Empirical Methods in Finance
Homework 6: Solution

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Working with Factor Models

Assume you have been given three assets to invest in, in addition to the market portfolio. From a historical regression of excess asset returns on the excess market return for $t = 1, \ldots, T$, you have:

$$
\begin{aligned}
R_{1t}^e &= 0.01 + 0.9R_{mt}^e + \hat{\varepsilon}_{1t} \\
R_{2t}^e &= -0.015 + 1.2R_{mt}^e + \hat{\varepsilon}_{2t} \\
R_{3t}^e &= 0.005 + 1.0R_{mt}^e + \hat{\varepsilon}_{3t}
\end{aligned}
$$

The sample mean excess return on the market is, $\bar{R}_m^e = 0.05$; the sample standard deviation of excess market returns is 15%. Thus, the market Sharpe ratio is $1/3$. Finally, the sample variance-covariance matrix of residual returns, $\varepsilon_t = \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \varepsilon_{3t} \end{bmatrix}'$, is:

$$
var(\hat{\varepsilon}_t) = \hat{\Sigma}_\varepsilon = \begin{bmatrix} 0.1^2 & 0 & 0 \\ 0 & 0.15^2 & 0 \\ 0 & 0 & 0.05^2 \end{bmatrix}
$$

1. What is the sample mean, standard deviation, and Sharpe ratio of these three assets?

   Suggested solution:

Assume all the residuals are mean-zero. Then we have

$$\begin{aligned}
\mathbb{E}(R^e_{1t}) &= 0.01 + 0.9\bar{R}^e_m = 0.055 \\
\mathbb{E}(R^e_{2t}) &= -0.015 + 1.2\bar{R}^e_m = 0.045 \\
\mathbb{E}(R^e_{3t}) &= 0.005 + 1.0\bar{R}^e_m = 0.055
\end{aligned}$$

Also as we know that factor returns and residuals are uncorrelated,

$$\begin{aligned}
var(R^e_{1t}) &= 0.9^2 \times 0.15^2 + 0.1^2 = 0.028225 \\
var(R^e_{2t}) &= (1.2)^2 \times 0.15^2 + 0.15^2 = 0.0549 \\
var(R^e_{3t}) &= 0.15^2 + 0.05^2 = 0.025
\end{aligned}$$

Thus,

$$\begin{aligned}
s.d.(R^e_{1t}) &= 0.17 \\
s.d.(R^e_{2t}) &= 0.23 \\
s.d.(R^e_{3t}) &= 0.16
\end{aligned}$$

Given mean of excess return and standard deviation, we can get the Sharpe ratio as

$$\begin{aligned}
S(R^e_{1t}) &= \frac{0.055}{0.17} = 0.324 \\
S(R^e_{2t}) &= \frac{0.045}{0.23} = 0.196 \\
S(R^e_{3t}) &= \frac{0.055}{0.16} = 0.344
\end{aligned}$$

2. For each of the three assets, construct the market-neutral versions by hedging out the market risk. For each of these three hedged asset returns, give the sample average return, standard deviation, and Sharpe ratio.

   Suggested solution:

   For each of the assets, we buy one unit of it and then hedge out the market risk by shorting corresponding units of market portfolio. We use h in the superscript to represent these hedged portfolios. Then the expected return would just be

$$\mathbb{E}\left[R_{1t}^{e,h}\right] = 0.01$$
$$\mathbb{E}\left[R_{2t}^{e,h}\right] = -0.015$$
$$\mathbb{E}\left[R_{3t}^{e,h}\right] = 0.005$$

and the s.d. would just be s.d. of residuals correspondingly:

$$s.d.\left(R_{1t}^{e,h}\right) = 0.1$$
$$s.d.\left(R_{2t}^{e,h}\right) = 0.15$$
$$s.d.\left(R_{3t}^{e,h}\right) = 0.05$$

Then we can get the Sharpe ratio to be

$$SR\left(R_{1t}^{e,h}\right) = \frac{0.01}{0.1} = 0.1$$
$$SR\left(R_{2t}^{e,h}\right) = \frac{-0.015}{0.15} = -0.1$$
$$SR\left(R_{3t}^{e,h}\right) = \frac{0.005}{0.05} = 0.1$$

3. Calculate the maximum Sharpe ratio you can obtain by optimally combining the three hedged assets. Give the math behind this calculation.

