Black - Scholes

$$C_0(S_0, K, \sigma, r, T, \delta) = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) \qquad (17)$$

$$d_1 = \frac{\ln(S_0/K) + (r - \delta + \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}} \qquad (18)$$

$$d_2 = d_1 - \sigma \sqrt{T} \qquad N(-x) = 1 - N(x)$$

Replicating $\quad C_0 = \Delta S_0 + B$

$$\Delta = e^{-\delta T} N(d_1) \qquad \text{risky} \qquad (20)$$

$$B = -K \cdot e^{-rT} N(d_2) \qquad \text{borrowing} \qquad (21)$$

$$\beta_{option} = \frac{\Delta S_0}{\Delta S_0 + B} \beta_{stock}$$

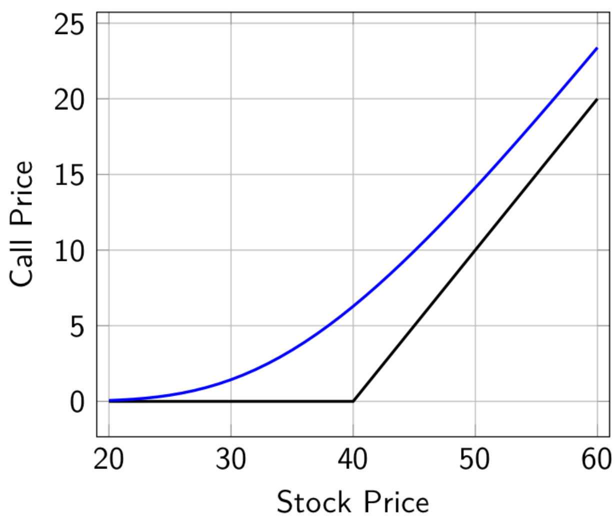
high stock beta → high discount rate

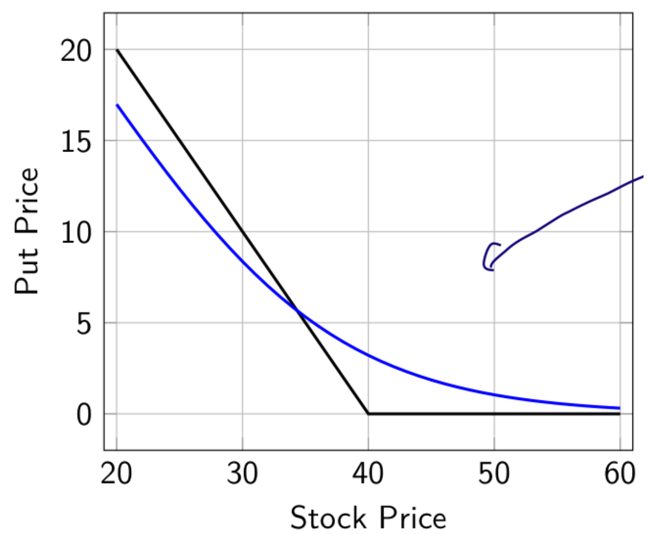
larger payoff offset by high discount

Put Option:

$$P_0 = -S_0 e^{-\delta T} N(-d_1) + K e^{-rT} N(-d_2) \qquad (23)$$

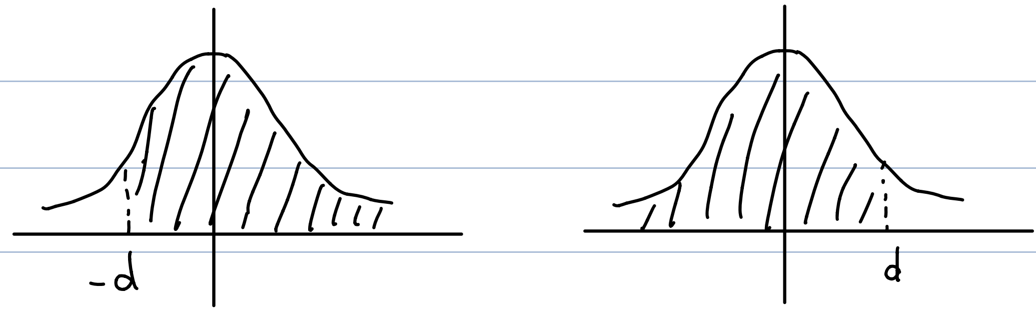
Put - Call Parity $\quad P_0 = C_0 + K e^{-rT} - S e^{-\delta T} \qquad (24)$





