$\mathbb{ITYM}$ 2021 - Problem 10: Non-nilpotent Graphs of Groups.

Team France

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Introduction

In this section, we consider a group G with a neutral element e.

Proposition 1. Let x be an element of G. x is a vertex of $\Gamma(G)$ if and only if there is $y \in G$ such as $\langle x, y\rangle$ is not nilpotent.

Proof.

$$\begin{aligned} x \text{ is not a vertex of } \Gamma(G) &\iff x \in nil(G) \\ &\iff x \in \bigcap_{y \in G} nil_G(y) \\ &\iff \forall y \in G, x \in nil_G(y) \\ &\iff \forall y \in G, \langle x, y\rangle \text{ is nilpotent} \end{aligned}$$

Proposition 2. e is not a vertex of $\Gamma(G)$.

Proof.

$$\forall a \in G, \langle a, e\rangle = \{a^n, n \in [\![1, ord(a)]\!]\}$$

and therefore, because two products of the same element commute, $\langle a, e\rangle$ is abelian, and therefore nilpotent. According to Proposition 1, this means that e is not a vertex of $\Gamma(G)$.

Proposition 3. For all element $a \in G$ different of the identity, there is a $n \in \mathbb{N}^*$ such as a^n is of prime order .

Proof. With the unique factorization theorem, there is a decomposition of $ord(a)$ such as :

$$ord(a) = p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_m^{e_m}$$

with $p_1 < p_2 < \cdots < p_m$ prime numbers and $e_1, e_2, ..., e_m$ positive integers.
Setting $n = p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_m^{e_m - 1}$:

$$ord(a) = n \times p_m$$

And

$$\begin{aligned} a^{ord(a)} &= e \\ a^{n \times p_m} &= e \\ (a^n)^{p_m} &= e \end{aligned}$$

So a^n is of prime order.

Dihedral group

For two natural numbers a and b, in the dihedral group of order $2n$ there is the following relations :

1. $x^a x^b = x^{a+b}$
2. $yx^a x^b = yx^{a+b}$
3. $x^a yx^b = yx^{b-a}$
4. $yx^a yx^b = x^{b-a}$

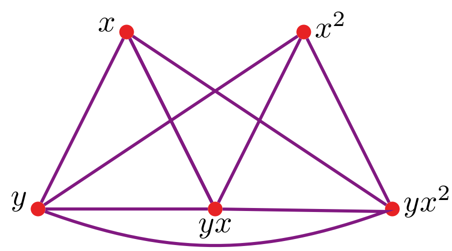
where all exponents are taken modulus n. The first two points comes from the associativity of the law. The second is a consequence of the relation : $xyxy = 1$:

$$\begin{aligned} xyxy = 1 &\iff xyx = y \quad (\text{because } y^2 = 1) \\ &\iff yxy = x^{-1} \\ &\iff yxy(yxy)^{b-1} = x^{-b} \\ &\iff yx^b y = x^{-b} \end{aligned}$$

The last relation comes from the third, by composing by y on the left.
That means that each element of D_n can be written as a power of x or as y times a power of x. We call the first kind of element rotations and the second reflections.
Since $yx^a yx^a = x^{a-a} = 1$, a reflection yx^a is of order 2 ; and a rotation x^a has the same order than a in $(\mathbb{Z}/n\mathbb{Z}, +)$, which is $\dfrac{n}{a \wedge n}$.
Note that for $n = 2^p$, $p \in \mathbb{N}^*$, D_n is nilpotent. This comes from Lagrange's theorem, which states that the order of each element of a group divides the order of the group. If n is a power of 2, then the order of D_n is also a power of 2, thus the order of each element is a power of 2. In particular, there is no elements in D_n of coprime order, therefore D_n is nilpotent. In this case, $\Gamma(D_n)$ is empty. Therefore, for all nilpotent group N, $\Gamma(N) \simeq \Gamma(D_n)$.
Now, we assume that n is not a power of 2.

Proposition 4. For odd n, $\Gamma(D_n)$ contains all the rotations different from the identity, and all the reflections, with the following edges :

- each rotation is connected to all reflections
- each reflection is connected to all rotations and to all reflections different from itself.



