Lecture 1: Time Series Properties of Stock Market Returns

Lars A. Lochstoer
UCLA Anderson School of Management

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Overview of Lecture 1

Time series properties of stock market returns

- Moments
- Distributions
- A simple benchmark model of returns
- Risk metrics

Prices and Returns: A Brief Review

P_t denotes the price of an asset at time t

D_t denotes the dividend of an asset at time t

simple gross return:

$$1 + R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}$$

simple net return:

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1$$

Where R_{t+1} is short hand for $R_{t,t+1}$; the return from holding the asset from time t to time $t+1$

Multi-Period Returns

suppose we buy an asset at t, we reinvest all of the dividend payments in the same risky asset and we sell at $t+k$

suppose $k=2$

$$1 + R_{t,t+2} = \frac{P_{t+1} + D_{t+1}}{P_t} \times \frac{P_{t+2} + D_{t+2}}{P_{t+1}} = (1 + R_{t+1}) \times (1 + R_{t+2})$$

then the k-period return is:

$$1 + R_{t,t+k} = (1 + R_{t+1}) \times (1 + R_{t+2}) \times \ldots \times (1 + R_{t+k})$$

Log Returns

continuously compounded or log return:

$$r_t = \log(1 + R_t) = \log\left(\frac{P_t + D_t}{P_{t-1}}\right),$$

log returns are easy to work with, especially over longer holding periods, because they're additive!

multi-period log return:

$$r_{t,t+k} = \log(1 + R_{t,t+k}) = r_{t+1} + r_{t+2} + \ldots + r_{t+k}$$

statistically, it's easier to work with log returns because of additivity

A simple example

Buy a stock at \$85 and sell at \$87.50

the **gross return** is:

$$1 + R_{t+1} \;=\; \frac{87.5}{85} \;=\; 1.0294$$

the **simple net return** is:

$$R_{t+1} \;=\; \frac{87.5}{85} - 1 \;=\; 0.0294$$

continuously compounded or log return:

$$r_t \;=\; \log\left(\frac{87.5}{85.0}\right) \;=\; 0.0289$$

Stylized facts

a **stylized fact** is something that is generally true but not always.

- Let's look at some stylized facts about stock market returns
- We will use the simple log-normal model as a benchmark

$$r_t \sim N\left(\mu, \sigma^2\right)$$

- Let's see how well it does at fitting:
  - daily returns
  - monthly returns
- In what ways are assumptions in the benchmark model violated?

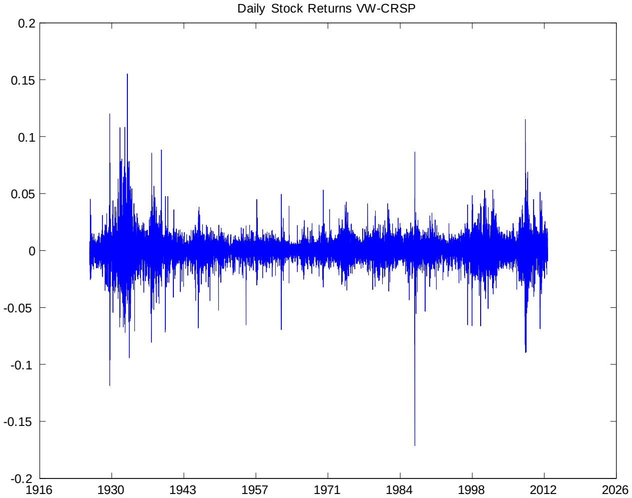
First: A brief review of the log-normal distribution

Gross returns $(1 + R_t)$ are lognormally distributed if the natural logarithm (r_t) is normally distributed

- Since, $1 + R_t = e^{r_t}$ this implies gross returns can never go negative
  - Limited liability!
- If gross returns are lognormal, then cumulative gross returns are also lognormal
  - Easiest to see using sums of the log returns. A sum of normally distributed variables is also normal
  - E.g., if returns are i.i.d. (*identically, independently distributed*; in this case uncorrelated normal with mean μ and variance σ^2)

$$r_{t,t+k} = r_{t+1} + r_{t+2} ... + \ r_{t+k} \sim N\left(k\mu, k\sigma^2\right)$$

