Fixed Income Week 6 Notes*

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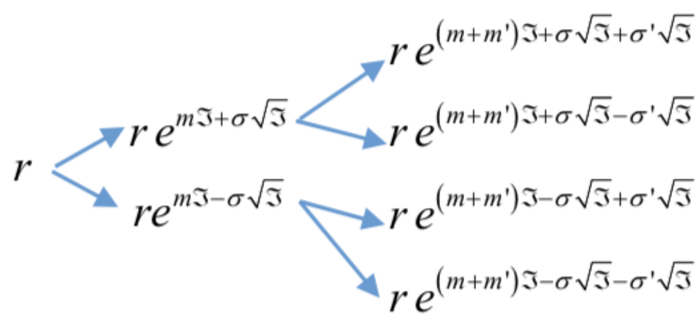
Homework 6

BDT/ tree review

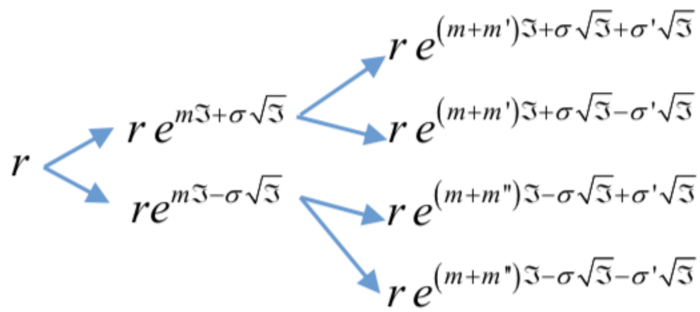
- This model is about the Black, Derman, Toy model that we discussed in class (BDT)
- The dynamics of the short rate are:
- $dln(r_t) = m(t)dt + \sigma(t)dZ_t$
- The drift and diffusion terms are deterministic functions of time
- Let τ denote an interval of time (sorry it looks a little different in the pictures). We usually implement the model in a binomial tree:



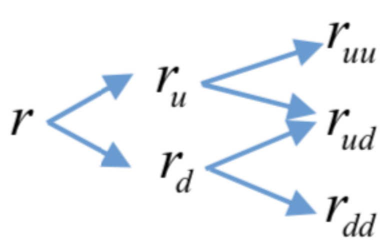
- Since non-recombining trees become difficult to handle, we impose a condition such that the trees become recombining. One approach to do this would be to say:
- $re^{(m+m')\tau+\sigma\sqrt{\tau}-\sigma'\sqrt{\tau}} = re^{(m+m')\tau-\sigma\sqrt{\tau}+\sigma'\sqrt{\tau}}$
- meaning $\sigma = \sigma'$

*I thank the former TAs for this course, Ye Wang, Rafael Porsani, Matthias Fleckenstein and Mindy X. Zhang. These notes are heavily based on their notes. All errors are my own. Email me at james.oneill.phd@anderson.ucla.edu for any errors, corrections or suggestions.

- But then the model becomes too simplistic for our uses. Therefore, we change the model so that the drifts are different for lower and upper nodes:



- Now, for the tree to be recombining, we require:
- $(m+m')\tau+\sigma\sqrt{\tau}-\sigma'\sqrt{\tau}=(m+m'')\tau-\sigma\sqrt{\tau}+\sigma'\sqrt{\tau}$
- $(m'-m'')\sqrt{\tau}=2(\sigma-\sigma')$
- With a recombining tree we write it as the following to simplify:

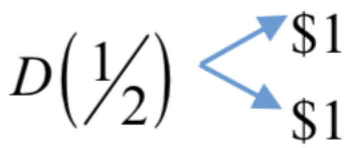


Hints for the homework:

- For each step, you will have a tree for interest rates AND a separate tree for cash flows
- As an example, let's do this for $T=0.5$. The $\frac{1}{2}$ year rate r ris known today. Therefore:
  1. Interest rate tree:

     $$r_0$$

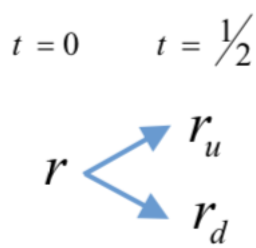
  2. Cash flow tree:



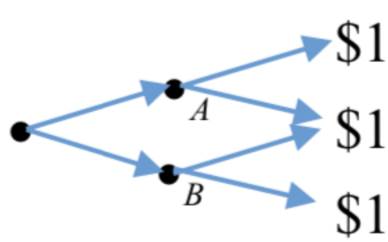
3. In either scenario a riskfree $\frac{1}{2}$ year bond pays you $1 in half a year from now.
4. In either scenario, you will discount this by using the rate r. So with probability $\frac{1}{2}$, the PV of $1 is 1/2*($1/(1+r/2)), and with probability $\frac{1}{2}$, the PV of $1 is 1/2*($1/(1+r/2))
5. Furthermore, the PV of $1 $= (1 + r/2)^{-1} = D(0.5)$. This should match the value for D(0.5) given in 'pfilea.xls'.

- Next, let us do this for $T = 1$. In $\frac{1}{2}$ year, the short rate could be r_u or r_d.

