Week 9 - Linear Algebra

1. Use Sage to solve the following system of equations:

$$\begin{cases} 10x + 2y = 0 \\ 2x - y = 154 \end{cases}$$

2. **TICKABLE** Note that the above system of equations is equivalent to the following systems of equations:

$$\begin{cases} 10a + 2b = 0 \\ 2a - b = 154 \end{cases}$$

$$\begin{cases} 10m + 2n = 0 \\ 2m - n = 154 \end{cases}$$

In essense the only thing that defines the system of equations is the cofficients:

$$\begin{pmatrix} 10, 2, 0 \\ 2, -1, 154 \end{pmatrix}$$

We can of course seperate the right hand side of our equation and perhaps include those elements in a vector. Our system can now be represented as:

$$\begin{pmatrix} 10, 2 \\ 2, -1 \end{pmatrix} \begin{pmatrix} 0 \\ 154 \end{pmatrix}$$

Let us attempt to represent the above system in Sage.

The following defines `b` as a vector:

```
b = vector(0,154)
```

The representation of coefficients is a well defined mathematical object called a `matrix`. The following code defines `A` as a matrix:

```
A = matrix([[10, 2], [2, -1]])
```

If we define a vector `X` as a vector of the symbolic variables:

```
X = vector([x, y])
```

We can **multiply** `A` by `X`:

```
A * X
```

Verify that $X = (x_0, y_0)$ where (x_0, y_0) is the solution to our system of equations (obtained in (1)).

3. **TICKABLE** In linear algebra (you will study this next semester) a matrix equation is an equation of the form:

$$AX = b$$

or

$$XA = b$$

If we define `A` and `b` as in question 2 we can solve this equation quite simply using the `solve_right` or `solve_left` methods. The following obtains a solution to the equation $AX = b$:

```
A.solve_right(b)
```

Note that `A\b` is shorthand for `A.solve_right`

Use the above to solve the following system of equations using matrix notation:

$$\begin{cases} 4x - 2y + 3z = 10 \\ -x - 5y - 8z = 9 \\ x + y + z = 1 \end{cases}$$

4. For reasons that will become clear, the following definition of matrix multiplication is required:

$$(AB)_{ij} = \sum_{j'} \sum_{i'} A_{ij'} B_{i'j}$$

For 2×2 matrices this is equivalent to:

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

As an example create the following two matrices in Sage:

```
A = matrix([[1,2],[3,4]])
B = matrix([[7,8],[9,10]])
```

Attempt to multiply these matrices by hand and carry out their multiplication in Sage:

```
A*B
```

Repeat the exercise by multiplying the following pairs of matrices:

1. $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix}$
2. $A = \begin{pmatrix} 0 & 144 \\ -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ -1 & -17 \end{pmatrix}$
4. $A = \begin{pmatrix} 0 & -1 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1/3 & 1/3 \\ -1 & 0 \end{pmatrix}$

5. **TICKABLE** The previous exercise shows that when considering matrix multiplication there exists a matrix which does not have a multiplicative affect: "the identity matrix".

   The identity matrix of size $n \times n$ is denoted by $\mathbb{I}_n$. The following Sage code gives $\mathbb{I}_n$:

   ```
   identify_matrix(4)
   ```

   Note also, that the previous exercise showed that we can sometimes find a matrix B such that $AB = \mathbb{I}_n$. Finding such a matrix is refered to as 'invering' A and if certain properties hold (you will see this in further details next semester) this matrix is denoted A^{-1}.

   If we recall the matrix equation $AX = b$ and if we assume that A^{-1}exists then multiplying both sides by A^{-1} gives:

   ```
   $$A^{-1}AX=A^{-1}b\Rightarrow \mathbb{I}_nX=A^{-1}b=X=A^{-1}b$$
   ```

   In Sage we can obtain A^{-1} (if it exists) with the following code:

   ```
   A.inverse()
   ```

   Thus another approach to solving $AX = b$ is:

   ```
   A.inverse()*b
   ```

   Use this approach to solve the systems of equations we have considered so far.

6. Enter the following matrices in to a list. Invert all of them.
7. Plotting something?
8. Solve a large number of systems of linear equations
9. Reading in data for a big system of linear equations
10. Creating a big linear system and solve it.