UCLA Anderson School of Management
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Problem Set 5

These exercises do not need to be turned in for credit.

1 Futures vs. Forwards

Futures prices are equal to forward prices as long as interest rates are not stochastic. Here is a proof of this statement (in practice there are many other reasons besides stochastic interest rates that explain why futures and forwards differ—*market liquidity, counterparty credit risk, delivery options*—but let's abstract from them for the moment).

Assume that there are no dividends, let $r > 0$ denote the continuously compounded interest rate **per day** and denote by f_t and F_t the futures and forward prices on day t for delivery of the underlying T days from now.

Consider the dynamic futures strategy:

- Day 0: enter a long position in e^r contracts
- Day 1: close the position, invest the gain or loss in the bank until day T and reopen a new position in e^{2r} contracts
- Day 2: close the position, invest the gain or loss in the bank until day T and reopen a new position in e^{3r} contracts
- ...
- Day $T-1$: close the position, invest the gain or loss in teh bank until day T and reopen a new position in e^{rT} contracts
- Day T: close the position and realize your total profit or loss

Recall that entering a futures position entail no initial cost and that the profit or loss from a position in one contract between days t and $t+1$ is realized at the end of day $t+1$ and given by $f_{t+1} - f_t$.

a. Compute the terminal pay-off of the **dynamic** futures strategy.

b. Compare the terminal value of the dynamic futures strategy to that of a static strategy which enters a long position in e^{rT} forward contracts on Day 0 and holds on to this position until the last day. What's the conclusion?

c. Why does the argument break down with sochastic interest rates?

1 a. Prior to increasing his position on day $k \leq N-1$ the investor holds e^{rk} contracts and realizes a gain equal to

$$G_k = \underbrace{e^{rk}}_{\text{Size of position}} \times \underbrace{(f_k - f_{k-1})}_{\text{Gain/loss on 1 contract}} \tag{1}$$

Capitalizing until the terminal date and summin over dates shows that the terminal payoff is

$$\sum_{k=1}^{N} e^{r(N-k)} G_k = e^{rN}(f_N - f_{N-1}) + \sum_{k=1}^{N-1} e^{rN}(f_k - f_{k-1}) \tag{2}$$

$$= e^{rN}(S_T - f_0) \tag{3}$$

where the last equality follows from the fact that the terminal value of the futures price is the spot price: $f_N = S_T$.

b. Comparison with static strategy in forward contract

- No initial cost. Its terminal value is given by $e^{rN}(S_T - F_{0,T})$
- Going long in the futures strategy and short in the forward strategy implies no cost and generates the riskless payoff

$$e^{rN}(S_t - f_0) - e^{rN}(S_T - F_{0,T}) = e^{rN}(F_{0,T} - f_0) \tag{4}$$

- Since the cost is zero, absence of arbitrage requires that the payoff is also zero. Thus $f_0 = F_{0,T}$.

c. With stochastic interest rates the futures strategy's gain at k would be

$$G_k = e^{\sum_{i=1}^{k} r_i}(f_k - f_{k-1}) \tag{5}$$

where r_i is the interest rate between time i and $i+1$. Notice the difference with (1). The terminal payoff is then

$$e^{\sum_{i=1}^{N} r_i}(f_N - f_{N-1}) + \sum_{k=1}^{N-1} e^{\sum_{i=k}^{N-1} r_{i+1} G_k} = e^{\sum_{i=1}^{N} r_i}(S_T - f_0) \tag{6}$$

On day 0, it is impossible to know the interest rates $(r_i)_{i>1}$ that will apply at future dates. Consequently, it is impossible to go short the right number of forward contracts and so the argument breaks down with stochastic interest rates.

2 Margin Account

Suppose that you can trade 3 stocks, A, B, and C, each of which has a current price of \$100:

- A is a non-dividend-paying stock
- B pays a discrete \$10 dividend in 3 months
- C pays dividends at a continuously compounded dividend yield of 3%

Consider a 6-month futures contract on stock A with futures price \$102 and size $\$250 \times A$. Suppose the futures is settled every 2 months, there is a 10% initial margin requirement and a maintenance margin set at 80% of the initial margin level. The continuously compounded risk-free rate is 5%.

a. What is the notional value of one contract in 6-month dollars? Suppose that you want to acquire a \$510,000 position in stock A in 6 months. How many futures contracts do you need to buy? What is the multiplier of your position?

b. Suppose you buy these contracts. What is the initial margin required by the clearinghouse? What is your maintenance margin?

c. Explain how the margin account would evolve if the futures price is \$90 in 2 months, \$105 in 4 months, and \$95 at maturity. State the mark-to-market gain or loss on your position, the margin call (if any), and the margin balance. Do not forget to take the interests on the margin into account.

d. What is your 6-month profit? (Do not forget to take the future value of the margin calls into account). What is the 6-month profit on the **forward contract**? Explain why you get different results.

