

Problem 1.

$$1. \quad P(W < W_0 - \text{VaR}) = 1 - C.$$

$$\text{cdf} = 1 - e^{-\lambda x} = 1 - C.$$

$$x = -\frac{1}{\lambda} \cdot \log C.$$

$$\therefore x = W_0 - \text{VaR}.$$

$$\therefore \text{VaR} = W_0 + \frac{1}{\lambda} \cdot \log(C)$$

$$\therefore \lambda = \frac{1}{W_0}$$

$$\therefore \text{VaR} = W_0 (1 + \log C).$$

$$W_0 = 200, \quad C = 99.9\%$$

$$\text{VaR} = 199.8$$

$$2. \quad P(W > W_0 + \text{VaR}) = 1 - C$$

$$\text{cdf} = 1 - e^{-\lambda x} = C.$$

$$x = -\frac{1}{\lambda} \log(1 - C)$$

$$\therefore x = W_0 + \text{VaR}$$

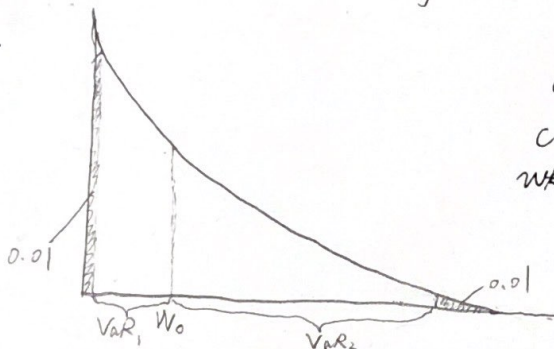
$$\therefore \text{VaR} = -W_0 - W_0 \cdot \log(1 - C)$$

$$= -W_0 (1 + \log(1 - C))$$

$$W_0 = 200, \quad C = 99.9\%$$

$$\text{VaR} = 1181.55$$

3.



Because of the asymmetry of exponential distribution, when you short the asset, its value could increase a lot more than it could decrease when you buy it. Therefore: $\text{VaR}_2 = 1181.55 \gg 199.8$.

$$4. \quad \text{ES} = W_0 - E[W | W \leq W_0 - \text{VaR}]$$

$$= W_0 - \frac{\int_{-\infty}^{W_0 - \text{VaR}} W \cdot f(W) dW}{\int_{-\infty}^{W_0 - \text{VaR}} f(W) dW}$$

In which:

$$\int_{-\infty}^{W_0 - \text{VaR}} f(W) dW = 1 - C.$$

\therefore For exponential distribution:

$$f(W) = 0 \text{ for } W < 0$$

$$\therefore \int_{-\infty}^{W_0 - \text{VaR}} W \cdot f(W) dW = \int_0^{W_0 - \text{VaR}} W \cdot f(W) dW.$$

$$f(W) = \lambda \cdot e^{-\lambda W}.$$

$$\text{Denote } A = \int_0^{W_0 - \text{VaR}} W \cdot f(W) dW.$$

Integrating by parts,

$$A = -W \cdot e^{-\lambda W} \Big|_0^{W_0 - \text{VaR}} + \int_0^{W_0 - \text{VaR}} e^{-\lambda W} dW.$$

$$= (\text{VaR} - W_0) \cdot e^{\lambda(\text{VaR} - W_0)} + \left(-\frac{1}{\lambda}\right) \cdot e^{-\lambda W} \Big|_0^{W_0 - \text{VaR}}$$

$$= (\text{VaR} - W_0) \cdot e^{\lambda(\text{VaR} - W_0)} + \left[\left(-\frac{1}{\lambda}\right) \cdot e^{\lambda(\text{VaR} - W_0)} + \frac{1}{\lambda} \right]$$

$$= (\text{VaR} - W_0) \cdot e^{(\text{VaR} - W_0)/W_0} + \frac{1}{W_0} - W_0 \cdot e^{(\text{VaR} - W_0)/W_0}$$

$$4. \therefore ES = W_0 - \frac{A}{1-c}$$

$$\text{in which } A = (VaR - 2W_0) e^{(VaR - W_0)/W_0} + W_0$$

$$\text{For question 1: } ES_1 = 199.9$$

~~For question 2: ES_2~~

$$\text{For question 2: } ES = E[W | W \geq W_0 + VaR] - W_0$$

$$ES = \frac{\int_{W_0 + VaR}^{\infty} W \cdot f(W) \cdot dW}{1-c} - W_0$$

same as question 1 above.

