UCLA Anderson School of Management

Daniel Andrei, Derivative Markets 237D, Winter 2014

MFE – Final Exam

March 2014

Date: ______________________________

Your Name: ______________________________

Your Equiz.me email address: ______________________________

Your Signature:[1] ______________________________

- This exam is open book, open notes. You can use a calculator or a computer, but be sure to show or explain your work.
- You are not allowed to communicate with anyone (verbally, in writing, or electronically), except for me, during the exam period.
- You may present calculations in non-reduced form (e.g., as "$e^{0.095} - 1$").
- If you are stuck on something, make an assumption, tell me what it is, and do the best you can. I give partial credit if you provide enough correct information.
- If you got the wrong answer and failed to show your work, there is no way to obtain partial credit.

TIME LIMIT: 3 hours

TOTAL POINTS: 100

[1]As a member of the UCLA Anderson academic community, the highest standards of academic behavior are expected of you. It is your responsibility to make yourself aware of these standards (specifically regarding plagiarism, individual work, and team work) and adhere to them.

By signing the exam: (*i*) you certify your presence, and (*ii*) you state that you neither gave nor received help on the exam.

1 (18 points) Consider a structured investment product available from UBS. Its payoff in 4 years is given by

$$S_0 + 0.75 \times \max\left[S_4 - S_0, 0\right] \tag{1}$$

where S_t denotes the level of the S&P500 at time t. Suppose the interest rate is $r = 6\%$ (continuously compounded), the index volatility is $\sigma = 30\%$, the S&P500 index today is $S_0 = 1300$, and the dividend yield is 3%.

a. (4 points) Find the value at time 0 of the payoff expressed in (1).

b. (4 points) Assume that the bank has sold 100 such products to retail clients and would like to delta-hedge its exposure to movements in the S&P500. How many units of the index are needed to achieve this hedge?

c. (5 points) Explain how to synthetically create this structured product by using a **forward contract** on the S&P500 index and a **put option** instead of a call option. Both the forward and the put option have maturity $t = 4$. The put option has a strike of S_0.

d. (5 points) Consider a new structured product whose payoff is

$$\$1196 + \lambda \times \max\left[S_4 - S_0, 0\right] \tag{2}$$

Find the value of λ such that this structured product has the same initial price with the one from point *(a)*.

2 (18 points) When answering questions *(a)* thru *(d)* refer to the following table:

Quarter	Oil forward price	Gas swap price	Zero-coupon bond price
1	21	2.2500	0.9852
2	21.1	2.4236	0.9701
3	20.8	2.3503	0.9546
4	20.5	2.2404	0.9388

a. (4 points) Construct the set of swap prices for oil for 1 through 4 quarters.

b. (4 points) What is the swap price of a 2-quarter oil swap with the first settlement occurring in the third quarter?

c. (5 points) What are gas forward prices for quarters 1 and 2?

d. (5 points) What is the fixed rate in a 4-quarter interest rate swap?

3 (18 points) Consider a portfolio worth \$90 million invested entirely in the S&P500. The S&P500 stands at 900. To protect against market downturns, the manager of the portfolio would like to use **index futures** to create a synthetic put option with a strike price of \$87 million. The risk-free rate is 5% per annum, the dividend yield is 3% per annum, and the volatility of the portfolio is 25% per annum. The futures contract matures in $T_F = 9$ months whereas the put option to be replicated matures in $T = 6$ months. **Each index futures contract is on one unit of the index.**

a. (4 points) What is the price of the put option? What is the delta of the put option?

b. (4 points) How many units **of the index** need to be sold in order to replicate the put option?

c. (5 points) For a futures position on an asset providing a dividend yield at rate δ, the delta is $e^{(r-\delta)T_F}$, where T_F is the maturity of the futures. What is the alternative required position **in futures contracts** for replicating the put option?

d. (5 points) What happens with the delta of the required put option if the value of the portfolio reduces to \$89 million after 1 day? How should be the position in futures contracts adjusted? **No calculations are required here, just explain intuitively how those numbers will change.**

4 (18 points) Short questions

a. (4 points) Consider a non-dividend paying stock. Derive the delta of a European call from the delta of the corresponding European put.