$$P(X > -d) = P(X < d) = N(d)$$

$$\log S_T \sim N(\log S_0 + (r - \tfrac{1}{2}\sigma^2)T, \ \sigma^2 T)$$



Call : $\mathbb{P}^Q(S_T \geq K) = N(d_2)$

mean of the stock price $= E^Q[S_T] = S_0 \cdot e^{(r-\delta)T}$

$\mathbb{P}^Q[S_T \leq \underline{S_0 \cdot e^{(r-\delta)T}}] = N(\frac{1}{2}\sigma\sqrt{T}) > 0.5$

把 price 代入 d_2 的 K. $P[S_T \leq Price] = 1 - N[d_2] = N[-d_2]$

Put : $P(S_T \leq K) = 1 - N(d_2)$

Unconditional Volatility $\sigma = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^2}$

Volatility Swaps

$(\sigma_R - K_{vol}) \times N$ ← receive N dollars for every point

σ_R : realized stock vol (floating)

K_{vol} : annualized delivery price (fixed)

Replicating Variance Swaps

A portfolio of options of all strikes, weighted in inverse proportion to the square of the strike level ⇒ An exposure to variance that independent of stock price

VIX : Volatility index

estimates expected volatility by averaging the weighted prices of put and calls over a wide range of K's

VIX strong negative correlation to the SP 500

Change in	Name	Call	Put
S_t	Δ (Delta)	$\Delta^C = \frac{\partial C}{\partial S_t} \in [0,1]$	$\Delta^P = \frac{\partial P}{\partial S_t} \in [-1,0]$
τ	Θ (Theta)	$\Theta^C = \frac{\partial C}{\partial \tau} > 0$	$\Theta^P = \frac{\partial P}{\partial \tau} > 0$
σ	ν (Vega)	$\nu^C = \frac{\partial C}{\partial \sigma} > 0$	$\nu^P = \frac{\partial P}{\partial \sigma} > 0$
r	ρ (Rho)	$\rho^C = \frac{\partial C}{\partial r} > 0$	$\rho^P = \frac{\partial P}{\partial r} < 0$
S_t	Γ (Gamma)	$\Gamma^C = \frac{\partial \Delta^C}{\partial S_t}$	$\Gamma^P = \frac{\partial \Delta^P}{\partial S_t}$

Delta : \$1 increase in Stock ⇒ Value of option ↑ \$delta

Delta-hedging ⇒ Short (100×) delta shares of stock (long call)

Buy (−100×) delta shares of stock (long put)

Input	Call Option		Put Option	
$S_t \uparrow$	$C_t \uparrow$	$\Delta_{Call} > 0$	$P_t \downarrow$	$\Delta_{Put} < 0$
	$\Delta_{Call} \uparrow$	$\Gamma_{Call} > 0$	$\Delta_{Put} \uparrow$	$\Gamma_{Put} > 0$
	$\Omega_{Call} \geq 1$		$\Omega_{Put} \leq 0$	
$\sigma \uparrow$	$C_t \uparrow$	$Vega_{Call} > 0$	$P_t \uparrow$	$Vega_{Put} > 0$
$t \uparrow$	C_t gen. $\downarrow$	θ_{Call} gen. < 0	P_t ambig.	θ_{Put} any
$r \uparrow$	$C_t \uparrow$	$\rho_{Call} > 0$	$P_t \downarrow$	$\rho_{Put} < 0$
$\delta \uparrow$	$C_t \downarrow$	$\Psi_{Call} < 0$	$P_t \uparrow$	$\Psi_{Put} > 0$