$$A = -W \cdot e^{-\lambda W} \Big|_{W_0 + VaR}^{\infty} + \left(-\frac{1}{\lambda}\right) \cdot e^{-\lambda W} \Big|_{W_0 + VaR}^{\infty}$$

$$= (2W_0 + VaR) \cdot e^{-(W_0 + VaR)/W_0}$$

$$\therefore ES_2 = \frac{A}{1-c} - W_0$$

$$\text{in which: } A = (2W_0 + VaR) \cdot e^{-(W_0 + VaR)/W_0}$$

$$ES_2 = 1381.56$$

Untitled

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4/4/2019

Problem 2

1.

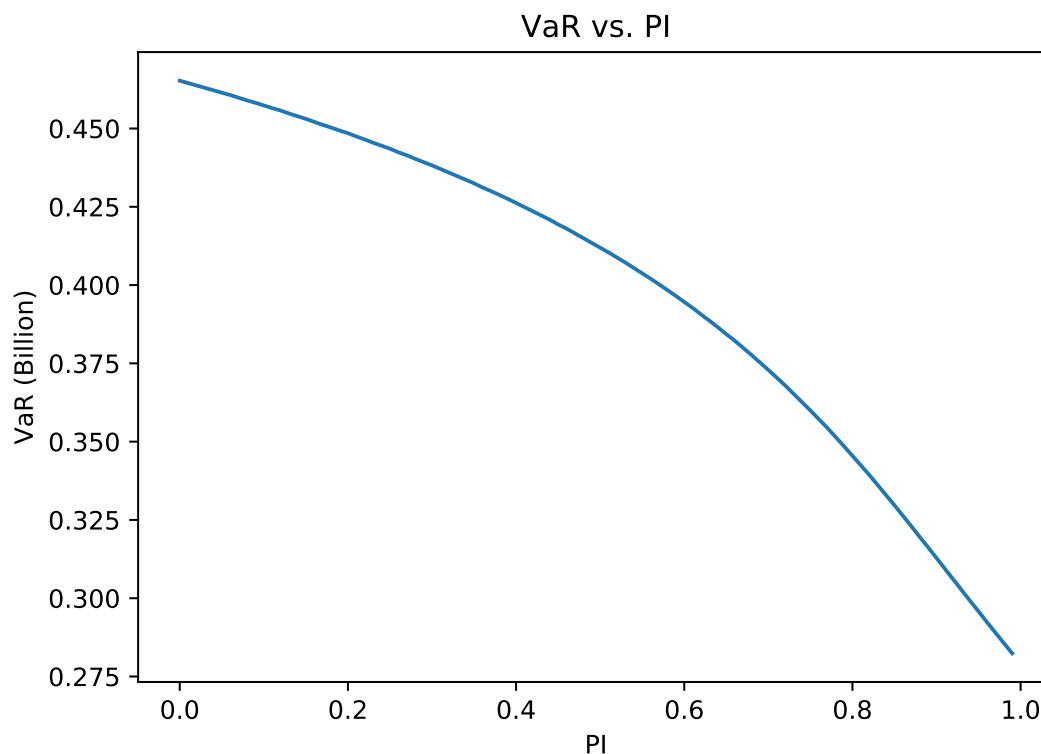
VaR for your view is 0.46526957480816816

VaR for partner view is 0.27916174488490086

VaR for common view is 0.372900390625

With higher volatility, the VaR will be higher.

With PI between 0 and 1, common view will be ranging between your view and partner's view.



2.

The VaR of common view is ranging from that of your view 0.4653 to that of partner view 0.2792.

The higher the probability PI, the VaR of common view will be more towards that of partner view.

