$d(AB) = AdB + BdA + dAdB$

Feynman Kac:

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} - r(x)F = -h(x)$$
$$F(T,x) = \Phi(x)$$

$dX = \mu(t,X)dt + \sigma(t,X)dW$

$$F(t,x) = E_t\left[\int_t^T e^{-\int_t^s r(X_u)du} h(X_s)ds + e^{-\int_t^T r(Xu)du}\Phi(X_T)\right]$$

Geometric Brownian Motion:

$dX_t = \alpha X_t dt + \sigma X_t dW_t$ $\quad$ *X is log normal*

$$logX_T \sim N\left\{logX_t + \left(\alpha - \frac{1}{2}\sigma^2\right)(T-t);\ \sigma^2(T-t)\right\}$$

if $T = t, t = 0$:

$$logX_t \sim N\left\{logX_0 + \left(\alpha - \frac{1}{2}\sigma^2\right)t;\ \sigma^2 t\right\}$$

$E_t(X_T) = X_t e^{\alpha(T-t)}$

$Var(X_T) = X_t^2\left[e^{(2\alpha-\sigma^2)(T-t)} - e^{2\alpha(T-t)}\right]$

$E_0(X_t) = X_0 e^{\alpha t}$

$Var(X_t) = X_0\left[e^{(2\alpha-\sigma^2)t} - e^{2\alpha t}\right]$

For a* log *normal distribution:

$X \sim N\{\mu; \sigma^2\}$ $\qquad$ $E(X) = e^{\mu + \frac{1}{2}\sigma^2}$

$Var(X) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$ $\qquad$ $E(X^2) = e^{2\mu+2\sigma^2}$

$E(X^n) = E^{n\mu + \frac{1}{2}n^2\sigma^2}$

$Var(X) = E(X^2) - \left(E(X)\right)^2$

Moment generating function of normal:

$M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ $\qquad$ $E(X) = M_x'(0) = \mu$

$E(X^2) = M_x''(0) = \sigma^2 + \mu$ $\qquad$ $E(X^n) = M_x^{(n)}(0)$

Risk free asset is that it has no diffusion dW term:

$dB(t) = rB(t)dt$

$dS(t) = \alpha Sd(t)t + \sigma S(t)d\overline{W}(t)$

No Arbitrage approach to Black Schole

$\Pi(t) = F(t,S)$

$d\Pi = \alpha_\pi \Pi dt + \sigma_\pi \Pi(t)d\overline{W}(t)$

$$\alpha_\pi(t) = \frac{F_t + \alpha SF_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{F}$$

$$\sigma_\pi(t) = \frac{\sigma SF_s}{F}$$

Consider a portfolio based on: 1. The stock, 2. The derivative asset:

$dV = V[u_s\alpha + u_\pi\alpha_\pi]dt + V[u_s\sigma + u_\pi\sigma_\pi]d\overline{W}$

$u_s + u_\pi = 1$ $\qquad$ $u_s\alpha + u_\pi\alpha_\pi = r$ $\qquad$ $u_s\sigma + u_\pi\sigma_\pi$

$$u_s = \frac{\sigma_\pi}{\sigma_\pi - \sigma} \qquad u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma}$$

$$u_s = \frac{SF_s}{SF_s - F} \qquad u_\pi = \frac{-F}{SF_s - F}$$

Black Scholes Equation:

$$F_t + rSF_s + \frac{1}{2}S^2\sigma^2 F_{ss} - rF = 0$$
$$F(T,s) = \Phi(s)$$

Under Q-Measurement, Risk Neutral Valuation:

$dS = rSdt + S\sigma dW$

$F(t,s) = e^{-r(T-t)}E_{t,s}^Q\left[\Phi\left(S(T)\right)\right]$

By Ito's lemma:

$S(T) = s \cdot e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W(T) - W(t))}$

$$z = \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W(T) - W(t)\right)$$

$$z \sim N\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t);\ \sigma^2(T-t)\right\}$$

$$F(t,s) = e^{-r(T-t)}\int_{-\infty}^{\infty}\Phi(se^z)f(z)dz$$

$$E\left(g(x)\right) = \int_{-\infty}^{\infty} g(x)f(x)dx \quad f(x) \text{ is the pdf of } x$$

$$pdf\ of\ normal: \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E^Q[\max(se^z - K, 0)] = 0 \cdot Q(se^z < K) + \int_{\log\left(\frac{K}{s}\right)}^{\infty}(se^z - K)f(z)dz$$