> \$1

> Percentage

one day

Gamma - Neutrality

$$\frac{\gamma_{K=40,\ T-t=91}}{\gamma_{K=45,\ T-t=120}} = 1.1213$$

buy 1.1213 45- option for every 40- option sold

Portfolio W: $\Delta + \gamma$

$$C^{(1)} + N_S S + N_2 C^{(2)}$$

$$(1)\quad \frac{\partial W}{\partial S} = \frac{\partial C^{(1)}}{\partial S} + N_S + \frac{\partial C^{(2)}}{\partial S} \cdot N_2 = 0 \quad , \quad \frac{\partial^2 W}{\partial S^2} = 0$$

$$\gamma_1 = \frac{\partial^2 C^{(1)}}{\partial S^2} \quad , \quad \gamma_2 = \frac{\partial^2 C^{(2)}}{\partial S^2} \Rightarrow \gamma_1 + N_2 \gamma_2 = 0 \Rightarrow N_2 = -\frac{\gamma_1}{\gamma_2}$$

$$\Delta^{(1)} + N_S - \frac{\gamma_1}{\gamma_2} \Delta^{(2)} = 0 \quad N_S = -\Delta^{(1)} + \frac{\gamma_1}{\gamma_2} \Delta^{(2)}$$

$\Delta + N_s - \gamma_2 \Delta' = 0 \quad , \quad N_s = -\Delta + \gamma_2 \Delta'$

Perpetual Options

$$h_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$h_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$C_{perpetual} = (H_c - K)(S/H_c)^{h_1}, \quad \text{where } H_c = K\frac{h_1}{h_1 - 1}$$

$$P_{perpetual} = (K - H_p)(S/H_p)^{h_2}, \quad \text{where } H_p = K\frac{h_2}{h_2 - 1}$$

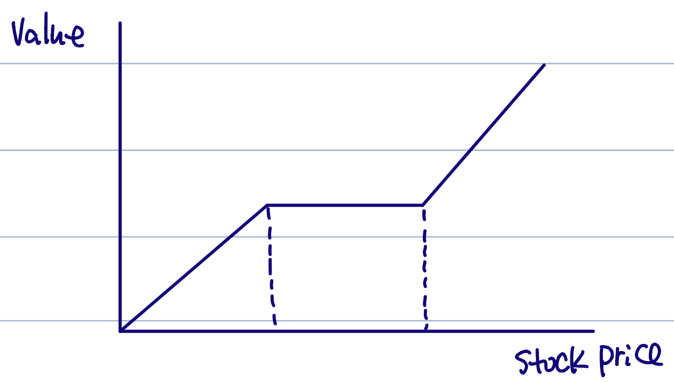
H_c / H_p: 到达即可 exercise

Collars in Acquisitions

Δ Stock $= 1$,

γ stock $= 0$,

Vega Stock $= 0$



Stock Price Jump : magnitude Y $\Rightarrow Y \cdot S$ after-jump price

Y lognormally distributed $\quad \ln(Y) \sim N(\alpha_J, \sigma_J^2)$

λ: annualized probability $\quad h$: time interval $\quad \lambda h$

probability (Jump σ $(t, t+\Delta)) = 1 - e^{-\lambda\Delta} \approx \lambda\Delta$

$$\log(S_T) - \log(S_t) \sim N\left((r - \tfrac{1}{2}\sigma^2 - \lambda k)(T-t) + i\alpha_J ; \sqrt{\sigma^2(T-t) + i\sigma_J^2}\right)$$

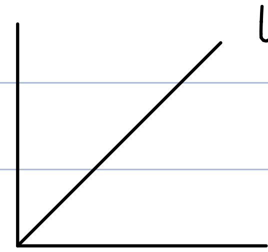
risk neutral density & second derivative of call

$$C(K) = e^{-r(T-t)} \int_{S_T=K}^{\infty} (S_T - K)\, p(S_T, T \mid S_t, t)\, dS_T$$

$$\frac{\partial C}{\partial K} = -e^{-r(T-t)} \int_{S_T=K}^{\infty} p(S_T, T \mid S_t, t)\, dS_T$$

