### ITYM 2021 - Problem 6: Binomial Coefficients and Prime Numbers

Presented by Philémon France

## Compoundness: definitions

An integer  $n \ge 2$  is S-compound if for each  $1 \le k \le n-1$ ,  $\binom{n}{k}$  is divisible by one number from S (a set of integers).

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### Example of *S*-compoundness

```
n = 0:
n = 1 : 1 1
n = 2: 1 2 1
n = 3: 1 3 3 1
n = 4: 1 4 6 4 1
n = 5: 1 5 10 10 5 1
n = 6: 1 6 15 20 15 6 1
n = 7: 1 7
            21 35 35 21 7
```

Example with n = 6 and  $S = \{3, 5\}$ : n is S-compound

#### Useful tool: Lucas's Theorem

Let p a prime number. If  $n = \overline{a_\ell} \ \overline{a_{\ell-1} \ \dots \ a_0}^p$  and  $k = \overline{b_\ell} \ \overline{b_{\ell-1} \ \dots \ b_0}^p$  then

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#### Claim:

The only  $\{p\}$ -compound integers are the powers of p.

 $\triangleright$  we note  $n=\overline{a_\ell}\ \overline{a_{\ell-1}\ \dots\ \overline{a_0}^p}$  and we use Lucas's Theorem with specific k's to show that we must have  $a_0=a_1=\dots=a_{\ell-1}=0$  and  $a_\ell=1$ , which precisely means that  $n=p^\ell$ 

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## 1. b) 1-compoundness

#### **Direct corollary**:

an integer  $n \geqslant 2$  is 1-compound if and only if  $n = p^{\ell}$  for a certain prime number p and a certain  $\ell \geqslant 1$ .

S is a set of  $\ell \geqslant 1$  prime numbers.

**Proposition**: There exist infinitely many *S*-compound integers  $n \ge 2$ .

Main idea: S-compoundness is preserved by inclusion.

Therefore to prove that there are infinitely many S-compound integers  $n \ge 2$ , it's sufficient to find  $S' \subseteq S$  s.t. there are infinitely many S'-compound integers  $n \ge 2$ .

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We set  $A = \{n \ge 2 \mid \forall p \in S, p \nmid n\}$ , and we show

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# 3. a) Around 2-compoundness when $n = p^{\alpha} + 1$

**Proposition**: If  $n = p^{\alpha} + 1$  with p a prime number and  $\alpha \geqslant 1$ , then n is 2-compound.

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# 3. b) Around 2-compoundness when $n < p_s^{\alpha_s} + q(n)$

Here we have the prime factorisation  $n = \prod_{i=1}^{s} p_i^{\alpha_i}$  with  $p_1^{\alpha_1} < \ldots < p_s^{\alpha_s}$ .

We denote by q(n) the largest prime less than n.

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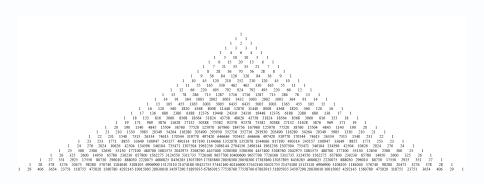
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## Thank for listening!



### When $n < q(n) + p_s^{\alpha_s}$ :

By symmetry, since  $\binom{n}{k} = \binom{n}{n-k}$ , it is sufficient to satisfy the divisibility condition for  $k \le n/2 \le n-k$ .

⊳ if q(n) > n - k, then q(n) is coprime to k! and (n - k)! and divides n!, so it divides  $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ 

ho if  $q(n) \leqslant n-k$ , then  $n-q(n) \geqslant k$  so  $k < p_s^{\alpha_s}$  and so by Lucas's Theorem :

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#### Another useful tool:

#### **Kummer's Theorem**

Let p a prime and non-negative integers  $n \ge k$ . Then  $\nu_p \binom{n}{k}$  is the number of carries when adding k and n-k is base p.

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