$\mathbb{ITYM}$ 2021 - Problem 7:
Proper Numbering Of graphs.

Team France

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Introduction

In this problem, we only consider connected graphs. That is sufficient for the general case, because if a connected component of a graph has no proper vertex k-numbering, then the whole graph has no proper vertex k-numbering, and if each connected component has a proper vertex k-numbering, then each proper vertex k-numbering of components can be used to create a proper vertex k-numbering of the whole graph. That's why we assume that the graphs we deal are connected.
Moreover, we consider only graphs with at least 1 edge, because graph with no edges are not interesting.

For a vertex v, we denote by $s_G(v)$ the sum of the labels on all the edges with an endpoint v in the graph G. When the context is clear, we denote it by $s(v)$.

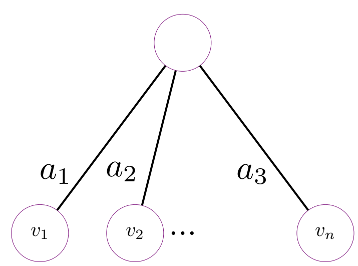
Proposition 1. Let $G = (V, E)$ be a graph and e an edge of G such that for all $e' \in E$, $\lambda(e) \geq \lambda(e')$. The smallest intergrer k for which there exists a proper k-numbering of G is strictly greater than $\lambda(e)$.

Proof. Assume, by contradiction, that there is a proper $\lambda(e)$-numbering of G, denoted ν, and take u, v two adjacent vertices of G, such that $\lambda((u, v)) = \lambda(e)$. Consequently, $|\nu(u) - \nu(v)| \geq \lambda(e)$.
Assume, without loss of generality, that $\nu(u) \geq \nu(v)$, so $\nu(u) - \nu(v) \geq \lambda(e)$ and thus $\nu(u) \geq \lambda(e) + \nu(v)$. Since $\nu(v) \geq 1$, we have $\nu(u) \geq \lambda(e) + 1$, which is a contradiction. Thus, there is no proper $\lambda(e)$-numbering of G.

Greedy algorithm

Proposition 2. For a vertex w of a graph G with at least 2 edges, there is an integer in $[\![1, 2S(G) - 1]\!]$ which can be used to label w without violating the required condition for a proper vertex k-numbering.

Proof. Denote by $v_1, v_2, ..., v_n$ the labels of neighbours of w which are already labelled ; denote by $a_1, ..., a_m$ the integers $\lambda((v_1, w)), ..., \lambda((v_n, w))$.



1: Illustration of the situation

By definition, the sum of a_i is smaller than or equal to $S(G)$.

Then, let E be the set of integers which belong to $[\![1, 2S(G) - 1]\!]$ which can be used to label w without violating the required condition.
For an edge labelled a_i, integers which violate the condition are integers x such that :

$$|v_i - x| < a_i$$

Therefore, for each edge a_i, integers which belong to $]\!]v_i - a_i, v_i + a_i[\![$ violate the condition. Because this interval contains $2a_i - 1$ elements, and because E is the union of all sets of integers which violate the condition, we have $\text{card}(E) \leq \sum_{i=1}^{n} 2a_i - 1$
But there is the relation :

$$\sum_{i=1}^{n} a_i \leq S(G)$$

So

$$\sum_{i=1}^{n} (2a_i) \leq 2S(G)$$

$$\sum_{i=1}^{n}(2a_i - 1) \leq 2S(G) - \sum_{i=1}^{n} 1 = 2S(G) - n \quad (1)$$

Now, distinguish three cases.

1. If $n = 0$ then w has no neighbour which is already numbered. So, assign 1 to w does not violate the condition, and 1 is in $[\![1, 2S(G) - 1]\!]$, as desired.

2. If $n = 1$ then since G is connected and has at least 2 edges, w or his neighbour is connected to an other vertex, so $S(G) \geq a_1 + 1$. Thus $2a_1 - 1 \leq 2S(G) - 3 < 2S(G) - 1$. So, there are fewer numbers which violate the condition than numbers in $[\![1, 2S(G) - 1]\!]$. That means that there exists a integer in $[\![1, 2S(G) - 1]\!]$ which can be used to label w without violating the required condition for a proper vertex k-numbering.