$$\frac{\partial^2 C}{\partial K^2} = e^{-r(T-t)} \quad p(K, T \mid S_t, t)$$

Put option



long index

$$P(K) = \int_0^K (K-S) f(s)\, ds$$

$$P_K = \int_0^K f(s)\, ds$$

$$P_{KK} = f(K)$$

$$\text{Vega}_c = e^{-\delta\tau} \cdot S_t\, \phi(d_1) \sqrt{\tau} \qquad \phi : \text{normal pdf}$$

$$\Delta_c = e^{-\delta\tau}\, \Phi(d_1) \qquad \Phi : \text{normal cdf}$$

$$d_1 = \Phi^{-1}(e^{\delta\tau} \Delta_c)$$

$$S_t = K \cdot e^{-(r-\delta)\tau + \sigma\sqrt{\tau}\,(\Phi^{-1}(e^{\delta\tau}\Delta_c) - \frac{1}{2}\sigma\sqrt{\tau})}$$

$$\text{Vega}_c = K \cdot e^{-r\tau + \sigma\sqrt{\tau}(\Phi^{-1}(e^{\delta\tau}\Delta_c) - \frac{1}{2}\sigma\sqrt{\tau})}\, \phi(\Phi^{-1}(e^{\delta\tau}\Delta_c)) \sqrt{\tau}$$

5. Feynman Kac Theorem

$F(t,x)$ solves
$$\begin{cases} \frac{\partial F}{\partial t} + \mu(t,x) \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x) \frac{\partial^2 F}{\partial x^2} - rF(t,x) = -h(x) \\ \text{such that } F(T,x) = \Phi(x) \end{cases}$$

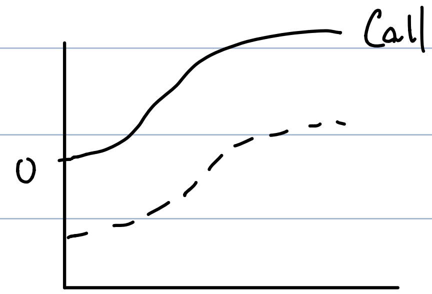
$F(t,x)$ solves
$$\begin{cases} F(t,x) = E_t\left[\int_t^T e^{-\int_t^s r(X_u)\,du} h(X_s)\, ds + e^{-\int_t^T r(X_u)\,du}\, \Phi(X_T)\right] \\ \text{for } dX_s = \mu(s, X_s)\, ds + \sigma(s, X_s)\, dW_s \end{cases}$$

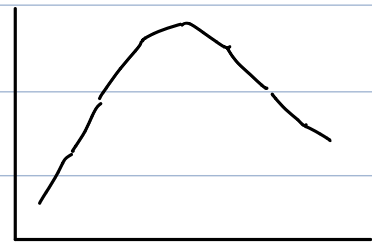
$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2$$