Daily stock market returns



Daily log returns for CRSP Value-Weighted market index 01/1925 - 12/2012

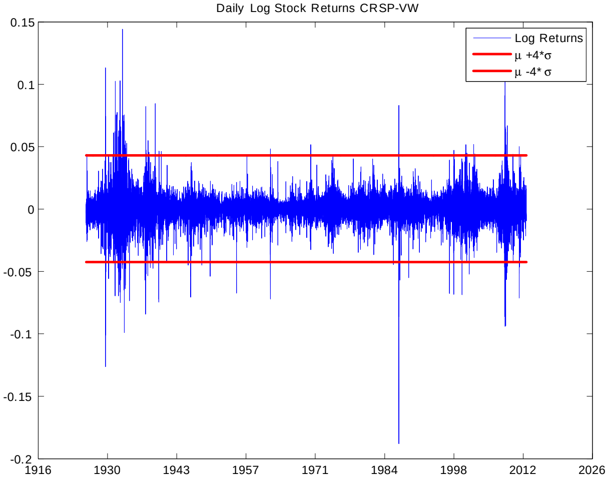
Daily Log-Normal Returns

if returns are log-normal, we should not see any returns greater than 4-σ's:

$$\begin{aligned} \Pr\left(\frac{r_t-\mu}{\sigma}<-4\right) &\approx 0 \\ \Pr\left(\frac{r_t-\mu}{\sigma}>4\right) &\approx 0 \end{aligned}$$

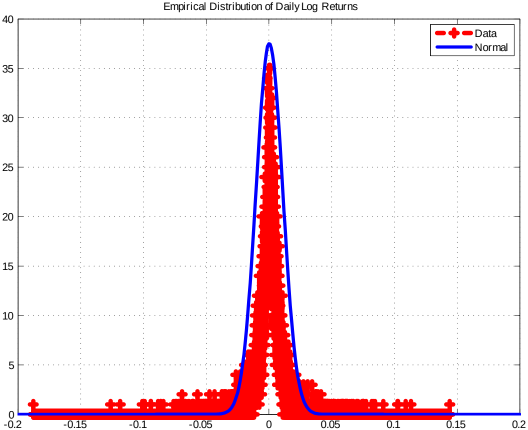
- but we see plenty of those in the data...(see next slide)
- returns are not lognormal (at least not with a constant variance!)

Daily Stock Market Returns



Daily log returns for CRSP Value-Weighted market index 01/1925 - 12/2012
μ is 0.0346%; σ is 1.07%

Empirical Distribution of Daily Stock Market Returns



Empirical and Normal density for daily log returns

- Daily returns have *fat tails*.
  - The empirical distribution is *leptokurtic*, meaning the *excess kurtosis* is greater than zero.
- Is it *skewed*? Let's first remind ourselves what these <u>moments</u> are...

Quick reminder of "moments"

- The j'th moment of a random variable X is

$$m'_j = E\left[X^j\right] = \int_{-\infty}^{\infty} x^j f(x)\,dx$$

- The j'th *central* moment of a random variable X is

$$m_j = E\left[(X - \mu_X)^j\right] = \int_{-\infty}^{\infty} (x - \mu_X)^j f(x)\,dx$$

1. The first moment is the mean, μ_X
2. The second central moment is the variance of X

$$m_2 = E\left[(X - \mu_X)^2\right] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)\,dx$$

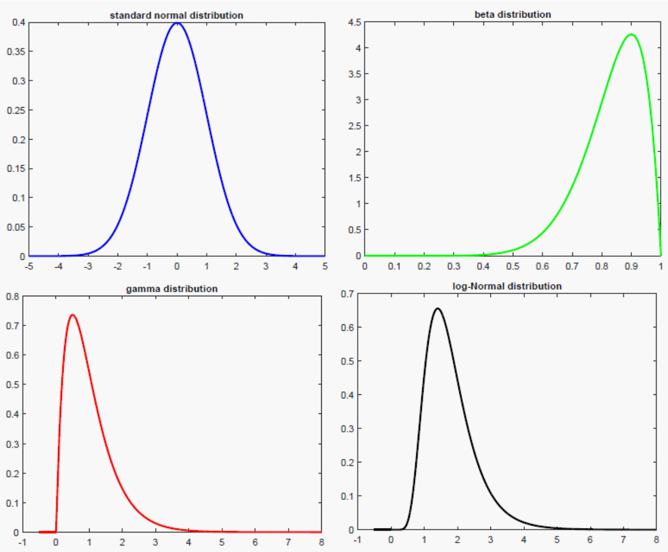
Skewness

the **skewness** of a random variable X is defined as the third central moment normalized by the standard deviation:

$$S(X) = E\left[\frac{(X - \mu_X)^3}{\sigma^3}\right]$$

- Skewness is a measure of the degree of asymmetry of a distribution.
  - If the left tail (tail at small end of the distribution) is more pronounced than the right tail (tail at the large end of the distribution), the function is said to have negative skewness.
  - If the reverse is true, it has positive skewness.
  - If the two are equal, it has zero skewness.
  - The Normal distribution is symmetric and therefore has zero skewness.

