MFE 409 Lecture 2b
Measuring Value-at-Risk

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Lecture Objectives

Measuring Value-at-Risk:

- How to judge validity of a VaR estimate?
- Historical approach
- Model-building approach
- How to get a measure for a given approach but also how to choose an appropriate approach

Outline

Model-Building Approach

- The main alternative to historical simulation is to make assumptions about the probability distributions of the returns on the market variables
- Sometimes called the variance-covariance approach

Normal Model

- Simplest and often-used assumption: normal distribution

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- VaR has a simple expression:

$$\text{VaR} = -\mu - \sigma \times z(c)$$

- Portfolios of normal returns are also normally distributed

- Estimation of normal distributions very developed

Portfolios

- With multivariate normal returns, portfolio returns are normally distributed
- Assume:
  - Each asset return R_i is normally distributed with mean 0 and variance σ_i^2
  - Pairwise correlations: ρ_{ij}
  - investment in each asset α_i

$$\underbrace{\Delta P}_{\text{portfolio gain}} = \sum_i \alpha_i R_i \sim \mathcal{N}(0, \sigma_P^2)$$

$$\sigma_P^2 = \sum_i \sum_j \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}$$

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- Multiply by $-z(c)$ to obtain VaR

Example: Imperfect Hedge

- Previous example:
  - Long position EUR 10m, $M_t =$ USD/EUR $= \$1.436$, $\sigma_M = 0.65\%$
  - Dollar position $14.36m
  - VaR$= \$217,204$
- Suppose you want to hedge with Japanese Yens: $\sigma_J = 0.69\%$, $\rho_{MJ} = 0.2775$
  - What Yen position do you choose to hedge as well as possible?
  - What is your hedged VaR?

$14.36 \times R_M + \alpha R_J$

$\alpha_1 R_1 + \alpha_2 R_2$

Variance of the portfolio:

$$\alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 + 2\alpha_1\alpha_2 \sigma_1\sigma_2 \rho_{12}$$

Minimize w.r.t. α_2

$$0 = 2\alpha_2 \sigma_2^2 + 2\alpha_1 \sigma_1\sigma_2\rho_{12}$$

$$\alpha_2 \sigma_2^2 = -\alpha_1 \sigma_1\sigma_2\rho_{12}$$

$$\alpha_2 = -\alpha_1 \frac{\sigma_1\sigma_2\rho_{12}}{\sigma_2^2} = -\alpha_1 \frac{\sigma_1\rho_{12}}{\sigma_2} = -\alpha_1 \underbrace{\frac{\mathrm{cov}(R_1,R_2)}{\mathrm{var}(R_2)}}_{\beta_{12}}$$

$$= -3.72$$

Compute variance of hedged portfolio

$$\alpha_1^2\sigma_1^2 + \alpha_1^2 \frac{\sigma_1^2\rho_{12}^2 \cancel{\sigma_2^2}}{\cancel{\sigma_2^2}} - 2\alpha_1^2 \frac{\sigma_1\rho_{12}}{\cancel{\sigma_2}} \sigma_1\cancel{\sigma_2}\rho_{12} = \alpha_1^2\sigma_1^2\left[1+\rho_{12}^2-2\rho_{12}^2\right]$$

$$= \alpha_1^2\sigma_1^2\left[1-\rho_{12}^2\right]$$

Imperfect Hedge

- Want to hedge a position R_p using a hedging instrument R_h
- Optimal hedging position:

$$\alpha_{\text{hedge}} = -\rho \frac{\sigma_p}{\sigma_h}$$

- Variance of the hedged portfolio:

$$\text{Minimum variance} = \sigma_p^2 (1 - \rho^2)$$

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- VaR of the hedged portfolio:

$$\text{Minimum VaR} = \text{VaR}_p \sqrt{1 - \rho^2}$$

  - Only depends of correlation ρ

Volatility

- Often, volatility is defined as standard deviation of *log return*

$$\log\left(\frac{P_{t+1}}{P_t}\right)$$

- In risk management, typically the standard deviation of *simple return*

$$\frac{P_{t+1}}{P_t} - 1$$

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- Some conventions:
  - Only count trading days: $\sigma_{\text{yr}} = \sigma_{\text{day}} \times \sqrt{252}$
  - Variance rate: σ^2

