$\mathbb{ITYM}$ 2021 - Problem 1:
Power Sums of Distances.

Team France

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Proposition 1. If P is the set of a regular n-gone n, and if $k = 2$, then $F_2(P, c)$ is a circle when $c \geq n$, and an empty set else.

Proof. Choose a basis such as A_j have for affix the j-th roots of unity. Then coordinates of A_j are $\left(\cos\dfrac{2j\pi}{n}; \sin\dfrac{2j\pi}{n}\right)$. Let X be a point of coordinates (x, y).

$$\begin{aligned}
X \in F_k(P,c) &\iff \sum_{j=0}^{n-1} \sqrt{\left(x - \cos\left(\frac{2j\pi}{n}\right)\right)^2 + \left(y - \sin\left(\frac{2j\pi}{n}\right)\right)^2}^{\,2} = c \\
&\iff \sum_{j=0}^{n-1} \left(x - \cos\left(\frac{2j\pi}{n}\right)\right)^2 + \left(y - \sin\left(\frac{2j\pi}{n}\right)\right)^2 = c \\
&\iff \sum_{j=0}^{n-1} \left(x^2 + y^2\right) - 2x \sum_{j=0}^{n-1} \left[\cos\left(\frac{2j\pi}{n}\right)\right] - 2y \sum_{j=0}^{n-1} \left[\sin\left(\frac{2j\pi}{n}\right)\right] + \sum_{j=0}^{n-1} \left[\cos^2\left(\frac{2j\pi}{n}\right) + \sin^2\left(\frac{2j\pi}{n}\right)\right] = c
\end{aligned}$$

Examining each term, observe that :

$$\sum_{j=0}^{n-1} \left(x^2 + y^2\right) = nx^2 + ny^2$$

And $\cos^2(a) + \sin^2(a) = 1$, so

$$\sum_{j=0}^{n-1} \left[\cos^2\left(\frac{2j\pi}{n}\right) + \sin^2\left(\frac{2j\pi}{n}\right)\right] = n$$

Moreover, the sum of the n-th roots of unity equal 0, so :

$$\sum_{j=0}^{n-1}\left[\cos\left(\frac{2j\pi}{n}\right)+i\sin\left(\frac{2j\pi}{n}\right)\right]=0$$

Taking the real part :

$$\sum_{j=0}^{n-1}\cos\left(\frac{2j\pi}{n}\right)=0$$

And the imaginary part :

$$\sum_{j=0}^{n-1}\sin\left(\frac{2j\pi}{n}\right)=0$$

Thus :

$$nx^2+ny^2+n=c$$

Dividing by n :

$$x^2+y^2=\frac{c-n}{n}$$

If $c\geq n$, so $\frac{c-n}{n}\geq 0$ and :

$$x^2+y^2=\sqrt{\frac{c-n}{n}}^2$$

which is the equation of the circle of center $(0;0)$ and of radius $\sqrt{\frac{c-n}{n}}$.
Else, $\frac{c-n}{n}<0$. But as $x^2+y^2\geq 0$, $F_2(P,c)$ is empty.