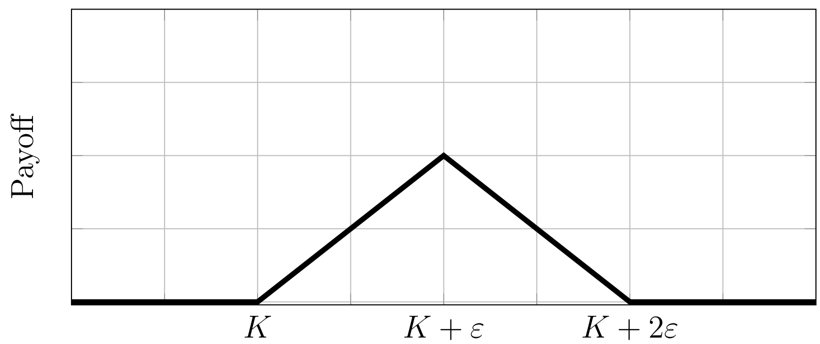
UCLA Anderson School of Management
Daniel Andrei, Derivative Markets MGMTMFE 406, Winter 2018

Problem Set 3

These exercises do not need to be turned in for credit.

1 Butterfly Spread & Risk-Neutral Density

A butterfly spread consists of a portfolio of options yielding the following payoff:



All options involved in this payoff are assumed to have the same time to maturity and the same underlying. The stock is not paying dividends ($\delta = 0$).

a. Indicate how this payoff profile may be obtained using only call options.

b. How many units of this portfolio need to be bought so that the surface determined by the payoff of the butterfly spread is always equal to 1?

c. Using iteratively the approximation of a derivative whereby, for some function $f(x)$ and some small ε,

$$\frac{\partial f}{\partial x} \sim \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

obtain that in the limit of ε converging to 0, the Butterfly spread equals $\partial^2 C / \partial K^2$.

d. For the Black-Scholes call option, compute this expression to establish a link between the second derivative of call options and the risk neutral density denoted $p(S_t, T \mid S_t, t)$.

e. We assume that $S_0 = 100$, $T - t = 0.5$. Use the following call prices and hedge ratios

$$C(S_0, T-t, K=95) = 10.97,\ \frac{\partial C}{\partial S_t}(S_0, T-t, K=95) = 0.63$$
$$C(S_0, T-t, K=100) = 8.45,\ \frac{\partial C}{\partial S_t}(S_0, T-t, K=100) = 0.54$$
$$C(S_0, T-t, K=105) = 6.39,\ \frac{\partial C}{\partial S_t}(S_0, T-t, K=105) = 0.45$$

We also assume the interest rate equals zero.

(i) This table indicates that all things remaining equal, an increase in the strike price lowers the hedge ratio. What is the economic intuition for this?

(ii) What is the initial value of the butterfly spread?

(iii) For one unit of butterfly spread sold, indicate the amount of risky asset that needs to be traded (bought or short-sold). Indicate the amount of risk-free asset that needs to be borrowed or placed as to start a dynamic trading strategy.

1 a. Purchase $C(K)$ and $C(K+2\varepsilon)$ and sell 2 units of $C(K+\varepsilon)$.

b. Surface is ε^2 so one needs $1/\varepsilon^2$.

c.

$$\frac{\partial C}{\partial K} = \frac{C(K+\varepsilon) - C(K)}{\varepsilon} \tag{1}$$
$$\frac{\partial^2 C}{\partial K^2} = \frac{\frac{\partial C}{\partial K}(K+\varepsilon) - \frac{\partial C}{\partial K}(K)}{\varepsilon} = \frac{C(K+2\varepsilon) - 2C(K+\varepsilon) + C(K)}{\varepsilon^2} \tag{2}$$

d.

