Convergence of sequences

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1 Introduction

In this report, I will look at bounded sequences, laws of limits and how to find the limit of a sequence using Sage and also by using the epsilon definition.

2 Bounded sequences

A sequence $(a_n)_{n=1}^{\infty}$ is bounded above if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. The sequence $(a_n)_{n=1}^{\infty}$ is bounded below if there exists $K \in \mathbb{R}$ such that $a_n \geq K$ for all $n \in \mathbb{N}$.

2.1 Example

The sequence $(a_n) = 2n\ \forall n \in \mathbb{N}$ is bounded below $(K = 2)$ however it is not bounded above therefore it is not bounded.

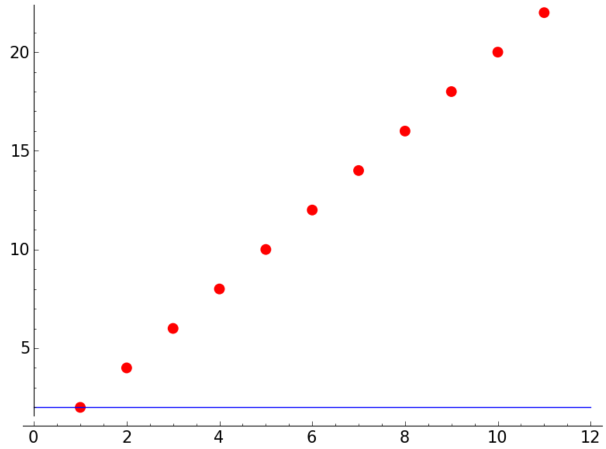


Figure 1: The sequence $(a_n) = 2n$

2.2 Example

If we take the sequence $(a_n) = (-1)^n\ \forall n \in \mathbb{N}$, we can see that it is bounded above $(M = 1)$ and bounded below $(K = -1)$ therefore it is a bounded sequence.

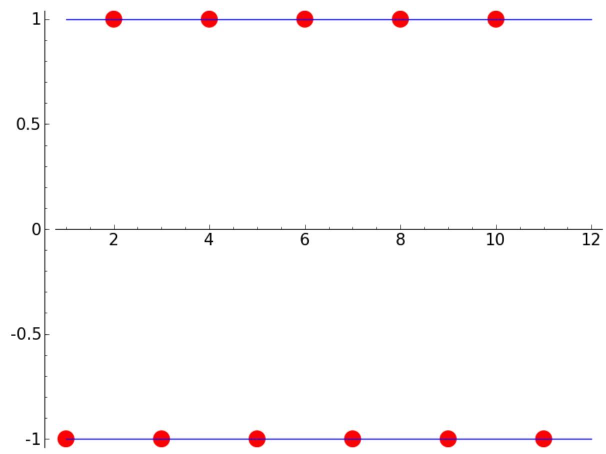


Figure 2: The sequence $(a_n) = (-1)^n$

3 Sequences which tend to a finite limit

3.1 Proving limits using epsilon

A sequence $(a_n)_{n=1}^{\infty}$ is said to tend to a limit of a if the following statement is true ....

For all $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N(\epsilon)$. If this statement is true then we say $\lim_{n\to\infty} a_n = a$. [?].

3.1.1 Example

Let $(a_n)_{n=1}^{\infty} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots)$.

So $a_n = (\frac{1}{2})^n \ \forall n \in \mathbb{N}$

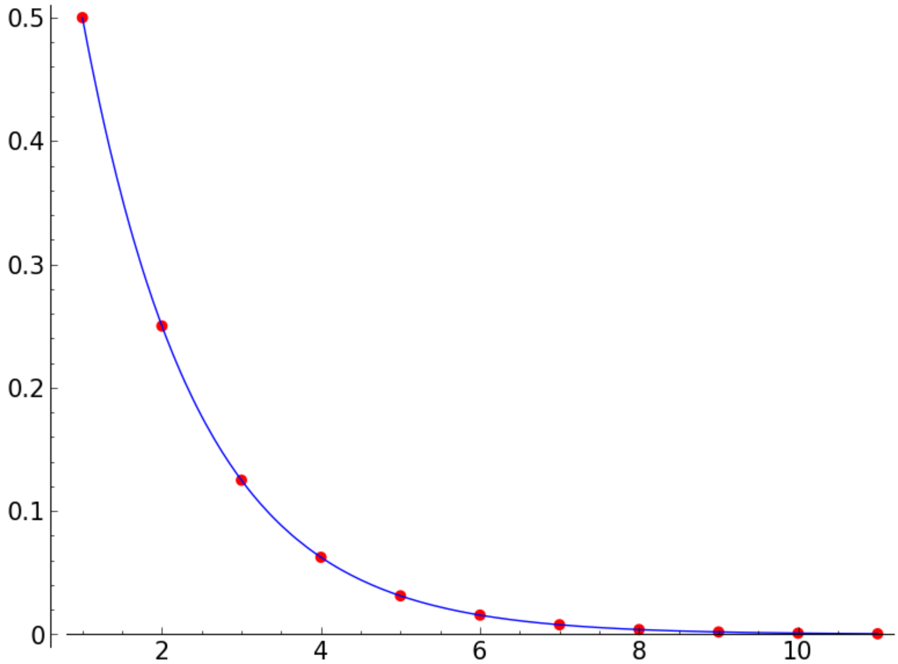


Figure 3: The sequence $(a_n) = (\frac{1}{2})^n$

We can see that this sequence tends to 0 but how can we prove it?

