UCLA Anderson School of Management
Daniel Andrei, Derivative Markets 237D, Winter 2015

MFE – Midterm

February 2015

Date: ______________________________

Your Name: ______________________________

Your email address: ______________________________

Your Signature:[1] ______________________________

- This exam is open book, open notes. You can use a calculator or a computer, **but be sure to show or explain your work**.
- You are not allowed to communicate with anyone (verbally, in writing, or electronically), except for me, during the exam period.
- You may present calculations in non-reduced form (e.g., as "$e^{0.095} - 1$").
- If you are stuck on something, make an assumption, tell me what it is, and do the best you can. I give partial credit if you provide enough correct information.

TIME LIMIT: 2 hours

TOTAL POINTS: 100

[1]As a member of the UCLA Anderson academic community, the highest standards of academic behavior are expected of you. It is your responsibility to make yourself aware of these standards (specifically regarding plagiarism, individual work, and team work) and adhere to them.
By signing the exam: (*i*) you certify your presence, and (*ii*) you state that you neither gave nor received help during the examination.

1 **Arbitrage I** (12 points)

a. (5 points) What is an arbitrage opportunity?

b. (7 points) Consider European call and put options on the stock price of company MFE, with strike price $K = 24$ and maturity 3 months for both options. Assume $S_0 = 25$, $C_0 = 2$, $P_0 = 1.5$, $r = 0$, and $\delta = 0$. Do you see any arbitrage opportunities? Explain. What steps would you take to take advantage of any mispricing?

1 *Answers:*

a. *(See reading "The Arbitrage Principle in Financial Economics," by Hal Varian)* Consider a portfolio that pays off non-negative amounts in every state in the future. This portfolio must be valuable to individuals, so if no "free lunches" exist, this portfolio must have a non-negative cost. If not, then we have an arbitrage opportunity. Formally:

$$\textit{If } Rx \geq 0 \textit{ then we must have } px \geq 0 \tag{1}$$

where R is the payoff matrix of assets, x is a portfolio, and p is the vector of asset prices.

b. Use put-call parity to find any mispricing:

$$0.5 = C_0 - P_0 < S_0 - K = 1 \tag{2}$$

This implies mispricing (namely, the call is underpriced and the put is overpriced). The strategy is to sell the expensive and buy the cheap:

Position	**Cash-flow at time 0**	**Value at maturity**
Sell a stock	25	$-S_T$
Sell a put	1.5	$-\max(0, 24 - S_T)$
Buy a call	-2	$\max(0, S_T - 24)$
Lend \$24.5	-24.5	24.5
NET	0	0.5

As can be seen, the cost of putting together a portfolio at time 0 is 0, and the payoff of that portfolio is positive at time T. That is, we found a pure arbitrage opportunity.

2 Arbitrage II (5 points)

The price of a stock today is \$90.37. You observe that the price of an European call with strike $K = 95$ and maturity 1 year is exactly the same as the price of an European put with the same strike and maturity. You also observe that:

$$C(80, 1) - P(80, 1) = \$13.57 \tag{3}$$

where $C(K, T)$ is the price of an European call with strike K and maturity T years, and $P(K, T)$ is the price of an European put with strike K and maturity T years.

a. (5 points) How much is $C(110, 1) - P(110, 1)$? Justify your answer.

2 *Answer:*

a. There are several ways to solve this problem. Consider a butterfly built with $C(80, 1)$, $C(95, 1)$, and $C(110, 1)$. We know that we can obtain the same payoff with put options. That is:

$$C(80, 1) - 2C(95, 1) + C(110, 1) = P(80, 1) - 2P(95, 1) + P(110, 1) \tag{4}$$

Given the information above, it follows that:

