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104.272 Discrete Mathematics, Group 1 (Professor Gittenberger)
13. Exercise, Due 29 January, 2020

121. (a) Determine explicitly the coefficients a_n of the power series

$$A(z) = 5(1-3z)^{1/3}.$$

Proof. Notice that we can do this by computing the Taylor series of $A(z)$. Therefore we need to compute the n^{th} derivative of $A(z)$

$$A^{(n)}(z) = 5\binom{1/3}{n}(1-3z)^{(1/3)-n}(-3)^n$$

If we compute the Taylor's expansion at 0, we get:

$$\sum_{k=0}^{\infty} \frac{5\binom{1/3}{k}}{k!}(-3)^k z^k$$

Therefore the coefficients are given by

$$a_k = \frac{5\binom{1/3}{k}}{k!}(-3)^k$$

□

(b) Determine the generating function $B(z) = \sum_{n\geq 0} b_n z^n$ of the sequence

$$b_n = \sin(2n).$$

Hint: $\sin(x) = (e^{ix} - e^{-ix})/(2i)$.

Proof. We take the hint and recall that the power series for $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$.
Therefore $\sin(x) = \left(\sum_{k=0}^{\infty} \frac{i^k x^k}{k!} - \sum_{k=0}^{\infty} \frac{(-ix)^k}{k!}\right)/(2i)$. Notice that for even indices, the power series vanishes since $(ik)^{2r} = (-ik)^{2r}$ for any $r \in \mathbb{Z}$. Therefore we can restrict to consider only odd indices. Also notice that $1/i = -i$
Consider $i^{2r+2}(-1)x^{2r+1}(1/i) - i^{2r+2}x^{2r+2}(1/i) = (-1)^r x^{2r+1} - (-1)^r(-1)x^{2r+1}$ $= 2(-1)^r x^{2r+1}$. Therefore $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$.
This implies that $\sin(2n) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2^{2k+1} \cdot n^{2k+1}}{(2k+1)!}$.

□

122. Suppose that a password can be made up of lower-case letters $a - z$, upper-case letters $A - Z$, and numbers $0 - 9$. Determine, with the help of the principle of inclusion and exclusion, the number of passwords of length 10 that contain at least one lower-case letter, at least one upper-case letter, and at least one number.

Remark: It is not necessary to calculate the precise number; it is sufficient to write it as the sum/difference of some appropriate numbers.

Proof. Notice that there are 26 letters in the standard english alphabet and 10 digits. The total number of passwords of length 10 is 62^{10}.

Let A be the set of passwords of length 10 that contain at least one lower-case letter , B be the set of passwords of length 10 that contain at least one upper-case letter, and C be the set of passwords of length 10 that contain at least one number.

From the principle of inclusion and exclusion we get that

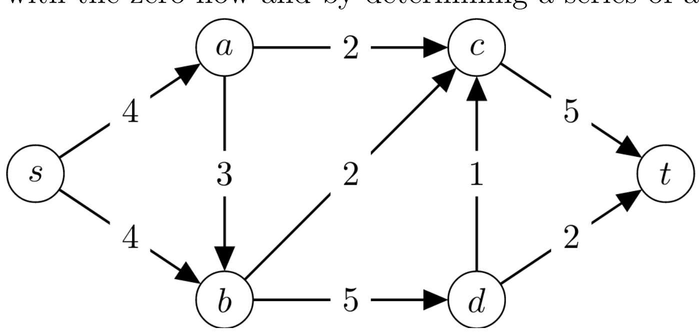
$$|A \cap B \cap C| = |S| - |\bar{A}| - |\bar{B}| - |\bar{C}| + |\bar{A} \cap \bar{B}| + |\bar{A} \cap \bar{C}| + |\bar{B} \cap \bar{C}| - |\bar{A} \cap \bar{B} \cap \bar{C}|$$

Therefore it suffices to compute each of the right hand terms individually.

