MFE 409 Lecture 1b
Value-at-Risk

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Lecture Objectives

Understanding **Value-at-Risk**

- How to compute it
- What is it useful for?
- What are its limitations?
- Some alternatives

Measuring Risk

- Defining and managing risk is one of the most important issues firms are facing in their daily operations
- Especially important for financial institutions that rely on leverage.
- Find an answer to the question:

  "What is realistically the worst that could happen over one day, one week, or one year?"

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Value-at-Risk

- **Value-at-risk (VaR)** is an answer to the question above where "realistically" is defined by finding an outcome that is so bad that anything worse is highly unlikely.
- Value-at-Risk is the realistically worst case outcome in the sense that anything worse only happens with probability less than some fixed level (such as 1%).

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Definition. Let W be a random variable. The *Value-at-Risk* at confidence level c relative to base level W_0 is the smallest non-negative number denoted by VaR such that

$$\text{Prob}(W < W_0 - \text{VaR}) \leq 1 - c$$

VaR solves $\text{Prob}(W < W_0 - \text{VaR}) = 1 - c$

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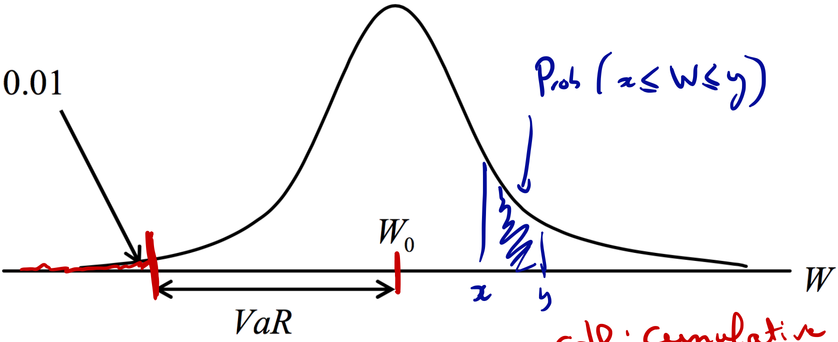
Typically:

- W is the value of a portfolio at some point in the future (1 day, 1 month, 1 year)
- W_0 gives some base level: often current value of the portfolio
- Confidence level c gives concrete mean to what "worst case" means: 99%, 99.9%

Value-at-Risk

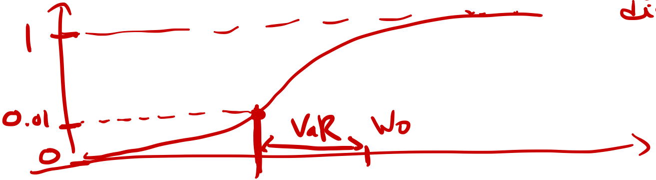
pdf: probability distribution function



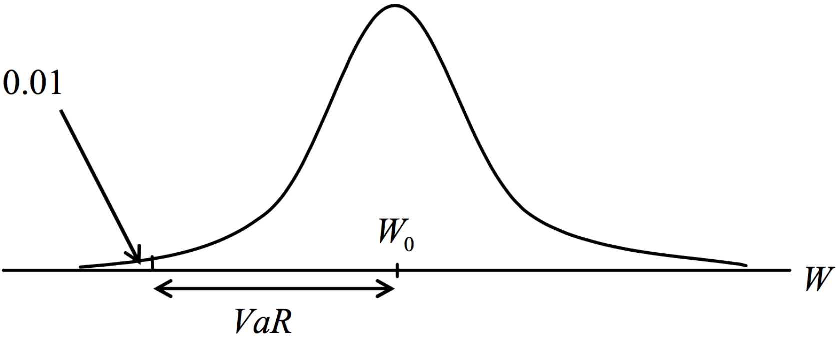
$Prob(x \leq W \leq y)$

cdf: cumulative distribution function

$F(x) = Prob(W \leq x)$



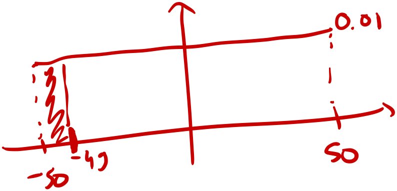
Value-at-Risk



- Bottom point: c-quantile, $F^{-1}(1-c)$ if F is cumulative distribution function

