UCLA Anderson School of Management
Daniel Andrei, Derivative Markets MGMTMFE 406, Winter 2018

Problem Set 2

These exercises do not need to be turned in for credit.

1 Consider a binomial model with $T = 4$, $r = 0.02$, $h = 0.25$, $u = e^{rh+0.2\sqrt{h}}$, $d = e^{rh-0.2\sqrt{h}}$, and $S_0 = 100$. Consider also a put with maturity T and strike $K = 90$ and denote by $v_t(s)$ the price of this option at date t $(0 \leq t \leq T)$ in the state where the stock price is s.

a. Compute the risk-neutral probability of an increase in the stock price.

b. Develop an algorithm for computing v_t recursively. In particular, write a formula for v_t in terms of v_{t+1}.

c. Apply the algorithm you developed in the above question to compute the value of the put option at the initial date and at all points on the tree.

d. Provide a formula for the number of shares $\Delta_{t+1}(s)$ that should be held in the replicating portfolio between t and $t+1$ when the stock is worth s at date t.

e. Compute the value at date 3 and when the stock price equals 91.851, i.e., $v_3^{\text{call}}(91.851)$, of a call option with maturity T and strike $K = 90$.

1 a. The risk-neutral probability satisfies:

$$p^* = \frac{e^{rh} - d}{u - d} = 0.475 \tag{1}$$

b. We have

$$v_T(S_T) = \max[K - S_T, 0] \tag{2}$$
$$v_{T-1}(S_{T-1}) = e^{-rh}[qv_T(uS_{T-1}) + (1-q)v_T(dS_{T-1})] \tag{3}$$

and so on. thus, the recursion can be written as

$$v_t(S_t) = e^{-rh}[qv_{t+1}(uS_t) + (1-q)v_{t+1}(dS_t)] \tag{4}$$

for all $0 \leq t \leq T-1$.

c. The stock prices are

					152.196
				137.026	124.608
			123.368	112.187	102.020
		111.071	101.005	91.851	83.527
	100	90.937	82.696	75.201	68.386
Time:	0	1	2	3	4

The put prices are then

					0
				0	0
			0	0	0
		0.923	1.766	3.381	6.473
	3.354	5.585	9.094	14.350	21.614
Time:	0	1	2	3	4

d. The number of shares of the stock to be held satisfies the recursion given by

$$\Delta_{t+1}(S_t) = \frac{v_{t+1}(uS_t) - v_{t+1}(dS_t)}{(u-d)S_t} \tag{5}$$

for $t = 0, 1, ..., T-1$.

e. The value of the call at time 3 when $S_3 = 91.851$ can be computed using the put/call parity

$$v_3^{\text{call}}(91.851) = v_3(91.851) + S_3 - Ke^{-rh} = 5.681 \tag{6}$$

2 Consider a binomial model with $T = 3$, $r = 0.02$, $h = 0.25$, $u = e^{rh+0.1\sqrt{h}}$, $d = e^{rh-0.1\sqrt{h}}$, and $S_0 = 100$.

a. Compute the risk-neutral probability of an increase in the stock price.

b. Determine the initial price P_0^a of an American put with maturity T and payoff function $g_P(s) = \max[112 - s, 0]$.

c. Determine the initial price C_0^a of an American call with maturity T and payoff function $g_C(s) = \max[s - 112, 0]$.

d. Determine the initial price S_0^a of an American straddle with maturity T and payoff function $g_S(s) = g_P(s) + g_C(s)$. Explain why you find $S_0^a < P_0^a + C_0^a$.

e. Determine the initial price A_0^a of an American Asian put option which pays off the nonnegative amount

$$G_t = \max\left[100 - \frac{1}{t+1}\sum_{k=0}^{t} S_k, 0\right]$$

when exercised at time t. What portfolio should you hold between times zero and one in order to hedge such a derivative?