$F(t,s) = sN\left(d_1(t,s)\right) - e^{-r(T-t)}KN\left(d_2(t,s)\right)$

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\log\left(\frac{s}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}$$

$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t}$

For a forward, $\Phi(S_T) = S_T - K$

$f(t) = e^{r(T-t)}S_t$ $\qquad$ $K = S_0 e^{rT}$

Replication Method to Black Sholes:

$u^0 + u^* = 1$ $\quad$ u^0: *Bond weight*, u^*: *Stock weight*

$dV = V\{u^0 r + u^*\alpha\}dt + Vu^*\sigma d\overline{W}(t)$

$V(T) = \Phi\left(S(t)\right)$

$V(t) = F(t,S)$ $\quad$ *By Ito's Lemma*:

$$dV = \left\{F_t + \alpha SF_s + \frac{1}{2}\sigma^2 S^2 F_{ss}\right\}dt + \sigma SF_s d\overline{W}$$

$$dV = V\left\{\frac{F_t + \alpha SF_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{F}\right\}dt + V\frac{SF_s}{F}\sigma d\overline{W}$$

$$u^* = \frac{SF_s}{F} \qquad u^0 = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} = 1 - \frac{SF_s}{F}$$

$$h^0 = \frac{u^0 V}{B} = \frac{F - SF_s}{B} \qquad h^* = \frac{u^* V}{S} = F_s$$

The Greeks:

$$\Delta = \frac{\partial P}{\partial S} = N(d_1)$$

$$\Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}} \quad \varphi(d_1) \text{ is the pdf of } N(0,1)$$

$$\rho = \frac{\partial P}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2)$$

$$\Theta = \frac{\partial P}{\partial t} = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

$$v = \frac{\partial P}{\partial \sigma} = s\varphi(d_1)\sqrt{T-t}$$

Put-Call Parity:

$P(t,s) + s = Ke^{-r(T-t)} + c(t,s)$

Multidimensional Black Sholes

$$dS_i = \alpha_i S_i dt + S_i\sum_{j=1}^{n}\sigma_{ij}\,d\overline{W}_i(t)$$

σ *is* $n \times n$ *matrix* $\{\sigma_{ij}\}$

$dF = F \cdot \alpha_F dt + F \cdot \sigma_F d\overline{W}$

If 2 dimension:

$$dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S_1}\alpha_1 S_1 + \frac{\partial F}{\partial S_2}\alpha_2 S_2 + \frac{1}{2}\frac{\partial^2 F}{\partial S_1^2}S_1^2(\sigma_{11}^2 + \sigma_{12}^2) + \frac{1}{2}\frac{\partial^2 F}{\partial S_2^2}S_2^2(\sigma_{21}^2 + \sigma_{22}^2) + \frac{\partial^2 F}{\partial S_1 \partial S_2}\ S_1 S_2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})\right]dt + \left[\frac{\partial F}{\partial S_1}S_1\sigma_1 + \frac{\partial F}{\partial S_2}S_2\sigma_2\right]dW$$

$$\alpha_F = \frac{1}{F}\left[F_t + \sum_1^n \alpha_i S_i F_i + \frac{1}{2}tr\{\sigma^T D[S]F_{ss}D[S]\sigma\}\right]$$

$$\sigma_F = \frac{1}{F}\sum_1^n S_i F_i \sigma_i$$

$$F_{ss} = \left\{\frac{\partial^2 F}{\partial s_i \partial s_j}\right\}_{i,j=1}^n$$

$$u_B = 1 - \left(\sum_1^n u_i + u_F\right)$$

$$dV = V\left[\sum_1^n u_i\frac{dS_i}{S_i} + u_F\frac{dF}{F} + u_B\frac{dB}{B}\right]$$

$$dV = V\left[\sum_1^n u_i(\alpha_i - r) + u_F(\alpha_F - r) + r\right]dt + V\left[\sum_1^n u_i\sigma_i + u_F\sigma_F\right]d\overline{W}$$

$$\sum_1^n u_i\sigma_i + u_F\sigma_F = 0$$

$$\begin{bmatrix}\alpha_1 - r & \dots & \alpha_n - r & \alpha_F - r \\ \sigma_1^T & \dots & \sigma_n^T & \sigma_F^T\end{bmatrix}\begin{bmatrix}u_S \\ u_F\end{bmatrix} = \begin{bmatrix}\beta \\ 0\end{bmatrix}$$