Skewness: Examples



Kurtosis

the **kurtosis** of a random variable X is defined as the fourth central moment normalized by the standard deviation:

$$K(X) = E\left[\frac{(X-\mu_X)^4}{\sigma^4}\right]$$

- Kurtosis is a measure of the heaviness of the tails of the distribution.
  - The Normal distribution has kurtosis of 3.
  - **Excess Kurtosis** is kurtosis in excess of 3, $K(X) - 3$

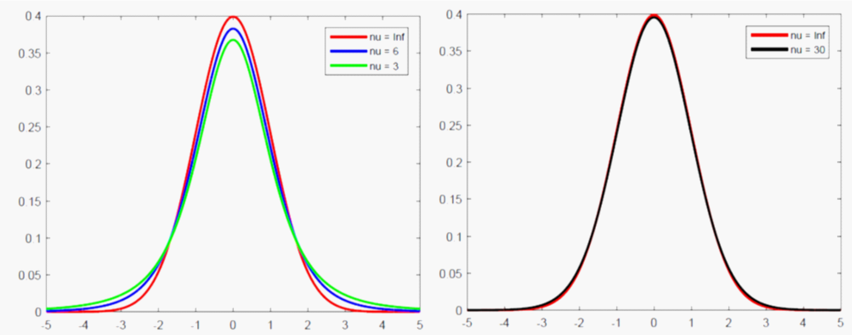
Example: Student's t distribution

Let $X \sim t\left(\mu, \sigma^2, v\right)$

- The parameter v is the **degrees of freedom**
- The mean, variance, skewness, and kurtosis of this distribution are:
  - $E[X] = \mu$ if $v > 1$
  - $V[X] = \sigma^2$ if $v > 2$
  - $S[X] = 0$ if $v > 3$
  - $K[X] = 3 + \frac{6}{v-4}$ if $v > 4$
- The Student's t distribution has heavier tails than a Normal
- as $v \to \infty$, the Student's t distribution converges to a Normal

Student's t vs. Normal distributions

- nu $= \nu$ (degrees of freedom)



Example: The Bernoulli distribution

Consider a Bernoulli distribution

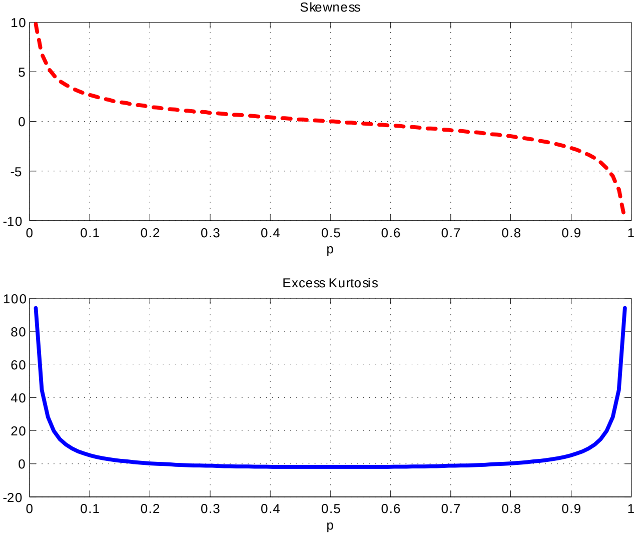
- the probability mass function is

$$P(x) = \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{if } x = 1 \end{cases}$$

- Example: flip a coin
- The variance, skewness, and excess kurtosis

$$\begin{aligned} V(X) &= p(1-p) \\ S(X) &= \frac{1-2p}{\sqrt{p(1-p)}} \\ K(X)-3 &= \frac{6p^2-6p+1}{p(1-p)} \end{aligned}$$

Example: The Bernoulli distribution



Skewness Test

- Suppose we have data on asset log returns $\{r_1, r_2, ..., r_T\}$
- Test whether series is skewed with the following null hypothesis

$$H_0\colon\ S_r = 0$$

- Compute the following **t-ratio** or **t-statistic**:

$$t = \frac{\hat{S}\left(r\right)}{\sqrt{6/T}}$$

- Reject the null hypothesis at the 5% level if $|t| > 1.96$ if the sample is large
  - under the null, t is asymptotically (as $T \to \infty$) standard normal

Kurtosis Test

- Suppose we have data on asset log returns $\{r_1, r_2, ..., r_T\}$
- Test whether series has fat tails with the following null hypothesis

$$H_0\colon\ K_r - 3 = 0$$

- Compute the following **t-ratio** or **t-statistic**:

$$t = \frac{\hat{K}(r) - 3}{\sqrt{24/T}}$$

- Reject the null hypothesis at the 5% level if $|t| > 1.96$ if the sample is large
  - under the null, t is asymptotically (as $T \to \infty$) standard normal

Jarque and Bera (1980) Normality Test

we can combine the skewness and kurtosis test to test normality.

