Privacy-Preserving Statistical Learning and Testing

Huanyu Zhang

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Table of Contents

Introduction and Motivation

Old Problems, New Challenges

Classical statistical learning and testing problem:

- Distribution learning
  - Estimating the bias of a coin
- Hypothesis testing
  - Testing whether a coin is fair
- Property estimation
  - Estimating the Shannon entropy

Small domain, many samples, asymptotic analysis

The Era of Big Data

2.5 quintillion(2.5×10^{18}) bytes of data are generated everyday[1].

Huge success for ML and statistics, but new challenges.

[1] Data Never sleeps 6.0 by Domo, 2018

Modern Challenges

Large domain, small sample

- Distributions over large domains/high dimensions
- Expensive data
- **Sample complexity**

Privacy

- Samples contain **sensitive information**
- Perform testing or learning while **preserving privacy**

Privacy

Data may contain **sensitive** information.

Medical studies:

- Learn behavior of genetic mutations
- Contains health records or disease history

Navigation:

- Suggests routes based on aggregate positions of individuals
- Position information indicates users' residence

Private Inference

We want to explore **privacy-sample complexity tradeoff**.

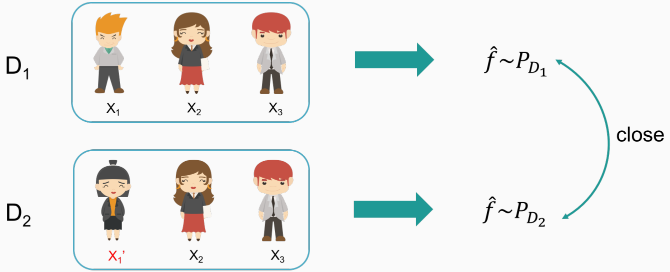
- Sample complexity of non-private algorithm
- Additional cost due to privacy

Question: Is privacy expensive, cheap or even free?

Differential Privacy (DP) [Dwork et al., 2006]

$\hat{f}$ is ε-DP for any X^n and Y^n, with $d_{Ham}(X^n, Y^n) \leq 1$, for all measurable S,

$$\frac{\Pr\left(\hat{f}(X^n) \in S\right)}{\Pr\left(\hat{f}(Y^n) \in S\right)} \leq e^{\varepsilon}.$$



DP is widely adopted by the industry, e.g., Microsoft, and Google.

From Non-private Algorithm to Private Algorithm

Sensitivity. The *sensitivity* of a non-private estimator f is

$$\Delta_{n,f} := \max_{d_{Ham}(X^n, Y^n) \leq 1} \left| f(X^n) - f(Y^n) \right|.$$

Laplace Mechanism [Dwork et al., 2006]:

- Design a non-private estimator with **low sensitivity**
- Privatize this estimator by adding Laplace noise $X \sim Lap(\Delta_{n,f}/\varepsilon)$

Our Results

This talk will contain the following two works:

- Jayadev Acharya, Ziteng Sun, **Huanyu Zhang**, Differentially Private Testing of Identity and Closeness of Discrete Distributions, Spotlight presentation at NeurIPS 2018.
- Jayadev Acharya, Gautam Kamath, Ziteng Sun, **Huanyu Zhang**, INSPECTRE: Privately Estimating the Unseen, ICML 2018.

Differentially Private Identity Testing

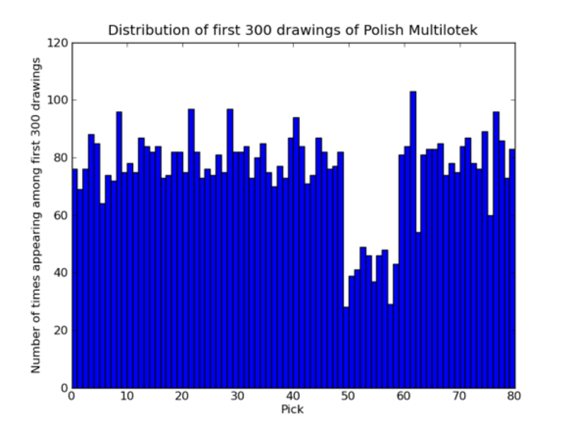
Motivating Example

Polish lottery Multilotek

- Choose “uniformly” at random distinct 20 numbers out of 1 to 80.
- Is the lottery fair?

Motivating Example

No! Probability of $50 - 59$ too small!



