price of an n-period zero-coupon bond.
  - $p_t^{(n)} = \log(P_t^{(n)})$ denotes the log price
  - the yield of an n-period zero-coupon bond is:

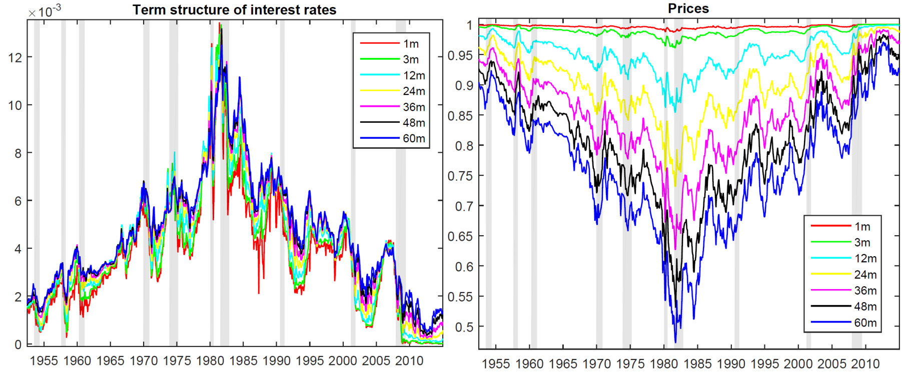
$$y_t^{(n)} \equiv -\frac{1}{n} p_t^{(n)}$$

  - the holding period return is:

$$hpr_{t+1}^{(n)} \equiv p_{t+1}^{(n-1)} - p_t^{(n)}$$

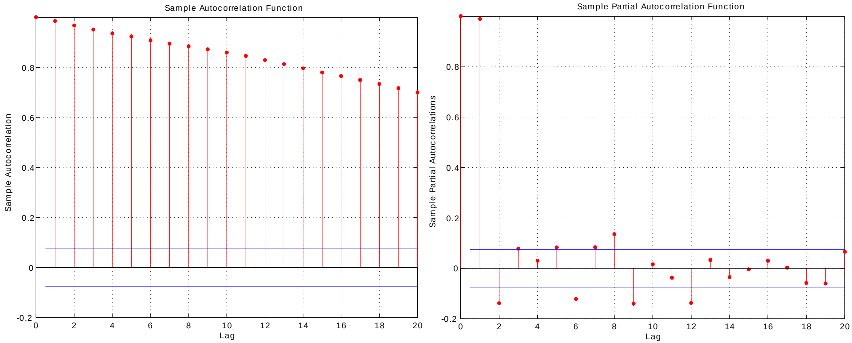
  - the short term interest rate $y_t^{(1)}$ is given special notation r_t

Term Structure of Interest Rates



CRSP Fama-Bliss Zero-Coupon Bond Data. Sample: 1952.6-2014.12. Monthly data. Left: yields. Right: prices

ACF and PACF of 1 month yield



CRSP Fama-Bliss Zero-Coupon Bond Data. Sample: 1952.6-2014.12. Monthly data. Yield on one month zero-coupon bond $y_t^{(1)}$.

- ACF is persistent. PACF drops off after 1 month.

Bond Pricing: Vasicek (1977) model

- discrete time models of bond pricing
- examine the simplest model: a single factor model
- the single factor g_t follows an AR(1):

$$g_{t+1} = (1-\phi)\mu + \phi g_t + \sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathrm{N}(0,1)$$

- Vasicek models the short-term interest rate (short rate) as a linear function of the single factor g_t given by

$$r_t = \delta_0 + \delta_1 g_t$$

- the short rate is the factor $r_t = y_t^{(1)} = g_t$ under the assumption that $\delta_0 = 0$ and $\delta_1 = 1$.

No arbitrage and Pricing Kernels

- let X_{t+1} denote the payoff or future cash flow of an asset

  In a market with no arbitrage, there exists a strictly positive random variable M_{t+1} such that, for any payoff X_{t+1}, the price of the asset is given by

$$P_t = E_t \left[M_{t+1} X_{t+1} \right]$$

- The r.v. M_{t+1} is called the **stochastic discount factor**.
- This equation generalizes the idea that future cash flows X_{t+1} are discounted back to the present but.....the rate at which we do the discounting is random.