Estimating Volatility

- Assume today is date t and we have data for n past dates
- Unbiased estimates
  - Mean

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_{t-i}$$

  - Volatility

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n} \left(R_{t-i} - \bar{R}\right)^2$$

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- Risk management practice:
  - Assume $\bar{R} = 0$: mean small relative to standard deviation for one day
  - Replace $n-1$ by n $\longleftrightarrow$ MLE for variance

Estimating Volatility: Maximum Likelihood

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- Likelihood for one observation

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-R_i^2}{2\sigma^2}\right)$$

- Log likelihood

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\left[-\log(\sigma^2) - \frac{R_{t-i}^2}{\sigma^2}\right]$$

- First-order condition w.r.t. σ^2

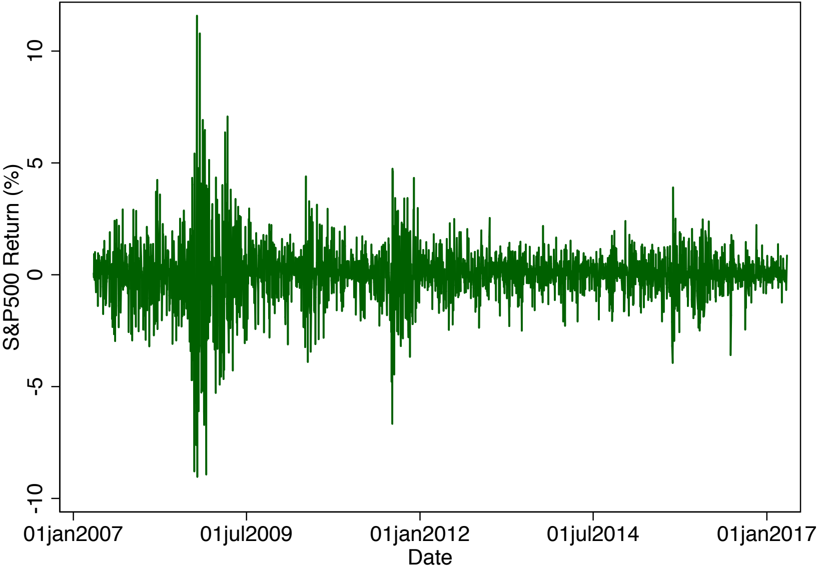
$$0 = -\frac{n}{\sigma^2} + \frac{1}{\sigma^4}\sum_{i=1}^{n} R_{t-i}^2$$

- Estimator

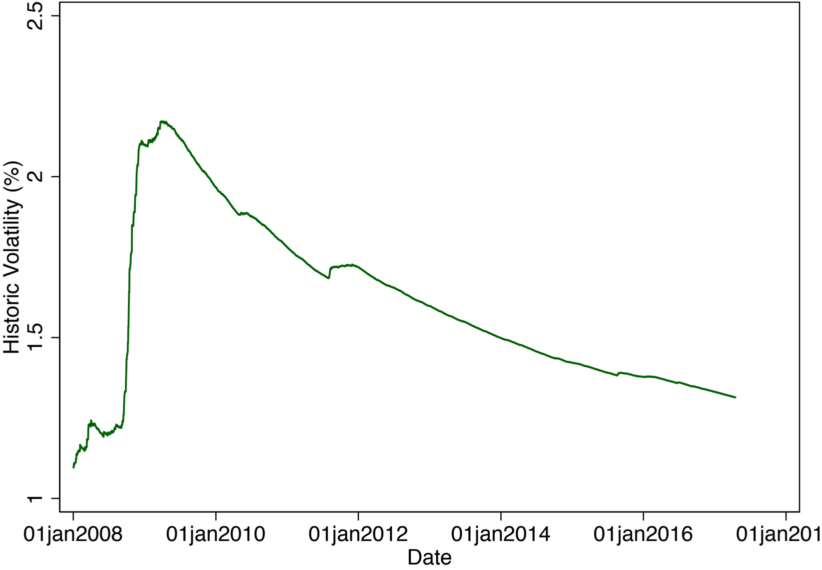
$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} R_{t-i}^2$$

