Option Markets MGMTMFE 406

Binomial Option Pricing (weeks 3 and 4)

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During these two classes (6 hours) we will learn:

- how to build binomial trees
- how to price European & American options using binomial trees
- how to price options with "exotic" payoffs using binomial trees
- how to price path-dependent options using binomial trees
- how to price convertible bonds using binomial trees
- how to embed the notion of credit risk into a binomial tree
- how to apply derivatives theory to the operation and valuation of real investment projects (real options)
- how to use simulation to price American options

Outline

Outline

Discrete-Time Option Pricing: The Binomial Model

- Until now, we have looked only at some basic principles of option pricing
  - We examined payoff and profit diagrams, and upper/lower bounds on option prices
  - We saw that with put-call parity we could price a put or a call based on the prices of the combinations on instruments that make up the synthetic version of the put or call.
- What we need to be able to do is price a put or a call without the other instrument.
- In this section, we introduce a simple means of pricing an option.

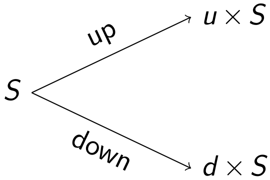
Discrete-Time Option Pricing: The Binomial Model (cont'd)

- The approach we take here is called the **binomial tree**.
- The word "binomial" refers to the fact that there are only two outcomes (we let the underlying price move to only one of two possible new prices).
- It may appear that this framework oversimplifies things, but the model can eventually be extended to encompass all possible prices.

Outline

The Binomial Option Pricing Model

- The binomial option pricing model assumes that, over a period of time, the price of the underlying asset can move only up or down by a specified amount—that is, the asset price follows a binomial distribution:



A Simple Example

- XYZ does not pay dividends and its current price is \$41. In one year the price can be either \$59.954 or \$32.903, i.e., $u = 1.4623$ and $d = 0.8025$.
- Consider a European call option on the stock of XYZ, with a \$40 strike an 1 year to expiration. The continuously compounded risk-free interest rate is 8%.
- We wish to determine the option price:

$$
\begin{tikzcd}[column sep=large]
 & \begin{array}{l} uS = 59.954 \\ C_u = \max\left[0, 59.954 - 40\right] = 19.954 \end{array} \\
\begin{array}{c} S = 41 \\ \boxed{C = ?} \end{array} \arrow[ur] \arrow[dr] & \\
 & \begin{array}{l} dS = 32.903 \\ C_d = \max\left[0, 32.903 - 40\right] = 0 \end{array}
\end{tikzcd}
$$

A Simple Example (cont'd)

- Let us try to find a portfolio that mimics the option **(replicating portfolio)**.
- We have two instruments: shares of stock and a position in bonds (i.e., borrowing or lending).
- To be specific, we wish to find a portfolio consisting of Δ shares of stock and a dollar amount B in borrowing or lending, such that the portfolio imitates the option whether the stock rises or falls.
- The value of this replicating portfolio at maturity is:

$$\begin{cases} 59.954\Delta + e^{0.08}B & = 19.954 \\ 32.903\Delta + e^{0.08}B & = 0 \end{cases} \tag{1}$$

A Simple Example (cont'd)

- The unique solution of this system of 2 equations with 2 unknowns is

$$\Delta = 0.738, \; B = -22.405, \tag{2}$$

  i.e., buy 0.738 shares of XYZ and borrow \$22.405 at the risk-free rate.
- In computing the payoff for the replicating portfolio, we assume that we sell the shares at the market price and that we repay the borrowed amount, plus interest.
- Thus, we obtain that the option and the replicating portfolio have the same payoff: \$19.954 if the stock price goes up and \$0 if the stock price goes down.

A Simple Example (cont'd)

- By the **law of one price**, positions that have the same payoff should have the same cost.
- The price of the option must be

$$C = \underbrace{0.738 \times \$41}_{risky} - \underbrace{\$22.405}_{risk-free} = \$7.839 \tag{3}$$

Arbitraging a Mispriced Option

- Suppose that the market price for the option is \$8 instead of \$7.839 (the option is overpriced).
- We can sell the option and buy a synthetic option at the same time **(buy low and sell high)**. The initial cash flow is

$$\$8.00 - \$7.839 = \$0.161 \tag{4}$$

and there is no risk at expiration:

	Stock Price in 1 Year	
	\$32.903	\$59.954
Written call	\$0	-\$19.954
0.738 purchased shares	\$24.271	\$44.225
Repay loan of \$22.405	-\$24.271	-\$24.271
Total payoff	\$0	\$0

Arbitraging a Mispriced Option (cont'd)

- Suppose that the market price for the option is \$7.5 instead of \$7.839 (the option is underpriced).
- We can buy the option and sell a synthetic option at the same time **(buy low and sell high)**. The initial cash flow is

$$\$7.839 - \$7.5 = \$0.339 \tag{5}$$

and there is no risk at expiration:

	Stock Price in 1 Year	
	\$32.903	\$59.954
Purchased call	\$0	\$19.954
0.738 short-sold shares	-\$24.271	-\$44.225
Sell T-bill	\$24.271	\$24.271
Total payoff	\$0	\$0

A Remarkable Result

- So far we have not specified the probabilities of the stock going up and down.
- In fact, probabilities were not used anywhere in the option price calculations.
- This is a remarkable result: **Since the strategy of holding Δ shares and B bonds replicates the option whichever way the stock moves, the probability of an up or down movement in the stock is irrelevant for pricing the option**.

The Binomial Solution

- Suppose that the stock has a continuous dividend yield of δ, which is reinvested in the stock. Thus, if you buy one share at time 0 and the length of a period is h, at time h you will have $e^{\delta h}$ shares.
- The up and down movements of the stock price reflect the **ex-dividend** price.
- We can write the stock price as uS_0 when the stock goes up and dS_0 when the stock goes down. We can represent the tree for the stock and the option as follows:

$$\begin{tikzcd}[row sep=small]
 & \begin{matrix} uS_0 \\ C_1^u \end{matrix} \\
\begin{matrix} S_0 \\ C_0 \end{matrix} \arrow[ur] \arrow[dr] & \\
 & \begin{matrix} dS_0 \\ C_1^d \end{matrix}
\end{tikzcd}$$

The Binomial Solution (cont'd)

- If the length of a period is h, the interest factor per period is e^{rh}.
- A successful replicating portfolio will satisfy

$$\begin{cases} \Delta \times S_0 \times u \times e^{\delta h} + B \times e^{rh} & = C_1^u \\ \Delta \times S_0 \times d \times e^{\delta h} + B \times e^{rh} & = C_1^d \end{cases} \tag{6}$$

- This is a system of two equations in two unknowns Δ and B. Solving for Δ and B gives

$$\begin{cases} \Delta & = e^{-\delta h} \frac{C_1^u - C_1^d}{S_0(u-d)} \\ B & = e^{-rh} \frac{C_1^d u - C_1^u d}{u-d} = e^{-rh} \left(C_1^u - \Delta S_0 u e^{\delta h} \right) \end{cases} \tag{7}$$

The Binomial Solution (cont'd)

- Given the expressions (7) for Δ and B, we can derive a simple formula for the value of the option. The cost of creating the option is the net cash required to buy the shares and bonds. Thus, the cost of the option is

$$\begin{aligned} C_0 &= \Delta S_0 + B \\ &= e^{-rh}\left(C_1^u \frac{e^{(r-\delta)h} - d}{u-d} + C_1^d \frac{u - e^{(r-\delta)h}}{u-d}\right) \end{aligned} \tag{8}$$

- Note that **if we are interested only in the option price, it is not necessary to solve for Δ and B**; that is just an intermediate step. If we want to know only the option price, we can use equation (8) directly.