No arbitrage and Pricing Kernels

- This is a fundamental equation in asset pricing

$$P_t = E_t[M_{t+1} X_{t+1}]$$

  as it can be applied to any asset (stocks, bonds, options).

- for example, a stock with current price P_t will pay dividends D_{t+1} and have price P_{t+1} which implies

$$P_t = E_t[M_{t+1}(P_{t+1} + D_{t+1})] \qquad X_{t+1} \; = \; P_{t+1} + D_{t+1}$$

- an asset pricing model specifies the evolution of the stochastic discount factor M_{t+1}

No arbitrage and Pricing Kernels

- What about zero coupon bonds?

- so, the price of a one-period zero-coupon bond would be

$$P_t^{(1)} = E_t[M_{t+1}] \qquad X_{t+1} \; = \; \$1$$

- the price of a two-period zero-coupon bond would be:

$$P_t^{(2)} = E_t[M_{t+1} P_{t+1}^{(1)}] \qquad X_{t+1} \; = \; P_{t+1}^{(1)}$$

- alternatively, we could write it as

$$\begin{aligned} P_t^{(2)} &= E_t[M_{t+1} P_{t+1}^{(1)}] \; = \; E_t[M_{t+1} E_{t+1}(M_{t+2})] \\ &= E_t[M_{t+1} M_{t+2}] \qquad \text{since } X_{t+2} \; = \; \$1 \end{aligned}$$

No arbitrage and Pricing Kernels

- What about n-period zero coupon bonds?

- the price of an n-period zero-coupon bond would be:

$$P_t^{(n)} = E_t[M_{t+1} P_{t+1}^{(n-1)}] \qquad X_{t+1} \; = \; P_{t+1}^{(n-1)}$$

- or, by recursive substitution like the last slide, we get

$$P_t^{(n)} = E_t[M_{t+1} M_{t+2} \dots M_{t+n}] \qquad X_{t+n} \; = \; \$1$$

- By observing zero-coupon bonds of different maturities, we learn about how financial markets discount future cash flows to the present.

Bond Pricing

- an asset pricing model specifies the evolution of the stochastic discount factor M_{t+1}
- in the Vasicek (1977) model, we assume that M_{t+1} is driven by the factor g_t

$$\begin{aligned} \log M_{t+1} &= -r_t - \frac{1}{2}\lambda_t^2 - \lambda_t \varepsilon_{t+1} \\ \lambda_t &= \lambda_0 + \lambda_1 g_t \end{aligned}$$

  where λ_t is called the **market price of risk**.

- Notice that the r.v. ε_{t+1} is the same shock as

$$g_{t+1} = (1-\phi)\mu + \phi g_t + \sigma\varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathrm{N}(0,1)$$

Price of a 1-period bond

- the one-period bond has payoff $X_{t+1} = \$1$
- the price of a one-period zero-coupon bond would be

$$\begin{aligned} P_t^{(1)} &= E_t[M_{t+1}] = E_t\left[\exp(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t\varepsilon_{t+1})\right] \\ &= \exp(-r_t - \frac{1}{2}\lambda_t^2)E_t\left[\exp(-\lambda_t\varepsilon_{t+1})\right] \\ &= \exp(-r_t - \frac{1}{2}\lambda_t^2)\exp(\frac{1}{2}\lambda_t^2) \end{aligned}$$

- where we used the moment-generating function of a Gaussian.

