$\mathbb{ITYM}$ 2021 - Problem 9: **Wobbly Tables.**

Team France

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Abstract

This document is dealing with the ninth problem of the ITYM 2021.
The problem consist of placing a table on the floor of a room. The floor is parametrised by a surface $z = f(x, y)$, where $f : [-1, 1]^2 \to \mathbb{R}$ is a continuous map.
For now, the document is dealing just with the first question. But a lot of elements of the first question help to do the next questions.

Contents

Introduction

Question 1

In this question, we will try to ground some tables on the different posssible floors.

0.1 Square

The table is a square with sides of length $\sqrt{2}$.

0.1.1 The horizontal plane

The floor is the function

$$f_1 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto 0 \end{cases}$$

We can put each legs:

- $P_1 = (-1, 0, 0)$
- $P_2 = (1, 0, 0)$
- $P_3 = (0, -1, 0)$
- $P_4 = (0, 1, 0)$

$P_1P_2P_3P_4$ is a square with sides of length $\sqrt{2}$ and the table is *grounded*:

$$\forall i \in [1,4],\ z_i = f(x_i, y_i)$$

0.1.2 The inclined plane

We will now generalise with an inclined plane defined as follow:

$$f_2 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto ax + by, \text{ with } a, b \in \mathbb{R} \end{cases}$$

The plane Π of the square $P_1P_2P_3P_4$ should be $ax + by - z = 0$
Firstly we put P_1 in the corner. The coordinates of P_1 are now : $(1, 1, a+b)$
Now, we will take care of P_2 We want:

- (1) $P_1P_2 = \sqrt{2}$
- (2) $y_2 = y_1 = 1$
- (3) $z_2 = f(x_2, y_2) = ax_2 + by_2$

From the first equality, we obtain:

$$\sqrt{(1-x_2)^2 + (1-y_2)^2 + (a+b-ax_2-by_2)^2} = \sqrt{2}$$

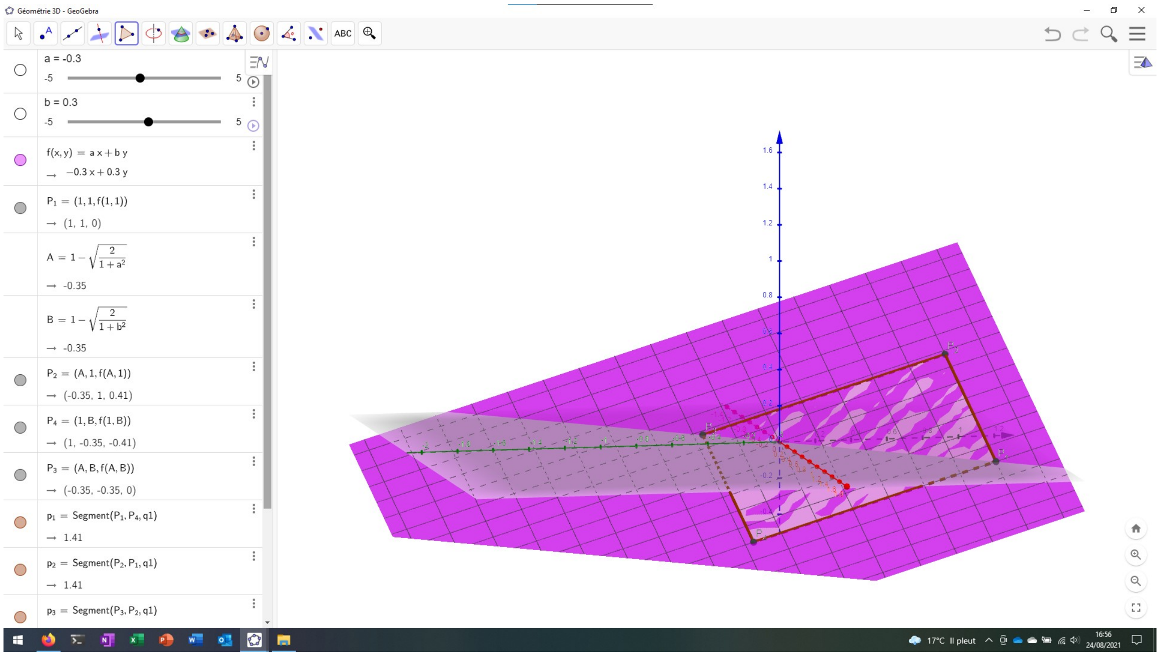
so, $(1-x_2)^2 + (1-y_2)^2 + (a+b-ax_2-by_2)^2 = 2$
but, with $y_2 = 1$, we can reduce:

$$(1-x_2)^2 + (a-ax_2)^2 = 2$$

and we have: $(1-x_2)^2 \times (a^2+1) = 2$
$1 - x_2 = \frac{\sqrt{2}}{\sqrt{a^2+1}}$ car $a^2 + 1 \geq 0$
we finally have $x_2 = 1 - \frac{\sqrt{2}}{\sqrt{a^2+1}}$
With the same process for P_4, where:

- (1) $P_1P_4 = \sqrt{2}$
- (2) $x_4 = x_1 = 1$
- (3) $z_4 = f(x_4, y_4) = ax_4 + by_4$

We obtain: $P_4 = (1, 1 - \frac{\sqrt{2}}{\sqrt{b^2+1}}, a + b \times (1 - \frac{\sqrt{2}}{\sqrt{b^2+1}})$
And the final point is $P_3 = (1 - \frac{\sqrt{2}}{\sqrt{a^2+1}}, 1 - \frac{\sqrt{2}}{\sqrt{b^2+1}}, a \times (1 - \frac{\sqrt{2}}{\sqrt{a^2+1}}) + b \times (1 - \frac{\sqrt{2}}{\sqrt{b^2+1}})$



1: Square on an inclined plane

0.1.3 Horse Saddle

$$f_3 : \left\{ \begin{array}{r} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto sxy, \text{ with } s \in \mathbb{R}^* \end{array} \right.$$

We can put the legs like this:

1. $P_1 = (0, 1, 0)$
2. $P_2 = (0, -1, 0)$
3. $P_3 = (-1, 0, 0)$
4. $P_4 = (1, 0, 0)$

And we have a *grounded* table because:

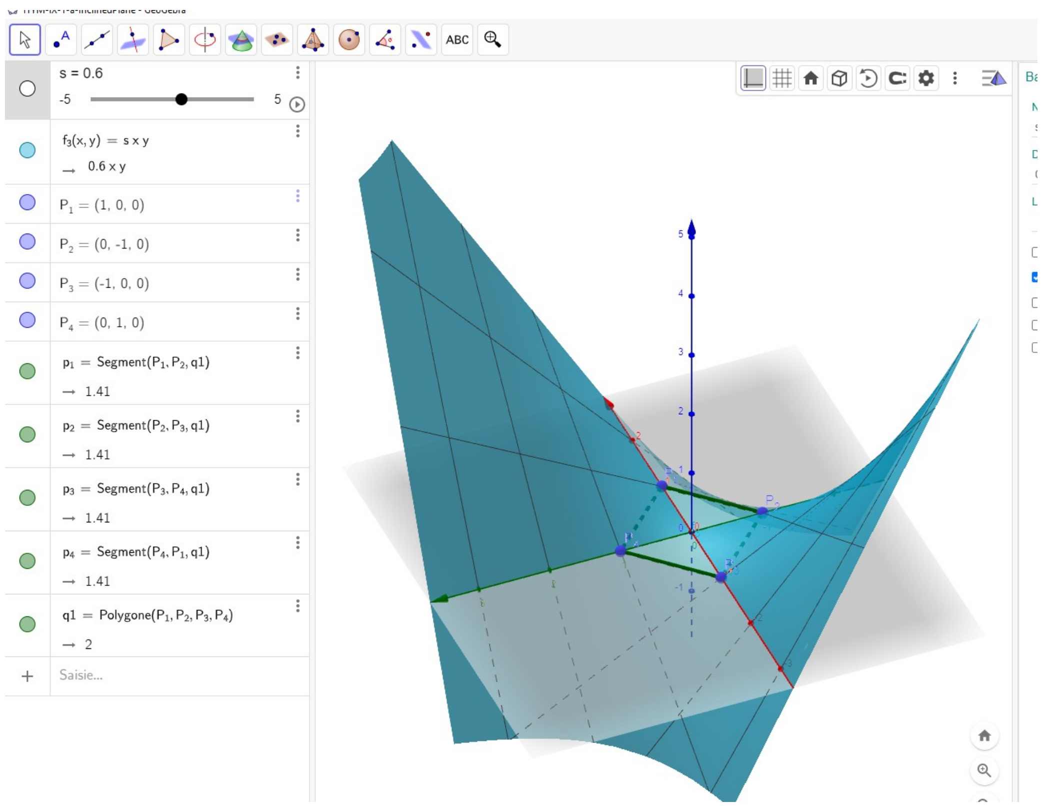
$$\forall i \in [1, 4],\ z_i = f(x_i, y_i)$$