- the Jarque-Bera test statistic is given by:

$$JB = \frac{\hat{S}(r)^2}{6/T} + \frac{\left(\hat{K}(r) - 3\right)^2}{24/T}$$

  which is χ^2-distributed with 2 degrees of freedom
- let CHI_α denote the $(1-\alpha) \times 100^{th}$ quantile of $\chi^2(2)$
- reject null if $JB > CHI_\alpha$
  - the sum of two squared standard normals is χ^2 with 2 degrees of freedom

Sample moments of daily stock market returns

Portfolio	*S&P500*	*CRSP − VW*	*CRSP − EW*
		Daily log Returns	
std	1.03	1.07	1.06
skewness	−1.03	−0.44	−0.07
t − test	[−63.92]	[−27.41]	[−4.59]
kurtosis	30.53	20.42	26.86
t − test	[852.74]	[539.50]	[739.05]
JB − test	731247.51	291810.13	546216.30
p-value JB	0.00	0.00	0.00

Sample Moments of log returns (in percentage points). Daily data. Sample is 01/1925 - 12/2012. T-tests in brackets.

- how does the t-test work?
  - set $\alpha = 0.05$ (this is referred to as the significance level)
  - reject the null hypothesis at the $\alpha = 0.05$ significance level if $|t| > 1.96$
  - this is a two-sided test

Sample moments of daily stock market returns

Portfolio	*S&P500*	*CRSP − VW*	*CRSP − EW*
		Daily log Returns	
std	1.03	1.07	1.06
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Sample Moments of Daily log returns (in percentage points). Daily data. Sample is 01/1925 - 12/2012. T-tests in brackets.

- how does the χ^2-test work?
  - set $\alpha = 0.05$ (this is referred to as the significance level)
  - let $\text{CHI}_\alpha = 5.99$ denote the 95-th quantile of the $\chi^2(2)$
  - reject null H0 if $\text{JB} > \text{CHI}_\alpha = 5.99$ which is the value of the $\chi^2(2)$
  - this is a one-sided test

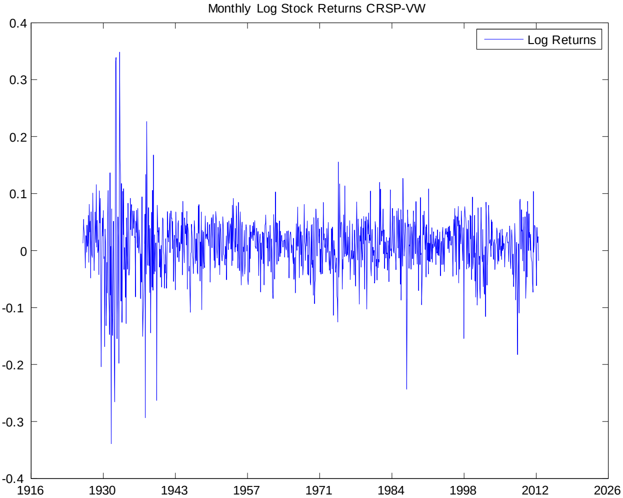
Summary: Daily Log Stock Market Returns

Consider the simple log-normal model as a benchmark

$$r_t \sim N\left(\mu, \sigma^2\right)$$

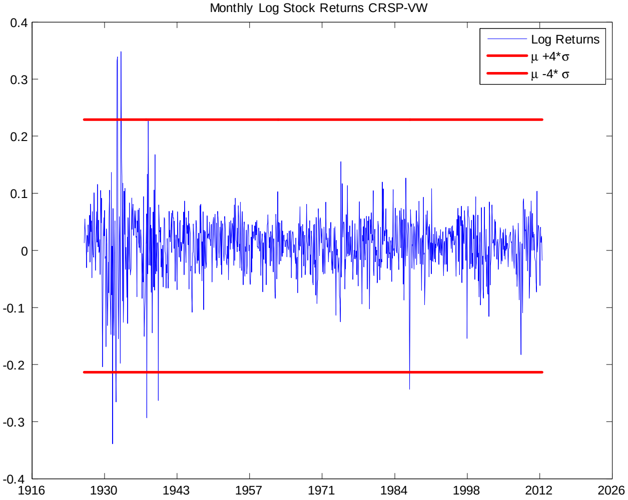
- Not a good model for daily log-returns r_t!
  - data is negatively skewed
  - data has high kurtosis: very heavy tails!
    $\Longrightarrow$ prices move too much from day-to-day to be normally distributed
  - From the time series plots, the variance σ^2 does not appear to be the same each time period.
    $\Longrightarrow$ time-varying variance σ_t^2
  - Clearly, the **identically distributed** assumption fails.