$$P_t^{(1)} = \exp(-r_t)$$

- the short-rate $r_t = g_t$ is the single-factor. We can write this as:

$$y_t^{(1)} \equiv -\log(P_t^{(1)}) \Rightarrow y_t^{(1)} = r_t = g_t$$

Price of a 2-period bond

- the two-period bond has payoff: $X_{t+1} = P_{t+1}^{(1)}$
- the price of a two-period zero-coupon bond would be:

$$\begin{aligned} P_t^{(2)} &= E_t[M_{t+1}P_{t+1}^{(1)}] = E_t\left[\exp(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t\varepsilon_{t+1})\exp(-r_{t+1})\right] \\ &= \exp(-r_t - \frac{1}{2}\lambda_t^2)E_t\left[\exp(-\lambda_t\varepsilon_{t+1})\exp(-[(1-\phi)\mu - \phi g_t + \sigma\varepsilon_{t+1}])\right] \\ &= \exp(-r_t - \frac{1}{2}\lambda_t^2 - (1-\phi)\mu + \phi g_t)E_t\left[\exp(-[\lambda_t + \sigma]\varepsilon_{t+1})\right] \end{aligned}$$

- using the normal MGF and the fact that $\lambda_t = \lambda_0 + \lambda_1 g_t$, we get:

$$P_t^{(2)} = \exp(-r_t - (1-\phi)\mu + \sigma\lambda_0 + \frac{1}{2}\sigma^2 - (\phi - \sigma\lambda_1)g_t)$$

- log prices and yields are linear functions of g_t

Price of a n-period bond

The Price of an *n*-period zero-coupon bond is:

$$P_t^{(n)} = \exp\left(\bar{a}_n + \bar{b}_n g_t\right)$$

where

$$\begin{aligned}
\bar{a}_n &= \bar{a}_{n-1} - \delta_0 + \bar{b}_{n-1}\left[(1-\phi)\,\mu - \sigma\lambda_0\right] + \frac{1}{2}\sigma^2 \bar{b}_{n-1}^2, \\
\bar{b}_n &= \bar{b}_{n-1}\left[\phi - \sigma\lambda_1\right] - \delta_1,
\end{aligned}$$

with initial conditions $\bar{a}_1 = 0$ and $\bar{b}_1 = -1$. (Proof on last slide)

- This implies yields are:

$$y_t^{(n)} = a_n + b_n g_t \qquad a_n = -\frac{1}{n}\bar{a}_n \qquad b_n = -\frac{1}{n}\bar{b}_n$$

- a_n and b_n are difference equations fit through the *cross-section* of yields.

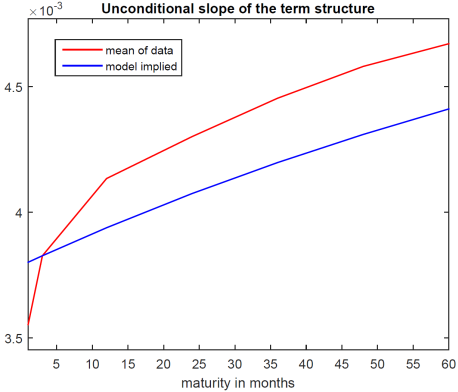
Estimation

- monthly U.S. zero coupon Fama-Bliss data from CRSP.
- 1,3,12,24,36,48,60 month yields
- Model estimated by maximum likelihood
- short rate $r_t = g_t$ is the one-month yield $y_t^{(1)}$

Parameter	Model	Sample moment	Model-implied
μ	$mean(y_t^{(1)})$	0.00355	0.00362
$\frac{\sigma^2}{1-\phi^2}$	$var(y_t^{(1)})$	5.976e-06	8.368e-06
ϕ	$\rho_{12}(y_t^{(1)})$	0.9756	0.9895

- market prices of risk: $\lambda_0 = -0.1144$ and $\lambda_1 = -10.741$

US Yield Curve



CRSP Fama-Bliss Zero-Coupon Bond Data. Sample: 1952.6-2014.12.

