A Study Of Polygonal Numbers

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1 Introduction

In this report I will be exploring simple polygonal numbers in an attempt to discover a relationship between sequential polygonals.
A regular polygonal number is a collection of points, starting with one, that can be formed together to create a shape that is convex, equiangular and equilateral, commonly known as a regular polygon. In Figure 1 we can see the simplist polygonal numbers, triangular numbers.

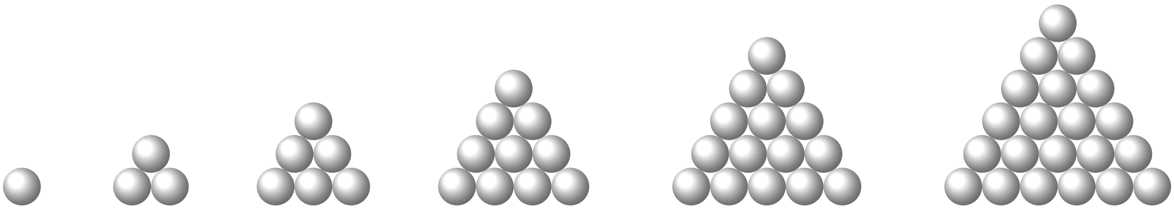


Figure 1: Triangular Number Sequence

2 Exploration of simple polygonal sequences

Being the simplist, trianglular numbers are the foundation of polygonal numbers and hold some incredible properties, which unfortunately I do not have enough time to explore in detail. However, I have written some code in Sage that plots any amount of triangular numbers to a graph. We can use this to start to explore their relationships with other polygonals. Grabowski [2] shows us that the nth triangular number is given by $\frac{n(n+1)}{2}$, $n \in \mathbb{Z}_+$, which I have used in the code below:

```
def triangular(n):
    return ((n * (n + 1)) / 2)
lst = []
for k in range(0, 101):
    lst.append(triangular(k))
list_plot(lst)
```

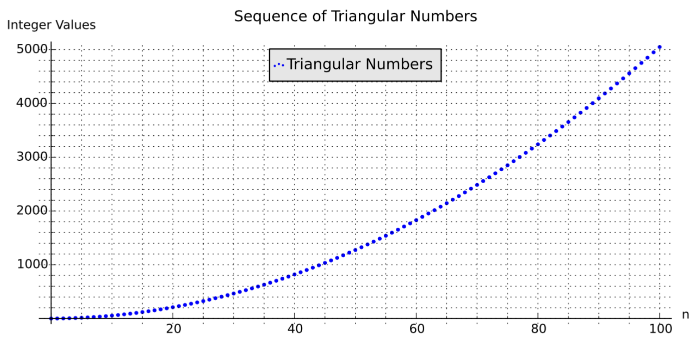


Figure 2: Triangular Number Graph

Due to limitation of space, I have ommitted the code comments and plotting details. Please refer to the published code for full details.
In the above code, I have plotted the first 100 triangular numbers to a graph, which is shown in Figure 2. Admittedly we do not see very much from this graph alone. Be that as it may, I called triangular numbers the foundation of polygonals for a reason, so let's build from them.

The next step we are going to take is to look at square numbers. The nth square number has the simplist formular of all polygonals, and is given by n^2, $n \in \mathbb{Z}_+$. Once again I have written some code in Sage that will plot us any amount of square numbers to a graph:

```
def square(n):
    return (n ^ 2)
lst2 = []
for k in range(0, 101):
    lst2.append(triangular(k))
list_plot(lst2)
```

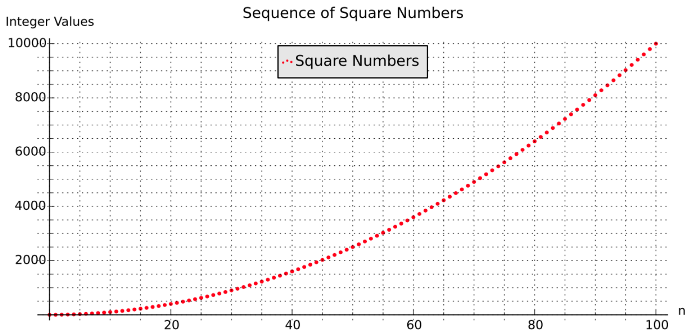


Figure 3: Square Number Graph

Again, due to limitation of space, I have ommitted the code comments and plotting details. Please refer to the published code for full details.
Immediately, we can spot a visual relationship between Figure 2 and Figure 3. We also notice that square numbers, as probably expected, seem to be larger. Is this the case for every n? The most efficient way of checking this is by plotting the two polygonals on the same graph. Some simple Sage code that sums together the two list plots gives the graph shown in Figure 4.

