Week 7 - Symbolic Calculus

Using Sage we can carry out various operations from Calculus. This week we will investigate how to:

- Carry out limits in Sage;
- Carry out differentiation in Sage;
- Carry out integration in Sage.

1. Last week we saw how to define a function in Sage:

   f(x) = x ˆ 3 + 3 * x + sin(x)

   To obtain the variables of a function we can use the `variables` method:

   print f.variables()

   Try this with a function of more than one variable:

   f(x, y) = x * y + x ˆ 2 + y ˆ 2

2. In calculus the following definition of a limit is well know:

   > $\lim_{x\to a} f(x) = L$ iff $\forall\ \epsilon > 0\ \exists\ \delta$ such that $\forall\ x$: $|x - a| < \delta \Rightarrow |f(x) - L| \leq \epsilon$.

   Let us calculate the limit of $f(x) = \frac{3x^2}{x^3+x-1}$ as $x \to 1$.
   First of all let us plot $f(x)$:

   ```
   plot(f(x), x, .5, 10)
   ```

   The following code obtains $\lim_{x\to 1} f$:

   ```
   f.limit(x=1)
   ```

   We can also obtain the same result using the `limit` method:

   ```
   limit(f,x=1)
   ```

   Note that $f(1) = \lim_{x\to 1} f(x)$:

   ```
   f(1)
   ```

   This implies that f is continuous at 1.

3. **TICKABLE** Plot $f(x) = \frac{3x^2}{x^3+x-1}$ using the default options:

   ```
   plot(f)
   ```

We see that Sage is plotting extremely high values at the discontinuity due to a root of the denominator which seems to be around $x = .7$. We can plot our function either side of that point and combine them. We do this by creating plot objects:

```
p = plot(f, x, 0.8, 10)
type(p)
p += plot(f, x, -10, .6)
type(p)
p.show()
```

and identify (use the `solve` function or the `roots` method, and maybe the `denominator` method on f) α: the root of the denominator of f. Obtain $\lim_{x\to\alpha+} f(x)$ and $\lim_{x\to\alpha-} f(x)$. Directions of limits can be obtained using the following code:

```
limit(f, x=??, dir="plus")
limit(f, x=??, dir="minus")
```

4. There are various algebraic relationships on limits:

    1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
    2. $\lim_{x\to a}[f(x) \times g(x)] = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$
    3. $\lim_{x\to a}[f(x)/g(x)] = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ (if $\lim_{x\to a} g(a) \neq 0$)

    We can verify the first identity with the following Sage code for a particular example:

    ```
    f(x) = exp(x)
    g(x) = sin(x)
    var('a')
    L1 = limit(f(x) + g(x), x = a)
    L2 = limit(f(x), x = a) + limit(g(x), x = a)
    bool(L1 == L2)
    ```

Note that we use the `bool` class to convert the symbolic equation `L1==L2` to a boolean variable. Verify with some example functions the other two relationships above.

5. $\lim_{x\to 0} \frac{sin(x)}{x}$
6. $e^(x)$
7. Basic differentiation
8. Limiting definition of a derivative
9. Plotting the limiting definition of a derivative

10. Visualising the limiting definition of a derivative
11. Differentiation rules
12. Basic integration
13. Integration by parts
14. Riemann integration
15. Numerical integration
16. Integrate polynomials in a data file