$$C(K) = e^{-r(T-t)} \int_{S_T=K}^{\infty} (S_T - K) p(S_T, T \mid S_t, t) dS_T \tag{3}$$
$$\frac{\partial C}{\partial K} = -e^{-r(T-t)} \int_{S_T=K}^{\infty} p(S_T, T \mid S_t, t) dS_T \tag{4}$$
$$\frac{\partial^2 C}{\partial K^2} = e^{-r(T-t)} p(K, T \mid S_t, t) \tag{5}$$

where $p(S_T, T \mid S_t, t)$ is the risk-neutral probability density functin of S_T. This shows that the probability density function is given by

$$p(K, T \mid S_t, t) = e^{r(T-t)} \frac{\partial^2 C}{\partial K^2} = e^{r(T-t)} \frac{C(K+2\varepsilon) - 2C(K+\varepsilon) + C(K)}{\varepsilon^2} \tag{6}$$

or a position of $e^{r(T-t)}/\varepsilon^2$ units of the butterfly. The area under the "spike" of this position is 1. See also Hull (8^{th} edition), Appendix 19A.

e. (i) As the strike price increases the option value decreases and it is out of the money, so its value is going to zero in the limit.

(ii) Initial value is $C(95) = 2C(100) + C(105) = 0.46$

(iii) The dynamic trading strategy dictates that initial value $= \delta_0 \times S_0 + \alpha_0$. We have $\delta_0 = \delta_{95} - 2\delta_{100} + \delta_{105} = 0$. For the given figures it appears that the full amount is invested in the riskless asset. No unit of risky asset needs to be purchased. In practice one could expect that some positive fraction of the risky asset needs to be held.

2 Portfolio of Options, Greeks

A financial institution currently holds the following portfolio of over-the-counter options on the stock XYZ:

	Position	Δ	Γ	Vega
Call A	-1,000	0.50	2.20	1.80
Call B	-500	0.80	0.60	0.20
Put C	-2000	-0.40	1.30	0.70
Call D	-500	0.70	1.80	1.40

In addition it is assumed that a traded option on XYZ is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

a. What position in the traded option and the stock of XYZ would make the portfolio both gamma and delta neutral?

b. What position in the traded option and the stock of XYZ would make the portfolio both vega and delta neutral? What is the gamma of the global portfolio?

c. Now assume that another option is made available with a delta of 0.1, a gamma of 0.5, and a vega of 0.6. How can the portfolio be made delta, gamma, and vega neutral?

2 a. Call p^E the price of the traded option and S the stock price at time t. The delta, gamma, and vega of the portfolio V are

$$\Delta_V = -1000 \times 0.5 - 500 \times 0.8 + 2000 \times 0.4 - 500 \times 0.7 = -450 \quad (7)$$

$$\Gamma_V = -1000 \times 2.2 - 500 \times 0.6 - 2000 \times 1.3 - 500 \times 1.8 = -6000 \quad (8)$$

$$\text{Vega}_V = -1000 \times 1.8 - 500 \times 0.2 - 2000 \times 0.7 - 500 \times 1.4 = -4000 \quad (9)$$

The total value (including the investment in the stock and in the traded option) of the portfolio $\bar{V}$ satisfies

$$\bar{V}_t = V_t + \pi_{1t} S_t + \pi_{2t} p_t^E \quad (10)$$

The delta and gamma neutral conditions are consequently

$$\frac{\partial \bar{V}}{\partial S} = -450 + \pi_{1t} + 0.6\pi_{2t} = 0 \tag{11}$$

$$\frac{\partial^2 \bar{V}}{\partial S^2} = -6000 + 1.5\pi_{2t} = 0 \tag{12}$$

The solutions are $\pi_{1t} = -1950$ and $\pi_{2t} = 4000$.

b. Performing similar computations, the number of stocks π_{1t} and the number of traded options π_{2t} that make the portfolio both vega and delta neutral are $\pi_{1t} = -2550$ and $\pi_{2t} = 5000$.