3. If $n > 1$, then $2S(G) - n \leq 2S(G) - 2$. Replacing that in (1) :

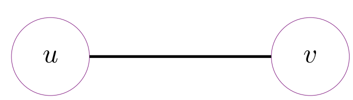
$$\sum_{i=1}^{n}(2a_i - 1) \leq 2S(G) - 2 < 2S(G) - 1$$

Therefore, there are fewer numbers which violate the condition than numbers in $[\![1, 2S(G) - 1]\!]$. This means that there exists an integer in $[\![1, 2S(G) - 1]\!]$ which can be assigned without violate the condition.

In all three cases, there is a integer in $[\![1, 2S(G) - 1]\!]$ which can be used to label w without violating the required condition for a proper vertex k-numbering, as desired.

Corollary 3. The greedy algorithm (which consists in giving to each vertex in an arbitrary order the least possible number which doesn't violate the condition on edges) always produces a proper vertex $2S(G)$-numbering for every graph G, and especially a proper vertex $(2S(G) - 1)$-numbering if G have at least 2 edges.

Proof. If G has at least two edges, Proposition 2 ensures that during the implementation of the greedy algorithm, there is always an integer in $[\![1, 2S(G) - 1]\!]$ which doesn't violate the required condition for a proper k-numbering. This leads the algorithm to produce a proper vertex $(2S(G) - 1)$-numbering. If G has less than 2 edge, because we assume G to be connected and to have at least 1 edge, it means that G is the graph drawn on figure 2. This graph has a proper vertex $2S(G)$-numbering, defining by $\nu(u) = 1$ and $\nu(v) = \lambda((u, v)) + 1$.



2: The only connected graph with 1 edge.

Proper vertex $(S(G)+1)$-numbering

In this section, we consider a graph $G_0 = (V, E, \lambda)$ of order $n > 1$. By convention, if e is not an edge of G_0, we say that $\lambda(e) = 0$.

Proposition 4. G_0 has a proper vertex $S(G_0) + 1$-numbering.

Proof. Consider the following algorithm : denote by v_0 a vertex of G_0 such that $s_{G_0}(v_0) = S(G_0)$, and give to v_0 the number $s_G(v_0)+1$. Then, denote by G_1 the induced subgraph of G_0 without v_0, and without all edges which are connected to v_0. Denote by v_1 a vertex of G_1 such that $s_{G_1}(v_1) = S(G_1)$, and give to v_1 the number $s_{G_1}(v_1)$. Similarly, define $G_2, ..., G_{n-1}$, define $v_2, ..., v_{n-1}$ and give to each v_i the number $s_{G_i}(v_i) + 1$.
It's easy to see that this algorithm produce a vertex $S(G_0) + 1$-numbering, so the only thing to show is that this numbering respect the condition on edges.

Let a, b be two integers of $[\![0, n-1]\!]$, such that $a < b$. Then :

1. $s_{G_a}(v_a) \geq s_{G_a}(v_b)$
2. $s_{G_b}(v_b) = s_{G_a}(v_b) - \sum\limits_{i=a}^{b-1} \lambda((v_b, v_i))$
3. $s_{G_a}(v_b) \geq s_{G_b}(v_b)$

The first point comes from the definition of v_a : $s_{G_a}(v_a) = S(G_a)$. The second relation comes from the definition of G_b : it is the induced subgraph of G_a without all edges which are connected to an edge of $\{v_a, ..., v_{b-1}\}$. The third point is a consequence of the second.
Now, look at the absolute value of the difference between numbers assign to two vertices v_a and v_b :

$$\begin{aligned} |s_{G_a}(v_a) + 1 - s_{G_b}(v_b) - 1| &= s_{G_a}(v_a) - s_{G_b}(v_b) \quad \text{(point 1 and 3)} \\ &= \underbrace{s_{G_a}(v_a) - s_{G_a}(v_b)}_{\text{positive, by point 1.}} + \sum_{k=a}^{b-1} \lambda((v_k, v_b) \quad \text{(point 2)} \\ &\geq \sum_{k=a}^{b-1} \lambda((v_k, v_b) \\ &\geq \lambda((v_a, v_b)) \end{aligned}$$

Thus the condition on edges is respected.

