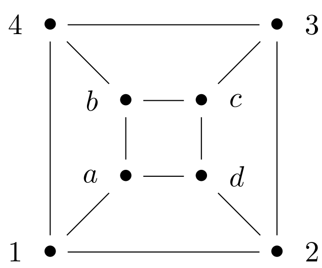
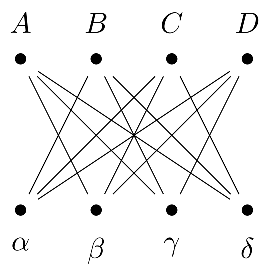
1st EXERCISE
104.272 Discrete Mathematics

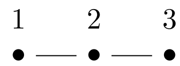
(1) A simple undirected graph is called cubic if each of its vertices has degree 3.
  (a) Find a cubic graph with 6 vertices!
  (b) Is there a cubic graph with an odd number of vertices?
  (c) Prove that for all $n \geq 2$ there exists a cubic graph with $2n$ vertices!

(2) Use a suitable graph theoretical model to solve the following problems:
  (a) Show that in every city at least two of its inhabitants have the same number of neighbours!
  (b) A group of friends goes (separately) on holidays. Each of them sends a postcard to three members of the group. When is it possible that every member of the group receives postcards from precisely those friends to whom he/she sent postcards?

(3) Are the following two graphs isomorphic?





(4) (a) Compute the number of walks of length ℓ from i to j in the graph 

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  (b) How could you use the adjacency matrix to compute the number of triangles (i.e., cycles of length three) in a (loopless) graph? Perform the computation for two graphs of your choice on four vertices.

(5) Let $G = (V, E)$ be a simple graph with at least five vertices. The complement $\bar{G}$ of G is the graph with the same vertex set as G, where two vertices are adjacent if and only if they are not adjacent in G.

Show that at least one of G and $\bar{G}$ contains a cycle. Furthermore, characterise all trees T such that $\bar{T}$ is also a tree.

(6) Prove that the following statements are all equivalent.
  (a) G is a tree, i.e., G is connected and has no cycles.
  (b) Every two vertices of G are connected by a unique path.
  (c) G is connected and $|V| = |E| + 1$.
  (d) G is a minimally connected graph, i.e., every edge is a bridge.
  (e) G is a maximally acyclic graph, i.e., adding any edge yields a cycle.

(7) Show that every graph with at least as many edges as vertices contains a cycle.

(8) Let T be a tree without vertices of degree 2. Show that T has more leaves than internal nodes using the handshaking lemma.

(9) Let G be a graph with at least two vertices. Prove or disprove:
  (a) Deleting a vertex of maximal degree Δ cannot increase the average degree.
  (b) Deleting a vertex of minimal degree δ cannot decrease the average degree.

(10) Let G_n be the graph whose vertices are the permutations of $\{1, 2, \ldots, n\}$. Two vertices in G_n are adjacent if they differ by swapping two numbers next to each other. Show that G_n is connected.

Example G_3 :