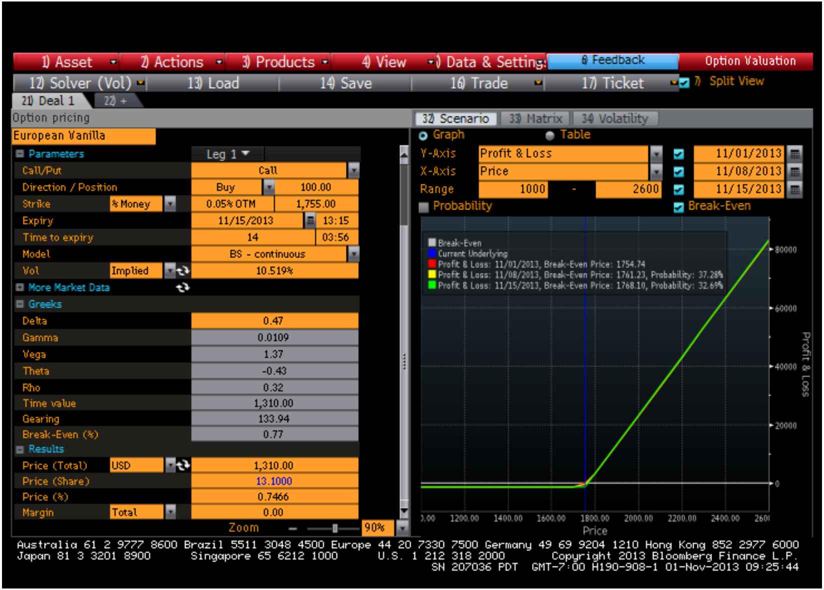
The Binomial Solution (cont'd)

- The assumed stock price movements, u and d, should not give rise to arbitrage opportunities. In particular, we require that

$$d < e^{(r-\delta)h} < u \tag{9}$$

- Note that because Δ is the number of shares in the replicating portfolio, it can also be interpreted as the sensitivity of the option to a change in the stock price. If the stock prices changes by \$1, then the option price, $\Delta S + B$, changes by Δ.

Delta of a Call Option on S&P 500



The Binomial Solution, Special Case: $\delta = 0$ and $h = 1$

- The solution for Δ and B reduces to

$$\begin{cases} \Delta & = \frac{C_1^u - C_1^d}{S_0(u-d)} \\ B & = e^{-r}\frac{C_1^d u - C_1^u d}{u-d} = e^{-r}\left(C_1^u - \Delta S_0 u\right) \end{cases} \tag{10}$$

- The option price further simplifies to

$$C_0 = \Delta S_0 + B = e^{-r}\left(C_1^u \frac{e^r - d}{u-d} + C_1^d \frac{u - e^r}{u-d}\right) \tag{11}$$

Problem 10.1.a: Let $S = \$100$, $K = \$105$, $r = 8\%$ (continuously compounded), $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and the number of binomial periods $n = 1$. What are the premium, Δ, and B for a European call?

Problem 10.1.a: Let $S = \$100$, $K = \$105$, $r = 8\%$ (continuously compounded), $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and the number of binomial periods $n = 1$. What are the premium, Δ, and B for a European call?

Using the formulas given in (7), we calculate the following values:

$$\Delta = 0.5$$
$$B = -38.4316$$
$$\text{Call price} = 11.5684$$

Problem 10.21: Suppose that $u < e^{(r-\delta)h}$. Show that there is an arbitrage opportunity. Now suppose that $d > e^{(r-\delta)h}$. Show again that there is an arbitrage opportunity.

Problem 10.21: Suppose that $u < e^{(r-\delta)h}$. Show that there is an arbitrage opportunity. Now suppose that $d > e^{(r-\delta)h}$. Show again that there is an arbitrage opportunity.

- If $u < e^{(r-\delta)h}$, we short a **tailed position** of the stock and invest the proceeds at the interest rate. There is an arbitrage opportunity:

	$t = 0$	$state = d$	$state = u$
Short stock	$+e^{-\delta h}S$	$-d \times S$	$-u \times S$
Lend money	$-e^{-\delta h}S$	$+e^{(r-\delta)h}S$	$+e^{(r-\delta)h}S$
Total	0	>0	>0

- If $d > e^{(r-\delta)h}$, we buy a **tailed position** of the stock and borrow at the interest rate. There is an arbitrage opportunity:

	$t = 0$	$state = d$	$state = u$
Buy stock	$-e^{-\delta h}S$	$+d \times S$	$+u \times S$
Borrow money	$e^{-\delta h}S$	$-e^{(r-\delta)h}S$	$-e^{(r-\delta)h}S$
Total	0	>0	>0

A Two-Period Binomial Tree

- We can extend the previous example to price a 2-year option, assuming all inputs are the same as before.

$$
\begin{array}{ccc}
 & & \begin{array}{l} S_2^{uu} = 87.669 \\ C_2^{uu} = 47.669 \end{array} \\
 & \begin{array}{l} S_1^{u} = 59.954 \\ C_1^{u} = ? \end{array} & \\
\begin{array}{l} S_0 = 41 \\ C_0 = ? \end{array} & & \begin{array}{l} S_2^{ud} = 48.114 \\ C_2^{ud} = 8.114 \end{array} \\
 & \begin{array}{l} S_1^{d} = 32.903 \\ C_1^{d} = ? \end{array} & \\
 & & \begin{array}{l} S_2^{dd} = 26.405 \\ C_2^{dd} = 0 \end{array}
\end{array}
$$

- Note that an up move followed by a down move (S_2^{ud}) generates the same stock price as a down move followed by an up move (S_2^{du}). This is called a **recombining tree**.

A Two-Period Binomial Tree (cont'd)

- To price the option when we have two binomial periods, we need to work **backward** through the tree.
- Suppose that in period 1 the stock price is $S_1^u = \$59.954$. We can use equation (11) to derive the option price:

$$C_1^u = e^{-r}\left(C_2^{uu}\frac{e^r - d}{u - d} + C_2^{ud}\frac{u - e^r}{u - d}\right) = \$23.029 \tag{12}$$

- Using equations (10), we can also solve for the composition of the replicating portfolio:

$$\Delta = 1,\ B = -36.925 \tag{13}$$

i.e., buy 1 share of XYZ and borrow \$36.925 at the risk-free rate, which costs $1 \times \$59.954 - \$36.925 = \$23.029$.

A Two-Period Binomial Tree (cont'd)

- Suppose that in period 1 the stock price is $S_1^d = \$32.903$. We can use equation (11) to derive the option price:

$$C_1^d = e^{-r}\left(C_2^{ud}\frac{e^r - d}{u-d} + C_2^{dd}\frac{u-e^r}{u-d}\right) = \$3.187 \tag{14}$$

- Using equations (10), we can also solve for the composition of the replicating portfolio:

$$\Delta = 0.374,\ B = -9.111 \tag{15}$$

i.e., buy 0.374 shares of XYZ and borrow \$9.111 at the risk-free rate, which costs $0.374 \times \$32.903 - \$9.111 = \$3.187$.

A Two-Period Binomial Tree (cont'd)

- Move backward now at period 0. The stock price is $S_0 = 41$. We can use equation (11) to derive the option price:

$$C_0 = e^{-r}\left(C_1^u \frac{e^r - d}{u - d} + C_1^d \frac{u - e^r}{u - d}\right) = \$10.737 \tag{16}$$

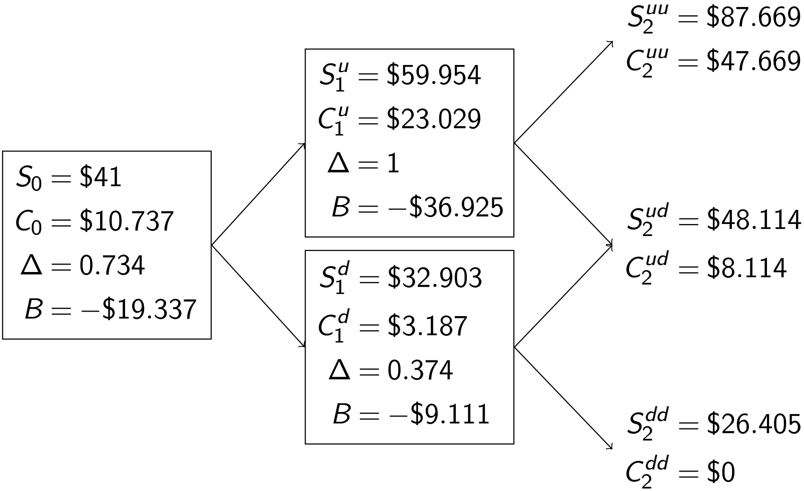
- Using equations (10), we can also solve for the composition of the replicating portfolio:

$$\Delta = 0.734, \;\; B = -19.337 \tag{17}$$

i.e., buy 0.734 shares of XYZ and borrow \$19.337 at the risk-free rate, which costs $0.734 \times \$41 - \$19.337 = \$10.737$.