The plot credits to "Statistics vs Big Data" by Constantinos Daskalakis.

Identity Testing (IT), Goodness of Fit

- $[k] := \{0, 1, 2, ..., k-1\}$
- q : a **known** distribution
- Given $X^n := X_1 \ldots X_n$ independent samples from **unknown** p
- **Is** $p = q$**?**
- Tester: $\mathcal{A} : [k]^n \rightarrow \{0, 1\}$, which satisfies the following:

With probability at least 2/3,

$$\mathcal{A}(X^n) = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } |p - q|_{TV} > \alpha \end{cases}$$

- **Sample complexity:** Smallest n where such a tester exists

Previous Results

Non-private:

$S(IT) = \Theta\left(\frac{\sqrt{k}}{\alpha^2}\right)$ [Paninski, 2008]

- Lower bound intuition: **Birthday Paradox**

ε-**DP algorithms:** $S(IT, \varepsilon) = O\left(\frac{\sqrt{k}}{\alpha^2} + \frac{\sqrt{k \log k}}{\alpha^{3/2}\varepsilon}\right)$ [Cai et al., 2017]

Problem: based on a χ^2-test, which has **high sensitivity**.

Our Results

Theorem

$$S(IT,\varepsilon) = \Theta\left(\frac{\sqrt{k}}{\alpha^2} + \max\left\{\frac{k^{1/2}}{\alpha\varepsilon^{1/2}}, \frac{k^{1/3}}{\alpha^{4/3}\varepsilon^{2/3}}, \frac{1}{\alpha\varepsilon}\right\}\right).$$

Our Results

Theorem

$$S(IT, \varepsilon) = \Theta\left(\frac{\sqrt{k}}{\alpha^2} + \max\left\{\frac{k^{1/2}}{\alpha\varepsilon^{1/2}}, \frac{k^{1/3}}{\alpha^{4/3}\varepsilon^{2/3}}, \frac{1}{\alpha\varepsilon}\right\}\right).$$

- When $\varepsilon \to \infty$, $S(\mathrm{IT}, \varepsilon) = \Theta\left(\frac{\sqrt{k}}{\alpha^2}\right)$.
- When k is large, $S(\mathrm{IT}, \varepsilon) = \Theta\left(\frac{\sqrt{k}}{\alpha^2} + \frac{k^{1/2}}{\alpha\varepsilon^{1/2}}\right)$, which is strictly better than the previous result!

Our Results

Theorem

$$S(IT, \varepsilon) = \Theta\left(\frac{\sqrt{k}}{\alpha^2} + \max\left\{\frac{k^{1/2}}{\alpha\varepsilon^{1/2}}, \frac{k^{1/3}}{\alpha^{4/3}\varepsilon^{2/3}}, \frac{1}{\alpha\varepsilon}\right\}\right).$$

New algorithms for achieving upper bounds

New methodology to prove lower bounds for hypothesis testing

Reduction from Identity Testing to Uniformity Testing

Uniformity Testing (UT): Identity testing when q is a uniform distribution over $[k]$.

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Reduction from Identity Testing to Uniformity Testing

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We proved this also hold for the private case: Up to constant factors,

$$S(IT, \varepsilon) = S(UT, \varepsilon)$$

It would be sufficient to only consider uniformity testing.

Warm Up - Binary Case (Non-private)

Let $q = B(0.5)$, $p = B(b)$. Test whether $b = 0.5$ or α away.

Algorithm (hard threshold):

1. Let $M_1(X^n)$ be the number of 1's in the samples,
2. If $\frac{1}{n}\left|M_1(X^n) - \frac{n}{2}\right| \leq \frac{\alpha}{2}$, **output** $b = 0.5$,
3. Else, **output** $b \neq 0.5$.

Analysis:

- **Expectation Gap**:

$$\mathbb{E}_{X^n \sim B(0.5+\alpha)}\left[M_1(X^n)\right] - \mathbb{E}_{X^n \sim B(0.5)}\left[M_1(X^n)\right] \geq \alpha n.$$

- **Variance** of $M_1(X^n)$: $\mathrm{Var}\left(M_1(X^n)\right) = O(n)$.
- By Chebyshev's inequality, the sample complexity is $O\left(\frac{1}{\alpha^2}\right)$.

Warm Up - Binary Case (Private)

Let $q = B(0.5)$, $p = B(b)$. Test whether $b = 0.5$ or α away.