2 a. The risk-neutral probability satisfies:

$$p^* = \frac{e^{rh} - d}{u - d} = 0.4875 \tag{7}$$

b. The initial price of the put is $P_0^a = 12$ and the option should be exercised at all nodes except u and uu. The values for the American put option are

				0
			2.69	5.28
		6.91	11	15.44
	12	16.40	20.61	24.63
Time:	0	1	2	3

c. The initial price of the call is $C_0^a = 0.678$ and the option should never be exercised. The values for the American call option are

				5.94
			2.88	0
		1.3975	0	0
	0.678	0	0	0
Time:	0	1	2	3

d. The initial price of the straddle is $S_0^a = 12.3948$ and, like the American put, the option should be exercised at all nodes except u, uu. The values for the straddle are

				5.94
			5.58	5.28
		8.31	11	15.44
	12.39	16.40	20.61	24.63
Time:	0	1	2	3

Since the American call should never be exercised, the optimal exercise rule of the American straddle implies that upon exercising you lose the difference between the value of the call and its intrinsic value. These loses percolate back to the initial node and imply that the American straddle is worth less than the portfolio of American options.

e. The extended stock price tree, the tree for the arithmetic mean of the stock price, and the tree for the American Asian put option are given below. The initial price of the option is $A_0^a = 1.65$.

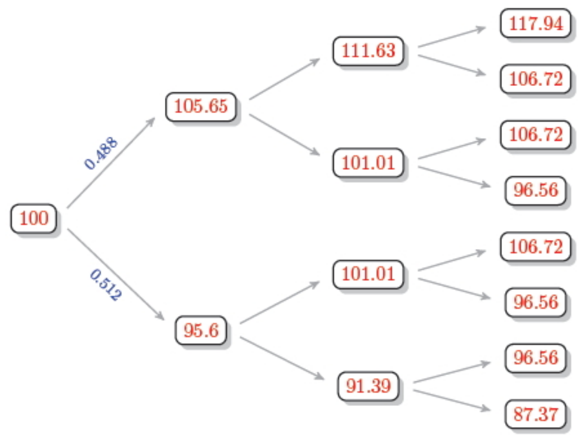

Figure 1: Stock price

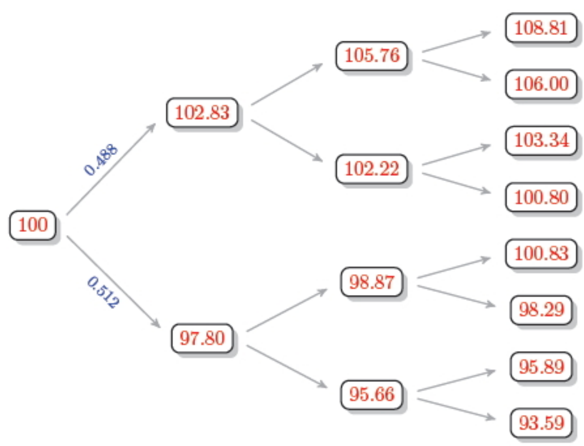

Figure 2: Arithmetic mean of the stock price

The portfolio that should be held at time 0 in order to hedge the option is

$$\Delta = \frac{0 - 3.23}{105.65 - 95.6} = -0.32 \tag{8}$$

$$B = 1.65 + 0.32 \times 100 = 33.80 \tag{9}$$

3 Suppose $S_0 = 55$, $K = 50$, $r = 0.06$, $u = 1.285233$, and $d = 0.809822$. A **barrier option** has a payoff that depends upon whether the price of the underlying asset reaches a specified level (over the whole life of the option), called a **barrier** B. The stock does not pay dividends ($\delta = 0$). Consider a 3-period binomial tree.

