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TU Wien, Winter 2019
104.272 Discrete Mathematics, Group 1 (Professor Gittenberger)
7. Exercise, Due 27 November, 2019

61. A derangement is a permutation without any fixed point. Determine the exponential generating function $\sum_{n\geq 0} \frac{d_n}{n!} z^n$, where d_n is the number of derangements of $\{1, 2, \ldots, n\}$.

    Then show that

$$d_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

Proof. Notice that a permutation $\mathcal{P}$ partitions its elements into fixed points and non fixed points. Therefore, we can view a permutation as a partitional product of a derangement and the identity permutation.

Then $\mathcal{P} = \mathcal{D} \star \mathcal{I}$.

The symbolic method yields the result $\hat{P}(z) = \hat{D}(Z) \cdot \hat{I}(z)$.

Notice that $\hat{P}(z) = \sum_{n\geq 0} \frac{n!}{n!} z^n = \sum_{n\geq 0} z^n = \frac{1}{1-z}$.

Notice also that $\hat{I}(z) = \sum_{n\geq 0} \frac{z^n}{n!} = e^x$.

It follows that $\hat{D}(z) = e^{-z} \cdot \frac{1}{1-z}$ □

62. An involution is a permutation π such that $\pi \circ \pi = \mathrm{id}_M$ where $M = \{1, 2, \ldots, n\}$ Let $\mathcal{I}$ be the set of involutions. Determine the exponential generating function $I(z)$ of $\mathcal{I}$.

Proof. Notice that an involution consists of a product of cycles of length 2 and 1. Simply notice that if x is not a fixed point, then $x \mapsto \pi(x) \mapsto x$ which is a cycle of length 2.

Notice then, that an involution is a set of size n made of cycles of size 2 or 1. This leads to the following combinatorial construction:

$$\mathcal{I} = SET(\mathcal{F} + \mathcal{P})$$

Where $\mathcal{I}$ is an involution, $\mathcal{F}$ is a fixed point (cycle of size 1) and $\mathcal{P}$ is a pair (cycle of length 2).

Notice that there is only one possible fixed point of size 1 and only one possible pair of size 2. This implies that $\hat{F}(z) = z$ and $\hat{P}(z) = \frac{z^2}{2}$, are the EGFs for $\mathcal{F}$ and $\mathcal{P}$ respectively.

The symbolic combinatorial method now yields the following EGF for involutions. $\hat{I}(z) = e^{z + \frac{z^2}{2}}$ □

63. Use exponential generating functions to determine the number a_n of **ordered** choices of n balls such that there are 2 or 4 red balls, an even number of green balls and an arbitrary number of blue balls.

Proof. Notice that the ordered choices of this sort can be specified as a partitional product of balls of each color. Let $\mathcal{R}, \mathcal{G}, \mathcal{B}$ be the combinatorial categories of red balls, green balls and blue balls respectively. Let $\mathcal{O}$ be the combinatorial category of ordered choices as specified previously. Notice that the following definition is provided by the problem specification.

$$\mathcal{O} = \mathcal{R} \star \mathcal{G} \star \mathcal{B}$$

.

Notice that since there must be either 2 or 4 red balls, then the EGF for $\mathcal{R}$ is given by

$$\hat{R}(z) = \frac{z^2}{2} + \frac{z^4}{4!}$$

And respectively $\hat{G}(z) = \sum_{n \geq 0} \frac{z^{2n}}{(2n)!} = \frac{1}{2} \sum_{n \geq 0} [1 + (-1)^n] \cdot \frac{z^n}{n!} = \frac{1}{2} e^z + e^{-z}$

Also $\hat{B} = e^z$

Applying the symbolic combinatorial method yields the following equation

$$\hat{O}(z) = (\frac{z^2}{2} + \frac{z^4}{4!}) \frac{1}{2} (e^z + e^{-z}) e^z$$

$$\hat{O}(z) = (\frac{z^2}{2} + \frac{z^4}{4!}) \frac{1}{2} (e^{2z} + 1)$$

$$\hat{O}(z) = \frac{1}{2} (\frac{z^2 e^{2z}}{2} + \frac{z^4 e^{2z}}{4!}) + \frac{1}{2} (\frac{z^2}{2} + \frac{z^4}{4!})$$