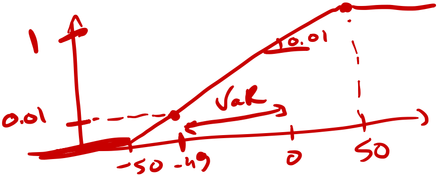
Example: Uniform distribution

- All outcomes between a loss of \$50 million and a gain of \$50 million are equally likely for a one-year project



$$VaR = 0 - (-49)$$
$$= \$49m$$

- The VaR for a one-year time horizon and a 99% confidence level is



Example: Uniform distribution

- All outcomes between a loss of \$50 million and a gain of \$50 million are equally likely for a one-year project

- The VaR for a one-year time horizon and a 99% confidence level is \$49 million

Rationale for Value-at-Risk

- It captures an important aspect of risk in a single number
- Easy to understand
- Two broad motivations
  - Measure of potential extreme loss
  - Capital to hold against possible failure

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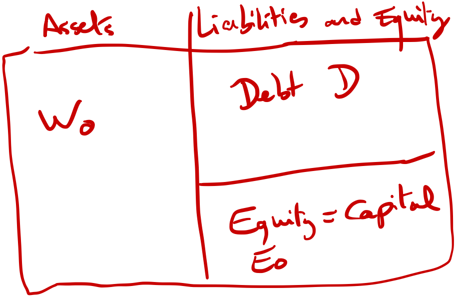
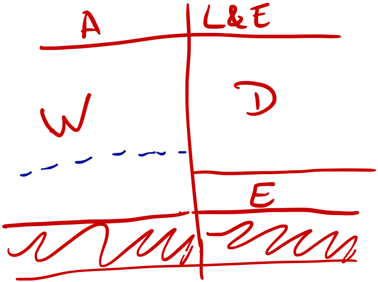
A Measure of Capital

- Example:
  - W measures value of total assets of the firm in 10 days
  - W_0 is today's value of the firm's assets
  - Firm remains solvent as long as W does not fall below $W_0 -$ capital.





$W_0 = D + E_0$

Bankrupt if:

$W_0 - W \geq E_0$

$\iff W \leq W_0 - E_0$

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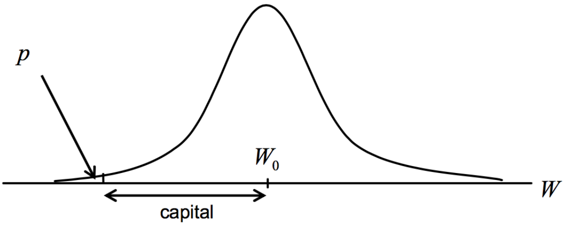

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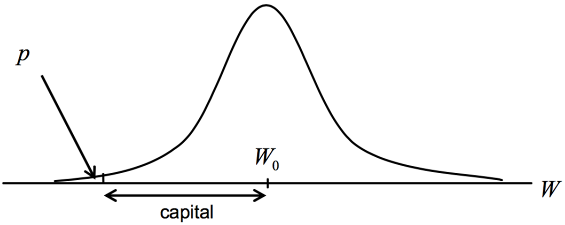


- 1% 10-day VaR: amount of capital to hold so that firm goes bankrupt with probability 1% in the next 10 days

Regulatory Capital

- Regulators have traditionally used VaR to calculate the capital they require banks to keep
- The market-risk capital has been based on a 10-day VaR estimated where the confidence level is 99%
- Credit risk and operational risk capital are based on a one-year 99.9% VaR