3. We can use simulation to compute the VaR of portfolio. With millions of draws from the gamma distribution of σ , we can compute the VaR for each σ . After that, we take the average of the VaRs. And the average VaR is the VaR of the portfolio.

Problem 3.

1. Let $X_t = \log S_t$.

By Ito's lemma: $dX_t = \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial S} dS + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} (dS)^2$

$$dX_t = \mu - \frac{1}{2} \sigma^2 dt + \sigma dW_t.$$

$$\log S_T = X_T \sim N(X_0 + (\mu - \frac{1}{2} \sigma^2)T, \sigma^2 T).$$

If X is a random variable with quantile c equal to x_0 , then the quantile c of $g(X)$ is $g(x_0)$ if g is a monotone function.

$$P(S_T < S_0 - \text{VaR}) = 1 - c. \Leftrightarrow$$

$$P(\log S_T < \log(S_0 - \text{VaR})) = 1 - c = 0.01$$

$$\frac{\log(S_0 - \text{VaR}) - (X_0 + (\mu - \frac{1}{2} \sigma^2)T)}{\sigma \sqrt{T}} = -2.327. \quad T = \frac{10}{252}, S_0 = 50$$

$$\text{VaR} = 50 \cdot (1 - e^{\frac{5}{176}(\mu - \frac{1}{2} \sigma^2) - 0.4635 \cdot \sigma}).$$

Input $\mu = 0.07$ $\sigma = 0.16$. $\Rightarrow \text{VaR} = 3.468$

2. Let x denote the value you want to borrow by bonds.

$$\frac{100 - x \times (e^{0.02 \times \frac{10}{252}} - 1)}{\text{VaR}} \times 50 = x + 100 \quad \Rightarrow x \approx \cancel{1426.57} \quad 1326.57 \text{ million}$$

Therefore, in your portfolio: Long 1426.57 million stock and
Short 1326.57 million bonds.

Problem 3

3.

c0 = 1.7108771746126656

Due to the property that if X is a random variable with quantile c equal to x_0 , then the quantile c of $g(X)$ is $g(x_0)$ if g is a monotone function.

Because the price of call is a monotone function of stock price, $P(C_T < C_0 - VaR_c) = 1 - c$ is equivalent to $P(C(S_T) < C(S_0 - VaR_s)) = 1 - c$.

Therefore, $C_0 - VaR_c = C(S_0 - VaR_s)$

VaR of call is 1.376634175835873

$$\frac{100 - x \times (e^{2\% \times \frac{10}{252}} - 1)}{1.3766} \times 1.7109 = x + 100$$

$$x = 24.26 \text{million}$$

In your portfolio, long 124.26 million ATM call options and short 24.26 million bonds.

4.

p0 = 1.461501134246781

Because stock has the mean of return larger than 0, so the expectation of stock price will be higher than S_0 in 10 days. The VaR of put will be lower if we short put instead of long it. So that we can invest more on put within the same VaR constrain.

Same as call option, $P(P_T > P_0 + VaR_p) = 1 - c$ is equivalent to $P(P(S_T) > P(S_0 - VaR_s)) = 1 - c$.

