

Hw2

Question 4

Suppose $Z(t) = f(W, t) = W_t^k$, so $Z(0) = 0$.

Apply Ito's Lemma:

$$\begin{aligned}dZ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial W} dW + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} (dW)^2 \\&= kW_t^{k-1} dW + \frac{1}{2} k(k-1) W_t^{k-2} dt \\Z(t) &= k \int_0^t W_s^{k-1} dW_s + \frac{1}{2} k(k-1) \int_0^t W_s^{k-2} ds + Z(0) \\E(Z(t)) &= kE\left(\int_0^t W_s^{k-1} dW_s\right) + \frac{1}{2} k(k-1) \int_0^t E(W_s^{k-2}) ds \\E(W_t^k) &= 0 + \frac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s) ds \\\beta_k(t) &= \frac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s) ds\end{aligned}$$