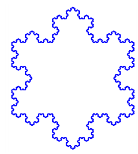
The Finite Within Infinity

K Phillips

December 9, 2015



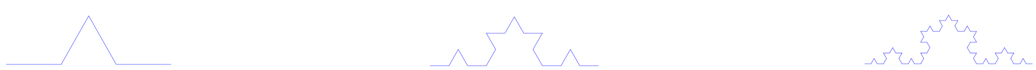
1 Introduction

A fractal can be defined as a geometrical figure which is self-similar across any scale. One of the first fractals to be discovered was the Koch Curve which first appeared in 1904 paper by Helge Von Koch [1]. The basic curve starts with a straight line, but can be used to form a snowflake by having the starting shape be an equilateral triangle. The snowflake is a curve that if unfolded would be an infinite circle, so the perimeter of the Snowflake is infinite. However the area bounded by the line is finite and can be calculated which will be shown in this report.

2 Constructing the Curve

The Koch curve is designed by taking a straight line into thirds, then replacing the middle third of the line with an equilateral triangle and repeating the process for every straight line on the curve such that there are technically no more straight lines as each one is made up of points. The construction of this, is done by first creating a function to divide a line segment into thirds and adding an equilateral triangle to the middle third, then having a second function which uses the previous function on every line and repeats for the desired amount of iterations [2]. Using these two functions the curve can be shown iteration by iteration by adding the code below. [3]

```
for i in range(1, 3):
    p = koch( [ [ (0,0),(1,0) ]], i ).show(axes=False,aspect_ratio=1)
```

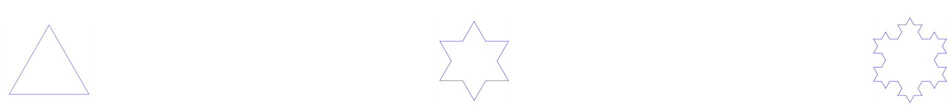


(a) First iteration (b) Second iteration (c) Third iteration

Figure 1: First three iterations of the Koch Curve

The curve is then constructed about an equilateral triangle, forming the snowflake, the first three iterations of which are shown below.

```
for i in range(1, 3):
kochsFractal( [ [ (0, 0), (1/2,sqrt(3/4))], [ (1/2,sqrt(3/4)), (1, 0) ],[ (1,0),(0,0) ]],
    i).show(axes=False,aspect_ratio=1,)
```



(a) Starting Equilateral Triangle (b) First Iteration (c) Second Iteration

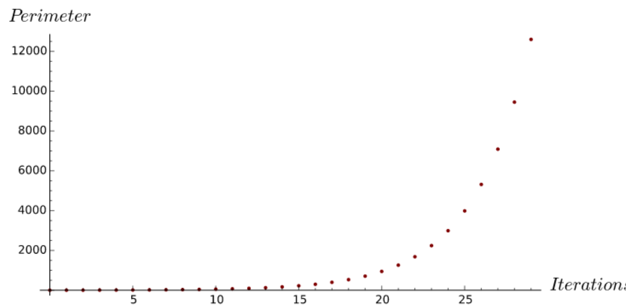
Figure 2: The starting figure and first two iterations of the Koch Snowflake.

3 Infinite Perimeter

If the iterations are repeated an infinite number of times, the fractal is formed. If we start with an equilateral triangle of side length S we break down each side into thirds and subtract the middle third while adding what equates to two thirds the length in the form of a triangle giving the next iteration to be four thirds the previous perimeter. [4]

$$P_{n+1} = (4/3)P_n \tag{1}$$

Using the above equation for the triangle with side length 1, $P_0 = 3$ we get the recursive code:




```
def perimeter(n):
    if n == 0:
        return 3
    return 4/3 * perimeter(n-1)
```

The above code can be used to plot a graph showing the trend of the perimeter as the number of iterations increases. This shows that as $\lim_{n\to\infty}(P_\infty) = \infty$, which is an infinite perimeter.

4 Finite Area

The informal proof of there being finite area inside the curve is to place the curve within a polygon of known finite area and perimeter.As the iterations of the curve increase the area never exceeds that of the polygon and thus must be finite. The are of the polygon can also be a rough approximation of the area within the snowflake. To increase the accuracy of the approximation by bringing the perimeter of the polygon closer and closer to the curve by increasing the number of sides. However a more accurate way to calculate the area would be by calculating the new area added on by each iteration to get an infinite series, the sum of which can be calculated using sage. Again presuming to start with an equilateral triangle of side length S.[5]

- Iteration: 0, Area: $\frac{\sqrt{3}S}{4}$
- Iteration: 1, Area: $\frac{\sqrt{3}S}{4} + 3 \times \frac{\sqrt{3}(\frac{S}{3})^2}{4}$
  - The new area being added on here is three smaller equilateral triangles with base length one third the size of the original.
- Iteration: 2, Area: $\frac{\sqrt{3}S}{4} + 3 \times \frac{\sqrt{3}(\frac{S}{3})^2}{4} + 3 \times 4 \times \frac{\sqrt{3}(\frac{S}{9})^2}{4}$
  - The new area for each side of the original triangle, four smaller triangles one third the base length of the previous triangle.