2 a. The notional value of one contract is the size of one contract times the current futures price:

$$\text{Notional value} = \$250 \times 102 = \$25,500 \tag{7}$$

Since we pay \$25,500 per futures contract at expiration, we need to buy 20 futures contracts. The multiplier of our position is $20 \times 250 = \$5,000$.

b. The clearinghouse requires that you pledge 10% of your position: $510,000 \times 0.1 = \$51,000$. Your maintenance margin is $0.8 \times 51,000 = \$40,800$.

c. After two months, the loss we incur is the difference between the past and current futures price times the multiplier:

$$(90 - 102) \times 5,000 = -\$60,000 \tag{8}$$

Accounting for 2-months interests and withdrawing the loss, the margin balance is

$$51,000 \times e^{0.05 \times 2/12} - 60,000 = -\$8,573.2 \tag{9}$$

Since we have to maintain a margin above \$40,800, we get a margin call for

$$51,000 - (-8,573.2) = \$59,573.2 \tag{10}$$

The table below summarizes how the margin account evolves in the next 6 months:

Months	Gain/Loss	Margin Call	Margin Balance
0	–	–	51,000
2	-60,000	59,573.2	51,000
4	75,000	–	126,427
6	-50,000	–	77,484.7

d. Our 6-month profit is obtained by subtracting from the final margin the future value of the initial margin and also the future value of the margin call:

$$77,484.7 - 51,000e^{0.05\times 6/12} - 59,573.2e^{0.05\times 4/12} = -\$35,380.8 \tag{11}$$

Since the forward is only settled at maturity, your 6-month profit on the forward contract is

$$(95 - 102) \times 5,000 = -\$35,000 \tag{12}$$

The difference between the two profits is due to interest payments on the margin account. Since we lost money, we founded losses and this further magnifies our loss. The interest we pay on mark-to-market losses are:

$$\left[(90 - 102)e^{0.05\times 4/12} + (105 - 90)e^{0.05\times 2/12} + (102 - 105)e^{0.05\times 0/12}\right] = -\$380.8 \tag{13}$$

3 Futures Overlay

Suppose that you can trade 3 stocks, A, B, and C, each of which has a current price of \$100:

- A is a non-dividend-paying stock
- B pays a discrete \$10 dividend in 3 months
- C pays dividends at a continuously compounded dividend yield of 3%

Suppose you manage a portfolio of stocks, the current value of which is \$250,000. The continuously compounded risk-free rate is 5%.

a. Assume that you are fully invested in stock A. How many shares of stock A do you currently hold?

b. You wish to temporarily convert your position from stock A to bonds over the next 6 months. Explain the transactions you undertake to obtain the desired result using the 6-month forward from the previous exercise. Report the cash-flows of your position if stock A trades in 6 months at \$60 and \$120.

3 a. Since stock A trades at \$100 and we are fully invested, we hold

$$\frac{250,000}{100} = 2,500 \text{ shares of stock } A \tag{14}$$

b. To transfer our stocks to bonds, we sell short 2,500 forward contracts. The resulting cash-flow is that of a synthetic bond created through cash-and-carry:

$$250,000 \times e^{0.05\times 0.5} = \$256,329 \tag{15}$$

	Cash Flows		
Transaction	Time 0	6 months $S_A = 60$	6 months $S_A = 120$
Long stock (2,500 units)	-250,000	150,000	300,000
Short forward (2,500 units)	0	106,329	-43,671
Total	-250,000	256,329	256,329

4 Options on Futures I

Consider a European put futures option on crude oil. The time to the option's maturity is 4 months, the current futures price is \$20, the exercise price is \$20, the risk-free interest rate is 9% per annum, and the volatility of the futures price is 25% per annum.

a. Assume that the maturity of the futures contract is 4 months. What is the price of the European put futures option?

b. Assume that the maturity of the futures contract is 1 year. What is the price of the European put futures option?

4 a. Black's formula for an European put is

$$P_0 = e^{-rT}\left[KN(-d_2) - F_0N(-d_1)\right] \tag{16}$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} \tag{17}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{18}$$

In this case we have $F_0 = 20$, $K = 20$, $r = 0.09$, $T = 4/12$, $\sigma = 0.25$, so that $d_1 = 0.07216$ and $d_2 = -0.07216$, $N(-d_1) = 0.4712$, $N(-d_2) = 0.5288$. The put price is therefore given by:

$$P_0 = e^{-0.09\times 4/12}\left(20 \times 0.5288 - 20 \times 0.4712\right) = 1.12 \tag{19}$$

b. The price of the European put option does not depend on the maturity of the futures contract.

