Relationship Between Product And Sum

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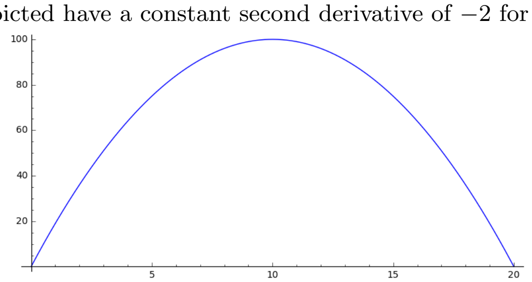
1 Introduction

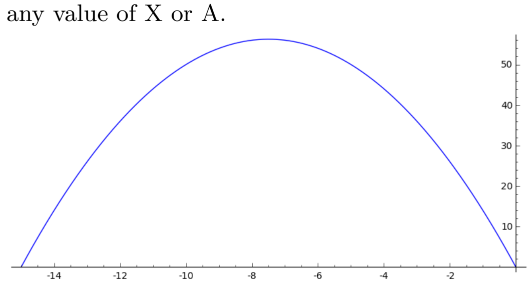
In this report I will examine for a constant sum the relationship between the difference of addends [2] and their product.

2 Two addends

This section will deal with cases where $X + Y = A$ where X and Y are variables and A is a constant. It is clear that in this case Y will be equal to $A - X$. Therefore the difference ($|X - Y|$) will equal $|X - (A - X)| = |2X - A|$. It is also true that the product (XY) equals $X(A - X) = XA - X^2$.

The maximum value of the product can be found by differentiating the product with respect to X. Then you find the value of X when the derivative equals zero. When you differentiate $XA - X^2$ with respect to X, you will always get $A - 2X$. This makes it clear that the value of the maximum product will always occur when $X = A/2$ for any real value of A. This value of X is also when the difference of the addends will equal zero. This also means the maximum value of the product will always be $A^2/4$. We can plot the value of the product of the addends for any value of X. The curves depicted have a constant second derivative of -2 for any value of X or A.





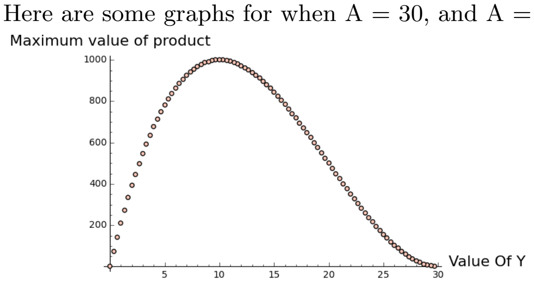
These are graphs of X against the value of the product for when A = 20 and A = -15 . These graphs are visually similar. The graphs are symmetrical as expected since the second derivative is constant. Some interesting conclusions can be gathered in the case when there are two addends such as $f'(0) = A = -f'(A)$ and $f'(A) = 2|A - X|$.