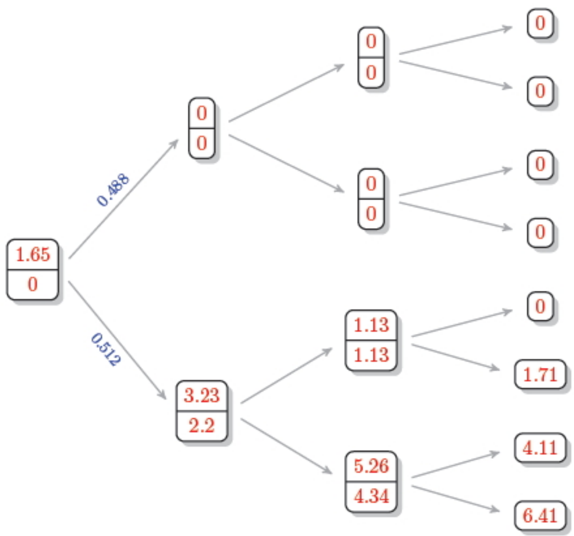

Figure 3: American Asian put option value (top) and intrinsic value (bottom)

a. Assuming the barrier is $B = 90$, find the price of the **up-and-in call**: a call option that comes into existence if the barrier is touched.

b. Is the up-and-in call more or less expensive than a European call? Comment.

c. Assuming the barrier is $B = 40$, find the price of a **down-and-out put**: a put option that goes out of existence if the barrier is touched.

d. A **lookback put** pays $S_T^* - S_T$ at maturity, where $S_t^* = \max_{\tau \le t} S_\tau$ denotes the **running maximum** of the stock price. With such an option the holder achieves perfect timing. This option requires that you keep track of both the current stock price S and its maximum S^*. Find the option price.

e. Is the lookback put more or less expensive than an American put? Explain.

3 a. At expiration, write $I_3^{\text{path}} = 1$ if the option is **knocked-in** and $I_3^{\text{path}} = 0$ otherwise. See Figure 4.

b. The up-and-in call is less expensive than a European call since certain paths on which the option would normally be in-the-money are discarded due to the knock-in barrier not being hit.

c. At expiration, write $I_3^{\text{path}} = 1$ if the option is **knocked-out** and $I_3^{\text{path}} = 0$ otherwise. See Figure 5.