- $|S| = 62^{10}$
- $|\bar{A}| = 36^{10}$, since we have only 26 characters for upper case and 10 digits.
- $|\bar{B}| = 36^{10}$, since we have only 26 characters for lower case and 10 digits.
- $|\bar{C}| = 52^{10}$, since we have only 26 characters for lower case and 26 for upper case.
- $|\bar{A} \cap \bar{B}| = 10^{10}$, since no letters are allowed, only digits.
- $|\bar{A} \cap \bar{C}| = 26^{10}$, since only upper case letters are allowed.
- $|\bar{B} \cap \bar{C}| = 26^{10}$, since only lower case letters are allowed.

□

123. (a) Determine a maximum flow from s to t on the following network by starting with the zero flow and by determining a series of augmenting paths.



Proof. We find the following sequence of augmenting paths:

- $s \to a \to c \to t$ of weight 2
- $s \to a \to b \to c \to t$ of weight 2
- $s \to b \to d \to t$ of weight 2
- $s \to b \to d \to c \to t$ of weight 1

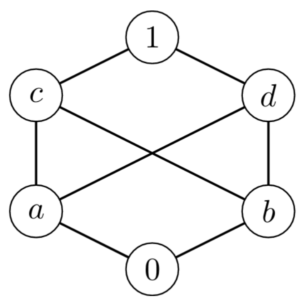
Notice that this flow saturates the incoming edges to t. Therefore it is a maximal flow.

□

(b) Does the maximum flow change, if the ede (a, b) (with capacity 3) is cut? Explain your answer!

Proof. Removing edge (a, b) changes the max flow, since then there exists a cut $(s, b), (a, c)$ with capacity 6. It follows from the max-flow min-cut theorem that the max flow should be less or equal to 6. However, the max flow with (a, b) was 7. □

124. (a) Determine the value of the Möbius function $\mu(0, 1)$ of the following partial order:



Proof. Notice that $\mu(x, x) = 1$. Therefore $\mu(c, 1) = \mu(d, 1) = -1$.
Also notice that $\mu(a, 1) + \mu(c, 1) + \mu(d, 1) + \mu(1, 1) = 0$. Therefore $\mu(a, 1) = 1$. Notice that also $\mu(b, 1) = 1$, since they have the same strict upper bounds.
Now consider the following $\mu(0, 1) + \mu(a, 1) + \mu(b, 1) + \mu(c, 1) + \mu(d, 1) + \mu(1, 1) = 0 = \mu(0, 1) + 1 + 1 - 1 - 1 + 1$. Therefore $\mu(0, 1) = -1$ □

(b) Consider the same partial order, but without the relations (a, d) and (b, c), and determine the corresponding value of the Möbius function $\mu(0, 1)$.

Proof. As usual $\mu(1, 1) = 1$ and $\mu(c, 1) = \mu(d, 1) = -1$. Now the difference lies with $\mu(a, 1) + \mu(c, 1) + \mu(1, 1) = \mu(b, 1) + \mu(d, 1) + \mu(1, 1) = 0$. This shows that $\mu(a, 1) = \mu(b, 1) = 0$.
This shows that $\mu(0, 1) + \mu(c, 1) + \mu(d, 1) + \mu(1, 1) = 0$. Therefore $\mu(0, 1) = 1$. □

125. (a) Which of the following two polynomials is irreducible over $\mathbb{Z}_3$:

$$f(x) = x^4 + x + 1, \quad g(x) = x^3 + 2x^2 + 1.$$

Proof. Notice that f is not irreducible since 1 is a root, and $g(x)$ is irreducible since it is of degree 3 without a root. □

(b) Determine all solutions of the following system of congruences:

$$2x^3 \underset{3}{\equiv} 1$$
$$15x \underset{21}{\equiv} 12$$

Hint: Reduce the given system into a system of the form $x \underset{m_i}{\equiv} a_i$

Proof. Notice that in $\mathbb{Z}_3$ it holds that $x^3 = x$, therefore the first equation is equivalent to $2x \underset{3}{\equiv} 1$. Also, since 3 is prime, we can further reduce it to $x \underset{3}{\equiv} 2$.

