Problem Set 5

MGMTMFE 403-2 Stochastic Calculus

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Problem 1

We use $V(t) = F(t, S(t))$ donate the portfolio which is to replicate the call option:

$$dV(t) = V(t)\{\mu^0(t)r + \mu^*(t)\alpha(t, S(t))\}dt + V(t)\mu^*(t)\sigma(t, S(t))d\bar{W}(t)$$
$$\mu^0(t) + \mu^*(t) = 1$$

$$\begin{aligned}
\mu_c &= \mu^0 r + \mu^* \mu \\
&= \mu^0 r + (1 - \mu^0)\mu \\
&= \mu + \mu^0(r - \mu)
\end{aligned}$$

$$\begin{aligned}
\mu^0 &= \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} \\
F_t &= \Theta = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) \\
F_{ss} &= \Gamma = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}
\end{aligned}$$

Using the above 3 equations:

$$\begin{aligned}
\mu^0 &= \frac{-\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) + \frac{1}{2}\sigma^2 S^2 \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}}{rF} \\
&= \frac{-rKe^{-r(T-t)}N(d_2)}{rF}
\end{aligned}$$

$$\begin{aligned}
&\because r > 0, K > 0, N(d_2) > 0, F > 0 \\
&\therefore \mu^0 < 0 \\
&\therefore \mu^* = 1 - \mu^0 > 1 \\
&\because r < \mu, \mu^0 < 0, \mu^* > 1 \\
&\therefore \mu_c = u + \mu^0(r - \mu) > \mu \\
&\therefore \sigma_c = \mu^*\sigma > \sigma
\end{aligned}$$

$$\begin{aligned}
\frac{\mu_c - r}{\sigma_c} &= \frac{\mu + (1 - \mu^*)(r - \mu) - r}{\mu^*\sigma} \\
&= \frac{\mu + r - \mu - r\mu^* + \mu\mu^* - r}{\mu^*\sigma} \\
&= \frac{\mu - r}{\sigma}
\end{aligned}$$

Problems 2

Put-Call parity:

$$p(t,s) = Ke^{-r(T-t)} + c(t,s) - s$$
$$\frac{\partial p}{\partial s} = \frac{\partial c}{\partial s} - 1$$

By Black-scholes formula:

$$c(t,s) = sN(d_1) - e^{-r(T-t)}KN(d_2)$$
$$\frac{\partial c}{\partial s} = N(d_1)$$
$$\therefore \frac{\partial p}{\partial s} = N(d_1) - 1$$