1. Interest rate tree:



2. Cash flow tree:



3. At node A:

$$PV = \frac{\frac{1}{2}(\$1)}{1+(\frac{r_u}{2})} + \frac{\frac{1}{2}(\$1)}{1+(\frac{r_u}{2})} = \frac{(\$1)}{1+(\frac{r_u}{2})}$$

4. At node B:

$$PV = \frac{\frac{1}{2}(\$1)}{1+(\frac{r_d}{2})} + \frac{\frac{1}{2}(\$1)}{1+(\frac{r_d}{2})} = \frac{(\$1)}{1+(\frac{r_d}{2})}$$

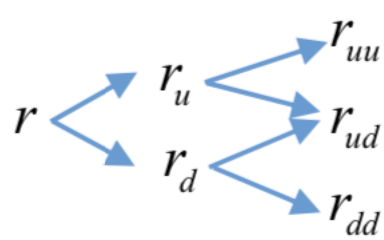
5. For the time 0 value, you need to discount this further using the rate r. Therefore:

$$D(1) = \left(\frac{\frac{1}{2}(\$1)}{1+(\frac{r_u}{2})} + \frac{\frac{1}{2}(\$1)}{1+(\frac{r_d}{2})}\right)\frac{1}{1+\frac{r}{2}}$$

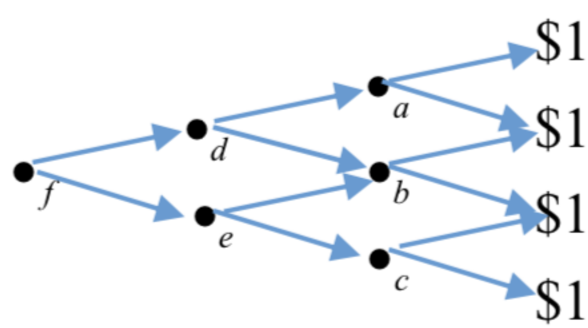
6. Solve for r_u and r_d such that the value of $D(1)$ matches the given value in the data file.

- Let's do this one more time for $T = 1.5$. At time $T = 1$, the interest rate could be r_{uu}, r_{ud} or r_{dd}.

1. Interest rate tree:



2. Cash flow tree:



3. At node A:

$$= \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{uu}}{2})} + \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{uu}}{2})}$$

4. At node B:

$$= \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{ud}}{2})} + \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{ud}}{2})}$$

5. At node C:

$$= \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{dd}}{2})} + \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{dd}}{2})}$$

6. At node D:

$$= \left(\frac{\frac{1}{2}(\$1)}{(1+\frac{r_{uu}}{2})} + \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{ud}}{2})}\right)\frac{1}{1+\frac{r_u}{2}}$$

7. At node E:

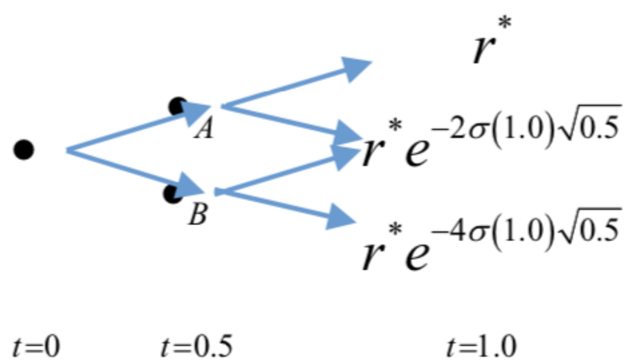
$$= \left(\frac{\frac{1}{2}(\$1)}{(1+\frac{r_{ud}}{2})} + \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{dd}}{2})}\right)\frac{1}{1+\frac{r_d}{2}}$$

8. At node F:

$$= \frac{1}{1+\frac{r}{2}}\left(\left(\frac{\frac{1}{2}(\$1)}{(1+\frac{r_{uu}}{2})} + \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{ud}}{2})}\right)\frac{1}{1+\frac{r_u}{2}} + \left(\frac{\frac{1}{2}(\$1)}{(1+\frac{r_{ud}}{2})} + \frac{\frac{1}{2}(\$1)}{(1+\frac{r_{dd}}{2})}\right)\frac{1}{1+\frac{r_d}{2}}\right)$$

9. Solve for r_{uu} , r_{ud}, r_{dd} so that this matches $D(1.5)$.

- The Final trick is to realize that at each node, the following relationship holds:



- So $r_{uu} = r^*$, $r_{ud} = r^* e^{-2\sigma(1)\sqrt{0.5}}$, and $r_{ud} = r^* e^{-4\sigma(1)\sqrt{0.5}}$
- Thus, at each node you only need to solve for r^* since the value of $\sigma(1.0)$, $\sigma(1.5)$, and so on are given to you in the spreadsheet voldat.xls.
- Thus to implement this all:

1. For each maturity build your cash flow tree (say maturity = T)
2. For this maturity build the interest rate tree
3. Set some initial value of r^* at this node.
4. Use the relationship above to fill out remaining node values at time T using relevant σ
5. Using your initial value of r^* solve for D(T+1/2) and adjust r^* until you get a matched value.
6. As a final step compute expected value of r (the short rate) – simply the probability weighted value at each time step.