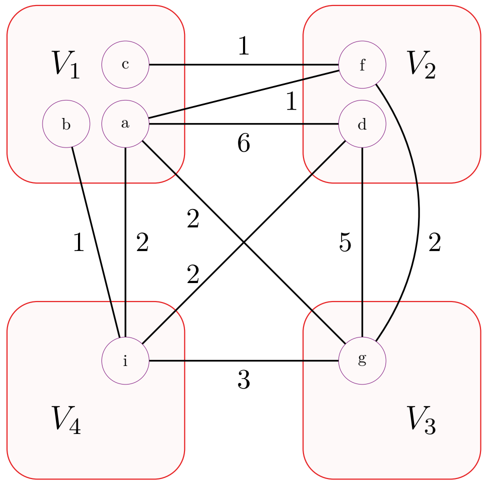
Bounds for the minimum k such that G has a proper vertex k-numbering

In this section, we say that a graph is c-colorable if he has a proper vertex c-coloring.

Proposition 5. Let n be an integer greater or equal than 2. Let $G = (V, \{e_1, e_2, ..., e_n\})$ be a graph such that $\lambda(e_1) \geq \lambda(e_2) \geq ... \geq \lambda(e_n)$.

If G is c-colorable, it has a proper vertex $(1 + \sum\limits_{i=1}^{c-1} \lambda(e_i))$-numbering.

Proof. For two set of vertices V et V', we denote by $\max(V, V')$ the largest $\lambda(e)$ where e is an edge joining a vertex of V and a vertex of V'.
Colour G with c colours, and consider the following partition of V : choose an arbitrary colour, and denote by V_1 the set of all vertices of G which are of this colour. Then, denote by V_2 the set of all vertices of the same colour such that for all edge e, $max(V_1, V_2) \geq \lambda(e)$. Similarly, for each $i \in [\![3, c]\!]$ define V_i as the set of all vertices of the same colour such that $max(V_{i-1}, V_i)$ is greater or equal than all $\lambda(e)$, where e is an edge which is not incident to a vertex of a set V_j.



3: An example of partition of a 4-colorable graph.

Consider the following numbering :

$$\forall v \in V_i, \nu(v) = 1 + \sum_{k=1}^{i-1} \max(V_k, V_k + 1)$$

For two vertices u and v which belong respectively to two different sets V_i and V_j (with $i > j$),

$$\begin{aligned} |\nu(u) - \nu(v)| &= \left| 1 + \sum_{k=1}^{i-1} \max(V_k, V_k + 1) - 1 - \sum_{k=1}^{j-1} \max(V_k, V_k + 1) \right| \\ &= \sum_{k=j}^{i-1} \max(V_k, V_k + 1) \\ &\geq \max(V_j, V_{j+1}) \\ &\geq \max(V_i, V_j) \end{aligned}$$

So ν is a proper vertex numbering.
In addition, edges of G have been sorted such that $\lambda(e_1) \geq \lambda(e_2) \geq ... \geq \lambda(e_n)$. So, for all different edges $f_1, ..., f_i$,

$$\sum_{k=1}^{i} \lambda(e_k) \geq \sum_{k=1}^{i} \lambda(f_k)$$

That's why

$$\forall x, \nu(x) \leq 1 + \sum_{k=1}^{c-1} \max(V_k, V_k + 1) \leq \sum_{k=1}^{c-1} \lambda(e_k)$$

Therefore, ν is a proper vertex $(1 + \sum_{i=1}^{c-1} \lambda(e_i))$-numbering, as desired.