Suggested Solution:
By construction, the mean variance efficient portfolio will give us the maximum Sharpe ratio. To do that, we first solve the following objective function in Lagrangian form

$$\min_{w} \frac{1}{2} w' \hat{\Omega} w - k(w' \bar{R}^e - m)$$

where w is the weight of each asset, $\hat{\Omega}$ is the variance-covariance matrix for these assets, $\bar{R}^e$ is mean excess return, m is the target return of the portfolio and k is the Lagrange multiplier. The first order condition gives us that

$$w^{MVE} = k \hat{\Omega}^{-1} \bar{R}^e \tag{1}$$

Thus we have

$$
\begin{aligned}
\bar{R}^e_{MVE} &= (w^{MVE})'\bar{R}^e = k\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e \\
var(R^e_{MVE}) &= (w^{MVE})'\hat{\Omega}w^{MVE} \\
&= k^2\bar{R}^{e\prime}\hat{\Omega}^{-1}\hat{\Omega}\hat{\Omega}^{-1}\bar{R}^e \\
&= k^2\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e
\end{aligned} \tag{2}
$$

Then we can get the highest possible Sharpe ratio to be

$$
SR^2 = \frac{(k\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e)^2}{k^2\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e} = \bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e \tag{3}
$$

In our case, we have that

$$
\begin{aligned}
\bar{R}^e &= \begin{bmatrix} 0.01 & -0.015 & 0.005 \end{bmatrix} \\
\hat{\Omega} &= \begin{bmatrix} 0.1^2 & 0 & 0 \\ 0 & 0.15^2 & 0 \\ 0 & 0 & 0.05^2 \end{bmatrix}
\end{aligned}
$$

Plug these back into equation (3), we have that the maximized SR^2 is 0.03, resulting in the maximum Sharpe ratio of $SR = 0.17$.

4. Given your result in (3), what is the maximum Sharpe ratio you can obtain by combining these three assets with the market portfolio?

   Suggested Solution:
   As we know that in the factor model world, the maximum portfolio Sharpe ratio squared will be the sum of maximum Sharpe ratio squared of hedged stock returns (calculate in part 3) plus factor Sharpe ratio squared. Thus, the maximum Sharpe ratio that can be obtained here is $\sqrt{0.03 + (1/3)^2} = 0.38$

5. You have been told to form a portfolio today, assuming the historical estimates given above are the true values also going forward, that (a) provides the maximum (expected) Sharpe ratio of returns and (b) has an (expected) volatility of 15%. You can invest in the three assets, as well as the market portfolio.

   (a) Give the portfolio weights (really, the loadings on each of these in total four assets since each asset is an excess return) that achieves objectives (a) and (b).

   Suggested solution:

To obtain the maximum Sharpe ratio, we mush have the portfolio to be mean-variance efficient. We then follow equation (2) and solve for the Lagrangian constant that

$$s.d. = k\sqrt{\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^{e}}$$
$$k = \frac{s.d.}{SR} = 0.39$$

To get the weight, we first simply add market to the three hedged portfolio and then back out the weight for each stock. For these four portfolios, we have that

$$\bar{R}^e = \begin{bmatrix} 0.01 & -0.015 & 0.005 & 0.05 \end{bmatrix}$$
$$\hat{\Omega} = \begin{bmatrix} 0.1^2 & 0 & 0 & 0 \\ 0 & 0.15^2 & 0 & 0 \\ 0 & 0 & 0.05^2 & 0 \\ 0 & 0 & 0 & 0.15^2 \end{bmatrix}$$

Now we use equation (1) to solve for the weight,

$$\begin{aligned} w^{MVE} &= k\hat{\Omega}^{-1}\bar{R}^e \\ &= \begin{bmatrix} 0.39 \\ -0.26 \\ 0.78 \\ 0.87 \end{bmatrix} \end{aligned}$$

The first three would be the weights for the three assets. We then back out the weight for the market

$$w_m = -0.9 \times 0.39 + 0.26 \times 1.2 - 0.78 + 0.87 = 0.051$$

(b) Give the expected excess return, standard deviation, and Sharpe ratio of this portfolio. When you are evaluating variance and covariances, recall that the variance of each asset includes a systemic component (relative to β_i) in addition to the residual covariance matrix given above.

Suggested solution:

The expected return is

$$\mathbb{E}(R) = \left(w^{MVE}\right)' \bar{R}^e = 0.055$$

And by construction, the standard error is 15% and the Sharpe ratio is 0.38.

6. Next, you run Fama-MacBeth regressions of the three asset returns on their market betas and an intercept.

   (a) Give the mean return, standard deviation, and Sharpe ratio of the factor-mimicking portfolio that is implied by the regression.

