Solutions to Sample Quizz 2 (b)

1. The PDE is given by

$$\begin{aligned}\frac{\partial F}{\partial t}+r\frac{\partial F}{\partial S}S+\frac{\partial^2 F}{\partial S^2}\sigma^2 S^{2\gamma}-rF &= 0\\ F(T,S_T) &= \max(S_T-K,0)\end{aligned}$$

The risk neutral dynamics are given by

$$dS_t = rS_t dt + \sigma S_t^{\gamma} dW_t$$

2. At time $t_1$ the present value of a payment equal to $K$ at time $T$ is given by

$$Ke^{-r(T-t_1)}.$$

Therefore the price of the derivative is given by

$$\begin{aligned}\Pi &= e^{-rt_1}E^Q Ke^{-r(T-t_1)}1_{\{S_{t_1}>K^*\}}\\ &= e^{-rT}K\overset{Q}{\Pr}(S_{t_1}>K^*)\end{aligned}$$

Under the risk neutral measure we have that

$$\frac{dS_t}{S_t}=r_t dt+\sigma dW_t$$

or

$$\log(S_{t_1})=\log(S_0)+\left(r-\frac{\sigma^2}{2}\right)t_1+\sigma W_{t_1}$$

and therefore

$$\overset{Q}{\Pr}(S_{t_1}>K^*)=N\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)t_1}{\sigma\sqrt{t_1}}\right)$$

where $N(.)$ is the cumulative normal. Combining terms gives

$$\Pi=e^{-rT}KN\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)t_1}{\sigma\sqrt{t_1}}\right)$$

3. Suppose that $C_0$ is the price of the call option at time 0. Then the forward price of a call option is given by

$$0=e^{-rt_1}\left(E_0^Q(C_{t_1})-K^*\right)$$

Since under the risk neutral measure $Q$ all assets grow in expectation at the rate of interest, we have

$$E_0^Q(C_{t_1})=C_0e^{rt_1}.$$

Therefore

$$K^*=C_0e^{rt_1}.$$