0.1.4 Sphere

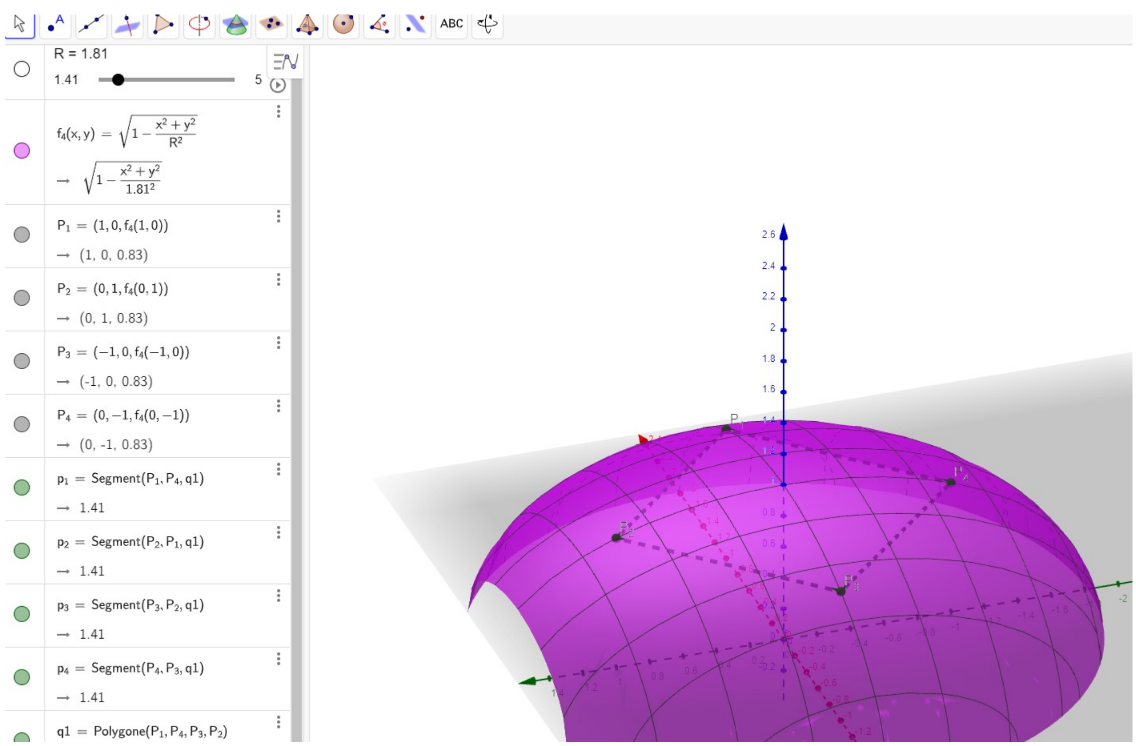
$$f_4 : \left\{ \begin{array}{r} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto \sqrt{1 - \frac{x^2+y^2}{R^2}}, \text{ with } R > \sqrt{2} \end{array} \right.$$

1. $P_1 = (0, 1, z)$
2. $P_2 = (0, -1, z)$
3. $P_3 = (-1, 0, z)$
4. $P_4 = (1, 0, z)$

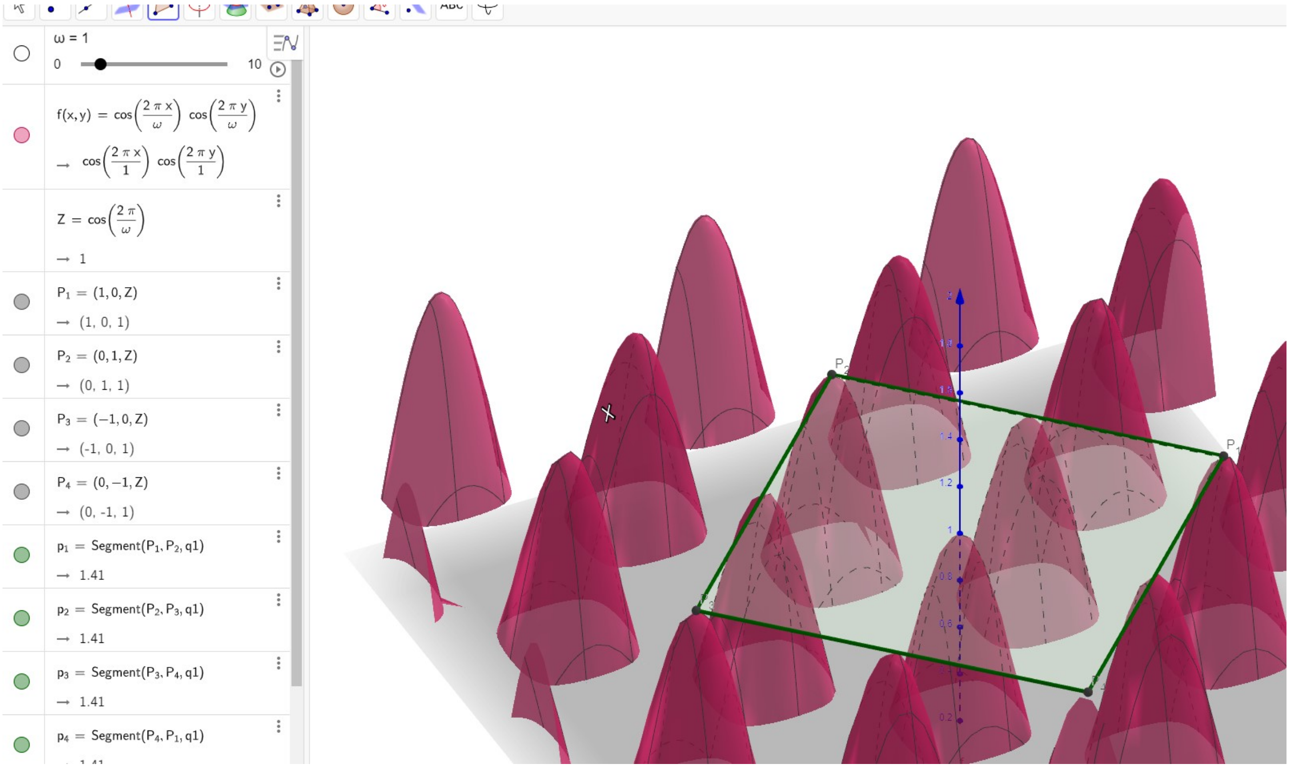
With $z = \sqrt{1 - \frac{1}{R^2}} = \sqrt{\frac{(R+1)(R-1)}{R^2}}$, the table is grounded.



2: Square on a saddle



3: Square on a sphere



4: Square on a double periodic surface

0.1.5 Double periodic surface

$$f_5 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto cos(\frac{2\pi x}{\omega})cos(\frac{2\pi y}{\omega}), \text{ with } \omega > \sqrt{2} \end{cases}$$

1. $P_1 = (0, 1, z)$
2. $P_2 = (0, -1, z)$
3. $P_3 = (-1, 0, z)$
4. $P_4 = (1, 0, z)$

We can see that $\forall X \in \mathbb{R}$, $cos(-X) = cos(X)$ and $cos(0) = 1$ With these properties, $z = cos(\frac{2\pi}{\omega})$, the table is grounded.

0.2 Rectangle

Now we have a **rectangle** with sides r and s such that $r^2 + s^2 = 4$. This implicate that $r \leq 2$ and $s \leq 2$ which is logical bedcause the surface is defined as a square area of dimensions 2×2.

0.2.1 The horizontal plane

This surface is a sub-case of the next surface with $a = 0$ and $b = 0$.