   The gamma of the portfolio is

$$\Gamma_{\bar{V}} = -6000 + 5000 \times 1.5 = 1500. \tag{13}$$

c. The system of equations is

$$-450 + \pi_{1t} + 0.6\pi_{2t} + 0.1\pi_{3t} = 0 \tag{14}$$

$$-6000 + 1.5\pi_{2t} + 0.5\pi_{3t} = 0 \tag{15}$$

$$-4000 + 0.8\pi_{2t} + 0.6\pi_{3t} = 0 \tag{16}$$

   where π_{1t} is the number of stocks, π_{2t} is the number of first options, and π_{3t} is the number of the second options. The solution is $\pi_{1t} = -1710$, $\pi_{2t} = 3200$, $\pi_{3t} = 2400$.

3 Exploring the Black-Scholes formula

In this exercise we consider European options written on an asset with constant volatility σ and constant dividend yield δ. The interest rate is r and the time to expiration of all options is given by τ.

a. Compute the unique level of the strike such that $C = P$.

b. Compute the Δ-symmetric strike, that is, the unique level of the strike such that $\Delta_C = -\Delta_P$.

c. An $x\Delta$ call is a call whose delta is equal to $x\%$. Show that for fixed τ and S the strice of the $x\Delta$ call is given by

$$K_C(x) = Se^{\left(r-\delta+\sigma^2/2\right)\tau-\sigma\sqrt{\tau}\Phi^{-1}\left(e^{\delta\tau}x/100\right)} \tag{17}$$

   where Φ^{-1} is the inverse of the normal cdf. How does $K_C(x)$ vary with x? What is the formula for the strike of the $x\Delta$ put?

d. Show that

$$\text{Vega}_C(t, S) = v(t, \Delta_C(t, S)) \tag{18}$$

for some function v to be determined. Plot vega as a function of delta for the case where $\delta = 0$, $T = 1$, $K = 1$ and interpret your result.

e. Under what condition(s) is it possible to make a European call on a stock both gamma and vega neutral by adding a position in only one other option?

3 a. By the put-call parity

$$0 = S_t e^{-\delta\tau} - Ke^{-r\tau} \tag{19}$$

$$\Rightarrow K = S_t e^{(r-\delta)\tau} \tag{20}$$

b. We have to solve

$$e^{-\delta\tau}\Phi(d_1) = e^{-\delta\tau}\Phi(-d_1) \tag{21}$$

where Φ is the normal cdf. Hence

$$d_1 = 0 \tag{22}$$

$$\Rightarrow K = S_t e^{(r-\delta+\sigma^2/2)\tau} \tag{23}$$

c. We have

$$\Delta_C = e^{-\delta\tau}\Phi(d_1) = \frac{x}{100} \tag{24}$$

$$\Rightarrow d_1 = \Phi^{-1}\left(e^{\delta\tau}\frac{x}{100}\right) \tag{25}$$

$$\Rightarrow K_C(x) = Se^{\left(r-\delta+\sigma^2/2\right)\tau-\sigma\sqrt{\tau}\Phi^{-1}\left(e^{\delta\tau}x/100\right)} \tag{26}$$

Figure 1 shows that the strike price decreases with x. In other words, the higher the delta is, the more in-the-money the call option should be.

For the $x\Delta$ put, we have

$$\Delta_P = e^{-\delta\tau}\Phi(-d_1) = -\frac{x}{100} \tag{27}$$

$$\Rightarrow d_1 = -\Phi^{-1}\left(-e^{\delta\tau}\frac{x}{100}\right) \tag{28}$$

$$\Rightarrow K_P(x) = Se^{\left(r-\delta+\sigma^2/2\right)\tau+\sigma\sqrt{\tau}\Phi^{-1}\left(-e^{\delta\tau}x/100\right)} \tag{29}$$

