Week 8 - Linear Algebra

1. Use Sage to solve the following system of equations:

$$\begin{cases} 10x + 2y = 0 \\ 2x - y = 154 \end{cases}$$

2. **TICKABLE** Note that the above system of equations is equivalent to the following systems of equations:

$$\begin{cases} 10a + 2b = 0 \\ 2a - b = 154 \end{cases}$$

$$\begin{cases} 10m + 2n = 0 \\ 2m - n = 154 \end{cases}$$

   In essense the only thing that defines the system of equations is the cofficients:

$$\begin{pmatrix} 10, 2, 0 \\ 2, -1, 154 \end{pmatrix}$$

   We can of course seperate the right hand side of our equation and perhaps include those elements in a vector. Our system can now be represented as:

$$\begin{pmatrix} 10, 2 \\ 2, -1 \end{pmatrix} \begin{pmatrix} 0 \\ 154 \end{pmatrix}$$

   Let us attempt to represent the above system in Sage.

   The following defines `b` as a vector:

```
b = vector(0,154)
```

   The representation of coefficients is a well defined mathematical object called a `matrix`. The following code defines `A` as a matrix:

```
A = matrix([[10, 2], [2, -1]])
```

   If we define a vector `X` as a vector of the symbolic variables:

```
X = vector([x, y])
```

   We can **multiply** `A` by `X`:

```
A * X
```

Verify that $X = (x_0, y_0)$ where (x_0, y_0) is the solution to our system of equations (obtained in (1)).

3. **TICKABLE** In linear algebra (you will study this next semester) a matrix equation is an equation of the form:

$$AX = b$$

or

$$XA = b$$

If we define `A` and `b` as in question 2 we can solve this equation quite simply using the `solve_right` or `solve_left` methods. The following obtains a solution to the equation $AX = b$:

```
A.solve_right(b)
```

Note that `A\b` is shorthand for `A.solve_right`

Use the above to solve the following system of equations using matrix notation:

$$\begin{cases} 4x - 2y + 3z = 10 \\ -x - 5y - 8z = 9 \\ x + y + z = 1 \end{cases}$$

4. For reasons that will become clear, the following definition of matrix multiplication is required:

$$(AB)_{ij} = \sum_{j'} \sum_{i'} A_{ij'} B_{i'j}$$

For 2×2 matrices this is equivalent to:

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

As an example create the following two matrices in Sage:

```
A = matrix([[1,2],[3,4]])
B = matrix([[7,8],[9,10]])
```

Attempt to multiply these matrices by hand and carry out their multiplication in Sage:

```
A*B
```

Repeat the exercise by multiplying the following pairs of matrices:

1. $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix}$
2. $A = \begin{pmatrix} 0 & 144 \\ -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ -1 & -17 \end{pmatrix}$
4. $A = \begin{pmatrix} 0 & -1 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1/3 & 1/3 \\ -1 & 0 \end{pmatrix}$

5. **TICKABLE** The previous exercise shows that when considering matrix multiplication there exists a matrix which does not have a multiplicative affect: "the identity matrix".

   The identity matrix of size $n \times n$ is denoted by $\mathbb{I}_n$. The following Sage code gives $\mathbb{I}_n$:

```
identify_matrix(4)
```

   Note also, that the previous exercise showed that we can sometimes find a matrix B such that $AB = \mathbb{I}_n$. Finding such a matrix is refered to as 'invering' A and if certain properties hold (you will see this in further details next semester) this matrix is denoted A^{-1}.

   If we recall the matrix equation $AX = b$ and if we assume that A^{-1}exists then multiplying both sides by A^{-1} gives:

$$A^{-1}AX = A^{-1}b \Rightarrow \mathbb{I}_nX = A^{-1}b \Rightarrow X = A^{-1}b$$

   In Sage we can obtain A^{-1} (if it exists) with the following code:

```
A.inverse()
```

   Thus another approach to solving $AX = b$ is:

```
A.inverse()*b
```

   Use this approach to solve the systems of equations we have considered so far.

6. **TICKABLE** Recalling your basic python knowledge. Lists can be used to hold any sort of object. Obtain a list of the inverses of the following matrices (when the inverse exists, you might need to look up information on `try` and `except`):

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & -\frac{1}{2} & 2 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & 0 \\ -1 & 0 & -2 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 1 & 1 \\ -2 & 0 & 1 & 2 & 2 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & -1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

For every matrix in this list and the original list obtain the result of the `det` method. This gives the **determinant** of the matrices. It is a very important quantity that will be explained next semester.

7 **TICKABLE** The `random_matrix` command can be used to obtain a random matrix:

```
random_matrix(ZZ, 5) # Gives a random square matrix of size 5 in $\mathbb{Z}$
random_matrix(QQ, 5) # Gives a random square matrix of size 5 in $\mathbb{Z}$

Using this attempt to conjecture a connection between the determinant of a matrix and it's i
```

8. **TICKABLE** The file W08_D01.txt contains 4 columns of data:

   `a, b, c, d, f, g`

   For each row of data, obtain the solution to the system of equations:

$$\begin{cases} ax + by = c \\ dx + fy = g \end{cases}$$

   Write to file a new data set containing the following columns:

```
A, B, C, D
```

Where A is the number of the original data set, B and C are the solutions to the system of equation in question: $B = x, C = y$. D is the 'norm of the solution vector': $D = \sqrt{C^2 + B^2}$. Write the data to file in such a way as it is sorted by D (attempt to do this without the inbuilt `sort` method).

If there is no solution to the system of equations set `B=C=D=False`. The data set is a randomly sampled set of problems, how often does a solution exist? What is the mean value of D?

9. The file W08_D02.txt contains 20 rows and 20 columns of numbers. This file represents a matrix. Invert the matrix, what is it's determinant? What is the determinant of it's inverse?