0.2.2 The inclined plane

$$f_2 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto ax + by, \text{ with } a, b \in \mathbb{R} \end{cases}$$

First we set $P_1 = (1, 1, f_2(1, 1)) = (1, 1, a + b)$

We want to place P_2 according to these hypotheses:

1. $P_1P_2 = r$
2. $y_1 = y_2 = 1$
3. $z_2 = f(x_2, y_2) = ax_2 + by_2$

From the first equality, we deduce:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = r$$

Using (1) and (2), we have, and by squaring each sides:

$$(x_1 - x_2)^2 + (a + b - ax_2 - b)^2 = r^2$$

i.e.

$$(1 - x_2)^2 + a^2(1 - x_2)^2 = r^2$$

so by factorising by $(1 - x_2)^2$, we obtain:

$$(1 - x_2)^2(1 + a^2) = r^2$$

We can divide each sides by $(1 + a^2) > 0$:

$$(1 - x_2)^2 = \frac{r^2}{1 + a^2}$$

$$x_2 = 1 - \sqrt{\frac{r^2}{1 + a^2}}$$

In conclusion, we have the point $P_2 = (1 - \sqrt{\frac{r^2}{1+a^2}}, 1, a(1 - \sqrt{\frac{r^2}{1+a^2}}) + b)$
We can apply the same process for P_3 with these hypotheses:

1. $P_1P_4 = s$
2. $x_1 = x_4 = 1$
3. $z_4 = f(x_4, y_4) = ax_4 + by_4$

From the first equality, we deduce:

$$\sqrt{(x_1 - x_4)^2 + (y_1 - y_4)^2 + (z_1 - z_4)^2} = s$$

Using (1) and (2), we have, and by squaring each sides:

$$(y_1 - y_4)^2 + (a + b - a - b \times y_4)^2 = s^2$$

i.e.

$$(1 - y_4)^2 + b^2(1 - y_4)^2 = s^2$$

so by factorising by $(1 - y_4)^2$, we obtain:

$$(1 - y_4)^2(1 + b^2) = s^2$$

We can divide each sides by $(1 + b^2) > 0$:

$$(1 - y_4)^2 = \frac{s^2}{1 + b^2}$$

$$y_4 = 1 - \sqrt{\frac{s^2}{1+b^2}}$$

In conclusion, we have the point $P_4 = (1, 1 - \sqrt{\frac{s^2}{1+b^2}}, a + b \times (1 - \sqrt{\frac{s^2}{1+b^2}}))$
For P_3, the coordinates are: $P_3 = (1 - \sqrt{\frac{r^2}{1+a^2}}, 1 - \sqrt{\frac{s^2}{1+b^2}}, a(1 - \sqrt{\frac{r^2}{1+a^2}}) + b \times (1 - \sqrt{\frac{s^2}{1+b^2}}))$
We finaly have our square. We can note $R = 1 - \sqrt{\frac{r^2}{1+a^2}}$ and $S = 1 - \sqrt{\frac{s^2}{1+b^2}}$ To sum up, the coordiantes of the points are:

- $P_1(1, 1, a+b)$
- $P_2(R, 1, aR+b)$
- $P_3(R, S, aR+bS)$
- $P_4(1, S, a+bS)$

0.2.3 The horse saddle

We search $(a, b) \in {\mathbb{R}_+}^2$ such that:

- $P_1 = (a, b, f_3(a, b))$
- $P_2 = (b, a, f_3(b, a))$
- $P_3 = (-a, -b, f_3(-a, -b))$
- $P_4 = (-b, -a, f_3(-b, -a))$

Because in this situation, we have:

$$f_3(a, b) = f_3(b, a) = f_3(-b, -a) = f_3(-a, -b) = sab$$

So the table is grounded and we have a rectangle.
Now we can search a and b depending on r and s
We know that:

1. $P_1P_2 = r$
2. $P_1P_4 = s$

This is equivalent to:

$$\begin{cases} (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = r^2 \\ (x_4 - x_1)^2 + (y_4 - y_1)^2 + (z_4 - z_1)^2 = s^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} (b-a)^2 + (a-b)^2 = r^2 \\ (-b-a)^2 + (-a-b)^2 = s^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2(a-b)^2 = r^2 \\ 2(a+b)^2 = s^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2(a^2 - 2ab + b^2) = r^2 \ (L_1) \\ 2(a^2 + 2ab + b^2) = s^2 \ (L_2) \end{cases}$$

$L_1 \leftarrow L_1 + L_2$
$L_2 \leftarrow L_2 - L_1$

$$\Leftrightarrow \begin{cases} 4(a^2 + b^2) = r^2 + s^2) \\ 8ab = s^2 - r^2 \end{cases}$$

If $r^2 = s^2$, we return to the case of the square. Here, $r \neq s$, so $s^2 - r^2 \neq 0$
Then, $a \neq 0$ and $b \neq 0$ due to the second equality.
So $a = \frac{s^2-r^2}{8b}$ (we have $b \neq 0$).
We can substitute a in the first equality, we obtain:

$$(E) : 4 \times \left[\left(\frac{s^2 - r^2}{8b} \right)^2 + b^2 \right] = r^2 + s^2$$

$$\begin{aligned} \text{b is solution of (E)} &\Leftrightarrow \frac{(s^2 - r^2)^2}{16 \times b^2} + 4 \times b^2 - (r^2 + s^2) = 0 \\ &\Leftrightarrow \frac{(s^2 - r^2)^2}{16 \times b^2} + 4 \times b^2 - (r^2 + s^2) = 0 \\ &\Leftrightarrow 4 \times b^4 - (r^2 + s^2) \times b^2 + \frac{(s^2 - r^2)^2}{16} = 0 \quad (b \neq 0) \end{aligned}$$

Let's set $B = b^2$

$$b \text{ is solution of (E)} \Leftrightarrow B \text{ is solution of (E')}$$

With: $(E') : 4 \times B^2 - (r^2 + s^2) \times B + \frac{(s^2-r^2)^2}{16} = 0$ Let Δ be the discriminant of this equation. We have:

$$\Delta = (r^2 + s^2)^2 - 4 \times 4 \times \frac{(s^2 - r^2)^2}{16}$$

$$\Delta = (r^2 + s^2)^2 - (s^2 - r^2)^2$$

$$\Delta = (r^2 + s^2 + s^2 - r^2) \times (r^2 + s^2 - s^2 + r^2)$$

$$\Delta = (2 \times s^2) \times (2 \times r^2)$$

$$\Delta = (2rs)^2$$

We have $\Delta \geq 0$ and $\sqrt{\Delta} = 2rs$
The solutions are:

$$B = \frac{r^2 + s^2 \pm (2rs)}{8}$$

$$\Leftrightarrow b^2 = \frac{(r \pm s)^2}{8}$$

$$\Leftrightarrow b = \frac{r \pm s}{2\sqrt{2}} \quad (b > 0)$$

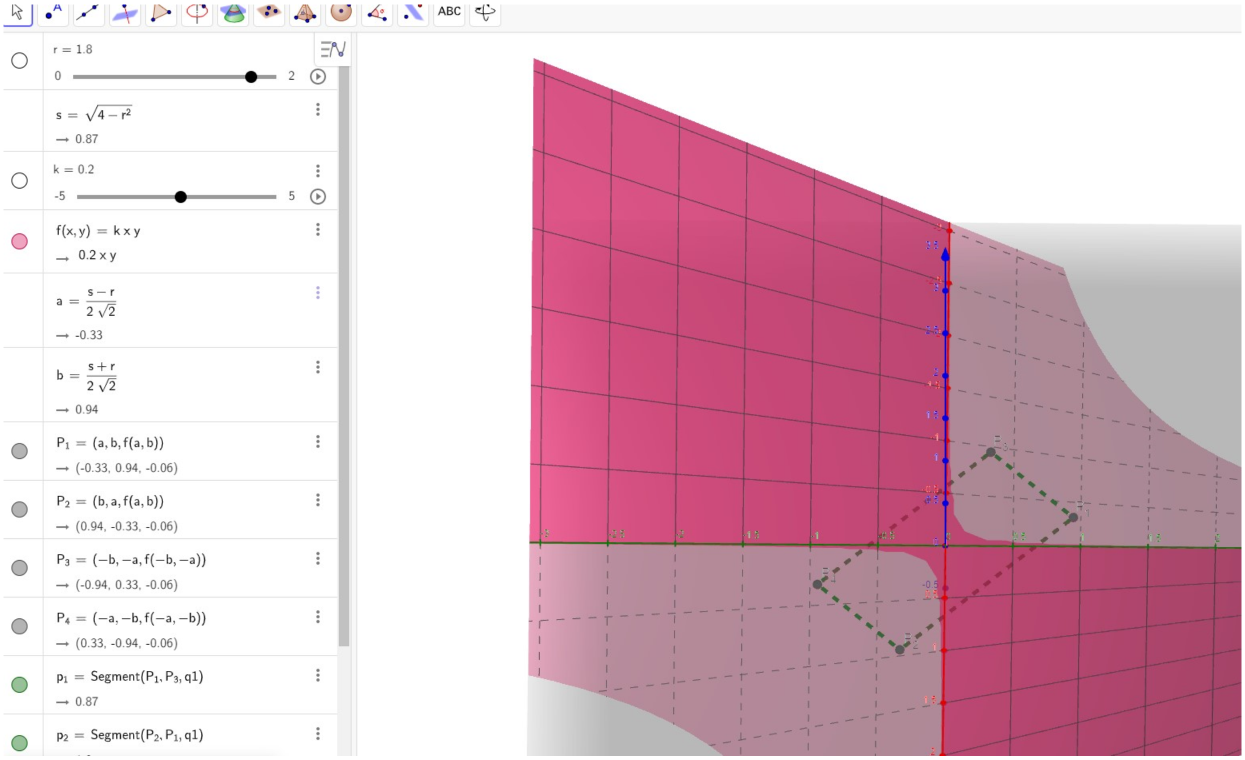
Then, $a = \frac{1}{8b} \times (s^2 - r^2)$

