

**TU Wien, Winter 2019**

**104.272 Discrete Mathematics, Group 1 (Professor Gittenberger)**

**7. Exercise, Due 27 November, 2019**

61. A derangement is a permutation without any fixed point. Determine the exponential generating function  $\sum_{n \geq 0} \frac{d_n}{n!} z^n$ , where  $d_n$  is the number of derangements of  $\{1, 2, \dots, n\}$ .

Then show that

$$d_n = n! \sum_{k=0}^n (-1)^k \frac{1}{k!}$$

*Proof.* Notice that a permutation  $\mathcal{P}$  partitions its elements into fixed points and non fixed points. Therefore, we can view a permutation as a partitional product of a derangement and the identity permutation.

Then  $\mathcal{P} = \mathcal{D} \star \mathcal{I}$ .

The symbolic method yields the result  $\hat{P}(z) = \hat{D}(Z) \cdot \hat{I}(z)$ .

Notice that  $\hat{P}(z) = \sum_{n \geq 0} \frac{n!}{n!} z^n = \sum_{n \geq 0} z^n = \frac{1}{1-z}$ .

Notice also that  $\hat{I}(z) = \sum_{n \geq 0} \frac{z^n}{n!} = e^z$ .

It follows that  $\hat{D}(z) = e^{-z} \cdot \frac{1}{1-z}$  □

62. An involution is a permutation  $\pi$  such that  $\pi \circ \pi = \text{id}_M$  where  $M = \{1, 2, \dots, n\}$ . Let  $\mathcal{I}$  be the set of involutions. Determine the exponential generating function  $I(z)$  of  $\mathcal{I}$ .

*Proof.* Notice that an involution consists of a product of cycles of length 2 and 1. Simply notice that if  $x$  is not a fixed point, then  $x \mapsto \pi(x) \mapsto x$  which is a cycle of length 2.

Notice then, that an involution is a set of size  $n$  made of cycles of size 2 or 1. This leads to the following combinatorial construction:

$$\mathcal{I} = \text{SET}(\mathcal{F} + \mathcal{P})$$

Where  $\mathcal{I}$  is an involution,  $\mathcal{F}$  is a fixed point (cycle of size 1) and  $\mathcal{P}$  is a pair (cycle of length 2).

Notice that there is only one possible fixed point of size 1 and only one possible pair of size 2. This implies that  $\hat{F}(z) = z$  and  $\hat{P}(z) = \frac{z^2}{2}$ , are the EGFs for  $\mathcal{F}$  and  $\mathcal{P}$  respectively.

The symbolic combinatorial method now yields the following EGF for involutions.  $\hat{I}(z) = e^{z + \frac{z^2}{2}}$  □

63. Use exponential generating functions to determine the number  $a_n$  of **ordered** choices of  $n$  balls such that there are 2 or 4 red balls, an even number of green balls and an arbitrary number of blue balls.

*Proof.* Notice that the ordered choices of this sort can be specified as a partitioned product of balls of each color. Let  $\mathcal{R}, \mathcal{G}, \mathcal{B}$  be the combinatorial categories of red balls, green balls and blue balls respectively. Let  $\mathcal{O}$  be the combinatorial category of ordered choices as specified previously. Notice that the following definition is provided by the problem specification.

$$\mathcal{O} = \mathcal{R} \star \mathcal{G} \star \mathcal{B}$$

Notice that since there must be either 2 or 4 red balls, then the EGF for  $\mathcal{R}$  is given by

$$\hat{R}(z) = \frac{z^2}{2} + \frac{z^4}{4!}$$

And respectively  $\hat{G}(z) = \sum_{n \geq 0} \frac{z^{2n}}{(2n)!} = \frac{1}{2} \sum_{n \geq 0} [1 + (-1)^n] \cdot \frac{z^n}{n!} = \frac{1}{2} e^z + e^{-z}$