Monthly stock market returns



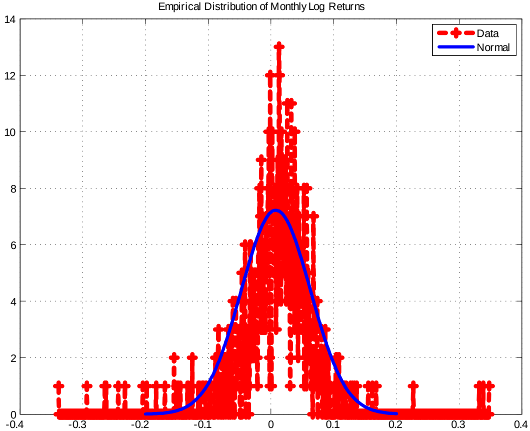
Monthly log returns for CRSP Value-Weighted market index 01/1925 - 12/2012

Monthly stock market returns



Monthly log returns for CRSP Value-Weighted market index 01/1925 - 12/2012
μ is 0.77%; σ is 5.52%

Empirical Distribution of Monthly Stock Market Returns



Empirical and Normal density for monthly log returns

Sample moments of monthly stock market returns

Portfolio	*S&P500*	*CRSP − VW*	*CRSP − EW*
		Monthly log Returns	
std	5.52	5.53	6.73
skewness	−0.53	−0.46	0.03
t − test	[−6.91]	[−6.00]	[0.44]
kurtosis	10.72	10.73	13.55
t − test	[50.78]	[50.81]	[69.36]
JB − test	[2626.42]	[2617.71]	[4811.08]
p-value JB	0.00	0.00	0.00

Sample Moments of monthly log returns (in percentage points). Sample is 01/1925 - 12/2012. T-tests in brackets.

Summary: Monthly Log Stock Market Returns

Consider the simple log-normal model as a benchmark

$$r_t \sim N\left(\mu, \sigma^2\right)$$

- Monthly log returns r_t are closer to this benchmark model, but still not great
  - data is negatively skewed
  - kurtosis > 3: still heavier tails than the Normal
  - the variance σ^2 still appears to vary over time, though not as much as with daily data
  - the **identically distributed** assumption still fails.

Portfolio Theory

classic portfolio theory: investors trade off mean against variance when choosing the efficient portfolio

- makes sense when returns are normally distributed.
- mean and variance are sufficient statistics for the normal distribution

  $\Longrightarrow$ Sharpe ratio is sufficient performance statistic: the efficient portfolio is the one with the highest possible Sharpe ratio

in the data, we saw evidence of non-normality.

when returns are not normally distributed, investors may care about higher order moments (e.g. skewness, kurtosis)

what role do higher order moments have in investing?

Tail Risk

- you can buy and hold the market portfolio: buy a market index fund.
- even with a market index, you are exposed to tail risk.
- evidence of tail risk in passively managed stock market index
- even more evidence of tail risk in actively managed portfolios
    1. writing out-of-the-money put options: excess kurtosis of 16.64 reported by Broadie, Chernov, and Johannes (2009).
    2. currency carry trade: Brunnermeier, Nagel, and Pedersen (2009)
    3. momentum portfolios in stocks: Daniel, Jagannathan, and Kim (2012)
    4. other hedge fund strategies look like writing out of the money put options Jurek and Stafford (2015)

Long and short positions

long position: buy an asset and hold it for n periods

short position: sell an asset you do not have (by borrowing it) and then buy it back after n periods

Zero-cost portfolios

zero-cost portfolios: start with \$0 and build a portfolio

for example, we borrow \$1 at the risk-free rate and invest it in the risky asset; that's a self-financing portfolio

you borrow \$1 in the risk-free asset and go long in asset i, earning a return:

$$R^e_{i,t} = R_{i,t} - R_{0,t}$$

this is referred to as an **excess return**

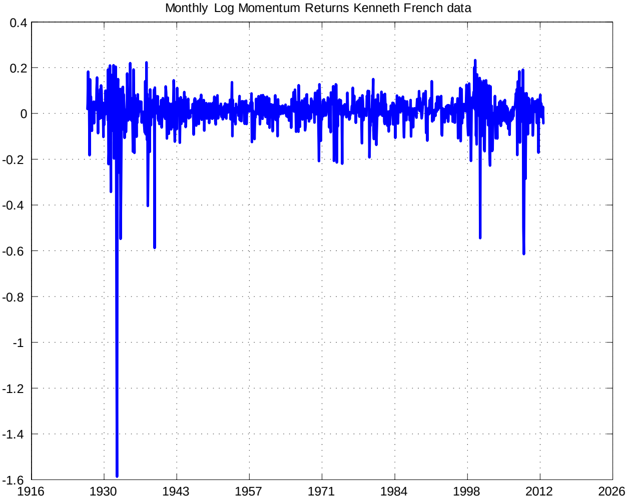
but you could short other assets too, not just the risk-free asset

- you short \$1 of asset j and you go long in \$1 of asset i, earning a return:

$$R^e_{ij,t} = R_{i,t} - R_{j,t}$$

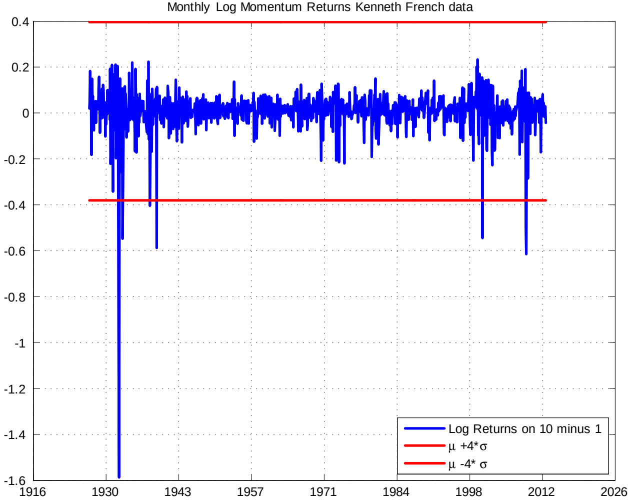
- E.g., Momentum: long 10th decile prior return sorted portfolio, short 1st decile prior return sorted portfolio

Monthly Momentum Stock Returns



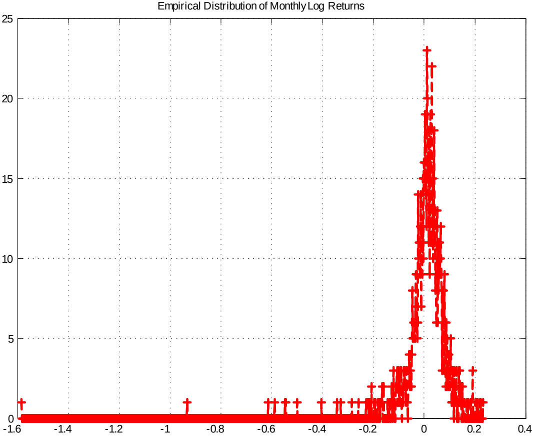
Monthly log returns - Kenneth French 10 momentum portfolios (10-minus-1) . 01/1927-7/2015. $\hat{K}$ is 75.92. $\hat{S}$ is -5.924.

Monthly Momentum Stock Returns



Monthly log returns - Kenneth French 10 momentum portfolios (10-minus-1) . 01/1927-7/2015. μ is 0.84%. σ is 9.37%.

Empirical Distribution of Monthly Log Momentum Returns



Empirical density for monthly log returns

Sharpe Ratio: Sharpe (1965)

- the Sharpe Ratio is defined as the expected return of an asset R_{it} relative to a benchmark asset R_{jt} divided by the standard deviation of R_{it}

$$SR = \frac{E\left[R_{it} - R_{jt}\right]}{Std\left(R_{it}\right)}$$

- historically, the benchmark asset is the risk-free rate R_{ft}.
- the Sharpe Ratio measures the 'reward' per unit 'risk' of an asset.
- the Sharpe ratio is simple & intuitive but inadequate if higher order moments matter.

Mean-variance analysis

standard mean-variance analysis assumes agents only care about the first two moments

- if returns are heavy tailed, we cannot simply rely on the Sharpe Ratio as a measure of performance
- some strategies deliver SRs > 1, but this comes with lots of tail risk.

