Eidgenössische Technische Hochschule Zürich
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On Products of Symmetries fixing a Quadric and their minimal Length

Master Thesis

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Abstract

A point P outside a *quadric* $\mathbf{P}\mathrm{Q}$ of a projective space gives rise to an involution $\hat{\sigma}_P$ which exchanges the two intersection points of $\mathbf{P}\mathrm{Q}$ with every line $\mathfrak{g}$ through P. Maps of the form $\hat{\sigma}_P$ are called *symmetries* and by $\hat{\mathrm{O}}$ we denote the group of products of symmetries. The minimal amount of symmetries needed to represent a given map $\hat{\pi} \in \hat{\mathrm{O}}$ is called the *length of* $\hat{\pi}$.

This thesis gives a formula for the length of $\hat{\pi}$ and an optimal bound depending on the signature of the quadratic space.

We will see that every symmetry $\hat{\sigma}_P$ is induced by a simple isometry on the underlying orthogonal space (V, f). An extended version of the theorem of Cartan-Dieudonné-Scherk gives the length for every isometry in the extended orthogonal group. Using this theorem we can derive a formula for the length of any map in $\hat{\mathrm{O}}$.

Lastly, if the index of the quadratic space $(\mathbb{R}^n, \mathrm{Q})$ is at most 1, we geometrically show that every $\hat{\pi} \in \hat{\mathrm{O}}$ can be written as a product of at most n symmetries.

Contents

Chapter 1

Symmetries of the unit Circle

Consider the unit circle $\mathcal{C} = \{(x,y)^\intercal \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. It is well known that every line $l : a \cdot x + b \cdot y + c = 0$ intersects $\mathcal{C}$ in at most two points. This allows us to define the symmetry at $P \in \mathbb{R}^2 \setminus \mathcal{C}$,

$$\hat{\sigma}_P : \mathcal{C} \to \mathcal{C}; \quad X \mapsto Y \quad \text{where } \overline{PX} \cap \mathcal{C} = \{X, Y\}. \tag{1.1}$$

So if the line $\overline{PX}$ is a secant, then $\hat{\sigma}_P$ exchanges the two intersection points X and Y. However, if $\overline{PZ}$ is a tangent of $\mathcal{C}$, we get $\hat{\sigma}_P(Z) = Z$ which can be seen in Fig. 1.1.

Notice that $\hat{\sigma}_P$ is an involution, $\hat{\sigma}_P^2 = \mathrm{id}_C$, and, therefore, a bijection.

Consider the group $\hat{\mathrm{O}}$ which is generated by $\{\hat{\sigma}_P : P \in \mathbb{R}^2 \setminus \mathcal{C}\}$. Hence, every map $\hat{\pi} \in \hat{\mathrm{O}}$ is of the form $\hat{\pi} = \hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_k}$ for points $P_1, \ldots, P_k \in \mathbb{R}^2 \setminus \mathcal{C}$. The goal of this thesis is to study $\hat{\mathrm{O}}$ and to give the minimal number of symmetries needed to represent a given map $\hat{\pi} \in \hat{\mathrm{O}}$.

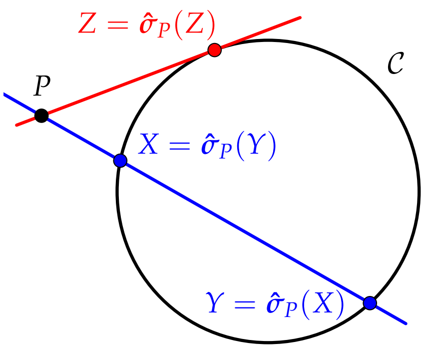

Figure 1.1: The symmetry $\hat{\sigma}_P$ exchanges the points X, Y and fixes Z.

1.1 Real projective plane

However, this will be easier if we change from affine to projective geometry. The set of all subspaces of $\mathbb{R}^3$

$$\mathbf{P}^2\mathbb{R} = \{U \subset \mathbb{R}^3 : U \text{ is a subspace}\}$$

is called the *real projective plane*. The 1-dimensional, 2-dimensional subspaces in $\mathbf{P}^2\mathbb{R}$ are called *points*, *lines* respectively. One says a point P lies on the line $\mathfrak{g}$ if $P \subset \mathfrak{g}$.

For two distinct point $P, Q \in \mathbf{P}^2\mathbb{R}$ there is a unique line $\overline{PQ} := Q \oplus P \in \mathbf{P}^2\mathbb{R}$ which goes through P and Q. On the other hand, two distinct lines $\mathfrak{g}, \mathfrak{h} \in \mathbf{P}^2\mathbb{R}$ will always intersect themselves in the unique point $\mathfrak{g} \cap \mathfrak{h} \in \mathbf{P}^2\mathbb{R}$.

The map $i : (x, y)^\intercal \in \mathbb{R}^2 \mapsto (x, y, 1)^\intercal \in \mathbb{R}^3$ embeds $\mathbb{R}^2$ bijectively on the plane $\{(x, y, z)^\intercal \in \mathbb{R}^3 : z = 1\} \subset \mathbb{R}^3$. Notice that $i(g)$ is well-defined if we view the line g as the set of points which lie on g.

Hence, for a point $(x, y)^\intercal$ or a line g of $\mathbb{R}^2$, one can uniquely define the point $[x, y, 1] := \langle i((x, y)^\intercal) \rangle$ or the line $\mathfrak{g} := \langle i(g) \rangle$ in $\mathbf{P}^2\mathbb{R}$. It is not immediately obvious that $\langle i(g) \rangle$ is 2-dimensional. We leave this to be proven by the reader and instead reason visually with Fig. 1.2.

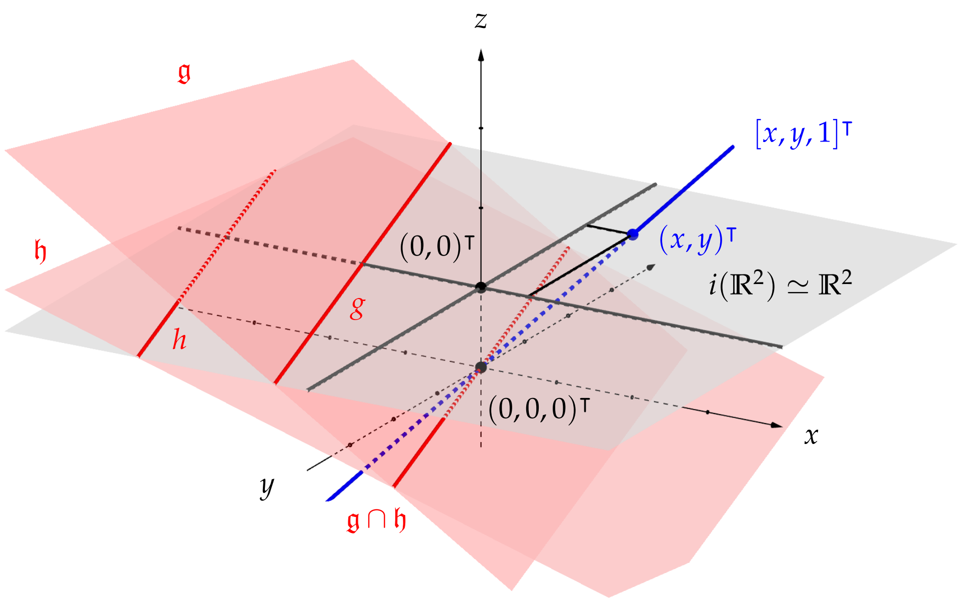

Figure 1.2: The point (x, y) and the lines g, h of $\mathbb{R}^2$ are identified with the point $[x, y, 1]^\intercal$ and the lines $\mathfrak{g}, \mathfrak{h}$ of $\mathbf{P}^2\mathbb{R}$. Since g, h are parallel, the intersection $\mathfrak{g} \cap \mathfrak{h}$ is an infinite point.

This correspondence extends the geometry of $\mathbb{R}^2$ to the geometry of $\mathbf{P}^2\mathbb{R}$. Indeed, if $(x,y)^\intercal$, $[x,y,1]$ and g, $\mathfrak{g}$ are corresponding points and lines respectively, then

$$(x,y)^\intercal \in g \Leftrightarrow [x,y,1]^\intercal \subset \mathfrak{g}. \tag{1.2}$$

All points and lines in $\mathbf{P}^2\mathbb{R}$ which intersect the plane $i(\mathbb{R}^2)$ are called *finite* and otherwise *infinite*. One easily sees that exactly the finite points and lines have counterparts in $\mathbb{R}^2$.

Investigating infinite points and lines, one can algebraically show that:

- Every infinite point is of the form $[x,y,0]$.
- A finite line goes trough exactly one infinite point.
- Two distinct finite lines $\mathfrak{g},\mathfrak{h}$ are parallel if and only if $\mathfrak{g}\cap\mathfrak{h}$ is an infinite point.
- There is exactly one infinite line $\{(x,y,z)^\intercal \in \mathbb{R}^3 : z = 0\}$ in $\mathbf{P}^2\mathbb{R}$. It is called the *line at infinity* and consist of all infinite points.

Again we refer to Fig. 1.2 for a visual proof.

The correspondence (1.2) shows that finite points and lines in $\mathbf{P}^2\mathbb{R}$ can be viewed as points and lines of $\mathbb{R}^2$. However, since we can not depict infinite points and lines with $\mathbb{R}^2$, we have to keep the rules above in mind if we want to work with them.

Remark 1.1 *Instead of defining $\mathbf{P}^2\mathbb{R}$ as the set of subspaces of $\mathbb{R}^3$ and then embed $\mathbb{R}^2$ into it, one can add infinite points and lines to $\mathbb{R}^2$ and define axioms similar to the rules above. This construction yields a geometry which is equivalent to $\mathbf{P}^2\mathbb{R}$. See [1] by Lenz for a rigours argument.*

1.2 The group of PC

Recall that we want to study symmetries $\hat{\sigma}_P$ of a circle. Under the correspondence (1.2), points on the unit circle $\mathcal{C}$ are mapped to the *projective circle*

$$\begin{aligned} \mathbf{PC} &= \{[x,y,1]^\intercal \in \mathbf{P}^2\mathbb{R} : x^2+y^2-1=0\} \\ &= \{[x,y,z]]^\intercal \in \mathbf{P}^2\mathbb{R} : x^2+y^2-z^2=0\}. \end{aligned}$$

Notice, we can extend the definition of the symmetry at P, (1.1), to all points $P \in \mathbf{P}^2\mathbb{R} \setminus \mathbf{PC}$. Indeed, since there are no infinite points on $\mathbf{PC}$, lines still intersect $\mathbf{PC}$ in at most two points.

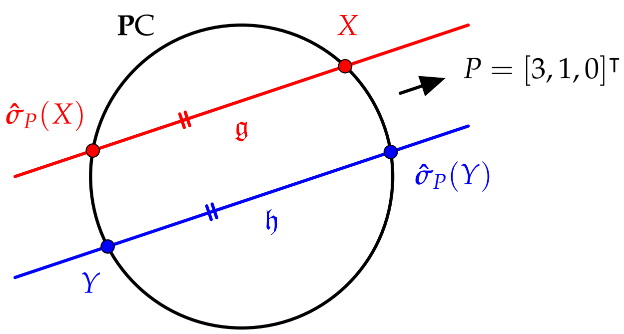

Figure 1.3: If P is an infinite point, for example, $P = [3,1,0]^\intercal$, the lines $\mathfrak{g} = \overline{XP}$, $\mathfrak{h} = \overline{YP}$ are parallel.

However, we now introduce symmetries at infinite points. This has the effect that we allow 'parallel' symmetries in the affine setting of $\mathbb{R}^2$, as seen in Fig. 1.3.

Nevertheless, we will see in Theorem 6.1 that a symmetry at an infinite point is in fact a product of 3 symmetries at finite points. Hence, the group $\hat{\mathrm{O}}$ does not change if we include them as generators and we redefine

$$\hat{\mathrm{O}} = \{\hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_k} : k \in \mathbb{N}, P_1, \ldots, P_k \in \mathbf{P}^2\mathbb{R} \setminus \mathbf{PC}\}.$$

One natural question is to ask how many symmetries are needed to express a given map $\hat{\pi} \in \hat{\mathrm{O}}$. Hence, we want to find the *length of* $\hat{\pi}$

$$\mathrm{l}(\hat{\pi}) = \min\{k \in \mathbb{N} : \hat{\pi} = \hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_k}\}.$$

Theorem 10, [2, p. 8], by Halbeisen, Hungerbühler, and Schiltknecht shows that $\mathrm{l}(\hat{\pi}) \leq 2$ for all $\hat{\pi} \in \hat{\mathrm{O}}$.

Theorem 1.2 (Halbeisen, Hungerbühler, and Schiltknecht) *If* **PC** *is the projective circle of* $\mathbf{P}^2\mathbb{R}$ *and*

$$\hat{\pi} = \hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_k} \in \hat{\mathrm{O}}$$

a product of symmetries, then there are two points $P, Q \in \mathbf{P}^2\mathbb{R} \setminus \mathbf{PC}$ *such that* $\hat{\pi} = \hat{\sigma}_P \circ \hat{\sigma}_Q$. *Additionally, if* $\hat{\pi} \neq \mathrm{id}$ *the line* $\overline{PQ}$ *is unique and for every* $P' \in \overline{PQ}$ *there exist exactly one* $Q' \in \overline{PQ}$ *such that* $\hat{\pi} = \hat{\sigma}_{P'} \circ \hat{\sigma}_{Q'}$.

Example 1.3 *Theorem 1.2 has a neat visualisation. For example if $k = 4$, then for points $P_1, \ldots, P_4 \in \mathbf{P}^2\mathbb{R} \setminus \mathbf{PC}$ one can construct $P_5, P_6 \in \mathbf{P}^2\mathbb{R} \setminus \mathbf{PC}$ such that $\hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_4} = \hat{\sigma}_{P_6} \circ \hat{\sigma}_{P_5}$. See [2] for the construction.*

Let $X_1 \in \mathbf{PC}$ and define inductively $X_{i+1} = \hat{\sigma}_{P_i}(X_i)$ for all $1 \leq i < 6$. Therefore,

$$(\hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_5})(X_1) = X_6 \Rightarrow \hat{\sigma}_{P_6}(X_6) = X_1$$

since $\hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_6} = \mathrm{id}_{\mathbf{PC}}$. The definition of a symmetry at P shows that

$$\hat{\sigma}_P(X) = Y \Leftrightarrow P \in \overline{XY}$$

for all $X, Y \in \mathbf{PC}$ where we denote the tangent at X by $\overline{XX}$. Hence, the side $\overline{X_i X_{i+1}}$ of the hexagon $X_1, \ldots, X_6$ goes through P_i for all i (with cyclically read indices). This is visualised in Fig. 1.4.

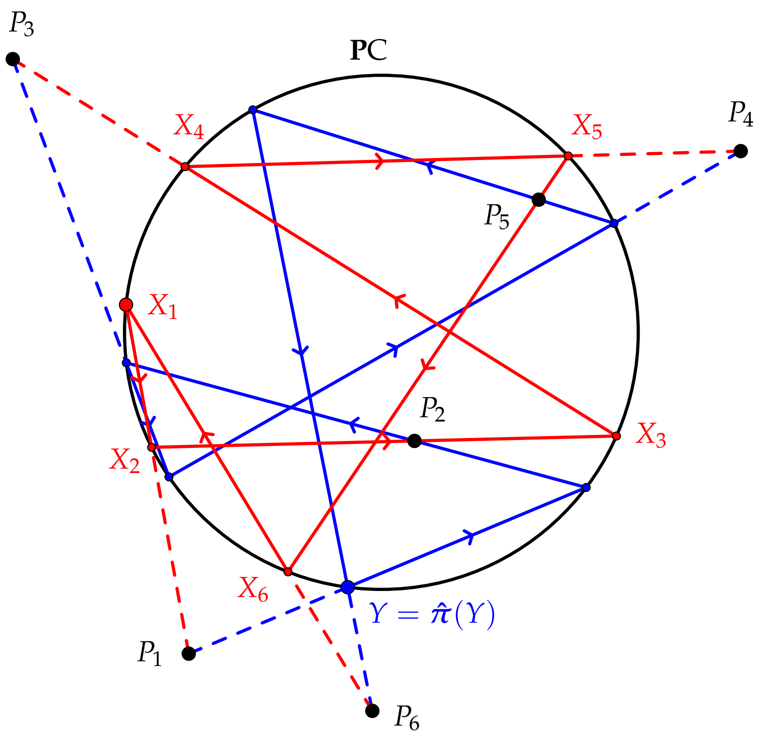

Figure 1.4: The sides of the hexagon $X_1, \ldots, X_6$ go through the points $P_1, \ldots, P_6$. Additionally, for any point $Y \in \mathbf{PC}$ we have $\hat{\pi}(Y) = Y$ for $\hat{\pi} = \hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_6}$.

1.3 Quadrics

So far we have only considered plane geometry. However, one can define symmetries $\hat{\sigma}_P$ for quadrics of higher dimensional spaces. Let

$$\mathcal{Z} = \{(x_1, x_2, x_3)^\intercal \in \mathbb{R}^3 : x_1^2 + y_2^2 = 1\} \subset \mathbb{R}^3$$

be the *infinite cylinder* of $\mathbb{R}^3$. One easily sees that a line l of $\mathbb{R}^3$ intersects $\mathcal{Z}$ in at most 2 points or is contained in $\mathcal{Z}$.

Hence, we can define symmetries $\hat{\sigma}_P$ of $\mathcal{Z}$ as in (1.1). The symmetry $\hat{\sigma}_P$, which is illustrated in Fig. 1.5, exchanges points of $\mathcal{Z}$ that lie on a secant through P and fixes points of $\mathcal{Z}$ that lie on a tangent through P.

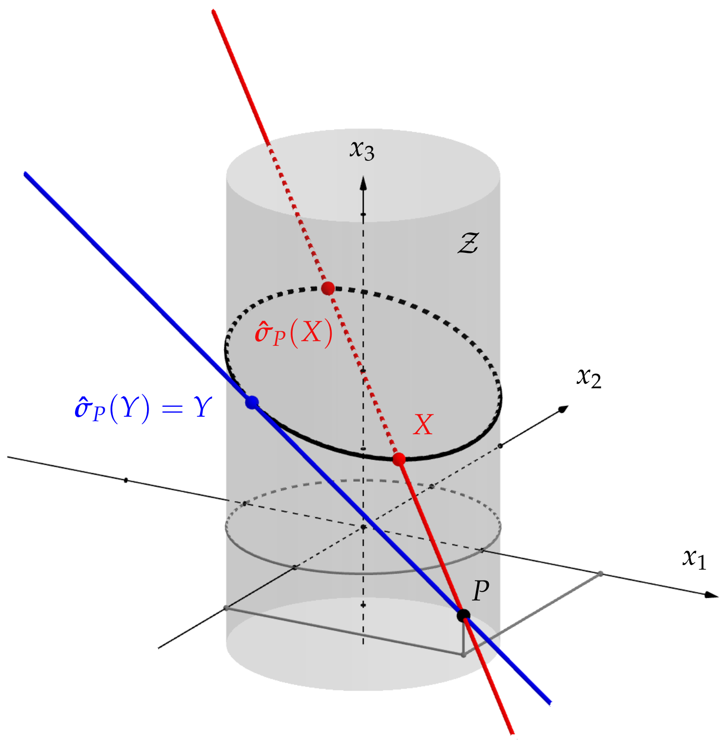

Figure 1.5: A point P outside of the cylinder $\mathcal{Z}$ defines a symmetry $\hat{\sigma}_P$ on $\mathcal{Z}$.

Hence, we want to study quadrics in the *n-dimensional real projective space*

$$\mathbf{P}^n\mathbb{R} = \{U \subset \mathbb{R}^{n+1} : U \text{ is a subspace}\}.$$

Consider a homogeneous polynomial of degree 2

$$Q : \mathbb{R}^{n+1} \to \mathbb{R}; \quad (x_1, \ldots, x_n)^\intercal \mapsto \sum_{i \le j \le n} \alpha_{ij} x_i x_j \quad \text{for } \alpha_{ij} \in \mathbb{R}.$$

The *quadric of* Q is the set

$$\mathbf{P}\mathrm{Q} = \{\langle x\rangle : x \in \mathbb{R}^{n+1} \text{ and } \mathrm{Q}(x) = 0\} \subset \mathbf{P}^n\mathbb{R}.$$

Note that if $\langle x\rangle \in \mathbf{P}\mathrm{Q}$, then we have $\mathrm{Q}(x') = 0$ for all $x' \in \langle x\rangle$. Quadrics are not restricted to spaces over $\mathbb{R}$ and we will consider arbitrary fields $\mathcal{K}$.

Example 1.4 *Similar to Section 1.1 one can embed the cylinder $\mathcal{Z}$ into $\mathbf{P}^3\mathbb{R}$. Consider the polynomial*

$$\mathrm{Q}_{\mathcal{Z}} : \mathbb{R}^4 \to \mathbb{R}; \quad (x_1, x_2, x_3, x_4)^\intercal \mapsto x_1^2 + x_2^2 - x_4^2.$$

For finite points we have $[x_1, x_2, x_3, 1]^\intercal \in \mathbf{P}\mathrm{Q}_{\mathcal{Z}}$ if and only if $(x_1, x_2, x_3)^\intercal \in \mathcal{Z}$. However, $\mathbf{P}\mathrm{Q}_{\mathcal{Z}}$ now contains an infinite point. Indeed, $[0, 0, 1, 0]^\intercal \in \mathbf{P}^3\mathbb{R}$ lies on $\mathbf{P}\mathrm{Q}_{\mathcal{Z}}$.

The embedding of $\mathbb{R}^3$ into $\mathbf{P}^3\mathbb{R}$ is similar to 1.1. However, there is more than one infinite line in $\mathbf{P}^3\mathbb{R}$. As we only consider projective geometry, we will not elaborate on this embedding and refer to [1] by Lenz instead.

As we will see in Chapter 3, a line in $\mathbf{P}^n\mathcal{K}$ intersects $\mathbf{P}\mathrm{Q}$ in at most two points or is entirely contained in $\mathbf{P}\mathrm{Q}$. Hence, the symmetry $\hat{\boldsymbol{\sigma}}_P$ of $\mathbf{P}\mathrm{Q}$ at a point $P \in \mathbf{P}^n\mathcal{K} \setminus \mathbf{P}\mathrm{Q}$ is well-defined. Similar to Section 1.2 $\hat{\mathrm{O}}$ denotes the group of all products of symmetries and $\mathrm{l}(\hat{\boldsymbol{\pi}})$ denotes the minimal number of symmetries needed to represent $\hat{\boldsymbol{\pi}} \in \hat{\mathrm{O}}$ as a product of symmetries.

The goal of this thesis is to find a formula for $\mathrm{l}(\hat{\boldsymbol{\pi}})$ and give an optimal bound for $\mathrm{l}(\hat{\boldsymbol{\pi}})$ depending on the quadric if $\mathrm{char}(\mathcal{K}) \neq 2$.

This needs some time and uses all of Chapters 2 to 5.

- Chapter 2 gives a rigorous definition of projective spaces and their collineations.
- Chapter 3 proves the stated properties of quadrics. Secondly, we will see that every symmetry $\hat{\boldsymbol{\sigma}}_P$ is the restriction of a collineation $\boldsymbol{\sigma}_P$ that is induced by an isometry σ_P.
- Chapter 4 investigates orthogonal spaces and shows that an isometry is simple if and only if it is of the form σ_P. Further, we proof an extended version of the Cartan-Dieudonné-Scherk theorem which gives an optimal way to represent any isometry $\pi \in \mathrm{O}^*$ as a product of simple isometries.
- Finally, in Chapter 5 we give the formula for the length of a map in $\hat{\mathrm{O}}$ and find an optimal bound for $\mathrm{l}(\pi)$ depending on the quadratic space.

Since the previous chapters mostly use theory of linear algebra, Chapter 6 geometrically rediscovers the bound for $\mathrm{l}(\hat{\boldsymbol{\pi}})$ if the quadratic space $(\mathbb{R}^n, \mathrm{Q})$ has index at most 1.

Chapter 2

Projective Spaces

There are two different ways to introduce projective geometry.

Firstly, we can start with an incidence relation $\in$ between a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$. So if $A \in \mathfrak{g}$, we say *the point A lies on the line $\mathfrak{g}$* or *the line $\mathfrak{g}$ goes through the point A*. The triple $(\mathcal{P}, \mathcal{L}, \in)$ is a *projective space* if it satisfies the following conditions:

P1 If $A \neq B \in \mathcal{P}$, then $\exists! \, \overline{AB} \in \mathcal{L}$ with $A, B \in \overline{AB}$.

P2 If $A, B, C, D \in \mathcal{P}$ distinct and $\overline{AB} \cap \overline{CD} = E \in \mathcal{P}$, then $\overline{AD} \cap \overline{BC} = F \in \mathcal{P}$.

P3 For any $\mathfrak{g} \in \mathcal{L}$ there are $A, B, C \in \mathfrak{g}$ distinct.

P4 There are $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}$ with $\mathfrak{g} \cap \mathfrak{h} = \emptyset$.

The intersection of two lines, $\mathfrak{g} \cap \mathfrak{h} = E$, means $E \in \mathfrak{g}, \mathfrak{h}$. Notice, by *P1* the point E is unique if $\mathfrak{g}, \mathfrak{h}$ are distinct.

On the other hand, one can consider the inclusion relation $\subset$ between the set $\mathbf{P}_0 V$ of all 1-dimensional and the set $\mathbf{P}_1 V$ of all 2-dimensional subspaces of a vector space V. We will see that $(\mathbf{P}_0 V, \mathbf{P}_1 V, \subset)$ satisfies *P1* to *P4* and, therefore, it is a projective space, if $\dim(V) \geq 4$.

Although the first approach seems to have no connection with vector spaces, one can show otherwise as seen in [1, p. 89].

Theorem 2.1 (Second fundamental theorem of projective geometry) *A projective space $(\mathcal{P},\ \mathcal{L},\ \in)$ is isomorphic to $(\mathbf{P}_0 V, \mathbf{P}_1 V, \subset)$ for some non necessarily finite dimensional left vector space $(V, \mathcal{S})$ over a skew-field $\mathcal{S}$.*

Remark 2.2 *There are multiple theorems named 'fundamental theorem of projective geometry'. However, the numbering is not consistent in the literature.*

Therefore, it makes no difference whether we start with vector spaces or projective spaces. There are ways of defining quadrics only with incidence.

However, we want to take the route of vector spaces and quadrics defined by quadratic forms.

For a more rigorous introduction to incidence geometry, a proof of Theorem 2.1, and a definition of quadrics without coordinates, see the book [1] by Lenz.

Remark 2.3 *Despite the fact that Theorem 2.1 can give us an infinite dimensional left vector space $(V, \mathcal{S})$ over a skew-field $\mathcal{S}$, we will only consider finite dimensional vector spaces $(V, \mathcal{K})$ over a field $\mathcal{K}$.*

For example, if $\mathcal{S}$ is the quaternion skew-field which is a proper skew-field, then the equation $x^2 + 1 = 0$ has 3 distinct solutions, as $i^2 = j^2 = k^2 = -1$. This leads to lines intersecting a quadric in 3 points. So our definition of symmetry at a point does not make sense. The article [3] by Ostrom discusses some of the non-equivalent definitions of quadrics in projective space over proper skew-fields.

2.1 Notations and prerequisites

We assume that the reader is familiar with basic knowledge of *sets*, *functions*, *groups* and *fields*. In the following, we give some basic definitions and notations which are used throughout this thesis. We refer to [4] by Lang for a formal definition and rigorous introduction of groups and fields.

- If a set U is contained in a set V, we write $U \subset V$ and allow the possibility $U = V$. We use $U \subsetneq V$ if we want to indicate $U \neq V$.
- We denote by $f : U \to V$ the *map* or *function* $u \in U \mapsto f(u) \in V$. If $U' \subset U$ and $V' \subset V$ are subsets, we write $f(U') = \{f(u) : u \in U'\} \subset V$ for the *image of* U' and $f^{-1}(V') = \{u \in U : f(u) \in V'\}$ for the *preimage of* V'. The *image of* f is denoted by $\mathrm{im}(f) = f(U)$ and if $\mathrm{im}(f) = V$, we call f *surjective*. The map f is said to be *injective* if $f(u) = f(u')$ implies $u = u'$ for all $u, u' \in U$. If f is both injective and surjective, we say f is *bijective*. Additionally, if $g : V \to W$ is a second function, we write $f \circ g : U \to W$ for the *composition* $(f \circ g)(v) = g(f(v))$.
- A map $f : (G, \cdot) \to (H, *)$ between two groups is a *group homomorphism* if $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in G$. If f is bijective, we say G *is isomorphic to* H and write $G \simeq H$. The *kernel of* f, $\ker(f)$, is the set of all elements $x \in G$ for which $f(x)$ is equal to the unit element of H. As one has certainly seen, the *factor group* $G/\ker(f)$ is isomorphic to $\mathrm{im}(f)$ by the canonical map $x \cdot \ker(f) \mapsto f(x)$. We will use this without further explanation.
- For a field $(\mathcal{K}+, \cdot)$ we write K^* for the group $(\mathcal{K} \setminus \{0\}, \cdot)$ and denote the *characteristic of* $\mathcal{K}$ by $\mathrm{char}(\mathcal{K})$.

2.2 Vector spaces

We need some basic information on vector spaces before exploring projective spaces. A lot of the following theorems are well-known but not trivial. We will not prove them and instead refer to chapter II in [5] by Baer for the proofs and a detailed introduction of vector spaces.

Definition 2.4 *Let $(\mathcal{K}, +, \cdot)$ be a field and $(V, +)$ an abelian group. We say $(V, \mathcal{K})$ is a* vector space over $\mathcal{K}$ *if there is a composition $(\alpha, x) \in \mathcal{K} \times V \mapsto \alpha x \in V$ satisfying the following properties:*

V1 $\alpha(x + y) = \alpha x + \alpha y$,

V2 $(\alpha + \beta)x = \alpha x + \beta x$,

V3 $(\alpha \cdot \beta)x = \alpha(\beta x)$,

V4 $1x = x$,

where $x, y \in V$, $\alpha, \beta \in \mathcal{K}$ and by 1 is the unit element of $\mathcal{K}$.

The elements of $\mathcal{K}$ and V are called *scalars* and *vectors* respectively. The composition above is referred to as *scalar multiplication*. Furthermore, we will denote by 0, 1 the zero, unit of $\mathcal{K}$ and $\mathbf{0}$ the zero of V.

