Fixed Income Final Notes*

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1 Review of the lectures

1.1 Bond math

- The value of \$1 to be received at time T in the future is given by D(T)
- D(T) usually decreases with time
- D(T) is backed out using data from the government bond market
- Price of a bond given the D(T) function:

$$P = \frac{C}{2}\sum_{i=1}^{2T}(D(\frac{i}{2})) + 100D(T)$$

- Price the bonds given yield to maturity:

$$P = \frac{C}{2}\sum_{i=1}^{2T}\frac{1}{(1+\frac{y}{2})^{2i}} + \frac{100}{(1+\frac{y}{2})^{2T}}$$

- Use the D(T) function to compute spot rates:

$$r(T) = 2(\frac{1}{D(T)}^{\frac{1}{2T}} - 1)$$

- Forward rates:

$$(1 +_n f_m)^m = \frac{D(n)}{D(n+m)}$$

$$1 + m_n f_m = \frac{D(n)}{D(n+m)} if m < 1$$

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- Par rates:

$$= 2\frac{100 - 100D(T)}{\sum_{i=1}^{2T} D(\frac{i}{2})}$$

- Forward par rates:

$$= 2\frac{100D(N) - 100D(N+M)}{\sum_{i=1}^{2M} D(N+\frac{i}{2})}$$

- Other formulas:
  - PV of annuity: $\frac{C}{r}(1 - \frac{1}{1+r)^T})$
  - PV of perpetuity: $\frac{C}{r}$
  - PV of growing annuity: $\frac{C(1+g)}{(1+r)}(1 - (\frac{1+g}{1+r})^T)$
  - PV of growing perpetuity: $\frac{C}{r-g}$

- Price yield relationship:
  - p-y are inversely related
  - p-y relationship is not the same for all bonds (when yield go up by 1 basis points, percentage change in prices of all bonds is not the same)
  - p-y relationship is approximately symmetric for small changes in the yields
  - p-y relationship is not symmetric for large changes in the yield curve

- First order approximation of p-y relationship is done using:
  - DV01: Dollar value of 1 basis points change in the yield curve. To measure DV01 compute the price of the bond, given its yield- to-maturity. Then change the ytm of the bond by 1 basis point (increase/decrease). Recompute the price of the bond. The difference of these two values gives you the DV01 of the bond. Remember to take the absolute value of this difference
  - Duration: Macaulay's duration is computed using the formula:

$$MC = \frac{\sum_{i=1}^{N} iCF(i)D(i)}{\sum_{i=1}^{N} CF(i)D(i)}$$

  - Modified duration is computed as:

$$MD = \frac{MC}{1+\frac{y}{2}}$$

- Hedging using duration: Compute the duration of the bonds in your portfolio (remember to compute the weighted average duration)

- Price approximation using duration and convexity:

$$\frac{\Delta P}{P} = -MD\Delta y + \frac{1}{2}(Convexity)(\Delta y)^2$$

- Total return on a bond/bond portfolio:

  -
  $$\mathbb{E}[\frac{\Delta P}{P}] = \text{“}ytm\text{”} - MD\mathbb{E}[\Delta y] + \frac{1}{2}(Convexity)\mathbb{V}[(\Delta y)]$$

  - An analysis of this equation allows you to benchmark fund performance
  - Also shows what different "styles" can bond managers adopt

1.2 Interest Rate Models

- Merton model:

  - Given by: $dr_t = \alpha dt + \sigma dZ_t$
  - You are sitting at time 0 and know the short rate (say the 3-month rate) now which is r_0
  - The short rate at any time in the future is given by r_t
  - This rate will be the current rate r_0 plus a trend component α_t plus a random component.
  - Given that I know r_0 I can compute D(T) for any maturity T at time 0 as $D(T) = exp(-r_0 T - \frac{\alpha}{2}T^2 + \frac{\sigma^2 T^3}{6})$
  - Yield to maturity on zero coupon bonds (spot rates) are given by $ytm = r_0 + \frac{\alpha}{2}T - \frac{\sigma^2 T^2}{6}$
  - Analysis: Not a realistic model; Produces negative interest rates; Volatility constant; Futures always increase with T. Does not account for jumps in interest rates