b. (4 points) Consider a non-dividend paying stock. Show that the delta of an at-the-money European call in the Black-Scholes model is at least 1/2.

c. (4 points) Consider a non-dividend paying stock. What happens to the delta of an at-the-money European call as the time-to-maturity declines?

d. (6 points) Consider a non-dividend paying stock, $S(t)$. The initial stock price at $t = 0$ is $S(0) = 10$ and the continuously compounded interest rate is $r = 0.02$. The stock's volatility is 20%. At time $t = 0$, **you write** a one-year European option that pays

$$\begin{cases} 100 & \text{if } S(1)^2 > 100 \\ 0 & \text{otherwise} \end{cases}$$

You delta-hedge your position. Calculate the number of shares of the stock for your hedging program at time $t = 0$.

Hint: *The time-0 price of a binary option which pays X if $S(T) > K$ and 0 otherwise is*

$$X \times e^{-rT} N(d_2) \tag{3}$$

where d_2 is defined as in the Black-Scholes formula.

5 (18 points) A financial institution currently holds a portfolio of the following instruments:

- Long 200 shares of stock
- Long 200 puts with strike \$50 and maturity of three months
- Short 200 calls with strike \$60 and maturity three months

The stock is trading today at \$55. The stock is not paying any dividends. You are given the following information about several European options with various strikes:

Instrument	**Price**	**Delta**	**Gamma**	**Vega**	**Theta**	**Rho**
Call(K=50,T=1/4)	6.321	0.823	0.038	0.072	-0.015	0.097
Put(K=50,T=1/4)	0.700	-0.177	0.038	0.072	-0.008	-0.026
Call(K=55,T=1/4)	3.079	0.565	0.057	0.108	-0.019	0.070
Put(K=55,T=1/4)	2.396	-0.435	0.057	0.108	-0.011	-0.066
Call(K=60,T=1/4)	1.210	0.297	0.050	0.095	-0.015	0.038
Put(K=60,T=1/4)	5.465	-0.703	0.050	0.095	-0.007	-0.110

a. (3 points) What is the current value of your portfolio?

b. (5 points) What is the delta of your portfolio? The gamma? The vega? The theta? The rho?

c. (5 points) Suppose you want to make your portfolio gamma neutral. What is the cost of achieving this using the 55-strike call? What is the theta of your new total position?

d. (5 points) What is the cost if you used the 55-strike put? What is the theta of the new total position?

6 (10 points) Fill in the blanks in the following table under the assumption that the stock of IBM does not pay any dividends **over the next month**.

Security	**Strike**	**Maturity**	**Price**
IBM Stock	–	–	100
Zero coupon bond	–	1 month	0.99
Zero coupon bond	–	3 months	**??**
Forward price IBM stock	–	1 month	**??**
Forward price IBM stock	–	3 months	**??**
American Call on IBM	100	1 month	**??**
European Call on IBM	90	3 months	8.0
European Call on IBM	110	3 months	0.5
European Call on IBM	115	3 months	0.25
European Put on IBM	100	1 month	1.0
European Put on IBM	90	3 months	0.5
European Put on IBM	110	3 months	**??**
European Put on IBM	115	3 months	16.5

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Solutions to Final Exam (MFE)

1 a. The value of the payoff at time 0 is

$$S_0 \times e^{-rT} + 0.75\text{Call}(S_0, S_0, \sigma, r, T, \delta) = 1022.62 + 0.75 \times 326.67 = \$1267.62 \tag{4}$$

b. The bank is short 75 call options, each having a delta of $\Delta_C = 0.6133$. Thus, the bank should buy $75 \times 0.6133 = 46$ units of the index.

c. The payoff of the structured product after 4 years is

$$\begin{aligned} S_0 + 0.75C_T &= S_0 + 0.75\left(P_T + S_T - S_0\right) && (5)\\ &= S_0 + 0.75\left(P_T + S_T - F_{0,T} + F_{0,T} - S_0\right) && (6)\\ &= \underbrace{0.25S_0 + 0.75F_{0,T}}_{A} + \underbrace{0.75P_T}_{B} + \underbrace{0.75\left(S_T - F_{0,T}\right)}_{C} && (7)\end{aligned}$$

The first equality follows from the put-call parity. Part A of the payoff is an investment in the risk free asset, part B of the payoff is a long position of 0.75 at-the-money put options, and part C of the payoff is a long forward position of 0.75 the index (zero cost).

d. λ must solve

$$1267.62 = 1196 \times e^{-0.06\times 4} + \lambda \times 326.67 \tag{8}$$

for a solution of $\lambda = 1$.