$a = \frac{2\sqrt{2}}{8} \times \frac{s^2-r^2}{r \pm s}$

$a = \frac{1}{2\sqrt{2}} \times \frac{(s^2-r^2) \times (r \mp s)}{(r \pm s) \times (r \mp s)}$

$a = \frac{s \pm r}{2\sqrt{2}}$

In conclusion, we have: $(a, b) = (\frac{s+r}{2\sqrt{2}}, \frac{s-r}{2\sqrt{2}})$ ou $(a, b) = (\frac{s-r}{2\sqrt{2}}, \frac{s+r}{2\sqrt{2}})$



5: Rectangle on a saddle

0.2.4 The sphere

We have this surface:

$$f_4 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto \sqrt{1 - \frac{x^2+y^2}{R^2}}, \text{ with } R > \sqrt{2} \end{cases}$$

We name Ω the center of the rectangle. Let $x_\Omega = y_\Omega = 0$ By applying Pythagoras theorem in $P_1P_2P_4$, we obtain:

$$P_1P_2^2 + P_1P_4^2 = P_2P_4$$
$$\text{i.e. } s^2 + r^2 = (2P_2\Omega)^2$$
$$\text{i.e. } 4 = 4P_2\Omega^2$$

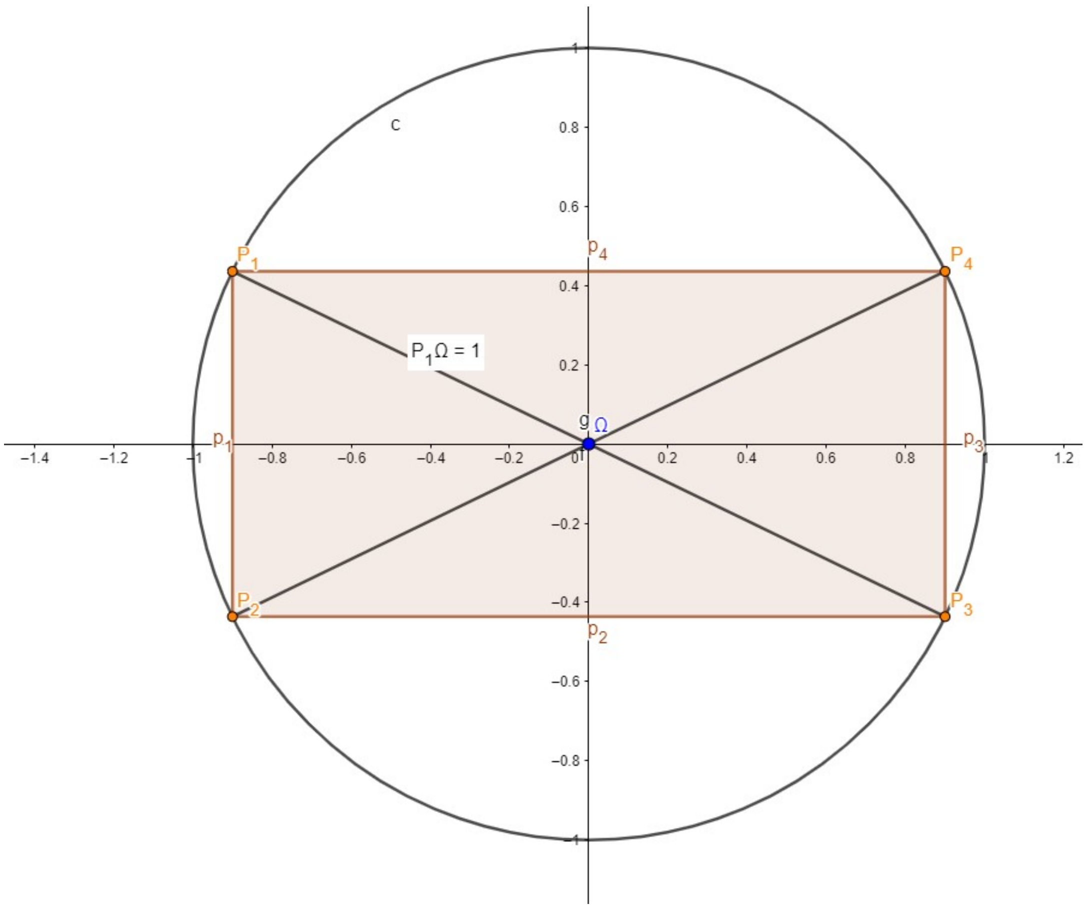
so, $P_2\Omega = 1$

The rectangle is in the circle $C : x^2 + y^2 = 1$ and $z = z_\Omega$
Now we want to calculate z for each leg of the table.
We want the leg on the surface defined as :

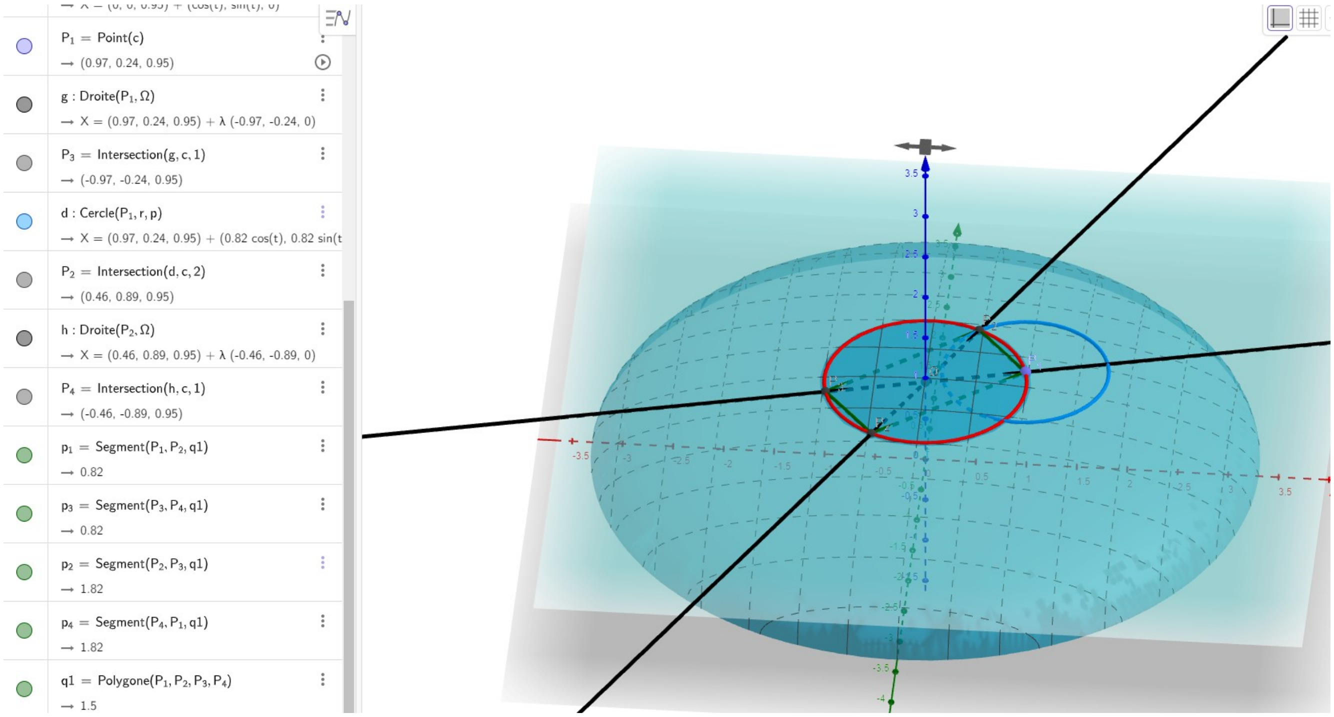
$$f_4(x,y) = \sqrt{1 - \frac{x^2+y^2}{R^2}}, \text{ with } R > \sqrt{2}$$

But we have $x^2 + y^2 = 1$

Then $z_1 = z_2 = z_3 = z_4 = \sqrt{1 - \frac{1}{R^2}} = \frac{\sqrt{(R+1)\times(R-1)}}{R}$



6: Rectangle



7: Rectangle on a Sphere

And we have our table that is *stabilised.*

0.2.5 The double periodic surface

$$f_5 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto cos(\frac{2\pi x}{\omega})cos(\frac{2\pi y}{\omega}), \text{ with } \omega > \sqrt{2} \end{cases}$$

Using the fact that *cosinus* is an **even** function, we can take the points:

1. $P_1 = (-\frac{r}{2}, -\frac{s}{2}, z_1)$
2. $P_2 = (\frac{r}{2}, -\frac{s}{2}, z_2)$
3. $P_3 = (\frac{r}{2}, \frac{s}{2}, z_3)$
4. $P_4 = (-\frac{r}{2}, \frac{s}{2}, z_4)$

And we have $z_1 = z_2 = z_3 = z_4 = cos(\frac{r\pi}{\omega}) \times \cos(\frac{s\pi}{\omega})$ So the table is stabilised.