σ_i^T *is column vector*

$$u_s = \begin{bmatrix}u_1 \\ u_2 \\ \dots \\ u_n\end{bmatrix} \qquad H = \begin{bmatrix}\alpha_1 - r & \dots & \alpha_n - r & \alpha_F - r \\ \sigma_1^T & \dots & \sigma_n^T & \sigma_F^T\end{bmatrix}$$

H must be singular

$$\alpha_i - r = \sum_{j=1}^n \sigma_{ij}\lambda_j, \quad i = 1, \dots, n,$$

$$\alpha_F - r = \textstyle\sum_{j=1}^n \sigma_{Fj}\,\lambda_j, \qquad \lambda = \begin{bmatrix}\lambda_1 \\ \dots \\ \lambda_n\end{bmatrix}$$

$\alpha - r1_n = \sigma\lambda$ $\qquad$ $\alpha_F - r = \sigma_F\lambda$

$$F_t + \sum_{i=1}^n r\,S_i F_i + \frac{1}{2}tr\{\sigma^T D[S]F_{ss}D[S]\sigma\} - rF = 0$$

$F = e^{-r(T-t)}E^Q\left[\Phi\left(S(T)\right)\right]$

Reducing the State Space

$$F(t, s_1, \dots, s_n) = s_n G\left(t, \frac{s_1}{s_n}, \dots, \frac{s_{n-1}}{s_n}\right)$$

$$z = \left(\frac{s_1}{s_n}, \dots, \frac{s_{n-1}}{s_n}\right)$$

$F_t(t,s) = s_n G_t(t,z)$

$F_i(t,s) = G_i(t,z), i = 1, \dots, n-1$

$$F_n(t,s) = G(t,z) - \sum_{j=1}^{n-1}\frac{s_j}{s_n}G_j(t,z)$$

$$F_{ij}(t,s) = \frac{1}{s_n}G_{ij}(t,z), \quad i,j = 1, \dots, n-1$$

$$F_{in}(t,s) = F_{ni}(t,s) = -\sum_{j=1}^{n-1} \frac{s_j}{S_n^2} G_{ij}(t,z),$$

$i = 1, \ldots, n-1$

$$F_{nn} = -\sum_{i,j=1}^{n-1} \frac{s_i s_j}{S_n^3} G_{ij}(t,z)$$

For two dimension:

$F_1 = G_z \qquad F_2 = G - zG_z$

$F_{11} = \frac{1}{S_2} G_{zz} \quad F_{22} = \frac{S_1^2}{S_2^3} G_{zz}$

$G_t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) z^2 G_{zz} = 0$

$F(t, s_1, s_2) = s_1 N(d_1(t,z)) - s_2 N(d_2(t,z))$

$$d_1(t,z) = -\frac{1}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}} \Big\{ \log z + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T-t) \Big\}$$

$d_2(t,z) = d_1(t,z) - \sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}$

Dividends

$\delta = \delta(S_{t-})$ *is a function of* S_{t-}

$S_t = S_{t-} - \delta(S_{t-})$

$F^0(T, S_T) = \Phi(S_T)$

$F^1(T_1^-, S_{T^-}) = F^0(T_1, S_{T^-} - \delta(S_{T^-}))$

$F(t, S_t) = e^{-r(T-t)} E^Q[\Phi(S_T)]$

$dS_t = rS_t dt + \sigma S_t dW_t$

if $\delta(S_{t^-}) = \delta S_{t^-}, \delta$ *is a constant*:

$F_\delta(t, S_t) = F(t, (1-\delta)^n S_t)$

n is the number of dividend points in the interval (t, T]

