1 The computation of the VIX

1.1 Properties of the derivatives of call an put prices with respect to K.

To save some notation, throughout these notes, assume that $r = \delta = 0$.

Let $f(S_T)$ denote the risk-neutral distribution of S_T. Then a call (C) and a put (P) with strike price K maturing at the same date T are defined as

$$C(K) \equiv \int_K^\infty (S_T - K) f(S_T)\, dS_T, \quad P(K) \equiv \int_0^K (K - S_T) f(S_T)\, dS_T$$

Accordingly,

$$\begin{aligned} C_K &= -\int_K^\infty f(S_T)\, dS_T, P_K = \int_0^K f(S_T)\, dS_T \qquad (1) \\ C_{KK} &= f(K), P_{KK} = f(K) \end{aligned}$$

Note that equation (1) implies that $-C_K$ is the value of a binary call (i.e., a contract that pays one dollar if $S_T > K$). Similarly P_K is value of a binary put, i.e., a contract that delivers one dollar if $S_T < K$.

1.2 The value of a portfolio of puts and calls

Let S_0 be the current value of the stock. Recall also that $r = \delta = 0$.

Define

$$V \equiv \int_0^{S_0} \frac{1}{K^2} P(K)\, dK + \int_{S_0}^\infty \frac{1}{K^2} C(K)\, dK$$

Then we have

$$\begin{aligned} \int_{S_0}^\infty \frac{1}{K^2} C(K)\, dK &= \left[-\frac{1}{K} C(K)\right]_{S_0}^\infty + \int_{S_0}^\infty \frac{1}{K} C_K(K)\, dK \\ &= \left[-\frac{1}{K} C(K)\right]_{S_0}^\infty + \left[\log(K) C_K(K)\right]_{S_0}^\infty - \int_{S_0}^\infty \log(K) C_{KK}(K)\, dK \\ &= \frac{1}{S_0} C(S_0) - \log(S_0) C_K(S_0) - \int_{S_0}^\infty \log(K) C_{KK}(K)\, dK \end{aligned}$$

and also

$$\begin{aligned} \int_{S_0}^\infty \frac{1}{K^2} P(K)\, dK &= \left[-\frac{1}{K} P(K)\right]_0^{S_0} + \int_0^{S_0} \frac{1}{K} P_K(K)\, dK \\ &= \left[-\frac{1}{K} P(K)\right]_0^{S_0} + \left[\log(K) P_K(K)\right]_0^{S_0} - \int_0^{S_0} \log(K) P_{KK}(K)\, dK \\ &= -\frac{1}{S_0} P(S_0) + \log(S_0) P_K(S_0) - \int_0^{S_0} \log(K) P_{KK}(K)\, dK \end{aligned}$$

Therefore

$$\begin{aligned} V &= \frac{1}{S_0}\left[C\left(S_0\right)-P\left(S_0\right)\right]+\log\left(S_0\right)\left[P_K\left(S_0\right)-C_K\left(S_0\right)\right]-\int_0^\infty \log\left(x\right)f\left(x\right)dx \\ &= \log\left(S_0\right)-\int_0^\infty \log\left(x\right)f\left(x\right)dx \\ &= -E^Q\left(\log\left(S_T\right)-\log\left(S_0\right)\right), \end{aligned} \tag{2}$$

where we have used the put call parity $C(S_0) - P(S_0) = S_0 - S_0 = 0$, and the properties of P_K, C_K, P_{KK}, C_{KK}.

1.3 Ito's Lemma and the portfolio V :

Recall that $r = \delta = 0$. Assume that

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t d\overline{W}_t$$

under the natural probability measure P. Under the risk neutral measure Q (and recalling $r = \delta = 0$) we have that

$$\frac{dS_t}{S_t} = \sigma_t dW_t.$$

Applying Ito's Lemma gives

$$d\log\left(S_t\right) = -\frac{1}{2}\sigma_t^2 dt + \sigma_t dW_t.$$

Integrating the above equation and taking expectations leads to

$$E^Q\left(\log\left(S_T\right)-\log\left(S_0\right)\right) = -\frac{1}{2}E^Q\int_0^T \sigma_t^2 dt. \tag{3}$$

Combining (3) with (2) implies that

$$2V = E^Q\int_0^T \sigma_t^2 dt.$$

2 The relation between implied volatility and the risk neutral distribution.

Suppose that we observe the call function $C(K)$. Letting $\sigma(K)$ denote the implied volatility associated with the strike price K, we have (by definition of the implied volatility) that

$$C\left(K\right) = C^{BS}\left(K, \sigma\left(K\right)\right), \tag{4}$$

where $C^{BS}(K, \sigma(K))$ is the Black Scholes formula. Differentiating both sides of (4) with respect to K leads to

$$C_K = C_K^{BS} + C_\sigma^{BS} \frac{d\sigma}{dK}.$$

Therefore

$$-C_K = -C_K^{BS} - C_\sigma^{BS} \frac{d\sigma}{dK}$$

or letting $B(K), B^{BS}(K)$ denote the value of a digital call (and the respective digital call under the Black Scholes model), we have that

$$B(K) = B^{BS}(K) - C_\sigma^{BS} \frac{d\sigma}{dK},$$

where we used (1). Therefore

$$\frac{d\sigma}{dK} = \frac{1}{C_\sigma^{BS}} \left[B^{BS}(K) - B(K)\right]$$

and the slope of the implied volatility curve $\left(sign\left(\frac{d\sigma}{dK}\right)\right)$ depends on whether the value of a binary call is higher or lower than its Black Scholes counterpart (computed with the volatility $\sigma(K)$).