Algorithm (soft threshold):

1. Let $Z(X^n) = M_1(X^n) - \frac{n}{2}$,
2. Generate $Y \sim B(\sigma\big(\varepsilon \cdot (|Z(X^n)| - \frac{\alpha n}{2})\big))$, σ sigmoid function,
3. If $Y = 0$, **output** $b = 0.5$,
4. Else, **output** $b \neq 0.5$.

Algorithm Analysis

Lemma

The Algorithm is ε-DP. It has error probability at most 0.1, with $O\left(\frac{1}{\alpha^2} + \frac{1}{\alpha\varepsilon}\right)$ samples.

Reminder: $Y \sim B(\sigma\left(\varepsilon \cdot \left(\left|M_1(X^n) - \frac{n}{2}\right| - \frac{\alpha n}{2}\right)\right))$

Proof idea:

- Privacy: For all $x, \gamma \in \mathbb{R}$, $\exp(-|\gamma|) \leq \frac{\sigma(x+\gamma)}{\sigma(x)} \leq \exp(|\gamma|)$.
- Sample complexity :
  1. Consider the case when $b = 0.5$,
  2. $Z(X^n) = O\left(\sqrt{n}\right)$ with high probability (**Chebyshev**),
  3. Given $n = O\left(\frac{1}{\alpha^2}\right)$, $\frac{\alpha n}{2} - |Z(X^n)| = O(\alpha n)$,
  4. Given $n = O\left(\frac{1}{\alpha\varepsilon}\right)$, $\varepsilon\left(|Z(X^n)| - \frac{\alpha n}{2}\right) < -1000$.
  5. Similar argument works for the case when $|b - 0.5| > \alpha$.

Upper Bound - General Case

Idea: Privatizing the statistic used by [Diakonikolas et al., 2017].

Let M_x be the number of samples of x,

$$S(X^n) := \frac{1}{2} \cdot \sum_{x=1}^{k} \left| \frac{M_x(X^n)}{n} - \frac{1}{k} \right|.$$

- **Sample optimal** in the non-private case.
- This statistic also has a **small sensitivity**!

Upper Bound - General Case

$S(X^n)$ has the following two properties:

- **Expectation gap [Diakonikolas et al., 2017]:**

  let $\mu(p) = \mathbb{E}_{X^n \sim p}[S(X^n)]$, if $d_{TV}(u[k], p) > \alpha$,

  $$\mu(p) - \mu(u[k]) \geq c\alpha^2 \min\left\{\frac{n^2}{k^2}, \sqrt{\frac{n}{k}}, \frac{1}{\alpha}\right\}.$$

- **Small sensitivity:**

  $\forall X^n,\ Y^n$ with $d_{Ham}(X^n, Y^n) \leq 1$, we have:

  $$|S(X^n) - S(Y^n)| \leq \min\left(\frac{1}{n}, \frac{1}{k}\right).$$

Upper Bound - General Case

Algorithm 1: Private Uniformity Testing

Input: ε, α, i.i.d. samples X^n from p

Let $Z(X^n)$ be defined as follows:

$$Z(X^n) := \begin{cases} k\Big(S(X^n) - \mu(u[k]) - \frac{1}{2}c\alpha^2 \cdot \frac{n^2}{k^2}\Big), & \text{when } n \le k, \\ n\big(S(X^n) - \mu(u[k]) - \frac{1}{2}c\alpha^2 \cdot \sqrt{\frac{n}{k}}\big), & \text{when } k < n \le \frac{k}{\alpha^2}, \\ n\big(S(X^n) - \mu(u[k]) - \frac{1}{2}c\alpha\big), & \text{when } n \ge \frac{k}{\alpha^2}. \end{cases}$$

Generate $Y \sim B(\sigma(\varepsilon \cdot Z(X^n)))$, σ is the sigmoid function.

if $Y = 0$, **return** $p = u[k]$, **else return** $p \neq u[k]$

Similar analysis also works here!

Lower Bound - Coupling Lemma

Lemma

Suppose there is a coupling between p and q over $\mathcal{X}^n$ (not necessarily i.i.d.), such that $\mathbb{E}\left[d_{Ham}(X^n, Y^n)\right] \leq D$.

Then, any ε-differentially private hypothesis testing algorithm satisfies

$$\varepsilon = \Omega\left(\frac{1}{D}\right).$$

Lower Bound - Binary Case

For any distribution p_1 and p_2 over $\mathcal{X}$ with $d_{TV}(p_1, p_2) = \alpha$, if we draw n samples i.i.d., there exists coupling with **expected Hamming distance** $O(\alpha n)$. Then we have $n = \Omega\left(\frac{1}{\alpha\varepsilon}\right)$.

