# Stochastic Calculus

Problem Set 7.

2. ATC 15.1

pricing function $F(t,x)$ , $\Phi(X(T))$.

prove proposition 15.4 ( $dF = rF\,dt + \{\ \}dW$ form)

→ dynamics of X under measure Q

$$dX = \underbrace{(\mu - \lambda\sigma)}_{=r} X_t \cdot dt + \sigma \cdot X_t \cdot dW_t \qquad (dX^2 = \sigma^2 \cdot X^2 \cdot dt)$$

→ Ito's formula on F

$$dF = \frac{\partial F}{\partial t} \cdot dt + \frac{\partial F}{\partial X} \cdot dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} dX^2$$

$$= \underbrace{\left(\frac{\partial F}{\partial t} + X\underbrace{(\mu - \lambda\sigma)}_{r}\frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2}\right)}_{= rF\ (\because \text{ given in the question})} dt + X\sigma \frac{\partial F}{\partial X} dW_t^Q \quad = rF\ dt + \sigma F_x \cdot dW^Q.$$

$$\therefore \frac{\partial F}{\partial t} + \underbrace{(\mu - \lambda\sigma)}_{= r} X \cdot \frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2} - rF = 0 \quad \text{(satisfying PDE)}$$

3.a) $F = C(t, X_t)$ , $G$ o $dC_t = r \cdot C_t \cdot dt + \sigma_c C_t \cdot d\overline{W_t}$

Ito's lemma $dF = a_F \cdot F \cdot dt + \sigma_F \cdot F \cdot d\overline{W_t}$

$$= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} \cdot dx_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \cdot dx^2$$

$$= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} (-\lambda(x_t - \bar{x}) \cdot dt + \sigma d\overline{W_t}) + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2} dt$$

$$= \underbrace{\left[ \frac{\partial F}{\partial t} - \lambda(x_t - \bar{x}) \cdot \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2} \right]}_{F} \frac{F}{} dt + \frac{F}{} \underbrace{\sigma \frac{\partial F}{\partial x}}_{F.} d\overline{W_t}$$

$$\therefore a_F = \frac{\frac{\partial F}{\partial t} - \lambda(x_t - \bar{x}) \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2}}{F_t}, \quad \sigma_F = \frac{\sigma \cdot \frac{\partial F}{\partial x}}{F.}$$

Form riskless portfolio with F and G.

$$dV = (\underbrace{u_F a_F + u_G a_G}_{\to r}) \cdot V \cdot dt + (\underbrace{u_F \sigma_F + u_G \sigma_G}_{\to 0}) \cdot V \cdot d\overline{W_t}$$

$\Rightarrow$ 1. $u_F + u_G = 1$
2. $u_F \sigma_F + u_G \sigma_G = 0$
$\Rightarrow u_F = -\frac{\sigma_G}{\sigma_F - \sigma_G}, \quad u_G = \frac{\sigma_F}{\sigma_F - \sigma_G}$

$$\therefore dV = V \left[ -\frac{\sigma_G}{\sigma_F - \sigma_G} (a_F dt + \sigma_F d\overline{W_t}) + \frac{\sigma_F}{\sigma_F - \sigma_G} (a_G \cdot dt + \sigma_G d\overline{W_t}) \right]$$

$$= V \cdot \left[ \frac{1}{\sigma_F - \sigma_G} (-a_F \sigma_G dt - \cancel{\sigma_F \sigma_G d\overline{W_t}} + a_G \cdot \sigma_F \cdot dt + \cancel{\sigma_F \sigma_G d\overline{W_t}}) \right]$$

$$= V \cdot \left[ \frac{a_G \sigma_F - a_F \sigma_G}{\sigma_F - \sigma_G} \right] dt$$

$$\frac{a_G \sigma_F - a_F \sigma_G}{\sigma_F - \sigma_G} = r.$$

From Q2, $\frac{a_F - r}{\sigma_F} = \lambda(t, X_t)$

$a_F = a_c = r$

$$\therefore \frac{r - r}{\sigma_F} = \boxed{0 = \lambda(t, X_t)}$$

$n$

3. b) Analytical price for the option with payoff (1). : $\max(X_T - K, 0)$

$dC_t = r \cdot C_t \cdot dt + \sigma_c \cdot C_t \cdot d\overline{W_t}$

let's put $Z_t = e^{-rt} \cdot C_t$ (discounted process of $C_t$).

$$\therefore dZ_t = -r \cdot e^{-rt} \cdot C_t \cdot dt + e^{-rt} \cdot dC_t$$
$$= -r \cdot e^{-rt} \cancel{C_t \cdot dt} + e^{-rt}(r \cancel{C_t\, dt} + \sigma_c \cdot C_t \cdot d\overline{W_t})$$
$$= \sigma_c \cdot e^{-rt} \cdot C_t \cdot d\overline{W_t}$$
$$= \sigma_c \cdot Z_t \cdot d\overline{W_t} \quad \text{(martingale)}.$$

$$\left.\begin{aligned} Z_T - Z_t &= \int_t^T \sigma_c \cdot Z_s \cdot d\overline{W_s} \\ Z_T &= Z_t + \int_t^T \sigma_c \cdot Z_s \cdot d\overline{W_s} \end{aligned}\right) \Rightarrow E(Z_T) = Z_t + \underbrace{E\left[\int_t^T \sigma_c \cdot Z_s \cdot d\overline{W_s}\right]}_{0}$$

$$\therefore E(Z_T) = E(e^{-rT} \cdot C_T) = e^{-rt} \cdot C_t$$

$-rt + rT \quad r(T-t)$

$$\therefore C_t = e^{-r(T-t)} \cdot E(C_T \mid X_t)$$
$$= \underline{e^{-r(T-t)} \cdot E(\max(X_T - K, 0) \mid X_t)}$$