Remark 2.5 *If we replace the field $\mathcal{K}$ with a skew-field $\mathcal{S}$ in Definition 2.4, one gets the definition for the* left vector space $(V, \mathcal{S})$ over $\mathcal{S}$.

Consider a fixed vector space $(V, \mathcal{K})$.

Definition 2.6 *A subset $U \subset V$ is called a* subspace of V *if $(U, \mathcal{K})$ is itself a vector space (with the restricted scalar multiplication).*

It is easy to see that U is a subspace if and only if $\alpha x + y \in U$ for all $\alpha \in \mathcal{K}$ and $x, y \in U$.

The next proposition summarises some important facts about subspaces.

Proposition 2.7 *Let V be a vector space and $U \subset V$ a subspace.*

i) *If $W \subset V$ is a subspace, the* sum of U and W,
$$U + W = \{u + w : u \in U, w \in W\},$$
is a subspace.

ii) *An arbitrary intersection of subspaces of V is a subspace of V.*

iii) *There is a subspace $W \subset V$ such that $U + W = V$ and $U \cap W = \{\mathbf{0}\}$.*

iv) *The factor space*
$$V/U = \{v + U : v \in V\}$$
with respect to the group $(V, +)$ is a vector space over $\mathcal{K}$.

If $U \cap W = \{\mathbf{0}\}$, we say the sum is *direct* and write $U \oplus W$ instead of $U + W$. In this case the representation for $v \in U \oplus W$ with $v = u + w$ and $u \in U, w \in W$ is unique. We see that for every subspace $U \subset V$ there is a *complementary subspace* $W \subset V$ with $U \oplus W = V$.

Now fix an arbitrary set $S \subset V$ of vectors.

Definition 2.8 *We denote by a* linear combination of elements of S *the finite sum* $\alpha_1 x_1 + \ldots + \alpha_r x_r$ *of distinct vectors* $x_1, \ldots, x_r \in S$ *for some scalars* $\alpha_1, \ldots, \alpha_r \in \mathcal{K}$. *If the sum is empty, we consider it to be equal to* $\mathbf{0}$.

One can easily show that $\langle S \rangle$, the set of all linear combinations of S, is a subspace of V. We say a subspace $U \subset V$ *is spanned by* S or S *generates* U if $U = \langle S \rangle$. The subspace $\langle S \rangle$ is the smallest subspace of V containing S.

If the set $S = \{s_1, \ldots, s_r\}$ is finite, we often write $\langle s_1, \ldots, s_r \rangle$ instead of $\langle S \rangle$.

Definition 2.9 *A set* $S \subset V$ *is called* independent *if* $\alpha_1 x_1 + \ldots \alpha_r x_r = \mathbf{0}$ *implies* $\alpha_1 = \ldots = \alpha_r = 0$ *for all linear combinations of* S. *Further, we say* S *is a* basis of V *if* S *is independent and* $\langle S \rangle = V$. *The* dimension of V *is denoted by* $\dim(V) = |S|$.

If S is a basis of V, every $x \in V$ can be uniquely written as a linear combination of S.

Proposition 2.10 *If* V *is a vector space, then the following holds true:*

i) *V has a basis.*

ii) *The dimension* $\dim(V)$ *does not depend of the choice of basis.*

iii) *If* $\dim(V) < \infty$, *we have*

$$\dim(U) \leq \dim(V), \tag{2.1}$$
$$\dim(U) = \dim(V) \Rightarrow U = V, \tag{2.2}$$
$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W), \tag{2.3}$$
$$\dim(V/U) = \dim(V) - \dim(U) \tag{2.4}$$

for any subspaces $U, W \subset V$.

We advise a study of [5] if one is not familiar with these statements.

Definition 2.11 *If* $\dim(V) < \infty$, V *is called* finite dimensional.

2.3 Semi-linear maps

Let $(V, \mathcal{K})$ and $(V', \mathcal{K}')$ be two vector spaces. Usually one studies linear maps between vector spaces, as they preserve the structure of vector spaces. However, our goal is to study maps, which identify subspaces of V with subspaces of V'.

Definition 2.12 *A map $\lambda : V \to V'$ is called* semi-linear with respect to a field isomorphism $\mu : \mathcal{K} \to \mathcal{K}'$ *if*

$$\lambda(x+y) = \lambda(x) + \lambda(y),$$
$$\lambda(\alpha y) = \mu(\alpha)\lambda(x)$$

holds for every $x, y \in V$ and $\alpha \in \mathcal{K}$. If $\mu = \mathrm{id}_{\mathcal{K}}$, then λ is called linear.

Remark 2.13 *A semi-linear map is a group homomorphism between the additive groups $(V, +)$ and $(V', +)$. Therefore, the sets*

$$\ker(\lambda) = \{x \in V : \lambda(x) = \mathbf{0}\} \subset V,$$
$$\mathrm{im}(\lambda) = \{\lambda(x) \in V' : x \in V\} \subset V'$$

are closed under addition. If $\alpha \in \mathcal{K}, x \in \ker(\lambda)$ and $\alpha' \in \mathcal{K}', x' \in \mathrm{im}(\lambda)$, we get

$$\lambda(\alpha x) = \mu(\alpha)\lambda(x) = \mathbf{0},$$
$$\alpha' x' = \alpha' \lambda(y) = \lambda(\mu^{-1}(\alpha')y)$$

for some $y \in V$ with $\lambda(y) = x'$. This shows that $\ker(\lambda)$, $\mathrm{im}(\lambda)$ are subspaces of V and V' respectively.

We now state some facts about semi-linear maps. They are well-known in the case of linear maps. However, they can easily be extended to semi-linear maps.

Remark 2.14 *Let λ be a semi-linear map and $x_1, \ldots, x_n$ be a basis of V. Consider $w_i = \lambda(x_i)$, then for any $x \in V$*

$$x = \sum_{i=1}^{n} \alpha_i x_i \in V \Rightarrow \lambda(x) = \sum_{i=1}^{n} \mu(\alpha_i) w_i$$

for some unique $\alpha_i \in \mathcal{K}$ which depend on x. This shows that

$$\lambda \textit{ injective} \Leftrightarrow w_1, \ldots, w_n \textit{ linearly independent},$$
$$\lambda \textit{ surjective} \Leftrightarrow \langle w_1, \ldots, w_n \rangle = V',$$
$$\lambda \textit{ bijective} \Leftrightarrow w_1, \ldots, w_n \textit{ is a basis of } V'.$$

With this in mind, the splitting

$$V \xrightarrow{p} V/\ker(\lambda) \xrightarrow{\lambda'} \mathrm{im}(\lambda) \xrightarrow{i} V',$$
$$x \overset{p}{\longmapsto} x + \ker(\lambda) \overset{\lambda'}{\longmapsto} \lambda(x) \overset{i}{\longmapsto} \lambda(x)$$

where p, i are the canonical projection, inclusion maps shows that λ' is bijective and semi-linear. Hence, $\dim(V/\ker(\lambda)) = \dim(\mathrm{im}(\lambda))$ *by our observation above. If* $\dim(V) < \infty$, *Proposition 2.10 now tells us*

$$\dim(V) = \dim(\ker(\lambda)) + \dim(\mathrm{im}(\lambda)),$$

which is well-known for linear maps.

Remark 2.15 *As with linear maps, the composition of two semi-linear maps is still a semi-linear map. Indeed, if* $\lambda_1 : V \to V'$, $\lambda_2 : V' \to V''$ *are semi-linear with respect to* μ_1, μ_2, *we have*

$$\begin{aligned}(\lambda_1 \circ \lambda_2)(\alpha x) &= \lambda_2(\mu_1(\alpha)\lambda_1(x)) \\ &= \mu_2(\mu_1(\alpha))\lambda_2(\lambda_1(x)) \\ &= (\mu_1 \circ \mu_2)(\alpha)(\lambda_1 \circ \lambda_2)(x).\end{aligned}$$

Therefore, $\lambda_1 \circ \lambda_2$ *is semi-linear with respect to the isomorphism* $\mu_1 \circ \mu_2$ *since one easily checks that* $(\lambda_1 \circ \lambda_2)(x + y) = (\lambda_1 \circ \lambda_2)(x) + (\lambda_1 \circ \lambda_2)(y)$.

On the other hand, if λ_1 *is bijective, then* λ_1^{-1} *is a semi-linear map with respect to* μ_1^{-1}. *We omit the proof, as it is similar to the linear case.*

These two facts show that the sets,

$$\begin{aligned}\Gamma\mathrm{L} &= \{\lambda : V \to V \textit{ bijective, semi-linear}\}, \\ \mathrm{GL} &= \{\lambda : V \to V \textit{ bijective, linear}\}\end{aligned}$$

are groups. They are called the general semi-linear group *and the* general linear group *respectively. We will write* $\Gamma\mathrm{L}(V)$, $\mathrm{GL}(V)$ *or* $\Gamma\mathrm{L}_n(\mathcal{K})$, $\mathrm{GL}_n(\mathcal{K})$ *if* $V = \mathcal{K}^n$, *whenever a distinction is needed.*

Further, we see that the map $p : \Gamma\mathrm{L} \to \mathrm{Aut}(\mathcal{K}); \lambda \mapsto \mu$ *which maps a semi-linear map* λ *to its automorphism* μ *is a group homomorphism. Its kernel is exactly* GL. *Since there is a semi-linear map for every automorphism* $\mu \in \mathrm{Aut}(\mathcal{K})$, *we see that* $\Gamma\mathrm{L} / \mathrm{GL} \simeq \mathrm{Aut}(\mathcal{K})$ *are isomorphic.*

2.4 Projective spaces

From now on we only consider finitely dimensional vector spaces. Otherwise, look at the book [5] by Baer for the theory and subtleties of infinitely dimensional projective spaces. We will follow the introduction of projective spaces by Artin in [6].

Definition 2.16 *Let* V *be a vector space. The* projective space of V, $\mathbf{P}V$, *denotes the set of all subspaces of* V. *With* $\mathbf{P}_i V$ *we mean the set of all* $i + 1$ *dimensional subspaces. The dimension of* $\mathbf{P}V$ *is equal to* $\dim(V) - 1$.

If $V = \mathcal{K}^{n+1}$, we write $\mathbf{P}^n\mathcal{K}$ for the *n dimensional projective space over* $\mathcal{K}$.

Remark 2.17 *If* $\mathbf{P}V$ *is an n-dimensional projective space, the elements of* $\mathbf{P}_0V$, $\mathbf{P}_1V$, $\mathbf{P}_2V$, $\mathbf{P}_{n-1}V$ *are* called points, lines, planes, hyperplanes *respectively. Since we want to have plenty of lines in a projective space, we usually restrict ourselves to the case* $n \geq 2$, *which means* $\dim(V) \geq 3$. *Otherwise, there is at most one line in our geometry.*

We think of the set $\{\mathbf{0}\} \in \mathbf{P}V$ as the empty set and, hence, if $U_1 \cap U_2 = \{\mathbf{0}\}$, we say U_1, U_2 are disjoint.

The inclusion relation defines an incidence relation on $\mathbf{P}V$. We say a subspace U_1 is incident with the subspace U_2 if and only if $U_1 \subset U_2$. If the dimension of V is big enough, we can show that $(\mathbf{P}_0 V, \mathbf{P}_1 V, \subset)$ satisfies axioms *P1* to *P4*.

Fix a vector space V with $\dim(V) \geq 3$.

P1: If $A, B \in \mathbf{P}_0 V$ are distinct points, then the line $A \oplus B \in \mathbf{P}_1 V$ satisfies the axiom. Uniqueness follows immediately by Proposition 2.10.

P2: Let $A, B, C, D \in \mathbf{P}_0 V$ be distinct points. If $A \oplus B = C \oplus D$, *P2* follows immediately. Otherwise, if $(A \oplus B) \cap (C \oplus D) = E \in \mathbf{P}_0 V$, we have

$$\dim(A + B + C + D) = \dim(A \oplus B) + \dim(C \oplus D) - \dim(E) = 3$$

which is symmetric in the points A, B, C, D. Hence, *P2* holds.

P3: Any line $\mathfrak{g} \in \mathbf{P}_1 V$ is spanned by a basis $\{x, y\}$ of two vectors. Then the points $\langle x \rangle, \langle x + y \rangle, \langle y \rangle \in \mathbf{P}_0 V$ are distinct and we have $\langle x \rangle, \langle x + y \rangle, \langle y \rangle \subset \mathfrak{g}$.

We need $\dim(V) \geq 4$ to show the last axiom.

P4: Since $\dim(V) \geq 4$, there exist four independent vectors $x, y, x', y' \in V$. Therefore, $\langle x, y \rangle \cap \langle x', y' \rangle = \{\mathbf{0}\}$ are two lines with the desired property.

Hence, $(\mathbf{P}_0 V, \mathbf{P}_1 V, \subset)$ is a 'projective space' as we encountered at the beginning of this chapter. However, with projective space, we always mean $\mathbf{P}V$ as defined in Definition 2.16.

Remark 2.18 *If* $\dim(V) = 3$*, then an intersection of two distinct lines* $\mathfrak{g}, \mathfrak{h} \in \mathbf{P}_1 V$ *satisfies*

$$\dim(\mathfrak{g} \cap \mathfrak{h}) = \dim(\mathfrak{g}) + \dim(\mathfrak{h}) - \dim(\mathfrak{g} + \mathfrak{h}) = 1$$

as $\dim(\mathfrak{g} + \mathfrak{h}) = 3$*. Therefore, two distinct lines always intersect each other in a unique point. The geometry* $(\mathbf{P}_0 V, \mathbf{P}_1 V, \subset)$ *is a* Desarguesian projective plane. *See [1] for more details.*

Remark 2.19 *The intersection and sum of two subspaces* $U, V \in \mathbf{P}V$ *can be expressed with just the inclusion relation in the following way:*

$$U = U_1 \cap U_2 \Leftrightarrow U \subset U_1, U_2 \text{ and } \forall U' \in \mathbf{P}V : U' \subset U_1, U_2 \Rightarrow U' \subset U,$$
$$U = U_1 + U_2 \Leftrightarrow U_1, U_2 \subset U \text{ and } \forall U' \in \mathbf{P}V : U_1, U_2 \subset U' \Rightarrow U \subset U'.$$

We now want to explore maps between projective spaces that are consistent with the inclusion relation. Let $\mathbf{P}V$ and $\mathbf{P}V'$ be projective spaces over the vector spaces V and V'.

Definition 2.20 *If* $\dim(V) = \dim(V')$*, then a bijective map* $\sigma : \mathbf{P}V \to \mathbf{P}V'$ *is called a* collineation *if*

$$U_1 \subset U_2 \Rightarrow \sigma(U_1) \subset \sigma(U_2) \tag{2.5}$$

for all $U_1, U_2 \in \mathbf{P}V$*.*

Remark 2.21 *The condition (2.5) in Definition 2.20 can be exchanged with*

$$\forall U_1, U_2 \in \mathbf{P}V : U_1 \subsetneq U_2 \Rightarrow \sigma(U_1) \subsetneq \sigma(U_2)$$

as σ *is bijective.*

Example 2.22 *If we have a bijective, semi-linear map* $\lambda : V \to V'$*, we can define a map* $\sigma : \mathbf{P}V \to \mathbf{P}V'$ *by setting* $\sigma(U) = \lambda(U)$ *for every* $U \in \mathbf{P}V$*. With Remarks 2.13 and 2.14 one can easily verify that* σ *is well defined and a collineation. We say* σ is the collineation induced by λ *or* λ induces the collineation σ*.*

The 'first fundamental theorem of projective geometry' will show that every collineation is induced by a suitable semi-linear map. However, we have to investigate collineations first.

Proposition 2.23 *Let* $\sigma : \mathbf{P}V \to \mathbf{P}V'$ *be a collineation between two n-dimensional projective spaces.*

i) *We have* $\dim(\sigma(U)) = \dim(U)$ *for any* $U \in \mathbf{P}V$*.*
ii) *The map* $\sigma^{-1} : \mathbf{P}V' \to \mathbf{P}V$ *is a collineation.*
iii) *If* $U_1, U_2 \in \mathbf{P}V$*, then* $\sigma(U_1 \cap U_2) = \sigma(U_1) \cap \sigma(U_2)$ *and* $\sigma(U_1 + U_2) = \sigma(U_1) + \sigma(U_2)$*.*

Proof *i)* Let $U \in \mathbf{P}V$ with $\dim(U) = j$. As one can extend any basis of U to a basis of V, we can find n+2 subspaces $U_i \in \mathbf{P}V$ such that

$$\{\mathbf{0}\} = U_0 \subsetneq U_1 \subsetneq \ldots \subsetneq U_{n+1} = V,$$

$U = U_j$ and $\dim(U_i) = i$. As σ is a collineation we get

$$\sigma(U_0) \subsetneq \sigma(U_1) \subsetneq \ldots \subsetneq \sigma(U_{n+1})$$

and $\dim(\sigma(U_i)) \geq i$ follows by Proposition 2.10. However, the reverse inequality holds as well because $\dim(\sigma(U_n)) \leq \dim(V') = n$. Therefore, we can conclude

$$\dim(\sigma(U)) = \dim(\sigma(U_j)) = j = \dim(U).$$

ii) Suppose $U_1, U_2 \in \mathbf{P}V$ and $\sigma(U_1) \subset \sigma(U_2)$. If we are able to show the inclusion $U_1 \subset U_2$, we see that σ^{-1} satisfies property 2.5 of collineations and we can conclude the statement.

As σ is bijective, we can find $W \in \mathbf{P}V$ such that $\sigma(U_1) \oplus \sigma(W) = \sigma(U_2)$ using Proposition 2.10. Further, we have

$$\sigma(U_1 \cap W) \subset \sigma(U_1) \cap \sigma(W) = \{\mathbf{0}\}$$

and, hence, $U_1 \cap W = \{\mathbf{0}\}$. We can now calculate

$$\begin{aligned} \dim(\sigma(U_1 \oplus W)) &= \dim(U_1 \oplus W) \\ &= \dim(U_1) + \dim(W) \\ &= \dim(\sigma(U_1)) + \dim(\sigma(W)) \\ &= \dim(\sigma(U_1) \oplus \sigma(W)) = \dim(\sigma(U_2)). \end{aligned}$$

This shows $\sigma(U_1 \oplus W) = \sigma(U_2)$ since

$$\sigma(U_2) = \sigma(U_1) \oplus \sigma(W) \subset \sigma(U_1 \oplus W).$$

Hence, $U_1 \oplus W = U_2$ and finally $U_1 \subset U_2$.

iii) Let $U_1, U_2 \in \mathbf{P}V$. Since σ is a collineation and $U_1 \cap U_2 \subset U_1, U_2$ trivially holds, the inclusion $\sigma(U_1 \cap U_2) \subset \sigma(U_1) \cap \sigma(U_2)$ follows. However, by *ii)* this has to hold for σ^{-1} as well. Hence, we see

$$\sigma^{-1}(\sigma(U_1) \cap \sigma(U_2)) \subset \sigma^{-1}(\sigma(U_1)) \cap \sigma^{-1}(\sigma(U_2)) = U_1 \cap U_2$$

and, therefore, the reverse inclusion

$$\sigma(U_1) \cap \sigma(U_2) \subset \sigma(U_1 \cap U_2)$$

holds as well. The statement with $+$ can be proven in a similar manner. □

Remark 2.24 *Proposition 2.23 shows that a collineation σ is determined if we know its effect on the points $\mathbf{P}_0V$. Since any $U \in \mathbf{P}V$ can be written as a direct sum of points $U = L_1 \oplus \ldots \oplus L_r$, we must have*

$$\sigma(U) = \sigma(L_1 \oplus \ldots \oplus L_r) = \sigma(L_1) \oplus \ldots \oplus \sigma(L_r).$$

We say the points $L_1, \ldots, L_k$ are *collinear* if there is a line $\mathfrak{g} \in \mathbf{P}V$ such that $\mathfrak{g}$ goes through all the points L_i. This is equivalent to $L_1, \ldots, L_k \subset \mathfrak{g}$. Proposition 2.23 now shows

$$L_1, \ldots, L_k \text{ collinear} \Leftrightarrow \sigma(L_1), \ldots, \sigma(L_k) \text{ collinear}$$

for any collineation σ. Hence, a collineation preserves collinear points.

This is especially true for $k = 3$. If $L_1, L_2, L_3 \in \mathbf{P}_0V$ are distinct points, we see

$$L_1 \subset L_2 \oplus L_3 \Rightarrow \sigma(L_1) \subset \sigma(L_2) \oplus \sigma(L_3). \tag{2.6}$$

Usually, collineations are defined as maps $\sigma : \mathbf{P}_0V \to \mathbf{P}_0V'$ satisfying (2.6) and, hence, they preserve collinearity. Similar to Remark 2.24, one can extend σ to a map on $\mathbf{P}V$. The following theorem, [6, p. 88], tells us that this extended map is indeed a collineation.

Theorem 2.25 (First fundamental theorem of projective geometry) *Assume $\sigma : \mathbf{P}_0 V \to \mathbf{P}_0 V'$ is a bijective map between points of two projective spaces with equal dimensions. If σ satisfies rule (2.6), then σ can be uniquely extended to a collineation $\sigma : \mathbf{P}V \to \mathbf{P}V'$. Further, there is a unique isomorphism $\mu : \mathcal{K} \to \mathcal{K}'$ and a semi-linear map $\lambda : V \to V'$ with respect to μ such that σ is the induced collineation by λ. If $\lambda_1 : V \to V'$ is another semi-linear map which induces σ, there is a fixed $\alpha \in \mathcal{K}^*$ with $\lambda_1(x) = \lambda(\alpha x)$ for all $x \in V$.*

We omit the proof for this theorem again and refer to [6] where a slightly generalised statement is proven for projective spaces $\mathbf{P}V$ of left vector spaces $(V, \mathcal{S})$.

Remark 2.26 *Similarly to Remark 2.15, we can study the group of collineations from $\mathbf{P}V$ to itself. We write*

$$\begin{aligned}
&\mathrm{P\Gamma L} = \{\sigma : \mathbf{P}V \to \mathbf{P}V \textit{ collineation}\},\\
&\mathrm{PGL} = \{\sigma : \mathbf{P}V \to \mathbf{P}V \textit{ collineation induced by a linear map}\} \subset \mathrm{P\Gamma L}
\end{aligned}$$

for the collineation group *and the* projective general linear group of $\mathbf{P}V$.

Theorem 2.25 allows us to write these groups as factor groups of $\Gamma\mathrm{L}$ *and* GL.

The map $j : \Gamma\mathrm{L} \to \mathrm{P\Gamma L}$ which maps λ to the collineation induced by λ is a group homomorphism. This can be seen easily as $j(\lambda_1 \circ \lambda_2)$ and $j(\lambda_1) \circ j(\lambda_2)$ act the same on the points of $\mathbf{P}V$ and, hence, they have to be equal by Remark 2.24. Further, one sees that the map

$$i : \mathcal{K}^* \to \Gamma\mathrm{L}; \quad \alpha \mapsto \lambda_\alpha(x) := \alpha x$$

is a homomorphism as well. Theorem 2.25 now shows $\mathrm{im}(i) = \ker(j)$ *and we get two isomorphism*

$$\Gamma\mathrm{L} \,/\, \mathcal{K}^* \simeq \mathrm{P\Gamma L}, \quad \mathrm{GL} \,/\, \mathcal{K}^* \simeq \mathrm{PGL}\,.$$

The second follows by considering the restriction of j to GL *and the fact that* $\lambda_\alpha \in \mathrm{GL}$ *for all* $\alpha \in \mathcal{K}^*$.

Lastly, the third isomorphism theorem [4, p. 17] and Remark 2.15 shows

$$\mathrm{P\Gamma L} \,/\, \mathrm{PGL} \simeq (\Gamma\mathrm{L} \,/\, \mathcal{K}^*)/(\mathrm{GL} \,/\, \mathcal{K}^*) \simeq \Gamma\mathrm{L} \,/\, \mathrm{GL} \simeq \mathrm{Aut}(\mathcal{K}).$$

For more details and the structure of these groups in the non-commutative case, we refer to Chapter II in [6].

Chapter 3

Quadrics and symmetries

We now want to introduce quadrics of a projective space in a rigours way. The results of this chapter are in the books [6], [7] by Artin and Mäurer.

3.1 Quadratic spaces

Let $(V, \mathcal{K})$ be an n-dimensional vector space.

Definition 3.1 *A map $f : V \times V \to \mathcal{K}$ is called* bilinear *if*

$$\begin{aligned} f(x+y,z) &= f(x,z) + f(y,z), \\ f(x,y+z) &= f(x,y) + f(x,z), \\ \alpha f(x,y) &= f(\alpha x, y) = f(x, \alpha y) \end{aligned}$$

holds for all $x, y, z \in V$ and $\alpha \in \mathcal{K}$. If we additionally have $f(x,y) = f(y,x)$ for all $x, y \in V$, we say f is symmetric.

A vector space with a fixed symmetric bilinear map (V, f) is called an *orthogonal space*.

In Section 1.3 we motivated the importance of homogeneous polynomials of degree 2. However, this definition depends on coordinates for a given basis. The following definition allows us study them in a coordinate-free approach.

Definition 3.2 *A map $\mathrm{Q} : V \to \mathcal{K}$ is a* quadratic form *if it satisfies the conditions*

$$\mathrm{Q}(\alpha x) = \alpha^2 \mathrm{Q}(X) \text{ for all } x \in V, \alpha \in \mathcal{K}, \tag{3.1}$$

$$f_{\mathrm{Q}}(x,y) = \mathrm{Q}(x+y) - \mathrm{Q}(x) - \mathrm{Q}(y) \text{ is bilinear.} \tag{3.2}$$

The map f_{Q} is the associated bilinear map of Q. *The tuple (V, Q) is called a* quadratic space.

We will show that every homogeneous polynomial of degree 2 defines a quadratic form of the vector space $\mathcal{K}^n$. Indeed, for some fixed $\alpha_{ij} \in \mathcal{K}$ the polynomial

$$Q : \mathcal{K}^n \to \mathcal{K}; \quad (x_1, \ldots, x_n)^\intercal \mapsto \sum_{i \le j \le n} \alpha_{ij} x_i x_j$$

clearly satisfies (3.2). Rule (3.1) can be seen after calculating

$$\begin{aligned} f_Q(x, y) &= Q(x + y) - Q(x) - Q(y) \\ &= \sum_{i \le j \le n} \alpha_{ij}(x_i + y_i)(x_j + y_j) - \sum_{i \le j \le n} \alpha_{ij} x_i x_j - \sum_{i \le j \le n} \alpha_{ij} y_i y_j \\ &= \sum_{i \le j \le n} \alpha_{ij}(x_i y_j + y_i x_j). \end{aligned}$$

Hence, Q is a quadratic form of $\mathcal{K}^n$ and $(\mathcal{K}^n, Q)$ a quadratic space.

Conversely, a finite dimensional quadratic space (V, Q) is always defined by a homogeneous polynomial. Indeed, the rule (3.2) shows

$$\begin{aligned} Q(\sum_{i=1}^{r} x_i) &= Q(\sum_{i=1}^{r-1} x_i) + Q(x_r) + f_Q(\sum_{i=1}^{r-1} x_i, x_r) \\ &= Q(\sum_{i=1}^{r-1} x_i) + Q(x_r) + \sum_{i=1}^{r-1} f_Q(x_i, x_r) \end{aligned}$$

for $x_1, \ldots, x_r \in V$. Using induction on r, we get

$$Q(\sum_{i=1}^{r} x_i) = \sum_{i=1}^{r} Q(x_i) + \sum_{i < j \le r} f_Q(x_i, x_j).$$

Choose a basis $e_1, \ldots, e_n$ of V. Hence, every vector $x \in V$ has a unique representation $x = \sum_{i=1}^{n} \alpha_i e_i$ for some $(\alpha_1, \ldots, \alpha_n)^\intercal \in \mathcal{K}^n$. The sum formula from above and rule (3.1) show that

$$\begin{aligned} Q(x) &= \sum_{i=1}^{n} \alpha_i^2 Q(e_i) + \sum_{i < j \le n} \alpha_i \alpha_j f_Q(e_i, e_j) \\ &= \sum_{i \le j \le n} \beta_{ij} \alpha_i \alpha_j \end{aligned}$$

for coefficients $\beta_{ii} = Q(e_i)$, $\beta_{ij} = f_Q(e_i, e_j)$. Therefore, the map

$$Q' : (\alpha_1, \ldots, \alpha_n)^\intercal \mapsto Q(\sum_{i=1}^{r} \alpha_i e_i)$$

is a homogeneous polynomial of degree two and $(\mathcal{K}^n, Q')$ is equal to (V, Q).

We now shift our attention back to Definition 3.2. If $(V\mathrm{Q})$ is a quadratic space, we immediately see that the associated bilinear map f_{Q} is symmetric and, hence, (V, f_{Q}) an orthogonal space. Further, a short calculation shows

$$f_{\mathrm{Q}}(x,x) = \mathrm{Q}(2x) - 2\mathrm{Q}(x) = 2\mathrm{Q}(x)$$

for all $x \in V$. Hence, the quadratic form Q differs only by a factor of 2 from the diagonal of a symmetric bilinear map. Indeed, if $\mathrm{char}(\mathcal{K}) \neq 2$, quadratic and orthogonal spaces are essentially the same.

Proposition 3.3 *If* $\mathrm{char}(\mathcal{K}) \neq 2$*, the map sending* (V, Q) *to* (V, f_{Q}) *is bijective.*

Proof If we have two quadratic forms Q, Q' such that $f_{\mathrm{Q}} = f_{\mathrm{Q}'}$ then

$$2\mathrm{Q}(x) = f_{\mathrm{Q}}(x,x) = f_{\mathrm{Q}'}(x,x) = 2\mathrm{Q}'(x)$$

for all $x \in V$. Since $\mathrm{char}(\mathcal{K}) \neq 2$, $\mathrm{Q} = \mathrm{Q}'$ and our map is injective.

On the other hand, if we start with an orthogonal space (V, f), we define $\mathrm{Q}_f(x) = \frac{1}{2} f(x,x)$ and see that

$$\begin{aligned} \mathrm{Q}_f(\alpha x) &= \frac{1}{2} f(\alpha x, \alpha x) = \alpha^2 \frac{1}{2} f(x,x) = \alpha^2 \mathrm{Q}_f(x) \\ f_{\mathrm{Q}_f}(x,y) &= \mathrm{Q}_f(x+y) - \mathrm{Q}_f(x) - \mathrm{Q}_f(y) \\ &= \frac{1}{2}(f(x+y, x+y) - f(x,x) - f(y,y)) = f(x,y) \end{aligned}$$

for all $x, y \in V$ and $\alpha \in \mathcal{K}$. Hence, Q_f is a quadratic form and (V, Q_f) is mapped to (V, f). This concludes the proposition. □

One calls the quadratic form $f_{\mathrm{Q}} = \frac{1}{2} f(x,x)$ the *associated quadratic form of* f.

