6^{th} EXERCISE
104.272 Discrete Mathematics

(51) Use generating functions to find a closed form expression for the sum $\sum_{k=0}^{n}(k^2+3k+2)$.

(52) Prove the following identity:

$$\sum_{n\geq 0}\binom{2n}{n}z^n = \frac{1}{\sqrt{1-4z}}.$$

(53) Compute

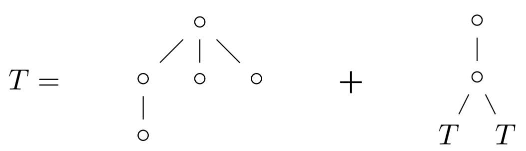
$$[z^n]\frac{z+z^2}{\sqrt[3]{1-2z}}.$$

(54) A t-ary tree is a plane rooted tree such that every node has either t or 0 successors. A node with t successors is called an internal node.

(a) How many leaves has a t-ary tree with n internal nodes?

(b) Moreover, let a_n be the number of t-ary trees with n internal nodes and $A(z)$ the generating function of this sequence. Find a functional equation for $A(z)$.

(55) Compute the number t_n of rooted trees with n nodes described by the following equation:



(56) Compute the number of plane rooted trees with n nodes.

(57) Let A be a regular $(n+2)$-gon with the vertices labelled 0, 1, $\ldots$, n and $n+1$. Triangulations are decompositions of A into n triangles such that the 3 vertices of each triangle are also vertices of A.

(a) Show that the set $\mathcal{T}$ of triangulations of regular polygons can be described as a combinatorial construction satisfying

$$\mathcal{T} = \{\varepsilon\} \cup \mathcal{T} \times \Delta \times \mathcal{T}$$

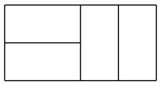
where Δ denotes the single triangle and ε denotes the empty triangulation (consisting of no triangle and corresponding to the case $n = 0$).

(b) What is the number of triangulations of A?

(58) Let $\mathcal{L}$ denote the set of words over the alphabet $\{a, b\}$ that contain exactly k occurrences of b. Find a closed formula for the number of words of length n in $\mathcal{L}$ (i.e. with exactly n letters) by finding a specification of $\mathcal{L}$ as a combinatorial construction and translating it into generating functions.

(59) Let d_n be the number of different ways one can tile a $2 \times n$ rectangle with 2×1 blocks (dominoes). Compute the associated generating function using a recursive combinatorial

construction. For example, here is a tiling of the 2×4 rectangle:



(60) Let $s_{n,k}$ be the Stirling numbers of the first kind, that is, the number of permutations of $\{1, 2, \ldots, n\}$ where the cycle representation has exactly k cycles.

(a) Prove the following identity:

$$s_{n,k} = \frac{n!}{k!} \sum \left(\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_k} \right),$$

where the sum is taken over all compositions of n, i.e. ways of writing n as the **unordered** sum of k natural numbers: $n = a_1 + a_2 + \cdots + a_k$.

(b) Use (a) to prove the following three identities:

$$s_{n,1} = (n-1)!, \qquad s_{n,n-1} = \binom{n}{2}, \qquad s_{n,n} = 1.$$

(c) Use also (a) to prove

$$\sum_{n,k} s_{n,k} \frac{z^n}{n!} u^k = e^{u \log \frac{1}{1-z}} = \frac{1}{(1-z)^u}.$$