S&P500: Returns

SP500Vix.xls



S&P500: Historic Volatility



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- Weighting scheme + long-run variance

$$\sigma_t^2 = \gamma V_L + \sum_{i=1}^{n} \alpha_i R_{t-i}^2$$

$$\text{with } 1 = \gamma + \sum_{i=1}^{n} \alpha_i$$

ARCH and EWMA

Engel

- ARCH(m), autoregressive conditional heteroskedasticity

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- EWMA, exponentially weighted moving average

$$\alpha_i = (1-\lambda)\lambda^i$$

$$\sigma_t^2 = \sum_{i=1}^{\infty} (1-\lambda)\lambda^{i-1} R_{t-i}^2$$

  - Simple volatility updating:

$$= (1-\lambda)R_{t-1}^2 + \lambda \underbrace{\sum_{i=2}^{\infty}(1-\lambda)\lambda^{i-2} R_{t-i}^2}_{\sigma_{t-1}^2}$$

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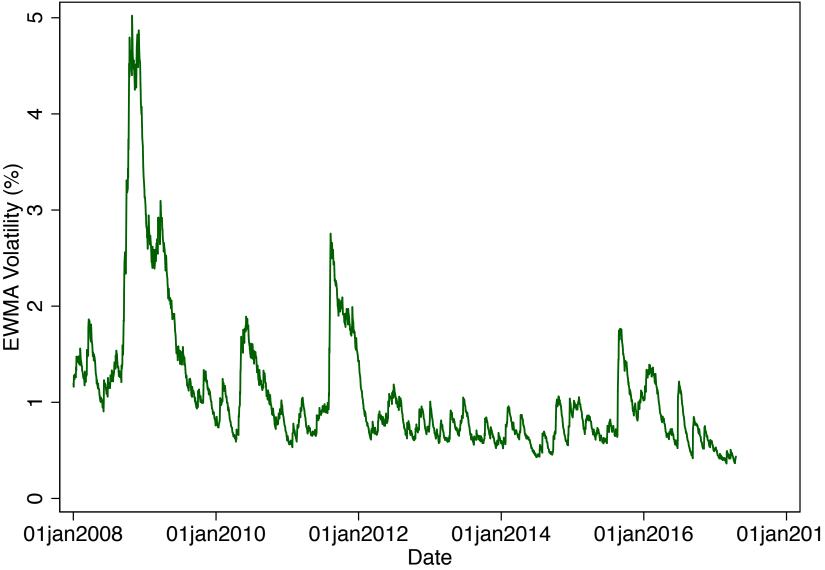
$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)R_t^2$$

  - RiskMetrics reported with $\lambda = 0.94$ until 2006

0.995

S&P500: EWMA Volatility



GARCH(1,1)

- GARCH(1,1), generalized autoregressive conditional heteroskedasticity

$$\sigma_t^2 = \underbrace{\gamma V_L}_{\omega} + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2$$

  - EWMA + long-run average
  - If $\gamma = 0$, EWMA
  - For stability, $\alpha + \beta < 1$

$\alpha + \beta + \gamma = 1$

MLE Estimation of GARCH(1,1)

$R_1 \sim N(0, \sigma_0^2)$

$R_2 \sim N(0, \sigma_1^2)$

$\vdots$

- Parameters: ω, α, β
- Log-likelihood

$$\sum_{i=1}^{n} \left[-\log(\sigma_{t-i}^2) - \frac{R_{t-i}^2}{\sigma_{t-i}^2} \right]$$