5 Options on Futures II

a. Consider a six-month European call option on the spot price of gold, that is, an option to buy one ounce of gold in the spot market in six months. The strike price is $1,200, the risk-free rate of interest is 5% per annum, and the volatility of gold is 20%. The spot price of gold today is $1,200. Your estimate for the lease rate of gold is 2% per annum. Find the price of the European call option.

b. Now assume that the six-month futures price of gold is $1,218.14 (observe that this gives a lease rate of 2%). What is the price of a European call option on the six-month futures price?

5 a. Use Black-Scholes Formula. The **lease rate** of a commodity is conceptually similar to a dividend yield (see pages 315-316 in McDonald). We have $d_1 = 0.176777$, $d_2 = 0.035355$, $N(d_1) = 0.570158$, $N(d_2) = 0.514102$, and thus

$$C_0 = e^{-0.02\times 0.5} \times S_0 \times N(d_1) - e^{-0.05\times 0.5} \times K \times N(d_2) = \$75.69 \tag{20}$$

b. Use Black's formula. We have $d_1 = 0.176777$, $d_2 = 0.035355$, $N(d_1) = 0.570158$, $N(d_2) = 0.514102$, and thus

$$C_0 = e^{-0.05\times 0.5} \times F_0 \times N(d_1) - e^{-0.05\times 0.5} \times K \times N(d_2) = \$75.69 \tag{21}$$

The big advantage of Black's model is that it avoids the need to estimate the **lease rate**, since the futures price already incorporates that information.

6 Arbitrage Bounds for Options on Futures

Show that, if C_0^A is the price of an American call option on a futures contract when the strike price is K and the maturity is T, and P_0^A is the price of an American put on the same futures contract with the same strike price and exercise date, then

$$F_0 e^{-rT} - K < C_0^A - P_0^A < F_0 - K e^{-rT} \tag{22}$$

where F_0 is the futures price and r is the risk-free rate. Assume that $r > 0$ and that there is no difference between forward and futures contracts.

6 Let us focus first on the first inequality. Consider a portfolio with the following positions:

a. Long an European call option, C_0^E

b. Short an American put option, P_0^A

c. An amount K invested at the risk-free rate

d. A short futures contract

e. An amount $F_0 e^{-rT}$ borrowed at the risk-free rate, where F_0 is the futures price

The possible values for this portfolio are:

	No exercise of the put option before maturity		Exercise of the put option before maturity (at some time $0 \leq t < T$)
	$F_T \geq K$	$F_T < K$	
Position a.	$F_T - K$	0	C_t^E
Position b.	0	$F_T - K$	$F_t - K$
Position c.	Ke^{rT}	Ke^{rT}	Ke^{rt}
Position d.	$-F_T + F_0$	$-F_T + F_0$	$-F_t + F_0$
Position e.	$-F_0$	$-F_0$	$-F_0 e^{-r(T-t)}$
TOTAL	> 0	> 0	> 0

Since all the possible values are positive, we have

$$C_0^E - P_0^A + K - F_0 e^{-rT} > 0 \tag{23}$$

$$C_0^E - P_0^A > F_0 e^{-rT} - K \tag{24}$$

and thus

$$C_0^A - P_0^A > F_0 e^{-rT} - K \tag{25}$$

Focus now on the second inequality. Consider a portfolio with the following positions:

a. Short an American call option, C_0^A

b. Long an European put option, P_0^E

c. An amount Ke^{-rT} borrowed at the risk-free rate

d. A long futures contract

e. An amount F_0 invested at the risk-free rate, where F_0 is the futures price

The possible values for this portfolio are:

	No exercise of the call option before maturity		Exercise of the call option before maturity (at some time $0 \leq t < T$)
	$F_T \geq K$	$F_T < K$	
Position a.	$K - F_T$	0	$K - F_t$
Position b.	0	$K - F_T$	P_t^E
Position c.	$-K$	$-K$	$-Ke^{-r(T-t)}$
Position d.	$F_T - F_0$	$F_T - F_0$	$F_t - F_0$
Position e.	$F_0 e^{rT}$	$F_0 e^{rT}$	$F_0 e^{rt}$
TOTAL	> 0	> 0	> 0

Since all the possible values are positive, we have

$$-C_0^A + P_0^E - Ke^{-rT} + F_0 > 0 \tag{26}$$

$$C_0^A - P_0^E < F_0 - Ke^{-rT} \tag{27}$$

and thus

$$C_0^A - P_0^A < F_0 - Ke^{-rT} \tag{28}$$