$$C(110, 1) - P(110, 1) = -(C(80, 1) - P(80, 1)) = -\$13.57 \tag{5}$$

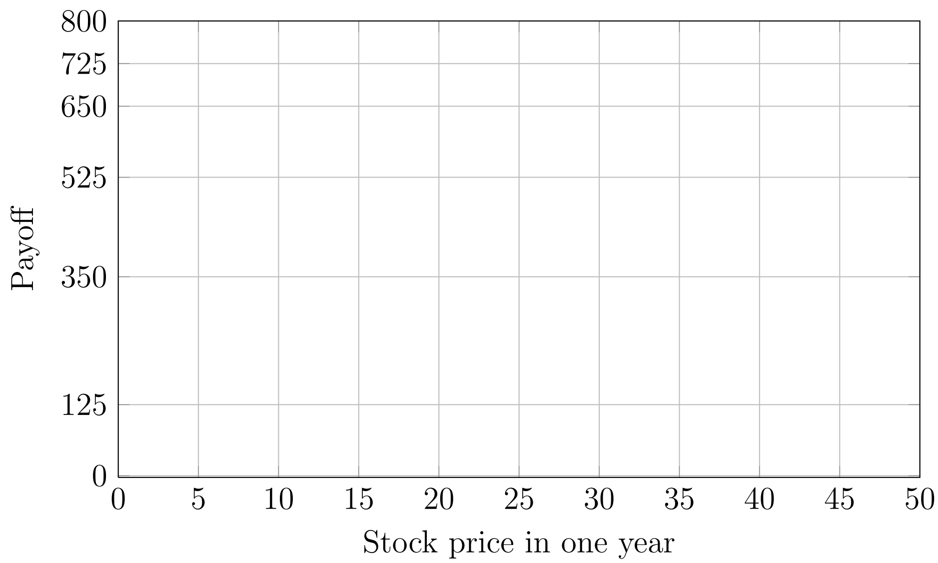
3 Binomial Model & Power Options (26 points)

An option whose payoff is based on the price of an underlying asset raised to a power is called a **power option**. Such an option has a nonlinear payoff at maturity. For this exercise, we will consider a power put option. Its payoff is:

$$\max\left[750 - S_T^2, 0\right] \tag{6}$$

where the maturity T is 1 year. That is, the option pays the strike price of \$750 less the **square** of the stock price if the owner of the option chooses to exercise it. For example, if the stock price is \$20, the claim when exercised pays \$750 − \$400 = \$350.

a. (5 points) Draw the payoff diagram of the power put option.



For the rest of this exercise, assume that the annual continuously compounded interest rate is $r = 10\%$, the annual dividend yield on the stock is $\delta = 0$, and there are 5 months between binomial nodes. Use the tree on the following page to answer the following questions. **Pay attention to the extra information provided in the tree.**

b. (7 points) What is the annualized volatility used to construct the tree?

Volatility:

c. (7 points) What is the risk-neutral probability of an up move?

Risk-neutral Probability:

d. (7 points) At each node in the tree, fill in the prices for the American and European versions of this option (when you exercise the American version at time t, you receive $\max\left[\$750 - S_t^2, 0\right]$, as at expiration). **Put an asterisk at each intermediary node where the American option is exercised.**