Multiple Factors

- the Vasicek (1977) model is a simple, one-factor model
- this model cannot capture the slope or curvature of yields, only the level of interest rates.
- we need a richer model with more factors, where we let g_t be a vector of factors $\Rightarrow$ vector autoregressive process
  - PCA of yields indicated we need three factors to explain yields.
- see, e.g., Ang and Piazzesi (2003), or just wait for Fixed Income class in spring

Moving Average Models

AR(infinity)

- in theory the true data generating process could be an AR(∞):

$$x_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \varepsilon_t$$

- implementation:
  - **infinite** number of parameters
- solution: constrain parameters

$$x_t = \phi_0 - \theta_1 x_{t-1} - \theta_1^2 x_{t-2} - \theta_1^3 x_{t-3} - \ldots + \varepsilon_t$$

where $\phi_i = -\theta_1^i, i \geq 1$

AR(infinity) to MA(1)

- solution: constrain parameters

$$x_t + \theta_1 x_{t-1} + \theta_1^2 x_{t-2} + \theta_1^3 x_{t-3} + \ldots = \phi_0 + \varepsilon_t$$

- this can be restated as an MA(1) model:

$$x_t = \phi_0(1 - \theta_1) + (1 - \theta_1 B)\varepsilon_t$$

  - MA(1) is a 'cheap' version of an AR(∞).

- general form of MA(1) model is:

$$x_t = \mu + (1 - \theta_1 B)\varepsilon_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

MA(q)

Definition

A **moving average process of order** q or MA(q) model is:

$$x_t = \mu + (1 - \theta_1 B - \ldots - \theta_q B^q)\varepsilon_t,$$

where $q > 0$

Stationarity

- consider the MA(1) model:

$$x_t = \mu + (1 - \theta_1 B)\varepsilon_t.$$

- compute the variance of an MA(1) model:

$$V\left[\mu + (1 - \theta_1 B)\,\varepsilon_t\right] = (1 + \theta_1^2)\sigma_\varepsilon^2.$$

- compute the variance of an MA(q) model:

$$V\left[\mu + (1 - \theta_1 B - ... - \theta_q B^q)\,\varepsilon_t\right] = (1 + \theta_1 + \theta_2^2 + \ldots + \theta_q^q)\sigma_\varepsilon^2.$$

Computing Autocovariances for MA(1)

- assume the unconditional mean $\mu = 0$
- pre-multiply the MA(1) model by r_{t-l}:

$$r_{t-j} r_t = r_{t-j} \varepsilon_t - \theta_1 r_{t-j} \varepsilon_{t-1}$$

- take expectations
- compute the auto-covariance of an MA(1) model:

$$\gamma_1 = -\theta_1 \sigma_\varepsilon^2, \ \gamma_j = 0, \ j > 1$$

- this implies the autocorrelations are:

$$\rho_1 = \frac{-\theta_1}{1 + \theta_1^2}, \ \rho_j = 0, \ j > 1$$

  - **the ACF is cut off after 1 lag**!

Computing Autocovariances for MA(2)

- the same argument implies that the autocorrelations of an MA(2) are:

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}, \qquad \rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \qquad \rho_j = 0,\ j > 2$$

  - the ACF is cut off after 2

Forecasting with MA(1)

- consider an MA(1) model:

$$r_{t+1} = \mu + \varepsilon_{t+1} - \theta_1 \varepsilon_t$$

- take conditional expectations:

$$\widehat{r}_t(1) = E_t\left[r_{t+1}\right] = \mu - \theta_1 \varepsilon_t$$

$$\widehat{r}_t(2) = E_t\left[r_{t+2}\right] = \mu$$

- the one-step ahead forecast error is given by:

$$v_t(1) = r_{t+1} - \widehat{r}_t(1) = \varepsilon_{t+1}$$

  - the variance of the one-step ahead forecast error is σ_ε^2

Forecasting with MA(1)

- consider an MA(1) model:

$$r_{t+1} = \mu + \varepsilon_{t+1} - \theta_1 \varepsilon_t$$

- take conditional expectations:

$$\widehat{r}_t(1) = E_t\left[r_{t+1}\right] = \mu - \theta_1 \varepsilon_t$$

$$\widehat{r}_t(2) = E_t\left[r_{t+2}\right] = \mu$$

- the two-step ahead forecast error is given by:

$$v_t(2) = r_{t+2} - \widehat{r}_t(2) = \varepsilon_{t+2} - \theta_1 \varepsilon_{t+1}$$

  - the variance of the two-step ahead forecast error is $(1 + \theta_1^2)\sigma_\varepsilon^2$
  - this is the unconditional variance

