4^{th} EXERCISE
104.272 Discrete Mathematics

(31) Conclude from the statement

a connected graph has an Eulerian circuit if and only if all its vertices have even degree

that a graph contains an Eulerian trail from u to v if and only if u and v are the only vertices of odd degree of the graph.

(32) (a) The line graph $\bar{G}$ of a simple undirected graph $G = (V, E)$ is the (simple) graph with vertex set E, with an edge between two vertices of $\bar{G}$, if and only if the corresponding edges are incident to a common vertex of G.
Show that the line graph of an Eulerian graph is Eulerian and Hamiltonian, and that the line graph of a Hamiltonian graph is also Hamiltonian. If $\bar{G}$ is Hamiltonian, can we conclude that G is Hamiltonian?

(b) Is a subdivision of an Eulerian graph Eulerian? Is a subdivision of a Hamiltonian graph Hamiltonian?

(33) For which m and n does the complete bipartite graph $K_{m,n}$ have a Hamiltonian cycle?

(34) Show that the n-dimensional hypercube is Hamiltonian, for $n \geq 2$.

(35) Prove that a graph G is bipartite if and only if each cycle in G has even length.

(36) Let M be a matching in a simple undirected graph $G = (V, E)$. A path P in G is *alternating* if every second edge of P belongs to M. An alternating path P is *extanding* if both the first and the last vertices of P are not covered by an edge in M.

Let P be an extending alternating path, and let $M \triangle P := (M \setminus P) \cup (P \setminus M)$. Prove that $M \triangle P$ is a matching, and that $|M \triangle P| = |M| + 1$.

(37) A graph is planar if there exists a drawing in the plane $\mathbb{R}^2$ such that no two edges intersect, except at their vertices.

Given such a drawing of a connected planar graph G, the dual G^* of G (with respect to the drawing) has as vertices the 'regions' (or faces) of the embedding. Two vertices of G^* are connected by an edge, if and only if the two corresponding regions are separated by an edge.

Show that G and G^* have the same number of spanning trees.

(38) Let G be an undirected simple graph and H a subgraph of G satisfying $H \cong K_n$ for some n. What can be said about the relationship between the chromatic numbers $\chi(G)$ and $\chi(H)$? Find a graph G with $\chi(G) = 3$ which does not admit K_3 as a subgraph.

For the following two exercises, use a suitable graph model to reformulate them as graph theoretical problems.

(39) Given a subset $A \subseteq \mathbb{R}^2$ with area a and two decompositions of A into subsets $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ such that all the sets A_i and B_i have the same area a/m. Prove that there exists a permutation π of $\{1, 2, \ldots, m\}$ such that for all $i = 1, \ldots, m$ we have $A_i \cap B_{\pi(i)} \neq \emptyset$.

(40) Given a set $A = \{a_1, \ldots, a_n\}$ with n elements and $B = \{B_1, \ldots, B_n\} \subseteq 2^A$. Prove that there exists an injective mapping $f : B \to A$ such that $f(B_i) \in B_i$ for all $i \in \{1, 2, \ldots, n\}$ if and only if for all $I \subseteq \{1, 2, \ldots, n\}$ it holds that $|\bigcup_{i \in I} B_i| \geq |I|$.