Maximum Drawdown is a popular alternative

Drawdown

the **drawdown** at time τ is defined as the decline from the peak (the running maximum) of asset value X_τ:

$$DD(\tau) = \max_{t \in (0,\tau]} [X_t] - X_\tau$$

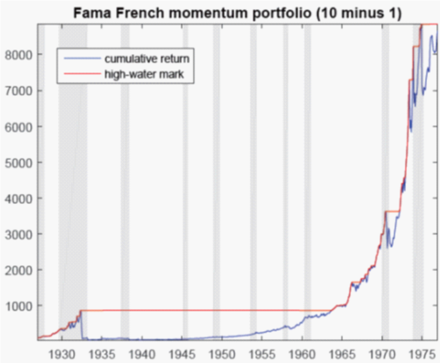
the **drawdown** can (equivalently) be calculated as

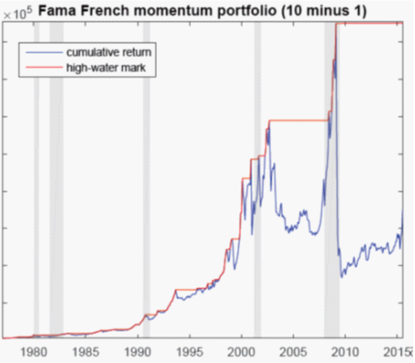
$$DD_t = \frac{HWM_t - P_t}{HWM_t}$$

where HWM_t is the the asset's peak or **high water mark**

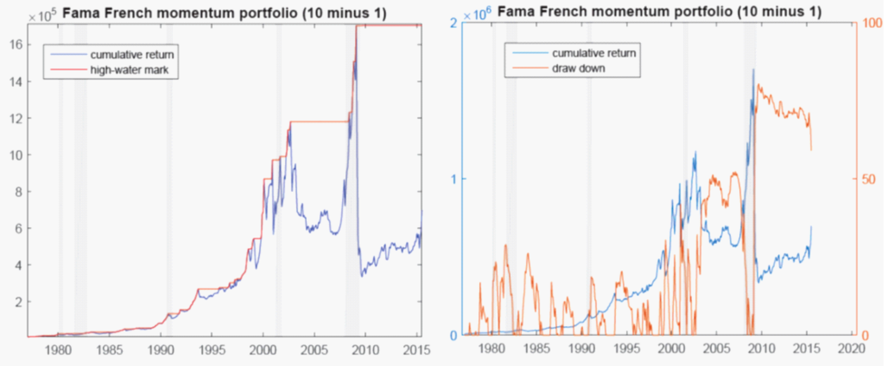
- the **high water mark** is the highest price the asset ever had.

Example: high water marks

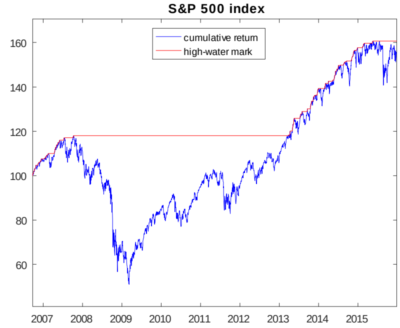


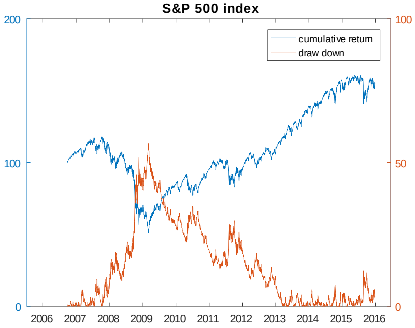


Example: drawdowns



Example: drawdowns for the S&P500 index





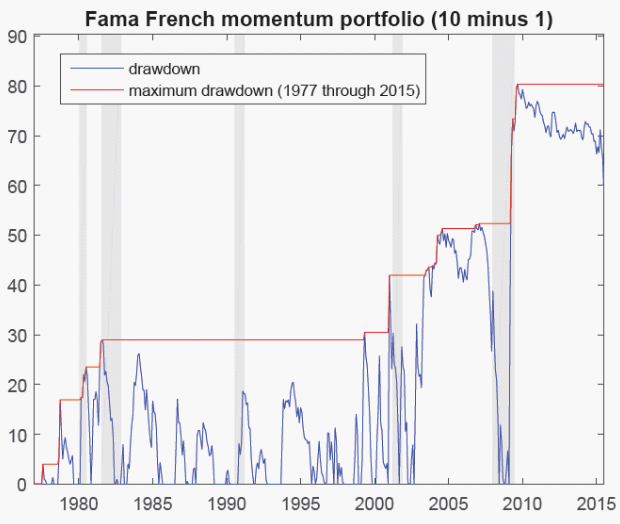
Maximum Drawdown

a commonly used measure in alternative investments is the maximum drawdown.

