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104.272 Discrete Mathematics, Group 1 (Professor Gittenberger)
8. Exercise, Due 4 December, 2019

71. Prove the following assertions:

(a) Every finite lattice has a 0-element and a 1-element.

Proof. We proceed by induction on the cardinality of the lattice.

Base

The only element in a singleton set $\{x\}$ lattice is both the 0-element and the 1-element

Induction Hypothesis

Assume that every finite lattice with k elements has a 0-element and a 1-element.

Inductive step

Let $x_1, \dots, x_{k+1}$ be the elements of the lattice. Notice that $x_1, \dots, x_k$ form a lattice with k elements. By applying the IH, then there is a 0-element and an 1-element for $x_1, \dots, x_k$. Let w_0 be the 0-element, and w_1 the 1-element. Notice that since $x_1, \dots, x_{k+1}$ form a lattice, then $w_1 \vee x_{k+1}$ is a 1-element of the whole lattice. Symmetrically $w_0 \wedge x_{k+1}$ is the 0-element of the whole lattice.

□

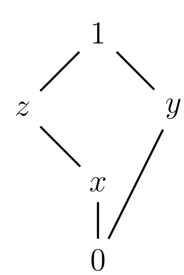
(b) In every lattice L we have $(x \wedge y) \vee y = y$ for all $x, y \in L$.

Proof. Notice that from definition $x \wedge y \leq y$, and $(x \wedge y) \vee y \geq y$. Also notice from the definitions of $\wedge$ and $\vee$, that they preserve the partial order on the lattice. Therefore $(x \wedge y) \vee y \leq y \vee y = y$. This implies that $(x \wedge y) \vee y \leq y \leq (x \wedge y) \vee y$. Since the lattice is also a poset, from antisymmetry we have that $y = (x \wedge y) \vee y$ □

(c) There exists a lattice such that the following implication is not true:

$$x \leq z \Rightarrow \forall y \ : \ x \vee (y \wedge z) = (x \vee y) \wedge z$$

Proof. Consider the lattice L given by the following Hasse diagram:



Notice that $y \wedge z = 0$, $x \vee y = 1$, $x \vee (y \wedge z) = x$, $(x \vee y) \wedge z = z$. In this lattice $x \neq z$. Therefore the equation does not hold. $\square$

72. Let $(P, \leq)$ be a finite poset. A subset $C \subseteq P$ is called a *chain* if $(C, \leq)$ is a linearly ordered set. A subset $A \subseteq P$ is called an *antichain* if no two elements of A are comparable with respect to $\leq$. A *chain cover* of P is a partition $P = C_1 \cup C_2 \cup \ldots \cup C_k$ in which all the C_i are chains. Dilworth's Theorem asserts that the size of any largest antichain is equal to the number of chains in a smallest chain cover.

    Use Dilworth's Theorem to prove that every poset with at least $rs + 1$ elements has either a chain with $r + 1$ elements or an antichain with $s + 1$ elements.

    Proof. Assume by elimination of the disjunction that there is no antichain with $s + 1$ elements in the lattice L, therefore if $\mathcal{C}^*$ is an antichain , then it has at most s elements. Notice that the smallest chain cover for L has at most s elements. Let $C_1, \ldots, C_s$ be the smallest chain cover for L. Applying the pigeonhole principle yields that there exists an i such that $|C_i| \geq r + 1$. Let C be any $r + 1$ element subset of C_i. Notice that C is a chain of L with $r + 1$ elements.

    $\square$

73. Two numbers x and y are called relatively prime if their greatest common divisor is 1. Let p, q, r be three distinct prime numbers and $m = pqr$. How many of the numbers $1, 2, \ldots, m$ are relatively prime to m.

    Proof. Notice that the numbers that relatively prime to $m = pqr$ are $[m] = \{1, \ldots, m\} \setminus (P \cup Q \cup R)$, where P, Q and R are the multiples of p, q and r respectively.

    We can compute this by using the inlcusion-exclusion principle for 3 sets (Exercise 70).