Let us recall that $\left[\frac{z^n}{n!}\right] \hat{F} \cdot \hat{G} = \sum_{k \geq 0}^{n} \binom{n}{k} \left[\frac{z^k}{k!}\right] \hat{F} \cdot \left[\frac{z^{n-k}}{(n-k)!}\right] \hat{G}$

Notice that since $\left[\frac{z^k}{k!}\right]\frac{z^2}{2}$ is only different from 0 when $k=2$ and $\left[\frac{z^k}{k!}\right]\frac{z^4}{4!}$ is only different from 0 when $k=4$, then $\left[\frac{z^n}{n!}\right]\frac{z^2e^{2z}}{2}=\binom{n}{2}2^{n-2}$ and $\left[\frac{z^n}{n!}\right]\frac{z^4e^{2z}}{4!}=\binom{n}{4}2^{n-4}$

Therefore $O(n)=\begin{cases}1/2+\binom{n}{2}2^{n-3}+\binom{n}{4}2^{n-5} & \text{if } n=2,4\\ \binom{n}{2}2^{n-3}+\binom{n}{4}2^{n-5} & \text{otherwise}\end{cases}$

□

64. Determine all solutions of the recurrence relation:

$$a_n-2na_{n-1}+n(n-1)a_{n-2}=2n\cdot n!,\quad n\geq 2,\quad a_0=a_1=1$$

Hint: Use exponential generating functions

Proof. The recurrence relation can be rewritten as follows:

$$a_n-2\binom{n}{1}a_{n-1}+2\binom{n}{2}a_{n-2}=2n\cdot n!$$

Notice that using the definition for the product of EGFs yields the following equation

$$\hat{A}(z)-2z\hat{A}(z)+z^2\hat{A}(z)=\sum_{n\geq 2}2nz^n$$

This implies that

$$\hat{A}(z)(1-2z+z^2)=\sum_{n\geq 2}2nz^n$$

This yields

$$\hat{A}(z)=\sum_{n\geq 2}2nz^n/(z-1)^2$$

It follows that

$$\hat{A}(z)=2(\frac{z}{1-z})'(1-z)^{-2}$$

This implies that $\hat{A}(z)=2z(1-z)^{-4}=2z(z-1)^{-4}$. Applying the negative binomial theorem yields $\hat{A}(z)=2z\sum_{n\geq 1}\binom{4+n-1}{n}z^n(-1^n)(-1^{-n})$ Therefore $\hat{A}(z)=\sum_{n\geq 2}2\binom{n+2}{n-1}z^n=\sum_{n\geq 2}2\binom{n+2}{3}z^n=\sum_{n\geq 2}\frac{(n+2)(n+1)(n-1)}{3}z^n.$

Since $\hat{A}(z)$ is an EGF, then $a_n = \begin{cases} 1 & \text{if } n = 0, 1 \\ \dfrac{(n-1)(n+2)!}{3} & \text{otherwise} \end{cases}$

$\square$

65. Let $S_{n,k}$ be the Stirling numbers of the second kind, that is, the number of partitions of the set $\{1, 2, \ldots, n\}$ into k (non-empty) subsets. Show the following formula:

$$\sum_{n,k} S_{n,k} \frac{z^n}{n!} u^k = e^{u(e^z - 1)}$$

.

Proof. First we will compute a formula for the Stirling numbers of the second kind.

Assume that $a_1 + a_2 + \ldots + a_k = n$. For now we will label each partition as $P_1, P_2, \ldots, P_k$ such that $|P_i| = a_i$, and later we will remove the labels.