Therefore, $P_0 + VaR_p = P(S_0 - VaR_s)$

VaR of put option is 2.130866119001105

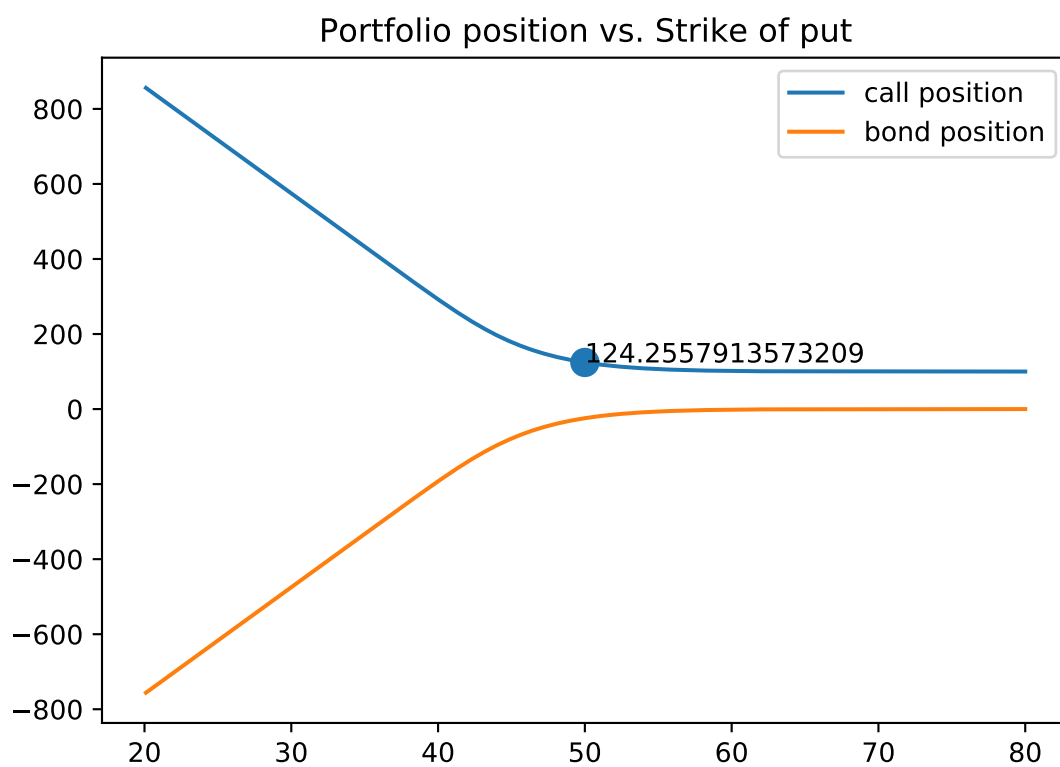
$$\frac{100 + x \times (e^{2\% \times \frac{10}{252}} - 1)}{2.1309} \times 1.4615 = x - 100$$

