2^{nd} EXERCISE
104.272 Discrete Mathematics

If not stated otherwise, we consider undirected graphs.

(11) A graph $H = (V', E')$ is an induced subgraph of $G = (V, E)$, if $V' \subseteq V$ and any edge in G connecting two vertices a, b in V' is in E'. Let G be a connected simple graph that does not have path or cycle with four vertices as an induced subgraph.

Show that G has a vertex adjacent to all other vertices.

Hint: consider a vertex of maximum degree u and assume that it is not connected with some vertex w. Consider the shortest path between u and w and show that there exists a vertex in this path whose degree is larger than that of u.

(12) Let G be a connected graph with an even number of vertices. Show that G has a spanning (but not necessarily connected) subgraph with all vertices of odd degree. Show that this is not necessarily the case for arbitrary graphs.

(13) Let T be a tree and let n_d be the number of vertices of degree d in T. Show that the number of leaves of T equals

$$2 + \sum_{d \geq 3} (d-2) n_d.$$

(14) Show that the number of spanning trees of the complete graph on n vertices K_n is n^{n-2}, using the matrix tree theorem. Hint: To compute the determinant of the resulting matrix, add all rows except the first one to the first row. Then add the first row of this new matrix to the other rows.

(15) Let G be a connected graph with n vertices. Let G_T be the graph having the spanning trees of G as vertices, with two vertices s and t being adjacent if and only if the corresponding spanning trees in G share precisely $n-2$ edges.

Show that G_T is connected. Hint: look at the proof of the correctness of Kruskal's algorithm given in the lecture.

(16) Show that all bases of a matroid $M = (E, S)$ have the same cardinality.

(17) Let $G = (V, E)$ be an undirected graph. Set $M_k(G) = (E, S)$ where

$$S = \{A \subseteq E : A = F \cup M, F \text{ acyclic and } |M| \leq k\}.$$

Show that $M_k(G)$ is a matroid.

(18) Let E_1 and E_2 be two disjoint sets. Moreover, assume that (E_1, S_1) and (E_2, S_2) are matroids. Define $S := \{X \cup Y \mid X \in S_1 \text{ and } Y \in S_2\}$. Prove that $(E_1 \cup E_2, S)$ is a matroid.

(19) Let J be the set of jobs $\{0, 1, 2, 3, 4\}$ and W be the set of workers $\{04, 0, 0123, 12\}$. Suppose that a job can be done if its number appears in the 'name' of the worker.

List all maximal sets of jobs that can be done simultaneously, i.e., the bases of the matroid considered in the lecture. Then use the greedy algorithm to find an optimal job assignment, where the priority of a job is given by its number.

(20) Let V be a vector space and E be a finite subset of V. Let I be the set of linearly independent subsets of E. Show that (E, I) is a matroid.