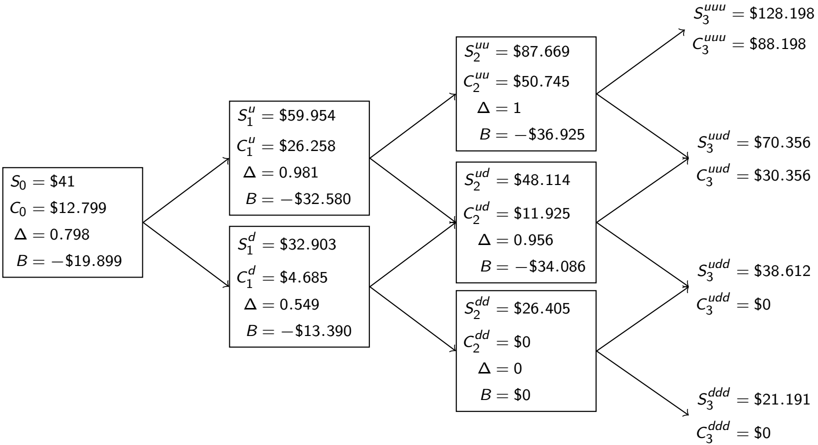
A Two-Period Binomial Tree (cont'd)

- The two-period binomial tree with the option price at each node as well as the details of the replicating portfolio is:



Many Binomial Periods: Three-Period Example

- Once we understand the two-period option it is straightforward to value an option using more than two binomial periods.
- The important principle is to work backward through the tree:



Self-Financing Strategy

- In the two-period binomial tree example, suppose that the stock moves from $S_0 = \$41$ to $S_1^u = \$59.954$.
- The replicating portfolio should be modified as follows:
  1. Buy $1 - 0.734 = 0.266$ shares of XYZ (increase the XYZ position from 0.734 shares to 1 share), which costs $0.266 \times \$59.954 = \$15.977$.
  2. Increase borrowing from $\$19.337 \times e^{0.08} = \$20.947$ to $\$36.925$, which yields $15.977.
- The amount necessary to buy shares is equal to the amount obtained from increased borrowing.
- Modifying the portfolio does not require additional cash. Thus, the replicating portfolio is **self-financing**.

Self-Financing Strategy (cont'd)

- If the stock moves from $S_0 = \$41$ to $S_1^d = \$32.903$, the replicating portfolio should be modified as follows:
  1. Sell $0.734 - 0.374 = 0.360$ shares of XYZ (decrease the XYZ position from 0.734 shares to 0.374 shares), which yields $0.360 \times \$32.903 = \$11.836$.
  2. Decrease borrowing from $\$19.337 \times e^{0.08} = \$20.947$ to $\$9.111$, which costs $\$11.836$.
- The amount obtained from selling shares is equal to the amount necessary to decrease borrowing.
- Modifying the portfolio does not require additional cash. Once again, the replicating portfolio is **self-financing**.

Outline

Put Options

- We compute put option prices using the same stock price tree and in the same way as call option prices.
- The only difference with an European put option occurs at expiration: Instead of computing the price as $\max[0, S - K]$, we use $\max[0, K - S]$.
- Here is a two-period binomial tree for an European put option with a \$40 strike:

Put Options (cont'd)

$$
\begin{tikzcd}
 & & \begin{array}{l} S_2^{uu} = \$87.669 \\ P_2^{uu} = \$0 \end{array} \\
 & \boxed{\begin{array}{r} S_1^u = \$59.954 \\ P_1^u = \$0 \\ \Delta = 0 \\ B = \$0 \end{array}} \arrow[ur] \arrow[dr] & \\
\boxed{\begin{array}{r} S_0 = \$41 \\ P_0 = \$3.823 \\ \Delta = -0.266 \\ B = \$14.749 \end{array}} \arrow[ur] \arrow[dr] & & \begin{array}{l} S_2^{ud} = \$48.114 \\ P_2^{ud} = \$0 \end{array} \\
 & \boxed{\begin{array}{r} S_1^d = \$32.903 \\ P_1^d = \$7.209 \\ \Delta = -0.626 \\ B = \$27.814 \end{array}} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{l} S_2^{dd} = \$26.405 \\ P_2^{dd} = \$13.595 \end{array}
\end{tikzcd}
$$

- The proof that the replicating portfolio is self-financing is left as an exercise.

Put Options (Using the Parity Relationship)

- For non-dividend paying stocks, the basic parity relationship for European options with the same strike price and time to expiration is

$$C_t - P_t = S_t - PV\,(\text{strike price}) \tag{18}$$

- We can use this relationship to find the put price at all nodes:

$S_0 = 41$
$C_0 = 10.737$
$\boxed{P_0 = 3.823}$

$S_1^u = 59.954$
$C_1^u = 23.029$
$\boxed{P_1^u = 0}$

$S_1^d = 32.903$
$C_1^d = 3.187$
$\boxed{P_1^d = 7.209}$

$S_2^{uu} = 87.669$
$C_2^{uu} = 47.669$
$\boxed{P_2^{uu} = 0}$

$S_2^{ud} = 48.114$
$C_2^{ud} = 8.114$
$\boxed{P_2^{ud} = 0}$

$S_2^{dd} = 26.405$
$C_2^{dd} = 0$
$\boxed{P_2^{dd} = 13.595}$

Problem 10.1.b: Let $S = \$100$, $K = \$105$, $r = 8\%$ (continuously compounded), $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and the number of binomial periods $n = 1$. What are the premium, Δ, and B for a European put?

Problem 10.1.b: Let $S = \$100$, $K = \$105$, $r = 8\%$ (continuously compounded), $T = 0.5$, and $\delta = 0$. Let $u = 1.3$, $d = 0.8$, and the number of binomial periods $n = 1$. What are the premium, Δ, and B for a European put?

Using the formulas given in (7), we calculate the following values:

$$\Delta = -0.5$$
$$B = 62.4513$$
$$\text{Put price} = 12.4513$$

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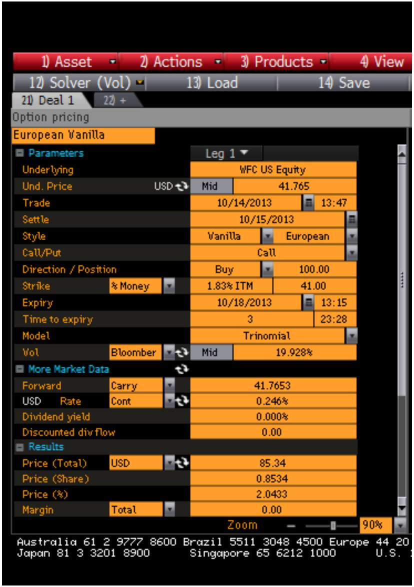

	A	B	C	D
1				
2		Und. Price	41.765	
3		Strike	41	
4		Vol	0.19928	
5				
6		USD Rate	0.00246	
7		Time to expiry	0.01096	(=4/365)
8		Dividend yield	0	
9		Style (E=0, A=1)	0	
10		Nb. Periods	100	
11		h	0.00011	(about 1h)
12				
13		Call Price	0.8552	
14				

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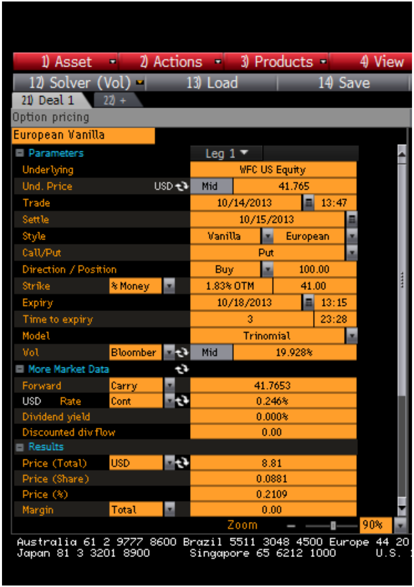


