MFE 409 Lecture 1b
Value-at-Risk

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Lecture Objectives

Understanding **Value-at-Risk**

- How to compute it
- What is it useful for?
- What are its limitations?
- Some alternatives

Measuring Risk

- Defining and managing risk is one of the most important issues firms are facing in their daily operations
- Especially important for financial institutions that rely on leverage.
- Find an answer to the question:

  "What is realistically the worst that could happen over one day, one week, or one year?"

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Value-at-Risk

- **Value-at-risk (VaR)** is an answer to the question above where "realistically" is defined by finding an outcome that is so bad that anything worse is highly unlikely.

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Definition. Let W be a random variable. The *Value-at-Risk* at confidence level c relative to base level W_0 is the smallest non-negative number denoted by VaR such that

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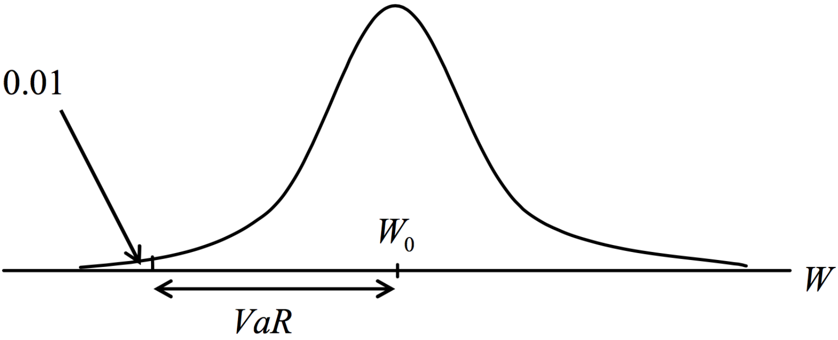
Typically:

- W is the value of a portfolio at some point in the future (1 day, 1 month, 1 year)
- W_0 gives some base level: often current value of the portfolio
- Confidence level c gives concrete mean to what "worst case" means: 99%, 99.9%

Value-at-Risk



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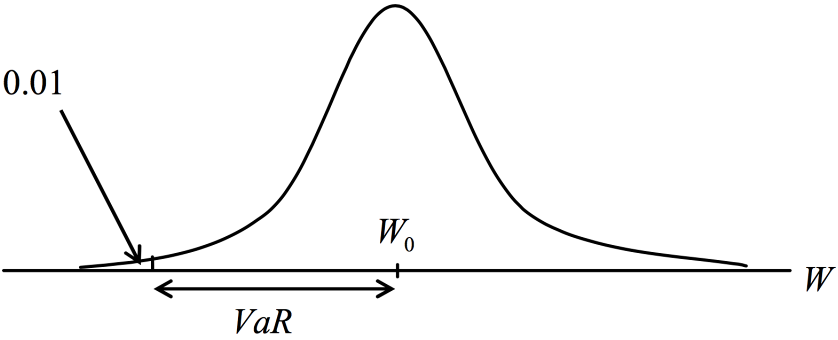

- Bottom point: c-quantile, $F^{-1}(1-c)$ if F is cumulative distribution function

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Rationale for Value-at-Risk

- It captures an important aspect of risk in a single number
- Easy to understand
- Two broad motivations
  - Measure of potential extreme loss
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A Measure of Capital

- Example:
  - W measures value of total assets of the firm in 10 days
  - W_0 is today's value of the firm's assets
  - Firm remains solvent as long as W does not fall below $W_0 - \text{capital}$.

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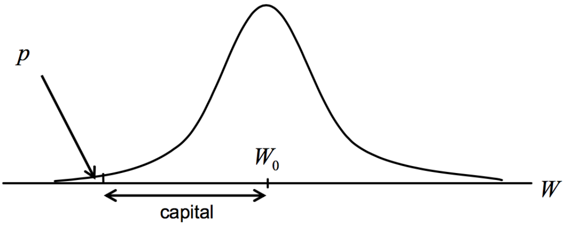

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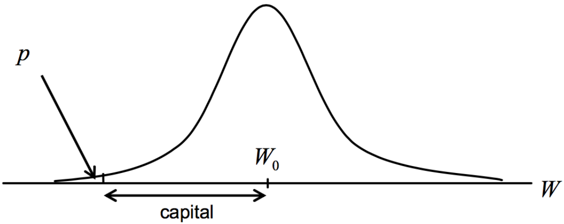

- 1% 10-day VaR: amount of capital to hold so that firm goes bankrupt with probability 1% in the next 10 days

Regulatory Capital

- Regulators have traditionally used VaR to calculate the capital they require banks to keep

- The market-risk capital has been based on a 10-day VaR estimated where the confidence level is 99%

- Credit risk and operational risk capital are based on a one-year 99.9% VaR

What is Special about Capital?