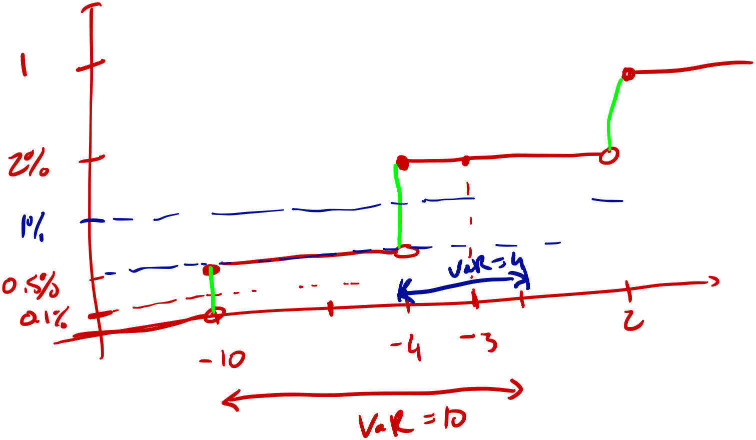
What is Special about Capital?

- If the firm or bank has limited liability, then it does not matter whether the firm goes bust just marginally, or whether it goes bust spectacularly, leaving a big shortfall

- The tail loss is not a concern for a firm with limited liability

Example

- Project A has:
  - 98% chance of leading to a gain of \$2 million
  - 1.5% chance of a loss of \$4 million
  - 0.5% chance of a loss of \$10 million
- The VaR with a 99.9% confidence level is ...
- What if the confidence level is 99.5%?
- What if it is 99%?



99.9% VaR = \$10m

99.5% VaR = \$4m

99% VaR = \$4m

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- The VaR with a 99.9% confidence level is \$10 million
- What if the confidence level is 99.5%? \$4 million
- What if it is 99%? \$4 million

Example

- Project A - B has:
  - 98% - 98% chance of leading to a gain of $2 million
  - 1.5% - 1.1% chance of a loss of $4 million
  - 0.5% - 0.9% chance of a loss of $10 million

Example

- Project A - B has:
  - 98% - 98% chance of leading to a gain of \$2 million
  - 1.5% - 1.1% chance of a loss of \$4 million
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- Which project is more risky?

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- Which project is more risky?

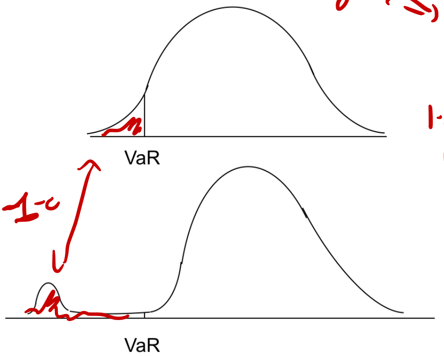
- Which project has a larger 99% VaR?

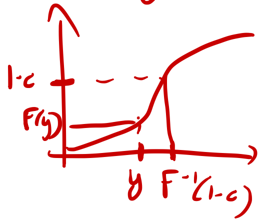
Limitation of VaR

- VaR does not capture the distribution of losses below the threshold
- Formally, any 2 distributions with same $F^{-1}(1-c)$ will have the same VaR

$y \iff F(y) = 1-c$

$1\% = 1-c$





Gaming VaR

- Banks are regulated based on VaR ...
- But would like to take more risk

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  - Selling puts

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- But would like to take more risk
- Taking extreme tail risk will not increase VaR
  - Selling puts
  - Selling disaster insurance

Expected Shortfall

- **Expected shortfall**: expected loss given loss larger than VaR

$$
\begin{aligned}
\mathrm{ES} &= W_0 - \mathbb{E}\left[W \middle| W \le W_0 - \mathrm{VaR}\right] \\
&= W_0 - \frac{\int_{-\infty}^{W_0-\mathrm{VaR}} W f(W)dW}{\underbrace{\int_{-\infty}^{W_0-\mathrm{VaR}} f(W)dW}_{\to 1-c}}
\end{aligned}
$$

$$\mathbb{E}(W) = \int_{-\infty}^{+\infty} w f(w)dw$$

$$\mathbb{E}(W \mid W \le a) = \frac{\int_{-\infty}^{a} w f(w)dw}{\int_{-\infty}^{a} f(w)dw}$$

$$\boxed{\mathrm{ES} \geqslant \mathrm{VaR}}$$

Expected Shortfall

- **Expected shortfall**: expected loss given loss larger than VaR

$$\text{ES} = W_0 - \mathbb{E}\left[W | W \leq W_0 - \text{VaR}\right]$$
$$= W_0 - \frac{\int_{-\infty}^{W_0 - \text{VaR}} W f(W) dW}{\int_{-\infty}^{W_0 - \text{VaR}} f(W) dW}$$