Also  $\hat{B} = e^z$

Applying the symbolic combinatorial method yields the following equation

$$\hat{O}(z) = \left(\frac{z^2}{2} + \frac{z^4}{4!}\right) \frac{1}{2} (e^z + e^{-z}) e^z$$

$$\hat{O}(z) = \left(\frac{z^2}{2} + \frac{z^4}{4!}\right) \frac{1}{2} (e^{2z} + 1)$$

$$\hat{O}(z) = \frac{1}{2} \left( \frac{z^2 e^{2z}}{2} + \frac{z^4 e^{2z}}{4!} \right) + \frac{1}{2} \left( \frac{z^2}{2} + \frac{z^4}{4!} \right)$$

Let us recall that  $\left[ \frac{z^n}{n!} \right] \hat{F} \cdot \hat{G} = \sum_{k \geq 0} \binom{n}{k} \left[ \frac{z^k}{k!} \right] \hat{F} \cdot \left[ \frac{z^{n-k}}{(n-k)!} \right] \hat{G}$

Notice that since  $\left[\frac{z^k}{k!}\right] \frac{z^2}{2}$  is only different from 0 when  $k = 2$  and  $\left[\frac{z^k}{k!}\right] \frac{z^4}{4!}$  is only different from 0 when  $k = 4$ , then  $\left[\frac{z^n}{n!}\right] \frac{z^2 e^{2z}}{2} = \binom{n}{2} 2^{n-2}$  and  $\left[\frac{z^n}{n!}\right] \frac{z^4 e^{2z}}{4!} = \binom{n}{4} 2^{n-4}$

Therefore  $O(n) = \begin{cases} 1/2 + \binom{n}{2} 2^{n-3} + \binom{n}{4} 2^{n-5} & \text{if } n = 2, 4 \\ \binom{n}{2} 2^{n-3} + \binom{n}{4} 2^{n-5} & \text{otherwise} \end{cases}$

□

64. Determine all solutions of the recurrence relation:

$$a_n - 2na_{n-1} + n(n-1)a_{n-2} = 2n \cdot n!, \quad n \geq 2, \quad a_0 = a_1 = 1$$

*Hint: Use exponential generating functions*

*Proof.* The recurrence relation can be rewritten as follows:

$$a_n - 2\binom{n}{1}a_{n-1} + 2\binom{n}{2}a_{n-2} = 2n \cdot n!$$

Notice that using the definition for the product of EGFs yields the following equation

$$\hat{A}(z) - 2z\hat{A}(z) + z^2\hat{A}(z) = \sum_{n \geq 2} 2nz^n$$

This implies that

$$\hat{A}(z)(1 - 2z + z^2) = \sum_{n \geq 2} 2nz^n$$

This yields

$$\hat{A}(z) = \sum_{n \geq 2} 2nz^n / (z - 1)^2$$

It follows that

$$\hat{A}(z) = 2\left(\frac{z}{1-z}\right)'(1-z)^{-2}$$

This implies that  $\hat{A}(z) = 2z(1-z)^{-4} = 2z(z-1)^{-4}$ . Applying the negative binomial theorem yields  $\hat{A}(z) = 2z \sum_{n \geq 1} \binom{4+n-1}{n} z^n (-1)^n (-1)^{-n}$ . Therefore

$$\hat{A}(z) = \sum_{n \geq 2} 2\binom{n+2}{n-1} z^n = \sum_{n \geq 2} 2\binom{n+2}{3} z^n = \sum_{n \geq 2} \frac{(n+2)(n+1)(n-1)}{3} z^n.$$

Since  $\hat{A}(z)$  is an EGF, then  $a_n = \begin{cases} 1 & \text{if } n = 0, 1 \\ \frac{(n-1)(n+2)!}{3} & \text{otherwise} \end{cases}$

□

65. Let  $S_{n,k}$  be the Stirling numbers of the second kind, that is, the number of partitions of the set  $\{1, 2, \dots, n\}$  into  $k$  (non-empty) subsets. Show the following formula:

$$\sum_{n,k} S_{n,k} \frac{z^n}{n!} u^k = e^{u(e^z-1)}$$

*Proof.* First we will compute a formula for the Stirling numbers of the second kind.

Assume that  $a_1 + a_2 + \dots + a_k = n$ . For now we will label each partition as  $P_1, P_2, \dots, P_k$  such that  $|P_i| = a_i$ , and later we will remove the labels.