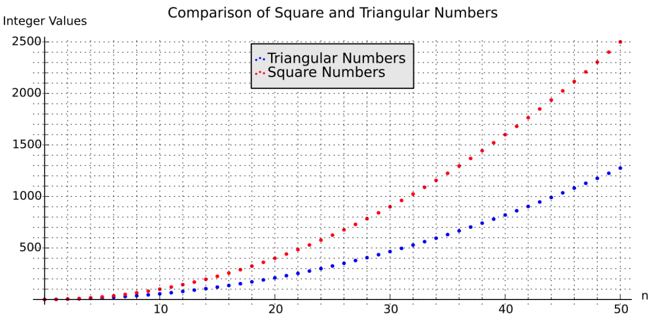


Figure 4: Square and Triangular Number Graph

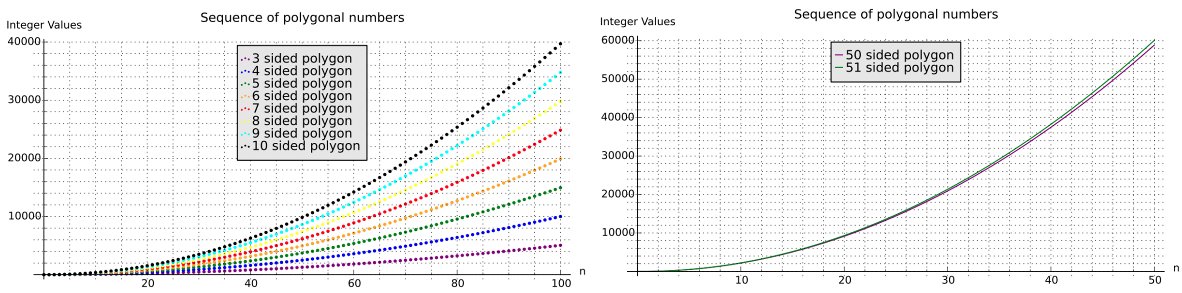
So now, using Figure 4 it is very easy to see that square numbers are larger than triangular for every value of n, exept for $n = 1$ at which point they are of course equal to 1. Hence we can write:
$Square_n \geq Triangular_n$, $\forall n$, $n \in \mathbb{Z}_+$
Now, we can certainly expect this relationship to hold for all sequential polygonals. But, to test this theory we will need to explore more complicated sequential polygons. Hopefully we can show that:
$(s + 1)gonals_n \geq (s)gonals_n$, $\forall n$, $n \in \mathbb{Z}_+$
Where s is the positive integer that defines our polygonal.

2.1 Sequential relationship

So the question we are asking ourselves now is: *Is the sequence of polygonals (nonstrictly) increasing?* In Grabowski's [2] work, we can see that the nth (s)gonal number is given by the following equation: $\frac{n^2(s-2)-n(s-4)}{2}$, $s, n \in \mathbb{Z}_+$. I have used this to write some Sage code that take S amount of polygonals, returns N amount of polygonal numbers and plots them to a single graph:

```
def polygonal(S, N):
    return ((((N ^ 2) * (S - 2)) - (N * (S - 4))) / 2)
dic = {}
for sides in S:
    dic[sides] = list()
for n in N:
    for polygon in S:
        dic[polygon].append(polygonal(polygon, n))
for e in dic.values():
    y += list_plot(sorted(e))
y.show()
```

Due to limitation of space, I have ommitted the code comments and plotting details. Please refer to the published code for full details.
This code is fundamental in helping us answer our question. It now means we can plot and compare any number of sequential polygonals. If we can use it to show that the sequence of polygonals is increasing, then we can build onto something much more interesting.



(a) Polygonals 3 - 10 (b) Polygonals 50 - 51

Figure 5: Sequential Relationships

Figure 5 shows us exactly what we were hoping to find. We can see that the sequence of polygonals is indeed increasing. I have double checked the result from from Figure 5a by plotting a graph with much larger polygonals in Figure 5b. The result holds.
Hence, we have shown that the enequality $(s+1)gonals_n \geq (s)gonals_n$, $\forall n$, $n \in \mathbb{Z}_+$ holds and we already knew that $1 \in$ all polygonals. Therefore, we can see that any $(s+1)$gonal number can be created by summing a finite amount (s)gonal numbers.