- If the firm or bank has limited liability, then it does not matter whether the firm goes bust just marginally, or whether it goes bust spectacularly, leaving a big shortfall

- The tail loss is not a concern for a firm with limited liability

Example

- Project A has:
  - 98% chance of leading to a gain of \$2 million
  - 1.5% chance of a loss of \$4 million
  - 0.5% chance of a loss of \$10 million
- The VaR with a 99.9% confidence level is ...
- What if the confidence level is 99.5%?
- What if it is 99%?

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Example

- Project A - B has:
  - 98% - 98% chance of leading to a gain of $2 million
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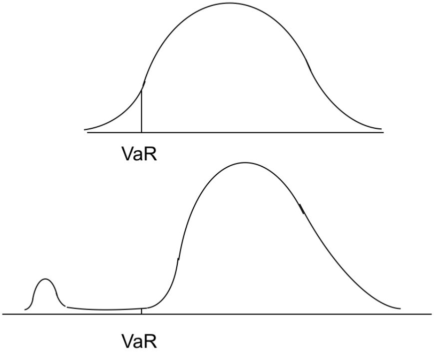
- Which project is more risky?
- Which project has a larger 99% VaR?

Limitation of VaR

- VaR does not capture the distribution of losses below the threshold
- Formally, any 2 distributions with same $F^{-1}(1-c)$ will have the same VaR



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- Taking extreme tail risk will not increase VaR
  - Selling puts
  - Selling disaster insurance

Expected Shortfall

- **Expected shortfall**: expected loss given loss larger than VaR

$$
\begin{aligned}
\mathsf{ES} &= W_0 - \mathbb{E}\left[W | W \leq W_0 - \mathsf{VaR}\right] \\
&= W_0 - \frac{\int_{-\infty}^{W_0 - \mathsf{VaR}} W f(W) dW}{\int_{-\infty}^{W_0 - \mathsf{VaR}} f(W) dW}
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- Also called C-VaR and Tail Loss
- Regulators have indicated that they plan to move from using VaR to using ES for determining market risk capital
- Two portfolios with the same VaR can have very different expected shortfalls

Example: Normal Distribution

- January 8, 2010
  - Position: EUR 10 million
  - Exchange rate M_t= USD/EUR $=$ \$1.436
  - Dollar position W_0 $=$ \$14.36 million
- Assume normal distribution for FX return

$$R_{M,t+1} \sim \mathcal{N}(\mu, \sigma)$$

  - Historically (in daily units), we find: $\sigma = 0.65\%$ and $\mu \approx 0$
- We want to compute the 99% 1-day Value-at-Risk

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  - 95% ES $= -(\mu_V - \sigma_V \times 2.0628)$
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- For normal distributions, close relation between VaR and ES: multiple of volatility

Role of Time

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- Autocorrelation ρ between losses on successive days, replace $\sqrt{T}$ by

$$\sqrt{T + 2(T-1)\rho + 2(T-2)\rho^2 + 2(T-3)\rho^3 + \ldots + 2\rho^{T-1}}$$

	T=1	T=2	T=5	T=10	T=50	T=250
ρ=0	1.0	1.41	2.24	3.16	7.07	15.81
ρ=0.05	1.0	1.45	2.33	3.31	7.43	16.62
ρ=0.1	1.0	1.48	2.42	3.46	7.80	17.47
ρ=0.2	1.0	1.55	2.62	3.79	8.62	19.35

Example: VaR for a Portfolio

- Positions: 10mil EUR, 1bil Yen
  - $J_t =$ USD/JPY $= 0.01078749$; USD/EUR $= 1.436$
  - Assume $R_{M,t+1}$ and $R_{J,t+1}$ jointly nomal with
  - $E(R_M) = E(R_J) \approx 0$, $\sigma_M = 0.65\%$, $\sigma_J = 0.69\%$
  - Corr$(R_M, R_J) = \rho_{MJ} = 0.2775$

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VaR for a Portfolio: Approximate approach

- An approximate approach that seems to works well is

$$\text{VaR}_{\text{total}} = \sqrt{\sum_i \sum_j \text{VaR}_i \text{VaR}_j \rho_{ij}}$$

  where VaR_i is the VaR for the i-th segment, $\text{VaR}_{\text{total}}$ is the total VaR, and ρ_{ij} is the coefficient of correlation between losses from the i-th and j-th segments

- Exact formula for normal distributions

VaR for a Portfolio: Exact approach

- Marginal VaR

$$\mathsf{DVaR}_i = \frac{\partial \mathsf{VaR}}{\partial x_i}$$

- Component VaR

$$\mathsf{CVaR}_i = x_i \frac{\partial \mathsf{VaR}}{\partial x_i}$$

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- Decomposition (Euler Theorem):

$$\mathsf{VaR} = \sum_i \mathsf{CVaR}_i$$

Using VaR for Capital Allocation

- You are the head of prop trading for an investment bank. You have to allocate capital between investing in FX or in fixed income. Last year FX invested \$100m and made 10% profits, while fixed income invested \$200m and made 5% profit. What do you do?