Forecasting with MA(2)

- consider an MA(2) model:

$$r_{t+1} = \mu + \varepsilon_{t+1} - \theta_1 \varepsilon_t - \theta_2 \varepsilon_{t-1}$$

- take conditional expectations:

$$\widehat{r}_t(1) = E_t\left[r_{t+1}\right] = \mu - \theta_1 \varepsilon_t - \theta_2 \varepsilon_{t-1}$$

$$\widehat{r}_t(2) = E_t\left[r_{t+2}\right] = \mu - \theta_2 \varepsilon_t$$

$$\widehat{r}_t(3) = E_t\left[r_{t+3}\right] = \mu$$

- the one-step ahead forecast error is given by:

$$v_t(1) = r_{t+1} - \widehat{r}_t(1) = \varepsilon_{t+1}$$

  - the variance of the one-step ahead forecast error is σ_ε^2

Forecasting with MA(2)

- consider an MA(2) model:

$$r_{t+1} = \mu + \varepsilon_{t+1} - \theta_1 \varepsilon_t - \theta_2 \varepsilon_{t-1}$$

- take conditional expectations:

$$\widehat{r}_t(1) = E_t\left[r_{t+1}\right] = \mu - \theta_1 \varepsilon_t - \theta_2 \varepsilon_{t-1}$$

$$\widehat{r}_t(2) = E_t\left[r_{t+2}\right] = \mu - \theta_2 \varepsilon_t$$

$$\widehat{r}_t(3) = E_t\left[r_{t+3}\right] = \mu$$

- the two-step ahead forecast error is given by:

$$v_t(2) = r_{t+2} - \widehat{r}_t(2) = \varepsilon_{t+2} - \theta_1 \varepsilon_{t+1}$$

  - the variance of the two-step ahead forecast error is $(1 + \theta_1^2)\sigma_\varepsilon^2$
  - this is smaller than the unconditional variance

Forecasting with MA(2)

- consider an MA(2) model:

$$r_{t+1} = \mu + \varepsilon_{t+1} - \theta_1 \varepsilon_t - \theta_2 \varepsilon_{t-1}$$

- take conditional expectations:

$$\widehat{r}_t(1) = E_t[r_{t+1}] = \mu - \theta_1 \varepsilon_t - \theta_2 \varepsilon_{t-1}$$

$$\widehat{r}_t(2) = E_t[r_{t+2}] = \mu - \theta_2 \varepsilon_t$$

$$\widehat{r}_t(3) = E_t[r_{t+3}] = \mu$$

- the three-step ahead forecast error is given by:

$$v_t(3) = r_{t+3} - \widehat{r}_t(3) = \varepsilon_{t+3} - \theta_1 \varepsilon_{t+2} - \theta_2 \varepsilon_{t+1}$$

  - the variance of the three-step ahead forecast error is $(1 + \theta_1^2 + \theta_2^2)\sigma_\varepsilon^2$
  - this is the unconditional variance

Maximum Likelihood

- MA(q) models can't be estimated using (conditional) least squares because the parameters are a non-linear function of the data
- MA(q) models are commonly estimated using Maximum Likelihood
- this involves assuming a parametric distribution for the shocks ε_t.
- Often, we assume ε_t are normally distributed.

ML Estimation of MA(1)

- conditional moments:

$$
\begin{aligned}
V\left[r_t | r_{t-1}\right] &= \sigma_\varepsilon^2, \\
E\left[r_t | r_{t-1}\right] &= \mu - \theta \varepsilon_{t-1}
\end{aligned}
$$

- hence, the density $p(r_t | \mathcal{F}_{t-1}; \boldsymbol{\theta})$ of the first observation is normal with the above (conditional) mean and variance

- suppose we assume that $\varepsilon_0 = 0$.
- then $\varepsilon_1 = r_1 - \mu$
- then $\varepsilon_2 = r_2 - \mu - \theta_1 \varepsilon_1$
- we can recursively calculate the sequence $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_t\}$

ML Estimation of MA(1)

