Prime Number Theory and the Distribution of the Primes

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1 What is a Prime?

A prime number is defined to be a positive number greater than 1 that has only two divisors, one and itself [5]. Every single natural number can be written as the product of prime numbers, with each number having its own unique combination of primes that multiply to make it.
This property means that primes have been described by mathematicians as the building blocks of all numbers [4], similar to the atomic elements in the study of chemistry.

2 Prime Distribution and The Prime Counting Function

Prime numbers have been studied by mathematicians for thousands of years. In 300BC Euclid proved in his book, The Elements, that there are infinitely many primes [1].
Over the centuries, as mathematicians tried and failed to create a formula that could produce an unending sequence of primes, they eventually turned their attention to the placement of the prime numbers along the number line. However, no matter how many primes they considered they found no discernible pattern in their positions [2].

Figure 1: All the primes less than 100 [6]

The above image has highlighted the prime numbers less than 100. Using the position of these primes, you can't predict when the next prime (which is 101) will occur.
Consider instead the question, how many primes are there less than a number x? This has been asked so frequently that is has been given its own notation, $\pi(x)$, which has nothing to do with the number π (3.141592...), it is just the notation used.
$\pi(x)$ is the prime counting function; it returns the number of prime numbers less than the search size x.

x	$\pi(x)$	$\pi(x)/x$
100 (10^2)	25	0.2500
10000 (10^4)	1,229	0.1229
1000000 (10^6)	78,498	0.0785
100000000 (10^8)	5,761,455	0.0570
10000000000 (10^{10})	455,052,511	0.0455
1000000000000 (10^{12})	37,607,912,018	0.0377
100000000000000 (10^{14})	3,204,941,750,802	0.0320

Table 1: $\pi(x)$ and $\pi(x)/x$ for chosen values of x

By looking at the last column in the table above, we can see that as we increase the value of x, the proportion of prime numbers, or the chance that a randomly picked number in the interval $[1, x]$ is prime, decreases. However, it never reaches zero because of Euclid's proof that there are infinitely many prime numbers [2].
So, although the primes have no discernible pattern along the natural number line, when we graph the prime counting function we see something very interesting.
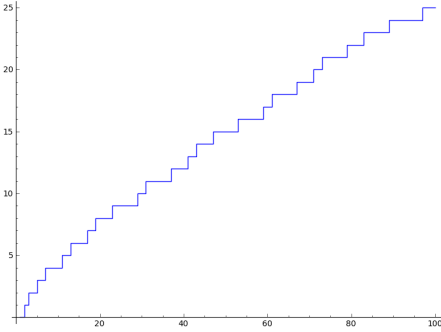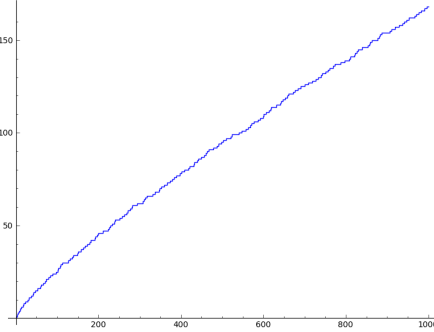

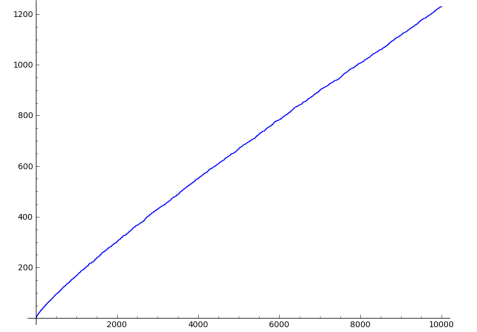Figure 2: $\pi(x)$ up to 100   Figure 3: $\pi(x)$ up to 1000   Figure 4: $\pi(x)$ up to 10000

The X axis plots the value x and the Y axis plots $\pi(x)$ for that value. As you increase the search size, or x, it can be seen that although the graph is locally irregular, there is certainly a trend to its values; something that wasn't expected from a set of numbers distributed so randomly amongst the natural numbers [3].