This pattern repeats for every available side a triangle one third the area is placed on it so the infinite series is geometric.

$$AREA_\infty = \frac{\sqrt{3}S}{4} + 3 \times \frac{\sqrt{3}(\frac{S}{3})^2}{4} + 3 \times 4 \times \frac{\sqrt{3}(\frac{S}{9})^2}{4} + 3 \times 4^2 \times \frac{\sqrt{3}(\frac{S}{27})^2}{4} + \cdots$$

In order to calculate the area, we can take out common factors to get an infinite geometric series the sum of which can be calculated. All the elements in the sum have $\frac{sqrt(3)S^2}{4}$ as a common factor which can be taken out, leaving the equation as:

$$AREA_\infty = \frac{\sqrt{3}S^2}{4}(1 + 3\frac{1}{3}^2 + 3 \times 4\frac{1}{3^2}^2 + 3 \times 4^2\frac{1}{3^3}^2 ...)$$

At which point we can use basic algebra to have a better looking series by swapping the index on the indices round on the $\frac{1}{3}$. Then it would also be simpler to multiply by $\frac{4}{4}$ to take the 4 into the fraction throughout the fraction, so we divide the common factor by 4 to give the following:

$$AREA_\infty = \frac{\sqrt{3}S^2}{16}(4 + 3 \times \frac{4}{9}^1 + 3 \times \frac{4}{9}^2 + 3 \times \frac{4}{9}^3 ...)$$

We can remove a factor of 3 from the second term onward to get a more simple geometric series

$$AREA_\infty = \frac{\sqrt{3}S^2}{16}(4 + 3(\frac{4}{9} + (\frac{1}{9})^2 + (\frac{4}{9})^3 ...))$$

In this calculation we have the geometric series of $\frac{4}{9}^n$ which using sage we can calculate $\sum_{n=1}^{\infty} \frac{4}{9}^n$,

```
import math
f(x) = ((4/9) ^ x)
k = sum(f(x), x, 1, oo)
k
```

This returns $\frac{4}{5}$ which can be put into the equation

$$AREA_{\infty} = \frac{\sqrt{3}S^2}{16}(4 + 3(\frac{4}{5})$$

Using the math library in sage this again can be calculated;

```
sqrt(3) / 16 * ( 4 + 3 * k)
```

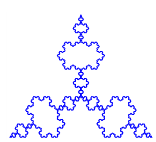
This gives the area finally to be:

$$AREA_{\infty} = \frac{2\sqrt{3}S^2}{5}$$

If we were to use an equilateral triangle of side length 1 and iterate the Koch curve an infinite number of times, we would have a Koch Snowflake of infinite perimeter with a finite area of $\frac{2\sqrt{(3)}}{5}$.

5 Final Comments

Using the informal proof seen in Chapter 4 can be used for other shapes to show a finite area within an infinite perimeter, a mock example of this would be to place a map of Wales within a rectangle as a rough approximation of the area and perimeter, then increase the number of sides of the polygon as you bring it closer to the coastline, to get a closer approximation of the area and a larger approximation of the perimeter. Once you get so close to the coastline, you eventually get such a large perimeter that it tends to infinity, but the area of Wales still remains finite. A less mathematical but still interesting way of seeing an (almost) infinite length bounding a finite area. Another mathematical way of seeing how finite curves interact with space would be to calculate the area within the Anti-Koch Curve as seen in Fig 3 as it would also be a finite area. My suggestion for this would to be to calculate the area iteration by iteration and then subtract the new areas the curve forms from the original triangle.



References

[1] Helge Von Koch. On a continuous curve without tangent constructible from elementary geometry. *Classics on Fractals (Westview Press, 2004) pp*, 25:45, 1993.

[2] 2d graphics. `http://www.norsemathology.org/wiki/index.php?title=2D_Graphics`, Accessed: 2015-12-04.

[3] Koch curve. `https://cloud.sagemath.com/projects/62ee05aa-fe61-4c83-85dc-d611bc6043c1/files/base%20code%20for%20curve.sagews`. Accessed: 2015-12-03.

[4] Perimeter. `https://cloud.sagemath.com/projects/62ee05aa-fe61-4c83-85dc-d611bc6043c1/files/Perimeter.sagews`. Accessed: 2015-12-03.

[5] Area. `https://cloud.sagemath.com/projects/62ee05aa-fe61-4c83-85dc-d611bc6043c1/files/area.sagews`. Accessed: 2015-12-03.