- Also called C-VaR and Tail Loss

  Conditional VaR

- Regulators have indicated that they plan to move from using VaR to using ES for determining market risk capital

- Two portfolios with the same VaR can have very different expected shortfalls

Example: Normal Distribution

- January 8, 2010
  - Position: EUR 10 million
  - Exchange rate M_t= USD/EUR = \$1.436
  - Dollar position W_0 = \$14.36 million
- Assume normal distribution for FX return

$$R_{M,t+1} \sim \mathcal{N}(\mu, \sigma)$$

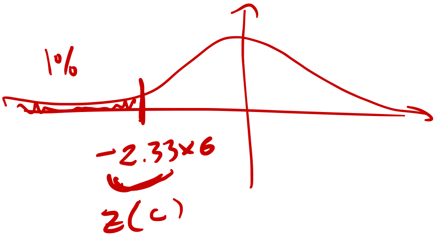
  - Historically (in daily units), we find: $\sigma = 0.65\%$ and $\mu \approx 0$
- We want to compute the 99% 1-day Value-at-Risk

Example: Normal Distribution

- ~~Loss~~ Gain distribution: $W - W_0 = \$14.36\text{m} \times R_{M,t+1} \sim \mathcal{N}(\mu_V, \sigma_V)$

$$\mu_V = 0$$

$$\sigma_V = 0.65\% \times 14.36 = 0.093$$



$$\text{VaR} = -(\mu + z(c)\sigma)$$

$$= 2.33 \times 0.09$$

$$= \$217{,}000$$

Example: Normal Distribution

- Loss distribution: $W - W_0 = -\$14.36\text{m} \times R_{M,t+1} \sim \mathcal{N}(\mu_V, \sigma_V)$
  - $\mu_V = \$14.36\text{m} \times \mu_M \approx 0$
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- Define $z(c)$: cutoff such that there is a probability c that a standard normal is larger than $z(c)$
  - $c = 99\% \rightarrow z(c) = -2.326$
  - $c = 95\% \rightarrow z(c) = -1.645$

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- We have $W - W_0 > \mu_V + z(c) \times \sigma_V$ with probability c

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- We have $W - W_0 > \mu_V + z(c) \times \sigma_V$ with probability c
  - 99% 1-day VaR $= -(-2.326 \times 14.36 \times 0.0065) = \$217,204$

VaR with Normal Distribution

- Assume $W - W_0 \sim \mathcal{N}(\mu, \sigma)$
- $VaR = -(\mu + z(c) \times \sigma_V)$

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- Assume $W - W_0 \sim \mathcal{N}(\mu, \sigma)$
- $VaR = -(\mu + z(c) \times \sigma_V)$
- Can also compute expected shortfall:
  - ES $= -\mu_V + \sigma_V \frac{e^{-z(c)^2/2}}{\sqrt{2\pi}(1-c)}$
  - 95% ES $= -(\mu_V - \sigma_V \times 2.0628)$
  - 99% ES $= -(\mu_V - \sigma_V \times 2.6649)$

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  - 99% ES $= -(\mu_V - \sigma_V \times 2.6649)$
- For normal distributions, close relation between VaR and ES: multiple of volatility

Role of Time

- Losses in successive days are independent, normally distributed, and have a mean of zero