Remark 3.4 *Consider the quadratic forms*

$$\mathrm{Q}_1((\alpha_1, \alpha_2)^\top) = \alpha_1^2 + \alpha_1\alpha_2, \quad \mathrm{Q}_2((\alpha_1, \alpha_2)^\top) = \alpha_1\alpha_2 + \alpha_2^2$$

of $\mathcal{K}^2$*. If* $\mathrm{char}(\mathcal{K}) = 2$*, they have the same associated bilinear function. Indeed,*

$$\begin{aligned} f_{\mathrm{Q}_1}((\alpha_1, \alpha_2)^\top, (\alpha_1', \alpha_2')^\top) &= 2\alpha_1\alpha_1' + \alpha_1\alpha_2' + \alpha_1'\alpha_2 = \alpha_1\alpha_2' + \alpha_1'\alpha_2, \\ f_{\mathrm{Q}_2}((\alpha_1, \alpha_2)^\top, (\alpha_1', \alpha_2')^\top) &= \alpha_1\alpha_2' + \alpha_1'\alpha_2 + 2\alpha_2\alpha_2' = \alpha_1\alpha_2' + \alpha_1'\alpha_2. \end{aligned}$$

Therefore, if $\mathrm{char}(\mathcal{K}) = 2$*, it is not enough to know* f_{Q} *for recovering* Q*.*

3.2 Quadrics

Let $\mathbf{P}V$ be the projective space over the quadratic space (V, Q). We will write f instead of f_{Q} to make formulas less cumbersome.

As $\mathrm{Q}(\alpha x) = \alpha^2 \mathrm{Q}(x)$, we can not meaningfully define Q on subspaces $U \in \mathbf{P}V$. However, if $\mathrm{Q}(x) = 0$ for some $x \in V$, then we have $\mathrm{Q}(x') = 0$ for all $x' \in \langle x \rangle$ since $x' = \alpha x$ for some $\alpha \in \mathcal{K}$. Therefore, we can say a point $X = \langle x \rangle \in \mathbf{P}_0 V$ is *singular* if $\mathrm{Q}(x) = 0$.

Definition 3.5 *The set of all singular points*

$$\mathbf{P}\mathrm{Q} = \{X \in \mathbf{P}_0 V : X \text{ singular}\} \subset \mathbf{P}_0 V$$

is the quadric of the quadratic form Q.

We will always denote vectors $x, y, z \in V$ with lower case letters, points $X, Y, Z \in \mathbf{P}_0 V$ with upper case letters and lines $\mathfrak{g}, \mathfrak{h} \in \mathbf{P}_1 V$ with Fraktur letters. If we write a point X and a vector x with the same letter, we assume $X = \langle x \rangle$ without always specifying this connection.

Remark 3.6 *If we have two distinct points $X, Y \in \mathbf{P}_0 V$, we usually write $\overline{XY}$ instead of $X \oplus Y$ for the line through X, Y. Further, we often think of lines $\mathfrak{g} \in \mathbf{P}_1 V$ as the set of all the points $X \in \mathbf{P}_0 V$ with $X \subset \mathfrak{g}$. So we write*

$$X \in \mathfrak{g}, \quad \mathfrak{g} \cap \mathbf{P}\mathrm{Q}, \quad \mathfrak{g} \subset \mathbf{P}\mathrm{Q}$$

instead of

$$X \subset \mathfrak{g}, \quad \{X \in \mathfrak{g} : X \in \mathbf{P}\mathrm{Q}\}, \quad \forall X \in \mathfrak{g} : X \in \mathbf{P}\mathrm{Q}$$

respectively.

Let $X, Y \in \mathbf{P}\mathrm{Q}$ be two distinct points on the quadric of Q. We want to find out whether there is another singular point $Z \in \overline{XY}$. Remember that Z can be written as $Z = \langle \alpha x + \beta y \rangle$ for some $\alpha, \beta \in \mathcal{K}$. Therefore, Z is singular if and only if

$$\begin{aligned} \mathrm{Q}(\alpha x + \beta y) &= \alpha^2 \mathrm{Q}(x) + \alpha\beta f(x, y) + \beta^2 \mathrm{Q}(y) \\ &= \alpha\beta f(x, y) = 0 \end{aligned}$$

because X, Y are singular. Therefore, if Z is distinct from X and Y, we have Z is singular if and only if $f(x, y) = 0$.

Hence, if $f(x, y) = 0$, every point $Z \in \overline{XY}$ is singular and we see $\overline{XY} \subset \mathbf{P}\mathrm{Q}$. On the other hand, if $f(x, y) \neq 0$, X, Y are the only singular points on $\overline{XY}$ and, hence, $\overline{XY} \cap \mathbf{P}\mathrm{Q} = \{X, Y\}$.

Therefore, a line $\mathfrak{g}$ is either contained in $\mathbf{P}\mathrm{Q}$ or intersects $\mathbf{P}\mathrm{Q}$ in at most two points. A line $\mathfrak{g}$ is called a

i) *secant* if $|\mathfrak{g} \cap \mathbf{P}\mathrm{Q}| = 2$,

ii) *tangent* if $|\mathfrak{g} \cap \mathbf{P}\mathrm{Q}| = 1$ or $\mathfrak{g} \subset \mathbf{P}\mathrm{Q}$,

iii) *interior line* if $\mathfrak{g} \subset \mathbf{P}\mathrm{Q}$,

iv) *exterior line* if $|\mathfrak{g} \cap \mathbf{P}\mathrm{Q}| = 0$.

Proposition 3.7 *Let $X, Y \in \mathbf{P}V$ be two distinct points and assume $X \in \mathbf{P}\mathrm{Q}$. Then*

i) $f(x,y) \neq 0 \Leftrightarrow \overline{XY}$ *is a secant,*

ii) $f(x,y) = 0 \Leftrightarrow \overline{XY}$ *is a tangent,*

iii) $f(x,y) = \mathrm{Q}(y) = 0 \Leftrightarrow \overline{XY}$ *is an interior line.*

Proof Every point $Z \in \overline{XY}$ that is distinct from X can be written in the form $Z = \langle \alpha x + y \rangle$ for some $\alpha \in \mathcal{K}$. Similar to the calculation earlier, Z is singular if and only if

$$\begin{aligned} \mathrm{Q}(\alpha x + y) &= \alpha^2 \mathrm{Q}(x) + \alpha f(x,y) + \mathrm{Q}(y) \\ &= \alpha f(x,y) + \mathrm{Q}(y) = 0. \end{aligned}$$

Hence, if $f(x,y) \neq 0$, Z is singular if and only if $\alpha = -\mathrm{Q}(y)/f(x,y)$ and we conclude $\overline{XY}$ is a secant.

On the other hand, if $f(x,y) = 0$, we have $\mathrm{Q}(\alpha x + y) = \mathrm{Q}(Y)$. This shows that $\overline{XY} \subset \mathbf{P}\mathrm{Q}$ if $\mathrm{Q}(y) = 0$ and $|\overline{XY} \cap \mathbf{P}\mathrm{Q}| = 1$ if $\mathrm{Q}(y) \neq 0$. This proves the first direction of all 3 statements.

As $\overline{XY}$ can not be a tangent and secant at the same time, the reverse implication of *i)*, *ii)* holds. Since $\overline{XY} \subset \mathbf{P}\mathrm{Q}$ implies $\mathrm{Q}(y) = 0$, we see *iii)* as well. $\square$

Remark 3.8 *If* $\mathrm{char}(\mathcal{K}) \neq 2$, *we can classify an arbitrary line $\overline{XY}$ with the discriminant $D = f(x,y)^2 - 4\mathrm{Q}(x)\mathrm{Q}(y)$. One can show that*

i) $D \neq 0$ *is a square in* $\mathcal{K} \Leftrightarrow \overline{XY}$ *is a secant,*

ii) $D = 0 \Leftrightarrow \overline{XY}$ *is a tangent,*

iii) D *is not a square in* $\mathcal{K} \Leftrightarrow \overline{XY}$ *is a exterior line.*

This can be done by first noting that Proposition 3.7 is consistent with the new characterisation. Let $X, Y \notin \mathbf{P}\mathrm{Q}$, then $4\mathrm{Q}(x) \neq 0$ and we can complete the square as below:

$$\begin{aligned} & 0 = \mathrm{Q}(\alpha x + y) = \alpha^2 \mathrm{Q}(x) + \alpha f(x,y) + \mathrm{Q}(y), && (3.3) \\ \Leftrightarrow \quad & 0 = 4\alpha^2 \mathrm{Q}(x)^2 + 4\alpha \mathrm{Q}(x) f(x,y) + 4\mathrm{Q}(x)\mathrm{Q}(y), && (3.4) \\ \Leftrightarrow \quad & 0 = (2\alpha \mathrm{Q}(x) + f(x,y))^2 - f(x,y)^2 + 4\mathrm{Q}(x)\mathrm{Q}(y), && (3.5) \\ \Leftrightarrow \quad & D = (2\alpha \mathrm{Q}(x) + f(x,y))^2. && (3.6) \end{aligned}$$

The proof is finished by calculating α for all cases of D.

3.3 Symmetries

Consider a non-singular point $P \in \mathbf{P}V$. As in Chapter 1 we define *the symmetry at* P as the map $\hat{\sigma}_P : \mathbf{P}\mathrm{Q} \to \mathbf{P}\mathrm{Q}$ with

$$\hat{\sigma}_P(X) = \begin{cases} X \text{ if } \overline{PX} \text{ is a tangent} \\ Y \text{ if } P \in \overline{XY} \text{ is a secant} \end{cases}.$$

This is well-defined by our considerations in Section 3.2. Further, we denote by

$$\hat{\mathrm{O}} = \{\hat{\sigma}_{P_1} \circ \ldots \circ \hat{\sigma}_{P_k} | k \in \mathbb{N},\, P_1, \ldots, P_k \in \mathbf{P}V \text{ non-singular}\}$$

the *group of the quadric* $\mathbf{P}\mathrm{Q}$.

Instead of studying $\hat{\pi} \in \hat{\mathrm{O}}$ directly, we will see that $\hat{\pi}$ is the restriction of a collineation.

If a vector $p \in V$ satisfies $\mathrm{Q}(p) \neq 0$, we define the *symmetry at* p as the map

$$\sigma_p : V \to V; \quad x \mapsto x - \mathrm{Q}(p)^{-1} f(x,p)p.$$

This function is clearly linear and for any $\alpha \in \mathcal{K}^*$ we have

$$\begin{aligned} \mathrm{Q}(\alpha p)^{-1} f(x, \alpha p)\alpha p &= \alpha^{-2}\mathrm{Q}(p)^{-1} f(x,p)\alpha^2 p \\ &= \mathrm{Q}(p)^{-1} f(x,p)p. \end{aligned}$$

Hence, if $P = \langle p \rangle = \langle p' \rangle$, then $\sigma_p = \sigma_{p'}$ and σ_P is well-defined for every non-singular point $P \in \mathbf{P}V$.

Remark 3.9 *If* $\mathrm{char}(\mathcal{K}) \neq 2$, *then* $\sigma_P = \sigma_{P'}$ *for two non-singular points* P, P' *implies* $P = P'$. *Indeed, if* $P = \langle p \rangle$ *and* $P' = \langle p' \rangle$, *then* $\sigma_p(p') = \sigma_{p'}(p')$ *gives us*

$$\mathrm{Q}(p)^{-1} f(p', p)p = \mathrm{Q}(p')^{-1} f(p', p')p' = 2p'.$$

Hence, $2p' \in P$ *which shows* $P = P'$.

We want to show that $\hat{\sigma}_P(X) = \sigma_P(X) = \{\sigma_P(x) : x \in X\}$ for all points $X \in \mathbf{P}\mathrm{Q}$. This needs the following lemma.

Lemma 3.10 *If* $p \in V$ *such that* $\mathrm{Q}(p) \neq 0$, *then the following is true:*

i) $\sigma_p(p) = -p$, ii) $\sigma_p^2(x) = x$, iii) $\mathrm{Q}(\sigma_p(x)) = \mathrm{Q}(x)$ *for all* $x \in V$.

Proof *i)* Recall $f(p,p) = 2\mathrm{Q}(p)$. Therefore

$$\sigma_p(p) = p - \mathrm{Q}(p)^{-1} f(p,p)p = p - 2p = -p.$$

If $\mathrm{char}(\mathcal{K}) = 2$, the statement is true as well since $p = -p$.

ii) As σ_p is linear, one concludes

$$\begin{aligned}\sigma_p(\sigma_p(x)) &= \sigma_p(x - \mathrm{Q}(p)^{-1}f(x,p)p)\\ &= \sigma_p(x) - \mathrm{Q}(x)^{-1}f(x,p)\sigma_p(p)\\ &= x - \mathrm{Q}(p)^{-1}f(x,p)p + \mathrm{Q}(x)^{-1}f(x,p)p = x.\end{aligned}$$

iii) Using the formula $\mathrm{Q}(x+y) = \mathrm{Q}(x) + \mathrm{Q}(y) + f(x,y)$, we get

$$\begin{aligned}\mathrm{Q}(\sigma_p(x)) &= \mathrm{Q}(x - \mathrm{Q}(p)^{-1}f(x,p)p)\\ &= \mathrm{Q}(x) + (-\mathrm{Q}(x)^{-1}f(x,p))^2\mathrm{Q}(p) + (-\mathrm{Q}(x)^{-1}f(x,p))f(x,p)\\ &= \mathrm{Q}(x) + \mathrm{Q}(x)^{-1}f(x,p)^2 - \mathrm{Q}(x)^{-1}f(x,p)^2 = \mathrm{Q}(x). \qquad \square\end{aligned}$$

Lemma 3.10 shows that $\sigma_p \in \mathrm{GL}$ since σ_p is an involution and, therefore, a bijection. Further, we see that σ_p preserves the quadratic form Q. A map $\pi \in \mathrm{GL}$ with this property is called an *isometry*. Clearly the set of all isometries

$$\mathrm{O} = \{\pi \in \mathrm{GL} : \pi \text{ isometry}\} = \{\pi \in \mathrm{GL} : \pi \circ \mathrm{Q} = \mathrm{Q}\} \subset \mathrm{GL}$$

is a subgroup of GL. It is called *the orthogonal group of* (V, Q).

Remark 3.11 *If π is an isometry, then for all $x, y \in V$ we have*

$$\begin{aligned}f(\pi(x), \pi(y)) &= \mathrm{Q}(\pi(x) + \pi(y)) - \mathrm{Q}(\pi(x)) - \mathrm{Q}(\pi(y))\\ &= \mathrm{Q}(x+y) - \mathrm{Q}(x) - \mathrm{Q}(y) = f(x,y)\end{aligned}$$

and, hence, π preserves the associated bilinear map f as well. However, not every map preserving f is necessarily an isometry if $\mathrm{char}(\mathcal{K}) = 2$.

The collineation $\boldsymbol{\sigma}_P$ induced by the linear map σ_P satisfies the following proposition.

Proposition 3.12 *If $X, P \in \mathbf{P}V$ are distinct points and P is non-singular, then*

i) $\boldsymbol{\sigma}_P(P) = P$,

ii) $P, X, \boldsymbol{\sigma}_P(X)$ *are collinear,*

iii) $\boldsymbol{\sigma}_P(X) = X \Leftrightarrow f(x,p) = 0$,

iv) $\boldsymbol{\sigma}_P(X) \neq X \Leftrightarrow f(x,p) \neq 0$,

v) $\boldsymbol{\sigma}_P(\mathbf{P}\mathrm{Q}) = \mathbf{P}\mathrm{Q}$.

Proof *i)* This follows immediately from Lemma 3.10.

ii) Since $\sigma_P(x) = x - \mathrm{Q}(p)^{-1}f(x,p)p \in P \oplus X$, we see collinearity as

$$\boldsymbol{\sigma}_P(X) = \langle \sigma_P(x) \rangle \subset P \oplus X = \overline{PX}.$$

iii) If $f(x,p) = 0$, we immediately get $\sigma_P(x) = x - \mathrm{Q}(p)^{-1} f(x,p)p = x$ and, therefore, $\boldsymbol{\sigma}_P(X) = X$. Conversely, if $\boldsymbol{\sigma}_P(X) = X$, there is $\alpha \in \mathcal{K}$ with

$$\alpha x - \sigma_P(x) = 0 \quad \Rightarrow \quad \alpha x - x = \mathrm{Q}(p)^{-1} f(x,p)p.$$

However, since P, X are distinct, the vectors x, p are linearly independent and we must have $f(x,p) = 0$.

iv) This is just the contraposition of *iii)*.

v) If X is singular, we get $\mathrm{Q}(\sigma_P(x)) = \mathrm{Q}(x) = 0$ using Lemma 3.10. Therefore, $\boldsymbol{\sigma}_P(X) \in \mathbf{P}\mathrm{Q}$ and $\boldsymbol{\sigma}_P(\mathbf{P}\mathrm{Q}) \subset \mathbf{P}\mathrm{Q}$. Equality follows by applying the collineation $\boldsymbol{\sigma}_P$ again. ☐

Remark 3.13 *Let $P \in \mathbf{P}V$ be a non-singular point and $\mathfrak{g}$ a line trough P. We want to show that there is at least one point $Z \in \mathfrak{g}$ with $\boldsymbol{\sigma}_P(Z) \neq Z$ if* $\mathrm{char}(\mathcal{K}) \neq 2$.

As there are at least 3 points on $\mathfrak{g}$, we find distinct points $X, Y \in \mathfrak{g} \setminus \{P\}$. If both of them are fixed by $\boldsymbol{\sigma}_P$, we see $f(p,x) = f(p,y) = 0$ by Proposition 3.12 iii). *However, this leads to a contradiction since $P \in \overline{XY}$ implies $p = \alpha x + \beta y$ for some $\alpha, \beta \in \mathcal{K}$ and, hence,*

$$\mathrm{Q}(p) = \frac{1}{2} f(p, \alpha x + \beta y) = 0.$$

Let $P, P' \in \mathbf{P}V$ be non-singular points and assume $\boldsymbol{\sigma}_P = \boldsymbol{\sigma}_{P'}$. If a point $X \in \mathbf{P}V$ is not fixed by $\boldsymbol{\sigma}_P$ and, hence, $\boldsymbol{\sigma}_{P'}(X) = \boldsymbol{\sigma}_P(X) = X' \neq X$, then $P, P' \in \overline{XX'}$ follows by Proposition 3.12 ii).

If $\dim(V) \geq 3$ *and* $\mathrm{char}(\mathcal{K}) \neq 2$, *there are at least two distinct lines $\mathfrak{g}, \mathfrak{h}$ trough P and both contain a point that is not fixed by $\boldsymbol{\sigma}_P$. Hence, $P' = \overline{XY} \cap \overline{X'Y'} = P$ and the symmetry $\boldsymbol{\sigma}_P$ is uniquely defined by P.*

We can now show that $\hat{\boldsymbol{\sigma}}_P$ is the restriction of the collineation $\boldsymbol{\sigma}_P$ to $\mathbf{P}\mathrm{Q}$. If $X \in \mathbf{P}\mathrm{Q}$, then Propositions 3.7 and 3.12 combined give us

$$\begin{aligned} \boldsymbol{\sigma}_P(X) = X &\Leftrightarrow f(x,p) = 0 \Leftrightarrow \overline{XP} \text{ is a tangent}, \\ \boldsymbol{\sigma}_P(X) \neq X &\Leftrightarrow f(x,p) \neq 0 \Leftrightarrow \overline{XP} \text{ is a secant}. \end{aligned}$$

By Proposition 3.12 *iii)* and *v)*, we have $\boldsymbol{\sigma}_P(X) \in \overline{PX} \cap \mathbf{P}\mathrm{Q}$. Therefore, if $\mathfrak{g}$ is a line through P, the collineation $\boldsymbol{\sigma}_P$ exchanges the two intersection points $\{X, Y\} = \mathfrak{g} \cap \mathbf{P}\mathrm{Q}$ whenever $\mathfrak{g}$ is a secant and fixes the point $\{X\} = \mathfrak{g} \cap \mathbf{P}\mathrm{Q}$ if $\mathfrak{g}$ is a tangent. Hence, the restriction $\boldsymbol{\sigma}_P|_{\mathbf{P}\mathrm{Q}}$ is exactly the symmetry $\hat{\boldsymbol{\sigma}}_P$ at P.

So instead of studying the product $\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\sigma}}_{P_1} \circ \ldots \circ \hat{\boldsymbol{\sigma}}_{P_r}$ of symmetries on $\mathbf{P}\mathrm{Q}$, we can investigate the isometry $\pi = \sigma_{P_1} \circ \ldots \circ \sigma_{P_r} \in \mathrm{O}$ of the quadratic space (V, Q) or the collineation $\boldsymbol{\pi} = \boldsymbol{\sigma}_{P_1} \circ \ldots \circ \boldsymbol{\sigma}_{P_r} \in \mathrm{PGL}$ of $\mathbf{P}V$.

Definition 3.14 *We call the subgroups*

$$\mathrm{O}_S = \{\sigma_{P_1} \circ \ldots \circ \sigma_{P_k} | k \in \mathbb{N}, P_1, \ldots, P_k \in \mathbf{P}V \text{ non-singular}\} \subset \mathrm{O},$$
$$\mathrm{PO}_S = \{\boldsymbol{\sigma}_{P_1} \circ \ldots \circ \boldsymbol{\sigma}_{P_k} | k \in \mathbb{N}, P_1, \ldots, P_k \in \mathbf{P}V \text{ non-singular}\} \subset \mathrm{PGL}$$

the symmetry group *or the* projective symmetry group of (V, Q) *respectively.*

Remark 3.15 *If we have an isometry $\pi \in \mathrm{O}_S$, then $\boldsymbol{\pi} \in \mathrm{PO}_S$ will always denote the collineation induced by π. Further, we write $\hat{\boldsymbol{\pi}} \in \hat{\mathrm{O}}$ for the restriction $\boldsymbol{\pi}|_{\mathbf{P}\mathrm{Q}}$ to the quadric $\mathbf{P}\mathrm{Q}$. Although the maps $\pi, \boldsymbol{\pi}, \hat{\boldsymbol{\pi}}$ have vastly different domains, they satisfy*

$$\langle \pi(x) \rangle = \pi(X) = \boldsymbol{\pi}(X) \quad \text{and} \quad \boldsymbol{\pi}(Y) = \hat{\boldsymbol{\pi}}(Y)$$

for all $x \in V$ with $\langle x \rangle = X \in \mathbf{P}_0 V$ and $Y \in \mathbf{P}\mathrm{Q}$ respectively.

Let $p : \mathrm{O}_S \to \mathrm{PO}_S$ be the map which sends the isometry $\pi \in \mathrm{O}_S$ to the collineation $\boldsymbol{\pi}$ induced by π. It is clearly surjective. Further, we have shown that the restriction $p : \mathrm{PO}_S \to \hat{\mathrm{O}}; \pi \mapsto \pi|_{\mathbf{P}\mathrm{Q}}$ is surjective as well. Hence, the composition $(p \circ r)$

$$\mathrm{O}_S \xrightarrow{p} \mathrm{PO}_S \xrightarrow{r} \hat{\mathrm{O}}; \quad \pi \overset{p}{\mapsto} \boldsymbol{\pi} \overset{r}{\mapsto} \boldsymbol{\pi}|_{\mathbf{P}\mathrm{Q}} \tag{3.7}$$

is a surjective group homomorphism. However, p, r must not be injective.

- Since both $\pm \mathrm{id}_V \in \mathrm{O}$ induce the identity collineation, the homomorphism p can not be injective if $-\mathrm{id}_V \in \mathrm{O}_S$.
- If there are no secants in $\mathbf{P}V$, then every symmetry $\boldsymbol{\sigma}_P$ fixes $\mathbf{P}\mathrm{Q}$ pointwise.

Example 3.16 *Let $V = \mathbb{R}^3$ and e_1, e_2, e_3 be the standard basis.*

If $\mathrm{Q}_1((x_1, x_2, x_3)^\intercal) = x_1^2 + x_2^2 - x_3^2$, one can show that

$$\sigma_{e_1} \circ \sigma_{e_2} \circ \sigma_{e_3} = -\mathrm{id}_V$$

and, hence, $\boldsymbol{\sigma}_{e_1} \circ \boldsymbol{\sigma}_{e_2} \circ \boldsymbol{\sigma}_{e_3} = \mathrm{id}_{\mathbf{P}V}$. This implies that the map $p : \mathrm{O}_S \to \mathrm{PO}_S$ can not be injective for the quadratic space (V, Q_1).

If $\mathrm{Q}_2((x_1, x_2, x_3)^\intercal) = x_1^2$, then $\mathbf{P}V$ has no secants. Hence, for every non-singular point $P \in \mathbf{P}V$ we have $\hat{\boldsymbol{\sigma}}_P = \mathrm{id}_{\mathbf{P}\mathrm{Q}}$. However, the collineation $\boldsymbol{\sigma}_{e_1} \in \mathrm{PO}_S(V, \mathrm{Q}_2)$ is non-trivial because $\boldsymbol{\sigma}_{e_1}(\langle e_1 + e_2 \rangle) = \langle -e_1 + e_2 \rangle \neq \langle e_1 + e_2 \rangle$. Therefore, the map $r : \mathrm{PO}_S \to \hat{\mathrm{O}}$ is not injective for the quadratic space (V, Q_2).

We will see that in lot of cases $-\mathrm{id}_V$ is indeed a product of symmetries. Hence, p can not be injective. However, if there is a vector $x_1 \in V$ with $\mathrm{Q}(x_1) \neq 0$, $\pm \mathrm{id}_V$ are the only isometries that induce the identity collineation.

Lemma 3.17 *Let* (V, Q) *be a quadratic space and assume there is a vector* $x_1 \in V$ *with* $\mathrm{Q}(x_1) \neq 0$*. If a isometry* $\pi \in \mathrm{O}$ *induces the identity collineation* $\mathrm{id}_{\mathbf{P}V} \in \mathrm{PGL}$*, then* $\pi = \pm\,\mathrm{id}_V$*.*

Proof Let $\langle x \rangle = X \in \mathbf{P}V$ be a point. Since π induces the identity collineation, we must have $\pi(x) = \alpha x \in X$ for some $\alpha \in \mathcal{K}$. As π is linear, we see $\pi(x') = \alpha x'$ for all $x' \in X$. This clearly holds for all points in $\mathbf{P}V$. However, the factor α might be different.

If $\langle y \rangle = Y \in \mathbf{P}V$ is a point distinct from X, we get $\beta, \gamma \in \mathcal{K}$ for the points $Y, \langle x + y \rangle$ with

$$\pi(y) = \beta y, \quad \pi(x+y) = \gamma(x+y).$$

Since x, y are linearly independent and π is linear, we get $\alpha = \gamma = \beta$ by comparing the scalars. Hence, $\pi(x) = \alpha x$ for all $x \in V$.

As π is an isometry, we see $\mathrm{Q}(x_1) = \mathrm{Q}(\pi(x_1)) = \mathrm{Q}(\alpha x_1) = \alpha^2 \mathrm{Q}(x_1)$ and, therefore, $\alpha = \pm 1$ because $\mathrm{Q}(x_1) \neq 0$. □

Remark 3.18 *If there is a vector* $x \in V$ *with* $\mathrm{Q}(x) \neq 0$*, then Lemma 3.17 shows that* $\ker(p) \subset \{\pm\,\mathrm{id}_V\}$ *and, hence, we get*

$$\mathrm{PO}_S \simeq \begin{cases} \mathrm{O}_S\,/\{\pm\,\mathrm{id}_V\} & \text{if } -\mathrm{id}_V \in \mathrm{O}_S \\ \mathrm{O}_S & \text{if } -\mathrm{id}_V \notin \mathrm{O}_S \end{cases}.$$

As we will see, if $\dim V \geq 3$, the existence of a secant is enough to ensure injectivity of the restriction r. The following theorem generalises Theorem 3.18. on p. 127 in [6].

Theorem 3.19 *Let* (V, Q) *be a quadratic space over* $\mathcal{K}$ *and* $\dim(V) \geq 3$*. If there is a secant in* $\mathbf{P}V$*, then any collineation* $\boldsymbol{\pi} \in \mathrm{PGL}$ *which*

i) *is induced by an isometry* $\pi \in \mathrm{O}$ *and*

ii) *fixes* $\mathbf{P}\mathrm{Q}$ *point-wise*

is the identity, $\boldsymbol{\pi} = \mathrm{id}_{\mathbf{P}V}$*. Further, we must have* $\pi = \pm\,\mathrm{id}_V$*.*

Proof As there is a secant in $\mathbf{P}V$, we have two singular points X, Y such that $\overline{XY}$ is a secant. Recall that if we write z, we implicitly mean $z \in V$ and $\langle z \rangle = Z \in \mathbf{P}_0 V$. Proposition 3.7 shows $f(x, y) \neq 0$. Therefore, by rescaling we can choose x, y in such a way that $f(x, y) = 1$. Since X, Y are singular and, hence, fixed by $\boldsymbol{\pi}$, we get

$$\pi(x) = \alpha x, \quad \pi(y) = \beta y \tag{3.8}$$

for some fixed $\alpha, \beta \in \mathcal{K}^*$.

As $\dim(V) \geq 3$, there is a vector $z' \notin \langle x, y \rangle$. Our goal is to show that $\boldsymbol{\pi}$ fixes Z' and every point on $\overline{XY}$. This will show that $\boldsymbol{\pi}$ is the identity collineation because Z' is an arbitrary point. Since $\mathrm{Q}(x+y) = f(x,y) = 1$, we can conclude the theorem with Lemma 3.17.