$S_2^{uu} = 44.2171$
$Eur =$
$Amer =$

$S_1^{u} = 29.7379$
$Eur =$
$Amer =$

$S_0 = 20$
$Eur =$
$Amer =$

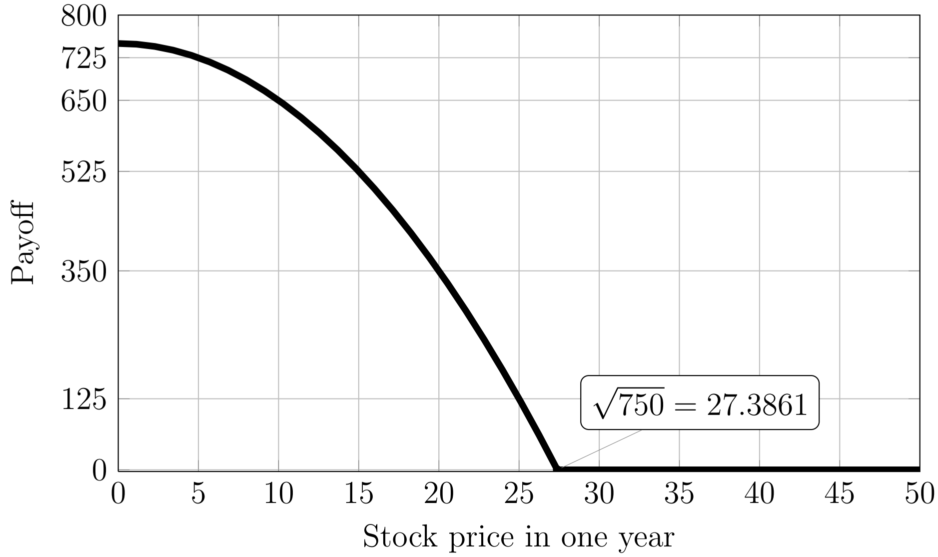
$S_2^{ud} = 21.7381$
$Eur =$
$Amer =$

$S_1^{d} = 14.6198$
$Eur =$
$Amer =$

$S_2^{dd} = 10.6869$
$Eur =$
$Amer =$

3 *Answers:*

a. Payoff:



b. From the tree, we observe that $u = 1.4869$ and $d = 0.7310$. The annualized volatility used to construct the tree is then:

$$\sigma = \frac{1}{2\sqrt{h}} \ln\left(\frac{u}{d}\right) = \frac{1}{2\sqrt{5/12}} \ln\left(\frac{1.4869}{0.7310}\right) = 0.55 \tag{7}$$

c. The risk-neutral probability of an up move is

$$p = \frac{e^{0.1\times 0.4167} - 0.7310}{1.4869 - 0.7310} = 0.4122$$

d. This claim is valued exactly like any other option. The expiration payoff is $\max\left[\$750 - S_T^2, 0\right]$. Then you work backward on the tree. For example, when the stock price is 29.7379, the price of the European claim is

$$e^{-.1\times 0.4167}\left[4122 \times 0 + (1 - 0.4122) \times 277.4558\right] = 156.4422$$

The value of the option at each node is given in the tree below. The American option is exercised when the stock price is $S_1^d = 14.6198$. The American option is more expensive than the European option.

$$
\begin{tikzcd}[column sep=large]
 & & \begin{array}{r} S_2^{uu} = 44.2171 \\ Eur = 0 \\ Amer = 0 \end{array} \\
 & \begin{array}{r} S_1^{u} = 29.7379 \\ Eur = 156.4422 \\ Amer = 156.4422 \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{r} S_0 = 20 \\ Eur = 325.8284 \\ Amer = 364.2173 \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{r} S_2^{ud} = 21.7381 \\ Eur = 277.4558 \\ Amer = 277.4558 \end{array} \\
 & \begin{array}{r} S_1^{d} = 14.6198 \\ Eur = 468.1779 \\ Amer = 536.2620^{*} \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{r} S_2^{dd} = 10.6869 \\ Eur = 635.7902 \\ Amer = 635.7902 \end{array}
\end{tikzcd}
$$

4 Black-Scholes, Implied Volatility & Risk-Neutral Probabilities (40 points)

The current value of a stock is $S_0 = 85$. The stock pays dividends at a continuously-compounded rate of $\delta = 1\%$ and the risk free interest rate is $r = 5\%$.

a. (5 points) Consider two European options: 1 call and 1 put both with strike price K and a 6-month maturity. The options have the same price. Find K.

K:

b. (7 points) Show that $d_1 = -d_2 = \frac{\sigma\sqrt{T}}{2}$ and that the price of the European call option can be written

$$C_0 = S_0 e^{-\delta T}\left[2N(d_1) - 1\right] \tag{8}$$

c. (7 points) The price of the European call option is \$9.5116704. Determine $N(d_1)$ and d_1. *(Hint: the Excel function NORM.S.INV(p) returns the inverse of the standard normal cumulative distribution; LOI.NORMALE.STANDARD.INVERSE(p) in French.)*

$N(d_1)$:

d_1:

d. (7 points) What is the implied volatility of the stock?

σ:

e. (7 points) Using the Black-Scholes model, compute the risk-neutral probability that the 6-months call ends up in-the-money. What about the 6-months put option?

