UCLA Anderson School of Management

Solutions to Quizz #1

Problem 1. Suppose that S_t follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \text{ with } S_0 = 1 \text{ and } \mu > 0, \sigma > 0 \text{ constants}$$

Use Ito's Lemma to derive a stochastic differential equation for $Y_t = S_t^\beta$ where $\beta > 0$. For any T, show that Y_T is lognormally distributed. Provide the mean and the variance of $\log(Y_T)$.

Solution: Using Ito's Lemma implies

$$dY_t = Y_t \left(\mu\beta + \frac{1}{2}\beta(\beta - 1)\sigma^2 \right) dt + \beta Y_t \sigma dW_t$$

Once again, applying Ito's Lemma

$$d\log Y_t = \beta \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \beta\sigma dW_t$$

Therefore

$$\log(Y_T) = N\left\{ \log(Y_t) + \beta\left(\mu - \frac{1}{2}\sigma^2\right)(T - t); \beta^2\sigma^2(T - t) \right\}$$

Problem 2. Solve the following Partial Differential Equation

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\mu x + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2 x^2 &= 0 \\ F(T, x) &= x \end{aligned}$$

Give an explicit expression for $F(t, x)$ for $t < T$.

Solution:

$$dx_t = \mu x_t dt + \sigma x_t dW_t$$

And so its distribution at time T is given by

$$\log x_T = N\left\{ \log x_t + \left(\mu - \frac{1}{2}\sigma^2\right)(T - t); \sigma^2(T - t) \right\}$$

Therefore the solution to F is

$$\begin{aligned} F &= E_t(x_T) \\ &= E_t e^{\log(x_T)} \\ &= x_t e^{\mu(T-t)} \end{aligned}$$

Problem 3. The goal of this exercise is to compute the distribution of the average value of brownian motion

$$\int_0^1 W_s ds$$

a)

$$\begin{aligned} d(tW_t) &= tdW_t + W_t dt \\ 1 \times W_1 - 0 \times W_0 &= \int_0^1 s dW_s + \int_0^1 W_s ds \end{aligned}$$

or

$$\int_0^1 W_s ds = \int_0^1 dW_s - \int_0^1 s dW_s = \int_0^1 (1-s)\, dW_s$$

b) Then derive the distribution of $\int_0^1 (1-s)\, dW_s$. Show that it is normal. What is its mean and variance?

b) As we have seen in the lecture notes, the distribution of any integral of the form $\int_0^1 f(s)\, dW_s$ is normal with mean zero. The Ito isommetry implies that the variane of the integral is

$$\int_0^1 (1-s)^2\, ds$$

The value of this integral is

$$\left[\frac{1}{3}(1-s)^3\right]_0^1 = \frac{1}{3}$$