UCLA ANDERSON SCHOOL OF MANAGEMENT
Daniel Andrei, Derivative Markets 237D, Winter 2015

MFE – Final Exam

March 2015

Date: ______________________________

Your Name: ______________________________

Your email address: ______________________________

Your Signature:[1] ______________________________

- This exam is open book, open notes. You can use a calculator or a computer, **but be sure to show or explain your work**.
- You are not allowed to communicate with anyone (verbally, in writing, or electronically), except for me, during the exam period.
- You may present calculations in non-reduced form (e.g., as "$e^{0.095} - 1$").
- If you are stuck on something, make an assumption, tell me what it is, and do the best you can. I give partial credit if you provide enough correct information.

TIME LIMIT: 2 hours

TOTAL POINTS: 100

[1] As a member of the UCLA Anderson academic community, the highest standards of academic behavior are expected of you. It is your responsibility to make yourself aware of these standards (specifically regarding plagiarism, individual work, and team work) and adhere to them.
By signing the exam: (*i*) you certify your presence, and (*ii*) you state that you neither gave nor received help during the examination.

1 **Warm-up Question** (7 points)

If we compare how the spot and futures evolves on a binomial tree, we observe a different solution for u and d in the case of future prices. That is:

$$u^S = e^{(r-\delta)h+\sigma\sqrt{h}} \tag{1}$$
$$d^S = e^{(r-\delta)h-\sigma\sqrt{h}} \tag{2}$$

for the spot price, and

$$u^F = e^{\sigma\sqrt{h}} \tag{3}$$
$$d^F = e^{-\sigma\sqrt{h}} \tag{4}$$

for the futures.

Prove that the risk-neutral probability p^* is the same for the futures and for the spot.

1 *Answer:*

We have

$$p^* = \frac{e^{(r-\delta)h} - d^S}{u^S - d^S} \quad \text{for the spot price, and} \tag{5}$$
$$p^{*F} = \frac{1 - d^F}{u^F - d^F} \quad \text{for the futures price.} \tag{6}$$

Equation (6): for the futures, the "dividend yield" equals the risk-free rate. Replacing u and d from above gives the same probability measure p^*.

2 Forwards and Swaps (15 points) Here is the forward curve for oil:

Maturity	Today=Spot	1 year	2 years	3 years	4 years
Price	44.34	56.62	59.31	62.19	70.00

The **continuously compounded annual** risk-free interest rates on zero-coupon bonds are: 1-year 1%, 2-years 2%, 3-years 3%, 4-years 4%.

a. (5 points) What is the **continuously compounded** lease rate on the 1-year oil forward contract?

1-year lease rate:

b. (5 points) What are the prices of zero-coupon bonds with maturities 1 year, 2 years, 3 years, and 4 years?

Prices of zero-coupon bonds:

c. (5 points) An industrial producer, IP Inc., needs to buy 1,000 barrels of oil in 1 year from today, 2,000 in 2 years, 3,000 in 3 years and 4,000 in 4 years from today. What is the 4-year swap price with these varying quantities?

4-year swap price:

2 *Answers:*

a. Use the formula

$$\delta_l = r - \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right) \tag{7}$$

The continuously compounded lease rate on the 1-year oil forward contract is -23.45%.

b.

Maturity	1 year	2 years	3 years	4 years
ZC Price	0.9900	0.9608	0.9139	0.8521

c. Use the formula

$$\bar{F} = \frac{\sum_{i=1}^{4} Q_{t_i} P(0,t_i) F_{0,t_i}}{\sum_{i=1}^{4} Q_{t_i} P(0,t_i)} \tag{8}$$

The 4-year swap price is $63.91.

3 Option Portfolio Hedging (18 points)

Assume that you own the portfolio of European options shown in the Table below

	Call #1	Call #2	Call #3	Call #4
Number of contracts	450	100	-400	-300
Days to maturity	120	60	60	30
Strike	105	85	100	100
Call Price	4.99	16.53	4.88	3.27
Delta	0.4843	0.9653	0.5843	0.5598
Gamma	0.0278	0.0076	0.0385	0.0550
Vega	0.2286	0.0311	0.1581	0.1131

Options are written on 1 unit of the underlying. Other parameters are: $S_0 = 100$, $\sigma = 0.25$, $r = 0.1$, and $\delta = 0$. The year has 365 days.

a. (6 points) How many units of the stock you need to buy/sell in order to delta-hedge your portfolio?

Units of stock:

b. (6 points) Suppose you want to delta-gamma hedge your portfolio by modifying your position in the call #4 (the one with strike 100 and maturity 30 days). What should be your new position in order to be gamma-hedged? How many units of the stock you need to buy/sell in order to be delta-gamma-hedged?