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- If you have access to leverage quantity of assets is not so important, rather quantity of capital mobilized.
- **Risk Adjusted Rate of Return on Capital (RAROC)**: profit per unit of necessary capital, i.e. profit per unit of VaR

$$\text{RAROC} = \frac{\text{Profit}}{\text{VaR}}$$

- Developed in the 1980s by Bankers Trust (taken over by Deutsche Bank) to develop internal capital budgeting system

Example: RAROC

- Let us compute RAROC for the two positions
  - Assume normal distribution with annual volatility 10% for FX and 4% for fixed income
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$$\begin{aligned} \mathsf{Var}\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) &= \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 + 2\mathsf{Cov}\left(\frac{1}{2}x_1, \frac{1}{2}x_2\right) \\ &= \frac{1}{2}\left(\sigma^2 + \mathsf{Cov}\left(x_1, x_2\right)\right) \\ &= \frac{1}{2}\sigma^2(1+\rho) \\ &\leq \sigma^2 \end{aligned}$$

- With normal distribution, also applies to Value-at-Risk: diversification reduces risk

VaR and Diversification

- Consider bonds with face value of 100 and default probability of 0.9% and 0 recovery. Assume defaults are independent across bonds and that the baseline level is $W_0 = 100$.
- What is the 99% VaR for one bond?
- What is the 99% VaR for two bonds?
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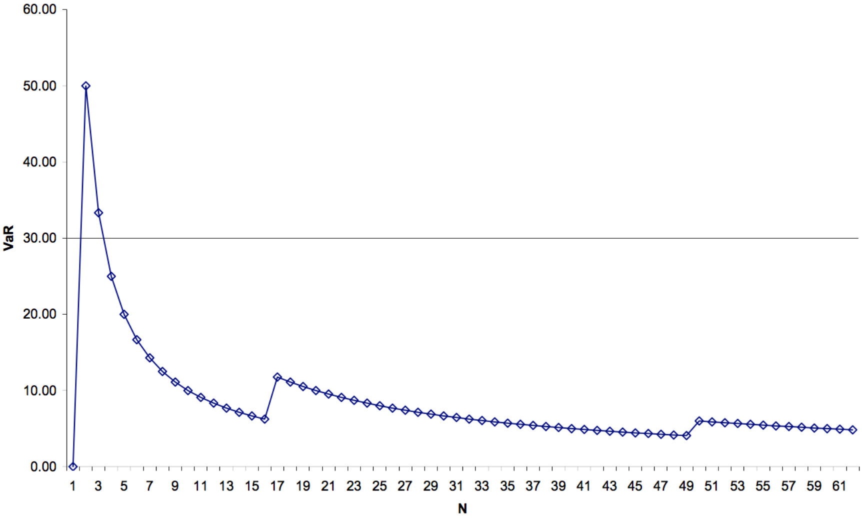

Coherent Risk Measures

- Define a coherent risk measure as the amount of cash that has to be added to a portfolio to make its risk acceptable
- Properties of coherent risk measure
  - If one portfolio always produces a worse outcome than another its risk measure should be greater
  - If we add an amount of cash K to a portfolio its risk measure should go down by K
  - Changing the size of a portfolio by a factor λ should result in the risk measure being multiplied by λ
  - The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged

Coherent Risk Measures

- Value-at-Risk
- Expected Shortfall

Coherent Risk Measures

- Value-at-Risk ✗
- Expected Shortfall ✓

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- Spectral measures
  - Spectral measures assigns weight to quantiles of the loss distribution
  - VaR assigns all weight to c-th percentile of the loss distribution
  - Expected shortfall assigns equal weight to all percentiles greater than the c-th percentile
  - For a coherent risk measure weights must be a non-decreasing function of the percentiles

Takeaways

- Value-at-Risk is:
  - a simple measure
  - used by regulators and practitioners to measure risk
  - which focuses on the extreme downside of a distribution
- It has some limitations
  - Does not capture the entire distribution of extreme losses
  - Does not always capture diversification
- Implications
  - If you want to monitor risk, know its limitations
  - Expected shortfall is a better behaved alternative
  - If you are constrained by it, know how to game it
- Next: how to measure VaR in the real world? We don't know the distribution of what will happen in the next few days!