- Vasicek model:

  - Given by: $dr_t = (\alpha - \beta r)dt + \sigma dZ_t$
  - Can think of this as: $r_1 = r_0 + (\alpha - \beta r_0)(1 - 0) + \sigma(Z_1 - Z_0)$

- You are sitting at time 0 and know the short rate (say the 3-month rate) now which is r0
- The short rate 1 period from now at time 1 is given by r_1
- This rate will be the current rate r_0 plus a trend component $(\alpha - \beta r_0)(1-0)$ plus a random component $\sigma(Z_1 - Z_0)$
- Given that I know r_0, I can compute the D(T) for any maturity T at time 0 as $D(T) = A(T)exp(-B(T)r_0)$
- Here:

$$A(T) = exp\left[\left(\frac{\sigma^2}{2\beta^2} - \frac{\alpha}{\beta}\right)T + \left(\frac{\alpha}{\beta^2} - \frac{\sigma^2}{\beta^3}\right)(1 - exp(-\beta T)) + \frac{\sigma^2}{4\beta^3}(1 - exp(-2\beta T))\right]$$

$$B(T) = \frac{1}{\beta}(1 - exp(\beta T))$$

- Yield to maturity on zero coupon bonds (spot rates) are given by: $ytm(T) = \frac{-ln(A(T))}{T} + \frac{B(T)}{T}r_0$
- Analysis: Interest rates can be negative; Constant volatility

- CIR model:
  - Given by: $dr = (\alpha - \beta r)dt + \sigma\sqrt{r}dZ$
  - Can think of this as: $r_1 = r_0 + (\alpha - \beta r_0)(1-0) + \sigma\sqrt{r_0}(Z_1 - Z_0)$
  - You are sitting at time 0 and know the short rate , which now which is r_0
  - The short rate 1 period from now at time 1 is given byr_1
  - This rate will be the current rate r_0 plus a trend component $(\alpha - \beta r_0)(1-0)$ plus a random component $\sigma\sqrt{r_0}(Z_1 - Z_0)$
  - Given that I know r_0, I can compute the D(T) for any maturity T at time 0 as $D(T) = A(T)exp(-B(T)r_0)$
  - For A(T) and B(T), see lecture notes.
  - Yield to maturity on zero coupon bonds (spot rates) are given by: $ytm(T) = \frac{-ln(A(T))}{T} + \frac{B(T)}{T}r_0$
  - Analysis: Interest rates cannot be negative; Non constant volatility

1.3 Fixed income derivatives

- Some general comments:
  - Fixed income derivatives are usually priced by simulation
  - In general, we will compute the cash flows for the given instrument
  - And discount the cash flows back to time 0 to compute the price
  - We will typically do this several hundred (more like several thousand) times. And take the average of these prices to compute the actual price that we should buy or sell this instrument at.
- Futures: $H_0 = \mathbb{E}[S_t]$
- Floating rate notes: Priced at par
- Swaps:
  - Generally involves exchange of fixed payments for floating payments
  - Floating leg is priced at par on reset date.
  - Rate on the fixed leg is also set so that it is priced at par
  - Forward swaps are swaps that start at some date in the future. If the swap starts at some date N in the future and lasts M years after that then the rate on the fixed leg is given by:

$$FSR = 2\frac{D(N) - D(N+M)}{\sum_{i=1}^{2M} D(N+\frac{i}{2})}$$

  - In order to price a swap, you need to use an interest rate model to compute future cash flows from the swap and discount the cash flows back to time zero. First, obtain the D(T) function/values you need to price the fixed leg. The value of the swap for the fixed rate receiver will be: $value_{fixed} - 1$ (per dollar of notional value), on a reset date.
- Caplets and Floorlets: Caplets are like European call options, Floorlets are like European put options