	A	B	C	D
1				
2		Und. Price	41.765	
3		Strike	41	
4		Vol	0.19928	
5				
6		USD Rate	0.00246	
7		Time to expiry	0.01096	(=4/365)
8		Dividend yield	0	
9		Style (E=0, A=1)	0	
10		Nb. Periods	100	
11		h	0.00011	(about 1h)
12				
13		Put Price	0.0891	

Outline

Non-Recombining Binomial Trees

- We show how to modify our binomial tree to deal with path dependent options
- We illustrate the ideas with **Asian options** (also called **Average Rate Options**)
- Using a non-recombining binomial tree will mean that at time $t = n$ there will be 2^n states of the world as opposed to $n + 1$ states with recombining trees
- This causes problems as the number of states is growing exponentially

Asian Option Example

- Let's go back to the previous example with two periods: $S_0 = \$41$, $u = 1.4623$, $d = 0.8025$, $K = 40$, $T = 2$, $h = 1$, and $r = 0.08$.
- Consider an Asian call option which pays-off the following non-negative amount at maturity:

$$G_T = \max\left[\frac{1}{3}\sum_{k=0}^{2} S_k - 40, 0\right] \tag{19}$$

- The two-period binomial tree with the option price at each node as well as the replicating portfolio is (details will be provided in class):

Asian Option Example (cont'd)

	A	B	C	D	E	F	G
1							
2		S	41				
3		K	40				
4		u	1.4623				87.6712
5		d	0.8025				62.8752
6		r	0.08				22.8752
7		T	2		59.9543		
8		delta	0		50.4772		
9					14.1243		
10		periods	2		0.3333		48.1133
11		h	1		-5.8605		49.6892
12							9.6892
13		Stock	41				
14		Mean	41				
15		Call	5.6886				
16		Shares	0.5124				48.1133
17		Cash	-15.3182				40.6719
18							0.6719
19					32.9025		
20					36.9513		
21					0.2640		
22					0.0310		26.4043
23					-0.7544		33.4356
24							0
25							
26							

Asian Option Example (cont'd)

- It is left as an exercise to price the option whose payoff depends on the **geometric average**:

$$G_T = \max\left[\left(\prod_{k=0}^{2} S_k\right)^{1/3} - 40, 0\right] \tag{20}$$

- See Chapter 14.2 in McDonald.

Outline

Uncertainty in the Binomial Model

- A natural measure of uncertainty about the stock return is the **annualized standard deviation of the continuously compounded stock return**, which we will denote by σ.
- If we split the year into n periods of length h (so that $h = 1/n$), the standard deviation over the period of length h, σ_h, is (assuming returns are uncorrelated over time)

$$\sigma_h = \sigma\sqrt{h} \tag{21}$$

- In other words, the standard deviation of the stock return is proportional to the **square root** of time.

Uncertainty in the Binomial Model (cont'd)

- We incorporate **uncertainty** into the binomial tree by modeling the up and down moves of the stock price. Without uncertainty, the stock price next period must equal:

$$S_{t+h} = S_t e^{(r-\delta)h} \tag{22}$$

- To interpret this, without uncertainty, the rate of return on the stock must be the risk-free rate. Thus, the stock price must rise at the risk-free rate less the dividend yield, $r - \delta$.
- We now model the stock price evolution as

$$\begin{aligned} uS_t &= S_t e^{(r-\delta)h} e^{+\sigma\sqrt{h}} \\ dS_t &= S_t e^{(r-\delta)h} e^{-\sigma\sqrt{h}} \end{aligned} \tag{23}$$

Uncertainty in the Binomial Model (cont'd)

- We can rewrite this as

$$\begin{aligned} u &= e^{(r-\delta)h+\sigma\sqrt{h}} \\ d &= e^{(r-\delta)h-\sigma\sqrt{h}} \end{aligned} \tag{24}$$

- Return has two parts, one of which is certain $[(r-\delta)\,h]$, and the other of which is uncertain and generates the up and down stock price moves $\left(\sigma\sqrt{h}\right)$.
- Note that if we set volatility equal to zero, we are back to (22) and we have $S_{t+h} = uS_t = dS_t = S_t e^{(r-\delta)h}$. **Zero volatility does not mean that prices are fixed; it means that prices are known in advance**.

Uncertainty in the Binomial Model (cont'd)

- In our example we assumed that $u = 1.4623$ and $d = 0.8025$. These correspond to an annual stock price volatility of 30%:

$$\begin{aligned} u &= e^{(0.08-0)\times 1+0.3\times\sqrt{1}} = 1.4623 \\ d &= e^{(0.08-0)\times 1-0.3\times\sqrt{1}} = 0.8025 \end{aligned} \tag{25}$$

- We will use equations (24) to construct binomial trees. This approach (called the **forward tree** approach) is very convenient because it never violates the no arbitrage restriction

$$d < e^{(r-\delta)h} < u$$

Problem 10.19: For a stock index, $S = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 3\%$, and $T = 3$. Let $n = 3$.

- What is the price of a European call option with a strike of \$95?
- What is the price of a European put option with a strike of \$95?

Problem 10.19: For a stock index, $S = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 3\%$, and $T = 3$. Let $n = 3$.

- What is the price of a European call option with a strike of $95?

	A	B	C	D
1				
2		S	100	
3		K	95	
4		sigma	0.3	
5		r	0.05	
6		Maturity (T)	3	
7		delta (d)	0.03	
8		optstyle	0	
9		Periods (n)	3	
10				
11		=BinomCall(C2,C3,C4,C5,C6,C7,C8,C9)		

The price of a European call option with a strike of 95 is $24.0058

- What is the price of a European put option with a strike of $95?

	A	B	C	D
1				
2		S	100	
3		K	95	
4		sigma	0.3	
5		r	0.05	
6		Maturity (T)	3	
7		delta (d)	0.03	
8		optstyle	0	
9		Periods (n)	3	
10				
11		=BinomPut(C2,C3,C4,C5,C6,C7,C8,C9)		

The price of a European put option with a strike of 95 is $14.3799

Outline

Risk-Neutral Pricing

- There is a probabilistic interpretation of the binomial solution for the price of an option (equation 8, restated below):

$$C_0 = e^{-rh}\left(C_1^u \frac{e^{(r-\delta)h}-d}{u-d} + C_1^d \frac{u-e^{(r-\delta)h}}{u-d}\right) \tag{26}$$

- The terms $\frac{e^{(r-\delta)h}-d}{u-d}$ and $\frac{u-e^{(r-\delta)h}}{u-d}$ sum to 1 and are both positive (this follows from inequality 9).
- Thus, we can interpret these terms as probabilities. Equation (8) can be written as

$$C_0 = e^{-rh}\left[p^* C_1^u + (1-p^*)\, C_1^d\right] \tag{27}$$

- This expression has the appearance of an expected value discounted at the risk-free rate. Thus, we will call $p^* \equiv \frac{e^{(r-\delta)h}-d}{u-d}$ the **risk-neutral probability** of an increase in the stock price.

Risk-Neutral Pricing (cont'd)

- Pricing options using risk-neutral probabilities can be done in one step, no matter how large the number of periods. In our example, $p^* = 0.4256$. For the one-period European call option we have:

Call Price in 1 Year	**Probability**
\$19.954	0.4256
\$0	0.5744

- The call price is

$$C_0 = e^{-0.08}(0.4256 \times \$19.954 + 0.5744 \times \$0) = \$7.839 \tag{28}$$

Risk-Neutral Pricing (cont'd)

- For the two-period European call option we have:

Call Price in 2 Years	Probability
\$47.669	$p^{*2} = 0.1811$
\$8.114	$2p^*(1-p^*) = 0.4889$
\$0	$(1-p^*)^2 = 0.3300$

- The call price is

$$\begin{aligned} C_0 &= e^{-0.08\times 2}(0.1811 \times \$47.669 + 0.4889 \times \$8.114 + 0.3300 \times \$0) \\ &= \$10.737 \end{aligned} \tag{29}$$

- The probability of reaching any given node is the probability of one path reaching that node times the number of paths reaching that node. For example, the probability of reaching the node $S_2^{ud} = \$48.114$ is $2p^*(1-p^*)$.
- It can be easily verified that the sum of probabilities in the table above is 1.