New position in call #4:

Units of stock:

c. (6 points) Suppose you want to delta-gamma-vega hedge your portfolio by modifying your positions in the calls #3 and #4. What should be the new positions in order to be gamma-vega-hedged? How many units of the stock you need to buy/sell in order to be delta-gamma-vega-hedged?

New position in call #3:

New position in call #2:

Units of stock:

3 *Answers:*

a. The delta of the portfolio is

$$\begin{aligned} \text{Portfolio delta } &= \sum_{i=1}^{4} \text{Number of contracts}_i \times \text{Delta}_i && (9) \\ &= -87.19 && (10) \end{aligned}$$

Thus, we need to buy 87.19 units of the stock in order to hedge the portfolio.

b. Replacing with x the position in the call #4, the gamma of the portfolio becomes $-2.1203 + 0.0550x$. In order to have a zero gamma, x should be equal to 38.53. The new delta of the portfolio is 102.33, thus we need to sell 102.33 units of the stock in order to be also delta-hedged.

c. Replacing with x and y the positions in the calls #3 and #4, one has to solve a system of two equations with two unknowns:

$$13.2706 + 0.0385x + 0.0550y = 0 \quad (11)$$

$$105.965 + 0.1581x + 0.1131y = 0 \quad (12)$$

The solution is $x = -995.37$ and $y = 454.77$. The new delta of the portfolio is -12.51, thus we need to buy 12.51 units of the stock in order to be also delta-hedged.

4 Strike from Delta (12 points)

Options are quoted by delta rather than strike in several OTC (over-the-counter) markets. This is a common quotation method in the OTC currency options market, where one typically asks for a delta and expects the salesperson to return a price as well as the strike. In these cases, one needs to find the strike that corresponds to a given delta.

Consider a European call option. All the Black-Scholes assumptions are satisfied. The strike can be derived from the delta analytically as follows:

$$K(\Delta) = S_0 e^{A+BN^{-1}(C)} \tag{13}$$

where $N^{-1}(\cdot)$ represents the *inverted cumulative normal distribution function* and A, B, and C are to be determined.

a. (6 points) Find A, B, and C. *Hint: C is a function of Δ.*

A:

B:

C:

b. (6 points) What should the strike be for a three-month ($T = 0.25$) call stock index option to get a delta of 0.25, with the stock index trading at $S_0 = 2000$? Other parameters are $\sigma = 0.5$, $r = 0.1$, and $\delta = 0.05$.

To answer this question you will need to compute the inverted cumulative normal distribution. You are provided here with 3 values for the inverted normal and you will have to pick the right one:

$$N^{-1}(0.2531) = -0.6646 \tag{14}$$
$$N^{-1}(0.2469) = -0.6843 \tag{15}$$
$$N^{-1}(0.2500) = -0.6745 \tag{16}$$

Strike:

4 *Answers:*

a. The delta of the call option is $e^{-\delta T}N(d_1)$. Thus

$$N(d_1) = e^{\delta T}\Delta \tag{17}$$
$$d_1 = N^{-1}\left(e^{\delta T}\Delta\right) \tag{18}$$
$$\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = N^{-1}\left(e^{\delta T}\Delta\right) \tag{19}$$

It follows that

$$K = S_0 e^{\left(r-\delta+\frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}N^{-1}\left(e^{\delta T}\Delta\right)} \tag{20}$$

and thus

$$A = \left(r - \delta + \frac{\sigma^2}{2}\right)T \tag{21}$$
$$B = -\sigma\sqrt{T} \tag{22}$$
$$C = e^{\delta T}\Delta \tag{23}$$

b. The correct value to use is $N^{-1}(0.2531) = -0.6646$. Replacing all the parameters in (20) gives $K = 2467.13$.

5 **Gap Option** (24 points)

A **gap call option** pays 0 if $S_T \leq K_1$ and $S_T - K_2$ if $S_T > K_1$. Its price is:

$$C_0 = S_0 e^{-\delta T} N(d_1) - K_2 e^{-rT} N(d_2) \tag{24}$$

where

$$d_1 = \frac{\ln \frac{S_0}{K_1} + (r - \delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{25}$$