3 A brief history of Fermat's Polygonal Theorem

In 1638, Pierre de Fermat took this relationship a whole lot further than I could and realised that this finite number equated to s. Such that any $(s+1)$gonal number is the sum of at most s (s)gonal numbers. Infact, he realised that this actually held for any positive integer. So, any positive integer is the sum of at most s (s)gonal numbers [3].

Fermat's proof of his theorem was never actually found, so to this day we can only truely say he realised the relationship. In September 1639 he wrote a letter to Merssene making the claim that he was 'the first to discover the very beautiful entirely general theorem' [1]. It wasn't until over 100 years later in 1770 the first steps in proving the theorem were made by Joseph Louis Lagrange. He was able to prove it for the square cases only. His theorem became known as 'Lagrange's four-square theorem' [1]. Just 26 years later, the theorem was proved for cases of the triangular form by Carl Friedrich Gauss. It wasn't actually until 1813, almost 200 years after Fermat's realisation, that the first proof for the general theorem was made. Augustin-Louis Cauchy was able to do this by building on work from Gauss' theorem [1].

4 Centred polygonal numbers

Due to this being such a short project, I was unable to fit in everything I would have liked. I started to dip my toes into centred polygonal numbers and ask the same questions. What I found was quite interesting.
Centred polygonal numbers are similar to regular ones. However, the collection of points is centred around the previous one, starting at 1. An example of centred polygonal numbers is shown in Figure 6, centred triangular numbers. Once again, we see by definition that $1 \in$ all centred polygonals.

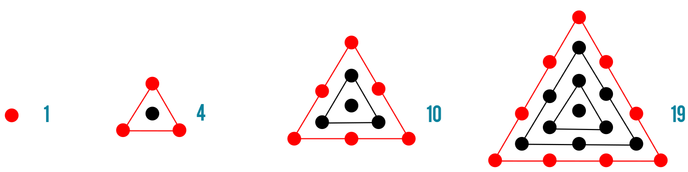


Figure 6: Centred Triangular Number Sequence

Just by looking at Figure 6 and forming the next few polygonals in our heads we can start to predict that the rule $(s+1)gonals_n \geq (s)gonals_n$ $\forall n$, $n \in \mathbb{Z}_+$, which we found for regular polygonals could be the same for centred polygonals.

The formular for the nth centred (s)gonal number is also found in Grabowski's [2] work, and is given by $\frac{sn}{2}(n-1)+1$, $s, n \in \mathbb{Z}_+$. By following the same process as before and tweaking our code to plot a graph that is shown in Figure 7, we can spot that the enequality once again holds.

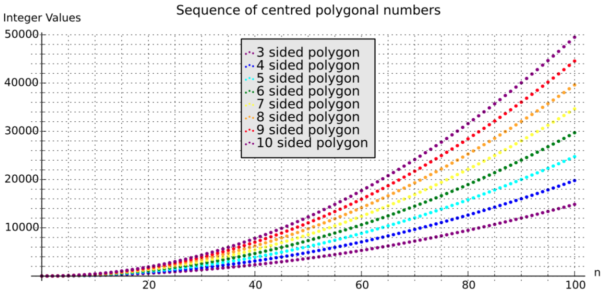


Figure 7: Centred Polygonals

So, could Fermat's theorem hold for centred polygonal numbers too? Let's quickly test it with a random number, let's say 7, using centred triangular numbers $[1, 4, 10, 19, ..]$.
The only way to produce 7 is $4 + 1 + 1 + 1$. Therefore, you need 4 (3)gonal numbers in this case. I continued to test this for a few more random positive integers and found that they can be summed by at most $(s + 1)$ centred (s)gonal numbers.
If I was to continue this project, I would research further in an attempt to show that any $\mathbb{Z}_+$ is the sum of at most $(s + 1)$ centred (s)gonal numbers.

References

[1] E. Deza. *Figurate Numbers*. World Scientific, 2012.

[2] Adam Grabowski. Polygonal numbers. *Formalized Mathematics*, 21(2):103–113, 2013.

[3] E.W. Weisstein. *CRC Concise Encyclopedia of Mathematics, Second Edition*. Taylor & Francis, 2002.