Notice that we can choose from  $\binom{n}{a_1}$  options for  $P_1$ . Now that we have picked the elements for  $P_1$ , we can proceed to pick the elements for  $P_2$ . We have  $\binom{n-a_1}{a_2}$  options for  $P_2$  and inductively we have

$$\binom{n - \sum_{j=1}^{i-1} a_j}{a_i}$$

different options for partition  $P_i$ . Therefore we have

$$\prod_{i=1}^k \binom{n - \sum_{j=1}^{i-1} a_j}{a_i}$$

This corresponds to  $\frac{n!}{a_1! a_2! \dots a_k!}$  (notice that expanding the product makes the rest of the terms cancel out).

Therefore we have

$$S_{n,k} = \frac{n!}{k!} \left( \sum_{a_1+a_2+\dots+a_k=n} \frac{1}{a_1! a_2! \dots a_k!} \right) \quad (1)$$

We added over all possible size assignments to the sets of the partitions  $P_i$ , and then we remove the set labels dividing by  $k!$ .

Now, recall that  $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ . This implies by substitution that  $e^{u(e^z-1)} =$

$\sum_{k=0}^{\infty} \frac{[u(e^z-1)]^k}{k!} = \sum_{k=0}^{\infty} \frac{u^k}{k!} [e^z - 1]^k$ . Notice that  $e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}$ . We can use this information to compute  $[e^z - 1]^k$  as a Cauchy product.

It follows that  $[z^n](e^z - 1)^k = \sum_{a_1 + a_2 + \dots + a_k = n} \frac{1}{a_1! a_2! \dots a_k!} = \frac{k!}{n!} S_{n,k}$ . This implies that  $\sum_{k=0}^{\infty} \frac{u^k}{k!} [e^z - 1]^k = \sum_{k=0}^{\infty} \sum_{n=0}^k S_{n,k} \frac{u^k}{n!} z^n$   $\square$

66. Prove the following representation for the Stirling numbers of the second kind:

$$S_{n,k} = \frac{1}{k!} \sum_{j=0}^k (-1)^k \binom{k}{j} j^n$$

*Remark:* compute first the generating function for  $\sum_{n=0}^{\infty} S_{n,k} \frac{z^n}{n!} = \frac{(e^z - 1)^k}{k!}$

*Proof.* Notice that  $(e^z - 1) = \sum_{k=1}^{\infty} \frac{z^k}{k!}$ , then  $\frac{(e^z - 1)^k}{k!} = \frac{1}{k!} \sum_{\sum a_i = n} \frac{1}{a_1! a_2! \dots a_n!} z^n = \sum_{n=0}^{\infty} S_{n,k} \frac{z^n}{n!}$

From the binomial theorem, it follows that  $(e^z - 1)^k = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} e^{zj} = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \frac{z^n j^n}{n!}$ . Since  $\frac{(e^z - 1)^k}{k!} = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n \frac{z^n}{n!}$  is the EGF for  $S_{n,k}$ , it follows that  $S_{n,k} = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$   $\square$

67. Let  $P$  be the set of all divisors of 12. Determine the Möbius function of  $(P, |)$

*Proof.* Notice that  $(\mathbb{N}^+, |) \cong (\mathbb{N} \times \mathbb{N} \times \dots, \leq)$  since every positive integer can be factorized uniquely as a product of powers of primes with only finitely many greater than 0 powers. Notice then that  $a = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n} | b = p_1^{b_1} p_2^{b_2} \dots p_n^{b_n}$  if and only if  $a_i \leq b_i$  for all  $i \in \{1, \dots, n\}$ .

Therefore the Möbius function for the divisibility order can be computed as a product of Möbius functions with respect to the usual order in  $\mathbb{N}$ .

$$\mu_{|}(a, b) = \prod_{i=1}^n \mu_{\leq}(a_i, b_i)$$

Recall that the Möbius function for the usual order in  $\mathbb{N}$  is given by

$$\mu_{\leq}(n, m) = \begin{cases} 0 & \text{if } n > m \vee m \geq n + 2 \\ 1 & \text{if } n = m \\ -1 & \text{if } m = n + 1 \end{cases}$$

Since  $\mu(n, n) = \delta(n, n) = 1$ ,  $\mu(n, n+1) + \mu(n+1, n+1) = \delta(n, n+1) = 0$  then  $\mu(n, n+1) = -1$  and notice also that  $\delta(n, n+2) = 0 = \delta(n+1, n+2) + \mu(n, n+2) = \mu(n, n+2)$ . Inductively  $\mu(n, m) = 0$  for any  $n+2 \leq m$ .