Let $z = z' - f(y,z')x - f(x,z')y \in \langle x, y, z' \rangle$. One can confirm that

$$f(x,z) = f(y,z) = 0 \tag{3.9}$$

and that x, y, z are linearly independent. We now have

$$f(x, y+z) = 1, \quad f(y, x+z) = 1$$

and since X, Y are singular, the lines

$$\mathfrak{g} = \langle x \rangle \oplus \langle y+z \rangle, \quad \mathfrak{h} = \langle y \rangle \oplus \langle x+z \rangle$$

are secants by Proposition 3.7. One easily sees that $\mathfrak{g}, \mathfrak{h}$ are distinct and intersect each other in the point $\mathfrak{g} \cap \mathfrak{h} = \langle x+y+z \rangle$. Since $\boldsymbol{\pi}$ fixes every singular point, it must fix secants. Hence, $\boldsymbol{\pi}(\mathfrak{g}) = \mathfrak{g}$ and $\boldsymbol{\pi}(\mathfrak{h}) = \mathfrak{h}$. This shows

$$\boldsymbol{\pi}(\langle x+y+z \rangle) = \boldsymbol{\pi}(\mathfrak{g} \cap \mathfrak{h}) = \boldsymbol{\pi}(\mathfrak{g}) \cap \boldsymbol{\pi}(\mathfrak{h}) = \mathfrak{g} \cap \mathfrak{h} = \langle x+y+z \rangle$$

and, therefore, $\pi(x+y+z) = \gamma(x+y+z)$ for some fixed $\gamma \in \mathcal{K}^*$.

We now see

$$\begin{aligned} \pi(z) &= \pi((x+y+z) - x - y) \\ &= \gamma(x+y+z) - \alpha x - \beta y \\ &= (\gamma - \alpha)x + (\gamma - \beta)y + \gamma z \end{aligned} \tag{3.10}$$

by linearity of π and (3.8). As π preserves f, Eqs. (3.8) to (3.10) combined show

$$\begin{aligned} 0 = f(x,z) &= f(\pi(x), \pi(z)) \\ &= f(\alpha x, (\gamma - \alpha)x + (\gamma - \beta)y + \gamma z) \\ &= f(\alpha x, (\gamma - \beta)y) \\ &= \alpha(\gamma - \beta) \end{aligned}$$

because $f(x,y) = 1$. Since $\alpha \neq 0$, we get $\beta = \gamma$. Similarly, we can deduce $\alpha = \gamma$ from $0 = f(y,z)$, hence, $\alpha = \beta = \gamma$. This shows that $\pi(v) = \alpha v$ for every vector $v \in \langle x, y, x+y+z \rangle = \langle x, y, z \rangle$. As $z' \in \langle x, y, z \rangle$, Z' is fixed by $\boldsymbol{\pi}$ and we can conclude the theorem. □

Remark 3.20 *If* $\dim(V) = 2$*, Theorem 3.19 must not be true. Consider the quadratic space* $V = \mathbb{R}^2$ *with* $\mathrm{Q}((x,y)^\intercal) = xy$ *and the map*

$$\pi_\alpha((x,y)^\intercal) = (\alpha x, \alpha^{-1} y)^\intercal \tag{3.11}$$

for some fixed $\alpha \in \mathbb{R}^*$*. One can easily see that* π *is an isometry which fixes the singular points* $\langle (1,0)^\intercal \rangle$*,* $\langle (0,1)^\intercal \rangle$*. There is only one line in* $\mathbf{P}\mathbb{R}$ *and it must be a secant. Therefore, all other conditions in Theorem 3.19 are met. However, if* $|\mathcal{K}| \neq 2$*,* π_α *can be non-trivial.*

We say a quadric $\mathbf{P}\mathrm{Q}$ of (V, Q) is *non-trivial*, if there are two distinct secants in $\mathbf{P}V$.

This is equivalent to the condition given in Theorem 3.19. Indeed, if there are two distinct lines, we must have $\dim(V) \geq 3$ and there is obviously a secant. On the other hand, the lines $\mathfrak{g}, \mathfrak{h}$ in the proof of Theorem 3.19 are distinct secants.

If $\mathbf{P}\mathrm{Q}$ is a non-trivial quadric, Theorem 3.19 shows that the restriction

$$r : \boldsymbol{\pi} \in \mathrm{PO}_S \overset{r}{\mapsto} \boldsymbol{\pi}|_{\mathbf{P}\mathrm{Q}} \in \hat{\mathrm{O}}$$

from (3.7) has $\ker(r) = \{\mathrm{id}_{\mathbf{P}V}\}$ and, therefore, r is an isomorphism. With the same reasoning as in Remark 3.18 we have shown.

Theorem 3.21 *Let* (V, Q) *be a quadratic space over* $\mathcal{K}$ *and assume the quadric* $\mathbf{P}\mathrm{Q}$ *is non-trivial, then*

$$\hat{\mathrm{O}} \simeq \mathrm{PO}_S \simeq \begin{cases} \mathrm{O}_S \,/\{\pm \mathrm{id}_V\} & \text{if } -\mathrm{id}_V \in \mathrm{O}_S \\ \mathrm{O}_S & \text{if } -\mathrm{id}_V \notin \mathrm{O}_S \end{cases}.$$

Similar to Chapter 1 we can define the *length of an isometry* $\pi \in \mathrm{O}_S = \mathrm{O}^*$*, a collineation* $\boldsymbol{\pi} \in \mathrm{PO}_S$ *and a map* $\hat{\boldsymbol{\pi}} \in \hat{\mathrm{O}}$ as

$$\begin{aligned} \mathrm{l}_{\mathrm{O}_S}(\pi) &= \min\{k \in \mathbb{N} : \pi = \sigma_{P_1} \circ \ldots \circ \sigma_{P_k}\}, \\ \mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) &= \min\{k \in \mathbb{N} : \boldsymbol{\pi} = \boldsymbol{\sigma}_{P_1} \circ \ldots \circ \boldsymbol{\sigma}_{P_k}\}, \\ \mathrm{l}_{\hat{\mathrm{O}}}(\hat{\boldsymbol{\pi}}) &= \min\{k \in \mathbb{N} : \hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\sigma}}_{P_1} \circ \ldots \circ \hat{\boldsymbol{\sigma}}_{P_k}\} \end{aligned}$$

respectively.

Remark 3.22 *If* $k = 0$*, we think of the product* $\sigma_{P_1} \circ \ldots \circ \sigma_{P_k}$ *as the identity map. Hence,* $\mathrm{l}_{\mathrm{O}_S}(\mathrm{id}_{\mathrm{O}_S}) = \mathrm{l}_{\mathrm{PO}_S}(\mathrm{id}_{\mathrm{PO}_S}) = \mathrm{l}_{\hat{\mathrm{O}}}(\mathrm{id}_{\hat{\mathrm{O}}}) = 0$*.*

Theorem 3.21 and Remark 3.18 give us a way to express $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$, $\mathrm{l}_{\hat{\mathrm{O}}}(\hat{\boldsymbol{\pi}})$ with the length $\mathrm{l}_{\mathrm{O}_S}(\pi)$.

Theorem 3.23 *Let (V, Q) be a quadratic space and assume there is a vector $x \in V$ with $\mathrm{Q}(x) \neq 0$. Then for an isometry $\pi \in \mathrm{O}_S$, the collineation $\boldsymbol{\pi}$ induced by π and $\hat{\boldsymbol{\pi}} = \boldsymbol{\pi}|_{\mathbf{P}\mathrm{Q}}$ we have*

$$\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) = \begin{cases} \min\{\mathrm{l}_{\mathrm{O}_S}(\pi), \mathrm{l}_{\mathrm{O}_S}(-\pi)\} & \text{if } -\mathrm{id}_V \in \mathrm{O}_S \\ \mathrm{l}_{\mathrm{O}_S}(\pi) & \text{if } -\mathrm{id}_V \notin \mathrm{O}_S \end{cases}$$

and $\mathrm{l}_{\hat{\mathrm{O}}}(\hat{\boldsymbol{\pi}}) = \mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$ if the quadric $\mathbf{P}\mathrm{Q}$ is non-trivial.

Proof Let $\pi \in \mathrm{O}_S$.

If $-\mathrm{id} \in \mathrm{O}_S$ then $-\pi \in \mathrm{O}_S$ and by Remark 3.18 we see that $\pm\pi$ are the only isometries which induce $\boldsymbol{\pi}$.

Therefore, $\boldsymbol{\pi}$ can be written as a product of $\mathrm{l}_{\mathrm{O}_S}(\pi)$ and as a product of $\mathrm{l}_{\mathrm{O}_S}(-\pi)$ symmetries of PO_S.

On the other hand, if $\boldsymbol{\pi} = \boldsymbol{\sigma}_{p_1} \circ \ldots \circ \boldsymbol{\sigma}_{p_k}$, either

$$\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k} \quad \text{or} \quad \pi = -\sigma_{p_1} \circ \ldots \circ \sigma_{p_k}$$

which shows that at least one of $\pm\pi$ can be written as a product of k symmetries. This shows the equality $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) = \min\{\mathrm{l}_{\mathrm{O}_S}(\pi), \mathrm{l}_{\mathrm{O}_S}(-\pi)\}$.

If $-\mathrm{id} \notin \mathrm{O}_S$, then Remark 3.18 shows that $\mathrm{PO}_S \simeq \mathrm{O}_S$. Since the isomorphism maps symmetries to symmetries, we see $\mathrm{l}_{\mathrm{O}_S}(\pi) = \mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$.

If $\mathbf{P}\mathrm{Q}$ is non-trivial, then Theorem 3.21 gives us $\mathrm{PO}_S \simeq \hat{\mathrm{O}}$ and we conclude with the same reasoning as before. □

This motivates us to study the subgroup $\mathrm{O}_S \subset \mathrm{O}$ and the length $\mathrm{l}_{\mathrm{O}_S}$ instead $\hat{\mathrm{O}}$ and $\mathrm{l}_{\hat{\mathrm{O}}}$. This will be the topic of the next chapter.

Chapter 4

Isometries of Orthogonal Spaces

Let (V, Q) be quadratic space and f_Q its associated bilinear map. Recall the definition of the orthogonal group

$$\mathrm{O}(V, \mathrm{Q}) = \{\pi \in \mathrm{GL} : \mathrm{Q}(\pi(x)) = \mathrm{Q}(x) \text{ for all } x \in V\}.$$

Usually, the name orthogonal group is reserved for the group

$$\mathrm{O}(V, f) = \{\pi \in \mathrm{GL} : f(\pi(x), \pi(y)) = f(x, y) \text{ for all } x, y \in V\}$$

of an orthogonal space (V, f). Similarly, one calls a map $\pi \in \mathrm{O}(V, f)$ an isometry instead of $\pi \in \mathrm{O}(V, \mathrm{Q})$.

However, if $\mathrm{char}(\mathcal{K}) \neq 2$, these two notions coincide. Indeed, we have already seen that $\mathrm{O}(V, f_\mathrm{Q}) \subset \mathrm{O}(V, \mathrm{Q})$ by Remark 3.18. On the other hand, since $f_\mathrm{Q}(x, x) = 2\mathrm{Q}(x)$, we get

$$\mathrm{Q}(\pi(x)) = \frac{1}{2} f_\mathrm{Q}(\pi(x), \pi(x)) = \frac{1}{2} f_\mathrm{Q}(x, x) = \mathrm{Q}(x)$$

for any $\pi \in \mathrm{O}(V, f_\mathrm{Q})$ and $x \in V$. Hence, $\mathrm{O}(V, f_\mathrm{Q}) = \mathrm{O}(V, \mathrm{Q})$. As any orthogonal space is induced by a quadratic space by Proposition 3.3, we see that there is no difference between these two notions.

The orthogonal group $\mathrm{O}(V, f)$ is well-studied and there is an extensive theory of trying to write an isometry $\pi \in \mathrm{O}(V, f)$ as a product of symmetries $\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k}$.

Firstly, Cartan and Diodonné have shown that every isometry can be written as a product of at most $\dim(V)$ symmetries if V is *non-singular*. Later, Scherk has answered the question of how many symmetries are needed for a given isometry $\pi \in \mathrm{O}$ in his article [8]. Usually, these theorems are stated with left vector spaces over skew-fields and sesquilinear maps instead of orthogonal spaces.

Since we are interested in the group $\mathrm{O}_S \subset \mathrm{O}(V, \mathrm{Q})$, we only consider orthogonal spaces and restrict ourselves to $\mathrm{char}(\mathcal{K}) \neq 2$. However, we want to include *singular* vector spaces, which are discussed by Götzky in [9]. Our chapter follows [9] and incorporates ideas of Artin [6] and Ellers [10].

4.1 Orthogonal spaces

Let (V, f) be an orthogonal space of a finite dimensional vector space $(V, \mathcal{K})$ with $\mathrm{char}(\mathcal{K}) \neq 2$ and denote by $\mathrm{Q}(x) = \frac{1}{2} f(x, x)$ the associated quadratic form.

Definition 4.1 *Two subspaces $U, W \subset V$ are said to be* orthogonal *if $f(u, w) = 0$ for all $u \in U$, $w \in W$. We write $U \perp W$ instead of $U \oplus W$ if U, W are orthogonal. Further, the* orthogonal subspace of U *is denoted by*

$$U^{\perp} = \{x \in V : f(x, u) = 0,\ \forall u \in U\}.$$

Additionally, we write $\mathrm{rad}(U) = U \cap U^{\perp}$ for the *radical of* U and say U *is non-singular* if $\mathrm{rad}(U) = \{\mathbf{0}\}$. Otherwise, U is called *singular*.

Remark 4.2 *Recall that we say a point $\langle x \rangle = X \in \mathbf{P}V$ is singular if $\mathrm{Q}(x) = 0$ by our definition in Section 3.2. However, this is equivalent to $\mathrm{rad}(X) = X \cap X^{\perp} = X$ and, therefore, these two definitions are compatible with each other.*

One easily sees that $U^{\perp}$ and, hence, $\mathrm{rad}(U)$ are a subspace of V. Further, the definition immediately shows

$$U \subset W \Rightarrow W^{\perp} \subset U^{\perp} \tag{4.1}$$

for any subspaces $U, W \subset V$. Therefore, we have $\mathrm{rad}(V) = V^{\perp} \subset U^{\perp}$ for all subspaces $U \subset V$. In other words, $\mathrm{rad}(V)$ is orthogonal to every subspace.

Remark 4.3 *If $R = \mathrm{rad}(V) \neq \{\mathbf{0}\}$, formulas are often more complicated. It will be useful to consider a complementary subspace U of R which means $R \oplus U = V$. Hence, we can write any $v \in V$ as $v = r + w$ with $r \in R$, $u \in U$. For a vector $x \in \mathrm{rad}(U)$ we have*

$$f(x, v) = f(x, r + u) = f(x, r) + f(x, u) = 0,$$

hence, $x \in R$. We conclude $x \in R \cap U = \{\mathbf{0}\}$ and see that U is non-singular and $V = R \perp U$.

As $U^{\perp} \subset V$ is a subspace, it makes sense to find out its dimension. If V is non-singular, the following formula holds:

$$\dim(U) + \dim(U^{\perp}) = \dim(V). \tag{4.2}$$

One has certainly seen this equation in the case of the dot product of $\mathbb{R}^n$. However, one can extend Eq. (4.2) to non-singular orthogonal spaces. For a proof we refer to Lemma A.1, which does not fit the style of this chapter and, hence, is found in Appendix A.

However, if $\mathrm{rad}(V) \neq \{\mathbf{0}\}$, we can derive a formula for $\dim(U^{\perp})$ by Eq. (4.2) and Remark 4.3. It is included in the following proposition.

Proposition 4.4 *If $U, W \subset V$ are subspaces and $R = \mathrm{rad}(V)$, we have:*

i) $(U+W)^{\perp} = U^{\perp} \cap W^{\perp}$,

ii) $(U+R)^{\perp} = U^{\perp}$,

iii) $\dim(U^{\perp}) = \dim(V) - \dim(U) + \dim(U \cap R)$,

iv) *if* $\mathrm{rad}(U) \subset R$, *then* $U + U^{\perp} = V$,

v) $(U^{\perp})^{\perp} = U + R$,

vi) $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$.

Proof *i)* The inclusion $(U+W)^{\perp} \subset U^{\perp} \cap W^{\perp}$ follows immediately by Eq. (4.1). On the other hand, for a vector $x \in U^{\perp} \cap W^{\perp}$ we have

$$f(x, u+w) = f(x,u) + f(x,w) = 0$$

for any $u + w \in U + W$. Hence, $x \in (U+W)^{\perp}$ follows and we conclude *i)*.

ii) As $R^{\perp} = V$, we see $(U+R)^{\perp} = U^{\perp} \cap V = U^{\perp}$ by *i)*.

iii) Let $R' = R \cap U$. We can find subspaces U', V' such that

$$U' \oplus R' = U, \quad U' \subset V', \quad V' \oplus R = V$$

by extending a basis of R'. Now Remark 4.3 shows that V' is non-singular and we can apply Eq. (4.2) on $U' \subset V'$. This gives us

$$\dim(U'^{\perp} \cap V') = \dim(V') - \dim(U'). \tag{4.3}$$

By *ii)* we have $U'^{\perp} = (U' \oplus R')^{\perp} = U^{\perp}$ and, hence,

$$U^{\perp} = U'^{\perp} = U'^{\perp} \cap (V' \oplus R) = (U'^{\perp} \cap V') \oplus R$$

since $R \subset U'^{\perp}$. Eq. (4.3) lets us conclude that

$$\begin{aligned}
\dim(U^{\perp}) &= \dim(U'^{\perp} \cap V') + \dim(R) \\
&= \dim(V') - \dim(U') + \dim(R) \\
&= (\dim(V) - \dim(R)) - (\dim(U) - \dim(R')) + \dim(R) \\
&= \dim(V) - \dim(U) + \dim(U \cap R).
\end{aligned}$$

iv) Since $\mathrm{rad}(U) \subset R$, we see $\mathrm{rad}(U) \subset U \cap R$. The reverse inclusion follows because $U \cap R \subset R \subset U^{\perp}$. By *iii)* we now have

$$\begin{aligned}\dim(U + U^{\perp}) &= \dim(U) + \dim(U^{\perp}) - \dim(U \cap U^{\perp}) \\ &= \dim(V) + \dim(U \cap R) - \dim(\mathrm{rad}(U)) = \dim(V).\end{aligned}$$

Hence, $U + U^{\perp} = V$.

v) If $u \in U$, then $f(x, u) = 0$ for all $x \in U^{\perp}$. Therefore, $u \in (U^{\perp})^{\perp}$ and we see $U + R \subset (U^{\perp})^{\perp}$ as $R \subset (U^{\perp})^{\perp}$ holds trivially. The equality follows since

$$\begin{aligned}\dim((U^{\perp})^{\perp}) &= \dim(V) - \dim(U^{\perp}) + \dim(U^{\perp} \cap R) \\ &= \dim(V) - (\dim(V) - \dim(U) + \dim(U \cap R)) + \dim(R) \\ &= \dim(U) - \dim(U \cap R)) + \dim(R) = \dim(U + R).\end{aligned}$$

vi) Notice that *i)* and *v)* show

$$\begin{aligned}(U^{\perp} + W^{\perp})^{\perp} &= (U^{\perp})^{\perp} \cap (W^{\perp})^{\perp} \\ &= (U + R) \cap (W + R) = (U \cap W) + R.\end{aligned}$$

Combining this with the fact that $R \subset U^{\perp}$ we get

$$\begin{aligned}(U \cap W)^{\perp} &= ((U \cap W) + R)^{\perp} = ((U^{\perp} + W^{\perp})^{\perp})^{\perp} \\ &= (U^{\perp} + W^{\perp}) + R = U^{\perp} + W^{\perp}. \qquad \square\end{aligned}$$

We have already seen that $f|_{\mathrm{rad}(V)} \equiv 0$. It will be useful to give a name to subspaces with this property.

Definition 4.5 *A vector $x \in V$ is called* isotropic *if $f(x, x) = 0$. Similarly, a subspace $U \in V$ is called* isotropic *if $f(x, y) = 0$ for all $x, y \in U$.*

One easily sees that a subspace $U \subset V$ is isotropic if and only if $\mathrm{rad}(U) = U$. Hence, if $\langle x \rangle = X \subset V$ is a 1-dimensional subspace it, is equivalent whether we say x is isotropic, X is isotropic or X is singular.

Generally, if $U \neq \{\mathbf{0}\}$ is isotropic, then U is singular as well. However, the reverse implication must not be true if $\dim(U) \geq 2$.

Remark 4.6 *Often one includes an additional property in Definition 4.5.*

A subspace $U \subset V$ is called totally isotropic *if $f(x, x) = 0$ for all $x \in U$. Clearly, an isotropic subspace is totally isotropic.*

If $\mathrm{char}(\mathcal{K}) \neq 2$, *the reverse implication is true as well. Indeed, for vectors x, y in an totally isotropic subspace U we have*

$$f(x, y) = \frac{1}{2}(f(x + y, x + y) - f(x, x) - f(y, y)) = 0$$

since $x + y \in U$. This is not true if $\mathrm{char}(\mathcal{K}) = 2$.

Remark 4.6 shows that a subspace U is *non-isotropic* if and only if U contains a non-isotropic vector.

If V is isotropic, then $f \equiv 0$ and we are left with only a vector space. So we will always assume that V is non-isotropic. This ensures that there are lots of non-isotropic vectors.

Proposition 4.7 *If a vector space V is non-isotropic, it has a basis consisting of non-isotropic vectors.*

Proof We will use induction on $\dim(V) = n$.

This is clearly true if $n = 1$. Assume the statement holds for all vector spaces with dimensions less than n.

Assume $\dim(V) = n$. As V is non-isotropic, there is a non-isotropic vector $x \in V$. Let $V' \subset V$ be a complementary subspace to $\langle x \rangle$. Therefore, $\dim(V') = n - 1$.

If V' is non-isotropic, we get a basis $x_1, \ldots, x_{n-1}$ of V' of non-isotropic vectors by the induction hypothesis. Hence, $x_1, \ldots, x_{n-1}, x$ has the desired property.

Otherwise, let $x_1, \ldots, x_{n-1}$ be a basis of V'. The basis must consist of isotropic vectors since V' is isotropic. A short calculation shows

$$f(\alpha x + x_1, \alpha x + x_1) = \alpha(\alpha f(x, x) + 2 f(x, x_1)) \neq 0$$

for any $\alpha \in \mathcal{K}$. Hence, $\alpha x + x_1$ is isotropic if and only if $\alpha = f(x, x_1)/\mathrm{Q}(x)$ or $\alpha = 0$. As $|\mathcal{K}| \geq 3$, we can choose $\alpha \in \mathcal{K}$ such that $\alpha x + x_1$ is non-isotropic. Hence, $V'' = \langle \alpha x + x_1, x_2, \ldots, x_{n-1} \rangle$ is non-isotropic and one easily confirms that $V = \langle x \rangle \oplus V''$. Hence, we can conclude as in the non-isotropic case. $\square$

4.2 Simple isometries

The focus for this section will be on isometries. Recall that (V, f) still denotes a finite-dimensional orthogonal space. As we have seen in Lemma 3.10, the symmetry

$$\sigma_p(x) = x - \mathrm{Q}(p)^{-1} f(x, p) p$$

is an isometry for any non-isotropic vector $p \in V$.

Notice that for any $r \in \mathrm{rad}(V)$ we have $f(r, p_i) = 0$ and, hence, $\sigma_{p_i}(r) = r$. Therefore, the radical $\mathrm{rad}(V)$ is fixed point-wise by symmetries. Therefore, for a product of symmetries $\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k} \in \mathrm{O}_S$, we have $\pi|_{\mathrm{rad}(V)} \equiv \mathrm{id}$.

This leads to the definition of the *extended orthogonal group*

$$\mathrm{O}^* = \{\pi \in \mathrm{O} : \pi|_{\mathrm{rad}(V)} = \mathrm{id}\} \subset \mathrm{O}.$$

We clearly have $\mathrm{O}_S \subset \mathrm{O}^*$. The goal of this chapter is to show that every $\pi \in \mathrm{O}^*$ can be written as a product of symmetries. Hence, $\mathrm{O}_S = \mathrm{O}^*$. On the way, we find the minimal number of symmetries needed to write π as a product of symmetries.

If $\dim(V)$ is small it is usually possible to give a precise description of $\mathrm{O}^*(V)$ and compute a decomposition for a given isometry in O^*.

Example 4.8 *Let (V, f) be a non-singular orthogonal space over $\mathcal{K}$ with dimension 2. We want to show that if there is an isotropic vector $\mathbf{0} \neq x \in V$, there exists a unique isotropic vector $y \in V$ with $f(x, y) = 1$.*

Since V is non-singular, there must be a vector $y' \in V$ such that $f(x, y') \neq 0$. Similarly to Proposition 3.7, we see that the vectors

$$y'' = f(x, y')y' - \mathrm{Q}(y')x, \quad y = \frac{1}{f(x, y')^2}y''$$

are isotropic and $f(x, y) = 1$. Notice that y is uniquely defined by x and $\langle x, y\rangle = V$. Such a pair is called a hyperbolic pair *and we write $H(x, y)$ instead of V.*

Let $\pi \in \mathrm{O} = \mathrm{O}^*$ be an arbitrary isometry of $H(x, y)$. It must map isotropic vectors to isotropic vectors. This means either

$$\pi(x) \in \langle x\rangle, \pi(y) \in \langle y\rangle \;\text{ or }\; \pi(x) \in \langle y\rangle, \pi(y) \in \langle x\rangle.$$

In the first case we have $\pi(x) = \alpha x$ and $\pi(y) = \beta y$ for some fixed $\alpha, \beta \in \mathcal{K}^*$. Since π is an isometry, we must have

$$1 = f(x, y) = f(\pi(x), \pi(y)) = \alpha\beta$$

and, hence, $\beta = 1/\alpha$. Therefore, $\pi = \pi_\alpha$ where

$$\pi_\alpha(x) = \alpha x, \quad \pi_\alpha(y) = \frac{1}{\alpha}y.$$

We can reduce the second case to the first because one easily checks that the map $\sigma \in \mathrm{GL}$ with $\sigma(x) = y$, $\sigma(y) = x$ is an isometry. Hence, the isometry $\pi \circ \sigma$ must be equal to π_α for some $\alpha \in \mathcal{K}^*$ and we get $\pi = \pi_\alpha \circ \sigma$.

This shows that the orthogonal group of $V = H(x, y)$ is equal to

$$\mathrm{O}(V) = \{\pi_\alpha \circ \sigma^b : \alpha \in \mathcal{K}^*, b \in \{0, 1\}\}.$$

Since $\{\mathrm{id}, \sigma\} \subset \mathrm{O}$ is abelian, it is normal and we can factor

$$\mathrm{O}(V)/\{\mathrm{id}, \sigma\} \simeq \{\pi_\alpha : \alpha \in \mathcal{K}^*\} \simeq \mathcal{K}^*$$

as $\pi_\alpha \circ \pi_\beta = \pi_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{K}^*$. Hence, $\mathrm{O}(V) \simeq \mathcal{K}^* \times \mathbb{Z}_2$.

Let $\alpha \in \mathcal{K}^*$ and consider $p = \alpha y - x \in V$. The vector p is non-isotropic since $\mathrm{Q}(p) = -\alpha$. If we evaluate the symmetry σ_p at x and y, we see

$$\sigma_p(x) = x + \frac{1}{\alpha} f(\alpha y - x, x)(\alpha y - x) = x + \alpha y - x = \alpha y,$$
$$\sigma_p(y) = y + \frac{1}{\alpha} f(\alpha y - x, y)(\alpha y - x) = y + \frac{1}{\alpha}(\alpha y - x) = \frac{1}{\alpha} x$$

and, therefore, $\sigma_p = \pi_\alpha \circ \sigma$. As $\sigma = \sigma_{y-x}$, we get $\pi_\alpha = \sigma_p \circ \sigma_{y-x}$ and see that every $\pi \in \mathrm{O}(V)$ can be written as a product of at most 2 symmetries.

The approach in Example 4.8 is not feasible for arbitrary orthogonal spaces. However, it is an important case and we need it later.

Firstly, we want to introduce and investigate some properties of isometries.

Definition 4.9 *Let $\pi \in \mathrm{O}^*$ and $\varphi = \pi - \mathrm{id}$. We denote the* fix *and* path *of π by*

$$\mathrm{Fix}(\pi) = \{x \in V : \pi(x) = x\} = \ker(\varphi) \subset V,$$
$$\mathrm{Path}(\pi) = \{\pi(x) - x : x \in V\} = \mathrm{im}(\varphi) \subset V.$$

The sets $\mathrm{Fix}(\pi), \mathrm{Path}(\pi) \subset V$ are clearly subspaces.

Example 4.10 *Let $\langle p \rangle = P \in \mathbf{P}V$ be a non-singular point and consider the symmetry $\sigma_P \in \mathrm{O}^*$. For $\varphi = \sigma_P - \mathrm{id}$ we have*

$$\varphi(x) = -\mathrm{Q}(p)^{-1} f(x, p) p \in \langle p \rangle.$$

Since $\varphi(p) = -2p$ we see

$$\mathrm{Path}(\sigma_P) = \mathrm{im}(\varphi) = \langle p \rangle = P, \quad \mathrm{Fix}(\sigma_P) = \ker(\varphi) = \langle p \rangle^\perp = P^\perp.$$

Notice that $V = P \perp P^\perp$ by Proposition 4.4 iv) and, hence, σ_P maps a point $x = p + p' \in P \perp P^\perp$ to $\sigma_P(x) = -p + p'$. Therefore, the symmetry at P reflects a vector x at the hyperplane $P^\perp$.

Definition 4.11 *An isometry $\pi \in \mathrm{O}^*$ is called* simple *if* $\dim(\mathrm{Path}(\pi)) = 1$ *and* $\mathrm{Path}(\pi) \not\subset \mathrm{rad}(V)$.

Example 4.10 shows that σ_p is simple for any non-isotropic vector $p \in V$.

Remark 4.12 *There are isometries $\pi \in \mathrm{O}^*$ such that* $\dim(\mathrm{Path}(\pi)) = 1$ *and* $\mathrm{Path}(\pi) \subset \mathrm{rad}(V)$. *They are called* radical transvections. *We will see one later.*

On the other hand, we can show that every simple isometry is of the form σ_p with help of the following lemma.

Lemma 4.13 *Let* $\pi \in \mathrm{O}^{*}$, $\varphi = \pi - \mathrm{id}$ *and* $R = \mathrm{rad}(V)$. *Then for any subspace* $U \subset V$ *we have*

i) $\varphi(U)^{\perp} = \varphi^{-1}(\mathrm{Path}(\pi) \cap \pi(U)^{\perp})$,

ii) $\mathrm{Path}(\pi)^{\perp} = \varphi^{-1}(\mathrm{Path}(\pi) \cap R)$,

iii) $\mathrm{Path}(\pi) \subset \mathrm{Fix}(\pi)^{\perp}$ *and* $\mathrm{Fix}(\pi) \subset \mathrm{Path}(\pi)^{\perp}$,

iv) $\dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Fix}(\pi)) = \dim(V)$.