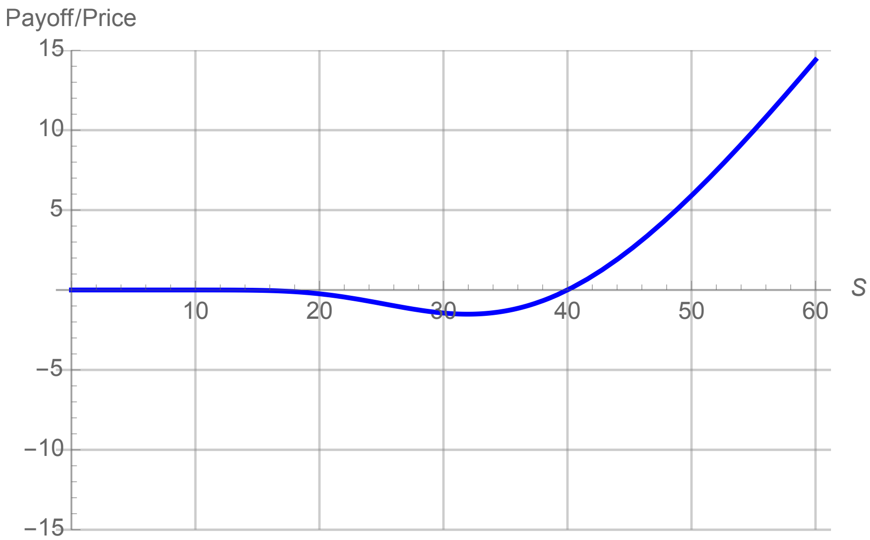
$$d_2 = d_1 - \sigma\sqrt{T} \tag{26}$$

Assume the following: $S_0 = 40$, $K_1 = 35$, $\sigma = 0.3$, $r = 0.05$, $T = 1$, and $\delta = 0.01$.

a. (4 points) Find K_2 such that the price of the gap option today is 0. This option is often referred to as a **pay-later option**.

K_2:

b. (4 points) The graph below shows the price of the option as a function of the stock price. One can see that if $S_0 = 40$ then the option price is zero. Draw the final payoff of the option. Clearly indicate K_1 and K_2 on the graph.



c. (4 points) Observe that a gap option can be replicated by the following portfolio: long one vanilla call option with strike K_1 and short one cash-or-nothing option which pays 0 if $S_T \leq K_1$ and $K_2 - K_1$ if $S_T > K_1$. **By using this property and equation** (24), find the price of the cash-or-nothing option today.

Price of cash-or-nothing option:

d. (4 points)* Compute the delta of the cash-or-nothing option, then the delta of the gap option.

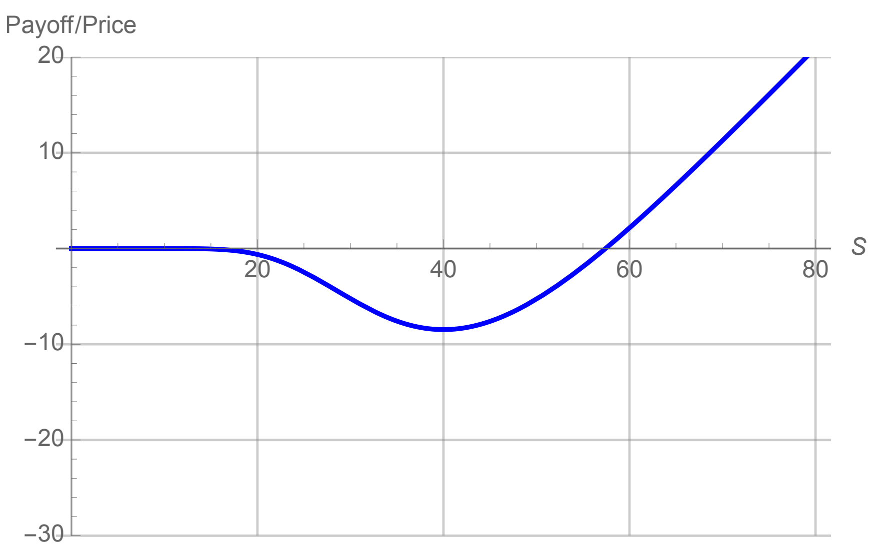
Delta of cash-or-nothing option:

Delta of gap option:

e. (4 points)* Find K_2 such that the delta of the gap option is zero.

K_2:

f. (4 points) The graph below shows the price of the option as a function of the stock price. One can see that if $S_0 = 40$ then the option's delta is zero. Draw the final payoff of the option. Clearly indicate K_1 and K_2 on the graph.