Corollary 6. Let $G = (V, E)$ be a bipartite graph and e an edge of G such that for all $e' \in E$, $\lambda(e) \geq \lambda(e')$. The smaller integer k for which G has a proper vertex k-numbering is equal to $\lambda(e) + 1$.

Proof. With proposition 5, G has a proper vertex $(\lambda(e)+1)$-numbering. Moreover, with proposition 1 there is no proper vertex $\lambda(e)$-numbering of G.

Corollary 7. Let G be a complete graph of order n for which each edge is labelled with an integer p, and k the smaller integer such that G has a proper vertex k-numbering. Then, $k=(n-1)p+1$.

Proof. As G is n-colorable, with proposition 5 G has a proper vertex $(1+\sum_{i=1}^{n-1} p)$-numbering, so $k \leq (n-1)p+1$.
Let's prove now that there is no smaller numbering. For that, denote by ν a proper vertex k-numbering of G . Sort vertices of G in an order $v_0, v_1, ..., v_{n-1}$ such that $\nu(v_0) \leq \nu(v_1) \leq ... \leq \nu(v_{n-1})$. By proposition 1, $v_0 = 1$. As v_0 and v_1 are connected by an edge labelled p, $v_1 \geq p+1$, and by induction, $v_i \geq ip+1$. Therefore, $k \geq (n-1)p+1$.

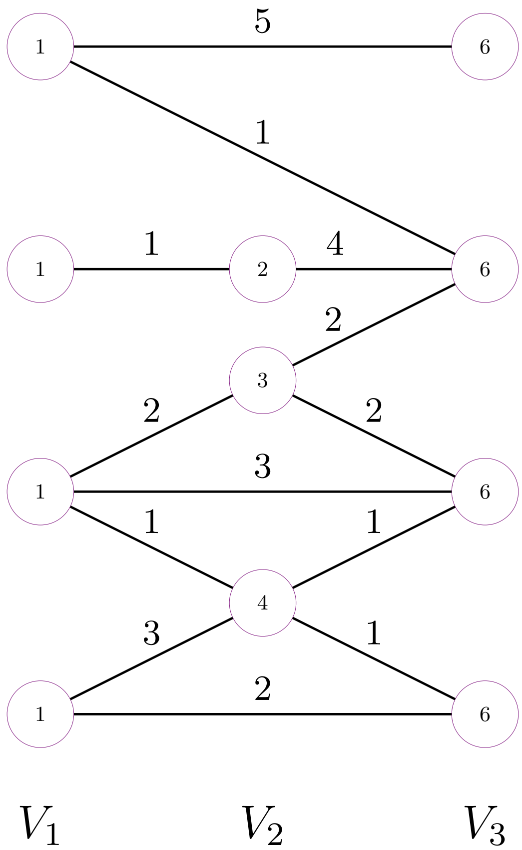
Note that this corollary leads complete graphs of order n with the same label p on each edge to not have any proper vertex $S(G)$-numbering, because in this case $S(G)=(n-1)p < (n-1)p+1$.

Proposition 8. Let $G=(V_1 \sqcup V_2 \sqcup V_3, E, \lambda)$ be a 3-colorable graph, such that a vertex of a set V_i is never connected to a vertex of V_i. For a vertex v and an integer i of $\{1,2,3\}$, denote by $w(v,i)$ the greatest label of (v,u) such that $u \in V_i$.
Let M be an integer. If for all vertex $v \in V_2$, $w(v,1)+w(v,3) < M$, then there exists a proper vertex M-numbering of G.
In particular, if for all vertex $v \in V_2$, $w(v,1)+w(v,3) < S(G)$, then there exists a proper vertex $S(G)$-numbering of G.