- Compute σ_{t-i}^2:
  - Initialize at $\sigma_0 = \sqrt{V_L} = \sqrt{\omega/(1-\alpha-\beta)}$
  - Use formula to iterate

$$\sigma_t^2 = \omega + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2$$

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    1. Remove time-varying volatility
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        - ⋆ Compare autocorrelations $c_k = \text{cor}(R_t^2/\sigma_t^2, R_{t-k}^2/\sigma_{t-k}^2)$

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- Ljung-Box Statistic

$$n \sum_{k=1}^{K} w_k c_k^2$$

$$w_k = \frac{n+2}{n-k}$$

- For $K = 15$, 95% threshold is 25

Volatility Forecasts

- If we want to forecast k days in the future:

$$\mathbb{E}_t\left[\sigma_{t+k}^2\right] = V_L + (\alpha + \beta)^k(\sigma_t^2 - V_L)$$

  - Exponential mean reversion

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- *Remark:* If we want to hedge volatility risk, need to consider how shocks today will affect volatility during the lifetime of the option

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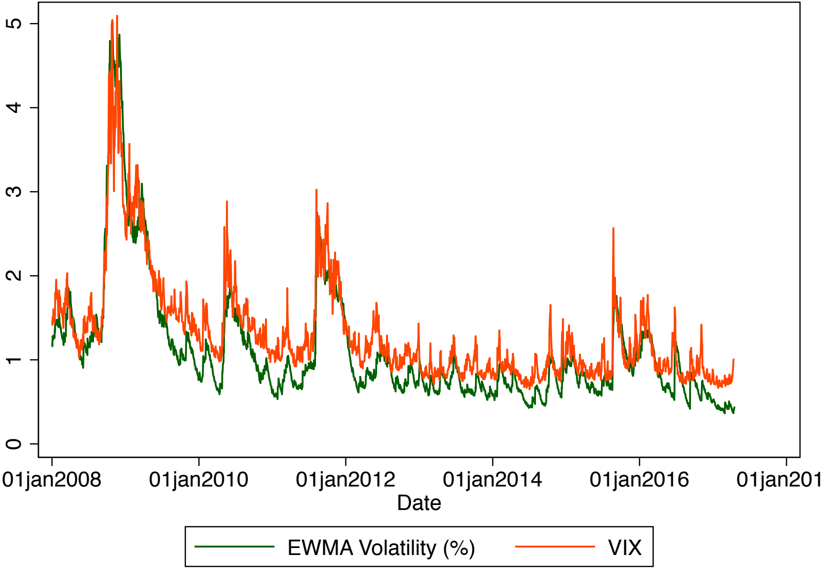
- Can use market prices to obtain expectation of future volatility
- Derivative contracts on volatility: VAR swaps, ...
- *Implied volatility* from calls and puts
  - Volatility so that Black-Scholes formula matches price

VIX



0
1
2
3
4
5
01jan2008
01jan2010
01jan2012
01jan2014
01jan2016
01jan201
Date
EWMA Volatility (%)
VIX

VIX

- The VIX index is published by CBOE
- It is no longer the Black and Scholes implied volatility
- But it is computed from a portfolio of options on the S&P500 index
  - It is deemed to better capture market “expected” volatility over the next 30 days without relying on any model

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- Why is VIX systematically higher than realized volatility?
  - Risk adjustment implicit in options, that make VIX higher than future realized volatility
  - It does not mean that market expectations are systematically too high

Non-Normal Assumptions

- Market information can be useful beyond the normal distribution

- Directly obtain measures of downside risks from options

Using Options to Infer Downside Risk

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- Because higher volatility implies a higher price, if OTM options have higher implied volatility than ATM option $\rightarrow$ market expects negative skewness
  - Since the crash of October 1987, OTM put options have a higher implied volatility than ATM put options.
  - Moreover, the difference in implied volatilities is time varying.