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$$T\text{-day VaR} = \text{1-day VaR} \times \sqrt{T}$$

$$T\text{-day ES} = \text{1-day ES} \times \sqrt{T}$$

Role of Time

- Losses in successive days are independent, normally distributed, and have a mean of zero

$$T\text{-day VaR} = 1\text{-day VaR} \times \sqrt{T}$$

$$T\text{-day ES} = 1\text{-day ES} \times \sqrt{T}$$

- Autocorrelation ρ between losses on successive days, replace $\sqrt{T}$ by

$$\sqrt{T + 2(T-1)\rho + 2(T-2)\rho^2 + 2(T-3)\rho^3 + \ldots + 2\rho^{T-1}}$$

$\mathrm{var}(R_t + R_{t+1})$
$= \mathrm{var}(R_t) + \mathrm{var}(R_{t+1})$
$+ 2\,\mathrm{cov}(R_t, R_{t+1})$
$= 2\,\mathrm{var}(R_t)$
$+ 2\rho\,\mathrm{var}(R_t)$
$= (2 + 2\rho)\,\mathrm{var}(R_t)$

	T=1	T=2	T=5	T=10	T=50	T=250
ρ=0	1.0	1.41	2.24	3.16	7.07	15.81
ρ=0.05	1.0	1.45	2.33	3.31	7.43	16.62
ρ=0.1	1.0	1.48	2.42	3.46	7.80	17.47
ρ=0.2	1.0	1.55	2.62	3.79	8.62	19.35

Example: VaR for a Portfolio

99% 1-day VaR

- Positions: 10mil EUR, 1bil Yen
  - $J_t =$ USD/JPY $= 0.01078749$; USD/EUR $= 1.436$
  - Assume $R_{M,t+1}$ and $R_{J,t+1}$ jointly nomal with
  - $E(R_M) = E(R_J) \approx 0$, $\sigma_M = 0.65\%$, $\sigma_J = 0.69\%$
  - Corr$(R_M, R_J) = \rho_{MJ} = 0.2775$

$$R_p = 14.36 \times R_{M,t+1} + 10.7 \times R_{J,t+1}$$

$$\sigma_p = \sqrt{14.36^2 \sigma_M^2 + 10.7^2 \sigma_J^2 + 2\rho \sigma_M \sigma_J \times 14.36 \times 10.7}$$

$= \$0.134m$

$$VaR = 2.32 \times \sigma_p = \$317{,}000$$

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  - Assume $R_{M,t+1}$ and $R_{J,t+1}$ jointly nomal with
  - $E(R_M) = E(R_J) \approx 0$, $\sigma_M = 0.65\%$, $\sigma_J = 0.69\%$
  - $\text{Corr}(R_M, R_J) = \rho_{MJ} = 0.2775$
- The change in portfolio value is:

$$W - W_0 = \$14.36\text{m} \times R_{M,t+1} + \$10.78\text{m} \times R_{J,t+1} \sim \mathcal{N}(0, \sigma_V)$$

  - $\sigma_V = \$134,445.20$
- 99% 1-day VaR$= 2.326 \times \sigma_V = \$312,719.40$

VaR for a Portfolio: Approximate approach

$$= \sqrt{\sum_i \mathrm{VaR}_i^2 + 2\sum_{i<j} \rho_{ij} \mathrm{VaR}_i \mathrm{VaR}_j}$$

- An approximate approach that seems to works well is

$$\mathrm{VaR}_{\mathrm{total}} = \sqrt{\sum_i \sum_j \mathrm{VaR}_i \mathrm{VaR}_j \rho_{ij}}$$

  where VaR_i is the VaR for the i-th segment, $\mathrm{VaR}_{\mathrm{total}}$ is the total VaR, and ρ_{ij} is the coefficient of correlation between losses from the i-th and j-th segments

- Exact formula for normal distributions

VaR for a Portfolio: Exact approach

$x_1, \ldots x_n \longrightarrow f(x_1, \ldots x_n)$

$\hookrightarrow$ VaR of portfolio

- Marginal VaR

$$\text{DVaR}_i = \frac{\partial \text{VaR}}{\partial x_i} \approx \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots x_n) - f(x_1, \ldots, x_i, \ldots x_n)}{\Delta x_i}$$