Notice that we can choose from $\binom{n}{a1}$ options for P_1. Now that we have picked the elements for P_1, we can proceed to pick the elements for P_2. We have $\binom{n-a_1}{a_2}$ options for P_2 and inductively we have

$$\binom{n - \sum_{j=1}^{i-1} a_j}{a_i}$$

different options for partition P_i. Therefore we have

$$\prod_{i=1}^{k} \binom{n - \sum_{j=1}^{i-1} a_j}{a_i}$$

This corresponds to $\dfrac{n!}{a_1!a_2!\ldots a_k!}$ (notice that expanding the product makes the rest of the terms cancel out).

Therefore we have

$$S_{n,k} = \frac{n!}{k!} \left(\sum_{a_1 + a_2 + \ldots + a_k = n} \frac{1}{a_1!a_2!\ldots a_k!} \right) \tag{1}$$

We added over all possible size assignments to the sets of the partitions P_i, and then we remove the set labels dividing by $k!$.

Now, recall that $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. This implies by substitution that $e^{u(e^z-1)} = \sum_{k=0}^{\infty} \frac{[u(e^z-1)]^k}{k!} = \sum_{k=0}^{\infty} \frac{u^k}{k!}[e^z - 1]^k$. Notice that $e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}$. We can use this information to compute $[e^z - 1]^k$ as a Cauchy product.

It follows that $[z^n](e^z-1)^k = \sum_{a_1+a_2+\ldots+a_k=n} \frac{1}{a_1!a_2!\ldots a_k!} = \frac{k!}{n!}S_{n,k}$. This implies that $\sum_{k=0}^{\infty} \frac{u^k}{k!}[e^z-1]^k = \sum_{k=0}^{\infty}\sum_{n=0}^{k} S_{n,k}\frac{u^k}{n!}z^n$ $\square$

66. Prove the following representation for the Stirling numbers of the second kind:

$$S_{n,k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}j^n$$

Remark: compute first the generating function for $\sum_{n=0}^{\infty} S_{n,k}\frac{z^n}{n!} = \frac{(e^z-1)^k}{k!}$

Proof. Notice that $(e^z-1) = \sum_{k=1}^{\infty}\frac{z^n}{n!}$, then $\frac{(e^z-1)^k}{k!} = \frac{1}{k!}\sum_{\sum a_i=n}\frac{1}{a_1!a_2!\ldots a_n!}z^n = \sum_{n=0}^{\infty} S_{n,k}\frac{z^n}{n!}$

From the binomial theorem, it follows that $(e^z-1)^k = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{zj} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{n=0}^{\infty}\frac{z^n j^n}{n!}$. Since $\frac{(e^z-1)^k}{k!} = \sum_{n=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n\frac{z^n}{n!}$ is the EGF for $S_{n,k}$, it follows that $S_{n,k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$ $\square$

67. Let P be the set of all divisors of 12. Determine the Möbius function of $(P, \mid)$

Proof. Notice that $(\mathbb{N}^+, \quad |) \cong (\mathbb{N}\times\mathbb{N}\times\ldots,\leq)$ since every positive integer can be factorized uniquely as a product of powers of primes with only finitely many greater than 0 powers. Notice then that $a = p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots p_n^{a_n} \mid b = p_1^{b_1}p_2^{b_2}\ldots p_n^{b_n}$ if and only if $a_i \leq b_i$ for all $i \in \{1,\ldots,n\}$.

Therefore the Möbius function for the divisibility order can be computed as a product of Möbius functions with respect to the usual order in $\mathbb{N}$.

$$\mu_{|}(a,b) = \prod_{i=1}^{n}\mu_{\leq}(a_i,b_i)$$

Recall that the Möbius function for the usual order in $\mathbb{N}$ is given by

$$\mu_{\leq}(n,m) = \begin{cases} 0 & \text{if } n > m \vee m \geq n+2 \\ 1 & \text{if } n = m \\ -1 & \text{if } m = n+1 \end{cases}$$

Since $\mu(n,n) = \delta(n,n) = 1$, $\mu(n,n+1) + \mu(n+1,n+1) = \delta(n,n+1) = 0$ then $\mu(n,n+1) = -1$ and notice also that $\delta(n,n+2) = 0 = \delta(n+1,n+2) + \mu(n,n+2) = \mu(n,n+2)$. Inductively $\mu(n,m) = 0$ for any $n+2 \le m$.