If we take $p_1 = B(0.5)$ and $p_2 = B(0.5 + \alpha)$, we get the exact lower bound for binary case.

Problem: This bound doesn't contain any dependency on k!

Lemma

Suppose there is a coupling between p and q over $\mathcal{X}^n$ (not necessarily i.i.d.), such that $\mathbb{E}\left[d_{Ham}(X^n, Y^n)\right] \leq D$.

Then, any ε-differentially private hypothesis testing algorithm satisfies

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Use LeCam's two-point method.
Construct two hypotheses and a coupling between them with small expected Hamming distance.

Lower Bound - Proof Sketch

- Design the following hypothesis testing problem,
  q: draw n i.i.d. samples from $u[k]$.
  p: a mixture of distributions:
  1. generate the set of $2^{k/2}$ distributions, where for each $\mathbf{z} \in \{\pm 1\}^{k/2}$,
  $$p_{\mathbf{z}}(2i-1) = \frac{1 + \mathbf{z}_i \cdot 2\alpha}{k}, \text{ and } p_{\mathbf{z}}(2i) = \frac{1 - \mathbf{z}_i \cdot 2\alpha}{k}.$$
  2. uniformly pick up one distribution, and generate n i.i.d. samples according to it.
- Bound the coupling distance of uniform to mixture,
$$\mathbb{E}\left[d_{Ham}(X^n, Y^n)\right] \leq C \cdot \alpha^2 \min\left\{\frac{n^2}{k}, \frac{n^{3/2}}{k^{1/2}}\right\}.$$
- Prove a lower bound by our coupling theorem.

Some Intuition when Sparse

- Consider the following two distribution:
  1. $p_1 = B(0.5)$,
  2. p_2 is a uniform mixture of $B(\frac{1}{2} - \alpha)$ and $B(\frac{1}{2} + \alpha)$.
- If we draw $(t \geq 2)$ samples, $d_{TV}(p_1, p_2) \leq 2t\alpha^2$ and the expected hamming distance is bounded by $2t^2\alpha^2$.
- Now we consider the coupling between p and q, for every pair of symbols, roughly appear $2n/k$ times in total.
- Therefore, the total coupling distance is $\frac{k}{2} \cdot \frac{4n^2\alpha^2}{k^2} = O\left(\frac{n^2\alpha^2}{k}\right)$.

Closeness Testing (CT), Two Sample Test

- $[k] = \{0, 1, 2, ..., k-1\}$ is a discrete set of size k.
- p, q two **unknown** distributions over $[k]$.
- $X^n = (X_1, X_2, .., X_n)$: n independent samples from p.
- $Y^n = (Y_1, Y_2, .., Y_n)$: n independent samples from q.
- Tester: $\mathcal{A} : [k]^n \times [k]^n \to \{0, 1\}$, which satisfies the following:

With probability at least 2/3,

$$\mathcal{A}(X^n, Y^n) = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } |p - q|_{TV} > \alpha \end{cases}$$

Closeness Testing (CT), Two Sample Test

- $[k] = \{0, 1, 2, ..., k-1\}$ is a discrete set of size k.
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- Tester: $\mathcal{A} : [k]^n \times [k]^n \to \{0, 1\}$, which satisfies the following:

With probability at least 2/3,

$$\mathcal{A}(X^n, Y^n) = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } |p - q|_{TV} > \alpha \end{cases}$$

$$S(CT) = \Theta\left(k^{2/3}/\alpha^{4/3} + \sqrt{k}/\alpha^2\right) \text{ [Chan et al., 2014]}$$

Our Results

Theorem

$$S(CT, \varepsilon) = O\left(\max\left\{\frac{k^{2/3}}{\alpha^{4/3}} + \frac{\sqrt{k}}{\alpha\sqrt{\varepsilon}}, \frac{\sqrt{k}}{\alpha^2} + \frac{1}{\alpha^2\varepsilon}\right\}\right).$$

- When $\varepsilon \to \infty$, $S(\mathrm{CT}, \varepsilon) = O\left(\frac{k^{2/3}}{\alpha^{4/3}} + \frac{\sqrt{k}}{\alpha^2}\right)$.
- When k is large, $S(\mathrm{CT}, \varepsilon) = \Theta\left(\frac{k^{2/3}}{\alpha^{4/3}} + \frac{\sqrt{k}}{\alpha\sqrt{\varepsilon}}\right)$.