0.3 Rhombus

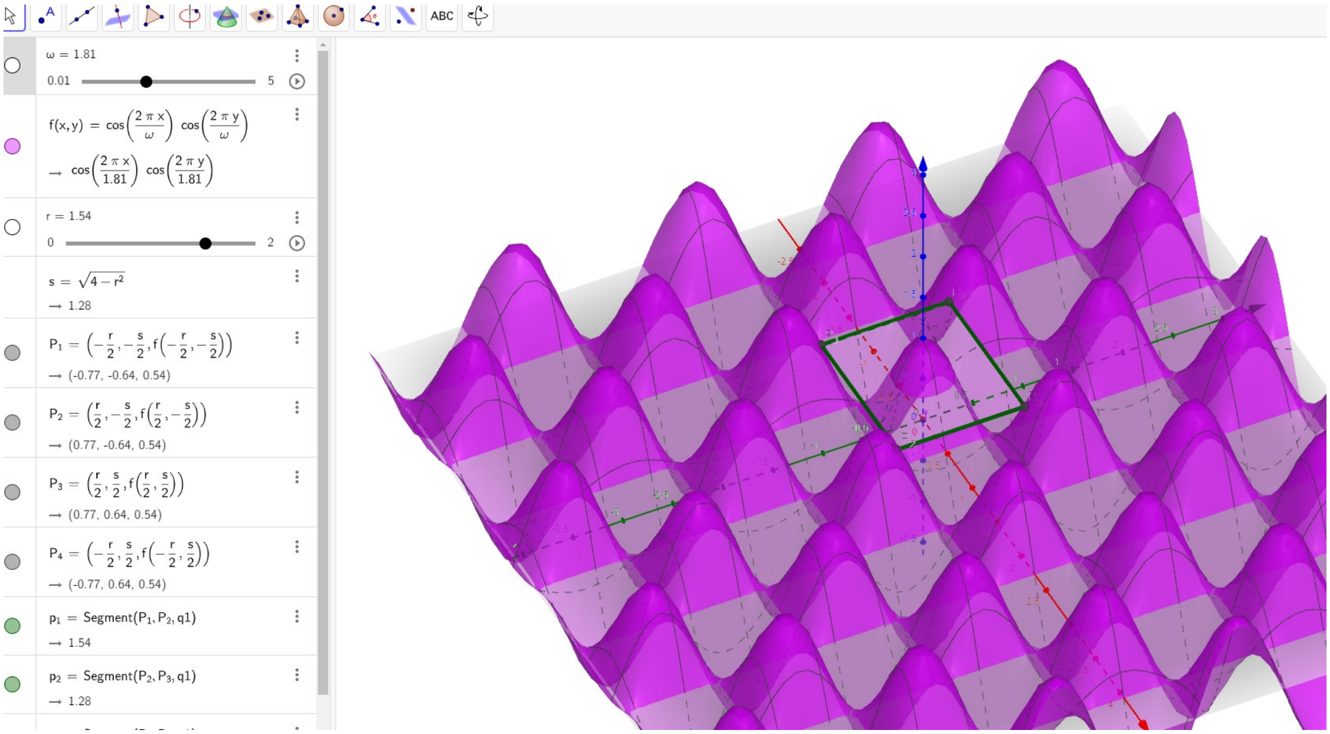
We have a rhombus with an axis of length 2 and an axis of lenght l, where $0 < l < 2$. Let Ω be the center of the rhombus.

0.3.1 Horizontal plane

$$f_1 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto 0 \end{cases}$$

Then we place the points as follow:

1. $P_1 = (-1, 0, 0)$



8: Rectangle on a double periodic surface

2. $P_2 = (0, -\frac{l}{2}, 0)$
3. $P_1 = (1, 0, 0)$
4. $P_4 = (0, +\frac{l}{2}, 0)$

And we have a stabilised rhombus on the horizontal plane.

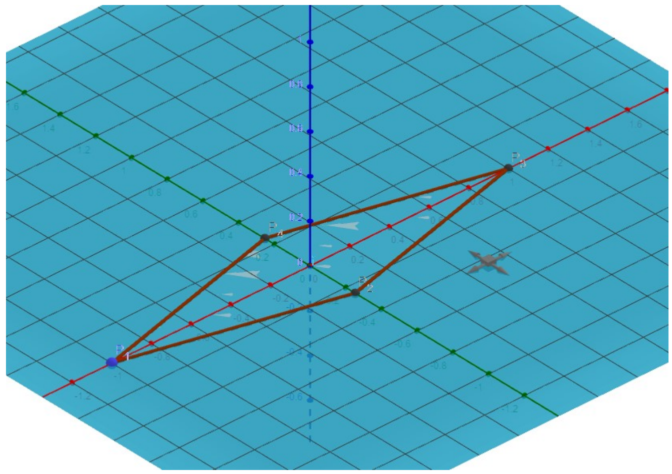
0.3.2 Inclined Plane

$$f_2 : \left\{ \begin{array}{r} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto ax+by, \text{ with } a,b \in \mathbb{R}^2 \end{array} \right.$$

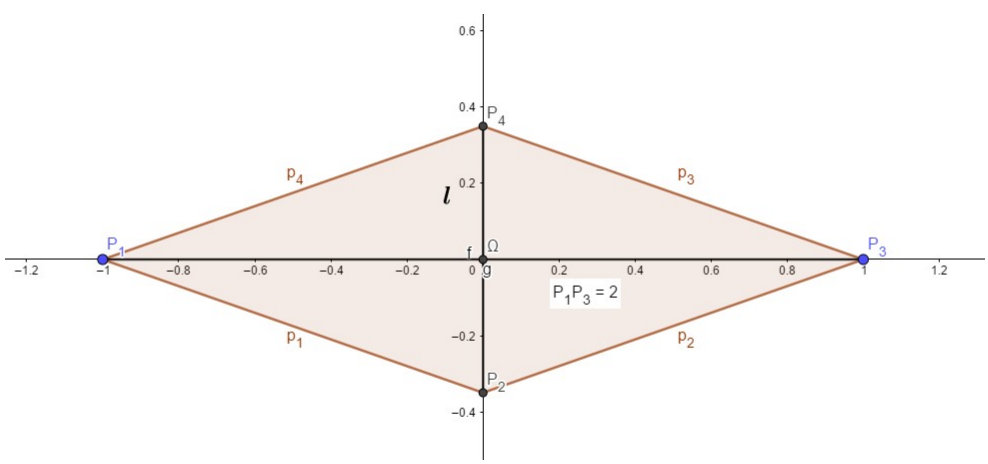
Let's name Ω the center of our rhombus.
We will place Ω on the (Oz) axis. Then $x_\Omega = y_\Omega = 0$.
Ω is on the floor, so $z_\Omega = f_2(x_\Omega, y_\Omega) = a \times 0 + b \times 0 = 0$
$\Omega = (0,0,0)$
We will search the coordinates of $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ that satisfies these propositions:

- $z_1 = f_2(x_1, y_1)$
- $z_2 = f_2(x_2, y_2)$
- $\Omega P_1 = 1$ because Ω is the middle of $[P_1 P_3]$ and $P_1 P_3 = 2$
- $\Omega P_2 = \frac{l}{2}$ because Ω is the middle of $[P_2 P_4]$ and $P_2 P_4 = l$
- $\overrightarrow{\Omega P_1} . \overrightarrow{\Omega P_2} = \overrightarrow{0}$

We obtain the system (S):



9: Rhombus on an horizontal plane



10: Rhombus

$$(S) : \begin{cases} z_1 = a \times x_1 + b \times y_1 \\ z_2 = a \times x_2 + b \times y_2 \\ (x_1 - x_\Omega)^2 + (y_1 - y_\Omega)^2 + (z_1 - z_\Omega)^2 = 1 \\ (x_2 - x_\Omega)^2 + (y_2 - y_\Omega)^2 + (z_2 - z_\Omega)^2 = \frac{l^2}{4} \\ (x_1 - x_\Omega) \times (x_2 - x_\Omega) + (y_1 - y_\Omega) \times (y_2 - y_\Omega) + (z_1 - z_\Omega) \times (z_2 - z_\Omega) = 0 \end{cases}$$