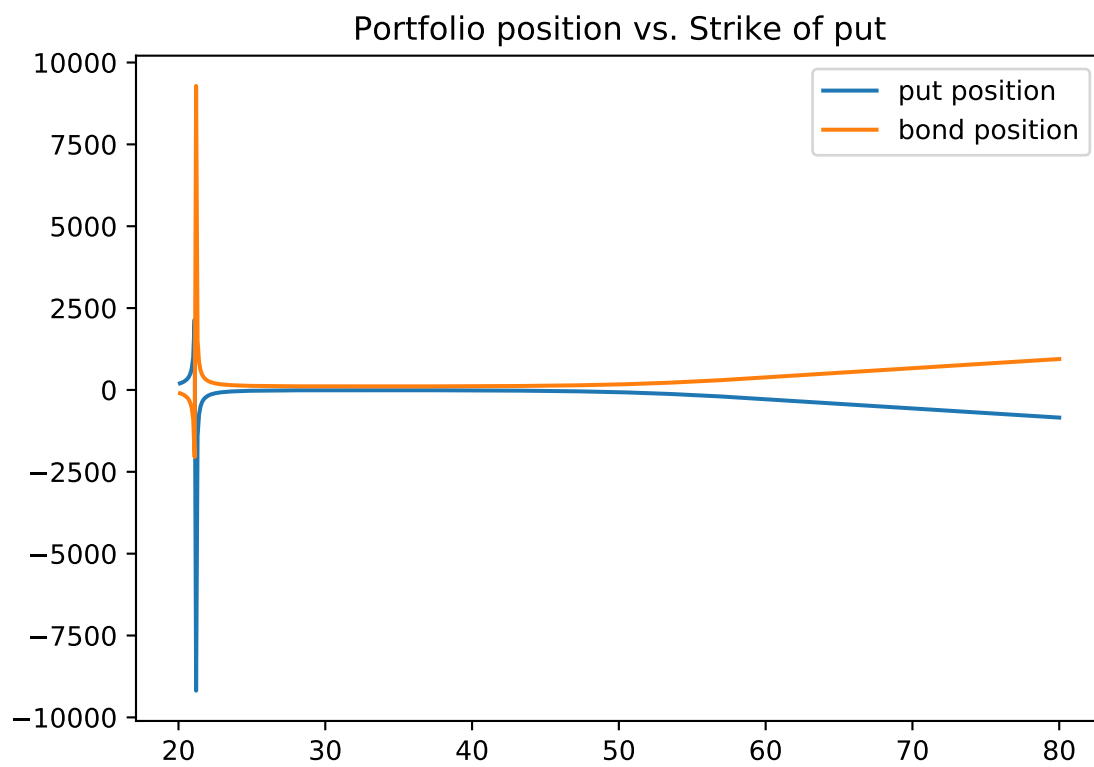
$$x = 168.68 \text{million}$$

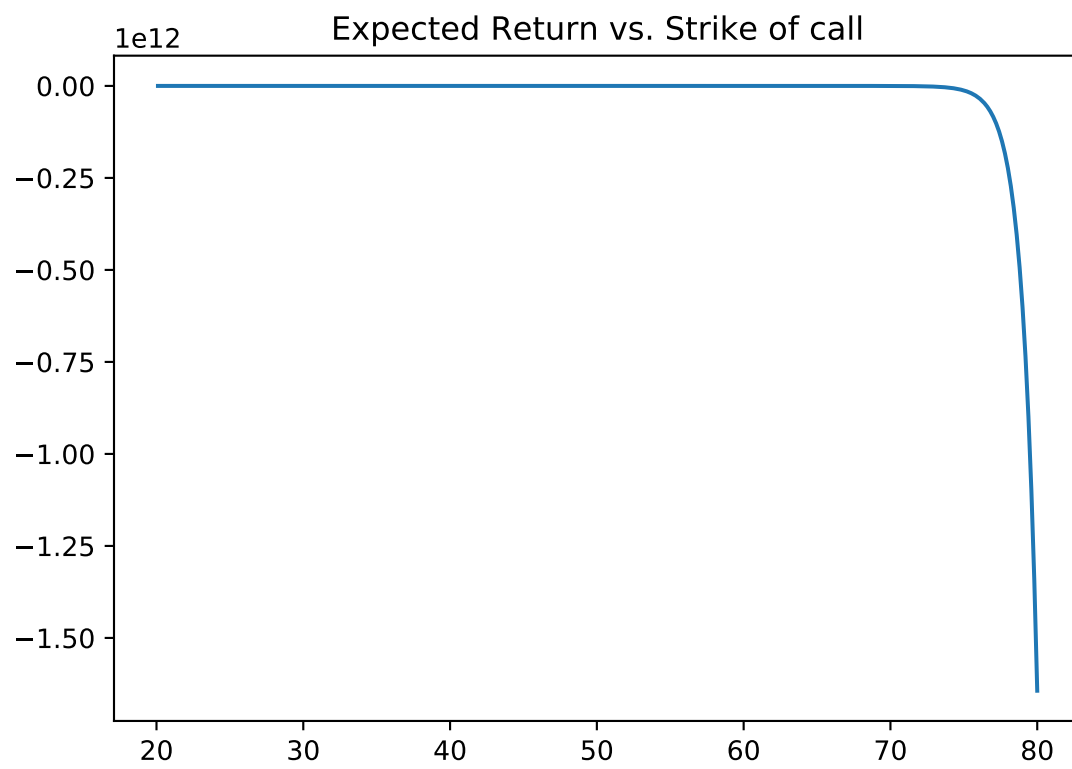
In your portfolio, short 68.68 million ATM put options and long 168.68 million bonds.