Continuous dividends:

Q-measure:

$dS = (r - \delta) S dt + \sigma S dW$

$F(t, S_t) = e^{-r(T-t)} E^Q[\Phi(S_T)]$

$F_\delta(t, s) = F_0(t, s e^{-\delta(T-t)})$

Forward with dividend:

$K = S_0 e^{(r-\delta)T}$

Foreign Exchange:

P measure:

$dX_t = X_t \alpha dt + X_t \sigma_X d\overline{W}_t$

Q measure:

$dX_t = X_t(r^d - r^f) dt + X_t \sigma_X dW_t$

This is similar to assuming that X_t is dividend paying asset.

Forward: $K = X_0 e^{(r^d - r^f)T}$

$F_\delta(t, S_t) = F(t, e^{-\delta(T-t)} S_t)$

$F(t, x) = x e^{-r^f(T-t)} N[d_1] - e^{-r^d(T-t)} K N[d_2]$

$$d_1(t,x) = \frac{1}{\sigma_X \sqrt{T-t}} \left\{ \log\left(\frac{x}{K}\right) + \left(r^d - r^f + \frac{1}{2}\sigma_X^2\right)(T-t) \right\}$$

$d_2(t,x) = d_1 - \sigma_X \sqrt{T-t}$

⑤ $X_t = \int_0^t g(s)\, dW \sim N\left(0, \int_0^t g^2(s)\, ds\right)$

$Z_t = e^{-\frac{1}{2}\eta^2 \int_0^t g^2(s) ds + \eta \int_0^t g(s) dW_s}$ martingale

$Z_0 = 1 = E(Z_t)$

⑥ $\beta_k(t) = E[W_t^k] \qquad \beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\, ds$

$Z = W_t^k$

⑦ $dX_t = -\eta(X_t - \bar{X})dt + \sigma dW_t$ 求 moment-generating function

$f(t,x) = e^{\alpha_0(t) + \alpha_1(t)\lambda} \Rightarrow \begin{cases} \alpha_0(t) = \lambda e^{-\eta\tau} \\ \alpha_1(t) = \lambda\bar{x}(1 - e^{-\eta\tau}) + \frac{\lambda^2\sigma^2}{4\eta}(1 - e^{-2\eta\tau}) \end{cases}$

⑧ $\left.\begin{array}{l} d\pi = r\pi(t, S_t)dt + \sigma_\pi \pi(t, S_t) dW_t \\ \sigma_\pi = \dfrac{\frac{\partial \pi}{\partial s}\sigma(t, S_t)}{\pi(t, S_t)} \end{array}\right\}$ Under Q

⑨ $\Phi(S_T) = S_T \qquad F(t, S_t) = S_t$

⑩ $\Phi(S_T) = \frac{1}{S_T} \qquad Z = \frac{1}{S_t} \qquad F(t, S_t) = \frac{1}{S_t} e^{(-2r+\sigma^2)(T-t)}$

Binomial Pricing

Arbitrage: $P(V(T;h)\geq 0)=1$ $\quad P(V(T;h)>0)>0$

Free of arbitrage for binomial pricing:

$$d \leq 1+R \leq u$$

Under Q:

$$q_u = \frac{(1+R)-d}{u-d}$$

$$q_d = \frac{u-(1+R)}{u-d}$$

replicating portfolio:

market is complete = all claims can be replicated

$X=\Phi(S_T)$ is reachable: can be replicated

In binomial: $u>d$ = completeness

$$V_0^h = \frac{1}{1+R}E^Q[X] = \frac{1}{1+R}\{q_u\Phi(u)+q_d\Phi(d)\}$$

multi-period

$$\Pi(0;X) = V_0 = \frac{1}{(1+R)^T}E_0^Q[\Phi(S_T)]$$

$$\Pi(0;X) = \frac{1}{(1+R)^T}\sum_{k=0}^{T}\binom{T}{k}q_u^k q_d^{T-k}\Phi(S_0u^kd^{T-k})$$

Stochastic Integrals

Brownian motion:

$W(0)=0$

$W(u)-W(t)$, $W(s)-W(r)$ independent $\quad (r<s\leq t<u)$

$W(t)-W(s)\sim N(0,t-s)$ $\quad (s<t)$

W has continuous trajectories

Properties:

$$E\int_a^b g(s)\,dW(s)=0$$

$$E\left[\left(\int_a^b g(s)\,dW(s)\right)^2\right] = E\int_a^b g^2(s)\,ds$$

Martingales:

$$\begin{cases} X \text{ adapted to } \mathcal{F}_t \\ E[|X_t|]<\infty \\ E[X_t\mid\mathcal{F}_s]=X(s) \quad (t>s)\end{cases}$$

martingale = no "dt" term

Itô's Lemma:

$$\int_0^t (dW)^2 = t \iff (dW)^2=dt$$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial f}{\partial x}(dX)^2$$

$dt^2=0$ $\quad dt\,dW=0$ $\quad dW^2=dt$

Geometric Brownian Motion

$dX_t = \alpha X_t\,dt + \sigma X_t\,dW_t$ $\quad Z=\log(X_t)$

$X_T = e^{(\alpha-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}$ $\quad E(X_T)=e^{\alpha(T-t)}$

multi-dimensional Itô's formula

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)\,dW_j(t)$$

$$df(t,X(t)) = \frac{\partial f}{\partial t}dt + \sum\frac{\partial f}{\partial x}dX + \frac{1}{2}\sum\sum\frac{\partial^2 f}{\partial x_i\partial x_j}dX_i\,dX_j$$

$dt^2=0$ $\quad dt\,dW=0$ $\quad d^2W_i=dt$ $\quad dW_i\,dW_j=0\ (i\neq j)$

product rule: $d(AB) = A\cdot dB + B\cdot dA + dA\cdot dB$

Stochastic Differential Equations

linear SDE $\quad dX_t = \alpha X_t\,dt + \sigma\,dW_t$

$$X_t = X_0e^{\alpha t} + \sigma\int_0^t e^{\alpha(t-s)}dW_s \quad \text{(normal distribution)}$$

Feynman-Kac Theorem

$$\begin{cases}\frac{\partial F(t,x)}{\partial t} + \frac{\partial F}{\partial x}\mu(t,x)F + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t,x) - r(x)F(t,x) = -h(x) \\ F(T,x)=\Phi(x)\end{cases}$$

$$\Rightarrow\begin{cases} dX = \mu(t,x)dt + \sigma(t,X)\,dW_t \\ F(t,x) = E_t\left[\int_t^T e^{-\int_t^s r(X_u)du}h(X_s)\,ds + e^{-\int_t^T r(X_u)du}\Phi(X_T)\right]\end{cases}$$

moment-generating functions

$\Phi(X_T) = e^{-\lambda X_T}$ $\quad f(t,x) = E[\Phi(X_T)\mid X_t=x]$

$E_t[df(t)]=0 \Rightarrow$ martingale

Portfolio Dynamics

Self-financing:

$$\begin{cases} V_t^h = \sum h_i(t)S_i(t) \\ dV^h(t) = \sum h_i(t)\,dS_i(t)\end{cases}$$

For self-financing no arbitrage:

$dV^h(t) = k(t)V^h(t)dt \iff k(t)=r(t)$

Black-Scholes Assumptions (complete)

- the derivative security can be bought and sold in market
- free of arbitrage

B-S equation

$$\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2S(t)^2\frac{\partial^2 F}{\partial S^2} - rF = 0$$

Boundary condition $\quad F(T,S_T)=\Phi(S_T)$

solving: $dS_t = rS_t\,dt + \sigma S_t\,dW_t$ (非真实)

$F(t,s) = e^{-r(T-t)}E_t^Q[\Phi(S_T)\mid S_t=s]$

Call option

$$F(t,S_t) = S_tN(d_1) - e^{-r(T-t)}KN(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right\}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Put option

$$P = e^{-r(T-t)}KN(-d_2) - SN(-d_1)$$

Forward pricing

$$0 = F(S_0,0) = e^{-rT}E^Q(S_T-K\mid S(0)=S_0) \Rightarrow K=S_0e^{rT}$$

Option on a forward

$$\Pi(S_t) = e^{r(T_1-T)}\underbrace{\left\{e^{-r(T-t)}E^Q\max[S_T-Ke^{-r(T_1-T)},0]\right\}}_{\text{European option with } K'=Ke^{-r(T_1-T)}}$$

call $\quad$ forward

$t \longrightarrow T \longrightarrow T_1$

行权 $\quad$ (forward 交割)