But $x_\Omega = y_\Omega = z\Omega = 0$

$$(S) \Leftrightarrow \begin{cases} z_1 = a \times x_1 + b \times y_1 \\ z_2 = a \times x_2 + b \times y_2 \\ x_1^2 + y_1^2 + (ax_1 + by_1)^2 = 1 \\ x_2^2 + y_2^2 + (ax_2 + by_2)^2 = \frac{l^2}{4} \\ x_1 \times x_2 + y_1 \times y_2 + (ax_1 + by_1) \times (ax_2 + by_2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} z_1 = a \times x_1 + b \times y_1 \\ z_2 = a \times x_2 + b \times y_2 \\ x_1^2 \times (a^2 + 1) + y_1^2 \times (b^2 + 1) + 2abx_1y_1 = 1 \\ x_2^2 \times (a^2 + 1) + y_2^2 \times (b^2 + 1) + 2abx_2y_2 = \frac{l^2}{4} \\ x_1x_2 \times (a^2 + 1) + y_1y_2 \times (b^2 + 1) + ab \times (x_1y_2 + x_2y_1) = 0 \end{cases}$$

Now we will take arbitrarily $y_1 = 0$:

$$(S) \Leftrightarrow \begin{cases} z_1 = a \times x_1 & (1) \\ z_2 = a \times x_2 + b \times y_2 & (2) \\ x_1^2 \times (a^2 + 1) = 1 & (3) \\ x_2^2 \times (a^2 + 1) + y_2^2 \times (b^2 + 1) + 2abx_2y_2 = \frac{l^2}{4} & (4) \\ x_1x_2 \times (a^2 + 1) + ab \times (x_1y_2) = 0 & (5) \end{cases}$$

From (3), we can deduce that: $x_1 = \pm\sqrt{\frac{1}{a+1}}$. We take x_1 negativ so $x_1 = -\sqrt{\frac{1}{a+1}} < 0$. We can see that $x_1 \neq 0$
From (5), we deduce: $abx_1y_2 = -x_1x_2(a^2 + 1)$, so $y_2 = \frac{-x_2 \times (a^2+1)}{ab}$ (we suppose that $a \neq 0$ and $b \neq 0$ and we know that $x_1 \neq 0$
x_2 remains to be found.
We will use (4): $x_2^2 \times (a^2 + 1) + y_2^2 \times (b^2 + 1) + 2abx_2y_2 = \frac{l^2}{4}$
By replacing the value of y_2 by $\frac{-x_2 \times (a^2+1)}{ab}$, we obtain:

$$x_2^2 \times (a^2 + 1) + x_2^2 \times \frac{(a^2 + 1)^2}{(ab)^2} \times (b^2 + 1) - 2abx_2^2 \times \frac{a^2 + 1}{ab} = \frac{l^2}{4}$$

$$\Leftrightarrow x_2^2 \times (a^2 + 1) + x_2^2 \times \frac{(a^2 + 1)^2}{(ab)^2} \times (b^2 + 1) - 2x_2^2 \times (a^2 + 1) = \frac{l^2}{4}$$

$$\Leftrightarrow x_2^2 \times (a^2 + 1) \times \left[1 + \frac{(a^2 + 1)}{(ab)^2} \times (b^2 + 1) - 2\right] = \frac{l^2}{4}$$

$$\Leftrightarrow x_2^2 \times (a^2 + 1) \times \left[\frac{a^2 \times b^2 + a^2 + b^2 + 1 - (ab)^2}{(ab)^2}\right] = \frac{l^2}{4}$$

$$\Leftrightarrow x_2^2 \times (a^2 + 1) \times \left[\frac{a^2 + b^2 + 1}{(ab)^2}\right] = \frac{l^2}{4}$$

$$\Leftrightarrow x_2^2 = \frac{l^2 \times (ab)^2}{4 \times (a^2 + 1) \times (a^2 + b2 + 1)}$$

$$\Leftrightarrow x_2 = \pm \frac{l \times ab}{2 \times \sqrt{(a^2+1) \times (a^2+b^2+1)}}$$

We will take $x_2 < 0$.
So $x_2 = -\frac{l \times ab}{2 \times \sqrt{(a^2+1) \times (a^2+b^2+1)}}$
Then:
$y_2 = +\frac{l \times ab}{2 \times \sqrt{(a^2+1) \times (a^2+b^2+1)}} \times \frac{(a^2+1)}{ab} = \frac{l \times \sqrt{(a^2+1)}}{2 \times \sqrt{(a^2+b^2+1)}}$
and:

$$\begin{aligned}
z_2 &= ax_2 + by_2 && (1)\\
&= \frac{-l \times (a^2 b)}{2 \times \sqrt{(a^2+1) \times (a^2+b^2+1)}} + \frac{l \times b \times \sqrt{(a^2+1)}}{2 \times \sqrt{(a^2+b^2+1)}} && (2)\\
&= \frac{l \times b}{2 \times \sqrt{(a^2+b^2+1)}} \times \left(\sqrt{a^2+1} - \frac{a^2}{\sqrt{a^2+1}} \right) && (3)\\
&= \frac{l \times b}{2 \times \sqrt{(a^2+b^2+1)}} \times \left(\frac{a^2+1-a^2}{\sqrt{a^2+1}} \right) && (4)\\
&= \frac{l \times b}{2 \times \sqrt{(a^2+b^2+1)}} \times \left(\frac{1}{\sqrt{a^2+1}} \right) && (5)\\
&= \frac{l \times b}{2 \times \sqrt{(a^2+b^2+1)(a^2+1)}} && (6)
\end{aligned}$$

Now we have the coordinates of P_1 and P_2:

1. $P_1 = \left(-\sqrt{\frac{1}{a+1}}, 0, -a \times \sqrt{\frac{1}{a+1}}\right)$
2. $P_2 = \left(-\frac{l \times ab}{2 \times \sqrt{(a^2+1) \times (a^2+b^2+1)}}, \frac{l \times \sqrt{(a^2+1)}}{2 \times \sqrt{(a^2+b^2+1)}}, \frac{l \times b}{2 \times \sqrt{(a^2+b^2+1)(a^2+1)}}\right)$

By symmetrie, we deduce:

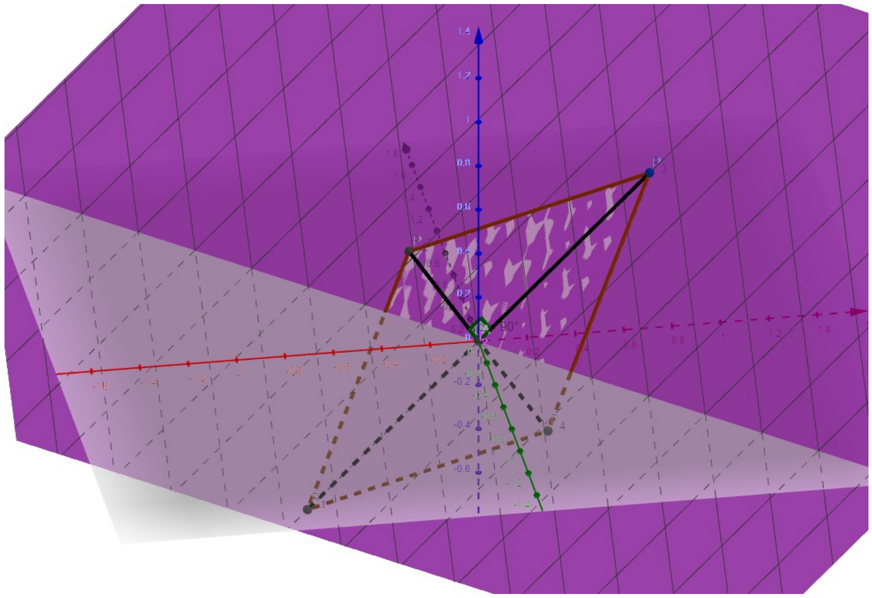
$$\begin{cases} (x_3, y_3, z_3) = (-x_1, -y_1, -z_1) \\ (x_4, y_4, z_4) = (-x_2, -y_2, -z_2) \end{cases}$$

so:

1. $P_3 = \left(\sqrt{\frac{1}{a+1}}, 0, a \times \sqrt{\frac{1}{a+1}}\right)$
2. $P_4 = \left(+\frac{l \times ab}{2 \times \sqrt{(a^2+1) \times (a^2+b^2+1)}}, -\frac{l \times \sqrt{(a^2+1)}}{2 \times \sqrt{(a^2+b^2+1)}}, -\frac{l \times b}{2 \times \sqrt{(a^2+b^2+1)(a^2+1)}}\right)$

And we have our stabilised rhombus.
In our demonstration we supposed that neither a and b were equaled to 0. If $a = b = 0$, we have a horizontal plane, which had already been done in the previous section.
Let see what happens if $b = 0$ (the case where $a = 0$ is symmetrical).
We have

$$f_2' : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x, y) \mapsto bx, \text{ with } a \in \mathbb{R}^* \end{cases}$$



