7^{th} EXERCISE
104.272 Discrete Mathematics

(61) A derangement is a permutation without any fixed point. Determine the exponential generating function $\sum_{n\geq 0} \frac{d_n}{n!} z^n$, where d_n is the number of derangements of $\{1, 2, \ldots, n\}$. Then show that

$$d_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

(62) An involution is a permutation π such that $\pi \circ \pi = \mathrm{id}_M$ where $M = \{1, 2, \ldots, n\}$. Let $\mathcal{I}$ be the set of involutions. Determine the exponential generating function $I(z)$ of $\mathcal{I}$.

(63) Use exponential generating functions to determine the number a_n of **ordered** choices of n balls such that there are 2 or 4 red balls, an even number of green balls and an arbitrary number of blue balls.

(64) Determine all solutions of the recurrence relation:

$$a_n - 2na_{n-1} + n(n-1)a_{n-2} = 2n \cdot n!, \quad n \geq 2, \quad a_0 = a_1 = 1.$$

Hint: Use exponential generating functions.

(65) Let $S_{n,k}$ be the Stirling numbers of the second kind, that is, the number of partitions of the set $\{1, 2, \ldots, n\}$ into k (non-empty) subsets. Show the following formula:

$$\sum_{n,k} S_{n,k} \frac{z^n}{n!} u^k = e^{u(e^z - 1)}.$$

(66) Prove the following representation for the Stirling numbers of the second kind:

$$S_{n,k} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

Remark: compute first the generating function for $\sum_n S_{n,k} z^n/n! = (e^z - 1)^k/k!$.

(67) Let P be the set of all divisors of 12. Determine the Möbius function of $(P, |)$.

(68) Let $(P, \leq)$ be the poset defined by $P = \{0, 1, 2, 3, 4\}$ and the three relations

$$0 \leq 1 \leq 4, \qquad 0 \leq 2 \leq 4, \qquad 0 \leq 3 \leq 4.$$

Compute all values $\mu(x, y)$ for $x, y \in P$.

(69) Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two locally finite posets and $(P, \leq)$ be defined by $P = P_1 \times P_2$ and for $(a, x), (b, y) \in P$:

$$(a, x) \leq (b, y) \Longleftrightarrow a \leq_1 b \wedge x \leq_2 y.$$

Show that $(P, \leq)$ is a poset and that the Möbius functions of P, P_1 and P_2 satisfy

$$\mu_P((a, x), (b, y)) = \mu_{P_1}(a, b) \cdot \mu_{P_2}(x, y).$$

(70) Draw the Hasse diagram of the poset $P := (2^{\{1,2,3\}}, \supseteq)$.

Let now $A_1, A_2, A_3 \subseteq M$. Use Möbius inversion on the poset P to show the following inclusion-exclusion principle

$$|M \setminus (A_1 \cup A_2 \cup A_3)| = |M| - |A_1| - |A_2| - |A_3| + \\ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|.$$