d. The vega and the delta of the call satisfy

$$\text{Vega}_C = e^{-\delta\tau}S_t\phi(d_1)\sqrt{\tau} \tag{30}$$

$$\Delta_C = e^{-\delta\tau}\Phi(d_1) \tag{31}$$

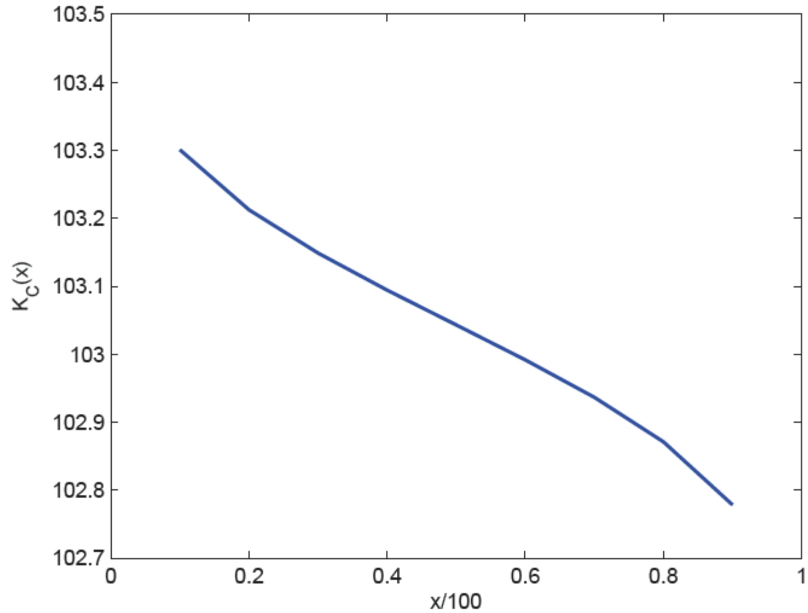


Figure 1: Strike price $K_C(x)$ against the delta of the call

where ϕ is the normal pdf and Φ is the normal cdf. The stock price S is obtained by inverting the delta of the call. We have

$$d_1 = \Phi^{-1}\left(e^{\delta\tau}\Delta_C\right) = \frac{\log\left(\frac{S_t e^{(r-\delta)\tau}}{K}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} \tag{32}$$

Thus, the stock price is written

$$S_t = Ke^{-(r-\delta)\tau+\sigma\sqrt{\tau}\left(\Phi^{-1}(e^{\delta\tau}\Delta_C)-\frac{1}{2}\sigma\sqrt{\tau}\right)} \tag{33}$$

Substituting Equations (32) and (33) in Equation (30) yields

$$\text{Vega}_C = Ke^{-r\tau+\sigma\sqrt{\tau}\left(\Phi^{-1}(e^{\delta\tau}\Delta_C)-\frac{1}{2}\sigma\sqrt{\tau}\right)}\phi\left(\Phi^{-1}\left(e^{\delta\tau}\Delta_C\right)\right)\sqrt{\tau} \tag{34}$$

Figure 2 shows the relationship between the vega and the delta of the call. The call option price is very sensitive to changes in the volatility only if the option is close to be at-the-money.

e. If π is the number of units of the other option, Vega_{new} its vega, and Γ_{new} its gamma, a gamma and vega neutral portfolio should satisfy

$$\text{Vega}_C + \pi\text{Vega}_{new} = 0 \tag{35}$$
$$\Gamma_C + \pi\Gamma_{new} = 0 \tag{36}$$