- hence, the log-likelihood function is:

$$
\begin{aligned}
\ln p(r_1, r_2, \dots, r_T; \boldsymbol{\theta}) &= \sum_{t=2}^{T} \ln p(r_t | r_{t-1}, \dots, r_1; \boldsymbol{\theta}) + \ln p(r_1; \boldsymbol{\theta}) \\
&= -\frac{1}{2} \sum_{t=1}^{T} \left(\ln(2\pi) + \ln(\sigma_\varepsilon^2) + \frac{(-\varepsilon_t)^2}{\sigma_\varepsilon^2} \right) \\
&\quad + \ln p(r_1; \boldsymbol{\theta})
\end{aligned}
$$

- choose parameters $\boldsymbol{\theta} = (\mu, \theta_1, \sigma_\varepsilon^2)$ to maximize the log-likelihood function

ACF and PACF

- *ACF* is useful for determining MA lag length:
  - autocorrelations are cut off at q for an $MA(q)$: $ACF(k) = 0$ for $k > q$
- *PACF* is useful for determining AR lag length
  - partial autocorrelations are cut off at p for an $AR(p)$: $PACF(k) = 0$ for $k > p$

ARMA Models

ARMA(p,q)

- certain processes can only be described by AR or MA models if we include lots of lags
  - unappealing (need to estimate lots of parameters)
- natural solution: ARMA(p, q) processes

ARMA(p,q)

- consider an $\mathrm{ARMA}(1,1)$ model:

$$r_t - \phi_1 r_{t-1} = \phi_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} \qquad \varepsilon_t \sim \mathrm{WN}(0, \sigma_\varepsilon^2)$$

  with $\theta_1 \neq \phi_1$
- the unconditional mean of an $\mathrm{ARMA}(1,1)$ has the same expression as an $\mathrm{AR}(1)$

$$E\left[r_t\right] = \frac{\phi_0}{1-\phi_1}$$

- we can re-write the process as:

$$r_t - \mu = \phi_1 (r_{t-1} - \mu) + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

- take expectations of $[r_t - \mu]^2$ to compute the variance:

$$V\left[r_t\right] = \phi_1^2 V\left[r_{t-1}\right] + \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2 - 2\phi_1 \theta_1 E\left[\varepsilon_{t-1}\left(r_{t-1} - \mu\right)\right]$$

ARMA(1,1)

- this reduces to:

$$V\left[r_t\right] = \phi_1^2 V\left[r_t\right] + \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2 - 2\phi_1 \theta_1 \sigma_\varepsilon^2$$

- collecting terms, we get:

$$V\left[r_t\right] = \sigma_\varepsilon^2 \frac{1 + \theta_1^2 - 2\phi_1 \theta_1}{1 - \phi_1^2}$$

- obviously, we need $\phi_1^2 < 1$
  - same stationarity requirement as for AR(1)

ACF of ARMA(1,1)

- to compute the auto-covariances:

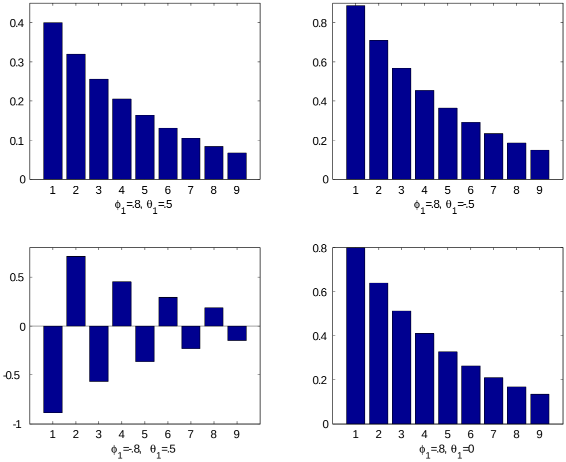
$$
\begin{aligned}
E\left[(r_t-\mu)\left(r_{t-j}-\mu\right)\right] &= \phi_1 E\left[(r_{t-1}-\mu)\left(r_{t-j}-\mu\right)\right] \\
&\quad + E\left[\varepsilon_t\left(r_{t-j}-\mu\right)\right] \\
&\quad - \theta_1 E\left[\varepsilon_{t-1}\left(r_{t-j}-\mu\right)\right]
\end{aligned}
$$