Conclusion

- We establish a general coupling method to prove lower bounds in DP.
- We derive the optimal sample complexity of DP identity testing for all parameter ranges.
- We also give the sample complexity of DP closeness testing, which is optimal in sparse case.

This work was accepted as spotlight presentation at NeurIPS 2018.

Differentially Private Property Estimation

Property Estimation

- p: unknown discrete distribution
- $f(p)$: some property of distribution, e.g. entropy
- α: accuracy
- **Input:** i.i.d. samples X^n from p
- **Output** $\hat{f} : X^n \to \mathbb{R}$ such that w.p. at least 2/3:

$$\left|\hat{f}(X^n) - f(p)\right| < \alpha.$$

- *Sample complexity:* least n to estimate $f(p)$

Private property estimation

Given i.i.d. samples from distribution p, the goals are:

- *Accuracy*: estimate $f(p)$ up to $\pm\alpha$ with probability $> \frac{2}{3}$
- *Privacy*: estimator must satisfy ε-DP

Private property estimation

Properties of interest:

- **Entropy**, $H(p)$: the Shannon entropy
- **Support Coverage**, $S_m(p)$: expected number of distinct symbols in m draws from p
- **Support Size**, $S(p)$: $\#$ symbols with non-zero probability

Support Coverage - Motivating Example

- Corbett collected butterflies in Malaya for 1 year.

1	2	3	4	5	6	7	...
118	74	44	24	29	22	20	...

- Number of seen species $= 118 + 74 + 44 + 24 + \dots$

How many new species can be found next year?

Main results

The cost of privacy in private property estimation is often **negligible**.

Main results

Theorem 1. Sample complexity of support coverage:

$$O\left(\frac{m\log(1/\alpha)}{\log m} + \frac{m\log(1/\alpha)}{\log(2+\varepsilon m)}\right).$$

Furthermore,

$$C(S_m, \alpha, \varepsilon) = \Omega\left(\frac{m\log(1/\alpha)}{\log m} + \frac{1}{\alpha\varepsilon}\right).$$

Privacy is free unless $\varepsilon < \frac{1}{\sqrt{m}}$. Similar bounds hold for other properties.

Laplace mechanism

Sensitivity. The *sensitivity* of an estimator f is

$$\Delta_{n,f} := \max_{d_{Ham}(X^n, Y^n) \leq 1} |f(X^n) - f(Y^n)| .$$

Our algorithms use *Laplace Mechanism* [Dwork et al., 2006].

- Compute a non-private estimator with **low sensitivity** [Acharya et al., 2017]
- Privatize this estimator by adding Laplace noise $X \sim Lap(\Delta_{n,f}/\varepsilon)$

Laplace mechanism (support coverage)

We borrow the following non-private estimator (SGT) [Orlitsky et al., 2016] with **low sensitivity**:

$$\hat{S}_m(X^n) = \Sigma_{i=1}^{n} \Phi_i \big(1 + (-t)^i \cdot \Pr(Z \geq i)\big),$$

where Φ is the profile of X^n, $Z \sim \text{Poi}(r)$ and $t = (m-n)/n$.

Lemma 1. When $t \geq 1$, the sensitivity of the estimator satisfies

$$\Delta\left(\frac{\hat{S}_m(X^n)}{m}\right) \leq \frac{2}{m} \cdot \Big(1 + e^{r(t-1)}\Big).$$

Lower Bound - Coupling Lemma

Lemma

Suppose there is a coupling between p and q over $\mathcal{X}^n$*, such that*

$$\mathbb{E}\left[d_{Ham}(X^n, Y^n)\right] \leq D$$

Then, any ε*-differentially private hypothesis testing algorithm must satisfy*

$$\varepsilon = \Omega\left(\frac{1}{D}\right)$$

Support Coverage - Lower bound

Consider the following two distributions:

- u_1 is uniform over $[m(1+\alpha)]$.
- u_2 is distributed over $m+1$ elements $[m] \cup \{\triangle\}$ where $u_2[i] = \frac{1}{m(1+\alpha)}, \forall i \in [m]$ and $u_2[\triangle] = \frac{\alpha}{1+\alpha}$.