increase by \$1

- Component VaR

$$\text{CVaR}_i = x_i \frac{\partial \text{VaR}}{\partial x_i}$$

$\rightarrow$ how much does VaR change if position in i increases by 1%

VaR for a Portfolio: Exact approach

- Marginal VaR

$$\mathrm{DVaR}_i = \frac{\partial \mathrm{VaR}}{\partial x_i}$$

- Component VaR

$$\mathrm{CVaR}_i = x_i \frac{\partial \mathrm{VaR}}{\partial x_i}$$

- Decomposition (Euler Theorem):

$$\mathrm{VaR} = \sum_i \mathrm{CVaR}_i$$

$f(\lambda x_1, \ldots, \lambda x_n) = \lambda f(x_1, \ldots, x_n)$

↳ differentiate wrt λ, set $\lambda = 1$: $x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots = f$

Using VaR for Capital Allocation

- You are the head of prop trading for an investment bank. You have to allocate capital between investing in FX or in fixed income. Last year FX invested \$100m and made 10% profits, while fixed income invested \$200m and made 5% profit. What do you do?

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- If you have access to leverage quantity of assets is not so important, rather quantity of capital mobilized.
- **Risk Adjusted Rate of Return on Capital (RAROC)**: profit per unit of necessary capital, i.e. profit per unit of VaR

Assets × VaR ≤ Equity

$$\text{RAROC} = \frac{\text{Profit}}{\text{VaR}}$$

$\text{Assets} \le \frac{\text{Equity}}{\text{VaR}}$

- Developed in the 1980s by Bankers Trust (taken over by Deutsche Bank) to develop internal capital budgeting system

Example: RAROC

100m 200m

- Let us compute RAROC for the two positions

10% profits 5% profits

  - Assume normal distribution with annual volatility 10% for FX and 4% for fixed income
  - We want to use annual 99.97% VaR ($z(99.97\%) = -3.4$)

RAROC

FX: $\dfrac{10\% \times 100m}{-10\% + 3.4 \times 10\% \times 100m} = 0.294 = 29.4\%$

Fixed income $\dfrac{5\%}{3.4 \times 4\%} = 0.367 = 36.7\%$

What do we do?

Example: RAROC

- Let us compute RAROC for the two positions
  - Assume normal distribution with annual volatility 10% for FX and 4% for fixed income
  - We want to use annual 99.97% VaR ($z(99.97\%) = -3.4$)
- FX
  - VaR $= 3.4 \times 0.1 \times 100\text{m} = \$34\text{m}$
  - RAROC $= 10\text{m}/34\text{m} = 29.4\%$
- Fixed Income
  - VaR $= 3.4 \times 0.04 \times 200\text{m} = \$27.2\text{m}$
  - RAROC $= 10\text{m}/27.2\text{m} = 36.8\%$

Example: RAROC

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  - Assume normal distribution with annual volatility 10% for FX and 4% for fixed income
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- FX
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  - RAROC $= 10\text{m}/34\text{m} = 29.4\%$
- Fixed Income
  - VaR $= 3.4 \times 0.04 \times 200\text{m} = \$27.2\text{m}$
  - RAROC $= 10\text{m}/27.2\text{m} = 36.8\%$
- Should tilt allocation towards Fixed Income even if it has lower return
- Similar to Sharpe Ratio, but with focus on downside

VaR and Diversification

- Diversification: 2 investments x_1 and x_2 with same mean and variance, correlation ρ

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$$\begin{aligned}
\text{Var}\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) &= \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 + 2\text{Cov}\left(\frac{1}{2}x_1, \frac{1}{2}x_2\right) \\
&= \frac{1}{2}\left(\sigma^2 + \text{Cov}\left(x_1, x_2\right)\right) \\
&= \frac{1}{2}\sigma^2(1+\rho) \\
&\leq \sigma^2
\end{aligned}$$

- With normal distribution, also applies to Value-at-Risk: diversification reduces risk

VaR and Diversification

- Consider bonds with face value of 100 and default probability of 0.9% and 0 recovery. Assume defaults are independent across bonds and that the baseline level is $W_0 = 100$.
- What is the 99% VaR for one bond? VaR $= 0$

