The Mathematics of

‘Twelve Days of Christmas’

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December 2014

Abstract

The popular Christmas carol ‘The 12 days of Christmas’ contains 12 verses, each describing the gifts given by the singers’ ’True Love’. The style of the song is cumulative, where in each verse, another gift is added and the frequency of the present also increases by 1. The following article explores the maths behind the song, including calculating the probability of receiving each gift individually.

Figure 1: An illustration showing the lyrics of the song.

1 Introduction

The mathematics behind the 12 days of Christmas is quite a basic concept, and there are others who have studied the problem mathematically [1]. But it is possible to develop the idea in more complex ways, some of which will be introduced below. The following code can be used to simulate the full lyrics of all 12 verses of the song, shown in Figure 1 <http://www.thepinterestproject.com/2011/12/on-first-day-of-christmas.html>.

```
day_number = '''1st 2nd 3rd 4th 5th 6th 7th 8th 9th 10th 11th 12th'''.split()
present_received = '''\
a Partridge in a Pear Tree.
2 Turtle Doves,
3 French Hens,
4 Calling Birds,
5 Gold Rings,
6 Geese A Laying,
7 Swans A Swimming,
8 Maids A Milking,
9 Ladies Dancing,
10 Lords A Leaping,
11 Pipers Piping,
12 Drummers Drumming'''.split('\n')

for n, day in enumerate(day_number, 1):
```

```
p = present_received[0:n][::-1]
print(('\nOn the %s day of Christmas my True Love gave to me,\n' % day) +
      '\n'.join(p[:-1]) + (' and\n' + p[-1] if n != 1
                           else p[-1].capitalize())
      )
```

I wanted to calculate the total number of presents one would receive after 12 days, and then produce a formula for the number of presents after n days. There are 2 ways in which these can be approached; either non-cumulatively, where on each day the total number of presents includes only those given on that particular day, or cumulatively, where the total number of presents includes those received on that day as well as all presents received on previous days.

1.1 Non-Cumulative Approach

$$x = \frac{n \times (n+1)}{2} \tag{1}$$

Equation 1 is the basic formula for this approach where x is the number of presents received that day and n is the 'nth day of Christmas'. So when n = 12, this would give you 78 presents.

1.2 Cumulative Approach

$$X = \frac{n \times (n+1) \times (n+2)}{6} \tag{2}$$

Equation (2) is the basic formula for the cumulative approach where X is the overall number of presents received and n is the 'nth day of Christmas'. When n = 12, you would get 364 presents - almost one for every day of the year!

Figure 2 represents these two equations graphically.

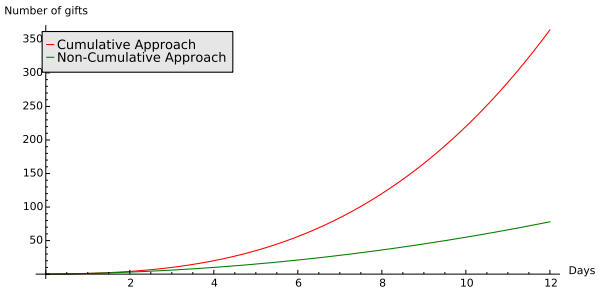

Figure 2: A graph to illustrate the difference between these two approaches.

2 Probabilities

2.1 What is the probability of choosing one particular gift?

Considering the cumulative approach, I can calculate the probability of choosing one particular gift from all the possible gifts given over the 12 days. We know that when n = 12, X = 354. The below code produces the probabilities of getting any of the presents, in numerical order.

```
days = [(e*(13-e)/364) for e in range(1,13)]
print days
```

2.2 What would happen if you didn't always get the expected amount of gifts?

Another interesting way of considering the maths behind the song, was to look at the probability of different events occurring if small details of the song were changed. The code below considers what would happen if we repeated day 3 of the song 200 times, where getting the expected amount had a probability of only 0.5, getting one more with a probability of 0.2 or one less than expected both with probability of 0.3.

```
from __future__ import division
import random
partridges = []
turtle_doves = []
french_hens = []
for day_three in range(200):
    partridges.append(random.choice([0,0,0,1,1,1,1,1,2,2]))
    turtle_doves.append(random.choice([1,1,1,2,2,2,2,2,3,3]))
    french_hens.append(random.choice([2,2,2,3,3,3,3,3,4,4]))
print sorted(partridges)
print sorted(turtle_doves)
print sorted(french_hens)
print 'The mean number of partidges is %s' % (sum(partridges)/len(partridges))
print 'The mean number of turtle doves is %s' % (sum(turtle_doves)/len(turtle_doves))
print 'The mean number of french hens is %s' % (sum(french_hens)/len(french_hens))
```

3 Exploring patterns in the song

Again cumulatively, there are a few parts of mathematics where the patterns for number of presents in the song are very obvious. Below is a matrix representing the numbers from Pascal's Triangle. The diagonal highlighted red contains the 'n' values, i.e the first day, second day etc. The next one down, highlighted in blue, is the non-cumulative amount of presents received on each day, i.e day 2 = 1 partridge + 2 turtle doves = 3. The bottom diagonal in green is the cumulative amount of presents, growing exponentially.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

These numbers are known as the triangle numbers and can be represented graphically using tetrahedrons. This would be a 3D format for the green diagonal in the above matrix.

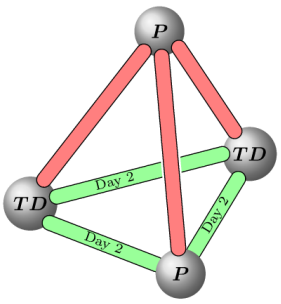

Figure 3: A tetrahedron illustrating the presents received when n = 2, where P = Partidges and TD = Turtle Doves

4 Conclusion

While the basic mathematics of this problem is quite simple, the idea has a lot of scope to become much more complex. If this project were to continue, I could look at involving a 3rd variable or perhaps assigning the relevant date to each of the days of Christmas, starting from December 25th.

Code used to help with this article is referenced below [2] and can also be found at:

- <http://tex.stackexchange.com/questions/22790>
- <http://www.texample.net/media/tikz/examples/TEX/christmas-tree-4.tex>.

A fully commented version of the code can be found at: <https://cloud.sagemath.com/projects/451b017b-124b-41ef-a6e0-dde03f4973ad/files/Codingfor12DaysofChristmas.sagews>

References

[1] Darrah Chavey. Songs and the analysis of algorithms. In *ACM SIGCSE Bulletin*, volume 28, pages 4–8. ACM, 1996.

[2] Wiki. The Twelve Days of Christmas (song) — WikiRosetta, The Free Encyclopedia",, 2014. [Online; accessed 10-December-2014].