5 *Answers:*

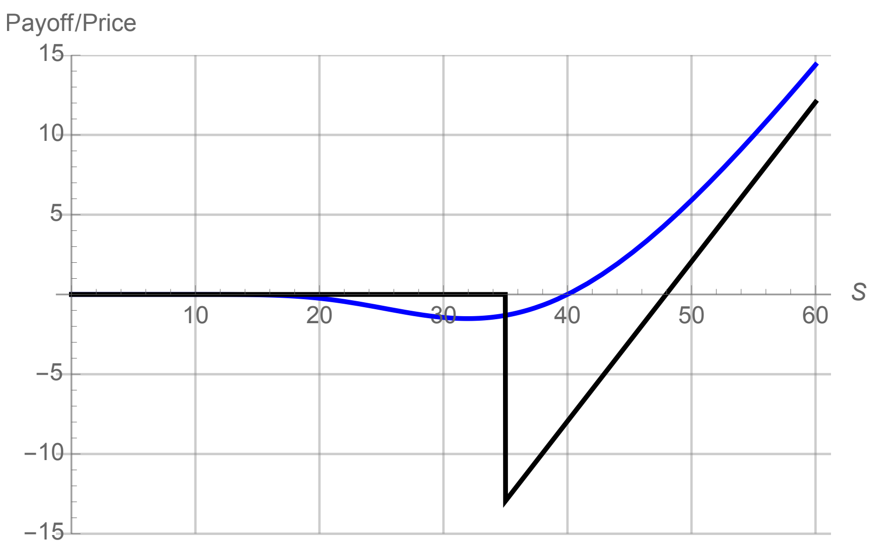
a. K_2 has to solve

$$S_0 e^{-\delta T} N(d_1) - K_2 e^{-rT} N(d_2) = 0 \tag{27}$$

Thus

$$K_2 = \frac{S_0 e^{-\delta T} N(d_1)}{e^{-rT} N(d_2)} = 47.9472 \tag{28}$$

b.



c. Denote by C_0 the vanilla call, G_0 the gap option, and B_0 the cash-or-nothing (binary) option. We have:

$$G_0 = C_0 - B_0 \tag{29}$$

and thus

$$B_0 = B_0 - C_0 \tag{30}$$

$$= (K_2 - K_1)\, e^{-rT} N(d_2) \tag{31}$$

d. The delta of the cash-or-nothing option is:

$$\Delta_B = (K_2 - K_1)\, e^{-rT} N'(d_2) \frac{1}{S_0 \sigma \sqrt{T}} = 0.3735 \tag{32}$$

The delta of the gap option is

$$
\begin{aligned}
\Delta_G &= \Delta_C - \Delta_B && (33)\\
&= e^{-\delta T} N(d_1) - (K_2 - K_1)\, e^{-rT} N'(d_2) \frac{1}{S_0 \sigma \sqrt{T}} && (34)\\
&= 0.3857 && (35)
\end{aligned}
$$

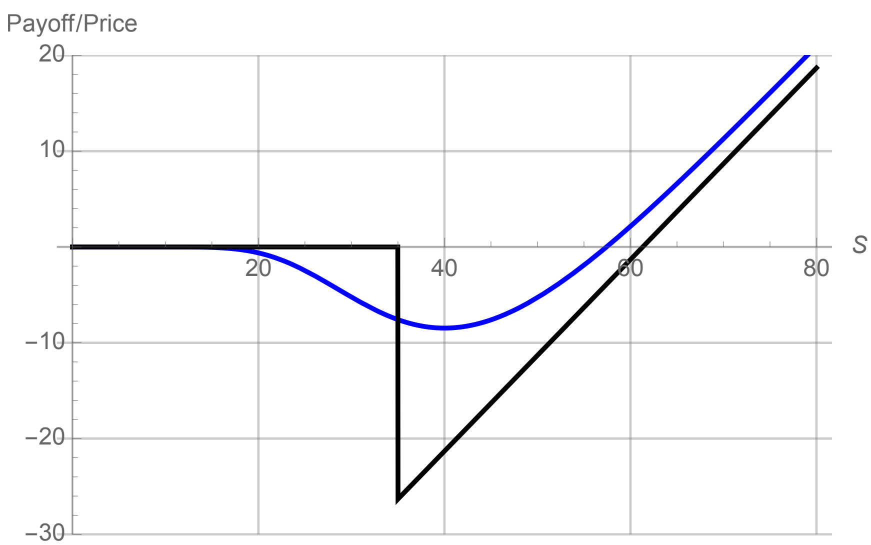
e. K_2 solves the equation

$$e^{-\delta T} N(d_1) - (K_2 - K_1)\, e^{-rT} N'(d_2) \frac{1}{S_0 \sigma \sqrt{T}} = 0 \tag{36}$$

and thus

$$K_2 = \frac{e^{-\delta T} N(d_1) + K_1 e^{-rT} N'(d_2) \frac{1}{S_0 \sigma \sqrt{T}}}{e^{-rT} N'(d_2) \frac{1}{S_0 \sigma \sqrt{T}}} = 61.3148 \tag{37}$$

f.