2 a. Use the formula

$$X = \frac{\sum_{i=1}^{n} P(0,i)F_{0,i}}{\sum_{i=1}^{n} P(0,i)} \tag{9}$$

where $n = 1, ..., 4$, $P(0,i)$ are the zero-coupon bond prices, and $F_{0,i}$ are the oil forward prices. We obtain the following per barrel swap prices:

Quarter	**Swap price**
1	21.0000
2	21.0496
3	20.9677
4	20.8536

b. We use formula (9). In particular, we calculate

$$X = \frac{\sum_{i=3}^{4} P(0,i)F_{0,i}}{\sum_{i=3}^{4} P(0,i)} \tag{10}$$

Therefore, the swap price of a 2-quarter oil swap with the first settlement occurring in the third quarter is \$20.6513.

c. We are asked to invert formula (9) (the swap prices are given and we want to back out the forward prices). We do so recursively. For a one-quarter swap, the swap price and the forward price are identical. Given the one-quarter forward price, the second quarter forward price solves the following equation

$$\frac{2.25 \times P(0,1) + X \times P(0,2)}{P(0,1) + P(0,2)} = 2.4236 \tag{11}$$

Doing so yields the following forward prices:

Quarter	**Forward price**
1	2.2500
2	2.5999

d. From the given zero-coupon bond prices, we can calculate the one-quarter forward interest rates. They are:

Quarter	**Forward interest rate**
1	0.0150
2	0.0156
3	0.0162
4	0.0168

Now, we can calculate the swap rate for 4 quarters according to the formula:

$$X = \frac{\sum_{i=1}^{4} P(0,i) r_0(i-1,i)}{\sum_{i=1}^{4} P(0,i)} \tag{12}$$

This yields a 4-quarter fixed swap rate of 0.0159, or 1.59%.

3 a. We have $d_1 = 0.33673$, $d_2 = 0.15996$, $N(d_1) = 0.6318$, $N(d_2) = 0.5635$ and thus

$$P = S_0 e^{-\delta T}\left[N(d_1) - 1\right] + K e^{-rT}\left[1 - N(d_2)\right] \tag{13}$$

$$= \$4.39 \text{ million} \tag{14}$$

The delta of the put option is

$$e^{-\delta T}\left[N(d_1) - 1\right] = -0.36268 \tag{15}$$

b. The portfolio is worth 100,000 times the index, thus we need to sell $0.36268 \times 100,000 = 36,268$ units of the index.

c. If H_A is the required position in the asset and H_F is the alternative required position in futures contracts, then

$$e^{(r-\delta)T_F} H_F = H_A \tag{16}$$

and thus $H_F = e^{-(r-\delta)T_F} H_A$. In our case, the number of futures contracts shorted should be $H_F = 35,728$. Intuitively, the delta of the futures in this case is $e^{(r-\delta)T_F} = 1.015113$ and thus less futures contracts are needed for delta hedging.

d. If the value of the portfolio reduces to \$89 million after 1 day, the delta of the required option becomes higher in magnitude (i.e., lower than -0.36268). Therefore, more futures contracts need to be sold.

4 a. For any given common strike and maturity, it is true that for European options

$$\Delta(\text{call}) - \Delta(\text{put}) = 1 \tag{17}$$

So, from knowledge of one delta, one can calculate the other one regardless of the (common) strike and maturity.

b. The delta of a call in the Black-Scholes model is given by $N(d_1)$, where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{18}$$

The call is at-the-money, so $S_0 = K$. Therefore

$$d_1 = \frac{1}{\sigma}\left(r + \frac{\sigma^2}{2}\right)\sqrt{T} > 0 \tag{19}$$

and thus $N(d_1) > 1/2$.

c. The delta of an at-the-money European call is given by $N(d_1)$, with d_1 specified in (19). The right-hand side of (19) decreases as time-to-maturity decreases. When d_1 decreases, $N(d_1)$ also decreases. This means that the delta of the call decreases as time-to-maturity declines.