1: The non-nilpotent graph of D_3

Proof. Let yx^a and x^b be two elements of D_n different of 1. Because n is odd, $b \wedge n$ is odd, and $\dfrac{n}{b \wedge n}$ is odd. yx^a is of order 2, so yx^a and x^b are of order coprime.
Assume, by contradiction, that x^a and yx^b commute. Then, since $yx^a x^b = yx^{a+b}$ and $x^b yx^a = yx^{a-b}$, you have $a + b \equiv a - b \quad [n]$. So there exists an integer k such as $2b = kn$. But $1 \leq b \leq n - 1$, so $2 \leq 2b \leq 2n - 2$. It follows that $k = 1$, and $2b = n$, so n is even : contradiction, therefore x^a and yx^b can't commute. Thus a group which contain yx^a and x^b is not nilpotent.
Consider every kind of subgroup generated by two distinct element different from the identity :

1. $\langle x^a, x^b \rangle$ is nilpotent, because it is abelian.
2. $\langle yx^a, x^b \rangle$ contains yx^a and x^b, and it has been proved that this subgroup is not nilpotent.
3. $\langle yx^a, yx^b \rangle$ contains $yx^b yx^a = x^{a-b}$. This element can't be the identity, because we assume yx^a distinct from yx^b. It has been proved that $\langle yx^a, x^{a-b} \rangle$ is not nilpotent, therefore $\langle yx^a, yx^b \rangle$ is not nilpotent too.

Finally, reflections are connected to all element different of the identity, and rotations different of the identity are connected to reflections.

Corollary 5. For odd n, $\Gamma(D_n)$:

1. is of order $2n - 1$;
2. is connected ;

3. has $\dfrac{3n(n-1)}{2}$ edges ;

4. is not Eulerian ;

5. is Hamiltonian.

Proof. $\Gamma(D_n)$ contains all the element of D_n without the identity, so $|\Gamma(D_n)| = |D_n| - 1 = 2n - 1$. $\Gamma(D_n)$ is connected because the complete bipartite graph $K_{n-1,n}$ is a connected subgraph of $\Gamma(D_n)$. For point 3, count the degree of each vertex : the $n-1$ rotations different from the identity are connected to each reflection, so they are of degree n ; the n reflections are connected to each rotation different from the identity and to each reflection different from itself, so they are of degree $n-1+n-1$. Therefore the number of edges of $\Gamma(D_n)$ is $\dfrac{3n(n-1)}{2}$. A rotation different from the identity is of degree n, which is assuming to be odd, so according to Euler's theorem on graphs, $\Gamma(D_n)$ is not Eulerian. Finally, the circuit $[(y,x),(x,yx),(yx,x^2),(x^2,yx^2),...,(x^{n-1},yx^{n-1}),(yx^{n-1},y)]$ is Hamiltonian, therefore $\Gamma(D_n)$ is Hamiltonian.

Proposition 6. Let n be an even positive integer which is not a power of 2, and i the greater odd integer which divides n. Then, $\Gamma(D_n)$ is connected and its order is greater or equal to $n + \varphi(n) + 1$, where φ is Euler's totient function.

Proof. Set $n = 2^v i$, where v is the 2-adic valuation of n. We have :

$$\mathrm{ord}(x^{(2^v)}) = \frac{n}{2^v \wedge n} = \frac{n}{2^v} = i$$

Since a reflection is of order 2, it means that $x^{(2^v)}$ and a reflection are of coprime order.
Suppose, by contradiction, that a reflection yx^a and $x^{(2^v)}$ commute. It follows that $yx^{b+2^v} = yx^{b-2^v}$, so $b + 2^v \equiv b - 2^v \quad [n]$ so it exists k in $\mathbb{Z}$ such as $2 \times 2^v = kn$, so $2 = ki$, so $i < 3$. But since n is not a power of 2, $i \geq 3$, so there is a contradiction. Therefore, a reflection and $x^{(2^v)}$ don't commute. Since they are of coprime order, it means that both are connected vertices of $\Gamma(D_n)$.
Moreover, if $a \wedge n = 1$, then a generate $(\mathbb{Z}/n\mathbb{Z}, +)$, so the rotation x^a generate the subgroup of rotations of D_n. It leads that for all reflection yx^b, $\langle x^a, yx^b \rangle$ contains all reflection, and in particular x^{2^v}. So $\langle x^a, yx^b \rangle$ is not nilpotent, and are connected. Additioning all reflections, the rotation x^{2^v} and all rotation x^a such that $a \wedge n = 1$, you have $|\Gamma(D_n)| \geq n + \varphi(n) + 1$
In addition, since groups generated by rotations are nilpotent, if another rotation is a vertex of D_n, it means that it is connected to a reflection, which belongs to a connected subgraph of $\Gamma(D_n)$, so $\Gamma(D_n)$ is connected.