Risk-Neutral Pricing (cont'd)

- For the three-period European call option we have:

Call Price in 3 Years	**Probability**
\$88.198	$p^{*3} = 0.0771$
\$30.356	$3p^{*2}(1-p^*) = 0.3121$
\$0	$3p^*(1-p^*)^2 = 0.4213$
\$0	$(1-p^*)^3 = 0.1896$

- The call price is

$$C_0 = e^{-0.08\times 3}\sum_{k=0}^{3}\frac{3!}{k!(3-k)!}p^{*k}(1-p^*)^{3-k}\max\left[S_0u^kd^{3-k}-K,0\right] \qquad (30)$$
$$= \$12.799$$

- It can be easily verified that the sum of probabilities in the table above is 1.
- It is left as an exercise to find the price of the two-period European put using risk-neutral probabilities.

Risk-Neutral Pricing (cont'd)

- For an arbitrary number of periods n, the price of an European call option is given by

$$C_0 = e^{-rT} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} p^{*k} \left(1 - p^*\right)^{n-k} \max\left[S_0 u^k d^{n-k} - K, 0\right] \tag{31}$$

- We will use this formula later on when talking about Black-Scholes: when the number of steps becomes great enough the price of the option appear to approach a limiting value. This value is given by the Black-Scholes formula.

Problem 11.12: Let $S = \$100$, $\sigma = 0.3$, $r = 0.08$, $T = 1$, and $\delta = 0$. Use equation (31) to compute the risk-neutral probability of reaching a terminal node and the price at that node for $n = 3$. Plot the risk-neutral distribution of year-1 stock prices.

Problem 11.12: Let $S = \$100$, $\sigma = 0.3$, $r = 0.08$, $T = 1$, and $\delta = 0$. Use equation (31) to compute the risk-neutral probability of reaching a terminal node and the price at that node for $n = 3$. Plot the risk-neutral distribution of year-1 stock prices.

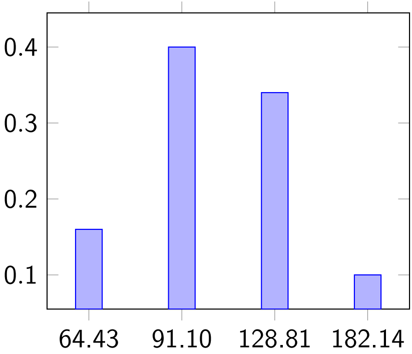
For $n = 3$, u and d are calculated as follows:

$$u = e^{(0.08-0)\times 1/3+0.3\times\sqrt{1/3}} = 1.2212$$

$$d = e^{(0.08-0)\times 1/3-0.3\times\sqrt{1/3}} = 0.8637$$

It follows that $p^* = 0.4568$, and

$n-k$	Stock price	Probability
0	182.14	0.0953
1	128.81	0.3400
2	91.10	0.4044
3	64.43	0.1603



Understanding Risk-Neutral Pricing

- A **risk-neutral** investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing.
- A **risk-averse** investor prefers a sure thing to a risky bet with an expected payoff equal to the value of the sure thing.
- Formula (27) suggests that we are discounting at the risk-free rate, even though the risk of the option is at least as great as the risk of the stock.
- Thus, the option pricing formula, equation (27), can be said to price options **as if** investors are risk-neutral.
- **Is this option pricing consistent with standard discounted cash flow calculations?**

Understanding Risk-Neutral Pricing (cont'd)

- Assume, as in our example, that a stock does not pay dividends and the length of a period is 1 year ($\delta = 0$ and $h = 1$).

Risk-Neutral Pricing

- The risk-free rate is the discount rate for any asset including the stock:

$$S_0 = e^{-r}\left[p^* u S_0 + (1 - p^*)\, d S_0\right]$$

Solving for p^* gives us

$$p^* = \frac{e^r - d}{u - d}$$

Standard DCF calculation

- Suppose that the continuously compounded expected return on the stock is α. If p is the true probability of the stock going up, p must be consistent with u, d, and α.

$$S_0 = e^{-\alpha}\left[p u S_0 + (1 - p)\, d S_0\right]$$

Solving for p gives us

$$p = \frac{e^\alpha - d}{u - d}$$

Understanding Risk-Neutral Pricing (cont'd)

Risk-Neutral Pricing

- We discount the option payoff at the risk-free rate, r.
- The option price is

$$C_0 = e^{-r}\left[p^* C_1^u + (1-p^*)\, C_1^d\right]$$

Standard DCF calculation

- At what rate do we discount the option payoff? Since an option is equivalent to holding a portfolio consisting of Δ shares of stock and B bonds, the expected return on this portfolio is

$$e^{\gamma} = \frac{S_0\Delta}{S_0\Delta + B}e^{\alpha} + \frac{B}{S_0\Delta + B}e^{r}$$

where γ is the discount rate for the option.
- The option price is

$$C_0 = e^{-\gamma}\left[pC_1^u + (1-p)\, C_1^d\right]$$

Understanding Risk-Neutral Pricing (cont'd)

- Are these two prices the same? **Yes** (proof left as an exercise).
- Note that it does not matter whether we have the "correct" value of α to start with.
- Any consistent pair of α and γ will give the same option price.
- Risk-neutral pricing is valuable because setting $\alpha = r$ results in the simplest pricing procedure.

Understanding Risk-Neutral Pricing (cont'd)

- We need to emphasize that **at no point are we assuming that investors are risk-neutral**.
- Rather, risk-neutral pricing is an **interpretation** of the formulas above.

Outline

American Options

- An **European** option can only be exercised at the expiration date, whereas an **American** option can be exercised at any time.
- Because of this added flexibility, an American option must always be at least as valuable as an otherwise identical European option:

$$C_{Amer}(S,K,T) \geq C_{Eur}(S,K,T)$$
$$P_{Amer}(S,K,T) \geq P_{Eur}(S,K,T)$$

- Combining these statements, together with the maximum and minimum option prices (from the first Section), gives us

$$S \geq C_{Amer}(S,K,T) \geq C_{Eur}(S,K,T) \geq \max[0, S - PV(Div) - PV(K)]$$
$$K \geq P_{Amer}(S,K,T) \geq P_{Eur}(S,K,T) \geq \max[0, PV(K) - S + PV(Div)]$$

American Call

- An American-style call option on a nondividend-paying stock should never be exercised prior to expiration (proof in class).
- For an American call on a dividend-paying stock it might be beneficial to exercise the option prior to expiration (by exercising the call, the owner will be entitled to dividend payments that she would not have otherwise received).
- Consider the previous example, with one exception: $\delta = 0.065$, i.e., XYZ stock has a continuous dividend yield of 6.5% per year.

American Call (cont'd)

- Because of dividends, early exercise is optimal at the node where the stock price is \$76.982.

$S_0 = \$41$
$C_0 = \$8.635$
$C_{0,NO} = \$8.635$
$C_{0,EX} = \$1$

$S_1^u = \$56.181$
$C_1^u = \$18.255$
$C_{1,NO}^u = \$18.255$
$C_{1,EX}^u = \$16.181$

$S_1^d = \$30.833$
$C_1^d = \$2.761$
$C_{1,NO}^d = \$2.761$
$C_{1,EX}^d = \$0$

$S_2^{uu} = \$76.982$
$\boxed{C_2^{uu} = \$36.982}$
$C_{2,NO}^{uu} = \$35.213$
$C_{2,EX}^{uu} = \$36.982$

$S_2^{ud} = \$42.249$
$C_2^{ud} = \$7.029$
$C_{2,NO}^{ud} = \$7.029$
$C_{2,EX}^{ud} = \$2.249$

$S_2^{dd} = \$23.187$
$C_2^{dd} = \$0$
$C_{2,NO}^{dd} = \$0$
$C_{2,EX}^{dd} = \$0$

$S_3^{uuu} = \$105.485$
$C_3^{uuu} = \$65.485$

$S_3^{uud} = \$57.892$
$C_3^{uud} = \$17.892$

$S_3^{udd} = \$31.772$
$C_3^{udd} = \$0$

$S_3^{ddd} = \$17.437$
$C_3^{ddd} = \$0$

$C_{\cdot,NO} =$ value of call if not exercised
$C_{\cdot,EX} =$ value of call if exercised

American Put

- When the underlying stock pays no dividend, a call will not be early-exercised, but a put might be.
- Suppose a company is bankrupt and the stock price falls to zero. Then a put that would not be exercised until expiration will be worth $PV(K)$, which is smaller than K for a positive interest rate.
- Therefore, early exercise would be optimal in order to receive the strike price earlier.
- This can also be shown by using a parity argument.
- Consider the previous example, with $\delta = 0.065$.