Notice that the second equation is equivalent to the expression $15x = 21y + 12$, since $\mathbb{Z}$ is an integral domain, then $5x = 7y + 4$. This reduces to the following system (by taking into account that $5 \cdot 3 = 1$ and that $3 \cdot 4 = 5$

$$x \underset{3}{\equiv} -1$$
$$x \underset{7}{\equiv} -2$$

This implies that $x = 5$ is a unique solution modulo 21. □

(c) With the help of the Euclidean algorithm, determine $\gcd(223, 25)$ and calculate $25^{-1} \bmod 223$

Proof. Notice that applying the Euclidean algorithm yields the following equation:

$$12 \cdot 223 - 107 \cdot 25 = 1$$

.

Therefore $25^{-1} = -107 = 116$ in $\mathbb{Z}_{223}$ □

(d) Determine all integer solutions x, y of the equation $25x + 223y = 1$

Proof. It follows from the equation that $25x \underset{223}{\equiv} 1$. Therefore $x \underset{223}{\equiv} 116$. This implies that $x = 116 + 223 \cdot \alpha$.

Replacing x in the equation yields

$$25(116 + 223 \cdot \alpha) + 223y = 1$$

Which then simplifies into $\begin{array}{l} x = 223 \cdot \alpha + 116 \\ y = 25 \cdot \alpha - 13 \end{array}$ where $\alpha \in \mathbb{Z}$

□

126. (a) Let $A(x)$ be the generating function of the sequence (a_n). What are the generating functions of the sequences (b_n) and (c_n), where :

$$b_n = \sum_{k=0}^{n} a_k, \quad c_n = na_{n-1}$$

Proof. Notice first that b_n is a Cauchy product of $A(x)$ and constant sequence 1 which is generated by $\frac{1}{1-x}$. Therefore $B(x) = \frac{A(x)}{1-x}$.

Also notice that if $A(x) = \sum_{n=0}^{\infty} a_n x^n$, then $A'(x) = \sum_{n=0}^{\infty} n \cdot a_n x^{n-1}$. Now consider $x^2 A'(x) = \sum_{n=0}^{\infty} n \cdot a_n x^{n+1} = \sum_{n=1}^{\infty} n \cdot a_{n-1} x^n$. Let $c = c_0$, then $C(x) = x^2 A'(x) + c$. □

(b) Let (d_n) be the sequence defined by the following relation:

$$d_n = 1 + \sum_{k=0}^{n} k d_{n-k} \quad (n \geq 1), \quad d_0 = 1$$

Determine the corresponding generating function $D(x)$.

Proof. Notice that $\sum_{n=0}^{\infty} nx^n$ has generating function $x\left(\frac{x}{1-x}\right)'$. It follows from the relation that:

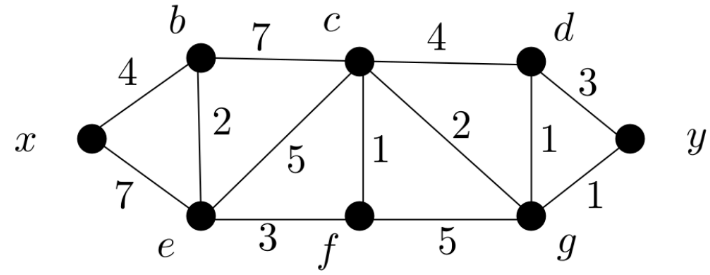
$$D(x) = 1 - \frac{xD(x)}{(1-x)^2}$$

.

Therefore $D(x)\left(1 + \frac{x}{(1-x)^2}\right) = 1$.

It follows that $D(x) = \left(1 + \frac{x}{(1-x)^2}\right)^{-1}$. □

127. Use Dijkstra's algorithm to determine $d(x, y)$ in the following graph:



Proof. First we start with infinite distances to every vertex, except for the starting point x. Then we take the minimum of the outgoing edges from x, which is (x, b) with weight 4. Therefore $w(b) = 4$.