Notice that this implies that

$$\mu|(a, b) = \begin{cases} 0 & \text{if } a \nmid b \vee b/a = p^2 z, p \in \mathcal{P}, z \in \mathbb{N}^+ \\ (-1)^k & \text{if } b/a = p_1 p_2 \dots p_k, p_i \in \mathcal{P} \end{cases}$$

Where  $\mathcal{P}$  is the set of positive prime numbers.

Notice that since divisibility is transitive then if  $a \mid b \mid 12$  then  $a \mid 12$ . This implies that the Möbius function for  $(\text{div}(12), \mid)$  is a restriction of the Möbius function for  $(\mathbb{N}^+, \mid)$ . Therefore it has the same definition.  $\square$

68. Let  $(P, \leq)$  be the poset defined by  $P = \{0, 1, 2, 3, 4\}$  and the three relations

$$0 \leq 1 \leq 4, \quad 0 \leq 2 \leq 4, \quad 0 \leq 3 \leq 4$$

Compute all values  $\mu(x, y)$  for  $x, y \in P$

*Proof.*  $\mu(n, n) = 1, \mu(1, 4) = \mu(2, 4) = \mu(3, 4) = \mu(0, 1) = \mu(0, 2) = \mu(0, 3) = -1$

Also notice that  $\delta(0, 4) = 0 = \sum_{z \in [0, 4]} \mu(z, 4) = \mu(0, 4) - 3 + 1$  this implies that

$$\mu(0, 4) = 2$$

$\mu(n, m) = 0$  in all the remaining cases.  $\square$

69. Let  $(P_1, \leq_1)$  and  $(P_2, \leq_2)$  be two locally finite posets and  $(P, \leq)$  be defined by  $P = P_1 \times P_2$  and for  $(a, x), (b, y) \in P$ :

$$(a, x) \leq (b, y) \Leftrightarrow a \leq_1 b \wedge x \leq_2 y$$

Show that  $(P, \leq)$  is a poset and that the Möbius functions of  $P, P_1$  and  $P_2$  satisfy

$$\mu_P((a, x), (b, y)) = \mu_{P_1}(a, b) \cdot \mu_{P_2}(x, y)$$

*Proof.*  $\leq$  is **Reflexive**:

Let  $(a, b) \in P$ . Since  $P_1$  and  $P_2$  are posets, then  $a \leq_1 a \wedge b \leq_2 b$ . From the definition  $(a, b) \leq (a, b)$ .

$\leq$  is **Antisymmetric**:

Assume that  $(a, b) \leq (c, d) \wedge (c, d) \leq (a, b)$ . From the definition of  $\leq$ , it follows that  $a \leq_1 c \leq_1 a \wedge b \leq_2 d \leq_2 b$ . Since  $P_1$  and  $P_2$  are posets, then it follows that  $a = c \wedge b = d$ . Therefore  $(a, b) = (c, d)$ , and  $\leq$  is antisymmetric.

$\leq$  is **Transitive**:

Assume that  $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3)$ . From the definition, then  $a_1 \leq_1 a_2 \leq_1 a_3 \wedge b_1 \leq_2 b_2 \leq_2 b_3$ . Since  $P_1$  and  $P_2$  are posets, it follows that  $a_1 \leq_1 a_3 \wedge b_1 \leq_2 b_3$ . From the definition, it follows that  $(a_1, b_1) \leq (a_3, b_3)$ .

This proves that  $(P, \leq)$  is a poset.

