

MFE 409 LECTURE 2B

MEASURING VALUE-AT-RISK

Valentin Haddad

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LECTURE OBJECTIVES

Measuring Value-at-Risk:

- How to judge validity of a VaR estimate?
- Historical approach
- Model-building approach
- How to get a measure for a given approach but also how to choose an appropriate approach

OUTLINE

1 MODEL-BUILDING

MODEL-BUILDING APPROACH

- The main alternative to historical simulation is to make assumptions about the probability distributions of the returns on the market variables
- Sometimes called the variance-covariance approach

NORMAL MODEL

- Simplest and often-used assumption: normal distribution

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- VaR has a simple expression:

$$\text{VaR} = -\mu - \sigma \times z(c)$$

- Portfolios of normal returns are also normally distributed
- Estimation of normal distributions very developed

PORTFOLIOS

- With multivariate normal returns, portfolio returns are normally distributed
- Assume:
 - ▶ Each asset return R_i is normally distributed with mean 0 and variance σ_i^2
 - ▶ Pairwise correlations: ρ_{ij}
 - ▶ investment in each asset α_i

$$\underbrace{\Delta P}_{\text{portfolio gain}} = \sum_i \alpha_i R_i \sim \mathcal{N}(0, \sigma_P^2)$$
$$\sigma_P^2 = \sum_i \sum_j \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}$$

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- Assume:
 - ▶ Asset return \mathbf{R} normally distributed with mean $\mathbf{0}$ and variance Σ
 - ▶ Investment vector α

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- Multiply by $-z(c)$ to obtain VaR

EXAMPLE: IMPERFECT HEDGE

- Previous example:

- ▶ Long position EUR 10m, $M_t = \text{USD/EUR} = \$1.436$, $\sigma_M = 0.65\%$
- ▶ Dollar position \$14.36m
- ▶ VaR= \$217,204

- Suppose you want to hedge with Japanese Yens: $\sigma_J = 0.69\%$,
 $\rho_{MJ} = 0.2775$

- ▶ What Yen position do you choose to hedge as well as possible?
- ▶ What is your hedged VaR?

IMPERFECT HEDGE

- Want to hedge a position R_p using a hedging instrument R_h
- Optimal hedging position:

$$\alpha_{\text{hedge}} = -\rho \frac{\sigma_p}{\sigma_h}$$

- Variance of the hedged portfolio:

$$\text{Minimum variance} = \sigma_p^2(1 - \rho^2)$$

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- VaR of the hedged portfolio:

$$\text{Minimum VaR} = \text{VaR}_p \sqrt{1 - \rho^2}$$

- Only depends of correlation ρ

VOLATILITY

- Often, volatility is defined as standard deviation of *log return*

$$\log \left(\frac{P_{t+1}}{P_t} \right)$$

- In risk management, typically the standard deviation of *simple return*

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- Some conventions:

- ▶ Only count trading days: $\sigma_{\text{yr}} = \sigma_{\text{day}} \times \sqrt{252}$
- ▶ Variance rate: σ^2

ESTIMATING VOLATILITY

- Assume today is date t and we have data for n past dates
- Unbiased estimates

- ▶ Mean

$$\bar{R} = \frac{1}{n} \sum_{i=1}^n R_{t-i}$$

- ▶ Volatility

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^n (R_{t-i} - \bar{R})^2$$

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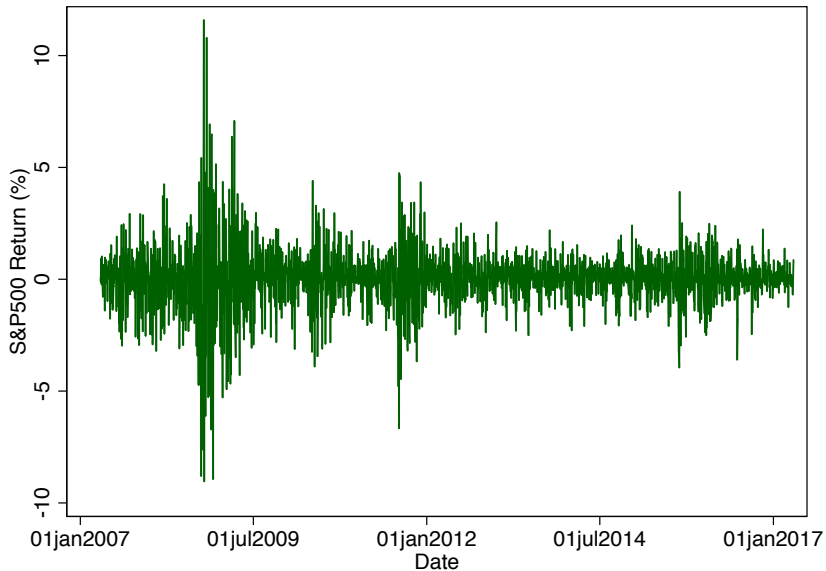
- Risk management practice:

- ▶ Assume $\bar{R} = 0$: mean small relative to standard deviation for one day
- ▶ Replace $n - 1$ by n

ESTIMATING VOLATILITY: MAXIMUM LIKELIHOOD

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S&P500: RETURNS



S&P500: HISTORIC VOLATILITY

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- Weighting scheme + long-run variance

$$\sigma_t^2 = \gamma V_L + \sum_{i=1}^n \alpha_i R_{t-i}^2$$

$$\text{with } 1 = \gamma + \sum_{i=1}^n \alpha_i$$

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- ▶ Simple volatility updating:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) R_t^2$$

- ▶ RiskMetrics reported with $\lambda = 0.94$ until 2006

S&P500: EWMA VOLATILITY

GARCH(1,1)

- GARCH(1,1), generalized autoregressive conditional heteroskedasticity

$$\sigma_t^2 = \underbrace{\gamma V_L}_{\omega} + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2$$

- ▶ EWMA + long-run average
- ▶ If $\gamma = 0$, EWMA
- ▶ For stability, $\alpha + \beta < 1$

MLE ESTIMATION OF GARCH(1,1)

- Parameters: ω, α, β

- Log-likelihood

$$\sum_{i=1}^n \left[-\log(\sigma_{t-i}^2) - \frac{R_{t-i}^2}{\sigma_{t-i}^2} \right]$$

- Compute σ_{t-i}^2 :

- Initialize at $\sigma_0 = \sqrt{V_L} = \sqrt{\omega/(1 - \alpha - \beta)}$

- Use formula to iterate

$$\sigma_t^2 = \omega + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2$$

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- Ljung-Box Statistic

$$n \sum_{k=1}^K w_k c_k^2$$
$$w_k = \frac{n+2}{n-k}$$

- For $K = 15$, 95% threshold is 25

VOLATILITY FORECASTS

- If we want to forecast k days in the future:

$$\mathbb{E}_t [\sigma_{t+k}^2] = V_L + (\alpha + \beta)^k (\sigma_t^2 - V_L)$$

- ▶ Exponential mean reversion

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- *Remark:* If we want to hedge volatility risk, need to consider how shocks today will affect volatility during the lifetime of the option

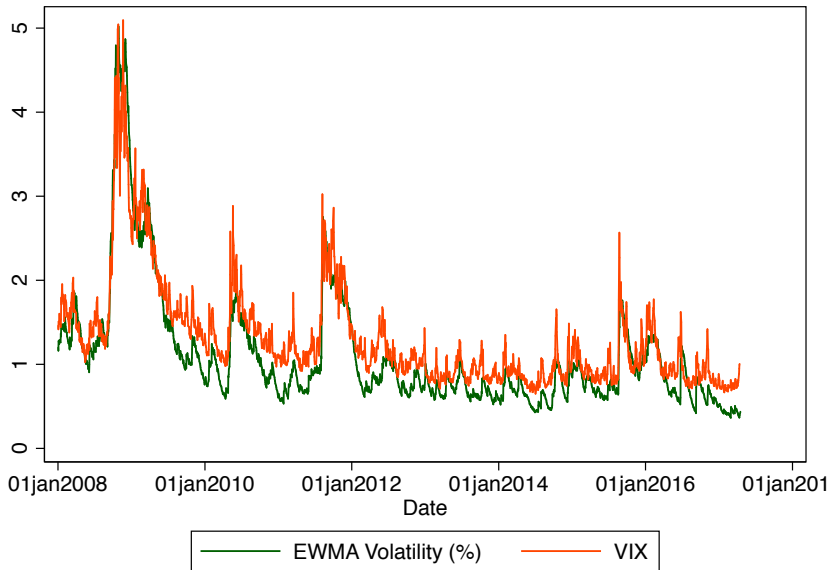
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- Derivative contracts on volatility: VAR swaps, ...
- *Implied volatility* from calls and puts
 - ▶ Volatility so that Black-Scholes formula matches price