Proof *i)* For $x, y \in V$ we have

$$f(\pi(x), \pi(y)) = f(\varphi(x), \varphi(y)) + f(\varphi(x), y) + f(x, \varphi(y)) + f(x, y)$$

and since π is an isometry, we see

$$\begin{aligned} f(\varphi(x), y) &= -f(\varphi(x), \varphi(y)) - f(x, \varphi(y)) \\ &= -f(\varphi(x) + x, \varphi(y)) = -f(\pi(x), \varphi(y)). \end{aligned}$$

This allows us to conclude

$$\begin{aligned} y \in \varphi(U)^{\perp} &\Leftrightarrow \forall x \in U : f(\varphi(x), y) = 0 \\ &\Leftrightarrow \forall x \in U : f(\pi(x), \varphi(y)) = 0 \\ &\Leftrightarrow \varphi(y) \in \pi(U)^{\perp} \\ &\Leftrightarrow y \in \varphi^{-1}(\mathrm{Path}(\pi) \cap \pi(U)^{\perp}). \end{aligned}$$

ii) This follows from *i)* if we set $U = V$. Indeed, $\pi(V)^{\perp} = V^{\perp} = R$.

iii) By *ii)* we see that

$$\mathrm{Path}(\pi)^{\perp} = \varphi^{-1}(\mathrm{Path}(\pi) \cap R) \supset \varphi^{-1}(\{\mathbf{0}\}) = \ker(\varphi) = \mathrm{Fix}(\pi).$$

Hence, Proposition 4.4 *v)* shows $\mathrm{Path}(\pi) \subset (\mathrm{Path}(\pi)^{\perp})^{\perp} \subset \mathrm{Fix}(\pi)^{\perp}$.

iv) This is a result of Remark 2.14, as $\varphi : V \to V$ is a linear map. □

We can now prove that there are no other simple isometries other than σ_p.

Proposition 4.14 *If* $\pi \in \mathrm{O}^{*}$ *is a simple isometry with* $\mathrm{Path}(\pi) = \langle p \rangle$, *then* p *is non-isotropic and we have* $\pi = \sigma_p$.

Proof First, assume $\mathrm{Path}(\pi) = \langle p \rangle$ is non-isotropic. Hence, p is non-isotropic and we clearly have $\mathrm{rad}(\langle p \rangle) = \{\mathbf{0}\}$. Therefore, $\langle p \rangle \oplus \langle p \rangle^{\perp} = V$ by Proposition 4.4 *iv)*. Additionally, Lemma 4.13 *iii)* and *iv)* show

$$\mathrm{Fix}(\pi) \subset \langle p \rangle^{\perp}, \quad \dim(\mathrm{Fix}(\pi)) = \dim(V) - \dim(\langle p \rangle) = \dim(\langle p \rangle^{\perp})$$

and, hence, $\mathrm{Fix}(\pi) = \langle p \rangle^{\perp}$.

Since $\pi(p) - p = \varphi(p) \in \langle p \rangle$, we must have $\pi(p) = \alpha p$ for some fixed $\alpha \in \mathcal{K}^*$. As π is an isometry, we see

$$\mathrm{Q}(p) = \mathrm{Q}(\pi(p)) = \mathrm{Q}(\alpha p) = \alpha^2 \mathrm{Q}(p) \Rightarrow \alpha = \pm 1$$

because $\mathrm{Q}(p) \neq 0$. If $\alpha = 1$, then $\pi = \mathrm{id}$ which is a contradiction to $\mathrm{Path}(\pi) = \langle p \rangle$. Therefore, $\alpha = -1$ and we can conclude

$$\begin{aligned} \pi(x) = \pi\Big(\underbrace{\frac{f(p,x)}{f(p,p)}p}_{\in\langle p\rangle} + \underbrace{x - \frac{f(p,x)}{f(p,p)}p}_{\in\langle p\rangle^\perp}\Big) &= x - 2\frac{f(p,x)}{f(p,p)}p \\ &= x - \mathrm{Q}(p)^{-1} f(x,p)p = \sigma_p(x). \end{aligned}$$

We still have to prove that $\mathrm{Path}(\pi) = \langle p \rangle$ can not be isotropic. Assume $\langle p \rangle$ is isotropic. Since π is simple, we have $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \{\mathbf{0}\}$ and, hence, get

$$\langle p \rangle \subset \langle p \rangle^\perp = \mathrm{Path}(\pi)^\perp = \mathrm{Fix}(\pi), \quad \dim(\mathrm{Fix}(\pi)) = n - 1$$

by Lemma 4.13 *ii), iv)*. Therefore, a vector $q \notin \langle p \rangle^\perp$ exists such that

$$f(p,q) \neq 0, \quad \pi(q) - q = \alpha p \in \mathrm{Path}(\pi)$$

for some fixed $\alpha \in \mathcal{K}$. However, as π is an isometry, we have

$$f(q,q) = f(\pi(q), \pi(q)) = f(q,q) + 2\alpha f(q,p) \Rightarrow \alpha = 0$$

and, hence, $\pi(x) = x$ for all vectors $x \in \langle p \rangle^\perp \oplus \langle q \rangle = V$. This is a contradiction to $\mathrm{Path}(\pi) = \langle p \rangle$ and we see that $\langle p \rangle$ must be non-isotropic. □

4.3 Theorem of Cartan-Dieudonné-Scherk

The importance of the following definition is not yet clear. However, it will reveal itself in this section.

Definition 4.15 *A map $\pi \in \mathrm{O}^*$ is called* singular *if* $\mathrm{Fix}(\pi)^\perp$ *is isotropic.*

The next theorem is due to Götzky in [9], which is an extended version of the Cartan-Dieudonné-Scherk theorem. It allows us to write any isometry in O^* as a product of symmetries.

Theorem 4.16 *Let (V, f) be a finite-dimensional, non-isotropic orthogonal space over a field $\mathcal{K}$ with $\operatorname{char}(\mathcal{K}) \neq 2$. Then every isometry*

$$\pi \in \mathrm{O}^* = \{\pi \in \mathrm{O} \ \pi|_{\mathrm{rad}(V)} = \mathrm{id}\}$$

can be written as a product of $\mathrm{l}(\pi)$ simple isometries where

$$\mathrm{l}(\pi) = \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap \mathrm{rad}(V))$$

if π is non-singular and

$$\mathrm{l}(\pi) = \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap \mathrm{rad}(V)) + 2$$

if π is singular. Additionally, if $\pi \neq \mathrm{id}$, it can not be written as a product of less than $\mathrm{l}(\pi)$ simple isometries.

Remark 4.17 *Notice that* $\mathrm{Fix}(\mathrm{id})^\perp = V^\perp = \mathrm{rad}(V)$ *is isotropic. Hence,* id *is singular and* $\mathrm{l}(\mathrm{id}) = 2$*. Since our definition of length satisfies* $\mathrm{l}_{\mathrm{O}_S}(\mathrm{id}) = 0$*, we just redefine* $\mathrm{l}(\pi) = 0$ *if* $\pi = \mathrm{id}$.

Theorem 4.16 gives us the length of an isometry.

Corollary 4.18 *If (V, f) is a finite-dimensional, non-isotropic orthogonal space over a field $\mathcal{K}$ with $\operatorname{char}(\mathcal{K}) \neq 2$, then $\mathrm{O}_S = \mathrm{O}^*$ and $\mathrm{l}_{\mathrm{O}_S}(\pi) = \mathrm{l}(\pi)$ for all $\pi \in \mathrm{O}_S$.*

We will need the rest of this chapter to prove Theorem 4.16. Our proof follows [9] closely. However, we want to improve on readability and intuition.

During the proof of Theorem 4.16 we will incrementally solve the following example. Hence, we advise to always check the calculations that are skipped.

Example 4.19 *Let $V = \mathbb{R}^3$ and $\mathrm{Q}((x, y, z)^\intercal) = xy$ be a quadratic space. Then*

$$f((x, y, z)^\intercal, (x', y', z')^\intercal) = xy' + x'y, \quad R = \mathrm{rad}(V) = \langle (0, 0, 1)^\intercal \rangle.$$

Denote by $e_1 = (1, 0, 0)^\intercal$, $e_2 = (0, 1, 0)^\intercal$, $e_3 = (0, 0, 1)^\intercal$ the standard basis of $\mathbb{R}^3$ and consider

$$\pi : V \to V; \quad \pi(v) = v - 4f(e_1, v)e_3.$$

One easily sees that π is linear, $\pi(e_3) = e_3$, $\pi^{-1}(v) = v + 4f(e_1, v)e_3$ and $f(\pi(v), \pi(w)) = f(v, w)$ for all $v, w \in V$. Therefore, π is an isometry in $\mathrm{O}^*(V)$. Check that

$$\mathrm{Path}(\pi) = \langle e_3 \rangle = R, \quad \mathrm{Fix}(\pi) = \langle e_1, e_3 \rangle, \quad \mathrm{Fix}(\pi)^\perp = \langle e_1, e_3 \rangle.$$

Hence, π is singular and $\mathrm{l}(\pi) = 1 + 1 + 2 = 4$.

4.3.1 Lower bound

Before we try to show that an isometry can be written as a product of symmetries, we want to prove that a product $\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k}$ must satisfy $\mathrm{l}(\pi) \leq k$. Hence, we need at least $\mathrm{l}(\pi)$ symmetries to express an isometry $\pi \in \mathrm{O}_S$. This is mainly a consequence of the following observation.

Lemma 4.20 *If $\pi, \pi' \in \mathrm{O}^*$, then the following inclusions hold:*

$$\begin{aligned} \mathrm{Fix}(\pi) \cap \mathrm{Fix}(\pi') &\subset \mathrm{Fix}(\pi \circ \pi'), \\ \mathrm{Path}(\pi \circ \pi') &\subset \mathrm{Path}(\pi) + \mathrm{Path}(\pi'). \end{aligned}$$

Proof If $x \in \mathrm{Fix}(\pi), \mathrm{Fix}(\pi)$, then $(\pi \circ \pi')(x) = x$ and, hence, $x \in \mathrm{Fix}(\pi \circ \pi')$. This shows the first inclusion.

For the second inclusion notice that

$$\pi \circ \pi' - \mathrm{id} = \pi \circ \pi' - \pi + \pi - \mathrm{id} = \pi \circ (\pi' - \mathrm{id}) + \pi - \mathrm{id}$$

and, hence, we see

$$\begin{aligned} \mathrm{Path}(\pi \circ \pi') &= (\pi \circ \pi' - \mathrm{id})(V) \\ &\subset (\pi' - \mathrm{id})(\pi(V)) + (\pi - \mathrm{id})(V) \\ &\subset \mathrm{Path}(\pi') + \mathrm{Path}(\pi). \end{aligned}$$ □

Lemma 4.20 allows us to contain the fix and path of an arbitrary product of symmetries.

Lemma 4.21 *If $p_1, \ldots, p_k \in V$ are non-isotropic vectors and $\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k}$, then we have*

$$\mathrm{Path}(\pi) \subset \langle p_1, \ldots, p_k \rangle, \quad \langle p_1, \ldots, p_k \rangle^\perp \subset \mathrm{Fix}(\pi).$$

Additionally, if $p_1, \ldots, p_k$ are linearly independent, then $\langle p_1, \ldots, p_k \rangle^\perp = \mathrm{Fix}(\pi)$.

Proof Since $\mathrm{Path}(\sigma_{p_i}) = \langle p_i \rangle$ and $\mathrm{Fix}(\sigma_{p_i}) = \langle p_i \rangle^\perp$, Lemma 4.20 shows

$$\begin{aligned} \mathrm{Path}(\pi) \subset \mathrm{Path}(\sigma_{p_1}) + \ldots + \mathrm{Path}(\sigma_{p_k}) &= \langle p_1, \ldots, p_k \rangle, \\ \langle p_1, \ldots, p_k \rangle^\perp = \mathrm{Fix}(\sigma_{p_1}) \cap \ldots \cap \mathrm{Fix}(\sigma_{p_k}) &\subset \mathrm{Fix}(\pi). \end{aligned}$$

The only thing left to prove is that $\mathrm{Fix}(\pi) \subset \langle p_1, \ldots, p_k \rangle^\perp$ if $p_1, \ldots, p_k$ are linearly independent. We will use induction on k for this statement.

If $k = 1$, then $\pi = \sigma_{p_1}$ and, hence, $\mathrm{Fix}(\pi) = \langle p_1 \rangle^\perp$.

Now, let $p_1, \ldots, p_k \in V$ be linearly independent and $\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k}$ for $k > 1$. Assume the statement holds for any map with less than k symmetries.

Therefore, we can apply the induction hypothesis on

$$\pi' = \sigma_{p_1} \circ \pi = \sigma_{p_2} \circ \ldots \circ \sigma_{p_k}$$

and get $\mathrm{Fix}(\pi') \subset \langle p_2, \ldots, p_k \rangle^{\perp}$.

Similar to the proof of Lemma 4.20, we see

$$\pi - \mathrm{id} = \sigma_{p_1} \circ (\pi' - \mathrm{id}) + \sigma_{p_1} - \mathrm{id}\,.$$

Hence, for any $x \in \mathrm{Fix}(\pi)$ we have

$$\mathbf{0} = (\pi - \mathrm{id})(x) = \underbrace{(\pi' - \mathrm{id})(\sigma_{p_1}(x))}_{\in \mathrm{Path}(\pi')} + \underbrace{(\sigma_{p_1} - \mathrm{id})(x)}_{\in \mathrm{Path}(\sigma_{p_1})}. \tag{4.4}$$

The first part of the proposition shows that $\mathrm{Path}(\pi') \subset \langle p_2, \ldots, p_k \rangle$ and $\mathrm{Path}(\sigma_{p_1}) = \langle p_1 \rangle$. However, since $p_1, \ldots, p_k$ are linearly independent, Eq. (4.4) can only be zero if

$$(\sigma_{p_1} - \mathrm{id})(x) = \mathbf{0} \text{ and } (\pi' - \mathrm{id})(\sigma_{p_1}(x)) = \mathbf{0}.$$

The first equation implies $x \in \mathrm{Fix}(\sigma_{p_1}) = \langle p_1 \rangle^{\perp}$ and, hence, $\sigma_{p_1}(x) = x$. Therefore, the second equation shows

$$(\pi' - \mathrm{id})(x) = \mathbf{0} \Rightarrow x \in \mathrm{Fix}(\pi') \subset \langle p_2, \ldots, p_k \rangle^{\perp}.$$

We can conclude the lemma as

$$\mathrm{Fix}(\pi) \subset \mathrm{Fix}(\sigma_{p_1}) \cap \mathrm{Fix}(\pi') \subset \langle p_1 \rangle^{\perp} \cap \langle p_2, \ldots, p_k \rangle^{\perp} = \langle p_1, \ldots, p_k \rangle^{\perp}. \qquad \square$$

Lemma 4.21 shows that for an isometry $\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k}$ we must have $\dim(\mathrm{Path}(\pi)) \leq k$. We can do better if $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) \neq \{\mathbf{0}\}$.

Proposition 4.22 *If $p_1, \ldots, p_k \in V$ are non-isotropic and $\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k}$, then $\mathrm{l}(\pi) \leq k$.*

Proof Let $R = \mathrm{rad}(V)$ and $U = \langle p_1, \ldots, p_k \rangle$. Proposition 4.4 and Lemma 4.21 show

$$U + R = (U^{\perp})^{\perp} \supset \mathrm{Fix}(\pi)^{\perp}. \tag{4.5}$$

By Proposition 4.4 and Lemma 4.13 we see

$$\begin{aligned} \dim(\mathrm{Fix}(\pi)^{\perp}) &= \dim(V) - \dim(\mathrm{Path}(\pi)) + \dim(R) \\ &= \dim(\mathrm{Path}(\pi)) + \dim(R) \end{aligned}$$

and, hence,

$$\begin{aligned}\dim(U) &= \dim(U+R) - \dim(R) + \dim(U \cap R) \\ &\geq \dim(\mathrm{Fix}(\pi)^{\perp}) - \dim(R) + \dim(\mathrm{Path}(\pi) \cap R) \\ &= \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap R)\end{aligned} \tag{4.6}$$

as $\mathrm{Path}(\pi) \subset U$.

Therefore, if π is non-singular, we already have $k \geq \dim(U) \geq \mathrm{l}(\pi)$.

If π is singular, the inclusion (4.5), $\langle p_1, \ldots, p_k \rangle \supsetneq \mathrm{Fix}(\pi)^{\perp}$, must be strict as p_i is non-isotropic, but $\mathrm{Fix}(\pi)^{\perp}$ only contains isotropic vectors. Hence, (4.6) must be strict as well. Further, Lemma 4.21 shows that $p_1, \ldots, p_k$ have to be linearly dependent since otherwise (4.5) would be an equality. Therefore,

$$k > \dim(U) > \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap R)$$

and we conclude $k \geq \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap R) + 2 = \mathrm{l}(\pi)$. □

4.3.2 Reduction

Our goal is to find a non-isotropic vector $q \in V$ such that $\pi' = \pi \circ \sigma_q$ satisfies $\mathrm{l}(\pi) > \mathrm{l}(\pi')$. If we have such a vector q, we can use induction to find a decomposition of π as a product of simple isometries.

Proving the existence of a suitable $q \in V$ is difficult. However, we can find promising candidates with the following proposition.

Proposition 4.23 *If $x, y \in V$ such that $\mathrm{Q}(x) = \mathrm{Q}(y)$ and $z = y - x$ is non-isotropic, then $\sigma_z(x) = y$.*

Proof Recall that $2\mathrm{Q}(x) = f(x,x)$ for all $x \in V$. Since $\mathrm{Q}(x) = \mathrm{Q}(y)$, we get

$$-f(x, y-x) = 2\mathrm{Q}(x) - f(y,x) = \mathrm{Q}(x) + \mathrm{Q}(y) - f(x,y) = \mathrm{Q}(y-x).$$

Hence, as $z = y - x$, we can conclude

$$\sigma_z(x) = x - \mathrm{Q}(z)^{-1} f(x,z) z = x + z = y.$$

□

If $q = \pi(p) - p \in \mathrm{Path}(\pi)$ is non-isotropic for some $p \in V$, then Proposition 4.23 shows that $\sigma_q(\pi(p)) = p$. Hence, p is a fixed point of $\pi' = \pi \circ \sigma_q$ and $p \in \mathrm{Fix}(\pi')$. However, we have $p \notin \mathrm{Fix}(\pi)$ as otherwise $q = \mathbf{0}$.

Lemma 4.20 yields

$$\mathrm{Fix}(\pi) \supsetneq \mathrm{Fix}(\pi) \cap \langle q \rangle^{\perp} = \mathrm{Fix}(\pi)$$

because $q \in \mathrm{Path}(\pi) \subset \mathrm{Fix}(\pi)^{\perp}$.

By Lemma 4.13 *iv)* we now see that

$$\dim(\mathrm{Path}(\pi')) < \dim(\mathrm{Path}(\pi)).$$

Additionally, we have $\dim(\mathrm{Path}(\pi') \cap \mathrm{rad}(V)) < \dim(\mathrm{Path}(\pi) \cap \mathrm{rad}(V))$ since

$$\mathrm{Path}(\pi') \cap \mathrm{rad}(V) \subset (\mathrm{Path}(\pi) + \langle q \rangle) \cap \mathrm{rad}(V) = \mathrm{Path}(\pi) \cap \mathrm{rad}(V)$$

by Lemma 4.20.

However, this does not yet show that $\mathrm{l}(\pi') < \mathrm{l}(\pi)$ since π' could be singular. This will be the hardest part of the proof.

Another complication arises if $\mathrm{Path}(\pi)$ is isotropic. Then there are no non-isotropic vectors q in $\mathrm{Path}(\pi)$. Hence, we can not proceed as in the beginning of this section. For example $\mathrm{Path}(\pi)$ is isotropic, if $\mathrm{Path}(\pi) \subset \mathrm{rad}(V)$ or π is singular. Indeed, if π is singular $\mathrm{Path}(\pi) \subset \mathrm{Fix}(\pi)^{\perp}$ is a subspace of an isotropic subspace.

However, we can find a suitable non-isotropic vector q for both cases with the next lemma.

Lemma 4.24 *Let* $\pi \in \mathrm{O}^{*}$ *and assume that there is a non-isotropic vector* $q \in V$ *such that* $q \notin \mathrm{Path}(\pi) + \mathrm{rad}(V)$. *Further, define* $\pi' = \pi \circ \sigma_p$, $\varphi = \pi - \mathrm{id}$ *and* $\varphi' = \pi' - \mathrm{id}$. *The following statements are true:*

i) *The map* $\tau : \mathrm{Path}(\pi') \to \mathrm{Path}(\pi)$; $\varphi'(x) \mapsto \varphi(x)$ *is a well-defined, surjective and linear.*

ii) $\mathrm{Path}(\pi') \cap \mathrm{rad}(V) \subset \mathrm{Path}(\pi) \cap \mathrm{rad}(V)$.

iii) *If* $q \notin \mathrm{Fix}(\pi)^{\perp}$, *then* $\mathrm{Path}(\pi') = \mathrm{Path}(\pi) \oplus \langle q \rangle$.

iv) *If* $q \in \mathrm{Fix}(\pi)^{\perp}$, *then* $\mathrm{Path}(\pi') \cap \mathrm{rad}(V) \subsetneq \mathrm{Path}(\pi) \cap \mathrm{rad}(V)$ *and* τ *is bijective.*

Proof Recall that $\mathrm{Path}(\pi) = \mathrm{im}(\varphi) = \varphi(V)$. We will regularly use the following observation to prove this lemma:

$$\varphi' = \pi' - \mathrm{id} = \pi \circ \sigma_q - \mathrm{id} = (\pi - \mathrm{id}) \circ \sigma_q + \sigma_q - \mathrm{id} = \varphi \circ \sigma_q + \sigma_q - \mathrm{id}\,.$$

Additionally, as $\sigma_p(x) = x - \mathrm{Q}(p)^{-1} f(x,p)p$, we see

$$\begin{aligned} \varphi'(x) &= \varphi(x) - \mathrm{Q}(q)^{-1} f(\varphi(x), q)q - \mathrm{Q}(q)^{-1} f(x,q)q \\ &= \varphi(x) - \mathrm{Q}(q)^{-1} f(\pi(x), q)q \end{aligned} \tag{4.7}$$

for all $x \in V$.

i) Assume $x \in \ker(\varphi')$, then Eq. (4.7) shows

$$\varphi(x) = \mathrm{Q}(q)^{-1} f(\pi(x), q) q \in \langle q \rangle.$$

However, as $q \notin \mathrm{Path}(\pi)$, we must have $\varphi(x) = \mathbf{0}$. Therefore, τ is well-defined. Clearly, τ is linear and surjective.

ii) Assume $\varphi'(x) \in \mathrm{rad}(V) = R$ for some $x \in V$, then

$$\varphi'(x) - \varphi(x) \in \mathrm{Path}(\pi) + R.$$

On the other hand, Eq. (4.7) shows that $\varphi'(x) - \varphi(x) \in \langle q \rangle$. Therefore, $\varphi'(x) - \varphi(x) = \mathbf{0}$ by the condition on q and $\varphi'(x) = \varphi(x) \in \mathrm{Path}(\pi) \cap R$ which concludes *ii)*.

iii) Assume $q \notin \mathrm{Fix}(\pi)^{\perp}$ and let $p = \pi^{-1}(q)$. The vector p must satisfy

$$p \notin \mathrm{Fix}(\pi)^{\perp}, \quad \mathrm{Fix}(\pi) \not\subset \langle p \rangle^{\perp}. \tag{4.8}$$

Indeed, if $p \in \mathrm{Fix}(\pi)^{\perp}$, then $f(p,x) = f(\pi(p), \pi(x)) = f(q,x) = 0$ for all $x \in \mathrm{Fix}(\pi)$, which is a contradiction to $q \notin \mathrm{Fix}(\pi)^{\perp}$. This shows the first part of (4.8). The second part can be seen as we have

$$\mathrm{Fix}(\pi) \subset \langle p \rangle^{\perp} \Rightarrow p \in (\langle p \rangle^{\perp})^{\perp} \subset \mathrm{Fix}(\pi)^{\perp}$$

by Proposition 4.4, which contradicts $p \notin \mathrm{Fix}(\pi)^{\perp}$. Hence,

$$\langle p \rangle^{\perp} + \mathrm{Fix}(\pi) = V. \tag{4.9}$$

Notice that Eq. (4.7) implies

$$\varphi'(x) = \varphi(x) \Leftrightarrow \pi(x) \in \langle q \rangle^{\perp} \Leftrightarrow x \in \langle p \rangle^{\perp}$$

and by (4.8) we see

$$\varphi'(\langle p \rangle^{\perp}) = \varphi(\langle p \rangle^{\perp}) = \varphi(\langle p \rangle^{\perp} + \mathrm{Fix}(\pi)) = \varphi(V). \tag{4.10}$$

Lastly, since $q \notin \mathrm{Fix}(\pi)^{\perp}$, we must have $\varphi'(\mathrm{Fix}(\pi)) = \langle q \rangle$ by Eq. (4.7). Combined with Eqs. (4.9) and (4.10), we conclude

$$\varphi(V) \oplus \langle q \rangle = \varphi'\big(\langle p \rangle^{\perp}\big) \oplus \varphi'\big(\mathrm{Fix}(\pi)\big) = \varphi'\big(\langle p \rangle^{\perp} + \mathrm{Fix}(\pi)\big) = \varphi'(V).$$

iv) Let $q \in \mathrm{Fix}(\pi)^{\perp}$ and $R = \mathrm{rad}(V)$.

We first show that τ is injective and, therefore, a bijection. If $\varphi(x) = \mathbf{0}$, then $x \in \mathrm{Fix}(\pi)$. However, since $q \in \mathrm{Fix}(\pi)^{\perp}$, Eq. (4.7) shows $\varphi'(x) = \varphi(x) = \mathbf{0}$. Hence, τ is injective.

Since the inclusion already holds by *ii)*, $\mathrm{Path}(\pi') \cap R \neq \mathrm{Path}(\pi) \cap R$ is enough to show *iv)*.

By Proposition 4.4 *v)* and Lemma 4.13 *ii)* we see

$$q \notin \mathrm{Path}(\pi) + R = ((\mathrm{Path}(\pi))^{\perp})^{\perp} = \varphi^{-1}(\mathrm{Path}(\pi) \cap R)^{\perp}.$$

Therefore, there must be a vector $z \in V$ with $\varphi(z) \in \mathrm{Path}(\pi) \cap R$ and $f(z, q) \neq 0$. Hence, Eq. (4.7) gives us

$$\varphi'(z) = \varphi(z) - \mathrm{Q}(q)^{-1} f(\varphi(z) + z, q) q = \varphi(z) - \mathrm{Q}(q)^{-1} f(z, q) q$$

and we can conclude *iv)* because

$$f(\varphi'(z), q) = -\mathrm{Q}(q)^{-1} f(z, q) f(q, q) = -2 f(z, q) \neq 0$$

and, therefore, $\varphi'(z) \notin \mathrm{Path}(\pi') \cap R$. □

If π is singular, then every non-isotropic vector $q \in V$ can not be in $\mathrm{Fix}(\pi)^{\perp}$. Therefore, Lemma 4.24 *iii)* shows that $\dim(\mathrm{Path}(\pi')) > \dim(\mathrm{Path}(\pi))$. However, since $\mathrm{l}(\pi)$ includes the term $+2$ if π is singular, we can still achieve $\mathrm{l}(\pi) > \mathrm{l}(\pi')$.

Proposition 4.25 *Let $\pi \in \mathrm{O}^*$ be a singular isometry. There is a non-isotropic vector $q \in V$ such that $\mathrm{l}(\pi) > \mathrm{l}(\pi')$ for $\pi' = \pi \circ \sigma_q$.*

Proof Write $R = \mathrm{rad}(V)$. Recall that $\mathrm{Path}(\pi) + R \subset \mathrm{Fix}(\pi)^{\perp}$ by Lemma 4.13 *iii)*. As $\mathrm{Fix}(\pi)^{\perp}$ is isotropic but V is non-isotropic, there is a non-isotropic vector $q \notin \mathrm{Fix}(\pi)^{\perp}$. Let $\pi' = \pi \circ \sigma_q$, then by Lemma 4.24 *iii)* we have

$$\mathrm{Path}(\pi) \oplus \langle q \rangle = \mathrm{Path}(\pi') \subset \mathrm{Fix}(\pi')^{\perp}$$

and we see that π' is non-singular. As $\mathrm{Path}(\pi') \cap R \subset \mathrm{Path}(\pi) \cap R$, we see that Lemma 4.24 *ii)* yields

$$\begin{aligned} \mathrm{l}(\pi) &= \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap R) + 2 \\ &\geq \dim(\mathrm{Path}(\pi')) - 1 + \dim(\mathrm{Path}(\pi') \cap R) + 2 \\ &> \dim(\mathrm{Path}(\pi')) + \dim(\mathrm{Path}(\pi') \cap R) = \mathrm{l}(\pi'). \end{aligned}$$ □

Example 4.26 *Let (V, Q) and π be as in Example 4.19. Choose*

$$q_4 = -e_1 - e_2 + e_3 = (-1, -1, 1)^{\intercal} \notin \mathrm{Fix}(\pi)^{\perp} = \langle e_1, e_3 \rangle$$

and notice $\mathrm{Q}(q_4) = 1$. If $\pi' = \pi \circ \sigma_{q_4}$, then a longer computation shows

$$\pi'(v) = v + f(e_2, v) q_4 + f(e_1, v)(q_4 - 4e_3).$$

Hence, we see

$$\begin{aligned} \mathrm{Path}(\pi') &= \langle q_4, q_4 - 4e_3 \rangle = \langle e_1 + e_2, e_3 \rangle, \\ \mathrm{Fix}(\pi') &= R, \quad \mathrm{Fix}(\pi')^{\perp} = V \end{aligned}$$

and conclude that π' is non-singular with $\mathrm{l}(\pi') = 3$.

Lemma 4.24 gives us a way to reduce π if $\mathrm{Path}(\pi) \cap R \neq \{\mathbf{0}\}$.