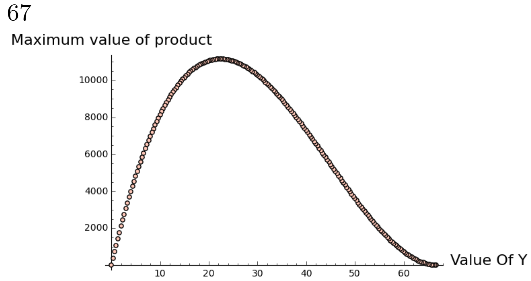
3 Three addends

This section will deal with cases where X + Y + Z = A where Z is also a variable. Here we would compare the product of XYZ and the average difference between X, Y and Z. Since Z equals $A-X-Y$, we can write the product as $XY(A-X-Y) = XYA-YX^2-XY^2$. The average value of the difference would be $(1/3)*(|X-Y|+|Y-Z|+|X-Z|)$ in terms of X, Y and A. Here I will do some coding to find properties of this relationship.

```
def threeaddends(A):
    """
    X + Y + Z equals A. This function finds the maximum value of the product, (XYZ),
    for different values of X, Y and Z. This function also creates a scatter graph of the
    maximum value of the product for different values of Y.
    """
    X = var('X')
    maximumproduct = 0 #  The maximum possible value for the product.
    maxy = 0 #  The value of Y for the maximum product.
    xforbiggestproduct = 0 #  The value of X corresponding to the biggest product value.
    p = scatter_plot([[0, 0]])
    """
    p is the scatter graph. This line introduces p so it can be added to later.
    The horizontal axes represents the value of Y.
    The vertical axes represents the maximum possible value of the product for that value of Y.
    [0, 0] is certainly a point since when Y = 0, the value of the product will always be zero.
    """
    for i in range(1, (3 * A)):
        Y = (1/3 * i) #  This lets me consider all Y values which are a multiple of 1/3.
        f(X) = (X * (A - X - (Y)) * (Y)) #  f(x) equals the function for the product, XYZ.
        g(X) = diff(f(X), X) #  g(x) is the differental of the function for the product.
        gxsolution = solve(g(X) == 0, X)
        """
        The maximum value of the product would happen at the turning point for f(x).
        The solution can be found by finding the value of X for when g(x) = 0.
        """
        xatgreatest = float(gxsolution[0].rhs())
        if f(xatgreatest) > maximumproduct:
               maximumproduct = f(xatgreatest)
               maxy = Y
               xforbiggestproduct = xatgreatest
        p = scatter_plot([[Y, f(xatgreatest)]]) + p #  Add to the scatter plot
    print (" The maximum value of the product is " + str(maximumproduct))
    print ("X equals " + str(xforbiggestproduct) + " and Y equals " + str(maxy))
    print (" The average difference equals " + str(averagedifference))
    p.show() #  Show the completed scatter graph.
```

Here are some graphs for when A = 30, and A = 67





You can see that these graphs are visually similar also. For these two values, the maximum value of the product consistently is achieved when X, Y and Z all equal A/3 and the average difference equals zero. I have tested with many other values of A which I am not including in this report. This property remains true for all real values of A that I have tested. This is similar to the reltionship when there are only two addends.

4 Imaginary Numbers

We have discussed cases in when the addends and the sum are real numbers. Now we are going to briefly explore cases with purely imaginary numbers with two addends. When X, Y and A are all imaginary numbers the relationship of the difference and the product would be similar to the case with real numbers. This is because the product of the addends will always be real since an imaginary number multiplied by another will always equal a real number [1]. The relationship significantly more complicated when A, X or Y have both real and imaginary parts.

However I can show by contradiction for complex numbers that it isn't always true that the maximum value of the product of the addends happens when the difference between the addends is zero [3]. $A = 10$, $X = 5 + 100i$, $Y = 5 - 100i$, difference $= 200i \neq 0$, $product = 10025$. When the difference is zero the product equals 25. $10025 > 25$.

5 Conclusion

Finding from this report suggest that if a real number has two or three real addends, the maximum value of the product of the addends would happen when the difference between the addends is minimised. While I have proved this conjecture in the case where there are two addends, I have not shown that this is also strictly true where there is three addends. I have only demonstrated that it is true for many values of A. Further work could find proof of this property if it exists. I hypothesis that product would always be the maximum when the average difference is minimised for any natural number of addends. I imagine that this would take a considerable amount of work to show, however. Further work can also explore the relationship where X, Y and A are complex numbers in more depth. A comparison between the value of the modulus for particular values related to complex numbers may be a good idea.

References

[1] Tikhon Jelvis. When multiplying an imaginary number by an imaginary number, the result is a real number. what does that mean?, [https://www.quora.com/when-multiplying-an-imaginary-number-by-an-imaginary-number-the-result-is-a-real-number-what-does-that-mean], 2014.

[2] Fact Monster. Terms used in equations, [http://www.factmonster.com/ipka/a0881931.html], 2015.

[3] Susan Schwartz. Properties of complex numbers, [http://www.wildstrom.com/susan/complexnumbersprop.pdf], 2015.