We must show that:
For all $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N(\epsilon)$.
First we start with the inequality $|(\frac{1}{2})^n - 0| < \epsilon$ and rearrange to make n the subject. We get $n > \frac{ln(\epsilon)}{ln(\frac{1}{2})}$.

Given any positive number $\epsilon > 0$ (no matter how small) we can choose $N(\epsilon)$ to be any natural number greater than $\frac{ln(\epsilon)}{ln(\frac{1}{2})}$. Then if $n \geq N(\epsilon)$, we have $|(\frac{1}{2})^n - 0| < \epsilon$ which proves that the sequence tends to 0.

For example ... Let $\epsilon = 0.001$
Then we have $\frac{ln(0.001)}{ln(0.5)} \approx 9.97$ so we can take $N(\epsilon) = 10$.
Now we can check: $|a_n - a| = (\frac{1}{2})^{10} \approx 0.00098$ which is less than $\epsilon = 0.001$ as required.

No matter how small ϵ gets, you can find some point in the sequence where all of the terms from that point onwards will be within the distance of ϵ from the limit value (0 in this case).

As ϵ gets smaller, the horizontal lines get closer together which means we will need a larger value of $N(\epsilon)$ to find a point where all of the remaining terms are within the horizontal lines.

3.2 Using Sage to represent this concept

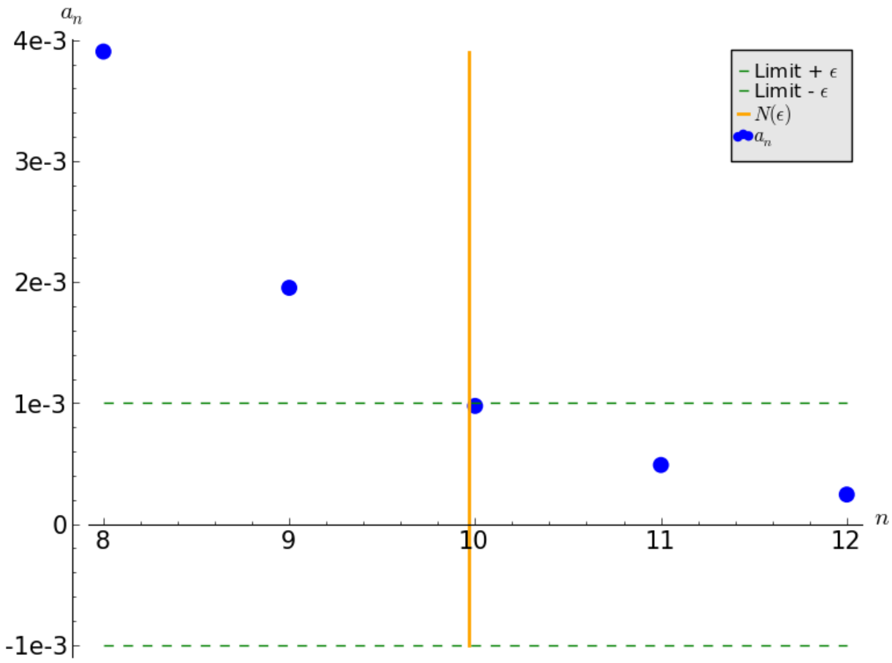
We can represent this concept visually using a function that I have designed on Sage. (A fully commented and more detailed version of the code can be found at `http://sage.maths.cf.ac.uk/home/pub/50`)

```
def Limitprover(f, e, z):
     b = limit(f(x), x=oo)
     epsilon = e
     p = solve(f(x) - b == epsilon, x)
     k = p[0].rhs()
     j = (float(k))
     data = [[r, f(r)] for r in srange(int(j - z + 1), int(j + z + 1) + 1, 1)]
```

```
    p1 = list_plot(data)
    p2 = plot(b + epsilon, x, j - z, j + z)
    p3 = plot(b -epsilon, x, j - z, j + z)
    p4 = line([(j, b - epsilon), (j, f(j - z))])
    return p1 + p2 + p3 + p4
```

We can then use this function to show the result from Example 3.1.1.

```
f(x) = (1/2)^x
Limitprover(f, 0.001, 2)
```



If a term a_n is between the horiztontal green lines then it must satisfy $|a_n - a| < \epsilon$.

Figure 4: A visual representation of Example 3.1.1

References

[1] Terence Tao. *Analysis I: Second Edition*. Hindustan Book Agency, 2009.