Proposition 4.27 *Let $\pi \in \mathrm{O}^*$ be a non-singular isometry and assume that the intersection $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) \neq \{\mathbf{0}\}$ is non-trivial. Then there is a non-isotropic vector $q \in V$ such that $\mathrm{l}(\pi) > \mathrm{l}(\pi')$ for $\pi' = \pi \circ \sigma_q$.*

Proof Write $R = \mathrm{rad}(V)$. As in the proof of Proposition 4.25 we have $\mathrm{Path}(\pi) + R \subset \mathrm{Fix}(\pi)^\perp$. Since $\mathrm{Fix}(\pi)^\perp$ is non-isotropic, there is a non-isotropic basis of $\mathrm{Fix}(\pi)^\perp$ by Proposition 4.7. This basis can not be a subset of $\mathrm{Path}(\pi) + R$ since

$$\begin{aligned}\dim(\mathrm{Path}(\pi) + R) &= \dim(\mathrm{Path}(\pi)) + \dim(R)\\ &\qquad - \dim(\mathrm{Path}(\pi) \cap R)\\ &< \dim(V) - \dim(\mathrm{Fix}(\pi)) + \dim(R)\\ &= \dim(\mathrm{Fix}(\pi)^\perp)\end{aligned}$$

by Proposition 4.4 *iii)* and Lemma 4.13 *iv)*. Therefore, at least one of the basis vectors, say $q \in \mathrm{Fix}(\pi)^\perp$, is non-isotropic and $q \notin \mathrm{Path}(\pi) + R$.

Let $\pi' = \pi \circ \sigma_q$, then Lemma 4.24 *iv)* shows

$$\begin{aligned}\dim(\mathrm{Path}(\pi') \cap R) &< \dim(\mathrm{Path}(\pi) \cap R),\\ \dim(\mathrm{Path}(\pi')) &= \dim(\mathrm{Path}(\pi)).\end{aligned}$$

Further, by Lemma 4.20 we see $\mathrm{Fix}(\pi') \supset \mathrm{Fix}(\pi) \cap \langle q \rangle = \mathrm{Fix}(\pi)$ and get equality as $\dim(\mathrm{Fix}(\pi')) = \dim(\mathrm{Fix}(\pi))$ by Lemma 4.13 *iv)*. Therefore, π' is non-singular and we conclude

$$\begin{aligned}\mathrm{l}(\pi) &= \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap R)\\ &> \dim(\mathrm{Path}(\pi')) + \dim(\mathrm{Path}(\pi') \cap R) = \mathrm{l}(\pi')\end{aligned}$$ □

Example 4.28 *Let (V, Q) and π' be as in Examples 4.19 and 4.26. Choose*

$$q_3 = e_1 - e_2 + e_3 = (1, -1, 1)^\intercal \notin \mathrm{Path}(\pi') + R = \langle e_1 + e_2, e_3 \rangle$$

and notice $\mathrm{Q}(q_4) = -1$. Since $\mathrm{Fix}(\pi')^\perp = V$, the vector q_3 satisfies the conditions in the proof of Proposition 4.27. If $\pi'' = \pi' \circ \sigma_{q_3}$, we can again compute

$$\begin{aligned}\pi''(v) &= v + f(e_2, v)(q_4 - q_3) + f(e_1, v)(q_4 + q_4 - 4e_3)\\ &= v - 2f(e_2, v)e_1 - 2f(e_1, v)(e_2 + e_3)\end{aligned}$$

by noticing $f(q_3, q_4) = 0$ and expanding $\sigma_{q_3}(\pi'(v))$. This shows that $\mathrm{l}(\pi'') = 2$ as

$$\mathrm{Path}(\pi'') = \langle e_1, e_2 + e_3 \rangle, \quad \mathrm{Fix}(\pi'') = \langle e_3 \rangle, \quad \mathrm{Fix}(\pi'')^\perp = V.$$

Propositions 4.25 and 4.27 show that we can reduce π if it is either singular or the intersection $\mathrm{Path}(\pi) \cap \mathrm{rad}(V)$ is non-trivial. This leaves us with the case that π is non-singular and $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \{\mathbf{0}\}$.

We can now use the idea in the beginning of Section 4.3.2. Recall that we have to find a non-isotropic vector $q \in \mathrm{Path}(\pi)$ such that $\pi \circ \sigma_q$ is non-singular. Showing the existence of q will take some time and can be done with the following lemmas.

Lemma 4.29 *Let $\pi \in \mathrm{O}^*$ with $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \{\mathbf{0}\}$ and $p_1, \ldots, p_k$ be a basis of $\mathrm{Path}(\pi)$. Then for $\varphi = \pi - \mathrm{id}$ and $H_i = \varphi(\langle p_i \rangle^\perp)$ we have:*

i) *$H_1, \ldots, H_k$ are distinct hyperplanes of $\mathrm{Path}(\pi)$ and $\bigcap_{i=1}^k H_i = \{\mathbf{0}\}$.*

ii) *If $\dim(\mathrm{Path}(\pi)) > 2$ and H_i isotropic for all i, then $\mathrm{Path}(\pi)$ is isotropic.*

iii) *If $\dim(\mathrm{Path}(\pi)) = 2$, then $\mathrm{Path}(\pi) = H_1 \oplus H_2$.*

Proof *i)* First, notice that $\langle p_i \rangle \subset \mathrm{Path}(\pi) \subset \mathrm{Fix}(\pi)^\perp$ by Lemma 4.13. Hence,

$$\ker(\varphi) = \mathrm{Fix}(\pi) \subset \langle p_i \rangle^\perp. \tag{4.11}$$

Since $p_i \notin \mathrm{rad}(V)$, we have $\dim \langle p_i \rangle^\perp = \dim(V) - 1$. The dimension formula of Remark 2.14 used on $\varphi|_{\langle p_i \rangle^\perp}$ shows

$$\begin{aligned}\dim(H_i) &= \dim(\langle p_i \rangle^\perp) - \dim(\ker(\varphi)) \\ &= \dim(V) - \dim(\mathrm{Fix}(\pi)) - 1 = \dim(\mathrm{Path}(\pi)) - 1.\end{aligned}$$

Therefore, $H_i \subset \varphi(V) = \mathrm{Path}(\pi)$ is a hyperplane.

The inclusion (4.11) implies

$$\varphi^{-1}(H_i) = \varphi^{-1}\big(\varphi(\langle p_i \rangle^\perp)\big) = \langle p_i \rangle^\perp$$

for all i. Since $\mathrm{Path}(\pi)^\perp = \varphi^{-1}(\mathrm{Path}(\pi) \cap \mathrm{rad}(V))$ by Lemma 4.13 *ii)*, we now have

$$\begin{aligned}\bigcap_{i=1}^k H_i = \varphi\Big(\bigcap_{i=1}^k \varphi^{-1}(H_i)\Big) &= \varphi\Big(\bigcap_{i=1}^k \langle p_i \rangle^\perp\Big) \\ &= \varphi\Big((\langle p_1 \rangle \oplus \ldots \oplus \langle p_k \rangle)^\perp\Big) \\ &= \varphi(\mathrm{Path}(\pi)^\perp) = \mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \{\mathbf{0}\}.\end{aligned}$$

Lastly, $H_1, \ldots, H_k$ are distinct as otherwise $\dim(\bigcap_{i=1}^k H_i) > 0$.

ii) If $\dim(\mathrm{Path}(\pi)) > 2$, part *i)* implies that the sum $H_i + H_j = \mathrm{Path}(\pi)$ and the intersection $U_{ij} = H_i \cap H_j \neq \{\mathbf{0}\}$ for all $i \neq j \leq k$.

Since H_i is isotropic for all i, we see $U_{ij} \subset \operatorname{rad}(\operatorname{Path}(\pi))$. Indeed, a vector $u_{ij} \in U_{ij}$ satisfies $f(u_{ij}, h_i + h_j) = 0$ for all $h_i + h_j \in H_i + H_j = \operatorname{Path}(\pi)$.

As $\dim(\operatorname{Path}(\pi)) > 2$, $\operatorname{Path}(\pi)$ can be written in the following way:

$$\begin{aligned} U_{12} + U_{13} + U_{23} &= H_1 \cap (H_2 + H_3) + (H_2 \cap H_3) \\ &= H_1 + (H_2 \cap H_3) \\ &= (H_1 + H_2) \cap (H_1 + H_3) = \operatorname{Path}(\pi). \end{aligned}$$

Hence, $\operatorname{Path}(\pi)$ is isotropic because

$$\operatorname{Path}(\pi) = U_{12} + U_{13} + U_{23} \subset \operatorname{rad}(\operatorname{Path}(\pi)) \subset \operatorname{Path}(\pi).$$

iii) This follows immediately from *i)*. □

With the help of Lemma 4.29 we can find a promising candidate $q \in \operatorname{Path}(\pi)$.

Lemma 4.30 *Assume* $\pi \in \mathrm{O}^*$ *is non-singular,* $\operatorname{Path}(\pi) \cap \operatorname{rad}(V) = \{\mathbf{0}\}$ *and* $\dim(\operatorname{Path}(\pi)) \geq 2$. *Then there is a vector* $p \in V$ *such that* $\operatorname{Path}(\pi) \cap \langle p \rangle^\perp$ *and* $\varphi(p)$ *are non-isotropic where* $\varphi = \pi - \operatorname{id}$.

Proof Lemma 4.13 *ii)* shows $\operatorname{Path}(\pi)^\perp = \varphi^{-1}(\{\mathbf{0}\}) = \ker(\varphi) = \operatorname{Fix}(\pi)$ and, therefore,

$$\operatorname{Fix}(\pi)^\perp = \operatorname{Path}(\pi) \oplus \operatorname{rad}(V)$$

by Proposition 4.4 *v)*. We now see that $\operatorname{Fix}(\pi)^\perp$ is isotropic whenever $\operatorname{Path}(\pi)$ is isotropic. However, since π is non-singular, $\operatorname{Path}(\pi)$ must be non-isotropic.

Proposition 4.7 shows that there is a non-isotropic basis $p_1, \ldots, p_k$ of $\operatorname{Path}(\pi)$.

Assume that for a fixed $i \leq k$ the subspace $H_i = \varphi(\langle p_i \rangle^\perp)$ is non-isotropic. Choose $\bar{p}_i \in V$ such that $\varphi(\bar{p}_i) = p_i$ and define $p = \pi(\bar{p}_i)$. By Lemma 4.13 *i)* we have

$$\begin{aligned} H_i = \varphi(\langle \varphi(\bar{p}_i) \rangle^\perp) &= \varphi(\varphi(\langle \bar{p}_i \rangle)^\perp) \\ &= \varphi\Big(\varphi^{-1}\Big(\operatorname{Path}(\pi) \cap \pi(\langle \bar{p}_i \rangle)^\perp\Big)\Big) \\ &= \operatorname{Path}(\pi) \cap \langle \pi(\bar{p}_i) \rangle^\perp = \operatorname{Path}(\pi) \cap \langle p \rangle^\perp. \end{aligned}$$

Further, $\varphi(p)$ is non-isotropic. Indeed, as $\pi \circ \varphi = \varphi \circ \pi$, we see

$$\varphi(p) = \varphi(\pi(\bar{p}_i)) = \pi(\varphi(\bar{p}_i)) = \pi(p_i)$$

and get $f(\varphi(p), \varphi(p)) = f(p_i, p_i) \neq 0$.

Therefore, if H_i is non-isotropic, p satisfies the lemma.

Now, assume $H_1, \ldots, H_k$ are isotropic for all $i \leq k$. If $\dim(\mathrm{Path}(\pi)) = k > 2$, then Lemma 4.29 *ii)* implies that $\mathrm{Path}(\pi)$ is isotropic which contradicts the observation in the beginning of the proof.

Therefore, we only have to check the case $\dim(\mathrm{Path}(\pi)) = 2$. By Lemma 4.29 *iii)* we have $\mathrm{Path}(\pi) = H_1 \oplus H_2$ and, hence, there is an isotropic basis h_1, h_2 of $\mathrm{Path}(\pi)$. Further, $\mathrm{Path}(\pi)$ can not be singular because this would imply $f(h_1, h_2) = 0$ and $\mathrm{Path}(\pi)$ would be isotropic again. Hence, by Example 4.8 we see that $\mathrm{Path}(\pi) = H(x, y)$ for a hyperbolic pair x, y.

We want to show that the restriction of π to $H(x, y)$ is an isometry of $H(x, y)$ which we characterised in Example 4.8.

First, notice that $\pi(\mathrm{Path}(\pi)) = \pi(\varphi(V)) = \varphi(\pi(V)) = \varphi(V)$ and, hence, the restriction $\pi|_{H(x,y)}$ is an isometry of $H(x, y)$. Secondly, $H(x, y)$ does not contain fixed vectors as

$$\{\mathbf{0}\} = \mathrm{Path}(\pi) \cap \mathrm{Path}(\pi)^{\perp} = \mathrm{Path}(\pi) \cap \mathrm{Fix}(\pi).$$

Now, Example 4.8 shows that $\pi|_{H(x,y)} = \pi_\alpha \circ \sigma^b$ for some $\alpha \in \mathcal{K}^*$, $b \in \{0, 1\}$ with $\pi_\alpha(x) = \alpha x$, $\pi_\alpha(y) = \frac{1}{\alpha} y$ and $\sigma(x) = y$, $\sigma(y) = x$.

Notice that $b = 0$ as otherwise $\pi|_{H(x,y)} = \sigma_{\alpha y - x}$ is a symmetry and has a fixed vector.

Therefore, $\pi|_{H(x,y)} = \pi_\alpha$ and $\alpha \neq 0$ as otherwise $\pi_\alpha = \mathrm{id}$. Consider the vector $p = x - \alpha y$, then

$$\varphi(p) = (\alpha x - y) - (x - \alpha y) = (\alpha - 1)(x + y).$$

is non-isotropic as $\mathrm{Q}(\varphi(p)) = (\alpha - 1)^2 \neq 0$. Further, one checks that $\mathrm{Path}(\pi) \cap \langle p \rangle^{\perp} = \langle x + \alpha y \rangle$ which is non-isotropic. Therefore p has the desired property. $\square$

The conditions of Lemma 4.30 for the vector $q = \varphi(p)$ is exactly what we need to ensure that $\pi \circ \sigma_q$ is non-singular.

Proposition 4.31 *Let $\pi \in \mathrm{O}^*$ be a non-singular isometry with $\mathrm{l}(\pi) > 1$. If $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \{\mathbf{0}\}$, then there is a non-isotropic vector $q \in V$ such that $\mathrm{l}(\pi) > \mathrm{l}(\pi')$ for $\pi' = \pi \circ \sigma_q$.*

Proof First, notice that $1 < \mathrm{l}(\pi) = \dim(\mathrm{Path}(\pi))$. Therefore, by Lemma 4.30 there is a vector $p \in V$ such that

$$q = \varphi(p), \quad \mathrm{Path}(\pi) \cap \langle p \rangle^{\perp} \tag{4.12}$$

are non isotropic for $\varphi = \pi - \mathrm{id}$.

Consider $\pi' = \pi \circ \sigma_q$, then Proposition 4.23 says that $\sigma_q(p) = \pi(p)$ because $q = \varphi(p) = \pi(p) - p$ and, hence, $\pi'(p) = p$. Since $q \in \mathrm{Path}(\pi) \subset \mathrm{Fix}(\pi)^\perp$ and $p \notin \mathrm{Fix}(\pi)$, we now see

$$\mathrm{Fix}(\pi') \supset (\mathrm{Fix}(\pi) \cap \langle q \rangle^\perp) + \langle p \rangle = \mathrm{Fix}(\pi) \oplus \langle p \rangle \tag{4.13}$$

by Lemma 4.20. Thus, $\dim(\mathrm{Fix}(\pi')) \geq \dim(\mathrm{Fix}(\pi)) + 1$ and Lemma 4.13 *iv)* shows $\dim(\mathrm{Path}(\pi')) + 1 \leq \dim(\mathrm{Path}(\pi))$.

On the other hand, Lemma 4.20 on $\pi = \pi' \circ \sigma_q$ gives us

$$\mathrm{Path}(\pi) \subset \mathrm{Path}(\pi') + \langle q \rangle$$

and, therefore, $\dim(\mathrm{Path}(\pi)) \leq \dim(\mathrm{Path}(\pi')) + 1$. This proves the equalities

$$\begin{aligned} \dim(\mathrm{Path}(\pi)) &= \dim(\mathrm{Path}(\pi')) + 1, & \mathrm{Path}(\pi) &= \mathrm{Path}(\pi') \oplus \langle q \rangle, \\ \dim(\mathrm{Fix}(\pi')) &= \dim(\mathrm{Fix}(\pi)) + 1, & \mathrm{Fix}(\pi') &= \mathrm{Fix}(\pi) \oplus \langle p \rangle. \end{aligned}$$

Hence, if π' is non-singular, we have proven

$$\mathrm{l}(\pi) = \dim(\mathrm{Path}(\pi)) + \dim(\{\mathbf{0}\}) > \dim(\mathrm{Path}(\pi')) = \mathrm{l}(\pi')$$

as $\mathrm{Path}(\pi') \cap \mathrm{rad}(V) \subset \mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \{\mathbf{0}\}$.

However, π' has to be non-singular, since

$$\begin{aligned} \mathrm{Fix}(\pi')^\perp &= (\mathrm{Fix}(\pi) \oplus \langle p \rangle)^\perp \\ &= \mathrm{Fix}(\pi)^\perp \cap \langle p \rangle^\perp \\ &= (\mathrm{Path}(\pi) \oplus \mathrm{rad}(V)) \cap \langle p \rangle^\perp \\ &= (\mathrm{Path}(\pi) \cap \langle p \rangle^\perp) \oplus \mathrm{rad}(V) \end{aligned}$$

is non isotropic by (4.12). □

The proof of Proposition 4.31 shows that π' is singular if $\mathrm{Path}(\pi) \cap \langle p \rangle^\perp$ is isotropic. Therefore, the condition in Lemma 4.30 is necessary.

Example 4.32 *Look again at (V, Q) and π'' from Examples 4.19, 4.26 and 4.28 and consider $p = -(e_1 + e_2 + e_3)/2$. We have to check that $q_2 = \pi''(p) - p$ and $\mathrm{Path}(\pi'') \cap \langle p \rangle^\perp$ are non-isotropic. One can show that*

$$\begin{aligned} q_2 &= e_1 + e_2 + e_3, \\ \mathrm{Path}(\pi'') \cap \langle p \rangle^\perp &= \langle e_1, e_2 + e_3 \rangle \cap \langle e_1 - e_2, e_3 \rangle = \langle -e_1 + e_2 + e_3 \rangle \end{aligned}$$

and, hence, p satisfies the condition of Lemma 4.30. Computing $\pi''' = \pi'' \circ \sigma_{q_2}$ is again left to the reader. For $v \in V$ one gets

$$\pi'''(v) = v + f(-e_1 + e_2 + e_3, v)(-e_1 + e_2 + e_3)$$

and we see that $\pi''' = \sigma_{q_1}$ with $q_1 = -e_1 + e_2 + e_3$. We clearly have $\mathrm{l}(\pi''') = 1$.

Finally, we prove the theorem.

Proof (of Theorem 4.16) We will use induction on $\mathrm{l}(\pi)$ to show that an isometry $\pi \in \mathrm{O}^*$ is a product of at most $\mathrm{l}(\pi)$ simple isometries.

If $\mathrm{l}(\pi) = 1$, we must have $\dim(\mathrm{Path}(\pi)) = 1$ and $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \{\mathbf{0}\}$. Hence, π is a simple isometry and we have $\pi = \sigma_{q_1}$ with $\langle q_1 \rangle = \mathrm{Path}(\pi)$ by Proposition 4.14.

Now assume $\mathrm{l}(\pi) > 1$ and that the induction hypothesis holds for all $\pi' \in \mathrm{O}^*$ with $\mathrm{l}(\pi') < \mathrm{l}(\pi)$.

In either of the cases

- π is singular,
- π is non-singular and $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) \neq \{\mathbf{0}\}$,
- π is non-singular and $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \{\mathbf{0}\}$

we find a non-isotropic vector $q_{k+1} \in V$ by Propositions 4.25, 4.27 and 4.31 such that $\mathrm{l}(\pi') < \mathrm{l}(\pi)$ for $\pi' = \pi \circ \sigma_{q_{k+1}}$.

Therefore, the induction hypothesis gives us

$$\pi' = \pi \circ \sigma_{q_{k+1}} = \sigma_{q_1} \circ \ldots \circ \sigma_{q_k} \Rightarrow \pi = \sigma_{q_1} \circ \ldots \circ \sigma_{q_{k+1}}$$

with $k + 1 \leq \mathrm{l}(\pi') + 1 \leq \mathrm{l}(\pi)$. This concludes the induction.

As we have seen in Proposition 4.22, every product $\pi = \sigma_{q_1} \circ \ldots \circ \sigma_{q_k}$ satisfies $k \geq \mathrm{l}(\pi)$ which concludes the theorem. $\square$

Example 4.33 *Consider (V, Q) and π from Examples 4.19, 4.26, 4.28 and 4.32. We have seen that $\pi(v) = v - 4f(e_1, v)e_3$ can be written as a product of simple isometries. In fact, we have*

$$\pi \circ \sigma_{q_4} \circ \ldots \circ \sigma_{q_1} = \mathrm{id} \Rightarrow \pi = \sigma_{q_1} \circ \ldots \circ \sigma_{q_4}$$

for $q_1 = (-1, 1, 1)^\intercal$, $q_2 = (1, 1, 1)^\intercal$, $q_3 = (1, -1, 1)^\intercal$, $q_4 = (-1, -1, 1)^\intercal$.

Recall that **PQ** *is the set of all 1-dimensional isotropic subspaces of V. If one takes $\langle e_1, e_2 \rangle$ as the line at infinity, then* **PQ** *is the union of the x and y-axis. Let $\boldsymbol{\pi}$ be the collineation induced by π. Fig. 4.1 shows how $\boldsymbol{\pi}$ acts on the quadric* **PQ**.

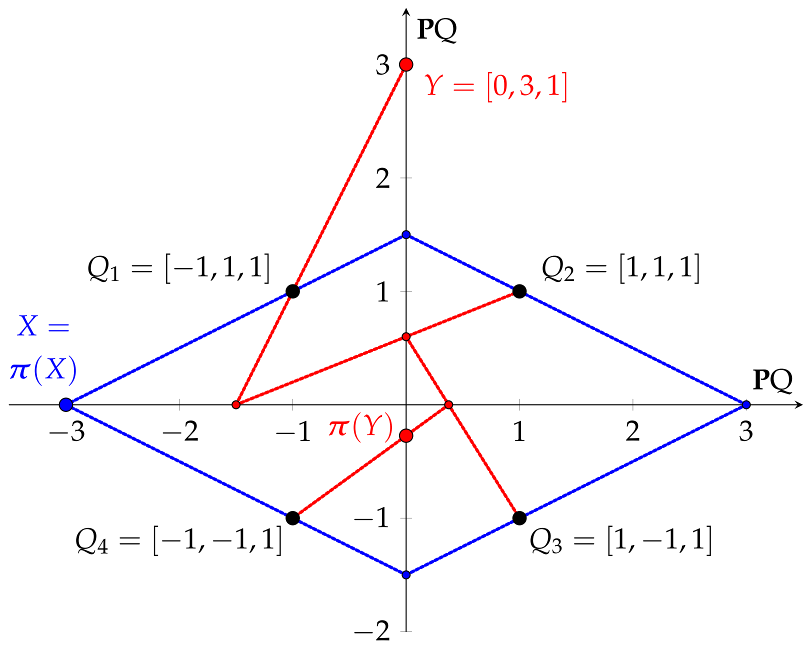

Figure 4.1: For all $x \in \mathbb{R}$ the point $X = [x, 0, 1]$ is fixed by $\boldsymbol{\pi}$. On the other hand, $\boldsymbol{\pi}([0,3,1]) = [0,3,-11] = [0,-\frac{3}{11},1] \neq [0,3,1]$.

Chapter 5

Maximal Length

Let (V, f) be a finite-dimensional orthogonal space and Q the associated quadratic form.

Recall that Theorem 4.16 gives us the formula

$$\mathrm{l}_{\mathrm{O}_S}(\pi) = \mathrm{l}(\pi) = \begin{cases} \mathrm{l}_0(\pi) & \text{if } \pi \text{ is non-singular} \\ \mathrm{l}_0(\pi) + 2 & \text{if } \pi \text{ is singular} \end{cases}$$

with $\mathrm{l}_0(\pi) = \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap \mathrm{rad}(V))$ for the length of an isometry $\pi \in \mathrm{O}_S(V) = \mathrm{O}^*(V)$. For this reason, we often write $\mathrm{l}(\pi)$ instead of $\mathrm{l}_{\mathrm{O}_S}(\pi)$.

In this chapter, we want to give an optimal bound for $\mathrm{l}_{\mathrm{O}_S}(\pi)$ depending on the given orthogonal space (V, f).

If $\pi \in \mathrm{O}_S$ is non-singular, Lemma 4.13 *iv)* shows

$$\begin{aligned} \mathrm{l}_{\mathrm{O}_S}(\pi) &= \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap \mathrm{rad}(V) \\ &\leq \dim(V) - \dim(\mathrm{Fix}(\pi)) + \dim(\mathrm{rad}(V)) \leq \dim(V) \end{aligned}$$

as $\mathrm{rad}(V) \subset \mathrm{Fix}(\pi)$. However, if π is singular, Example 4.19 shows that $\mathrm{l}_{\mathrm{O}_S}(\pi) > \dim(V)$ is possible. Therefore, we need a different approach for the bound if the isometry is singular.

As we have seen in Theorem 3.23, we can express $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$ and $\mathrm{l}_{\hat{\mathrm{O}}}(\hat{\boldsymbol{\pi}})$ with $\mathrm{l}(\pi)$. This will allow us to derive optimal bounds for $\mathrm{l}_{\mathrm{PO}_S}$ and $\mathrm{l}_{\hat{\mathrm{O}}}$ which will be part of this chapter as well.

5.1 Signature of an orthogonal space

There are two important ways to split an orthogonal space into a direct sum of smaller orthogonal subspaces. This lets us define the *index of V* and gives a bound on the dimension of an isotropic subspace of V. We follow chapter III in [6] by Artin.

Let (V, f) be an n-dimensional non-singular orthogonal space. If $U \subset V$ is a non-singular subspace, then by Proposition 4.4 we see

$$(U^{\perp})^{\perp} = U \quad \Rightarrow \quad \mathrm{rad}(U^{\perp}) = \mathrm{rad}(U) = \{\mathbf{0}\}$$

and, hence, $U^{\perp}$ is non-singular. Additionally, we have $V = U \perp U^{\perp}$.

This leads to two different splittings of V.

Firstly, as V is non-singular, there is a non-isotropic vector $p_1 \in V$ and, hence,

$$V = \langle p_1 \rangle \perp W \tag{5.1}$$

for $W = \langle p_1 \rangle^{\perp}$. Since W is non-singular, we can iterate this procedure with $(W, f|_W)$ as long as $W \neq \{\mathbf{0}\}$. Therefore, we get an *orthogonal splitting of* V,

$$V = \langle p_1 \rangle \perp \ldots \perp \langle p_n \rangle$$

with non-isotropic vectors $p_1, \ldots, p_n$.

Secondly, if there is an isotropic vector $x_1 \in V$, then similarly to Example 4.8 we find a vector $y_1 \in V$ such that x_1, y_1 is a hyperbolic pair. The subspace $H(x_1, y_1) = \langle x_1, y_1 \rangle \subset V$ is non-singular and, hence,

$$V = H(x_1, y_1) \perp W \tag{5.2}$$

with $W = H(x_1, y_1)^{\perp}$ which is non-singular. Therefore, we can first split V with (5.2) as long $W \neq \{\mathbf{0}\}$ contains isotropic vectors and then continue with (5.1) as long as $W \neq \{\mathbf{0}\}$ and get a *hyperbolic splitting of* V

$$V = \underbrace{H(x_1, y_1) \perp \ldots \perp H(x_h, y_h)}_{H} \perp \underbrace{\langle p_1 \rangle \perp \ldots \perp \langle p_a \rangle}_{A}$$

for some $h, a \in \mathbb{N}$ with $2h + a = n$.

The space (H, f) is called *hyperbolic* and since (A, f) does not contain isotropic vectors, we say A *is anisotropic*. Further, h is called the *index* of V.

This splitting is far from unique. However, $h, a \in \mathbb{N}$ are well-defined. This is a result of Witt's theorem and we refer to [6, p. 124] for the proof.

If a subspace $U \subset V$ is isotropic, we will see that one can choose $x_1, \ldots, x_h$ in such a way that $U \subset \langle x_1, \ldots, x_h \rangle$ and, hence, $\dim(U) \leq h$.

Proposition 5.1 *If (V, f) is a non-singular orthogonal space and $U \subset V$ isotropic, then there is a hyperbolic space*

$$H = H(x_1, y_1) \perp \ldots \perp H(x_k, y_k) \subset V$$

such that $x_1, \ldots, x_k$ is a basis of U.

Proof If $\dim(U) = 0$, the proposition is true. Assume $\dim(U) = k$ and that the statement holds for isotropic subspaces with dimensions less than k.

Let $x_k \in U$ and $y_k \in V$ such that x_k, y_k is a hyperbolic pair. By (5.2) we get $V = H(x_k, y_k) \perp W$ and $W = H(x_k, y_k)^{\perp}$ is non-singular. Notice that

$$x_k \notin W \text{ and } y_k \notin U + W$$

since $f(x_k, u + w) = 0$ for all $u + w \in U + W$ whereas $f(x_k, y_k) = 1$.

Hence, $W \subsetneq U + W \subsetneq V$ and we see

$$\begin{aligned}\dim(W) &< \dim(U+W)\\ &= \dim(U) + \dim(W) - \dim(U \cap W) < \dim(V).\end{aligned}$$

As $\dim(V) = \dim(W) + 2$, we must have $\dim(U \cap W) = \dim(U) - 1 = k - 1$. Therefore, the subspace $U' = U \cap W$ of the vector space W satisfies the induction hypothesis and there is a hyperbolic space

$$H' = H(x_1, y_1) \perp \ldots \perp H(x_{k-1}, y_{k-1}) \subset W$$

such that $x_1, \ldots, x_{k-1}$ is a basis of U'. Clearly, $H = H' \perp H(x_k, y_k)$ has the desired property as $U' \oplus \langle x_k \rangle \subset U$ have the same dimension. □

So far we have assumed that V is non-singular. If V is non-isotropic, then by Remark 4.3 we find a non-singular subspace W such that $V = W \perp R$ with $R = \mathrm{rad}(V)$. Clearly any basis $r_1, \ldots, r_r$ of R is orthogonal. Therefore, an orthogonal splitting of W gives rise to an orthogonal splitting of V

$$V = \underbrace{\langle p_1 \rangle \perp \ldots \perp \langle p_{n-r} \rangle}_{W} \perp \underbrace{\langle r_1 \rangle \perp \ldots \perp \langle r_r \rangle}_{R} \tag{5.3}$$

with non-isotropic vectors $p_1, \ldots, p_{n-r}$.