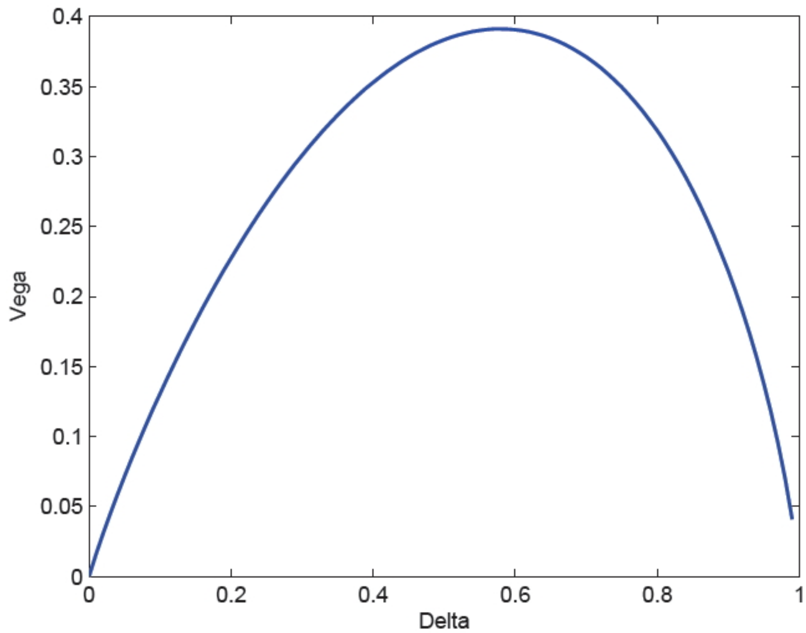


Figure 2: The vega of the call against the delta of the call

This system has a solution only if

$$\frac{\text{Vega}_C}{\text{Vega}_{new}} = \frac{\Gamma_C}{\Gamma_{new}} \tag{37}$$

4 Pricing Options & Risk-Neutral Probabilities

The current value of a stock is $S_0 = 85$. The stock pays dividends at a continuously-compounded rate of $\delta = 1\%$ and the risk free interest rate is $r = 4\%$.

Consider a set of 3 European options: 1 call and 1 put both with strike price $K = 80$ and a 6-month maturity, and 1 call with strike price $K = 90$, a 1-year maturity and a Black-Scholes price $C_{0,K=90} = 5.69957$.

a. What is the implied volatility of the stock?

b. Compute the price of the 80-strike 6-months call using the Black-Scholes formula.

c. Compute the put price using the put-call parity relation.

d. Using the Black-Scholes model, compute the risk-neutral probability that the 80-strike 6-months call ens up in-the-money. What about the put?

e. Using the Black-Scholes model, compute the risk-neutral probability that the stock price ends up **below its mean** at maturity.

f. Suppose a contract offers to pay \$1 if the stock price ends up between 80 and 90 in 6 months. How much would you be willing to pay for this contract?

4 a. The implied volatility solves the equation

$$BSP(S_0, K, r, T, \delta, \sigma) = 5.69957 \tag{38}$$

We find $\sigma = 0.2$.

b. $C_{0,K=80} = 8.31436$.

c. Using the put-call parity relation, we obtain

$$\begin{aligned} P_0 &= C_0 - S_0 e^{-\delta/2} + K e^{-r/2} && (39) \\ &= 2.15419 && (40) \end{aligned}$$

d. In the Black-Scholes model, the risk-neutral probability that the call ends up in the money is

$$\mathbb{P}^Q [S_T \geq K] = N(d_2) = 0.678689. \tag{41}$$

The probability that the put ends up in-the-money is

$$\mathbb{P}^Q [S_T \leq K] = 1 - \mathbb{P}^Q [S_T \geq K] = 0.321311. \tag{42}$$

e. Under the risk-neutral probability measure, the mean of the stock price is $\mathbb{E}^Q [S_T] = S_0 e^{(r-\delta)T}$. Setting $K = S_0 e^{(r-\delta)T}$, the probability that the stock ends up below its mean is given by

$$\mathbb{P}^Q \left[S_T \leq S_0 e^{(r-\delta)T}\right] = N\left(\frac{1}{2}\sigma\sqrt{T}\right) \tag{43}$$

Plugging numbers, we obtain $\mathbb{P}^Q \left[S_T \leq S_0 e^{(r-\delta)T}\right] = 0.528186$.