$\mathbb{P}^Q$ [Call ITM]:

$\mathbb{P}^Q$ [Put ITM]:

f. (7 points) Suppose a contract offers to pay \$100 if the stock price ends up above the strike price K (determined above) in 6 months, and 0 otherwise. How much would you be willing to pay for this contract?

Price of contract:

4 a. Use the put-call parity to determine the strike:

$$C_0 - P_0 = S_0 e^{-\delta T} - K e^{-rT} = 0 \tag{9}$$

and thus $K = e^{(r-\delta)T} = \$86.7171$.

b. The formula for d_1 is

$$d_1 = \frac{\log\left(\frac{S_0}{S_0 e^{(r-\delta)K}}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \tag{10}$$

$$= \frac{\sigma\sqrt{T}}{2} \tag{11}$$

Then, $d_2 = d_1 - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$. The price of the call option is then:

$$C_0 = S_0 e^{-\delta T} N(d_1) - S_0 e^{(r-\delta)T} e^{-rT} N(-d_1) \tag{12}$$
$$= S_0 e^{-\delta T} \left[2N(d_1) - 1\right] \tag{13}$$

c. $N(d_1) = 0.556231$, $d_1 = 0.14142$.

d. The implied volatility solves the equation

$$d_1 = \frac{\sigma\sqrt{T}}{2} \tag{14}$$

We find $\sigma = 0.4$.

e. In the Black-Scholes model, the risk-neutral probability that the call ends up in the money is

$$\mathbb{P}^Q\left[S_T \geq K\right] = N(d_2) = N(-d_1) = 0.4438. \tag{15}$$

The probability that the put ends up in-the-money is

$$\mathbb{P}^Q\left[S_T \leq K\right] = 1 - \mathbb{P}^Q\left[S_T \geq K\right] = N(d_1) = 0.5562. \tag{16}$$

f. Applying the risk-neutral valuation, the price of this contract satisfies

$$\$100 \times e^{-rT}\mathbb{E}^Q\left[\mathbb{1}_{\{S_t \geq K\}}\right] = \$100 \times e^{-0.05\times\frac{1}{2}} 0.4438 \tag{17}$$
$$= \$43.2812. \tag{18}$$

5 **Black-Scholes** (12 points)

a. (5 points) Assume a Black-Scholes world **with no dividends**. What happens to the price of a vanilla call option as volatility tends to infinity?

b. (7 points) Assume a Black-Scholes world **with dividends**. What happens to the price of a vanilla put option as volatility tends to infinity?

5 *Answers:*

a. When volatility tends to infinity we have

$$d_1 = \lim_{\sigma \to \infty} \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} = \infty \tag{19}$$

$$d_2 = \lim_{\sigma \to \infty} \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} = -\infty \tag{20}$$

The corresponding normal cdf of d_1 and d_2 will then be 1 and 0 respectively. So, as volatility tends to infinity the call option's price will tend to the spot price:

$$C_0 = S_0 e^{-0 \times T} N(d_1) - K e^{-r \times T} N(d_2) \to S_0 \tag{21}$$

b. For the put option, the corresponding normal cdf of $-d_1$ and $-d_2$ will be 0 and 1 respectively. So, as volatility tends to infinity the put option's price will tend to Ke^{-rT}:

$$P_0 = -S_0 e^{-\delta \times T} N(-d_1) + K e^{-r \times T} N(d_2) \to K e^{-rT} \tag{22}$$

6 **Volatility** (5 points)

a. (5 points) Explain the difference between "realized" or "historical" volatility and "implied" or "anticipated" volatility.

6 *Answer:*

a. *(See reading "Risk's Reward," by Kopin Tan)* "Realized" volatility refers to the standard deviation of stock returns. It is **backward looking**. "Implied" volatility is the option market's guess as to how volatile a security will be over the term of the option. It is a **forward looking** projection extracted from option prices. Some traders compare implied and historical volatilities to see if the market is anticipating a movement far bigger than any the stock has actually made.

A popular of implied volatility measure is VIX. It typically spikes when stocks prices go down, which has earned it the nickname "the fear gauge."