Greeks

Δ	$\frac{\partial C}{\partial S}$	$\Delta^C \in [0,1]$	$\Delta^P \in [-1,0]$	$\Delta P = \Delta C - 1$	$\Delta C = N(d_1)$
θ	$\frac{\partial C}{\partial t}$	$\theta^C > 0$	$\theta^P > 0$	τ: time to maturity	
ν	$\frac{\partial C}{\partial \sigma}$	$\nu^C > 0$	$\nu^P > 0$		
ρ	$\frac{\partial C}{\partial r}$	$\rho^C > 0$	$\rho^P < 0$		
Γ	$\frac{\partial^2 C}{\partial S^2}$	$\Gamma^C = \frac{\partial \Delta^C}{\partial S_t}$	$\Gamma^P = \frac{\partial \Delta^P}{\partial S_t}$		

Delta-hedging

For 1 call option, short $\Delta = N(d_1)$ shares of stock

In continuous time, Δ-hedging is riskless

In discrete time, gain from big movements if long call-option

$\Delta \uparrow$ as $S_t \uparrow$

Θ offsets Γ rent

if $r=0$ $\quad \Theta = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}$

Put-Call Parity

$$C + Ke^{-r(T-t)} = P + S$$

Multi-dimentional Black-Scholes

$dF = \mu_F F\,dt + \sigma_F F\,d\bar{W}_t \qquad$ "P"

$$\begin{cases} \frac{\partial F}{\partial t} + \sum_{i=1}^{n} r S_i F_{S_i} + \frac{1}{2}\mathrm{tr}\{\sigma' S F_{SS} S \sigma\} - rF = 0 \\ F(T, S_T) = \Phi(S_T) \end{cases}$$

$$S = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_n \end{bmatrix}$$

$$\Rightarrow \begin{cases} F(t, S_t) = e^{-r(T-t)} E_t^Q [\Phi(S_T) \mid S(t) = S_t] \\ dS_i = rS_i\,dt + \sigma S_i\,dW_i \end{cases}$$

"Q"

Admissible

$\int_0^t h_S(u)\,dS(u) \geq -\alpha$ for all $(t, T]$

First Fundamental theorem of asset pricing

no arbitrage $\Leftrightarrow$ exists an equivalent martingale measure

Second Fundamental Theore

Q is unique $\Leftrightarrow$ market is complete, all claims replicated

Girsanov Theorem

$dW^P = \phi_t\,dt + dW^Q$ (drift can be modified, not volatility)

多元: $dS = S\vec{\mu}\,dt + \sigma S(dW^Q + \vec{\phi}_t\,dt)$ Exists

$\vec{\phi}_t$: λ in the multi B-S model

Incomplete market and non-tradeble assets

$$\left.\begin{aligned} dF &= \alpha_F F\,dt + \sigma_F F\,d\bar{W}_t \\ dG &= \alpha_G G\,dt + \sigma_G G\,d\bar{W}_t \end{aligned}\right\} \frac{\alpha_F - r}{\sigma_F} = \frac{\alpha_G - r}{\sigma_G} = \lambda(t)$$

$$\frac{\partial F}{\partial t} + [\mu(X_t) - \lambda(X_t)\sigma(X_t)]\frac{\partial^2 F}{\partial X_t^2} = rF(X_t)$$

$$F(T, X_T) = e^{-r(T-t)} E^Q[\Phi(X_T)]$$

$$dX_t = \{\mu(X_t) - \lambda(X_t)\sigma(X_t)\}dt + \sigma(X_t)\,dW_t$$

if traded $\lambda = \frac{\mu - r}{\sigma}$

Dividend

discrete $\quad F_\delta(t, S_t) = F(t, (1-\delta)^n S_t)$

continuous $\quad F_\delta(t, S_t) = F(t, e^{-\delta(T-t)} S_t)$

Exchange option

$$\begin{cases} dS_1 = S_1 \mu\,dt + \sigma_1 S_1\,d\bar{W}_1 \\ dS_2 = \mu S_2\,dt + \sigma_2 S_2\,d\bar{W}_2 \end{cases}$$

