Machine Learning Exercise 3 - 2021

This exercise is composed of 2 parts:

1. probability theory part

2. coding part

Probability Theory Questions

1. Given a random sample $\{x_1, x_2, \dots, x_n\}$, derive the maximum likelihood estimator p of the Binomial distribution.

$$B(x,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

First for simplicity we take the ln of the function.

$$= ln\left[\binom{n}{x} p^x (1-p)^{n-x}\right] = ln\left(\binom{n}{x}\right) + ln(p) * x + ln(1-p) * (n-x)$$

$now\ we\ dereive$:

$$0 + x * \frac{1}{p} + x * \frac{1}{1-p} - n * \frac{1}{1-p}$$

set derivative to zero:

$$0 + x * \frac{1}{p} + x * \frac{1}{1-p} - n * \frac{1}{1-p} = 0 \Rightarrow x - np = 0 \Rightarrow p = \frac{x}{n}$$

2. A student wants to know her chances to pass and fail an exam if she studies and if she doesn't study. From last year's results, she sees that $P(pass) = 60\%$. She also found out that $P(studied|pass) = 95\%$, $P(studied|failed) = 60\%$. You can assume that every student either studied or didn't study, and either passed or failed.
   a. What is her probability of passing the exam if she studies?
      Define:
      A – pass, A^{-} - failed, B - $studied$, B^{-}did not $studied$.
      P(A) = 0.6, P(A^{-}) = 0.4, P(B|A) = 0.95, P (B| A^{-}) = 0.6, P(B^{-}|A) = 0.05
      We are looking for P(A|B) (Bayes theorem) = P(A) * P(B|A) / P(B) thus we need to find P(B) ,note that P(B|A)=0.95 = P(A∩B)/P(A) ⇒ P(A∩B) = 0.57 and P(B| A^{-})= 0.6 = P(A^{-}∩B)/P(A^{-}) ⇒ P(A^{-}∩B) = 0.24 thus P(B) = 0.81 now we can calculate the requested probability , P(A|B) = P(A) * P(B|A) / P(B) = 0.6* 0.95 / 0.81 = 0.703
   b. What is her probability of passing if doesn't study?
      P(A|, B^{-}) = P(A) * P(B^{-}|A) / P(B^{-})   = 0.6*0.05/0.19 = 0.158

3. Find 3 random variables X, Y, C such that:
   a. $X \perp Y \mid C$ (X and Y are independent given C).
   b. X and Y are not independent.
   c. X, Y are integers such that $3 \leq X, Y \leq 9$ and C is binary.
   d. The following conditions hold:
      i. $P(1 \leq X \leq 5) = 0.4$
      ii. $P(1 \leq Y \leq 5) = 0.4$
      iii. $P(C = 0) = 0.3$

   You need to specify the value of $P(X = x, Y = y, C = c)$. How many relevant values exist?

<u>We define:</u>

Let define C Such that p(C=0) =0.3, P(C=1) = 0.7, now define Y such that:

X$\in$ {3,4,5,6,7,8,9}
$P(3 \leq X \leq 9|C = 0) = 0$
$P(1 \leq X \leq 5|C = 1) = 4/7$
$P(6 \leq X \leq 8|C = 1) = 0$
$P(X = 9|C = 1) = 0.4$

Y $\in$ {3,4,5,6,7,8,9}
$P(3 \leq Y \leq 9|C = 1) = 0$
$P(1 \leq Y \leq 5|C = 0) = 1$
$P(6 \leq Y \leq 8|C = 0) = 0$
$P(Y = 9|C = 0) = 0$

Thus, X and Y are dependent:
P (X = 9) < P (X=9|Y=9)

$P(1 \leq X \leq 5) = P(1 \leq X \leq 5|C = 1) * P(C = 1) + P(1 \leq X \leq 5|C = 0) * P(C = 0) = 4/7 * 0.7 + 0 * 0.3 = 0.4$
$P(1 \leq Y \leq 5) = P(1 \leq Y \leq 5|C = 0) * P(C = 0) + P(1 \leq Y \leq 5|C = 1) * P(C = 1) = 1 * 0.3 + 0 * 0.7 = 0.4$
$P(C = 0) = 0.3$

4. The probability of Wolt arriving on time is 0.75.
   <u>Define:</u>
   A- On time, $\bar{A}$ late $P(A)=0.75$ and $P(A^-) = 0.25$
      a. What is the probability of having 2 on-time meals in a week (7 days)?
         This is the equivalent to have 2 success out of 7 in a Bernoulli trail thus we compute:
         $P(X=2) = \binom{7}{2} * 0.75^2 * 0.25^5$ = 0.01153
      b. What is the probability of having at least 4 on-time meals in a week?
         having at least 4 on-time meals is equal to (1-(0 meals on time + 1 meal on time + 2 meals on time + 3 meals on time)) = 1 – (P(X=0) +P(X=1) +P(X=2) + P(X=3)) = 1 – 0.07055099 = 0.92944901.
      c. A company of 100 employees recorded the number of on-time meals they had during a particular week and averaged their results. What do you expect the value of that average to be?
         as all "Wolters" are independent and each one as the probability of 0.75 to be on time, we use the Expected value formula of the binomial distribution i.e., n*P to get 5.25 on time meals per week and that is the average.

Coding exercise

Follow the instructions supplied for you in the MAP classifier Jupyter notebook.