Stock price is the present value of the underlying asset
but not the investment

Delta

$e^{-\delta T} N(d_1)$

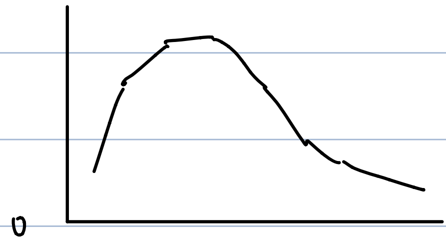


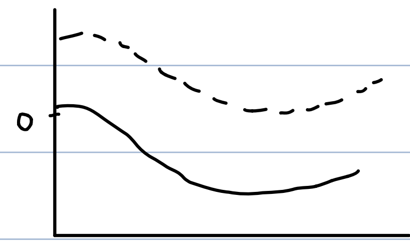


Vega

Gamma

$$\frac{e^{-\delta T - 0.5 d_1^2}}{S \sigma \sqrt{2\pi T}}$$

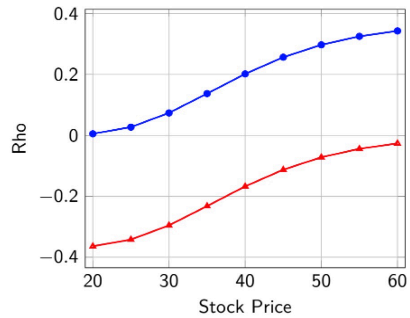


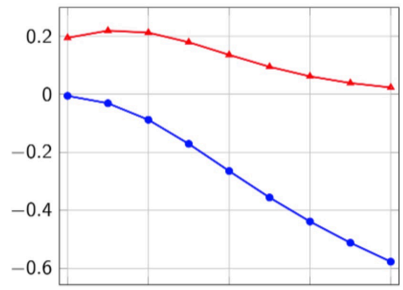


Theta θ

ρ Rho

$\rho = K \cdot T \cdot e^{-rT} N(d_2)$





Psi (Ψ)

$$\frac{d}{dx}\int_a^{g(x)} f(t)\, dt = f(g(x)) \cdot g'(x)$$

Strike price that

$C = P$: $\quad K = S_t e^{(r-\delta)\tau}$

$\Delta_c = -\Delta_p \quad K = S_t e^{(r-\delta+\sigma^2/2)\tau}$

$\Delta = x\%$

naked

Day	Stock	Call Position	Daily Profit
0	85	-415.718 $C \times 50$	–
1	89	-569.887	-154.169
2	81	-281.246	288.641

$=(89-85)\times$

$\times 50$

Option Delta	Stock Position	Daily Profit (Shares)	Daily Profit (Total)
0.723934	36.1967	–	–
0.820297	41.0148	144.787	-9.38262
0.60118	30.059	-328.119	-39.4775

For Gamma, Find $\frac{\gamma_{sell}}{\gamma_{buy}} = a$ (buy a shares for 1 share sold)

$\Rightarrow$ stock $= 50 \times a \times \Delta_{buy}$ 算上表

$$y = \int_{l(x)}^{u(x)} f(t)\,dt \qquad \frac{dy}{dx} = f(u(x))\,u'(x) - f(l(x))\,l'(x)$$

1 The computation of the VIX

1.1 Properties of the derivatives of call an put prices with respect to K.

To save some notation, throughout these notes, assume that $r = \delta = 0$.

Let $f(S_T)$ denote the risk-neutral distribution of S_T. Then a call (C) and a put (P) with strike price K maturing at the same date T are defined as

$$C(K) \equiv \int_K^\infty (S_T - K) f(S_T)\, dS_T, \quad P(K) \equiv \int_0^K (K - S_T) f(S_T)\, dS_T$$

Accordingly,

$$\begin{aligned} C_K &= -\int_K^\infty f(S_T)\, dS_T, P_K = \int_0^K f(S_T)\, dS_T \qquad (1) \\ C_{KK} &= f(K), P_{KK} = f(K) \end{aligned}$$

Note that equation (1) implies that $-C_K$ is the value of a binary call (i.e., a contract that pays one dollar if $S_T > K$). Similarly P_K is value of a binary put, i.e., a contract that delivers one dollar if $S_T < K$.

1.2 The value of a portfolio of puts and calls

Let S_0 be the current value of the stock. Recall also that $r = \delta = 0$.

Define

$$V \equiv \int_0^{S_0} \frac{1}{K^2} P(K)\, dK + \int_{S_0}^\infty \frac{1}{K^2} C(K)\, dK$$

Then we have

$$\begin{aligned} \int_{S_0}^\infty \frac{1}{K^2} C(K)\, dK &= \left[-\frac{1}{K} C(K)\right]_{S_0}^\infty + \int_{S_0}^\infty \frac{1}{K} C_K(K)\, dK \\ &= \left[-\frac{1}{K} C(K)\right]_{S_0}^\infty + [\log(K)\, C_K(K)]_{S_0}^\infty - \int_{S_0}^\infty \log(K)\, C_{KK}(K)\, dK \\ &= \frac{1}{S_0} C(S_0) - \log(S_0)\, C_K(S_0) - \int_{S_0}^\infty \log(K)\, C_{KK}(K)\, dK \end{aligned}$$