The candidates at this point are e with distance $4 + 2 = 6$ or c with distance $7 + 4 = 11$. Therefore we choose e and update $w(e) = 6$.

The candidates at step 3 are f with weight $6+3 = 9$ or c with weight 11 (either $6 + 5$ or $4 + 7$). Therefore we pick f and update $w(f) = 9$.

At this point the candidates are given by c with weight $9 + 1 = 10$ or g with weight $9 + 5 = 14$. We choose c and update $w(c) = 10$.

In the next step the candidates are g with $10 + 2 = 12$ or d with $10 + 4 = 14$. Vertex g has the least weight and $w(g)$ is updated to 12.

Finally the last candidates are d and y with weight $12 + 1$ both. We can either choose both simultaneously or sequentially. In either case $w(y) = d(x, y) = 13$ □

128. Let $M = \{1, \ldots, n\}$. Consider the graph G with

$$V(G) = 2^M = \{A \mid A \subseteq M\},$$

$$E(G) = \{A, B \mid A, B \in V(G), A \neq B, A \cap B \neq \varnothing\}.$$

Calculate $\alpha_0(G) := |V(G)|$ and $\alpha_1(G) := |E(G)|$.

Proof. Notice that $\alpha_0 = 2^n$. Now notice that each set $S \subseteq M$ has degree 2^{n-m} where $|S| = m$. There are $\binom{n}{m}$ sets of cardinality m. Therefore the sum of all degrees is $\sum_{k=0}^{n} \binom{n}{m} \cdot 2^{n-m}$. It follows from the handshaking theorem that:

$$\alpha_1 = 1/2 \cdot \sum_{k=0}^{n} \binom{n}{m} \cdot 2^{n-m}$$

□

129. Let R be an integral domain. Prove that $(a) = \{ra \mid r \in R\}$.

Proof. Let $c \in (a)$. It follows that $c = b \cdot a - d \cdot a = (b - d) \cdot a$ for $b, d \in R$. Notice that since $b - d \in R$, then $c = ra$.

Now, let $c = ra$ with $r \in R$. Notice that $a \in (a)$, and since it is an ideal, $ra \in (a)$.

It follows from the previous statements that $(a) = \{ra \mid r \in R\}$ $\square$

130. Alex and Leo are a couple, and they organize a party together with 4 other couples. There are a number of greetings but, naturally, nobody says hello to their own partner. At the end of the party Alex asks everyone how many people did they greet, receiving nine different answers. How many people did Alex greet and how many people did Leo greet?

Hints: Describe a graph that models the situation. Find out how many people did each member of a couple greet.

Proof. We model this with an non-directed graph (V, E) where V is given by the collection of people at the party, and $(a, b) \in E$ if the person represented by a greeted the person represented by b. Since there are 10 people in total, and each person does not greet their partner, then each person greets at most 8.

Since Alex got 9 different answers, these must be $\{0, \ldots, 8\}$. Notice that the person who greeted 8 persons is a couple to the person that didn't greet anyone, since it greeted the maximum of people it could greet. Let's call the social partner p_0^+ and the antisocial p_0^-. Notice that if we remove the couple p_0 from the graph, we are left with all greetings from $\{0, \ldots, 6\}$ outside of p_0. We can repeat the argument for p_0 in this subgraph, which yields p_1^+ and p_1^- who must be a couple. Notice that in G p_1^+ greeted 7 people and p_1^- greeted only 1 person.

We can now proceed inductively in order to assert that the couples are given by $(0, 8), (1, 7), (2, 6), (3, 5)$. However, notice that the person with 4 greetings must be coupled to the person with 4 greetings. Since all answers were different, one of the people that greeted 4 is not on the list of people that answered. Therefore, Alex must be the couple of the person that greeted 4. It follows that both Alex and Leo greeted 4 persons in total. $\square$