- for $j=1$, we get: $\gamma_1 = \phi_1\gamma_0 - \theta_1\sigma_\varepsilon^2$
- this implies that the ACF is given by:

$$
\rho_1 = \phi_1 - \theta_1\frac{\sigma_\varepsilon^2}{\gamma_0}
$$

$$
\rho_j = \phi_1\rho_{j-1}, \qquad j>1
$$

Autocorrelation for ARMA(1,1)



Autocorrelation Function for ARMA(1,1) processes.

PACF of ARMA(1,1)

- PACF does not die out at some lag
- slow decay (as is the case for MA models)

ARMA(p,q)

- consider an ARMA(p, q) model:

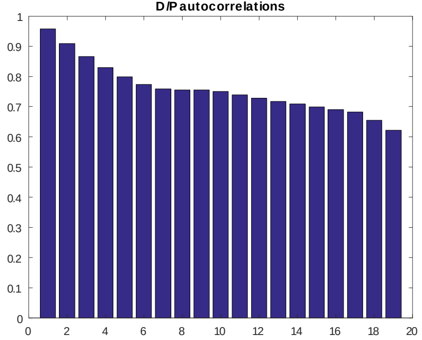
$$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}, \qquad \varepsilon_t \sim \mathrm{WN}(0, \sigma_\varepsilon^2)$$

- using the backshift operator

$$(1 - \phi_1 B - \ldots - \phi_p B^p) r_t = \phi_0 + (1 - \theta_1 B - \ldots - \theta_q B^q) \varepsilon_t$$

D/P autocorrelation function

- Revisiting the D/P ratio
  - Sample autocorrelation function:



- 'Drop off' for about first 4 lags, then stable...
  - Indicates a 4 lags of MA might be a good representation + 1 lag AR

D/P as ARMA(1,4)

```
ARIMA(1,0,4) Model:
   --------------------
   Conditional Probability Distribution: Gaussian
```

Parameter	Value	Standard Error	t Statistic
Constant	0.000484308	0.000550722	0.879406
AR{1}	0.980784	0.0158269	61.9696
MA{1}	0.103568	0.0573102	1.80715
MA{2}	-0.172191	0.0611522	-2.81577
MA{3}	-0.148333	0.0632631	-2.34469
MA{4}	-0.106098	0.0582711	-1.82076
Variance	7.50796e-06	1.64215e-07	45.7203

- Forecast by getting sample series of residuals, then plug in as needed for forecasts

$$\mu = \frac{0.000484}{1 - 0.9807}$$

$$\begin{aligned}
E_t\left[DP_{t+1}\right] &= \mu + 0.98\left(DP_t\text{-}\mu\right) + 0.10\varepsilon_t\text{-}0.17\varepsilon_{t-1}\text{-}0.14\varepsilon_{t-2}\text{-}0.11\varepsilon_{t-3}, \\
E_t\left[DP_{t+2}\right] &= E_t\left[E_{t+1}\left[DP_{t+2}\right]\right] \\
&= E_t\left[\mu + 0.98\left(DP_{t+1}\text{-}\mu\right) + 0.10\varepsilon_{t+1}\text{-}0.17\varepsilon_t\text{-}0.14\varepsilon_{t-1}\text{-}0.11\varepsilon_{t-2}\right] \\
&= \mu + 0.98\left(E_t\left[DP_{t+1}\right]\text{-}\mu\right)\text{-}0.17\varepsilon_t\text{-}0.14\varepsilon_{t-1}\text{-}0.11\varepsilon_{t-2}
\end{aligned}$$

etc.