This illustrates the effect of Jensen's inequality due to the log-normality of the price: the median is located below the mean $\Rightarrow \mathbb{P}^Q \left[S_T \leq S_0 e^{(r-\delta)T}\right] > 0.5$.

f. Applying the risk-neutral valuation, the price of this contract satisfies

$$\begin{aligned} e^{-rT}\mathbb{E}^Q \left[\mathbb{1}_{\{80 \leq S_t \leq 90\}}\right] &= e^{-rT} \left[\mathbb{P}^Q(S_T \leq 90) - \mathbb{P}^Q(S_T \leq 80)\right] && (44) \\ &= 0.316169. && (45) \end{aligned}$$

5 Hedging

The current value of a stock is $S_0 = 85$. The stock pays dividends at a continuously-compounded rate of $\delta = 1\%$ and the risk free interest rate is $r = 4\%$.

Consider a set of 3 European options: 1 call and 1 put both with strike price $K = 80$ and a 6-month maturity, and 1 call with strike price $K = 90$, a 1-year maturity and a Black-Scholes price $C_{0,K=90} = 5.69957$.

Suppose you are the writer of 50 calls with $K = 80$ and $T = 0.5$ and you wish to hedge your position. Assume that the stock price increases to 89 tomorrow and decreases to 81 in 2 days.

a. Compute the daily profits on your **naked position**.

b. Suppose you want to hedge your position by taking an offsetting position in shares. Explain how you delta-hedge your position.

c. You wish to correct your delta-hedged position to be gamma-neutral. To do so, suppose you use the 90-strike 1-year call. Compute the number of calls you need to buy for every 80-strike 6-months call you sell.

d. Assuming you do not dynamically re-adjust the number you computed under (c), compute the daily profits of your delta-hedged gamma-neutral strategy.

5 a.

Day	Stock	Call Position	Daily Profit
0	85	-415.718	–
1	89	-569.887	-154.169
2	81	-281.246	288.641

b.

Day	Daily Profit (Naked)	Option Delta	Stock Position	Daily Profit (Shares)	Daily Profit (Total)
0	–	0.723934	36.1967	–	–
1	-154.169	0.820297	41.0148	144.787	-9.38262
2	288.641	0.60118	30.059	-328.119	-39.4775

To obtain the numbers in the table above, we first compute the option delta at each date t, Δ_t. We then compute the daily profits on our stock position given by $50 \times \Delta_t \left(S_{t+1/365} - S_t\right)$. We finally obtain our daily delta-hedged profits by adding the daily naked profits and the daily profits on our stock position.

For large stock price moves (as for instance between days 1 and 2), our delta-hedged strategy loses money. This loss is due to delta moving with the stock price itself: the sensitivity of delta to stock price movements (the gamma) causes our position to lose money faster than our call position makes money. This effect is clearly apparent between days 1 and 2.

c. We first compute the gamma of each option:

$$\Gamma_{K=80,T=0.5} = 0.0274918 \quad (46)$$

$$\Gamma_{K=90,T=1} = 0.0232188 \quad (47)$$

We then compute the ratio of the gammas to obtain:

$$\frac{\Gamma_{K=80,T=0.5}}{\Gamma_{K=90,T=1}} = 1.18403 \tag{48}$$

and conclude that for each 80-strike call we sell, we must buy 1.18403 90-strike call.

d. The table below gives the **additional** number of shares (besides shares already involved in the delta-hedged strategy) you need to buy or sell, and the resulting total profit:

Day	Daily Profit (Delta-Hedge)	Stock Position (# add. shares)	Daily profit (on add. shares)	Daily Profit (90-strike call)	Daily Profit (Total)
0		-28.4695			
1	-9.38262	-33.8086	-113.878	123.931	0.670774
2	-39.4775	-22.8687	270.469	-227.974	3.01695

To obtain the numbers in the table above, we first compute the additional number of shares we need to sell (at each date) using:

$$50 \times \frac{\Gamma_{K=80,T=0.5}}{\Gamma_{K=90,T=1}} \times \Delta_{K=90} \tag{49}$$

then everything follows.

We observe that the delta-hedged gamma-neutral strategy makes positive profits at day 1 and 2, even for large price movements. The conclusion is that a gamma-neutral strategy helps hedging against large price moves, as opposed to the plain delta-hedged strategy. The reason is that gamma takes the sensitivity of delta to stock price movements into account.