American Put (cont'd)

- Early exercise is optimal at the node where the stock price is \$23.187.

$S_0 = \$41$
$P_0 = \$6.546$
$P_{0,NO} = \$6.546$
$P_{0,EX} = \$0$

$S_1^u = \$56.181$
$P_1^u = \$2.314$
$P_{1,NO}^u = \$2.314$
$P_{1,EX}^u = \$0$

$S_1^d = \$30.833$
$P_1^d = \$10.630$
$P_{1,NO}^d = \$10.630$
$P_{1,EX}^d = \$9.167$

$S_2^{uu} = \$76.982$
$P_2^{uu} = \$0$
$P_{2,NO}^{uu} = \$0$
$P_{2,EX}^{uu} = \$0$

$S_2^{ud} = \$42.249$
$P_2^{ud} = \$4.363$
$P_{2,NO}^{ud} = \$4.363$
$P_{2,EX}^{ud} = \$0$

$S_2^{dd} = \$23.187$
$\boxed{P_2^{dd} = \$16.813}$
$P_{2,NO}^{dd} = \$15.197$
$P_{2,EX}^{dd} = \$16.813$

$S_3^{uuu} = \$105.485$
$P_3^{uuu} = \$0$

$S_3^{uud} = \$57.892$
$P_3^{uud} = \$0$

$S_3^{udd} = \$31.772$
$P_3^{udd} = \$8.228$

$S_3^{ddd} = \$17.437$
$P_3^{ddd} = \$22.563$

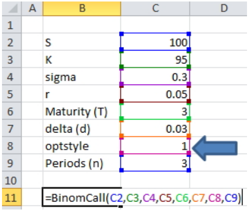
$P_{\cdot,NO}$ = value of put if not exercised
$PC_{\cdot,EX}$ = value of put if exercised

Problem 10.20: For a stock index, $S = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 3\%$, and $T = 3$. Let $n = 3$.

- What is the price of an American call option with a strike of \$95?
- What is the price of an American put option with a strike of \$95?

Problem 10.20: For a stock index, $S = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 3\%$, and $T = 3$. Let $n = 3$.

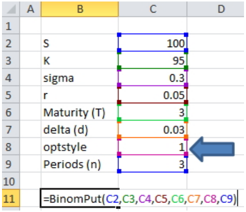
- What is the price of an American call option with a strike of \$95?



	A	B	C	D
1				
2		S	100	
3		K	95	
4		sigma	0.3	
5		r	0.05	
6		Maturity (T)	3	
7		delta (d)	0.03	
8		optstyle	1	
9		Periods (n)	3	
10				
11		=BinomCall(C2,C3,C4,C5,C6,C7,C8,C9)		

The price of an American call option with a strike of 95 is \$24.1650

- What is the price of an American put option with a strike of \$95?



	A	B	C	D
1				
2		S	100	
3		K	95	
4		sigma	0.3	
5		r	0.05	
6		Maturity (T)	3	
7		delta (d)	0.03	
8		optstyle	1	
9		Periods (n)	3	
10				
11		=BinomPut(C2,C3,C4,C5,C6,C7,C8,C9)		

The price of an American put option with a strike of 95 is \$15.2593

Understanding Early Exercise

- In deciding whether to early-exercise an option, the option holder compares the value of exercising immediately with the value of continuing to hold the option.
- Consider the cost and benefits of early exercise for a call option and a put option. By exercising, the option holder

Call Option

- Receives the stock and therefore receives future dividends
- Bears the interest cost of paying the strike price prior to expiration
- Loses the insurance implicit in the call (the option holder is protected against the possibility that the stock price will be less than the strike price at expiration).

Put Option

- Dividends are lost by giving up the stock
- Receives the strike price sooner rather than later
- Loses the insurance implicit in the put (the option holder is protected against the possibility that the stock price will be more than the strike price at expiration)

Outline

Stocks Paying Discrete Dividends

- It is reasonable to assume that a **stock index** pays dividends continuously
- **Individual stocks** pay dividends in discrete lumps, quarterly or annually. In addition, over short horizons it is frequently possible to predict the amount of the dividend
- How should we price an option when the stock will pay a known dollar dividend (or a known dividend yield) during the life of the option?
- The binomial tree can be adjusted to accommodate this case

Constant Dividend Yield

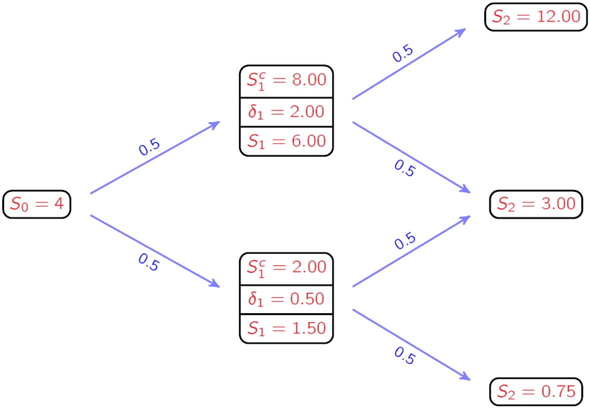
- A very simple way to incorporate dividends is to assume a constant dividend yield at the payment date
- Assume that

$$S_{t+1} = \left(1 - \delta \mathbb{1}_{\{t+1 \in \mathbb{D}\}}\right) S_t \xi_{t+1} \tag{32}$$

  where $\delta > 0$ is the constant dividend yield, $\mathbb{D} \subseteq \{1, ..., T\}$ is the set of dividend dates, $\xi_{t+1} \in \{u, d\}$, and the variable $\mathbb{1}_{\{t+1 \in \mathbb{D}\}}$ takes the value of 1 if $t+1$ is a dividend date and 0 otherwise.
- Everything works out the same as before, except for u and d, which now are defined $u = 1/d = e^{\sigma\sqrt{h}}$ and $p^* = \frac{e^{rh}-d}{u-d}$
- An important feature of this model is that the tree for the ex-dividend stock price S_t is **recombining**

Example

Two periods, $u = 1/d = 2$, $S_0 = 4$, $e^{rh} = 1.25$, $\delta = 0.25$, $\mathbb{D} = \{1\}$



Exercise: Find the price of an American call with strike $K = 2$

Discrete Dividends: Selected Readings

- Chapter 11.4 in McDonald
- Schroder, 1988, "Adapting the Binomial Model to Value Options on Assets with Fixed-Cash Payouts," *Financial Analyst Journal*, 44(6), 54-62

Outline

Convertible Bonds: Basics

- Bonds issued by a company where the holder has the option to exchange the bonds for a certain number (**conversion ratio**) of shares of the company's stock at certain times in the future
- Convertible bonds are typically **callable**: the issuer has the right to buy them back at certain times at predetermined prices. There is a **"tug-of-war"** going between the issuer and bondholders:
  - Bondholders decide whether to hold or convert (**maximizing value**)
  - The issuer decides whether to call (**minimizing value**)
- See also McDonald (3rd Edition), pages 479-485.

Convertible Bonds: Credit Risk

- Credit risk plays an important role in the valuation of convertible bonds: if credit risk is ignored, bond prices are overvalued.
- The stock price process can be represented by varying the usual binomial tree so that at each node there is:
  1. A probability of up movement:
  $$p_u = \frac{e^{(r-\delta)h} - d e^{-\lambda h}}{u - d} \tag{33}$$
  2. A probability of down movement:
  $$p_d = \frac{u e^{-\lambda h} - e^{(r-\delta)h}}{u - d} \tag{34}$$
  3. A probability of default:
  $$p_0 = 1 - e^{-\lambda h} \tag{35}$$

  where the variable λ is the default intensity.
- One can easily verify that $p_u + p_d + p_0 = 1$.