$\langle d\bar{W}_1, d\bar{W}_2 \rangle = \rho\,dt$

B-S

$$F_t + rS_1F_1 + rS_2F_2 + \frac{1}{2}\sigma_1^2 S_1^2 F_{11} + \frac{1}{2}\sigma_2^2 S_2^2 F_{22} + \rho S_1 S_2 F_{12} - rF = 0$$

$F = S_2 G(t, z) \qquad (z = \frac{S_1}{S_2})$

$$\Rightarrow \begin{cases} G_t(t,z) + \frac{1}{2}z^2 G_{zz}(t,z)(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) = 0 \\ G(T,z) = \max\{z-1, 0\} \end{cases}$$

$F = S_1 N(d_1) - S_2 N(d_2)$

$d_1 = \frac{1}{\sigma'}\{\ln\frac{S_1}{S_2} + \frac{1}{2}\sigma'^2\}$

$d_2 = d_1 - \sigma' \qquad \sigma' = \sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)(T-t)}$

Exchange rate

$dX_t = X_t(r^d - r^f)dt + \sigma_X X_t\,dW_t$

T-forward

$K = X_0 e^{(r^d - r^f)T}$

Call option

$F(t, X_t) = X_t e^{-r^f(T-t)} N(d_1) - e^{-r^d(T-t)} K N(d_2)$

$d_1 = \frac{1}{\sigma_X\sqrt{T-t}}\{\ln\frac{X}{K} + (r^d - r^f + \frac{1}{2}\sigma_X^2)(T-t)\}$

$d_2 = d_1 - \sigma_X\sqrt{T-t} \qquad$ (用$e^{-r^d(T-t)}$折现)

$S_0 \in [S^l, S^h]$ first hits S^h, 1 dollar

$\pi(t,x) = E^Q[e^{-rT^h} 1\{T^h < T^l\} \mid \mathcal{F}_t]$

$$\begin{cases} rS_t\frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial^2 S_t} - rF = 0 \\ F(S^h) = 1 \\ F(S^l) = 0 \end{cases}$$

guess $F(S_t) = C_1 S_t^{\alpha_1} + C_2 S_t^{\alpha_2}$

α_1, α_2: $r\alpha + \frac{1}{2}\sigma^2\alpha(\alpha-1) - r = 0$

$$C_1 = \frac{(S^h)^{-\alpha_1}}{1 - (\frac{S^h}{S^l})^{\alpha_2 - \alpha_1}}$$

$$C_2 = -\frac{(\frac{S^h}{S^l})^{-\alpha_1} \cdot (S^l)^{-\alpha_2}}{1 - (\frac{S^h}{S^l})^{\alpha_2 - \alpha_1}}$$

$$F = \frac{(\frac{S_t}{S^h})^{\alpha_1} - (\frac{S^h}{S^l})^{-\alpha_1} \cdot (\frac{S_t}{S^l})^{\alpha_2}}{1 - (\frac{S^h}{S^l})^{\alpha_2 - \alpha_1}}$$

---

① Look-back Put Option

历史最低价 pay-off，倒回算 max{ }

② $X_t = \int_0^t g(s)\,dW_s$ martingale

$E[X(t) \mid \mathcal{F}_u] = \int_0^u g(s)\,dW(s) + E[\int_u^t g(s)\,dW(s) \mid \mathcal{F}_u]$

$= X(u) + 0$

③ moment-generating function of brownian motion

$$E[e^{aW(t)}] = e^{\frac{1}{2}a^2 t} \qquad \sim N(0, \sqrt{t})$$

$Z(t) = e^{aW(t)} \qquad dZ(t) = aZ(t)\,dW + \frac{1}{2}a^2 Z(t)\,dt \qquad Z(0) = 1$

④ $\int_0^t W_s\,dW_s = \frac{1}{2}W^2(t) - \frac{1}{2}t$

$Z(t) = W^2(t)$