11: Rhomnbus on an inclined plane

We have $\Omega = (0,0,0)$. The system is now:

$$(S) : \begin{cases} z_1 = a \times x_1 \\ z_2 = a \times x_2 \\ (x_1 - x_\Omega)^2 + (y_1 - y_\Omega)^2 + (z_1 - z_\Omega)^2 = 1 \\ (x_2 - x_\Omega)^2 + (y_2 - y_\Omega)^2 + (z_2 - z_\Omega)^2 = \frac{l^2}{4} \\ (x_1 - x_\Omega) \times (x_2 - x_\Omega) + (y_1 - y_\Omega) \times (y_2 - y_\Omega) + (z_1 - z_\Omega) \times (z_2 - z_\Omega) = 0 \end{cases}$$

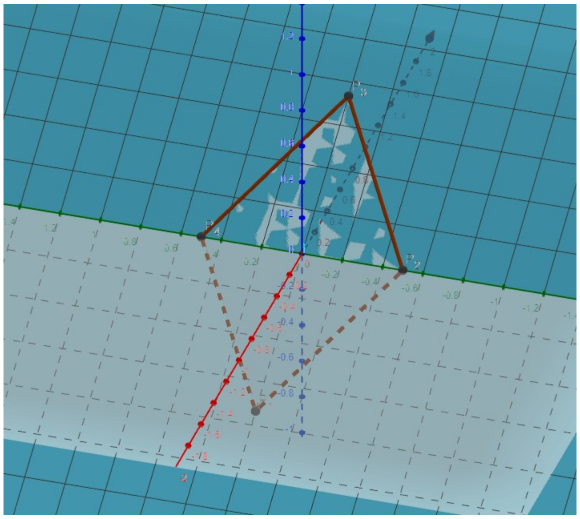
$$\Leftrightarrow \begin{cases} z_1 = a \times x_1 \\ z_2 = a \times x_2 \\ x_1^2 + y_1^2 + (ax_1)^2 = 1 \\ x_2^2 + y_2^2 + (ax_2)^2 = \frac{l^2}{4} \\ x_1 \times x_2 + y_1 \times y_2 + z_1 \times z_2 = 0 \end{cases}$$

We take $y_1 = 0$

$$(S) \Leftrightarrow \begin{cases} z_1 = a \times x_1 \\ z_2 = a \times x_2 \\ x_1^2 \times (1 + a^2) = 1 \\ x_2^2 + y_2^2 + (ax_2)^2 = \frac{l^2}{4} \\ x_1 \times x_2 + ax_1 \times ax_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} z_1 = a \times x_1 \\ z_2 = a \times x_2 \\ x_1^2 = \frac{1}{1+a^2} \quad a^2 + 1 > 0 \\ x_2^2 + y_2^2 + (ax_2)^2 = \frac{l^2}{4} \\ x_1 \times x_2 \times (a^2 + 1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} z_1 = a \times x_1 \\ z_2 = a \times x_2 \\ x_1 s = \pm\sqrt{\frac{1}{1+a^2}} \quad a^2 + 1 > 0 \\ x_2^2 + y_2^2 + (ax_2)^2 = \frac{l^2}{4} \\ x_1 = 0 \text{ or } x_2 = 0 \end{cases}$$



12: Rhombus on an inclined plane bis

$$\Leftrightarrow \begin{cases} z_1 = a \times x_1 \\ z_2 = a \times x_2 \\ x_1 = \pm\sqrt{\frac{1}{1+a^2}} \quad \text{so } x_1 \neq 0 \\ x_2 = 0 \\ y_2^2 = \frac{l^2}{4} \end{cases}$$

$$\Leftrightarrow \begin{cases} z_1 = a \times x_1 \\ z_2 = a \times x_2 \\ x_1 = \pm\sqrt{\frac{1}{1+a^2}} \\ x_2 = 0 \\ y_2 = \pm\frac{l}{2} \end{cases}$$

We will take $x_1 = -\sqrt{\frac{1}{1+a^2}}$ and $y_2 = -\frac{l}{2}$
We have our points:

1. $P_1 = (-\sqrt{\frac{1}{1+a^2}}, 0, -a \times \sqrt{\frac{1}{1+a^2}})$
2. $P_2 = (0, -\frac{l}{2}, 0)$
3. $P_1 = (\sqrt{\frac{1}{1+a^2}}, 0, a \times \sqrt{\frac{1}{1+a^2}} 0)$
4. $P_4 = (0, +\frac{l}{2}, 0)$

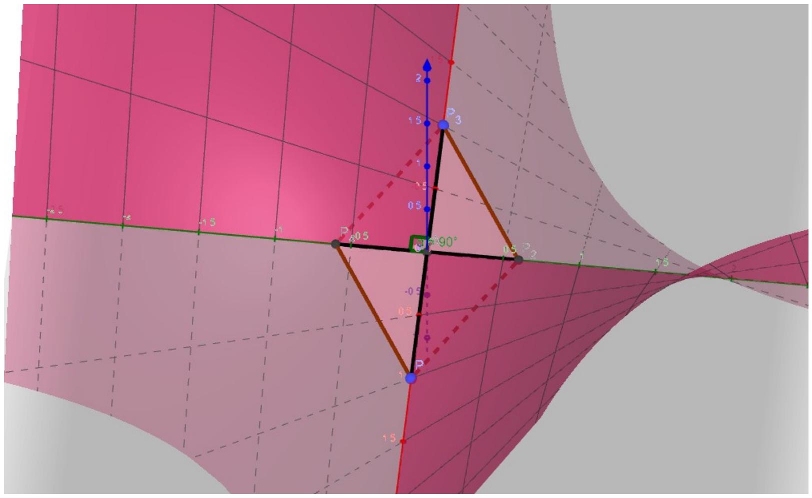
0.3.3 Horse Saddle

For the horse saddle it will much more simple.

$$f_3 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto sxy, \text{ with } s \in \mathbb{R}^* \end{cases}$$

If we place each leg on the (Ox) axis or the (Oy) axis, $f(x,y) = sxy = 0$
Then we place the points as follow:

1. $P_1 = (-1, 0, 0)$
2. $P_2 = (0, -\frac{l}{2}, 0)$



13: Rhombus on a saddle

3. $P_1 = (1, 0, 0)$
4. $P_4 = (0, +\frac{l}{2}, 0)$

And we have a stabilised rhombus on the saddle.

0.3.4 Sphere

$$f_4 : \begin{cases} [-1,1] \times [-1,1] \to \mathbb{R} \\ (x,y) \mapsto \sqrt{1 - \frac{x^2+y^2}{R^2}}, \text{ with } R > \sqrt{2} \end{cases}$$

Now we will prouve that it is impossible to have a stabilised table.
Let Ω be the center of our rhombus. Without loosing generality, we take $\Omega \in (Oz)$. Thus $x_\Omega = y_\Omega = 0$
We place $P_2 = (-\frac{l}{2}, 0, f_4(-\frac{-l}{2}, 0)$ and $P_4 = (\frac{l}{2}, 0, f_4(\frac{-l}{2}, 0)$.
$f_4(\frac{-l}{2}, 0) = \sqrt{1 - \frac{\frac{l^2}{4}}{R^2}} = \sqrt{1 - \frac{l^2}{4R^2}} = \frac{\sqrt{4R^2-l^2}}{2R} = \frac{\sqrt{(2R-l)(2R+l)}}{2R}$
We have necessarly $z_\Omega = z_2 = z_4 = \frac{\sqrt{(2R-l)(2R+l)}}{2R}$.
Now we will seek for P_1
We have the following hypothesis:

$$(S) \begin{cases} \overrightarrow{\Omega P_1}.\overrightarrow{\Omega P_4} = \overrightarrow{0} & (1) \\ \Omega P_1 = 1 & (2) \\ z_1 = f_4(x_1, y_1) = \sqrt{1 - \frac{x_1^2+y_1^2}{R^2}} & (3) \end{cases}$$

From the (1), we deduce:

$$(x_1 - x_\Omega) \times (x_4 - x_\Omega) + (y_1 - y_\Omega) \times (y_4 - y_\Omega) + (z_1 - z_\Omega) \times (z_4 - z_\Omega) = 0$$