   Suggested Solution:
   We run the Fama-MacBeth as

$$R_i^e = \lambda_0 + \lambda_1 \beta_i + \epsilon$$

   We know that

$$\begin{aligned}
\lambda_1 &= \frac{1}{N} \frac{\beta_i - \mathbb{E}(\beta_i)}{var(\beta_i)} R^e \\
&= \frac{1}{3} \times \frac{\begin{bmatrix} 0.9 - 1.03 & 1.2 - 1.03 & 1.0 - 1.03 \end{bmatrix}}{0.016} \begin{bmatrix} 0.055 \\ 0.045 \\ 0.055 \end{bmatrix} \\
&= -0.036
\end{aligned}$$

   Then for the factor mimicking portfolio, we have that the return should be -0.036. The weight for the factor would just be

$$w = \frac{1}{N} \frac{\beta_i - \mathbb{E}(\beta_i)}{var(\beta_i)}$$

   To show that this is the weight for the factor mimicking portfolio, we calculate the portfolio beta as

$$\begin{aligned}
\beta &= \frac{1}{3}\frac{(\beta_1 - \mathbb{E}(\beta_i))\beta_1}{var(\beta_i)} + \frac{1}{3}\frac{(\beta_2 - \mathbb{E}(\beta_i))\beta_2}{var(\beta_i)} + \frac{1}{3}\frac{(\beta_3 - \mathbb{E}(\beta_i))\beta_3}{var(\beta_i)} \\
&= \frac{\frac{1}{3}(\beta_1^2 + \beta_2^2 + \beta_3^2) - \frac{1}{3}\mathbb{E}(\beta_i)(\beta_1 + \beta_2 + \beta_3)}{var(\beta_i)} \\
&= \frac{\mathbb{E}(\beta_i^2) - (\mathbb{E}\beta_i)^2}{var(\beta_i)} \\
&= 1
\end{aligned}$$

Thus we have the weight to be

$$w = \begin{bmatrix} -\frac{20}{7} & \frac{25}{7} & -\frac{5}{7} \end{bmatrix}$$

To get the variance of this portfolio, we first calculate the covariance of these three assets:

$$\begin{aligned}
cov(R_{1t}^e, R_{2t}^e) &= 0.9 \times 1.2 \times var_M = 0.024 \\
cov(R_{1t}^e, R_{3t}^e) &= 0.9 \times var_M = 0.020 \\
cov(R_{2t}^e, R_{3t}^e) &= 1.2 \times var_M = 0.027
\end{aligned}$$

Then we can get the var-cov matrix as

$$\Sigma = \begin{bmatrix} 0.028 & 0.024 & 0.020 \\ 0.024 & 0.055 & 0.027 \\ 0.020 & 0.027 & 0.025 \end{bmatrix}$$

Now we have that

$$\begin{aligned}
var &= w'\Sigma w \\
&= 0.40
\end{aligned}$$

Thus the s.d. is 0.63 and the Sharpe ratio is -0.06.

(b) What is the correlation between this factor-mimicking portfolio return and the market portfolio?

Suggested Solution:

Let fmp denote the factor mimicking portfolio, we have that

$$
\begin{aligned}
corr &= \frac{cov(m, fmp)}{sd(m)sd(fmp)} \\
&= \frac{\mathbb{E}(-\frac{20}{7} \times 0.01 - \frac{25}{7} \times 0.015 - \frac{5}{7} \times 0.005 + R^e_{mt} - 0.036)(R^e_{mt} - 0.05)}{0.15 \times 0.63} \\
&= \frac{\mathbb{E}(R^e_{mt} - 0.12)(R^e_{mt} - 0.05)}{0.0945} \\
&= \frac{var(R^e_{mt}) + (\mathbb{E}R^e_{mt})^2 - 0.17\mathbb{E}R^e_{mt} + 0.12 \times 0.05}{0.0945} \\
&= 0.238
\end{aligned}
$$

(c) Do a PCA on the three assets. What is the variance of each of the three PCs, relative to the sum of their variances?

Suggested Solution:

To do the PCA, Then we can do the PCA using the eigenvector of the var-cov matrix,

$$\Sigma e_i = \lambda_i e_i$$

We have that

$$\lambda = \begin{bmatrix} 0.087 & 0.015 & 0.006 \end{bmatrix}$$

Thus the proportion that is explained by each of the PC is

$$\begin{bmatrix} 80.62\% & 14.05\% & 5.33\% \end{bmatrix}$$

(d) What are the portfolio weights of each PC?

Suggested Solution:

The portfolio weights of each PC would be the eigenvectors:

$$e_1 = \begin{bmatrix} 0.465 \\ 0.747 \\ 0.475 \end{bmatrix}; \ e_2 = \begin{bmatrix} 0.706 \\ -0.636 \\ 0.311 \end{bmatrix}; \ e_3 = \begin{bmatrix} -0.534 \\ -0.191 \\ 0.823 \end{bmatrix}$$

(e) Given (c), and (d), why is it that the correlation between the factor-mimicking portfolio and the market portfolio is not 1?

Suggested Solution:

The reason is that the factor mimicking portfolio weights are very different from the first PC, but more like the negative of the second PC. Thus, estimated betas correlate with loadings on a factor that is orthogonal to the market factor. The return to the factor-mimicking portfolio, then, are contaminated by this other factor. This is why the correlation between the beta-sorted portfolio and the market factor is low.