$x \cdot P(x)$

$dX_t = -\lambda(X_t - \bar{x})dt + \sigma \cdot d\overline{W_t}$

let's put $\boxed{Y_t = e^{\lambda t} \cdot X_t}$

$$\therefore dY_t = \lambda \cdot e^{\lambda t} \cdot X_t \cdot dt + e^{\lambda t} \cdot dX_t$$
$$= \lambda \cdot e^{\lambda t} \cdot \cancel{X_t} \cdot dt + e^{\lambda t}(-\lambda(\cancel{X_t} - \bar{x})dt + \sigma \cdot d\overline{W_t}).$$
$$= \lambda \cdot \bar{x} \cdot e^{\lambda t} dt + \sigma e^{\lambda t} \cdot d\overline{W_t}$$

$\lambda s = u \rightarrow \frac{du}{ds} = \lambda \Rightarrow ds = \frac{1}{\lambda} du. \Rightarrow \bar{x} \cdot \cancel{\lambda} \cdot \cancel{\frac{1}{\lambda}} \int_t^T e^u \cdot du = \underline{\bar{x}(e^{\lambda T} - e^{\lambda t})}$

$$Y_T - Y_t = \lambda \cdot \bar{x} \cdot \int_t^T e^{\lambda s} \cdot ds + \int_t^T \sigma \cdot e^{\lambda s} \cdot d\overline{W_s}$$
$$= \bar{x}(e^{\lambda T} - e^{\lambda t}) + \int_t^T \sigma \cdot e^{\lambda s} \cdot d\overline{W_s}$$

$$\therefore E(Y_T) = Y_t + \bar{x}(e^{\lambda T} - e^{\lambda t}) + \int_t^T \sigma \cdot e^{\lambda s} \cdot d\overline{W_s}$$

$$\therefore E(X_T) = e^{-\lambda(T-t)} \cdot X_t + \bar{x}(1 - e^{-\lambda(T-t)}) + \underbrace{E\left(\int_t^T \sigma \cdot e^{-\lambda(T-s)} \cdot d\overline{W_s}\right)}_{0}$$

$$\therefore X_T \sim N\left(\underline{e^{-\lambda(T-t)}(X_t - \bar{x}) + \bar{x}}, \sigma^2 \frac{1 - e^{-2\lambda(T-t)}}{2\lambda}\right). \qquad \sigma_x = \sigma \cdot \sqrt{\frac{1 - e^{-2\lambda(T-t)}}{2\lambda}}$$

$$\therefore C_t = e^{-r(T-t)} \int_{-\infty}^{\infty} \max(X_T - K, 0) \cdot f(x) \cdot dx = e^{-r(T-t)} \int_K^{\infty} (X_T - K) \cdot f(x) \cdot dx$$
$$= e^{-r(T-t)} \int_K^{\infty} X_T \cdot f(x) \cdot dx - K \cdot e^{-r(T-t)} \int_K^{\infty} f(x) \cdot dx \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{1}{2}\left(\frac{x - \mu_x}{\sigma_x}\right)^2}$$
$$= e^{-r(T-t)} \int_K^{\infty} X_T f(x) dx - K \cdot e^{-r(T-t)} \cdot N(d_2)$$

$$d_2 = \frac{e^{\lambda(T-t)}(X_t - \bar{x}) + \bar{x} - K}{\sigma \cdot \sqrt{(1 - e^{-2\lambda(T-t)})/2\lambda}}$$

$$E\left[\left(\int_t^T \sigma \cdot e^{-\lambda(T-s)} \cdot d\overline{W_s}\right)^2\right]$$
$$= E\left[\sigma^2 \int_t^T e^{-2\lambda(T-s)} \cdot ds\right]$$

$-2\lambda(T-s) = u.$

$\frac{du}{ds} = 2\lambda \rightarrow ds = \frac{1}{2\lambda} du$

$\Downarrow$

$$\sigma^2 \cdot \frac{1}{2\lambda} E\left[\int_t^T e^u \cdot du\right]$$
$$\parallel$$
$$\sigma^2 \frac{1}{2\lambda}\left[\underbrace{e^{-2\lambda(T-T)}}_{1} - e^{-2\lambda(T-t)}\right]$$
$$\parallel$$
$$\sigma^2 \cdot \frac{1 - e^{-2\lambda(T-t)}}{2\lambda}$$
$$\parallel$$
$$Var\left(\int_t^T \sigma \cdot e^{-\lambda(T-s)} \cdot dW_s\right)$$