But $x_\Omega = y_\Omega = 0$, $x_4 = \frac{l}{2}$, $y_4 = 0$ and $z_1 = z_\Omega$
Then,

$$x_1 \times \frac{l}{2} = 0$$

So, $x_1 = 0$
From (2), we deduce that:

$$(x_1 - x_\Omega)^2 + (y_1 - y_\Omega)^2 + (z_1 - z_\Omega)^2 = 1$$

But $x_1 = x_\Omega = 0$, $y_\Omega = 0$, $z_1 = \sqrt{1 - \frac{x_1^2+y_1^2}{R^2}}$ cf. (3) and $z_\Omega = \frac{\sqrt{(2R-l)(2R+l)}}{2R}$ Then,

$$y_1^2 + \left(\sqrt{1 - \frac{y_1^2}{R^2}} - \frac{\sqrt{(2R-l)(2R+l)}}{2R}\right)^2 = 1$$

$$\Leftrightarrow y_1^2 + \left(\frac{2\sqrt{R^2-y_1^2} - \sqrt{4R^2-l^2}}{2R}\right)^2 = 1$$

$$\Leftrightarrow y_1^2 + \frac{4(R^2-y_1^2) + (4R^2-l^2) - 4\sqrt{(R^2-y^2)\,(4R^2-l^2)}}{4R^2} = 1$$

$$\Leftrightarrow y_1^2 \times 4R^2 + 4(R^2-y^2) + (4R^2-l^2) - 4\sqrt{(R^2-y_1^2)\,(4R^2-l^2)} - 4R^2 = 0$$

If we pass the square root to the other side and then square each sides, we will get the equation (E):

$$\left(y_1^2 \times 4R^2 + 4(R^2-y^2) + (4R^2-l^2) - 4R^2\right)^2 = \left(4\sqrt{(R^2-y^2)\,(4R^2-l^2)}\right)^2$$

$$\Leftrightarrow y^4 \times (R^2-1)^2 + y^2 \times \left(4R^2 - l^2 - 2 \times \left(R^2 - \left(\frac{l}{2}\right)^2\right)(R^2-1)\right) + \left(\left(R^2 - \left(\frac{l}{2}\right)^2\right)^2 - 4R^4 + (Rl)^2\right) = 0$$

By plugging the equation in wolfram, we get :Wolfram Solution of E
You can have a glance on the solutions on the figure.
N.B.Why do we have four solutions ? Use sage math
We will take :

$$y_1 = -\sqrt{\frac{l^2 \times (R^2+1) + 4R^2 \times (R^2-1)}{4\,(R^4-2R^2+1)} - \frac{\sqrt{l^4R^2 - 4l^2R^6 - 4l^2R^2 + 16R^8 - 16R^6 + 16R^4}}{2 \times (R^4-2R^2+1)}}$$

So $z_1 = \sqrt{1 - \frac{\frac{l^2 \times (R^2+1) + 4R^2 \times (R^2-1)}{4\left(R^4-2R^2+1\right)} - \frac{\sqrt{l^4R^2 - 4l^2R^6 - 4l^2R^2 + 16R^8 - 16R^6 + 16R^4}}{2 \times (R^4-2R^2+1)}}{R^2}}$
And we have $P_1(0, y_1, z_1)$
We will stop the calculations here because it is too heavy and we will just prove the impossibility to have a grounded table.
P_2 and P_3 are distant from Ω of $\frac{l}{2} < 1$ whereas P_1 and P_3 are distant from Ω of 1, thus there are more far away. Then $z_1 < z_\Omega$ and $z_3 < z_\Omega$ because as the further away we get away from the center and the more lower we go. If we name d the distance from (Oz), we have $x^2 + y^2 = d^2$, so $z = \sqrt{1 - \frac{d^2}{R^2}}$. If d goes up, z goes down.
But here is the contradiction: it is impossible to have a segment that passes through three distinct points where the altitude of one of them is different from the other two.
Thus we cannot have a stabilised table.

0.4 Any convex quadrilateral

The section of a sphere by a plan is a circle. Then the quadrilateral has to be in a circle to stand on a sphere. If the convex quadrilateral cannot stand in a circle, he cannot stand on a sphere.
The convex quadrilateral is contained in a plane. Then any quadrilateral table is grounded on a plane.

Question 2

We will which of the above quadrilaterals could the table be wobbly or/and stabilised.

Solutions for the variable y

Step-by-step solution

$$y = -\sqrt{\left(\frac{l^2 R^2}{4\left(R^4 - 2R^2 + 1\right)} + \frac{l^2}{4\left(R^4 - 2R^2 + 1\right)} - \frac{\sqrt{l^4 R^2 - 4 l^2 R^6 - 4 l^2 R^2 + 16 R^8 - 16 R^6 + 16 R^4}}{2\left(R^4 - 2R^2 + 1\right)} - \frac{R^4}{R^4 - 2R^2 + 1} - \frac{R^2}{R^4 - 2R^2 + 1}\right)}$$

$$y = \sqrt{\left(\frac{l^2 R^2}{4\left(R^4 - 2R^2 + 1\right)} + \frac{l^2}{4\left(R^4 - 2R^2 + 1\right)} - \frac{\sqrt{l^4 R^2 - 4 l^2 R^6 - 4 l^2 R^2 + 16 R^8 - 16 R^6 + 16 R^4}}{2\left(R^4 - 2R^2 + 1\right)} - \frac{R^4}{R^4 - 2R^2 + 1} - \frac{R^2}{R^4 - 2R^2 + 1}\right)}$$

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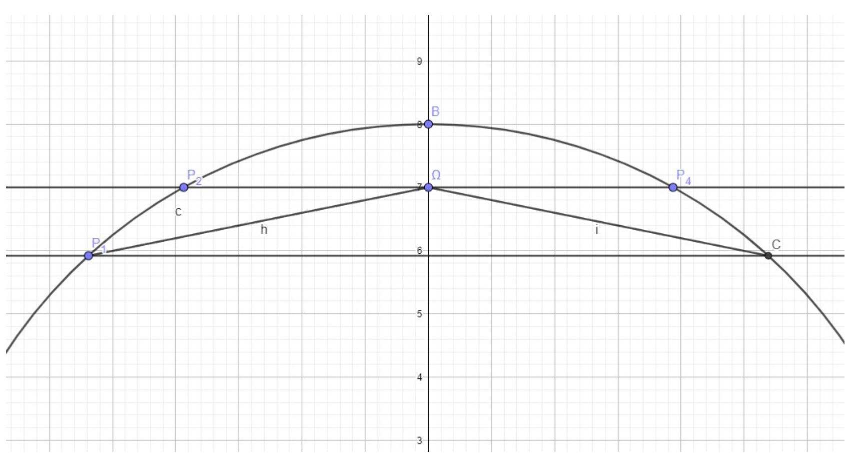
$$y = -\sqrt{\left(\frac{l^2 R^2}{4\left(R^4 - 2R^2 + 1\right)} + \frac{l^2}{4\left(R^4 - 2R^2 + 1\right)} + \frac{\sqrt{l^4 R^2 - 4 l^2 R^6 - 4 l^2 R^2 + 16 R^8 - 16 R^6 + 16 R^4}}{2\left(R^4 - 2R^2 + 1\right)} - \frac{R^4}{R^4 - 2R^2 + 1} - \frac{R^2}{R^4 - 2R^2 + 1}\right)}$$

$$y = \sqrt{\left(\frac{l^2 R^2}{4\left(R^4 - 2R^2 + 1\right)} + \frac{l^2}{4\left(R^4 - 2R^2 + 1\right)} + \frac{\sqrt{l^4 R^2 - 4 l^2 R^6 - 4 l^2 R^2 + 16 R^8 - 16 R^6 + 16 R^4}}{2\left(R^4 - 2R^2 + 1\right)} - \frac{R^4}{R^4 - 2R^2 + 1} - \frac{R^2}{R^4 - 2R^2 + 1}\right)}$$

14: Solutions from Wolfram Alpha for y



15: Slice of the 3D representation

0.5 Stabilised

On a plane, the table is always stabilised. Because the table is contained in a plane.
For a sphere, if the table is contained in a circle, then the table is stabilised.

0.6 Wobbly

For a sphere, if the table cannot be contained in a circle, then the table is wobbly. Because the section of a sphere by a plane is a circle.

Question 3

0.7 Inclined Plane

Our table is on the plane $P : ax + by - z = 0$. A normal vector of this plane is $\overrightarrow{n} = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$

And a normal vector of the horizontal plane $P' : z = 0$ is $\overrightarrow{n'} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

The angle α between the two planes is defined as follow:

$$\cos\alpha = \frac{|\overrightarrow{n}.\overrightarrow{n'}|}{||\overrightarrow{n}|| \times ||\overrightarrow{n'}||}$$

Here, we have:

$$\cos\alpha = \frac{|\overrightarrow{n}.\overrightarrow{n'}|}{||\overrightarrow{n}|| \times ||\overrightarrow{n'}||} = \frac{|-1|}{\sqrt{a^2+b^2+1} \times 1} = \frac{1}{\sqrt{a^2+b^2+1}}$$

Then, $\alpha = \arccos\left(\frac{1}{\sqrt{a^2+b^2+1}}\right)$