    Therefore $|[m] \setminus (P \cup Q \cup R)| = m - |P| - |Q| - |R| + |P \cap Q| + |P \cap R| + |Q \cap R| - |P \cap Q \cap R|$.

    Notice that there are $m/p = q \cdot r$ multiples of p. Therefore $|P| = q \cdot r$. Likewise $|Q| = p \cdot r$, $|R| = p \cdot q$.

    Also notice that $P \cap Q$ are the numbers that are multiples of both P and Q. Since p and q are primes (and therefore relatively primes to each other), then the common multiples of p and q correspond to the multiples of pq. This implies that $|P \cap Q| = \frac{m}{p \cdot q} = r$. By analogy $|P \cap R| = q$ and $|Q \cap R| = p$.

    Notice that since p, q, r are all primes (and therefore pairwise relatively prime), then the common multiples of p, q, r consist of the multiples of $pqr = m$. There is only m in $P \cap Q \cap R$.

By substituting the inclusion exclusion formula we get that

$$|[m] \setminus (P \cup Q \cup R)| = m - qr - pr - pq + r + q + p - 1$$

$$= pqr - qr - pr - pq + r + q + p - 1$$
$$= q(r(p-1)+1) - p(r-q+1) + r - 1$$

$\square$

74. Let a, b, c, d be integers. Prove:

(a) If $a|b$ and $a|c$, then for all integers x, y we have a $a|(xb + yc)$.

Proof. Since $a|b$, then $a \cdot z_1 = b$. Likewise $a \cdot z_2 = c$ for some $z_1, z_2 \in \mathbb{Z}$. Notice that $a(x \cdot z_1 + y \cdot z_2) = x \cdot b + y \cdot c$. This is the definition of $a|(xb + yc)$. $\square$

(b) If $\gcd(a, b) = 1$ and $c|a$ and $d|b$, then $\gcd(c, d) = 1$.

Proof. Let $\gcd(c, d) := m$. Then $m|c$ and $m|d$. From the transitivity of $|$, we have that $m|a$ and $m|b$. Since $\gcd(a, b) = 1$, this implies that $m|1$. Since the gcd is always a positive number, then $m = \gcd(c, d) = 1$ (since the only divisors of 1 are 1 and -1).

$\square$

(c) If $a|c$ and $b|c$ and $\gcd(a, b) = 1$, then $ab|c$.

Proof. Notice the equality $\gcd(a, b) \cdot \text{lcm}(a, b) = p \cdot q$. For this simply consider $a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$. Notice that $\gcd(a, b) = p_1^{min(a_1,b_1)} p_2^{min(a_2,b_2)} \dots p_k^{min(a_k,b_k)}$.
Respectively $\text{lcm}(a, b) = p_1^{max(a_1,b_1)} p_2^{max(a_2,b_2)} \dots p_k^{max(a_k,b_k)}$.
Notice that $min(p, q) + max(p, q) = p + q$ for any $p, q \in \mathbb{Z}$. Therefore $\gcd(a, b) \cdot \text{lcm}(a, b) = a \cdot b$.
Since $\gcd(a, b) = 1$, then $\text{lcm}(a, b) = a \cdot b$. Notice that c is a common multiple for a and b. Therefore $a \cdot b = \gcd(a, b)|c$ $\square$

75. Prove that if x and y are odd integers, then $2|(x^2 + y^2)$ but $4 \not| (x^2 + y^2)$.

Proof. Notice that x^2 is odd and y^2 is odd, therefore $x^2 + y^2$ is even (divisible by 2).

Notice that $x \underset{\text{mod } 4}{\equiv} 1$ or $x \underset{4}{\equiv} -1$ since x is odd. The same applies for y.