Notice that the Kronecker delta for  $P$ ,  $\delta_P = (\delta_{P_1} \circ \pi_1) \cdot (\delta_{P_2} \circ \pi_2)$  where  $\pi_1$  and  $\pi_2$  are the projections on the first and second coordinate respectively.

$$\text{Now notice that } \delta((a, b), (c, d)) = \sum_{(x, y) \in [(a, b), (c, d)]} \mu_P((x, y), (c, d))$$

$$\begin{aligned} \text{On the other hand, } \delta((a, b), (c, d)) &= \delta(a, c) \cdot \delta(b, d) = \sum_{x \in [a, c]} \mu_{P_1}(x, c) \sum_{y \in [b, d]} \mu_{P_2}(y, d) = \\ &\sum_{(x, y) \in [(a, b), (c, d)]} \mu_{P_1}(x, c) \cdot \mu_{P_2}(y, d) \end{aligned}$$

Applying the first equation yields

$$\sum_{(x, y) \in [(a, b), (c, d)]} \mu_P((x, y), (c, d)) = \sum_{(x, y) \in [(a, b), (c, d)]} \mu_{P_1}(x, c) \cdot \mu_{P_2}(y, d)$$

Since we are only considering locally finite posets, then it can be shown by induction over  $|(a, b), (c, d)|$  that  $\mu_P((x, y), (c, d)) = \mu_{P_1}(x, c) \cdot \mu_{P_2}(y, d)$  for any  $(x, y) \in [(a, b), (c, d)]$

$$\mu_P((a, x), (b, y)) = \mu_{P_1}(a, b) \cdot \mu_{P_2}(x, y)$$

□

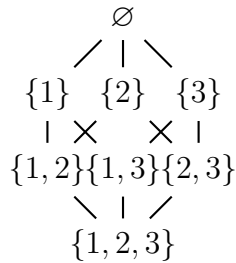
70. Draw the Hasse diagram of the poset  $P := (2^{\{1,2,3\}}, \supseteq)$ .

Let now  $A_1, A_2, A_3 \subseteq M$ . Use Möbius inversion on the poset  $P$  to show the following inclusion-exclusion principle

$$|M \setminus \bigcup_i A_i| = |M| - \sum_i |A_i| + \sum_{i \neq j} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|$$

*Proof.*

□



We consider the function  $f : P \rightarrow \mathbb{R}, f(I) = |M \cap \bigcap_{i \in I} A_i \cap (\bigcap_{j \notin I} M \setminus A_j)|$

The definition for  $S_f(I) = \sum_{J \supseteq I} f(J) = \sum_{J \supseteq I} |\bigcap_{i \in I} A_i \cap (\bigcap_{j \notin I} M \setminus A_j)|$ .

Notice that  $\bigcup_{J \supseteq I} (M \cap \bigcap_{i \in J} A_i \cap (\bigcap_{j \notin J} M \setminus A_j)) = M \cap \bigcap_{i \in I} A_i$  since one contention is trivial by definition and for any  $x \in M \cap \bigcap_{i \in I} A_i$ , if  $x \in A_j$ , then  $S_x = \{j \mid j \notin I, x \in A_j\}$ . It follows that  $x \in M \cap \bigcap_{i \in I \cup S_x} A_i \cap (\bigcap_{j \notin I \cup S_x} M \setminus A_j)$ .

Notice that if  $J \neq Q, J \supseteq I, Q \supseteq I$ , then  $\hat{J} \cap \hat{Q} = \emptyset$  for  $\hat{J} = M \cap \bigcap_{i \in J} A_i \cap (\bigcap_{j \notin J} M \setminus A_j)$  and  $\hat{Q} = M \cap \bigcap_{i \in Q} A_i \cap (\bigcap_{j \notin Q} M \setminus A_j)$ .

This shows that these are pairwise disjoint sets. Therefore  $S_f(I) = |\bigcap_{i \in I} A_i|$

Applying the Möbius inversion theorem, then  $f(\emptyset) = |M \setminus \bigcup_i A_i| = \sum_{J \subseteq \{1, \dots, m\}} (-1)^{|J|} \left| \bigcap_{j \in J} A_j \right|$

By grouping together sets with the same cardinality we get

$$\sum_{i=0}^m \sum_{|J|=i} (-1)^i \left| \bigcap_{j \in J} A_j \right| = |M| - |A_1| - |A_2| \dots - |A_m| + |A_1 \cap A_2| + \dots$$