VIX



VIX

- The VIX index is published by CBOE
- It is no longer the Black and Scholes implied volatility
- But it is computed from a portfolio of options on the S&P500 index
 - ▶ It is deemed to better capture market “expected” volatility over the next 30 days without relying on any model

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 - ▶ It is deemed to better capture market “expected” volatility over the next 30 days without relying on any model
- Why is VIX systematically higher than realized volatility?
 - ▶ Risk adjustment implicit in options, that make VIX higher than future realized volatility
 - ▶ It does not mean that market expectations are systematically too high

NON-NORMAL ASSUMPTIONS

- Market information can be useful beyond the normal distribution
- Directly obtain measures of downside risks from options

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- Because higher volatility implies a higher price, if OTM options have higher implied volatility than ATM option → market expects negative skewness
 - ▶ Since the crash of October 1987, OTM put options have a higher implied volatility than ATM put options.
 - ▶ Moreover, the difference in implied volatilities is time varying.

SKEWNESS INDEX

- CBOE publishes a Skew Index
- Implied expected skewness is computed from option prices with a more elaborate methodology than the Black and Scholes implied volatilities, but the logic is similar.
- Recall that high *negative* skewness imply high downside risk.
 - ▶ CBOE define the Skew Index as

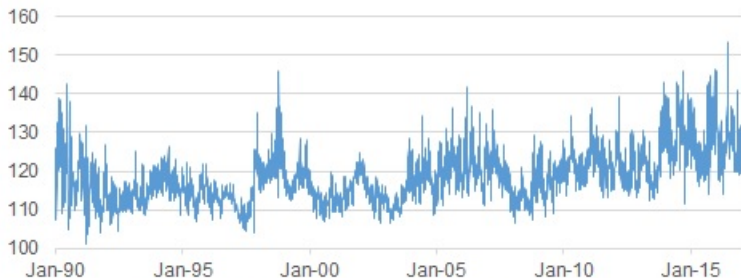
$$\text{Skew Index} = 100 - 10 \times \text{Implied Expected Skewness}$$

- ▶ Higher positive index \rightarrow higher downside risk

SKREW INDEX

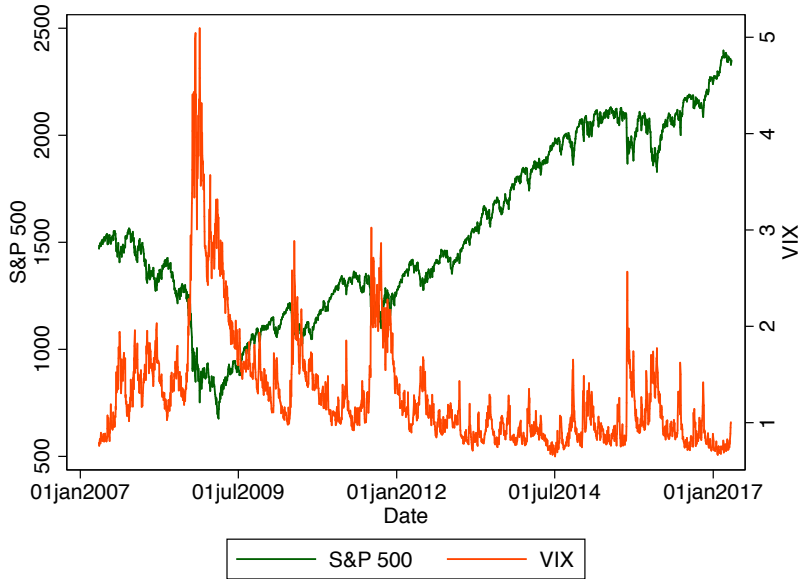
CBOE SKEW Index (SKEW)

(Jan. 1990 - Jan. 18, 2017)



Daily Closing Values. Source: www.cboe.com/SKEW

S&P500 AND VIX



MODEL-BUILDING

- To accurately model risk, necessary to understand interactions between different risks
- Lots of models, for each asset class
- Key question: what can go wrong?
- If model is too complex to compute VaR explicitly: Monte-Carlo simulations

MODEL-BUILDING vs. HISTORICAL SIMULATION

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- Model-building useful for
 - ▶ Large portfolios
 - ▶ Limited data
 - ▶ Taking account of nonlinearities
- Historical simulations useful for:
 - ▶ Non-normal situations
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- Historical simulations useful for:
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- Key trade-off: making more assumptions vs. using a small part of the data
 - ▶ Always the same in statistics!