Convertible Bonds: Example from Hull (8th Edition), pages 608-611:

- Consider a 9-month (0.75 years) zero-coupon bond issued by company XYZ with a face value of \$100
- Suppose that it can be exchanged for two shares of company XYZ's stock at any time during the 9 months
- Assume also that it is callable for \$113 at any time
- The initial stock price is \$50, its volatility is 30% per annum, there are no dividends, and the risk-free rate is 5%
- The default intensity λ is 1% per year. Suppose that in the event of a default the bond is worth \$40 (i.e., the "recovery rate" is 40%)
- Evaluate the convertible bond using a 3-period binomial tree

Solution (to be discussed in class)

	A	B	C	D	E	F	G	H
1								
2		S	50		u	1.1764		
3		Face value	100		d	0.8715		
4		Conversion ratio	2					
5		σ (volatility)	0.3		pu	0.4697		
6		λ (default prob)	0.01		pd	0.5278		
7		r	0.05		p0	0.0025		
8		T	0.75					81.41
9		delta	0					162.82
10		Callable for:	113					
11		periods	3			69.20		
12		h	0.25			138.40		
13						138.40		
14				58.82		113.00		60.31
15				117.64		Called		120.62
16				117.64				
17		50.00		113.00		51.27		
18		107.67		Called		108.18		
19		100.00				102.53		
20		107.67		43.58		108.18		44.68
21		Not called		101.68		Not called		100.00
22				87.15				
23				101.68		37.98		
24		Value of r-f bond:		Not called		98.61		
25		96.32				75.96		
26						98.61		33.10
27		Implied int. rate				Not called		100.00
28		-9.85%						
29				Default		Default		Default
30				0		0		0
31				40		40		40

Convertible Bonds: Extensions

- When interest is paid on the debt, it must be taken into account
- Parameters λ, σ, and r can be functions of time (this can be handled using a trinomial rather than a binomial tree)
- The default intensity could be a function of the stock price as well

Outline

Real Option = the right, but not the obligation, to make a particular business decision

- The application of derivatives theory to the **operation** and **valuation** of real investment projects
- A **call option** is the right to pay a **strike price** to receive the present value of a stream of future cash flows
- An **investment project** is the right to pay an **investment cost** to receive the present value of a future cash flow stream:

Investment Project		Call Option
Investment Cost	=	Strike Price
Present Value of Project	=	Price of Underlying Asset

Investment Under Uncertainty

- A project requires an initial investment of \$100.

$$\Rightarrow K = 100 \tag{36}$$

- The project is expected to generate a perpetual cash flow stream, with a first cash flow \$18 in one year, expected to grow at 3% annually. Assume a discount rate of 15%.

$$\Rightarrow \begin{cases} \text{Perpetual growing annuity} \Rightarrow PV = \frac{\$18}{0.15-0.03} = \$150 \\ \text{Static NPV} = \$150 - \$100 = \$50 \\ \text{Cont. compounded dividend yield } \delta = \ln\left(1 + \frac{\$18}{\$150}\right) = 0.1133 \end{cases} \tag{37}$$

- The cont. compounded risk-free rate is $r = 6.766\%$. The cash flows of the project are normally distributed with a volatility of $\sigma = 50\%$.

$$\Rightarrow \begin{cases} u = e^{(r-\delta)+\sigma} = 1.571 \\ d = e^{(r-\delta)-\sigma} = 0.5795 \end{cases} \tag{38}$$

Investment Under Uncertainty (cont.)

- Suppose the project can be delayed: we can decide whether to accept the project at time 0, 1, or 2.
- Should the project be accepted? If yes, when?
- Trade-off between three factors:
  1. Foregone initial cash flow: \$18
  2. Interest savings: $100(e^{0.06766} - 1) = \$7$

$$\Rightarrow \text{Loss of \$11 if delayed} \tag{39}$$

  3. **Value of preserved insurance: Is it more than the loss due to delay?**

Binomial tree for project cash flows & project value

$$
\begin{tikzcd}
 & & \begin{matrix}\$44.66\\ PV=\$372.15\end{matrix} \\
 & \begin{matrix}\$28.35\\ PV=\$236.27\end{matrix} \arrow[ur] \arrow[dr] & \\
\begin{matrix}\$18\\ PV=\$150\end{matrix} \arrow[ur] \arrow[dr] & & \begin{matrix}\$16.43\\ PV=\$136.91\end{matrix} \\
 & \begin{matrix}\$10.43\\ PV=\$86.92\end{matrix} \arrow[ur] \arrow[dr] & \\
 & & \begin{matrix}\$6.04\\ PV=\$50.36\end{matrix}
\end{tikzcd}
$$

- The risk-neutral probability of the project value increasing in any period, p^*, is given by

$$p^* = \frac{e^{r-\delta} - d}{u - d} = 0.3775 \tag{40}$$

Value of the investment option

$$PV = \$150 \quad Call_{EX} = \$50 \quad Call_{NO} = \$56.19$$

$$PV = \$236.27 \quad Call_{EX} = \$136.27 \quad Call_{NO} = \$124.22$$

$$PV = \$86.92 \quad Call_{EX} = \$0 \quad Call_{NO} = \$13.94$$

$$PV = \$372.15 \quad Call = \$272.15$$

$$PV = \$136.91 \quad Call = \$36.91$$

$$PV = \$50.36 \quad Call = \$0$$

- Notice that the initial value of the project option is \$56.19, which is greater than the static NPV of \$50.
- If we invest immediately, the project is worth \$50. The ability to wait increases that value by \$6.19.

Real Options in Practice

- The decision about whether and when to invest in a project $\sim$ **call option**
- The ability to shut down, restart, and permanently abandon a project $\sim$ **project + put option**
- Strategic options: the ability to invest in projects that may give rise to future options $\sim$ **compound option**
- Flexibility options: the ability to switch between inputs, outputs, or technologies $\sim$ **rainbow option**

Outline

Longstaff and Schwartz (RFS, 2001)

- Consider an American option with final expiration date T
- At any possible exercise time, the holder should compare the payoff from immediate exercise to the expected value from keeping the option alive
- The optimal decision is to exercise if the value of immediate exercise is positive and larger than the expected value of continuation
- In a simulation approach the problem is that we cannot simply use next period values to determine the expected pathwise value from continuation (this corresponds to assuming perfect foresight and would lead to biased price estimates)
- Longstaff and Schwartz suggest estimating the conditional expectation of the payoff from continuation using the cross-sectional information in the simulation

Longstaff and Schwartz (RFS, 2001)

- Consider an option that can be exercised at a discrete set of dates $t_1, t_2, ..., t_N$
- Such "Bermudan options" are widely traded
- The price of a true American option (which is continuously exercisable) is obtained by letting the number of exercise dates increase towards infinity

LSM simple example

- Consider an American put option on a non-dividend paying stock
- The option may be exercised at time 1,2, and 3, where time 3 is the final expiration date
- Strike is 1.10
- Risk-free rate is 0.06

Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	**1.03**
4	1.00	.93	.97	**.92**
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	**.90**
7	1.00	.92	.84	**1.01**
8	1.00	.88	1.22	1.34

- Simulate 8 sample paths under the risk-neutral probability measure

Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	**1.03**
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6	1.00	.76	.77	**.90**
7	1.00	.92	.84	**1.01**
8	1.00	.88	1.22	1.34

Cash Flow Matrix			
Path	t_1	t_2	t_3
1	–	–	–
2	–	–	–
3	–	–	.07
4	–	–	.18
5	–	–	–
6	–	–	.20
7	–	–	.09
8	–	–	–

- Simulate 8 sample paths under the risk-neutral probability measure
- Exercise decision at time 3 is trivial

Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	1.09	**1.08**	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	**1.07**	1.03
4	1.00	.93	**.97**	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	**.77**	.90
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Cash Flow Matrix			
Path	t_1	t_2	t_3
1	–	–	–
2	–	–	–
3	–	–	.07
4	–	–	.18
5	–	–	–
6	–	–	.20
7	–	–	.09
8	–	–	–