Symmetric Group

In this section, we consider permutation of a set S. We call *support* of a permutation σ all the element x of S such as $\sigma(x) \neq x$.
We write permutation in cycle notation, and use the following statement :

1. for all permutation σ, $\sigma \circ (a_1 \ a_2 \ ... \ a_k) \circ \sigma^{-1} = (\sigma(a_1) \ \sigma(a_2) \ \sigma(a_k))$

2. the order of a permutation is the LCM of the lengths of its cycles.

Proposition 7. Let σ be a permutation and $\tau = (a_1 \ a_2 \ ... \ a_k)$ a k-cycle. Then, σ and τ commute if and only if the supports of σ and τ are disjoint, or τ is a cycle in the decomposition into disjoint cycles of σ.

Proof. If the supports of σ and τ are disjoint, then σ and τ commute.
If τ is a cycle in the decomposition into disjoint cycles of σ, then since τ commute with itself, and τ commute with disjoint cycles of σ, so τ commute with σ.
If σ and τ commute, then since $\sigma \circ (a_1 \ a_2 \ ... \ a_k) = (\sigma(a_1) \ \sigma(a_2) \ ... \ \sigma(a_k)) \circ \sigma$, it means $(a_1 \ a_2 \ ... \ a_k) = (\sigma(a_1) \ \sigma(a_2) \ \sigma(a_k))$. That is true only if the supports of σ and τ are disjoint, or if τ is a cycle in cycle notation of σ.

Proposition 8. For $n \geq 7$, $\Gamma(S_n)$ is of order $n! - 1$, and $\Gamma(A_n)$ is of order $\frac{n!}{2} - 1$. Both are connected, and more precisely of diameter 2.

Proof. Let $n \geq 7$ be an integer, σ and σ' be two permutations of S_n different from the identity. By Proposition 3, there exists a product of σ which is of prime order. Denote it by π. Similarly, denote by π' a product of σ' of prime order. Denote by x and x' two different elements of the support of π and π' respectively, and y, y', y'' three different elements of S, different from x and x'.Now, distinguish 3 cases.

1. if the orders of π and π' are different of 3, then set $\tau = (x\ x'\ y)$. Assume, by contradiction, that τ and π commute. π is of order different from 3, so cycle notation of π doesn't contain 3-cycle. Nevertheless, supports of π and τ are not disjoint, so Proposition 7 states that π and τ doesn't commute. Since the order of π is prime, it means that $\langle \pi, \tau \rangle$ is not nilpotent. With the same reasoning, $\langle \pi', \tau \rangle$ is not nilpotent.

2. if the order of π (resp. π') equal 3 and the order of π' (resp. π) is different from 5, then for the same reason, the 5-cycle $\tau = (x\ x'\ y\ y'\ y'')$ doesn't commute with π or π', and $\langle \pi, \tau \rangle, \langle \pi', \tau \rangle$ are not nilpotent.

3. if the order of π (resp. π') equal 3 and the order of π' (resp. π) equal 5, then, as in the first two points, a 7-cycle τ which contains x and x' doesn't commute with π and π', so $\langle \pi, \tau \rangle, \langle \pi', \tau \rangle$ are not nilpotent.

Finally, there exists a permutation τ such that $\langle \pi, \tau \rangle$ and $\langle \pi', \tau \rangle$ are not nilpotent. Therefore, $\langle \sigma, \tau \rangle$ and $\langle \sigma', \tau \rangle$ are not nilpotent, so σ and σ' are vertices of $\Gamma(S_n)$. Moreover, in this graph, σ is connected to τ, and τ is connected to π', so there exists a path of length 2 which connect all permutation. It means that $\Gamma(S_n)$ is of diameter 2, and thus connected.
To conclude, σ is an odd cycle, so it is an even permutation. Thus, if π and π' are even permutation, the proof can be adapted for A_n.