Notice that this implies that

$$\mu_{|}(a,b) = \begin{cases} 0 & \text{if } a \nmid b \vee b/a = p^2 z,\ p \in \mathcal{P}, z \in \mathbb{N}^+ \\ (-1)^k & \text{if } b/a = p_1 p_2 \dots p_k, p_i \in \mathcal{P} \end{cases}$$

Where $\mathcal{P}$ is the set of positive prime numbers.

Notice that since divisibility is transitive then if $a \mid b \mid 12$ then $a \mid 12$. This implies that the Möbius function for $(div(12), \mid)$ is a restriction of the Möbius function for $(\mathbb{N}^+, \mid)$. Therefore it has the same definition. □

68. Let $(P, \le)$ be the poset defined by $P = \{0,1,2,3,4\}$ and the three relations

$$0 \le 1 \le 4, \quad 0 \le 2 \le 4, \quad 0 \le 3 \le 4$$

Compute all values $\mu(x,y)$ for $x, y \in P$

Proof. $\mu(n,n) = 1, \mu(1,4) = \mu(2,4) = \mu(3,4) = \mu(0,1) = \mu(0,2) = \mu(0,3) = -1$

Also notice that $\delta(0,4) = 0 = \sum_{z \in [0,4]} \mu(z,4) = \mu(0,4) - 3 + 1$ this implies that $\mu(0,4) = 2$

$\mu(n,m) = 0$ in all the remaining cases. □

69. Let $(P_1, \le_1)$ and $(P_2, \le_2)$ be two locally finite posets and $(P, \le)$ be defined by $P = P_1 \times P_2$ and for $(a,x), (b,x) \in P$:

$$(a,x) \le (b,y) \Leftrightarrow a \le_1 b \wedge x \le_2 y$$

Show that $(P, \le)$ is a poset and that the Möbius functions of P, P_1 and P_2 satisfy

$$\mu_P((a,x),(b,y)) = \mu_{P_1}(a,b) \cdot \mu_{P_2}(x,y)$$

Proof. $\le$ **is Reflexive**:

Let $(a,b) \in P$. Since P_1 and P_2 are posets, then $a \le_1 a \wedge b \le_2 b$. From the definition $(a,b) \le (a,b)$.

$\le$ **is Antisymmetric**:

Assume that $(a,b) \le (c,d) \wedge (c,d) \le (a,b)$. From the definition of $\le$, it follows that $a \le_1 c \le_1 a \wedge b \le_2 d \le_2 b$. Since P_1 and P_2 are posets, then it follows that $a = c \wedge b = d$. Therefore $(a,b) = (c,d)$, and $\le$ is antisymmetric.

$\leq$ **is Transitive**:

Assume that $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3)$. From the definition, then $a_1 \leq_1 a_2 \leq_1 a_3 \wedge b_1 \leq_2 b_2 \leq_2 b_3$. Since P_1 and P_2 are posets, it follows that $a_1 \leq_1 a_3 \wedge b_1 \leq_2 b_3$. From the definition, it follows that $(a_1, b_1) \leq (a_3, b_3)$.

This proves that $(P, \leq)$ is a poset.

Notice that the Kronecker delta for P , $\delta_P = (\delta_{P_1} \circ \pi_1) \cdot (\delta_{P_2} \circ \pi_2)$ where π_1 and π_2 are the projections on the first and second coordinate respectively.