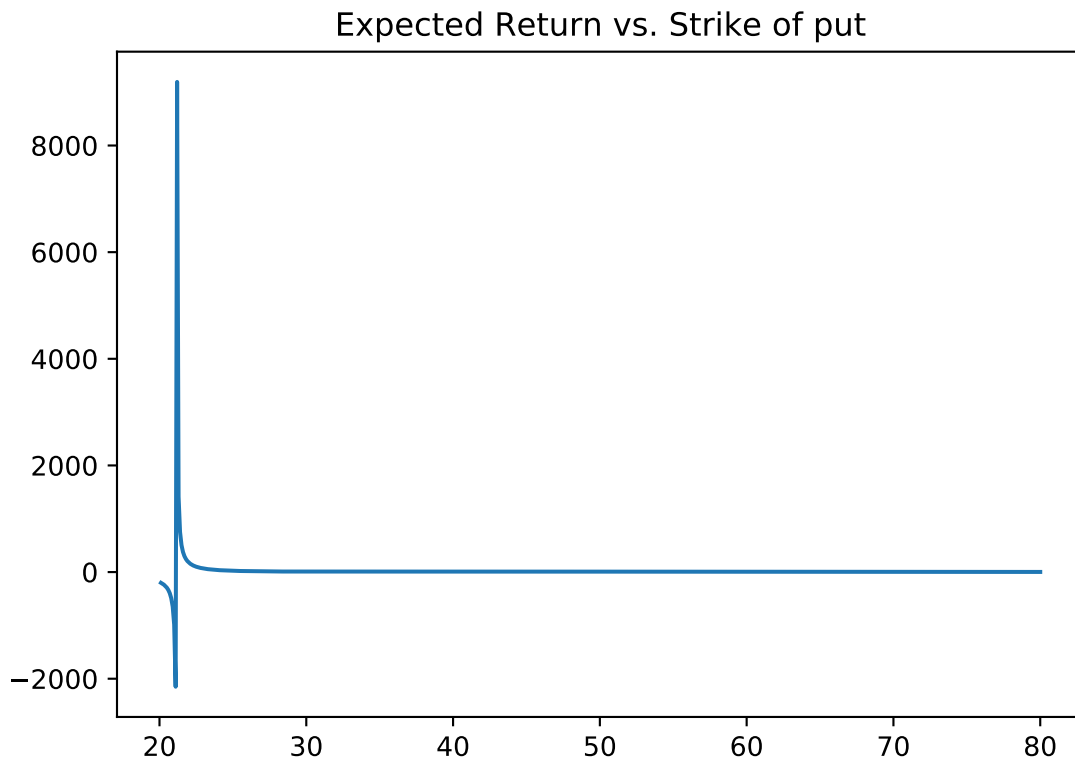
5.

Strike price of put with highest return is 21.200000000003342









According to the return plots, we should short put option with stike = 21.2.

In your portfolio, short put option 9185.47 million dollars and long 9285.47 million dollars. The expected return of this portfolio is 9192.76 million dollars.

6. With respect tp a constraint on expected shortfall, all the portfolio will have a low position on risky assets. And the return of all portfolio will be lower.

Python Code

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# Problem 2
# 1
from scipy.stats import norm
import numpy as np
import matplotlib.pyplot as plt
mu = 0.08
sigma1 = 0.12
sigma2 = 0.2
small = 0.00001
var = 2 * mu
c = 0.01
left = mu
right = 100
pi = 0.7
var_yours = mu - norm.ppf(0.01, loc = mu, scale = sigma2)
var_partner = mu - norm.ppf(0.01, loc = mu, scale = sigma1)
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def cdf(var, p = pi):
    return p * norm.cdf(mu - var, loc = mu, scale = sigma1) \
        + (1 - p) * norm.cdf(mu - var, loc = mu, scale = sigma2);

def value_at_risk(p, var = var, mu = mu):
    result = cdf(var)
    right = 100
    left = mu
    while (abs(result - c) > small):
        if result < c:
            right = var
        else:
            left = var
        var = (left + right) / 2
        result = cdf(var, p)
    return var

var_result = value_at_risk(pi)
print('VaR for your view is', var_yours)
print('VaR for partner view is', var_partner)
print('VaR for common view is', var_result)

