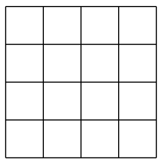
5^{th} EXERCISE
104.272 Discrete Mathematics

(41) In how many ways can the letters a, a, b, b, b, c, d, e be listed such that the letters c and d are not in consecutive positions?

(42) Let $p_n(k)$ be the number of permutations of $\{1, 2, \ldots, n\}$ having exactly k fixed points. Use the method of double counting to prove the identity $\sum_{k=0}^{n} k \cdot p_n(k) = n!$

(43) Let M be a non-empty set. Show that M has as many subsets with an odd number of elements as subsets with an even number of elements.

(44) How many rectangles are in the following figure?



(45) Let n be a positive integer and let $(a_1, \ldots, a_n)$ be a permutation of $\{1, 2, \ldots, n\}$. Define

$$A_k = \{a_i \mid a_i < a_k, i > k\} \text{ and } B_k = \{a_i \mid a_i > a_k, i < k\}$$

for $1 \leq k \leq n$. Prove that $\displaystyle\sum_{k=1}^{n} |A_k| = \sum_{k=1}^{n} |B_k|$.

(46) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, \ldots, 82\}$. All points of the plane having coordinates (x, y) which satisfy $(x, y) \in A \times B$ are colored with one of the colours red, green or blue. Prove that there exists a monochromatic rectangle.
Remark: A rectangle is called monochromatic if all its four vertices have the same colour.

(47) Let A be a 10 element subset of $\{1, 2, \ldots, 100\}$. Show that there are at least two disjoint subsets of A having the same sum.

(48) Let $n \in \mathbb{N}$. Prove the identities

$$k\binom{n}{k} = n\binom{n-1}{k-1} \text{ and } \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

by using only the combinatorial interpretation of the binomial coefficients.

(49) Let $n \in \mathbb{N}$. Prove the identity by using only the combinatorial interpretation of the binomial coefficients:

$$\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

(50) Prove that for all complex number x and all $k \in \mathbb{N}$ we have

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k},$$

where for a complex number x and a natural number k

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$