MA representation

- start from this expression:

$$(1 - \phi_1 B - \ldots - \phi_p B^p) r_t = \phi_0 + (1 - \theta_1 B - \ldots - \theta_q B^q) \varepsilon_t$$

- re-arranging this expression delivers an *MA* representation:

$$r_t = \frac{\phi_0}{(1 - \phi_1 B - \ldots - \phi_p B^p)} + \frac{(1 - \theta_1 B - \ldots - \theta_q B^q)}{(1 - \phi_1 B - \ldots - \phi_p B^p)} \varepsilon_t$$

- more succinctly:

$$r_t = \mu + \psi(B) \varepsilon_t$$

- stationarity: the solutions of $(1 - \phi_1 x - \ldots - \phi_p x^p) = 0$ should lie outside of the unit circle

Impulse-Response Function

- consider the MA representation:

$$r_t = \mu + \psi(B)\varepsilon_t$$

- this can be written out as:

$$r_t = \mu + \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \dots$$

  where $\{\psi_i\}$ is the *impulse response* function of the ARMA model.
- the coefficients $\{\psi_i\}$ are functions of the parameters $\{\phi_i\}$ and $\{\theta_i\}$
- the impulse response function shows the effect today of a shock k periods ago:

$$\frac{\partial r_t}{\partial \varepsilon_{t-k}} = \psi_k$$

Forecasting

- consider the MA representation:

$$r_t = \mu + \psi(B)\varepsilon_t$$

- the h-period ahead forecast :

$$\widehat{r}_t(h) = \mu + \psi_h \varepsilon_t + \psi_{h+1}\varepsilon_{t-1} + \dots$$

- the h-period ahead forecast error can be stated as:

$$v_t(h) = \varepsilon_{t+h} + \psi_1 \varepsilon_{t+h-1} + \psi_2 \varepsilon_{t+h-2} + \dots + \psi_{h-1}\varepsilon_{t+1}$$

- the variance of the h-step ahead forecast error is:

$$V\left[v_t\left(h\right)\right] = \left(1 + \psi_1^2 + \psi_2^2 + \dots + \psi_{h-1}^2\right)\sigma_\varepsilon^2$$

Variance of Forecast Error

- the variance of the *h*-step ahead forecast error is:

$$V\left[v_t\left(h\right)\right] = \left(1 + \psi_1^2 + \psi_2^2 + \ldots + \psi_2^h\right)\sigma_\varepsilon^2$$

  - non-decreasing function of forecast horizon
  - variance of forecast error converges to variance of process

$$V\left[v_t\left(h\right)\right] \rightarrow V\left[r_t\right]$$

  as $h \rightarrow \infty$

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Appendix

Proof: Bond Pricing

- We will guess and verify the solution. Guess the solution for maturity n as:

$$P_t^{(n)} = \exp(\bar{a}_n + \bar{b}_n g_t)$$

  where $\bar{a}_n$ and $\bar{b}_n$ are unknown coefficients to be determined.
- now, using our model for M_{t+1}, we verify that our guess was correct.

$$\begin{aligned} P_t^{(n)} &= E_t[M_{t+1} P_{t+1}^{(n-1)}] \\ &= E_t\left[\exp(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t \varepsilon_{t+1})\exp(\bar{a}_{n-1} + \bar{b}_{n-1} g_{t+1})\right] \\ &= \exp(\bar{a}_{n-1} - r_t - \frac{1}{2}\lambda_t^2 + (1-\phi)\mu \bar{b}_{n-1} + \bar{b}_{n-1}\phi g_t) E_t\left[\exp(-[\lambda_t - \bar{b}_{n-1}\sigma]\varepsilon_{t+1})\right] \end{aligned}$$

- use the normal MGF, $\lambda_t = \lambda_0 + \lambda_1 g_t$, and $r_t = \delta_0 + \delta_1 g_t$, we get:

$$P_t^{(n)} = \exp(\bar{a}_{n-1} - \delta_0 + \bar{b}_{n-1}[(1-\phi)\mu - \sigma\lambda_0] + \frac{1}{2}\sigma^2 \bar{b}_{n-1}^2 + (\bar{b}_{n-1}[\phi - \sigma\lambda_1] - \delta_1) g_t)$$

- This defines $\bar{a}_n$ and $\bar{b}_n$ based on our guess above as given in the earlier slides on the Vasicek model