Similarly, a hyperbolic splitting of W defines a hyperbolic splitting of V

$$V = \underbrace{H(x_1, y_1) \perp \ldots \perp H(x_h, y_h)}_{H} \perp \underbrace{\langle p_1 \rangle \perp \ldots \perp \langle p_a \rangle}_{A} \perp \underbrace{\langle r_1 \rangle \perp \ldots \perp \langle r_r \rangle}_{R}$$

with $h, a, r \in \mathbb{N}$, $2h + a + r = n$ and $H \perp A = W$.

We say V has *signature* $[h, a, r]$ and h is again called the *index of* V. It is not obvious that two different complementary space W, W' of R define the same signature. However, Theorem 1.3 in [6, p. 116] shows that W and W' are isometric (there is an isometry $\pi \in \mathrm{O}(V)$ with $\pi(W) = W'$). Hence, every hyperbolic splitting (5.1) of W gets bijectively mapped to a hyperbolic splitting of W'. Therefore, the signature is well-defined.

Finally, if a subspace $U \subset V$ is isotropic, it must satisfy $\dim(U) \leq h + r$. Indeed, $\dim(U) = \dim(U \cap W) + \dim(U \cap R) \leq h + r$.

5.2 Length of isometries in O_S

Let (V, f) be a non-isotropic orthogonal space with signature $[h, a, r]$ and write $R = \mathrm{rad}(V)$, $n = 2h + a + r = \dim(V)$ and $r = \dim(R)$.

For any isometry $\pi \in \mathrm{O}_S(V, f)$ we have

$$\begin{aligned} \dim(\mathrm{Path}(\pi)) = n - \dim(\mathrm{Fix}(\pi)) &\leq n - r \\ \dim(\mathrm{Path}(\pi) \cap R) &\leq \min\{n - r, r\} \end{aligned}$$

since $R \subset \mathrm{Fix}(\pi)$. Hence, if π is non-singular, then

$$\mathrm{l}(\pi) \leq \begin{cases} n & \text{if } 2r \leq n \\ 2(n-r) & \text{if } 2r > n \end{cases}. \tag{5.4}$$

We have already seen that the singular isometry in Example 4.19 does not satisfy the bound (5.4) since $\dim(V) = 3 < 4 = \mathrm{l}(\pi)$. Notice, the signature of (V, f) in Example 4.19 is $[1, 0, 1]$. We want to show that $[1, 0, 1]$ is the only signature which does not satisfy (5.4) for singular isometries.

The considerations of Section 5.1 show that any isotropic subspace $U \subset V$ satisfies $\dim(U) \leq h + r$. Hence, if $\pi \in \mathrm{O}_S$ is a singular isometry, we have $\dim(\mathrm{Fix}(\pi)^{\perp}) \leq h + r$. Proposition 4.4 *iii)* shows that

$$h + r \geq n - \dim(\mathrm{Fix}(\pi)) + r = \dim(\mathrm{Path}(\pi)) + r$$

and, therefore, $\dim(\mathrm{Path}(\pi)) \leq h$. Hence, $\mathrm{l}(\pi)$ is bounded by

$$\begin{aligned} \mathrm{l}(\pi) &= \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap R) + 2 \\ &\leq h + \min\{h, r\} + 2. \end{aligned}$$

Assume the signature $[h, a, r]$ satisfies $\min\{n, 2(n - r)\} < h + \min\{h, r\} + 2$.

If $2r > n$, we have $h < r$ and see that V must be isotropic since

$$2(n - r) < 2h + 2 \Rightarrow 2a + 2h < 2 \Rightarrow a = h = 0$$

If $2r \leq n$, we get

$$\begin{aligned} n < h + \min\{h, r\} + 2 &\Leftrightarrow o + h + r - \min\{h, r\} < 2 \\ &\Leftrightarrow o + \max\{h, r\} < 2. \end{aligned}$$

Therefore, only the signatures $[0, 1, 0]$, $[1, 0, 0]$, $[1, 0, 1]$ are possible for a non-isotropic orthogonal space V which does not satisfy the bound (5.4).

We can check them one by one.

i) If (V, f) has signature $[0,1,0]$, it has no singular isometries since there is no isotropic subspace.

ii) If (V, f) has signature $[1,0,0]$, Example 4.8 says that every isometry $\pi \in \mathrm{O}_S$ can be written as a product of two symmetries. Hence, id_V is the only isometry that is singular, otherwise, $\mathrm{l}(\pi) > 2$.

iii) If (V, f) has signature $[1,0,1]$, we have $\mathrm{l}(\pi) \leq 2h + 2 = 4$ for any singular isometry π. This bound is met by the isometry

$$\pi(v) = v - 4f(x, v)z$$

where $V = H(x, y) \perp \langle z \rangle$ is a hypberbolic splitting.

We have now shown the following theorem.

Theorem 5.2 *Let (V, f) be a non-isotropic orthogonal space over $\mathcal{K}$ with signature $[h, a, r]$ and $\mathrm{char}(\mathcal{K}) \neq 2$. For any isometry $\pi \in \mathrm{O}_S(V)$ we have*

$$\mathrm{l}(\pi) \leq \begin{cases} n & \text{if } 2r \leq n \\ 2(n-r) & \text{if } 2r > n \end{cases}$$

if $[h, a, r] \neq [1,0,1]$. Otherwise, $\mathrm{l}(\pi) \leq 4$.

Remark 5.3 *The bound of $\mathrm{l}(\pi)$ in Theorem 5.2 is optimal. Indeed, if (V, f) has signature $[h, a, r]$, consider an orthogonal splitting*

$$V = \langle p_1 \rangle \perp \ldots \perp \langle p_{n-r} \rangle \perp \underbrace{\langle p_{n-r+1} \rangle \perp \ldots \perp \langle p_n \rangle}_{\mathrm{rad}(V)}$$

of V with non-isotropic vectors $p_1, \ldots, p_{n-r}$.

If $n \geq 2r$, we can define π such that

$$\begin{aligned} \pi(p_i) &= p_i + p_{n-i} && \text{if } 1 \leq i \leq r, \\ \pi(p_i) &= -p_i && \text{if } r < i \leq n-r, \\ \pi(p_i) &= p_i && \text{if } n-r < i \leq n. \end{aligned}$$

One easily checks that $\pi \in \mathrm{O}^*(V) = \mathrm{O}_S(V)$, $\mathrm{Path}(\pi) = \langle p_{r+1}, \ldots, p_n \rangle$ and $\mathrm{Path}(\pi) \cap \mathrm{rad}(V) = \langle p_{n-r+1}, \ldots, p_n \rangle$. Hence, $\mathrm{l}(\pi) \geq n - r + r = n$.

If $n < 2r$, we can define π such that

$$\begin{aligned} \pi(p_i) &= p_i + p_{n-i} && \text{if } 1 \leq i \leq n-r, \\ \pi(p_i) &= p_i && \text{if } n-r < i \leq n. \end{aligned}$$

Again, one checks that $\pi \in \mathrm{O}^*(V) = \mathrm{O}_S(V)$, $\mathrm{Path}(\pi) = \langle p_{r+1}, \ldots, p_n \rangle$ and $\mathrm{Path}(\pi) \subset \mathrm{rad}(V)$. Hence, $\mathrm{l}(\pi) \geq 2(n-r)$.

5.3 Length of collineations in PO_S

Let (V, f) be a non-isotropic orthogonal space. We already established the formula

$$\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) = \begin{cases} \min\{\mathrm{l}(\pi), \mathrm{l}(-\pi)\} & \text{if } V \text{ is non-singular} \\ \mathrm{l}(\pi) & \text{if } V \text{ is singular} \end{cases}$$

in Theorem 3.23 as one easily checks

$$V \text{ is non-singular } \Leftrightarrow -\mathrm{id}_V \in \mathrm{O}^* = \mathrm{O}_S\,.$$

Hence, it satisfies the bound in Theorem 5.2 as $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) \leq \mathrm{l}(\pi)$. However, we can improve it if V is non-singular. For this, we need to introduce determinants.

Remark 5.4 *Let $(V, \mathcal{K})$ be an n-dimensional vector space. If one chooses a basis of V, every linear map $\pi \in \mathrm{GL}(V)$ can be represented by a $n \times n$-matrix with values in $\mathcal{K}$ by the usual construction. We assume that the reader is already familiar with the determinant of matrices. Hence, we can define* $\det(\pi)$ *as the determinant of its matrix. Since the determinant is multiplicative for matrices,* $\det(\pi)$ *does not depend on the chosen basis.*

We will not go into more detail about determinants. However, we will use the two following well-known facts.

i) *The map* $\det : \mathrm{GL}(V) \to \mathcal{K}^*; \pi \mapsto \det(\pi)$ *is a group homomorphism with respect to the composition and multiplication.*

ii) *Let $\pi \in \mathrm{GL}(V)$ be a linear map. If $b_1, \ldots, b_n$ is a basis of V such that $\pi(b_i) = \alpha_i b_i$ for some $\alpha_i \in \mathcal{K}$, then* $\det(\pi) = \prod_{i=1}^n \alpha_i$.

If one is not familiar with this topic, the book [4] by Lang gives a concrete definition and proves of the given properties.

We want to compute $\det(\pi)$ for an isometry $\pi \in \mathrm{O}_S(V)$ and connect it with $\mathrm{l}(\pi)$.

Firstly, consider σ_p for a non-isotropic vector $p \in V$. Let $b_1 = p$ and $b_2, \ldots, b_n$ be a basis of $\langle p \rangle^\perp = \mathrm{Fix}(\sigma_p)$. Since $V = \langle p \rangle \perp \langle p \rangle^\perp$, the vectors $b_1, \ldots, b_n$ are a basis of V and

$$\sigma_{b_1} = -b_1,\ \sigma_{b_2} = b_2,\ \ldots,\ \sigma_{b_n} = b_n.$$

Therefore, $\det(\sigma_p) = -1$ by Remark 5.4 *ii)*.

Let $\pi \in \mathrm{O}_S(V)$ and assume $\pi = \sigma_{q_1} \circ \ldots \circ \sigma_{q_k}$ is a product of k symmetries. By Remark 5.4 *i)* we see

$$\det(\pi) = \det(\sigma_{q_1}) \cdot \ldots \cdot \det(\sigma_{q_k}) = (-1)^k.$$

Hence, $\det(\pi) = \pm 1$ depending on k being even or odd.

By Theorem 4.16 π can be written as a product of $\mathrm{l}(\pi)$ symmetries and, hence,

$$\mathrm{l}(\pi) \text{ is even} \Leftrightarrow \det(\pi) = 1 \quad \Leftrightarrow k \text{ is even}, \tag{5.5}$$
$$\mathrm{l}(\pi) \text{ is odd} \Leftrightarrow \det(\pi) = -1 \Leftrightarrow k \text{ is odd}. \tag{5.6}$$

This allows us to improve the bound on $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$ if (V, f) is non-singular and $\dim(V)$ is odd.

Theorem 5.5 *Let (V, f) be a non-isotropic orthogonal space of dimension n. Further, let $\boldsymbol{\pi} \in \mathrm{PO}_S$ be a collineation which is induced by $\pi \in \mathrm{O}_S$. If V is non-singular, then $\mathrm{PO}_S \simeq \mathrm{O}_S / \{\pm \mathrm{id}_V\}$ and*

$$\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) = \min\{\mathrm{l}(\pi), \mathrm{l}(-\pi)\} \leq \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Otherwise, we have $\mathrm{PO}_S \simeq \mathrm{O}_S$, $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) = \mathrm{l}(\pi)$ and the same bound as in Theorem 5.2 applies.

Proof The equation for $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$ and both isomorphisms are a direct cause of the observation (5.3) applied to Theorem 3.23. Hence, we only have to show the bound of $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$ if V is non-singular.

Assume V is non-singular. We clearly have $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) \leq n$ by Theorem 5.2. If n is odd and $\mathrm{l}(\pi) = n$ for some $\pi \in \mathrm{O}_S$, we must have $\det(\pi) = -1$ by (5.6). Hence,

$$\det(-\pi) = \det(\pi) \cdot \det(-\mathrm{id}_V) = (-1)^{n+1} = 1$$

as $\det(-\mathrm{id}_V) = (-1)^n = -1$ by Remark 5.4 *ii)*. Therefore, $\mathrm{l}(-\pi)$ is even by (5.5) and we must have $\mathrm{l}(-\pi) \leq n-1$ which proves the bound in the non-singular case. $\square$

Remark 5.6 *Similar to Remark 5.3 we can ask ourselves if the bound in Theorem 5.5 is optimal. This is clearly the case if (V, f) is singular as $\mathrm{PO}_S \simeq \mathrm{O}_S$.*

We will show that if (V, f) is non-singular and of even dimension n, there is a map $\pi \in \mathrm{O}_S$ such that $\mathrm{l}(\pi) = \mathrm{l}(-\pi) = n$. This shows that the bound is optimal for both cases since we can use the same construction to get a map $\pi \in \mathrm{O}_S$ such that $\mathrm{l}(\pi) = \mathrm{l}(-\pi) = n-1$ in the odd case.

Assume $\langle x \rangle \perp \langle y \rangle \subset V$ and x, y are non-singular. Then for any $\alpha \in \mathcal{K}^*$ one can check that $\beta = -\mathrm{Q}(x)/(\alpha \mathrm{Q}(y))$ satisfies

$$\langle x + \beta y \rangle \subset \langle x + \alpha y \rangle^{\perp}.$$

Hence, if $\alpha \neq \beta$, the vector $x + \alpha y$ is non-isotropic as otherwise

$$x + \alpha y \in \mathrm{rad}(\langle x + \alpha y, x + \beta y\rangle) = \mathrm{rad}(\langle x, y\rangle) = \{\mathbf{0}\}.$$

Therefore, we have

$$\begin{aligned}(\sigma_{x+\alpha y} \circ \sigma_y)(x + \alpha y) &= \sigma_y(-x - \alpha y) = -x + \alpha y\\ (\sigma_{x+\alpha y} \circ \sigma_y)(x + \beta y) &= \sigma_y(x + \alpha y) = x - \beta y\end{aligned}$$

and, hence, by Lemma 4.13 the isometry $\pi = \sigma_{x+\alpha y} \circ \sigma_y$ satisfies

$$\begin{aligned}\mathrm{Path}(\pi) &= \langle -2x, -2\beta y\rangle = \langle x, y\rangle \Rightarrow \mathrm{Fix}(\pi) = \langle x, y\rangle^{\perp},\\ \mathrm{Path}(-\pi) &= \langle 2\alpha y, -2x\rangle = \langle x, y\rangle \Rightarrow \mathrm{Fix}(-\pi) = \langle x, y\rangle^{\perp},\end{aligned}$$

If $|\mathcal{K}| > 3$, we can always find $\alpha \in \mathcal{K}^*$ with $\alpha \neq \beta$. Indeed, if $\alpha = \beta$, then $\alpha^2 = -\mathrm{Q}(y)^{-1}\mathrm{Q}(x)$. However, there can be at most two solutions to this polynomial and, hence, there is an $\alpha \in \mathcal{K}^*$ with the desired properties if $|\mathcal{K}^*| > 2$.

As V is non-singular, the vectors $p_1, \ldots, p_n$ of an orthogonal splitting

$$V = \langle p_1\rangle \perp \ldots \perp \langle p_n\rangle$$

of V are non-isotropic. We can pair them up and define

$$\pi_i = \sigma_{p_{2i-1}+\alpha_i p_{2i}} \circ \sigma_{p_{2i}}$$

for all $i \leq n/2$ and some $\alpha_i \in \mathcal{K}^*$ just as above. Here we used the fact that n is even. The isometry $\pi = \pi_1 \circ \ldots \circ \pi_{n/2} \in \mathrm{O}_S$ has the desired property, as one easily sees

$$\mathrm{Path}(\pi) = \mathrm{Path}(-\pi) = V \;\Rightarrow\; \mathrm{l}(\pi) = \mathrm{l}(-\pi) = n$$

since $(\pi - \mathrm{id})(\langle p_{2i-1}, p_{2i}\rangle) = \langle p_{2i-1}, p_{2i}\rangle$. Therefore, if $|\mathcal{K}| > 3$, the bound in Theorem 5.5 is optimal.

5.4 Length of maps in Ô

Finally, we can give a bound for the length of a map $\hat{\pi} \in \hat{\mathrm{O}}$. If the quadric $\mathbf{P}\mathrm{Q}$ is non-trivial, we have $\hat{\mathrm{O}} \simeq \mathrm{PO}_S$ by Theorem 3.21.

Let (V, f) be an orthogonal space with signature $[h, a, r]$. Then

$$\begin{aligned}\mathbf{P}\mathrm{Q} \text{ is non-trivial } &\Leftrightarrow n \geq 3 \text{ and there is a secant in } \mathbf{P}V\\ &\Leftrightarrow n \geq 3 \text{ and } h \geq 1.\end{aligned}$$

The last implication is true because a hyperbolic subspace $H(x, y)$ is a secant in $\mathbf{P}V$ and, conversely, a secant satisfies all conditions in Example 4.8 and, hence, it is equal to $H(x, y)$ for a hyperbolic pair $x, y \in V$.

If $h = 0$, then $\mathbf{P}V$ has no secants and every σ_p induces the identity map on $\mathbf{P}\mathrm{Q}$. This shows that $\hat{\mathrm{O}} = \{\mathrm{id}_{\mathbf{P}\mathrm{Q}}\}$.

Further, if $V = H(x,y)$ is a hyperbolic space, Example 4.8 shows that π_α induces the identity on $\mathbf{P}\mathrm{Q}$ and σ swaps $\langle x\rangle$ and $\langle y\rangle$. Hence, $\hat{\mathrm{O}} = \{\mathrm{id}_{\mathbf{P}\mathrm{Q}}, \hat{\boldsymbol{\sigma}}\}$.

We have now shown the following theorem.

Theorem 5.7 *If (V,f) is orthogonal space over $\mathcal{K}$ with signature $[h,a,r]$ and* $\mathrm{char}(\mathcal{K}) \neq 2$, *then $\hat{\mathrm{O}}(V,f)$, the group of the quadric $\mathbf{P}\mathrm{Q}$, is equal to*

$$\hat{\mathrm{O}}(V,f) \simeq \begin{cases} \{\mathrm{id}_{\mathbf{P}\mathrm{Q}}\} & \text{if } h = 0 \\ \{\mathrm{id}_{\mathbf{P}\mathrm{Q}}, \hat{\boldsymbol{\sigma}}\} & \text{if } h = 1, n = 2 \\ \mathrm{PO}_S(V,f) & \text{if } h = 1, n > 2 \end{cases}.$$

If the signature of V equals $[1,0,0]$, then the map $\hat{\boldsymbol{\sigma}}$ exchanges the two singular points of $\mathbf{P}V$. Further, for $\hat{\boldsymbol{\pi}} \in \hat{\mathrm{O}}(V)$ the following bound is optimal

$$\mathrm{l}_{\hat{\mathrm{O}}}(\hat{\boldsymbol{\pi}}) \leq \begin{cases} 0 & \text{if } h = 0 \\ 1 & \text{if } h = 1, r = a = 0 \\ 4 & \text{if } h = r = 1, a = 0 \\ 2(n-r) & \text{if } 2r > n \\ n-1 & \text{if } r = 0, n \text{ odd} \\ n & \text{otherwise} \end{cases}.$$

The bounds follow directly by Theorems 5.2 and 5.5.

5.5 Subgroups of O_S

If we want to calculate $\mathrm{l}(\pi)$ for a given $\pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k} \in \mathrm{O}_S$, we have to calculate $\mathrm{Path}(\pi)$ and check whether $\mathrm{Fix}(\pi)^\perp$ is isotropic or not. For these calculations, one usually has to multiply the matrices of σ_{p_i}. However, we can give a bound on $\mathrm{l}(\pi)$ if we know the non-isotropic vectors $p_1, \ldots, p_k$.

Let (V,f) be a non-isotropic orthogonal space and $u_1, \ldots, u_k \in V$ non-isotropic vectors. For

$$\pi = \sigma_{u_1} \circ \ldots \circ \sigma_{u_k}, \quad U = \langle u_1, \ldots, u_k\rangle, \quad m = \dim(U), \tag{5.7}$$

the calculation in the proof of Proposition 4.22 shows

$$m = \dim(U) \geq \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap \mathrm{rad}(V))$$

which is strict if π is singular. Hence,

$$\mathrm{l}(\pi) \leq \begin{cases} m & \text{if } \pi \text{ is non-singular} \\ m+1 & \text{if } \pi \text{ is singular} \end{cases}. \tag{5.8}$$

By Remark 5.3 and Example 4.10 we can not improve the bound in the general case. However, we can relate it to $\pi|_U$.

For any $u, u' \in U$ we clearly have

$$\sigma_u(u') = \underbrace{\sigma_u(u') - u'}_{\in \mathrm{Path}(\sigma_u) = \langle u \rangle} + u' \in U$$

and, hence, $\sigma_u|_U$ is the symmetry at u in the vector space $(U, f|_U)$. This shows that for the subgroup

$$\mathrm{O}_S^U = \{\sigma_{u_1} \circ \ldots \circ \sigma_{u_k} : k \in \mathbb{N}, u_1, \ldots, u_k \in U \text{ non-isotropic}\} \subset \mathrm{O}_S\,.$$

the map

$$r : \mathrm{O}_S^U(V, f) \to \mathrm{O}_S(U, f|_U); \pi \mapsto \pi|_U$$

is well-defined and surjective.

The following lemma can be found in [11] by Götzky.

Lemma 5.8 *If $U \subset V$ satisfies* $\mathrm{rad}(U) \subset \mathrm{rad}(V)$*, then* $\mathrm{O}_S^U(V, f) \simeq \mathrm{O}_S(U, f|_U)$.

Proof We only have to show injectivity of r. Assume $\pi \in \mathrm{O}_S^U(V, f)$ with $\pi|U = \mathrm{id}_U$. Since Lemma 4.30 and Proposition 4.4 *iv)* show $U^\perp \subset \mathrm{Fix}(\pi)$ and $V = U + U^\perp$, we see

$$\pi(x) = \pi(u + u') = u + u' = x$$

for all $x = u + u' \in V$ with $u \in U$, $u' \in U^\perp$. Therefore, $\pi = \mathrm{id}_V$ and r is injective. □

If the isometry π in (5.7) satisfies $\mathrm{rad}(U) \subset \mathrm{rad}(V)$, then Lemma 5.8 shows that $\mathrm{l}(\pi) = \mathrm{l}(\pi|_U)$ where we view $\pi|_U$ as an isometry of $\mathrm{PO}_S(U, f|_U)$. Hence, the bound of Theorem 5.2 applies and we see that

$$\mathrm{l}(\pi) \leq \begin{cases} m & \text{if } 2r \leq m \\ 2(m - r) & \text{if } 2r > m \end{cases} \tag{5.9}$$

with $m = \dim(U)$, $r = \dim(\mathrm{rad}(U))$ as long as the signature of $(U, f|_U)$ is not equal to $[1, 0, 1]$.

Remark 5.9 *A similar statement for the case* $\mathrm{rad}(U) \not\subset \mathrm{rad}(V)$ *seems tricky and needs further investigation.*

Chapter 6

Geometric Approach

Until now, most of the chapters used methods of linear algebra. In this chapter, we want to argue geometrically and derive the bound for $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$ in Theorem 5.5 for quadratic spaces with an anisotropic hyperplane.

In the end, we revisit Theorem 1.2 from Chapter 1 and extend it to any non-singular quadratic space of dimension 3.

For this chapter (V, Q) always denotes a non-isotropic quadratic space over a field $\mathcal{K}$ with $\mathrm{char}(\mathcal{K}) \neq 2$ and $3 \leq \dim(V) < \infty$.

6.1 Products of 3 symmetries

In the Euclidean geometry of $\mathbb{R}^2$, it is well known that if 3 lines $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ intersect in a point P, then the product $\boldsymbol{\pi} = \boldsymbol{\sigma}_{\mathfrak{g}_1} \circ \boldsymbol{\sigma}_{\mathfrak{g}_2} \circ \boldsymbol{\sigma}_{\mathfrak{g}_3}$ of the 3 line reflections is again a reflection at a line $\mathfrak{h}$ through P. Compare with Fig. 6.1.

This can be extended to arbitrary quadratic spaces. However, the condition of the symmetries changes slightly.

Theorem 6.1 *Let (V, Q) be a quadratic space. If the points $P, Q, R \in \mathbf{P}V$ are collinear, non-singular points, then*

$$\boldsymbol{\sigma}_P \circ \boldsymbol{\sigma}_Q \circ \boldsymbol{\sigma}_R = \boldsymbol{\sigma}_S$$

for a unique non-singular point S which is collinear with P, Q, R.

Usually, Theorem 6.1 is proven by showing that for $\pi = \sigma_P \circ \sigma_Q \circ \sigma_R$ there is an orthogonal splitting $p_1, \ldots, p_n$ of V with $\pi(p_1) = -p_1$ and $\pi(p_i) = p_i$. Hence, π is a symmetry by a similar argument as in *Proposition* 4.14. See [1] by Lenz for the proof in this style.

However, we will prove Theorem 6.1 in two different ways. The first one uses the following lemma.

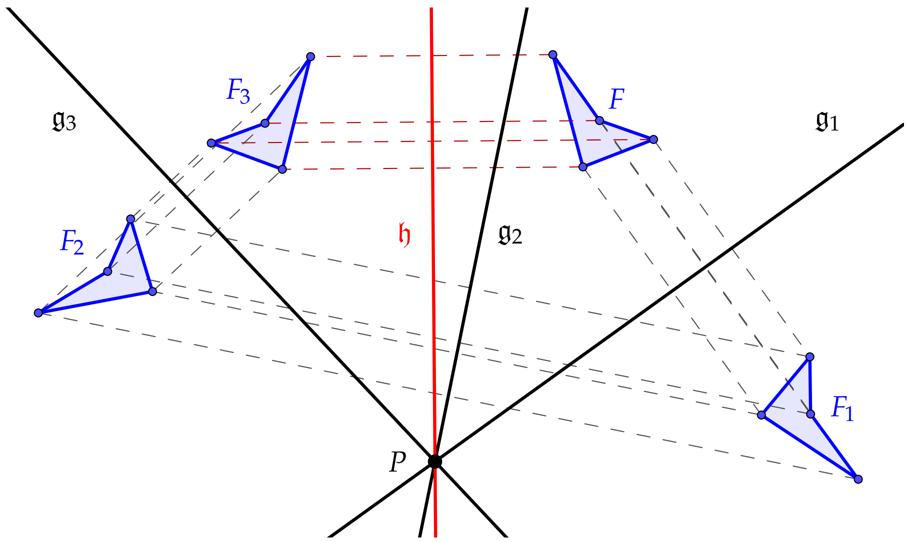

Figure 6.1: In the euclidean geometry of $\mathbb{R}^2$, the composition $\sigma_{\mathfrak{g}_1} \circ \sigma_{\mathfrak{g}_2} \circ \sigma_{\mathfrak{g}_3}$ of the line reflections at the lines $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ is equal to the line reflection at $\sigma_{\mathfrak{h}}$. Hence, for the quadrilateral F we have $(\sigma_{\mathfrak{g}_1} \circ \sigma_{\mathfrak{g}_2} \circ \sigma_{\mathfrak{g}_3})(F) = \sigma_{\mathfrak{h}}(F)$.

Lemma 6.2 (Lemma A.2) *Let (V, Q) be a quadratic space. If $x, y \in V$ are non-isotropic vectors, then $\sigma_x(y)$ is non-isotropic and*

$$\sigma_x \circ \sigma_y \circ \sigma_x = \sigma_{\sigma_x(y)}.$$

Additionally, if $x + y$ is non-isotropic, then $\mathrm{Q}(y)x + \mathrm{Q}(x)y$ is non-isotropic and

$$\sigma_x \circ \sigma_{x+y} \circ \sigma_y = \sigma_{\mathrm{Q}(y)x+\mathrm{Q}(x)y}.$$

The proof of Lemma 6.2 checks the given equalities. Look at Lemma A.2 for the calculation.

Proof (of Theorem 6.1) The uniqueness of S follows by Remark 3.13. Hence, we only have to show the existence of S.

If $P = Q$ or $Q = R$, then $S = R$, $S = P$ satisfy the theorem.

If $P = R$, Lemma 6.2 shows that $S = \hat{\sigma}_P(Q)$ is a solution.

If P, Q, R are distinct, then there are $p, r \in V$ such that $P = \langle p \rangle$, $Q = \langle p + r \rangle$ and $R = \langle r \rangle$ since $Q \subset P \oplus R$. Hence, the condition in Lemma 6.2 is satisfied, which shows that $S = \langle \mathrm{Q}(r)p + \mathrm{Q}(p)r \rangle$ is non-singular and

$$\sigma_P \circ \sigma_Q \circ \sigma_R = \sigma_S. \qquad \square$$

Secondly, we want to give a proof of Theorem 6.1 with the theory of Chapters 4 and 5.

Proof (Theorem 6.1) Let $\mathfrak{g} \in \mathbf{P}V$ be a line through the collinear points P, Q, R. If $\pi = \sigma_P \circ \sigma_R \circ \sigma_R \in \mathrm{O}_S$ satisfies $\mathrm{l}(\pi) = 1$, we are done as π must be simple and, hence, $\pi = \sigma_S$ with $S = \mathrm{Path}(\pi) \subset P + Q + R \subset \mathfrak{g}$ by Proposition 4.14. The uniqueness of S follows again by Remark 3.13.

Since $\det(\pi) = -1$, the length $\mathrm{l}(\pi)$ is odd and it is enough to show that $\mathrm{l}(\pi) \leq 2$.

Consider $U = P + Q + R \subset \mathfrak{g}$. We want to use the bound (5.8) in Section 5.5. If π is non-singular, we already have $\mathrm{l}(\pi) \leq \dim(U) \leq 2$.

