Problem Solutions for Lecture 2

1. Assume that spot rates are as follows:

Maturity	Spot Rate
1	5%
2	5.5%
3	6%
4	6.3%

Compute the prices of the following bonds:

(a) A zero-coupon bond with 3 years to maturity.
The price is
$$\frac{100}{(1+6\%)^3} = 83.96.$$

(b) A bond with coupon rate 6% and 2 years to maturity.
The price is
$$\frac{6}{1+5\%} + \frac{106}{(1+5.5\%)^2} = 100.95.$$

(c) A bond with coupon rate 8% and 4 years to maturity.
The price is
$$\frac{8}{1+5\%} + \frac{8}{(1+5.5\%)^2} + \frac{8}{(1+6\%)^2} + \frac{108}{(1+6.3\%)^4} = 106.11.$$

Assume that spot rates and YTMs are with annual compounding, coupon payments are annual, and par values are $100.

2. You observe prices for the following bonds:

Bond	Coupon rate (%)	Maturity	Price
X	4	6 months	100.98
Y	6	1 year	103.59

Coupon payments are semiannual.

Determine the 6-month spot rate ($r_{0.5}$) and the 1-year spot rate (r_1), both expressed as APRs with semiannual compounding.

To solve the problem, we use the formula:

$$P = \frac{\frac{c}{2}}{1 + \frac{r_{0.5}}{2}} + \frac{\frac{c}{2}}{\left(1 + \frac{r_1}{2}\right)^2} + \cdots + \frac{100 + \frac{c}{2}}{\left(1 + \frac{r_T}{2}\right)^{2T}}.$$

The 6-month spot rate $r_{0.5}$ solves

$$100.98 = \frac{100 + \frac{4}{2}}{1 + \frac{r_{0.5}}{2}}$$

and the 1-year spot rate r_1 solves

$$103.59 = \frac{\frac{6}{2}}{1 + \frac{r_{0.5}}{2}} + \frac{100 + \frac{6}{2}}{\left(1 + \frac{r_1}{2}\right)^2}.$$

Solving these equations, we find $r_{0.5} = 2.02\%$ and $r_1 = 2.35\%$.

3. It's lunchtime. You are thinking about this restaurant, *Obligation du Tresor*, where you have always wanted to go but never did because you thought it was too pricey. Luckily, you bump into your friend Jerry, the bond trader. Today Jerry is buying and selling the following bonds:

Bond	Coupon rate (%)	Maturity	Price
A	0	1 year	95.238
B	5	2 years	98.438
C	7	2 years	103.370

Coupon payments are annual, and bid-ask spreads are zero.

Is it possible to construct an arbitrage (and get a free lunch at *Obligation du Tresor*), given the bond prices? If so, what is the trading strategy that produces the arbitrage?

Hint: Given the prices of two bonds, determine whether the third bond is over- or under-priced.

We will determine whether bond C is over- or under-priced, given the prices of bonds A and B. To do this, we will construct a portfolio of bonds A and B that replicates the cash flows of bond C in years 1 and 2. We will examine whether today's market value of this replicating portfolio is equal to the price of bond C, or whether it is greater or smaller. In the latter two cases, bond C will be mispriced, and we can construct an arbitrage.

Suppose that the replicating portfolio consists of x_1 units of Bond A and x_2 units of Bond B. Equating the cash flows of Bond C to those of the replicating portfolio, we have

$$
\begin{array}{lrcl}
\text{Year 1:} & 100x_1 + 5x_2 & = & 7 \\
\text{Year 2:} & 105x_2 & = & 107
\end{array}
$$

Solving this system, we get $x_1 = 0.019$ and $x_2 = 1.019$. The replicating portfolio thus consists in buying $x_1 = 0.019$ units of bond A and $x_2 = 1.019$ units of bond B.

The market value of the replicating portfolio is $95.238x_1 + 98.438x_2 = 102.12$. This is smaller than the price of bond C, which means that bond C is mispriced. To construct the arbitrage, we sell bond C (which is "expensive") and buy the replicating portfolio (which is "cheap"). We thus need to sell one unit of bond C, buy $x_1 = 0.019$ units of bond A and $x_2 = 1.019$ units of bond B.