  $0.9\% < 1\%$

- What is the 99% VaR for two bonds? VaR $= 50$

  100 proba: $99.1\% \times 99.1\%$

  50 proba: $2 \times 99.1\% \times 0.9\% > 1\%$

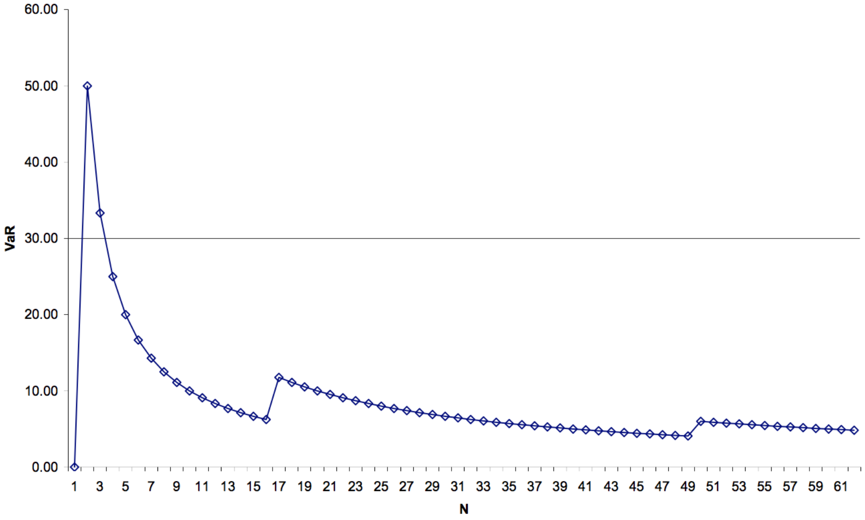
  0 proba: $0.9\% \times 0.9\% \approx 0$

- What is the 99% VaR for n bonds?

VaR and Diversification

- Consider bonds with face value of 100 and default probability of 0.9% and 0 recovery. Assume defaults are independent across bonds and that the baseline level is $W_0 = 100$.
- What is the 99% VaR for one bond?
  - VaR$= 0$
- What is the 99% VaR for two bonds?
  - Both pay, loss 0, with proba $0.991^2 = 0.982$
  - Both default, loss 100 with proba $0.009^2 = 8 \times 10^{-5}$
  - Last case, loss 50
  - VaR$= 50$
- What is the 99% VaR for n bonds?

VaR and Diversification



Coherent Risk Measures

- Define a coherent risk measure as the amount of cash that has to be added to a portfolio to make its risk acceptable
- Properties of coherent risk measure
  - If one portfolio always produces a worse outcome than another its risk measure should be greater
  - If we add an amount of cash K to a portfolio its risk measure should go down by K
  - Changing the size of a portfolio by a factor λ should result in the risk measure being multiplied by λ
  - The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged

Coherent Risk Measures

- Value-at-Risk
- Expected Shortfall

Coherent Risk Measures

- Value-at-Risk ✗
- Expected Shortfall ✓

Coherent Risk Measures

- Value-at-Risk ✗
- Expected Shortfall ✓
- Spectral measures
  - Spectral measures assigns weight to quantiles of the loss distribution
  - VaR assigns all weight to c-th percentile of the loss distribution
  - Expected shortfall assigns equal weight to all percentiles greater than the c-th percentile
  - For a coherent risk measure weights must be a non-decreasing function of the percentiles

Takeaways

- Value-at-Risk is:
  - a simple measure
  - used by regulators and practitioners to measure risk
  - which focuses on the extreme downside of a distribution
- It has some limitations
  - Does not capture the entire distribution of extreme losses
  - Does not always capture diversification
- Implications
  - If you want to monitor risk, know its limitations
  - Expected shortfall is a better behaved alternative
  - If you are constrained by it, know how to game it
- Next: how to measure VaR in the real world? We don't know the distribution of what will happen in the next few days!

Exponential distribution

long ← losses | gains →

short ← gains | losses →

$c_{BS} = S\,N(d_1) - Ke^{-rT}N(d_2)$

$c_{BS}(S_0, 3\text{month}) \rightarrow c_{BS}(S_{today}, 3m - 1\text{day})$