d. See Figure 6

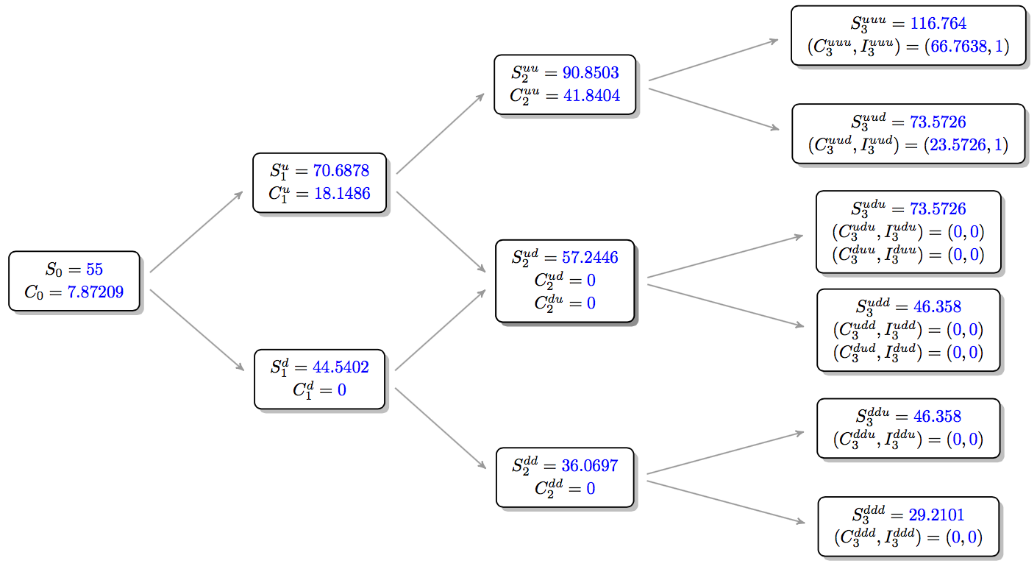

Figure 4: Up-And-In Call

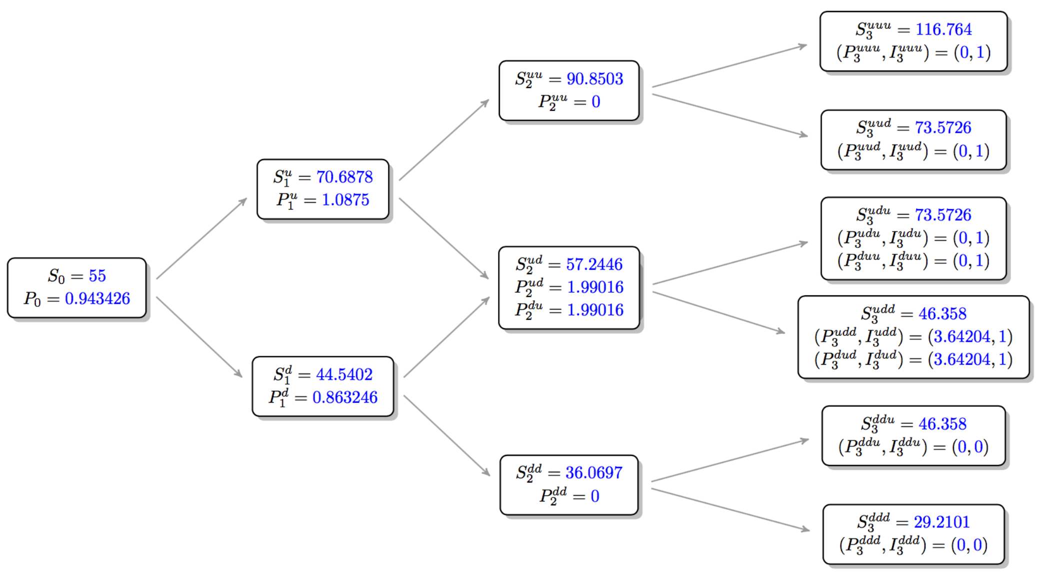

Figure 5: Down-And-Out Put

e. The lookback put is more expensive than an American put. The reason is that the holder of an American put has to time the market in real time, while the holder of a lookback put times the market after the fact.

4 Suppose we want to price options on a non-dividend paying stock, $S(t)$. The initial stock price at $t_0 = 0$ is $S(t_0) = 1$ and the continuously compounded interest rate is $r = 0.05$. We simulate 8 stock price paths under the risk-neutral measure, which are given Table 1