d. Note that $S(1)^2 > 100$ is equivalent to $S(1) > 10$. Thus, the option is a cash-or-nothing option with strike price 10. The time-0 price of the option is

$$100 \times e^{-rT} N(d_2) \tag{20}$$

To find the number of shares in the hedging program, we differentiate the price formula with respect to S:

$$\frac{\partial}{\partial S} 100e^{-rT} N(d_2) = 100e^{-rT} N'(d_2) \frac{\partial d_2}{\partial S} \tag{21}$$

$$= 100e^{-rT} N'(d_2) \frac{1}{S\sigma\sqrt{T}} \tag{22}$$

With $T = 1$, $r = 0.02$, $\delta = 0$, $\sigma = 0.2$, $S = S(0) = 10$, $K = 10$, we have

$$d_2 = \frac{\ln(S/K) + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} = 0 \tag{23}$$

and

$$100e^{-rT} N'(d_2) \frac{1}{S\sigma\sqrt{T}} = 100e^{-0.02} N'(0) \frac{1}{2} \tag{24}$$

$$= \frac{50e^{-0.02}}{\sqrt{2\pi}} \tag{25}$$

$$= 19.55 \tag{26}$$

5 a. Let m denote the number of positions in the underlying, n_1 denote the position in the 50-strike puts, and n_2 denote the position in the 60-strike calls. We have $m = 200$, $n_1 = 200$, and $n_2 = -200$.

The stock is trading at 55, the 50-strike put is worth 0.700 and the 60-strike call is worth 1.210. Thus, the value of the position is

$$V = m \times 55 + n_1 \times 0.700 + n_2 \times 1.210 = \$10,898 \tag{27}$$

b. The position greeks may be calculated using the greeks of the options and the stock: $\Delta = 105.140$, $\Gamma = -2.499$ Vega $= -4.725$, $\Theta = 1.348$, $\rho = -12.788$.

c. From the previous answer, the position gamma is -2.499. The option we are to use to help neutralize this quantity is the 55-strike call which has a gamma of 0.057. Thus, the number of the 55-strike call required to make the gamma zero (denoted, say n_3) is

$$n_3 = \frac{2.499}{0.057} = 43.65 \tag{28}$$

or a long position in approximately 44 calls. Since each of these calls costs \$3.079, the total cost of achieving gamma neutrality in this way is

$$43.65 \times 3.079 = 134.389 \tag{29}$$

Finally, the theta of each new call is -0.019. Thus, the new position theta is $\Theta_{new} = 1.348 + 43.65 \times (-0.019) = 0.533$.

d. If we make the position gamma neutral using the 55-strike put instead, the number of puts we would need is exactly the same since the gamma of the 55-strike call and the 55-strike put coincide. Since each of these puts costs \$2.396, the total cost of achieving gamma neutrality using the puts is

$$43.65 \times 2.396 = 104.57 \tag{30}$$

Since each of the puts has a theta of -0.011, the new position theta in this case would be $\Theta_{new} = 1.348 + 43.65 \times (-0.011) = 0.858$.

6 By the put-call parity

$$8 - 0.5 = 100 - 90B(0, 3M) - D_0(3M) \tag{31}$$
$$0.25 - 16.5 = 100 - 115B(0, 3M) - D_0(3M). \tag{32}$$

where $D_0(3M)$ is the present value of dividends paid by IBM between 1 and 3 months from now and $B(0, 3M)$ is the price of a zero-coupon bond with maturity 3 months.

This yields

$$B(0, 3M) = 0.95 \qquad D_0(3M) = 7. \tag{33}$$

The 1 month forward price is

$$F_0(1M) = \frac{100}{0.99} = 101.01. \tag{34}$$

The 3 month forward price is

$$F_0(3M) = \frac{1}{0.95}(100 - 7) = 97.89. \tag{35}$$

The 1 month American call with strike 100 is a European call (there are no dividends over the next month). By the put-call parity

$$\text{Call}(1M) = 1 + 100 - 100 \times 0.99 = 2 \tag{36}$$

The 3 months European put with strike 110 satisfies

$$\text{Put}(3M) = 0.5 - 100 + 110 \times 0.95 + 7 = 12 \tag{37}$$