Now notice that $\delta((a,b),(c,d)) = \sum\limits_{(x,y) \in [(a,b),(c,d)]} \mu_P((x,y),(c,d))$

On the other hand, $\delta((a,b),(c,d)) = \delta(a,c) \cdot \delta(b,d) = \sum\limits_{x \in [a,c]} \mu_{P_1}(x,c) \sum\limits_{y \in [b,d]} \mu_{P_2}(y,d) = \sum\limits_{(x,y) \in [(a,b),(c,d)]} \mu_{P_1}(x,c) \cdot \mu_{P_2}(y,d)$

Applying the first equation yields

$$\sum_{(x,y) \in [(a,b),(c,d)]} \mu_P((x,y),(c,d)) = \sum_{(x,y) \in [(a,b),(c,d)]} \mu_{P_1}(x,c) \cdot \mu_{P_2}(y,d)$$

Since we are only considering locally finite posets, then it can be shown by induction over $|[(a,b),(c,d)]|$ that $\mu_P((x,y),(c,d)) = \mu_{P_1}(x,c) \cdot \mu_{P_2}(y,d)$ for any $(x,y) \in [(a,b),(c,d)]$

$$\mu_P((a,x),(b,y)) = \mu_{P_1}(a,b) \cdot \mu_{P_2}(x,y)$$

□

70. Draw the Hasse diagran of the poset $P := (2^{\{1,2,3\}}, \supseteq)$.

Let now $A_1, A_2, A_3 \subseteq M$. Use Möbius inversion on the poset P to show the following inclusion-exclusion principle

$$|M \setminus \bigcup_i A_i| = |M| - \sum_i |A_i| + \sum_{i \neq j} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|$$

Proof. □

$$\begin{tikzcd}
 & \varnothing \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
\{1\} \arrow[d, no head] \arrow[dr, no head] & \{2\} \arrow[dl, no head] \arrow[dr, no head] & \{3\} \arrow[dl, no head] \arrow[d, no head] \\
\{1,2\} \arrow[dr, no head] & \{1,3\} \arrow[d, no head] & \{2,3\} \arrow[dl, no head] \\
 & \{1,2,3\} &
\end{tikzcd}$$

We consider the function $f : P \to \mathbb{R}, f(I) = |M \cap \bigcap_{i\in I} A_i \cap (\bigcap_{j\notin I} M \setminus A_i)|$

The definition for $S_f(I) = \sum_{J\supseteq I} f(J) = \sum_{J\supseteq I} |\bigcap_{i\in I} A_i \cap (\bigcap_{j\notin I} M \setminus A_i)|$.

Notice that $\bigcup_{J\supseteq I}(M \cap \bigcap_{i\in J} A_i \cap (\bigcap_{j\notin J} M \setminus A_i)) = M \cap \bigcap_{i\in I} A_i$ since one contention is trivial by definition and for any $x \in M \cap \bigcap_{i\in I} A_i$, if $x \in A_j$, then $S_x = \{j \mid j \notin I, x \in A_j\}$. It follows that $x \in M \cap \bigcap_{i\in I\cup S_x} A_i \cap (\bigcap_{j\notin I\cup S_x} M \setminus A_j)$.

Notice that if $J \neq Q, J \supseteq I, Q \supseteq I$, then $\hat{J} \cap \hat{Q} = \varnothing$ for $\hat{J} = M \cap \bigcap_{i\in J} A_i \cap (\bigcap_{j\notin J} M \setminus A_i)$ and $\hat{Q} = M \cap \bigcap_{i\in Q} A_i \cap (\bigcap_{j\notin Q} M \setminus A_i)$.

This shows that these are pairwise disjoint sets. Therefore $S_f(I) = |\bigcap_{i\in I} A_i|$

Applying the Möbius inversion theorem, then $f(\varnothing) = |M\setminus\bigcup_i A_i| = \sum_{J\subseteq\{1,\ldots,m\}} (-1)^{|J|} \left|\bigcap_{j\in J} A_j\right|$

By grouping together sets with the same cardinality we get

$$\sum_{i=0}^{m} \sum_{|J|=i} (-1)^i \left|\bigcap_{j\in J} A_j\right| = |M| - |A_1| - |A_2| \ldots - |A_m| + |A_1 \cap A_2| + \ldots$$