This implies that $x^2, y^2 \underset{\text{mod } 4}{\equiv} 1$. Therefore $x^2 + y^2 \underset{\text{mod } 4}{\equiv} 2$. This shows that $4 \not| x^2 + y^2$ $\square$

76. Prov that for every integer n, the number $n^2 - n$ is even and $n^3 - n$ is a multiple of 6.

Proof. Notice that for all $x \in \mathbb{Z}_2$ $x^2 = x$. This implies that $n^2 \underset{\text{mod } 2}{\equiv} n$. Therefore $n^2 - n \underset{\text{mod } 2}{\equiv} 0$. This implies that $n^2 - n$ is even.

Notice that in $\mathbb{Z}_6$ $0^3 = 0; 1^3 = 1; 2^3 = 2; 3^3 = 3; 4^3 = 4; 5^3 = 5$. Therefore for all $x \in \mathbb{Z}_6$, $x^3 = x$. This implies that $n^3 \underset{\text{mod } 6}{\equiv} n$. This implies that $6|n^3 - n$. □

77. Consider two integers a and b such that $\gcd(a, 4) = 2$ and $\gcd(b, 4) = 2$. Prove that in this case $\gcd(a + b, 4) = 4$.

Proof. Notice that since $\gcd(a, 4) = 2$, then $4 \not| a$. Since $2|a$, then $a \underset{\text{mod } 4}{\equiv} 2$. The same argument can be repeated for b.

Therefore $a + b \underset{\text{mod } 4}{\equiv} 0$. Therefore $\gcd(a + b, 4) = 4$. □

78. Prove that any two positive integers a, b satisfy $\gcd(a, b)\cdot$

Proof. Consider $a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$. Notice that $\gcd(a, b) = p_1^{min(a_1,b_1)} p_2^{min(a_2,b_2)} \dots p_k^{min(a_k,b_k)}$.

Respectively $\operatorname{lcm}(a, b) = p_1^{max(a_1,b_1)} p_2^{max(a_2,b_2)} \dots p_k^{max(a_k,b_k)}$.

Notice that $min(p, q) + max(p, q) = p + q$ for any $p, q \in \mathbb{Z}$. Therefore $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = a \cdot b$. □

79. Use the Euclidean algorithm to find two integers a and b such that $420a+546b = 42$

Proof.

$$546 = 420 \cdot 1 + 126$$

$$420 = 126 \cdot 3 + 42$$

Therefore $42 = 420 - 3 \cdot 126$, and $126 = 546 - 1 \cdot 420$

$$42 = 420 - 3 \cdot (546 - 1 \cdot 420)$$

$$= 4 \cdot 420 - 3 \cdot 546$$

□

80. Prove that there exist infinitely many prime numbers p which are solutions of the equation $p \equiv 3 \text{ mod } 4$

Hint: Assume that there are only finitely many such primes, say $p_1, \dots, p_n$, and consider the number $4p_1p_2 \dots p_n - 1$.

Proof. Let $z := 4p_1p_2\dots p_n - 1$. Notice that $4p_1p_2\dots p_n \underset{\text{mod } 4}{\equiv} 0$. Therefore $4p_1p_2\dots p_n - 1 \underset{\text{mod } 4}{\equiv} 3$. Notice that since $4p_1\dots p_n - (4p_1\dots p_n - 1) = 1$, then $\gcd(z, p_i) = 1$.

Therefore z is a product of powers of primes $q_1\dots q_m$ such that $\{p_1,\dots,p_n\} \cap \{q_1,\dots q_m\} = \varnothing$. Notice that since $p_1\dots p_n$ are the only primes that are equivalent to 3 modulo 4, then for $i \in \{1,\dots,m\}$ $q_i \underset{\text{mod } 4}{\not\equiv} 3$. Also since $z \underset{\text{mod } 4}{\equiv} 3$, then $i \in \{1,\dots,m\}$, $q_i \underset{\text{mod } 4}{\not\equiv} 2, 4$. Therefore for all $i \in \{1,\dots,m\}$, $q_i \underset{\text{mod } 4}{\equiv} 1$. Since z is a multiple of only q_i factors, then $z \underset{\text{mod } 4}{\equiv} 1$. This contradicts that $z \underset{\text{mod } 4}{\equiv} 3$. □