Assume π is singular, then (5.8) shows that

$$\mathrm{l}(\pi) = \dim(\mathrm{Path}(\pi)) + \dim(\mathrm{Path}(\pi) \cap \mathrm{rad}(V)) + 2 \leq 3.$$

Since $\mathrm{l}(\pi)$ must be odd, we only have to consider the case $\dim(\mathrm{Path}(\pi)) = 1$ and $\mathrm{Path}(\pi) \not\subset \mathrm{rad}(V)$. However, this implies that π is simple and by Proposition 4.14 we see that π is non-singular. □

Remark 6.3 *Let (V, f) be a non-singular orthogonal space. In [12] Gunther shows that* $\dim(\mathrm{Path}(\pi))$ *and, hence,* $\mathrm{l}(\pi)$ *are even for all singular isometries* $\pi \in \mathrm{O}_S$*. This allows for an extension of Theorem 6.1 of the following form.*

$$\text{If } \pi = \sigma_{p_1} \circ \ldots \circ \sigma_{p_k} \text{ is a product of an odd amount of symmetries and } p_1, \ldots, p_k \text{ are linearly dependent, then } \pi \text{ is a product of } k-2 \text{ symmetries.} \tag{6.1}$$

Indeed, if k is odd, then $\mathrm{l}(\pi)$ must be odd and, hence, non-singular by [12]. If now $p_1, \ldots, p_k$ are linearly dependent, then $k \geq 3$ and (5.8) says

$$\mathrm{l}(\pi) \leq \dim(\langle p_1, \ldots, p_k \rangle) < k.$$

This shows $\mathrm{l}(\pi) \leq k - 2$ as $\mathrm{l}(\pi)$ can not be even.

It would be interesting to see if (6.1) still holds for singular vector spaces and if one can derive this result constructively by Lemma 6.2.

Notice, Example 4.10 shows that (6.1) can not be true for $k = 4$.

6.2 Geometric reduction

We want to use Theorem 6.1 to derive a bound for $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi})$ if (V, f) contains an anisotropic hyperplane.

Assume the points $X_1, X_2, X_3, X_4 \in \mathbf{P}V$ lie on a plane. Hence, $\overline{X_1X_2}$ and $\overline{X_3X_4}$ intersect each other in a point $X \in \mathbf{P}V$. If X is non-singular, Theorem 6.1 shows that the product

$$\boldsymbol{\pi} = \boldsymbol{\sigma}_{X_1}\boldsymbol{\sigma}_{X_2}\boldsymbol{\sigma}_{X_3}\boldsymbol{\sigma}_{X_4} = \overbrace{\boldsymbol{\sigma}_{X_1}\boldsymbol{\sigma}_{X_2}\boldsymbol{\sigma}_{X}}^{\sigma_{Y_1}}\overbrace{\boldsymbol{\sigma}_{X}\boldsymbol{\sigma}_{X_3}\boldsymbol{\sigma}_{X_4}}^{\sigma_{Y_2}} = \boldsymbol{\sigma}_{Y_1}\boldsymbol{\sigma}_{Y_2}$$

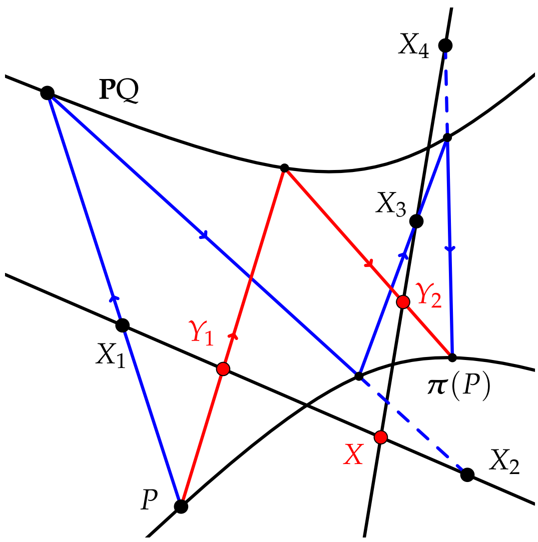

Figure 6.2: If $X = \overline{X_1X_2} \cap \overline{X_3X_4}$ is non-singular, we can reduce the composition $\pi = \sigma_{X_1}\sigma_{X_2}\sigma_{X_3}\sigma_{X_4}$ to $\pi = \sigma_{Y_1}\sigma_{Y_2}$.

can be shortened as X_1, X_2, X and X, X_3, X_4 are collinear. This is illustrated in Fig. 6.2. Notice that we write $\sigma_X\sigma_Y$ instead of $\sigma_X \circ \sigma_Y$ to save some space.

However, this idea falls short for two reasons. Firstly, as the collineation π from Example 4.33 has $\mathrm{l}_{\mathrm{PO}_S}(\pi) = 4$, we see that for all points $X_1, \ldots, X_4$ with $\pi = \sigma_{X_1}\sigma_{X_2}\sigma_{X_3}\sigma_{X_4}$ the intersection $\overline{X_1X_2} \cap \overline{X_3X_4}$ must be singular. Secondly, two lines do not have to intersect each other if $\dim(V) > 3$.

Therefore, we have to use a different technique. Assume $U \subset V$ is an anisotropic subspace and X, Y are non-singular points. If $\overline{XY}$ intersects U in a point Y', then Theorem 6.1 shows $\pi = \sigma_X\sigma_Y = \sigma_{X'}\sigma_{Y'}$ for a point $X' \in \overline{XY}$. This shows that we can choose one factor of π to be a symmetry at a point in U. This can be done inductively.

Lemma 6.4 *Let $U \subset W \subset V$ be subspaces such that $\dim(W) = \dim(U) + 1 > 1$ and assume U contains no singular points. Then a product $\pi = \sigma_{X_1} \circ \ldots \circ \sigma_{X_k}$ with non-singular points $X_1, \ldots, X_k \in W$ can be written as a product*

$$\pi = \sigma_X\sigma_{Y_1} \circ \ldots \circ \sigma_{Y_{k-1}}$$

with $Y_1, \ldots, Y_{k-1} \in U$ and $X \in W$.

Proof If $k = 1$, the statement holds trivially. Assume the statement is true for any product with less than k symmetries. If $\pi = \sigma_{X_1} \circ \ldots \circ \sigma_{X_k}$, then by the induction hypothesis there are $X' \in W$ and $Y_2, \ldots, Y_{k-1} \in U$ with

$$\pi' = \sigma_{X_2} \circ \ldots \circ \sigma_{X_k} = \sigma_{X'}\sigma_{Y_2} \circ \ldots \circ \sigma_{Y_{k-1}}.$$

If $X' \in U$, then $\pi = \sigma_{X_1}\sigma_{X'}\sigma_{Y_2} \circ \ldots \circ \sigma_{Y_{k-1}}$ satisfies the lemma. Otherwise, we have $U \oplus P = W$ and we see

$$\dim(\overline{X_1 X'} \cap U) = 2 + \dim(U) - \dim(W) = 1$$

if $X_1 \neq X'$. Hence, there is a point $Y_1 \in U \cap \overline{X_1 X'}$. By our assumption on U the point Y_1 is non-singular and, hence, Theorem 6.1 shows that $\sigma_{X_1}\sigma_{X'}\sigma_{Y_1} = \sigma_X$ for a point $X \in \overline{X_1 X'} \subset W$. Now we can write

$$\begin{aligned} \boldsymbol{\pi} = \boldsymbol{\sigma}_{X_1}\boldsymbol{\pi}' &= \boldsymbol{\sigma}_{X_1}\boldsymbol{\sigma}_{X'}\boldsymbol{\sigma}_{Y_2} \circ \ldots \circ \boldsymbol{\sigma}_{Y_{k-1}} \\ &= \boldsymbol{\sigma}_X\boldsymbol{\sigma}_{Y_1}\boldsymbol{\sigma}_{Y_2} \circ \ldots \circ \boldsymbol{\sigma}_{Y_{k-1}} \end{aligned}$$

in the desired form.

If $X_1 = X'$, then $\sigma_{X_1}\sigma_{X'} = \sigma_{Y_1}\sigma_{Y_1}$ for any $Y_1 \in U$ which concludes the induction step. $\square$

Lemma 6.4 allows us to reduce any $\boldsymbol{\pi} \in \mathrm{PO}_S$ if there is an anisotropic hyperplane in V.

Proposition 6.5 *Let (V, Q) be a n-dimensional quadratic space. Assume, there is a hyperplane $H \subset V$ which does not contain singular points, $H \cap \mathbf{PQ} = \{\mathbf{0}\}$. Then every product $\boldsymbol{\pi} \in \mathrm{PO}_S$ can be written as a product of at most n symmetries.*

Proof There are n subspaces $U_i \subset V$ such that

$$\{\mathbf{0}\} \subsetneq U_1 \subsetneq \ldots \subsetneq U_{n-1} = H \subsetneq U_n = V.$$

Clearly, $\dim(U_i) = i$. If $i < n$, U_i does not contain singular points as $U_i \subset H$. Hence, for all $i < n$ the subspaces $U_i \subset U_{i+1}$ satisfy the condition in Lemma 6.4.

If $\boldsymbol{\pi} = \boldsymbol{\sigma}_{A_1} \circ \ldots \circ \boldsymbol{\sigma}_{A_k}$ with $k > n$, then we can apply Lemma 6.4 $n-1$ times and successively we get

$$\begin{aligned} \boldsymbol{\pi} &= \boldsymbol{\sigma}_{A_1} \circ \ldots \circ \boldsymbol{\sigma}_{A_k} \\ &= \boldsymbol{\sigma}_{P_1}\boldsymbol{\sigma}_{B_1} \circ \ldots \circ \boldsymbol{\sigma}_{B_{k-1}}, \\ \Rightarrow \boldsymbol{\sigma}_{P_1}\boldsymbol{\pi} &= \boldsymbol{\sigma}_{B_1} \circ \ldots \circ \boldsymbol{\sigma}_{B_{k-1}} \\ &= \boldsymbol{\sigma}_{P_2}\boldsymbol{\sigma}_{C_1} \circ \ldots \circ \boldsymbol{\sigma}_{C_{k-2}}, \\ &\vdots \\ \boldsymbol{\sigma}_{P_{n-2}} \circ \ldots \circ \boldsymbol{\sigma}_{P_1}\boldsymbol{\pi} &= \boldsymbol{\sigma}_{P_{n-1}}\boldsymbol{\sigma}_{Z_1} \circ \ldots \circ \boldsymbol{\sigma}_{Z_{k-n+1}}, \\ \Rightarrow \boldsymbol{\sigma}_{P_{n-1}}\boldsymbol{\sigma}_{P_{n-2}} \circ \ldots \circ \boldsymbol{\sigma}_{P_1}\boldsymbol{\pi} &= \boldsymbol{\sigma}_{Z_1} \circ \ldots \circ \boldsymbol{\sigma}_{Z_{k-n+1}}. \end{aligned}$$

Hence, $P_i \in U_{n+1-i}$ and $Z_1, \ldots, Z_{k-n+1} \in U_1$. However, as $\dim(U_1) = 1$, we must have $Z_1, \ldots, Z_{k-n+1} = U_1 =: P$ and we see

$$\boldsymbol{\sigma}_{P_{n-1}}\boldsymbol{\sigma}_{P_{n-2}} \circ \ldots \circ \boldsymbol{\sigma}_{P_1}\boldsymbol{\pi} = \boldsymbol{\sigma}_P^{k-n+1} \Rightarrow \boldsymbol{\pi} = \boldsymbol{\sigma}_{P_1} \circ \ldots \circ \boldsymbol{\sigma}_{P_{n-1}}\boldsymbol{\sigma}_P^b$$

with $b \equiv k - n + 1 \mod 2$. This shows that π is a product of at most n symmetries. $\square$

Remark 6.6 *Notice that if $k-n$ is odd, the proof of Proposition 6.5 shows that $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) \leq n-1$. As we will see, this can be used to derive $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) \leq n-1$ for all $\boldsymbol{\pi} \in \mathrm{PO}_S$ if n is odd and V non-singular.*

Assume (V, Q) is non-singular and $P_1, \ldots, P_n$ is an orthogonal splitting of V. Then $\boldsymbol{\sigma}_{P_1} \circ \ldots \circ \boldsymbol{\sigma}_{P_n} = \mathrm{id}_{\mathbf{P}V}$. Indeed, if $P_i = \langle p_i \rangle$, then

$$\sigma_{P_j}(p_i) = \begin{cases} p_i & \text{if } i \neq j \\ -p_i & \text{if } i = j \end{cases} \quad \Rightarrow \quad (\sigma_{P_1} \circ \ldots \circ \sigma_{P_n})(p_i) = -p_i$$

for all $i \leq n$. Hence, $\sigma_{P_1} \circ \ldots \circ \sigma_{P_n} = -\mathrm{id}_V$ and $\boldsymbol{\sigma}_{P_1} \circ \ldots \circ \boldsymbol{\sigma}_{P_n} = \mathrm{id}_{\mathbf{P}V}$.

Therefore, if n is odd and $\boldsymbol{\pi} \in \mathrm{PO}_S$, then $\boldsymbol{\pi}$ can be written as a product of an even amount of symmetries by appending $\boldsymbol{\sigma}_{P_1} \circ \ldots \circ \boldsymbol{\sigma}_{P_n}$ whenever the number of symmetries of $\boldsymbol{\pi}$ was odd. Hence, we get $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) \leq n-1$.

Notice that the existence of a hyperplane H with no singular points in Proposition 6.5 is very restrictive. However, it can be achieved.

Example 6.7 *Let $V = \mathbb{R}^n$, $b \in \mathbb{R}$ and define the quadratic form*

$$\mathrm{Q}_b(x) = x_1^2 + \ldots + x_{n-1}^2 + b x_n^2$$

for $x = (x_1, \ldots, x_n)^\intercal$. One easily sees that $H = \langle x_1, \ldots, x_{n-1} \rangle$ is anisotropic and, hence, satisfies the condition in Proposition 6.5. The signature of (V, Q_b) is equal to $[0, n, 0]$, $[0, n-1, 1]$, $[1, n-2, 0]$ if $b > 0$, $b = 0$, $b < 0$ respectively.

The signature $[h, a, r]$ of V in Example 6.7 satisfies $h + r \leq 1$. This is necessary. Indeed, if $h + r > 1$, then there is an isotropic subspace $U \subset V$ with $\dim(U) = 2$. Hence, any hyperplane $H \subset V$ satisfies $H \cap U \neq \{\mathbf{0}\}$ and contains at least one singular point.

Generally, the condition $h + r \leq 1$ does not guarantee an anisotropic hyperplane. If $\mathcal{K}$ is algebraically closed, then any 2-dimensional subspace contains singular points by Remark 3.8. Hence, $\mathbb{C}^3$ with $\mathrm{Q}((x_1, x_2, x_3)^\intercal) = x_1^2 + x_x^2 + x_3^2$ has signature $[1, 1, 0]$ and does not contain an anisotropic hyperplane.

However, if $\mathcal{K} = \mathbb{R}$, then $h + r \leq 1$ is enough. We clearly have an anisotropic hyperplane if $h = 0$ and $r \leq 1$. Assume V is non-singular and has index 1, hence, the signature is $[1, n-2, 0]$. Consider the hyperbolic splitting

$$V = H(x, y) \perp \underbrace{\langle p_1 \rangle \perp \ldots \perp \langle p_{n-2} \rangle}_{A}.$$

The subspace A is anisotropic. We want to show that $\mathrm{Q}(p_1), \ldots, \mathrm{Q}(p_{n-2})$ have the same sign. Otherwise, if $\mathrm{Q}(p_i) < 0 < \mathrm{Q}(p_j)$ for some $i \neq j$, we see that $\mathbf{0} \neq p = \alpha p_i + p_j \in A$ with $\alpha = \sqrt{-\mathrm{Q}(p_j)/\mathrm{Q}(p_i)}$ is isotropic, which is a contradiction. Hence, $\mathrm{Q}(p_1), \ldots, \mathrm{Q}(p_{n-2})$ have the same sign. One easily checks that

$$H = \begin{cases} \langle x+y \rangle \perp A & \text{if } \mathrm{Q}(p_1), \ldots, \mathrm{Q}(p_{n-2}) > 0 \\ \langle x-y \rangle \perp A & \text{if } \mathrm{Q}(p_1), \ldots, \mathrm{Q}(p_{n-2}) < 0 \end{cases}$$

is an anisotropic hyperplane since $\mathrm{Q}(x-y) = -1 < 0 < 1 = \mathrm{Q}(x+y)$.

With these considerations, we rediscover a special case of the bound in Theorem 5.5 using Proposition 6.5 and Remark 6.6.

Corollary 6.8 *If* $(\mathbb{R}^n, \mathrm{Q})$ *is a non-singular quadratic space with index at most 1, then for any* $\boldsymbol{\pi} \in \mathrm{PO}_S$ *we have*

$$\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) \leq \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}.$$

6.3 Theorem 10

We want to extend Theorem 1.2 in Chapter 1 to collineations of non-singular quadratic spaces of dimension 3.

Theorem 6.9 *If* (V, Q) *is a non-singular quadratic space over* $\mathcal{K}$ *with* $\mathrm{char}(\mathcal{K}) \neq 2$ *and* $\dim(V) = 3$*, then every* $\boldsymbol{\pi} \in \mathrm{PO}_S$ *is a product of two symmetries* $\boldsymbol{\pi} = \boldsymbol{\sigma}_P \circ \boldsymbol{\sigma}_Q$*. Further, if* $\boldsymbol{\pi} \neq \mathrm{id}$ *the line* $\overline{PQ}$ *is unique and for every* $P' \in \overline{PQ}$ *there exist exactly one* $Q' \in \overline{PQ}$ *such that* $\boldsymbol{\pi} = \boldsymbol{\sigma}_{P'} \circ \boldsymbol{\sigma}_{Q'}$*.*

Proof As V is non-singular and $\dim(V) = 3$, Theorem 5.5 says $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) \leq 2$.

If $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) = 1$, then $\boldsymbol{\pi} = \boldsymbol{\sigma}_X$ for a point $X \in \mathbf{P}V$. Let $P, Q \in \mathbf{P}V$ be points such that $X \perp P \perp Q = V$ is an orthogonal splitting of V. This is possible because one can choose X in the first step (5.1) of the splitting. Therefore, $\boldsymbol{\pi} = \boldsymbol{\sigma}_X = \boldsymbol{\sigma}_P \circ \boldsymbol{\sigma}_Q$ by Remark 6.6.

This shows the existence of the points P, Q since $\boldsymbol{\pi} = \mathrm{id}_{\mathbf{P}V}$ if $\mathrm{l}_{\mathrm{PO}_S}(\boldsymbol{\pi}) = 0$ and, hence, we can choose any $P = Q$ non-singular.

If $\mathrm{id} \neq \boldsymbol{\pi} = \boldsymbol{\sigma}_P \circ \boldsymbol{\sigma}_Q = \boldsymbol{\sigma}_{P'} \circ \boldsymbol{\sigma}_{Q'}$, then P, Q and P', Q' must be distinct and, hence, linearly independent. Lemma 4.21 now shows

$$\pi = \sigma_P \circ \sigma_Q \Rightarrow \mathrm{Fix}(\pi)^\perp = P \oplus Q$$
$$\pi' = \sigma_{P'} \circ \sigma_{Q'} \Rightarrow \mathrm{Fix}(\pi')^\perp = P' \oplus Q'$$

and we can conclude $\overline{PQ} = \overline{P'Q'}$ if $\pi = \pi'$. Indeed, we have $\pi = \pm\pi'$ as they induce the same collineation. However, $\pi = -\pi'$ can not be the case, since $\det(\pi) = 1 \neq -1 = \det(-\pi')$.

If $P' \in \overline{PQ}$, then $\boldsymbol{\sigma}_{P'} \circ \boldsymbol{\sigma}_P \circ \boldsymbol{\sigma}_Q = \boldsymbol{\sigma}_{Q'}$ for a unique $\mathrm{Q}' \in \overline{PQ}$ by Theorem 6.1, which concludes the proof. $\square$

Remark 6.10 *If $\mathcal{K} = \mathbb{R}$ in Theorem 6.9, then we can use Corollary 6.8 instead of Theorem 5.5.*

If the quadric **P**Q is non-trivial, $\mathrm{PO}_S \simeq \hat{\mathrm{O}}$ holds and we immediately get the following corollary.

Corollary 6.11 *If* $\mathbf{P}V$ *is a non-trivial quadric of a non-singular 3-dimensional quadratic space over* $\mathcal{K}$ *with* $\mathrm{char}(\mathcal{K}) \neq 2$*, then every* $\hat{\boldsymbol{\pi}} \in \hat{\mathrm{O}}$ *is a product of two symmetries* $\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\sigma}}_P \circ \hat{\boldsymbol{\sigma}}_Q$*. Further, if* $\hat{\boldsymbol{\pi}} \neq \mathrm{id}$ *the line* $\overline{PQ}$ *is unique and for every* $P' \in \overline{PQ}$ *there exist exactly one* $Q' \in \overline{PQ}$ *such that* $\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\sigma}}_{P'} \circ \hat{\boldsymbol{\sigma}}_{Q'}$*.*

Remark 6.12 *In [13] Buekenhout considers an oval in a projective plane, that is a set of points* $\mathcal{O}$ *such that every line intersects* $\mathcal{O}$ *in at most two points and for every point* $P \in \mathcal{O}$ *there exists exactly one tangent through* P*. Hence, symmetries* $\hat{\boldsymbol{\sigma}}_P$ *for points* $P \notin \mathcal{O}$ *and the group* $\hat{\mathrm{O}}$ *are well-defined.*

Buekenhout shows in a geometric way that if the oval $\mathcal{O}$ *satisfies the Pascal property, then every* $\boldsymbol{\pi} \in \hat{\mathrm{O}}$ *can be written as a product of two symmetries. This can be used to prove Corollary 6.11 with purely geometric reasoning.*

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Appendix A

Appendix

A.1 Dimension of the orthogonal space

Let V be a finite-dimensional vector space over $\mathcal{K}$. The set

$$\hat{V} = \{f : V \to \mathcal{K} \text{ linear}\}$$

is *the dual space of* V. It is a vector space over $\mathcal{K}$ and a map $f \in \hat{V}$ is called a *linear functional*. We will use the fact that $\dim(V) = \dim(\hat{V})$ and refer to [5] for the proof and more details.

Lemma A.1 *If* (V, f) *is a non-singular finite-dimensional orthogonal space and* $U \subset V$ *a subspace, then* $\dim(V) = \dim(U) + \dim(U^{\perp})$.

Proof Consider the map $i : V \to \hat{V}$ which sends $x \mapsto f_x(y) = f(x, y)$. One easily sees that f_x is a linear functional and that i is a linear map. Since V is non-singular, we see

$$\begin{aligned} \ker(i) &= \{x \in V : f_x \equiv 0\} \\ &= \{x \in V : f(x, v) = 0, \forall v \in V\} = \mathrm{rad}(V) = \{\mathbf{0}\}. \end{aligned}$$

As V is finite-dimensional, we have $\dim(V) = \dim(\hat{V})$ and, hence, i is bijective.

Consider now the map $r : \hat{V} \to \hat{U}$ which sends the functional f to the restriction $f|_U$. This map is well-defined and linear. One can show that r is surjective. Indeed, if $\tilde{f} \in \hat{U}$, we can define $f \in \hat{V}$ such that $f|_U = \tilde{f}$ and $f|_W \equiv 0$ for a complementary subspace $U \oplus W = V$. One checks that $f \in \hat{V}$ and $r(f) = \tilde{f}$.

Since i and r are surjective, the composition $i \circ r$ is surjective as well and, hence, $\mathrm{im}(i \circ r) = \hat{U}$. On the other hand, we have

$$\ker(i \circ r) = \{x \in V : f(x, u) = 0, \forall u \in U\} = U^{\perp}$$

and we can conclude the lemma as Remark 2.14 shows

$$\dim(U^{\perp}) = \dim(V) - \dim(\hat{U}) = \dim(V) - \dim(U). \qquad \square$$

A.2 Calculation of a product of 3 symmetries

Lemma A.2 *If $x, y \in V$ are non-isotropic vectors, then $\sigma_x(y)$ is non-isotropic and*

$$\sigma_x \circ \sigma_y \circ \sigma_x = \sigma_{\sigma_x(y)}.$$

Additionally, if $x + y$ is non-isotropic, then $\mathrm{Q}(y)x + \mathrm{Q}(x)y$ is non-isotropic and

$$\sigma_x \circ \sigma_{x+y} \circ \sigma_y = \sigma_{\mathrm{Q}(y)x+\mathrm{Q}(x)y}.$$

Proof First, notice that the statement holds if $x = y$. Hence, assume $x \neq y$.

The first equation follows quite easily as for any $z \in V$ we have

$$\begin{aligned}(\sigma_x \circ \sigma_y \circ \sigma_x)(z) &= \sigma_x\Big(\sigma_x(z) - \mathrm{Q}(y)^{-1} f(\sigma_x(z), y)y\Big) \\ &= z - \mathrm{Q}(y)^{-1} f\big(\sigma_x(z), y\big)\sigma_x(y) \\ &= z - \mathrm{Q}\big(\sigma_x(y)\big)^{-1} f\big(z, \sigma_x(y)\big)\sigma_x(y) = \sigma_{\sigma_x(y)}(z)\end{aligned}$$

by first expanding σ_y and using the isometry property of σ_x. Notice that $\sigma_x(y)$ is non-isotropic since $\mathrm{Q}(\sigma_x(y)) = \mathrm{Q}(y)$.

For the second equation consider $w = \mathrm{Q}(y)x + \mathrm{Q}(x)y$. Then

$$\begin{aligned}\mathrm{Q}(w) &= \mathrm{Q}\big(\mathrm{Q}(y)x + \mathrm{Q}(x)y\big) \\ &= \mathrm{Q}(y)^2\mathrm{Q}(x) + \mathrm{Q}(x)^2\mathrm{Q}(y) + \mathrm{Q}(y)\mathrm{Q}(x)f(x, y) \\ &= \mathrm{Q}(x)\mathrm{Q}(y)\big(\mathrm{Q}(y) + \mathrm{Q}(x) + f(x, y)\big) \\ &= \mathrm{Q}(x)\mathrm{Q}(x + y)\mathrm{Q}(y) \neq 0 \qquad \text{(A.1)}\end{aligned}$$

and, hence, w is non-isotropic.

Recall that σ_x is linear. Let $z \in V$ be an arbitrary vector. By expanding the product $\sigma_x \circ \sigma_{x+y} \circ \sigma_y$, we get

$$\begin{aligned}
& (\sigma_x \circ \sigma_{x+y} \circ \sigma_y)(z) \\
= \;& \sigma_y\Big(\sigma_x(z) - \underbrace{Q(x+y)^{-1} f\big(\sigma_x(z), x+y\big)}_{=:I}(x+y)\Big) \\
= \;& \sigma_y\Big(z - \underbrace{(\frac{f(z,x)}{Q(x)} + I)}_{=:I_x} x - Iy\Big) \\
= \;& z - I_x x - Q(y)^{-1} f(z - I_x x, y) y + Iy \\
= \;& z - I_x x - \Big(\frac{f(z,y)}{Q(y)} - \frac{f(x,y)}{Q(y)} I_x - \big(I_x - \frac{f(z,x)}{Q(x)}\big)\Big) y && \text{(A.2)} \\
= \;& z - I_x x - \underbrace{\Big(\frac{f(z,y)}{Q(y)} + \frac{f(z,x)}{Q(x)} - (\frac{f(x,y)}{Q(y)} + 1) I_x\Big)}_{=:I_y} y \\
= \;& z - I_x x - I_y y && \text{(A.3)}
\end{aligned}$$

where we used $I = I_x - \frac{f(z,x)}{Q(x)}$ in step (A.2). Since σ_x is a linear isometry, we see

$$\begin{aligned}
Q(x) f\big(\sigma_x(z), x+y\big) &= Q(x) f\big(z, \sigma_x(x+y)\big) \\
&= Q(x) f\big(z, -x + y - Q(x)^{-1} f(y,x) x\big) \\
&= f\Big(z, y - \big(Q(x) + f(x,y)\big) x\Big). && \text{(A.4)}
\end{aligned}$$

We can now calculate I_x by combining (A.1) and (A.4). Indeed,

$$\begin{aligned}
\frac{Q(w)}{Q(y)} I_x &= Q(x) Q(x+y) \Big(\frac{f(z,x)}{Q(x)} + \frac{f\big(\sigma_x(z), x+y\big)}{Q(x+y)}\Big) \\
&= Q(x+y) f(z,x) + f\Big(z, y - \big(Q(x) + f(x,y)\big) x\Big) \\
&= f\Big(z, Q(x+y) x + Q(x) y - Q(x) x - f(y,x) x\Big) \\
&= f\Big(z, Q(y) x + Q(x) y\Big) = f(z,w). && \text{(A.5)}
\end{aligned}$$

Hence, $Q(w)I_x = Q(y)f(z,w)$ and combined with Eq. (A.1) we see

$$
\begin{aligned}
Q(w)I_y &= Q(w)\Big(\frac{f(z,y)}{Q(y)} + \frac{f(z,x)}{Q(x)} - (\frac{f(x,y)}{Q(y)} + 1)I_x\Big) \\
&= Q(x+y)\Big(Q(x)f(z,y) + Q(y)f(z,x)\Big) - \Big(\frac{f(x,y)}{Q(y)} + 1\Big)Q(y)f(z,w) \\
&= Q(x+y)f(z,w) - \big(f(x,y) + Q(y)\big)f(z,w) \\
&= \big(Q(x+y) - Q(y) - f(x,y)\big)f(z,w) \\
&= Q(x)f(z,w)
\end{aligned}
\tag{A.6}
$$

because $f(x,y) = Q(x+y) - Q(x) - Q(y)$. Rearranging Eqs. (A.5) and (A.6) gives us

$$
I_x = Q(w)^{-1}f(z,w)Q(y), \quad I_y = Q(w)^{-1}f(z,w)Q(x),
$$

which combined with (A.3) finally show

$$
\begin{aligned}
(\sigma_x \circ \sigma_{x+y} \circ \sigma_y)(z) &= z - I_x x - I_y y \\
&= z - Q(w)^{-1}f(z,w)\big(Q(y)x + Q(x)y\big) \\
&= z - Q(w)^{-1}f(z,w)w = \sigma_w(z).
\end{aligned}
$$

□

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Eigenständigkeitserklärung

Die unterzeichnete Eigenständigkeitserklärung ist Bestandteil jeder während des Studiums verfassten Semester-, Bachelor- und Master-Arbeit oder anderen Abschlussarbeit (auch der jeweils elektronischen Version).

Die Dozentinnen und Dozenten können auch für andere bei ihnen verfasste schriftliche Arbeiten eine Eigenständigkeitserklärung verlangen.

Ich bestätige, die vorliegende Arbeit selbständig und in eigenen Worten verfasst zu haben. Davon ausgenommen sind sprachliche und inhaltliche Korrekturvorschläge durch die Betreuer und Betreuerinnen der Arbeit.

Titel der Arbeit (in Druckschrift):

On Products of Symmetries fixing a Quadric and their minimal Length

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Ich nehme zur Kenntnis, dass die Arbeit mit elektronischen Hilfsmitteln auf Plagiate überprüft werden kann.

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Winterthur, 20.03.2023	A. Huber

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