- the **maximum drawdown** is defined as the largest drawdown over the entire history:

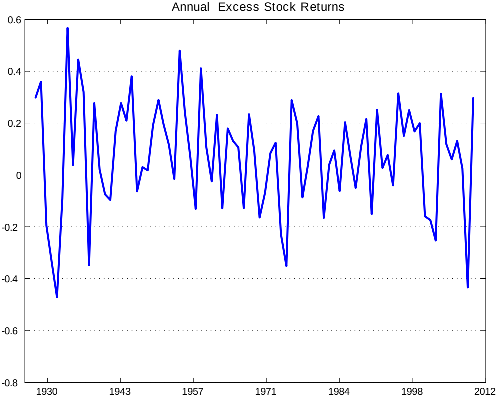
$$MDD\left(T\right) = \max_{t\in(0,T]}\left[\max_{t\in(0,\tau]}\left[X_t\right] - X_\tau\right]$$

Example: Maximum Drawdown



Monthly log Returns - Kenneth French 10 momentum portfolios (10-minus-1) . 01/1927-7/2015.

Annual Excess Returns



Annual Excess Returns - CRSP VW Index (NYSE-AMEX-NASDAQ). We use the 3-month T-bill yield from CRSP as the risk-free rate. 01/1925-12/2009

Annual returns: Some observations

Log excess VW CRSP returns (ln(1+R) - ln(1+Rf))

- Volatility is high
  - Highest return about +50%
  - Lowest return about -50%
  - Volatility of log excess returns: 20.1%
  - Mean of log excess returns: 5.3%
- Still negative skewness:
  - Skewness: -0.86 (t-stat: -3.28)
- No strong excess kurtosis
  - Kurtosis: 3.86 (t-stat vs null of 3: 1.63)
  - Intuition: crashes indistinguishable from the high volatility at annual horizons
  - Long-horizon investors may not care so much about a crash versus a pro-longed downturn: same effect for them

The Equity Premium

The equity premium is the return that equity investors expect to earn in excess of the risk-free rate:

$$E\left[R_{equity} - R_f\right]$$

The risk-free rate R_f is the rate you are guaranteed to get in advance with certainty.

- The return on the risk-free asset is known in advance and is not a random variable.
- In the real-world, there is no such thing as a risk-free asset.
- In practice, we use ultra-short U.S. treasury-bills as a proxy (default risk is low).

Estimating the Equity Premium: data

- collect annual stock return data: CRSP-VW index (NYSE-AMEX-NASDAQ)(1925-2009)
- the risk-free rate is the average 90-day T-bill rate from CRSP (1925-2009)
- The sample mean $\hat{\mu}$ of the excess returns is 7.50% per annum
  - our estimate of the population moment (the equity premium)

Estimating the Equity Premium: standard error

The sample mean $\hat{\mu}$ of the excess returns is 7.50% per annum

- how precise is this estimate of the mean? not very..
  - the standard error on the sample mean is (assuming returns are uncorrelated across time):
    $$\sqrt{E\left[\hat{\mu}-\mu\right]^2} \approx \frac{\hat{\sigma}}{\sqrt{T}} = 2.27\%$$
  - what matters here is the span of the data. See Merton (1980).
  - e.g., using daily data will not help us pin down the mean return on an asset (the volatility will decrease at a rate $\sqrt{T}$)
  - *we need* **longer** *data sets instead.* (THIS IS AN IMPORTANT POINT!)

Estimating the Volatility

- collect annual stock return data: CRSP-VW index (NYSE-AMEX-NASDAQ)(1925-2009)
- the risk-free rate is the average 90-day T-bill rate from CRSP (1925-2009)
- The sample standard deviation $\hat{\sigma}$ of the excess returns is 20.98% per annum
- How precise is this estimate? Quite precise.
  - the standard error of the standard deviation estimate is approximately

$$\sqrt{E\left[\hat{\sigma}-\sigma\right]^2} \approx \hat{\sigma}\sqrt{\frac{1}{2\left(T-1\right)}} = 1.35\%$$

  - **High frequency** data will help us estimate the volatility (see Merton, 1980; another IMPORTANT POINT).

Important insights

1. It is hard to estimate the unconditional mean of a volatile series
   - Need a LONG data-series, unless we have a *theory* we believe in
   - Recall: CAPM needs market risk premium is an input; we have to use long data series to infer its value
   - In *Multi-Factor Models,* we must estimate the risk premium of each factor in a similar manner – using long data series
2. Volatility (really, variances and covariances) are better estimated with higher frequency data
   - In theory, we could get incredibly precise estimates using tick-data
   - But, there are micro-structure issues (bid-ask spread, inventory effects, data errors)
   - Since betas (covariance over variance) are easier to measure, this fact is useful for getting expected returns on individual stocks