6 Put-Call Symmetry and Barrier Options (24 points)

In a Black-Scholes world, consider two European options written on the same underlying and with the same maturity T. The first option is a call with strike K and the second option is a put with strike $\frac{\left(S_0 e^{(r-\delta)T}\right)^2}{K}$. The Black-Scholes formula for each option is:

$$C_0 = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) \tag{38}$$

$$P_0 = -S_0 e^{-\delta T}\left[1 - N(\widehat{d}_1)\right] + \frac{\left(S_0 e^{(r-\delta)T}\right)^2}{K} e^{-rT}\left[1 - N(\widehat{d}_2)\right] \tag{39}$$

a. (4 points)* Show that $\widehat{d}_1 = -d_2$ and $\widehat{d}_2 = -d_1$

b. (4 points)* Show that

$$C_0 = \frac{K}{S_0 e^{(r-\delta)T}} P_0 \tag{40}$$

The relationship (40) is called **put-call symmetry** and states that a call with strike K when the spot is at S_0 must have the same value as $\frac{K}{S_0 e^{(r-\delta)T}}$ number of put options with strike $\frac{\left(S_0 e^{(r-\delta)T}\right)^2}{K}$.

The put-call symmetry (40) is useful for hedging and pricing barrier options. To see this, consider a **down-and-in call** option on an index with barrier $B = 120$ and strike $K = 140$.

This option expires worthless unless the index price reaches the barrier before expiry; if the index price hits the value B at some time prior to expiry then the option becomes a vanilla option with the appropriate payoff.

The index price today is $S_0 = 125$. Assume the following Black-Scholes parameters: $\sigma = 0.3$, $r = \delta = 0.1$, and $T = 1$ *(note that $r = \delta$ is particularly important here).*

We will try to find the price of the down-and-in call by constructing a **static hedge**.

c. (4 points) Find the price today of a vanilla put option with strike $\frac{B^2}{K} = 102.857$.

Price of vanilla put option:

d. (4 points) Say you are buying N such put options today. If the index price stays above the barrier until maturity, then the put options will expire worthless (same thing would happen with the down-and-in call option). If the asset price hits the barrier before expiration, then you can sell the puts and buy a vanilla call option with strike $K = 140$.

How many put options you will have to buy today in order to be able to buy exactly one vanilla call with strike $K = 140$ if the asset price hits the barrier? This position will perfectly replicate the down-and-in call option. Why is this called a "static hedge"?

Number of put options N:

e. (4 points) What is the price of the down-and-in call option today?

Price of the down-and-in call option:

f. (4 points) What is the price today of a vanilla call option with strike $K = 140$ (all other parameters are the same)? Interpret the difference between the price of the down-and-in call option and the vanilla call option.

Price of vanilla call option:

6 *Answers:*

a. We have

$$\widehat{d}_1 = \frac{\ln\left[\frac{S_0 K}{\left(S_0 e^{(r-\delta)T}\right)^2}\right] + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{41}$$

$$= \frac{-\ln\left(\frac{S0}{K}\right) - \left(r - \delta - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -d_2 \tag{42}$$

$$\widehat{d}_2 = \widehat{d}_1 - \sigma\sqrt{T} = -d_2 - \sigma\sqrt{T} = -d_1 \tag{43}$$

b. The put price becomes:

$$P_0 = -S_0 e^{-\delta T} N(d_2) + \frac{\left(S_0 e^{(r-\delta)T}\right)^2}{K} e^{-rT} N(d_1) \tag{44}$$

and thus

$$\frac{K}{S_0 e^{(r-\delta)T}} P_0 = -K e^{-rT} N(d_2) + S_0 e^{-\delta T} N(d_1) = C_0 \tag{45}$$

c. The put price is $P_0 = 4.7534$.

d. The trick is to have exactly one call option once the index touches the barrier. By using the put-call symmetry (40), the number of put options that we need to buy is $K/B = 1.1667$. Note that here it does not matter exactly when the index touches the barrier, because $r = \delta$.

This is called a "static hedge" because there is no need to dynamically rebalance this portfolio.

e. Since we have a replicating portfolio, by no arbitrage the price of the down-and-in call option should be:

$$C_{\text{down-and-in}} = 1.1667 \times 4.7534 = 5.5457 \tag{46}$$

f. A vanilla call would cost $C_{\text{vanilla}} = 8.5081$. This is more expensive than the down-and-in call, because it has a higher chance of being exercised.