Proof. Let v_1,v_2 and v_3 be three vertices belonging respectively to V_1, V_2, V_3, and consider the following numbering :

$$\begin{cases} \nu(v_1) = 1 \\ \nu(v_2) = 1 + w(v_2, 1) \\ \nu(v_3) = M \end{cases}$$

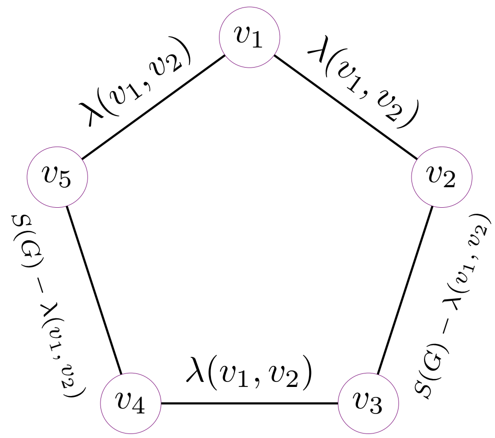
We have $|\nu(v_1) - \nu(v_3)| = M - 1 \geq \lambda((v_1, v_3))$, because G have at least 2 edges.
Moreover, $|\nu(v_1) - \nu(v_2)| = |1 - 1 - w(v_2, 1)| = w(v_2, 1) \geq \lambda((v_1, v_1))$.
Finally, $M > w(v_2, 3) + w(v_2, 1)$, so $M - 1 - w(v_2, 1) \geq w(v_2, 3)$, so $|\nu(v_3) - \nu(v_2)| \geq w(v_2, 3)$.
Therefore, ν is a proper vertex M-numbering of G.



4: Example of a proper $S(G)$-numbering of a tripartite graph.

Corollary 9. Let n be an odd integer and $G = (\{v_1, v_2, ..., v_n\}, \{(v_1, v_2), (v_2, v_3), ..., (v_n, v_1)\}, \lambda)$ be a labeled odd cycle. If there exists an edge e such that $\lambda(e) \neq \frac{S(G)}{2}$, then G has a proper vertex $S(G)$-numbering.

***Proof*.** Assume, without loss of generality, that $\lambda(v_1, v_2) \neq \frac{S(G)}{2}$. Assume, by contradiction, that for all vertex v, $s(v) = S(G)$. Then, it means that $\lambda(v_2, v_3) = S(G) - \lambda(v_1, v_2)$. Similarly, $\lambda(v_3, v_4) = \lambda(v_1, v_2)$, and by induction, since n is odd, $\lambda(v_k, v_1) = \lambda(v_1, v_2)$. It leads that $s(v1) = 2\lambda(v_1, v_2) < S(G)$, which is a contradiction. Thus, there exists a vertex v such that $s(v) < S(G)$.



5: Example when $n = 5$.

Consider the following colouring of G : assign a first colour to v, an other to all vertices which are an even distance from v, and a last to all vertices which are an odd distance to v. Then G is 3-colorable, and since $s(v) < S(G)$, then by Proposition 8, there exists a proper vertex $S(G)$-numbering of G.

Graphs which are labelled with the same integer

Proposition 10. If a graph G have a proper vertex k-numbering, then it is k-colorable.

Proof. Assume that G have a proper vertex k-numbering, denotes by ν. Then, for $i \in [\![1, k]\!]$ denote by V_1 the set of all vertex v such that $\nu(v) = i$. Two vertices which belong to the same P_i can't be connected, otherwise the label of their edge have to be greater than their absolute difference, which is 0. So you can colour each vertex of P_i with the same colour to obtain a proper k-colouring of G.

Proposition 11. Let G be a graph where all edges are labelled with 1. G have a proper vertex $S(G)$-numbering if and only if he is not a complete graph or an odd cycle.

Proof. By Proposition 5, if G is k-colorable, it has a proper vertex k-numbering. Moreover, by Proposition 10, if G have a proper vertex k-numbering, then it is k-colorable. It means that G have a proper vertex k-numbering if and only if he is k-colorable.
If we call D the maximum degree of G, we have $S(G) = D$. But Brooks'theorem states that the chromatic number of a connected graph is smaller that D, unless this graph is an odd cycle or a complete graph : in this case, the chromatic number equals $D + 1$. Therefore, G have a proper vertex $S(G)$-numbering if and only if he is not an odd cycle or a complete graph.