# 2
pi_array = np.arange(0,1,0.01)
var_array = np.zeros(100)
for i,v in enumerate(pi_array):
    var_array[i] = value_at_risk(v)
plt.plot(pi_array, var_array)
plt.xlabel('PI')
plt.ylabel('VaR (Billion)')
plt.title('VaR vs. PI')
plt.show()

# Problem 3

# 3
import math
def eur_call(S,T,r,sigma,K):
    d1 = (math.log(S/K) + (r + sigma * sigma / 2) * math.sqrt(T)) / (sigma * math.sqrt(T))
    d2 = d1 - sigma * math.sqrt(T)
    return S * norm.cdf(d1) - K * math.exp(-r * T) * norm.cdf(d2)

S = 50
T = 0.25
K = 50
r = 0.02
sigma = 0.16
c0 = eur_call(S,T,r,sigma,K)
print(c0)

# 4
def eur_put(S,T,r,sigma,K):

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    d1 = (math.log(S/K) + (r + sigma * sigma / 2) * math.sqrt(T)) / (sigma * math.sqrt(T))
    d2 = d1 - sigma * math.sqrt(T)
    return K * math.exp(-r * T) * norm.cdf(-d2) - S * norm.cdf(-d1)
p0 = eur_put(S,T,r,sigma,K)
print(p0)
p_at_var = eur_put(S - 3.468, T - 10/252, r, sigma, K)
print(p_at_var)
VaR_p = p_at_var - p0
print(VaR_p)

# 5
def call_position(S,T,r,sigma,K):
    c0 = eur_call(S,T,r,sigma,K)
    c_at_var = eur_call(S - 3.468, T - 10/252, r, sigma, K)
    var = c0 - c_at_var
    return 100 * (1 - var / c0) / (var / c0 + math.exp(0.02 * 10 / 252) - 1) + 100

def put_position(S,T,r,sigma,K):
    p0 = eur_put(S,T,r,sigma,K)
    p_at_var = eur_put(S - 3.468, T - 10/252, r, sigma, K)
    var = p0 - p_at_var
    return 100 * (var / p0 + 1) / (var / p0 - math.exp(0.02 * 10 / 252) + 1) - 100

put_returns = list()
call_returns = list()
stike = 0
S_exp = S * math.exp((mu - 0.5 * sigma * sigma) * 10 / 252)
put_positions = list()
call_positions = list()
max_return = 0
k_list = np.arange(80,20,-0.1)
for k in k_list:
    call_unit = call_position(S, T, r, sigma, k)
    call_positions.append(call_unit)
    put_unit = put_position(S, T, r, sigma, k)
    put_positions.append(-put_unit)
    c_0 = eur_call(S,T,r,sigma,k)
    p_0 = eur_put(S,T,r,sigma,k)
    c_end = eur_call(S_exp,T-10/252,r,sigma,k)
    p_end = eur_put(S_exp,T-10/252,r,sigma,k)
    retn_c = (c_end - p_0) * call_unit / c_0 - (call_unit - 100) * (math.exp(0.02 * 10/252) - 1)
    retn_p = (p_0 - p_end) * put_unit / p_0 + (put_unit + 100) * (math.exp(0.02 * 10/252) - 1)
    call_returns.append(retn_c)
    put_returns.append(retn_p)
    if retn_p > max_return:
        stike = k
        max_return = retn_p
print(stike)

plt.plot(k_list, call_positions, label = 'call position')
plt.plot(k_list, [-x + 100 for x in call_positions], label = 'bond position')
plt.scatter(50, call_positions[300], s = 100)
plt.annotate(str(call_positions[300]),xy=(50,call_positions[300]))

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plt.legend()
plt.title('Portfolio position vs. Strike of put')
plt.show()
plt.plot(k_list, put_positions, label = 'put position')
plt.plot(k_list, [-x + 100 for x in put_positions], label = 'bond position')
plt.legend()
plt.title('Portfolio position vs. Strike of put')
plt.show()
plt.plot(k_list, call_returns)
plt.title('Expected Return vs. Strike of call')
plt.show()
plt.plot(k_list, put_returns)
plt.title('Expected Return vs. Strike of put')
plt.show()

```