and also

$$\begin{aligned} \int_{S_0}^\infty \frac{1}{K^2} P(K)\, dK &= \left[-\frac{1}{K} P(K)\right]_0^{S_0} + \int_0^{S_0} \frac{1}{K} P_K(K)\, dK \\ &= \left[-\frac{1}{K} P(K)\right]_0^{S_0} + [\log(K)\, P_K(K)]_0^{S_0} - \int_0^{S_0} \log(K)\, P_{KK}(K)\, dK \\ &= -\frac{1}{S_0} P(S_0) + \log(S_0)\, P_K(S_0) - \int_0^{S_0} \log(K)\, P_{KK}(K)\, dK \end{aligned}$$

Therefore

$$\begin{aligned} V &= \frac{1}{S_0}\left[C\left(S_0\right)-P\left(S_0\right)\right]+\log\left(S_0\right)\left[P_K\left(S_0\right)-C_K\left(S_0\right)\right]-\int_0^{\infty}\log\left(x\right)f\left(x\right)dx \\ &= \log\left(S_0\right)-\int_0^{\infty}\log\left(x\right)f\left(x\right)dx \\ &= -E^Q\left(\log\left(S_T\right)-\log\left(S_0\right)\right), \end{aligned} \tag{2}$$

where we have used the put call parity $C\left(S_0\right)-P\left(S_0\right)=S_0-S_0=0$, and the properties of P_K, C_K, P_{KK}, C_{KK}.

1.3 Ito's Lemma and the portfolio V :

Recall that $r=\delta=0$. Assume that

$$\frac{dS_t}{S_t}=\mu dt+\sigma_t d\overline{W}_t$$

under the natural probability measure P. Under the risk neutral measure Q (and recalling $r=\delta=0$) we have that

$$\frac{dS_t}{S_t}=\sigma_t dW_t.$$

Applying Ito's Lemma gives

$$d\log\left(S_t\right)=-\frac{1}{2}\sigma_t^2 dt+\sigma_t dW_t.$$

Integrating the above equation and taking expectations leads to

$$E^Q\left(\log\left(S_T\right)-\log\left(S_0\right)\right)=-\frac{1}{2}E^Q\int_0^T\sigma_t^2 dt. \tag{3}$$

Combining (3) with (2) implies that

$$2V=E^Q\int_0^T\sigma_t^2 dt.$$

2 The relation between implied volatility and the risk neutral distribution.

Suppose that we observe the call function $C\left(K\right)$. Letting $\sigma\left(K\right)$ denote the implied volatility associated with the strike price K, we have (by definition of the implied volatility) that

$$C\left(K\right)=C^{BS}\left(K,\sigma\left(K\right)\right), \tag{4}$$

where $C^{BS}(K, \sigma(K))$ is the Black Scholes formula. Differentiating both sides of (4) with respect to K leads to

$$C_K = C_K^{BS} + C_\sigma^{BS} \frac{d\sigma}{dK}.$$

Therefore

$$-C_K = -C_K^{BS} - C_\sigma^{BS} \frac{d\sigma}{dK}$$

or letting $B(K), B^{BS}(K)$ denote the value of a digital call (and the respective digital call under the Black Scholes model), we have that

$$B(K) = B^{BS}(K) - C_\sigma^{BS} \frac{d\sigma}{dK},$$

where we used (1). Therefore

$$\frac{d\sigma}{dK} = \frac{1}{C_\sigma^{BS}} \left[B^{BS}(K) - B(K) \right]$$

and the slope of the implied volatility curve $\left(sign\left(\frac{d\sigma}{dK}\right)\right)$ depends on whether the value of a binary call is higher or lower than its Black Scholes counterpart (computed with the volatility $\sigma(K)$).

Butterfly $C(K_1) - 2C(K_2) + C(K_3) \geq 0$

$C(105) - C(100) < 0$

Asset price that max. Vega given strike X / strike given S_0

$S_0 = X \cdot e^{(\delta - r + \frac{\sigma^2}{2})T}$ $X = S_0 \cdot e^{(r - \delta + \frac{1}{2}\sigma^2)T}$