3 The Prime Number theorem: an approximation for $\pi(x)$

Despite there being no pattern to the distribution of the primes along the number line, the trend in the graph of $\pi(x)$ means we can look for functions that behave in the same way, and so can be used to approximate the value of $\pi(x)$.
The prime number theorem states that the number of primes not exceeding the number x (or $\pi(x)$) is asymptotic to $\frac{x}{log(x)}$ [1]. This would be written as:

$$\pi(x) \sim \frac{x}{log(x)}$$

This means that $\frac{x}{log(x)}$ is a good approximation for $\pi(x)$. The $\sim$ symbol means that:

$$\lim_{x \to \infty} \left(\frac{\pi(x)}{x/log(x)} \right) = 1$$

So, as you increase the value of x to infinity, the difference between $\pi(x)$ and $\frac{x}{log(x)}$ tends to zero, relative to the search size x.
This can be seen by graphing the two functions. Although there appears to be an increasing gap between the two functions, this gap is actually decreasing relative to the search size.
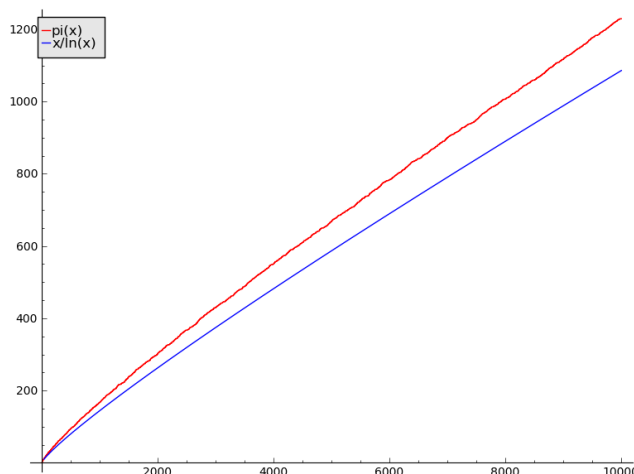
2

Figure 5: A plot of $\pi(x)$ and $\frac{x}{log(x)}$ for x up to 10,000

4   The Riemann Zeta Function

Although the prime number theorem provides a suitable approximation for $\pi(x)$, there is always room for improvement. The Riemann Zeta function, $\zeta(s)$ where s $\in \mathbb{C}$, is a function of complex numbers. Riemann's explicit formula for $\pi(x)$, uses a sum over all the real parts of the zeros of the zeta function to predict the position of the graph of $\pi(x)$. The zeros of the function are the complex numbers that cause the function to take the value of zero [4].

Riemann showed that the more zeros of the function you consider in his equation for $\pi(x)$, the more accurate the approximation becomes, much more so than $\frac{x}{log(x)}$.

Riemann hypothesised that all of the non-trivial zeros (the zeros that aren't easy to show) of the function, all have real value $\frac{1}{2}$ [4]. These are the zeros that are summed to create Riemann's approximation for $\pi(x)$ and also form the Riemann hypothesis, which has yet to be proved true.

This hypothesis is one of the unsolved clay millenium institute problems; it is considered one of the most important unsolved problem in mathematics [4] and has a 1 million pound prize for the first person to solve it.

So, despite the use of $\frac{x}{log(x)}$ as an approximation for $\pi(x)$ being relatively simple to understand, the more exact approximation using the zeta function plays a part in one of the greatest unsolved problems in mathematics.

Appendix

Code used to produce the graphs of $\pi(x)$ in this document are available at http://sage.maths.cf.ac.uk/home/pub/90

References

[1] Encyclopaedia Britannica. Prime number theorem. http://www.britannica.com/EBchecked/topic/476362/prime-number-theorem, 2013.

[2] Chris Caldwell. How many primes are there? http://primes.utm.edu/howmany.shtml, 2013.

[3] Brit Cruise. The prime number theorem. https://www.khanacademy.org/math/applied-math/cryptography/comp-number-theory/v/prime-number-theorem--the-density-of-primes, 2013.

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[5] Paul Glendinning. *Maths in Munutes*. Quercus, 2012.

[6] Mathcounts Notes. Prime numbers, mathcounts beginning level. http://mathcountsnotes.blogspot.co.uk/2012/05/prime-numbers-mathcounts-beginning.html, 2013.