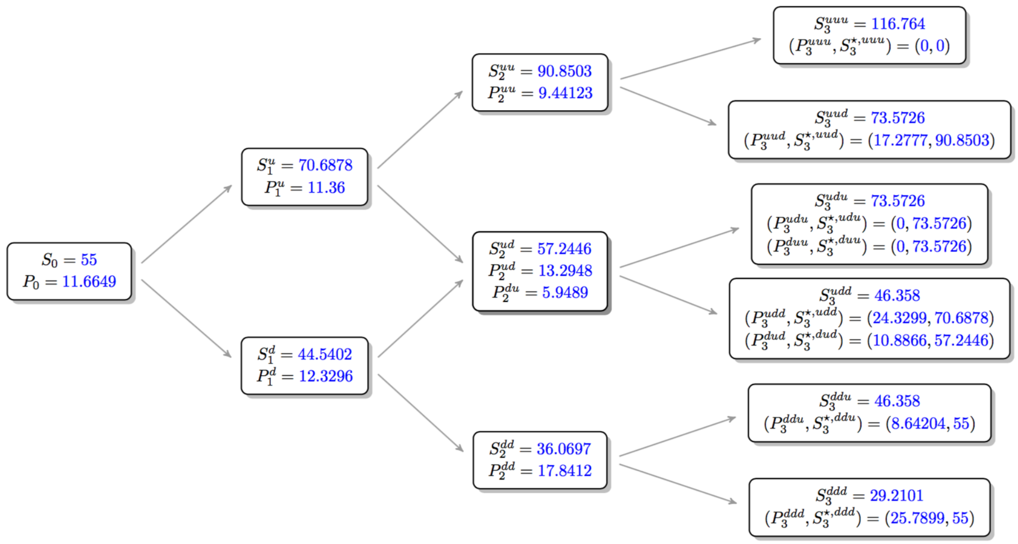

Figure 6: Lookback Put

Path	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
1	0.80	0.93	1.20
2	1.07	1.20	1.05
3	0.91	0.85	0.77
4	1.03	1.17	1.32
5	0.95	1.12	0.87
6	1.10	0.87	1.13
7	1.12	1.03	0.81
8	0.89	0.97	1.23

Table 1: Stock price paths

a. Compute the price of an up-and-in call option with strike $K = 1$, barrier at $S_b = 1.20$, and expiration at t_3. The barrier is monitored at times t_1, t_2, and t_3.

b. Compute the price of a down-and-out put option with strike $K = 1$, barrier at $S_b = 0.80$, and expiration at t_3. The barrier is monitored at times t_1, t_2, and t_3.

c. Compute the price of an Asian call option with strike $K = 1$ and a payoff at t_3 of $\max\left[\frac{1}{3}\sum_{i=1}^{3} S(t_i) - K; 0\right]$, i.e. one with discrete arithmetic averaging.

4 a. An up-and-in call option comes into existence if the barrier is hit. Hence, it gets knocked in on paths 1, 2, 4, and 8. The price is

$$\frac{0.20 + 0.05 + 0.32 + 0.23}{8} e^{-0.05\times 3} = 0.0861 \tag{10}$$

b. An down-and-out put option ceases to exist if the barrier is hit. Hence, it gets knocked out on paths 1 and 3. The price is

$$\frac{0.13 + 0.19}{8} e^{-0.05 \times 3} = 0.0344 \tag{11}$$

c. The price of the Asian call option is 0.0369.

5 Suppose we want to price options on a non-dividend paying stock, $S(t)$. The initial stock price at $t_0 = 0$ is $S(t_0) = 1$ and the instantaneous interest rate is $r = 0.05$. We simulate 8 stock price paths under the risk neutral measure, which are given in Table 2.

Path	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
1	1.18	1.28	1.51
2	0.89	1.18	1.32
3	1.12	1.43	1.34
4	0.78	0.74	0.87
5	1.07	0.97	1.33
6	0.91	0.95	0.94
7	1.24	1.06	1.05
8	0.94	0.85	1.04

Table 2: Stock price paths

a. Compute the price of a standard European put option with strike $K = 1$ expiring at t_3.

b. Compute the price of an American put option with strike $K = 1$, final expiration date t_3, and which is exercisable at times t_1, t_2, and t_3. Use the LSM approach of Longstaff and Schwartz (2001). This involves running a cross-sectional regression at times t_1 and t_2 of the realized discounted cash flows from continuation, which we denote as Y, on a constant, $S(t)$ and $S(t)^2$. The resulting conditional expectations functions are

$$\mathbb{E}_{t_1}[Y] = 23.73 - 54.56S(t_1) + 31.35S(t_1)^2 \tag{12}$$

$$\mathbb{E}_{t_2}[Y] = 3.93 - 8.77S(t_2) + 4.90S(t_2)^2 \tag{13}$$

5 a. The price of the European put option is

$$\frac{0.13 + 0.06}{8} e^{-0.05 \times 3} = 0.0204 \tag{14}$$

b. The price of the American put option is 0.0702.