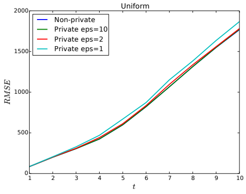
We know

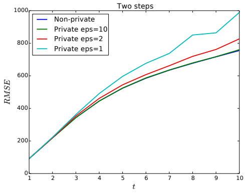
$$S_m(u_1) - S_m(u_2) = \Omega(\alpha m).$$

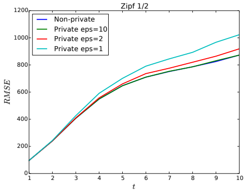
Moreover, their total variation distance is $\frac{\alpha}{1+\alpha}$. So the coupling distance is $\frac{m\alpha}{1+\alpha}$.

Support coverage estimation on synthetic data

- Given $n = 10000$ samples, then estimate the support coverage at $m = n \cdot t$, $t = 1, 2, \ldots$
- Comparison on performance (RMSE) of private and non-private estimator.

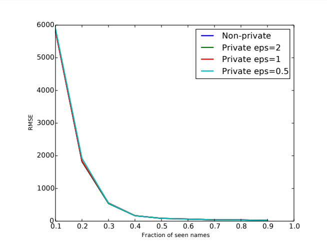


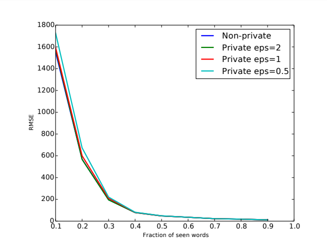




Support coverage estimation on real data

- Comparison on performance (RMSE) of private and non-private estimator
- The dataset: 2000 US Census data, and Hamlet





Conclusion

1. Our upper bounds show that the cost of privacy in these settings is often **negligible** compared to the non-private statistical task.
2. We derive lower bound for these problems by reducing them into binary hypothesis testing.
3. Our methods are realizable in practice, and we demonstrate their effectiveness on several synthetic and real-data examples.

This work was accepted by ICML 2018.

Future Work

Private Discrete Distribution Learning

- $[k] := \{0, 1, 2, ..., k-1\}$
- Distribution: $p = (p_1, ..., p_k)$
- Simplex in $\mathbb{R}^k$: $\Delta_k = \{(p_1, ..., p_k)\}$
- Given $X^n := X_1 \dots X_n$ independent samples from **unknown** p
- Estimator: $\mathcal{A} : [k]^n \to \mathbb{R}^k$, which satisfies the following:

$$\boxed{\text{With probability at least } 2/3,\ d_{TV}(\mathcal{A}(X^n), p) < \alpha.}$$

Sample complexity: Smallest n where such a estimator exists

Private Discrete Distribution Learning

Non-private: $S(DL) = \Theta\left(\frac{k}{\alpha^2}\right)$ (folklore).

- Lower bound intuition: reducing multiclass classification problem to learning problem
- Fano's inequality

ε-**DP algorithms:** $S(DL, \varepsilon) = O\left(\frac{k}{\alpha^2} + \frac{k}{\alpha\varepsilon}\right)$ [Diakonikolas, 2016].

Private Discrete Distribution Learning

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ε**-DP algorithms:** $S(DL, \varepsilon) = O\big(\frac{k}{\alpha^2} + \frac{k}{\alpha\varepsilon}\big)$ [Diakonikolas, 2016].

What is the sample complexity of private distribution learning?

Private Product Distribution Learning

- **Unknown** product distribution p over $\{0,1\}^d$
- Given $X^n := X_1 \ldots X_n$ independent samples from p
- Output product distribution $\hat{p}$, which satisfies the following:

$$\boxed{\text{With probability at least } 2/3,\ d_{TV}(\hat{p}, p) < \alpha.}$$

Private Product Distribution Learning

Non-private algorithm: $S(PL) = \Theta\left(\frac{d}{\alpha^2}\right)$ (folklore).

ε**-DP algorithms:**

- Laplace Mechanism: $S(DL, \varepsilon) = O\left(\frac{d}{\alpha^2} + \frac{d^{1.5}}{\alpha\varepsilon}\right)$.
- [Bun et al., 2019]:
  $S(DL, \varepsilon) = O\left(\frac{d}{\alpha^2} + \frac{d \log(d/\alpha)}{\alpha\varepsilon}\right)$, but not efficient.

Question:

- What is the sample complexity of private distribution learning?
- Does there exist any efficient algorithm?

Thank you!

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