Regression & Optimal Exercise				
Path	Y	X	EX	NO
1		1.08	.02	
2		–	–	
3		1.07	.03	
4		.97	.13	
5		–	–	
6		.77	.33	
7		.84	.26	
8		–	–	

- X is the stock price at time 2 for the paths that are in-the-money

Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	1.09	**1.08**	1.34
2	1.00	1.16	1.26	1.54
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6	1.00	.76	**.77**	.90
7	1.00	.92	**.84**	1.01
8	1.00	.88	1.22	1.34

Cash Flow Matrix			
Path	t_1	t_2	t_3
1	–	–	–
2	–	–	–
3	–	–	.07
4	–	–	.18
5	–	–	–
6	–	–	.20
7	–	–	.09
8	–	–	–

Regression & Optimal Exercise				
Path	Y	X	EX	NO
1	$.00 \times 0.94176$	1.08	.02	
2	–	–	–	
3	$.07 \times 0.94176$	1.07	.03	
4	$.18 \times 0.94176$	.97	.13	
5	–	–	–	
6	$.20 \times 0.94176$	.77	.33	
7	$.09 \times 0.94176$	.84	.26	
8	–	–	–	

- X is the stock price at time 2 for the paths that are in-the-money
- Y is the corresponding discounted cash flow if the option is not exercised

Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	1.09	**1.08**	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	**1.07**	1.03
4	1.00	.93	**.97**	.92
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Cash Flow Matrix			
Path	t_1	t_2	t_3
1	–	–	–
2	–	–	–
3	–	–	.07
4	–	–	.18
5	–	–	–
6	–	–	.20
7	–	–	.09
8	–	–	–

$$\mathbb{E}[Y] = -1.070 + 2.983X - 1.813X^2$$

Regression & Optimal Exercise				
Path	Y	X	EX	NO
1	$.00 \times 0.94176$	1.08	.02	
2	–	–	–	
3	$.07 \times 0.94176$	1.07	.03	
4	$.18 \times 0.94176$	.97	.13	
5	–	–	–	
6	$.20 \times 0.94176$	.77	.33	
7	$.09 \times 0.94176$	.84	.26	
8	–	–	–	

- X is the stock price at time 2 for the paths that are in-the-money
- Y is the corresponding discounted cash flow if the option is not exercised
- The expected continuation is approximated by the fitted value of a cross-sectional least squares

Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	1.09	**1.08**	1.34
2	1.00	1.16	1.26	1.54
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Path	t_1	t_2	t_3
1	–	–	–
2	–	–	–
3	–	–	.07
4	–	–	.18
5	–	–	–
6	–	–	.20
7	–	–	.09
8	–	–	–

$$\mathbb{E}[Y] = -1.070 + 2.983X - 1.813X^2$$

Regression & Optimal Exercise				
Path	Y	X	EX	NO
1	$.00 \times 0.94176$	1.08	.02	.0369
2	–	–	–	–
3	$.07 \times 0.94176$	1.07	.03	.0461
4	$.18 \times 0.94176$	.97	.13	.1176
5	–	–	–	–
6	$.20 \times 0.94176$	.77	.33	.1520
7	$.09 \times 0.94176$	.84	.26	.1565
8	–	–	–	–

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Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	1.09	**1.08**	1.34
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Path	t_1	t_2	t_3
1	–	–	–
2	–	–	–
3	–	–	.07
4	–	–	.18
5	–	–	–
6	–	–	.20
7	–	–	.09
8	–	–	–

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Path	t_1	t_2	t_3
1	–	–	–
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3	–	–	.07
4	–	.13	–
5	–	–	–
6	–	.33	–
7	–	.26	–
8	–	–	–

- X is the stock price at time 2 for the paths that are in-the-money
- Y is the corresponding discounted cash flow if the option is not exercised
- The expected continuation is approximated by the fitted value of a cross-sectional least squares
- Exercise the option if payoff is larger than the discounted expected value of keeping the option alive

Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	**1.09**	1.08	1.34
2	1.00	1.16	1.26	1.54
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8	–	–	–

Regression & Optimal Exercise				
Path	Y	X	EX	NO
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2		–	–	
3		–	–	
4		.93	.17	
5		–	–	
6		.76	.34	
7		.92	.18	
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- Repeat the procedure at time 1

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$$\mathbb{E}[Y] = -1.070 + 2.983X - 1.813X^2$$

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1	–	–	–
2	–	–	–
3	–	–	.07
4	–	.13	–
5	–	–	–
6	–	.33	–
7	–	.26	–
8	–	–	–

$$\mathbb{E}[Y] = 2.038 - 3.335X + 1.356X^2$$

Regression & Optimal Exercise				
Path	Y	X	EX	NO
1	$.00 \times 0.94176$	1.09	.01	
2	–	–	–	
3	–	–	–	
4	$.13 \times 0.94176$	.93	.17	
5	–	–	–	
6	$.33 \times 0.94176$	.76	.34	
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- Repeat the procedure at time 1

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Path	t_0	t_1	t_2	t_3
1	1.00	**1.09**	1.08	1.34
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8	–	–	–

$$\mathbb{E}[Y] = -1.070 + 2.983X - 1.813X^2$$

Regression & Optimal Exercise				
Path	Y	X	EX	NO
1	$.00 \times 0.94176$	1.08	.02	.0369
2	–	–	–	–
3	$.07 \times 0.94176$	1.07	.03	.0461
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5	–	–	–
6	–	.33	–
7	–	.26	–
8	–	–	–

$$\mathbb{E}[Y] = 2.038 - 3.335X + 1.356X^2$$

Regression & Optimal Exercise				
Path	Y	X	EX	NO
1	$.00 \times 0.94176$	1.09	.01	.0139
2	–	–	–	–
3	–	–	–	–
4	$.13 \times 0.94176$	.93	.17	.1092
5	–	–	–	–
6	$.33 \times 0.94176$	.76	.34	.2866
7	$.26 \times 0.94176$	.92	.18	.1175
8	$.00 \times 0.94176$	.88	.22	.1533

- Repeat the procedure at time 1

Stock Price Paths				
Path	t_0	t_1	t_2	t_3
1	1.00	**1.09**	1.08	1.34
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Cash Flow Matrix			
Path	t_1	t_2	t_3
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6	.34	–	–
7	.18	–	–
8	.22	–	–

- Repeat the procedure at time 1

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Path	t_0	t_1	t_2	t_3
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Path	t_1	t_2	t_3
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5	–	–	–
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Regression & Optimal Exercise				
Path	Y	X	EX	NO
1	$.00 \times 0.94176$	1.09	.01	.0139
2	–	–	–	–
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4	.17	–	–
5	–	–	–
6	.34	–	–
7	.18	–	–
8	.22	–	–

- Repeat the procedure at time 1
- Discount cash-flows back to time 0 and average over all 8 paths $\Rightarrow$ American put = .1144 (European = .0564)

Longstaff and Schwartz (RFS, 2001)

- An important point to understand is that the early exercise decision is based on the fitted regression in which the coefficients are common on each sample path; the decision is *not* based on the knowledge of the future prices along each sample path
- Clement, Lamberton, and Protter (2002) prove that the LSM algorithm converges (for a given number of basis functions) as $M \to \infty$
- The algorithm can be applied to many realistic cases (jump diffusion processes, American-Bermuda-Asian options,...)
- See Chapter 19.6 in McDonald or Chapter 26.8 in Hull (8^{th} edition)

Final Thoughts: Is the Binomial Model Realistic?

- Volatility is constant
  - There is ample evidence that volatility changes over time
- Large stock price movements do not occur
  - It appears that on occasion stocks move by a large amount (jumps)
- Returns are independent over time
  - There is strong evidence that stock returns are correlated across time, with positive correlations at the short to medium term (momentum) and negative correlation at long horizons (reversal).
- Continuous dividend yield δ
  - Stocks pay dividends in discrete lumps, quarterly or annually
  - In addition, over short horizons it is frequently possible to predict the amount of the dividend
  - The binomial tree can be adjusted to accommodate this case (see Chapter 11.4 in McDonald)

- Problem Set 2 is available