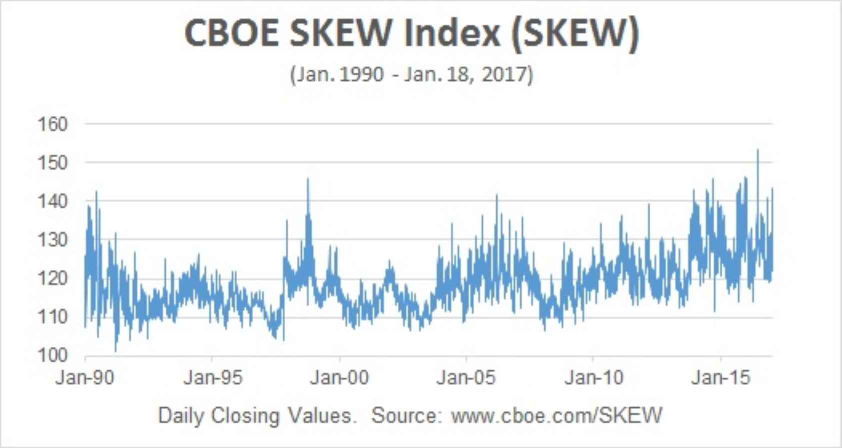
Skewness Index

- CBOE publishes a Skew Index
- Implied expected skewness is computed from option prices with a more elaborate methodology than the Black and Scholes implied volatilities, but the logic is similar.
- Recall that high *negative* skewness imply high downside risk.
  - CBOE define the Skew Index as

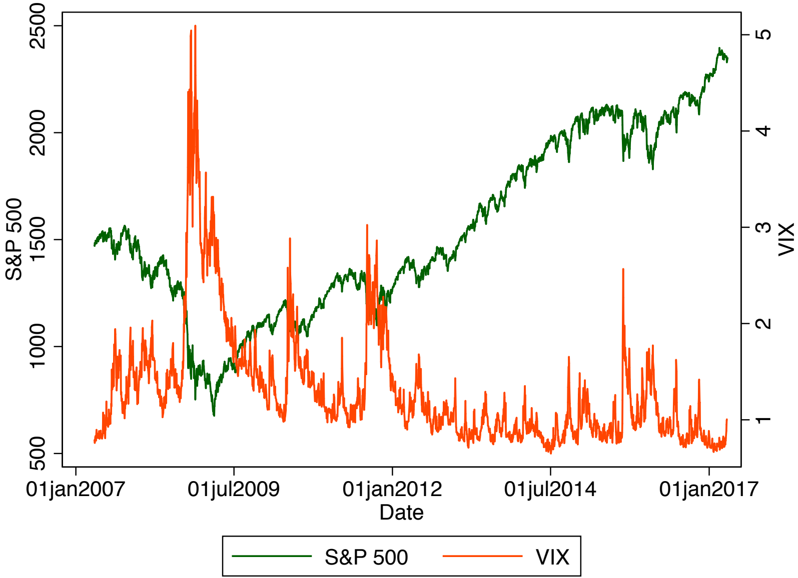
$$\text{Skew Index} = 100 - 10 \times \text{Implied Expected Skewness}$$

  - Higher positive index $\rightarrow$ higher downside risk

Skew Index


CBOE SKEW Index (SKEW)
(Jan. 1990 - Jan. 18, 2017)
160
150
140
130
120
110
100
Jan-90
Jan-95
Jan-00
Jan-05
Jan-10
Jan-15
Daily Closing Values. Source: www.cboe.com/SKEW

S&P500 and VIX



Model-Building

- To accurately model risk, necessary to understand interactions between different risks
- Lots of models, for each asset class
- Key question: what can go wrong?
- If model is too complex to compute VaR explicitly: Monte-Carlo simulations

Model-Building vs. Historical Simulation

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  - Large portfolios
  - Limited data
  - Taking account of nonlinearities
- Historical simulations useful for:
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- Historical simulations useful for:
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  - Unknown structure of investment performance
- Key trade-off: making more assumptions vs. using a small part of the data
  - Always the same in statistics!