Exercise sheet 4
Dynamical systems and bifurcation theory

Due date: 2022-12-22
Tasks: 5

In this exercise, you will study qualitative changes of dynamical systems over changes of their parameters. These changes in the qualitative behavior of the system are called *bifurcations*, "to divide into two, like a fork". The notion of a qualitative change has been made precise, the definitions are given below. The goals for this exercise are

- to familiarize yourself with the mathematical notation of bifurcation theory,
- to understand topological equivalence between systems,
- to know several basic bifurcations present in almost all dynamical systems in the world,
- to be able to visualize qualitative changes of a dynamical system in a bifurcation diagram, and
- to apply these ideas to crowd dynamics (the SIR model).

Why is this related to Machine Learning? Even though it is possible to analyze a given mathematical model formally regarding its bifurcations, there are many systems where such an analysis is not possible. The most difficult examples are systems in the real world, where you cannot study the behavior on pen and paper, only through observations. This is where Machine Learning can assist you, to produce data-driven bifurcation diagrams for real systems (or large, complex simulated systems such as the ones you studied with Vadere) from data. Note that algorithms in machine learning also have bifurcations (if you think about them as "dynamical systems"): if you change hyperparameters of an algorithm, the results can suddenly change drastically. You can read more about this here: [1]. What is even more important: to design and use good algorithms in Machine Learning, you still need to know about the basic theory behind them—otherwise, you are just blindly applying a tool with no hope of correctly interpreting its output.

Dynamical systems with bifurcations

A dynamical system is a set X of states $x \in X$ together with a combination of rules (the evolution operator) $\phi : I \times X \to X$ that change this state over the change of the parameter t, typically considered as "time" $t \in I$ in an index set I. Typically, for discrete dynamical systems, we choose $I \subseteq \mathbb{N}$, for continuous dynamical systems, $I \subseteq \mathbb{R}$. The combination of time, state space, and evolution operator defines the dynamical system and is often stated as a triple (I, X, ϕ). For a discrete system, the evolution by ϕ starting at an initial point $x_0 \in X$ is usually written

$$x_n = \phi(n, x_0),\ x_n \in X,\ n \in I. \tag{1}$$

Many descriptions of continuous dynamical systems do not directly specify the map ϕ, but its derivative with respect to time as a function $v : X \to TX$,

$$\left.\frac{d\phi(t,x)}{dt}\right|_{t=0} = v(x), \tag{2}$$

where TX denotes the tangent bundle of X, and $v(x) \in T_xX$ for all $x \in X$. The symbol T_xX denotes the local tangent space at x, with $TX = \cup_{x\in X} T_xX$. The map v is called *vector field*, as it associates a vector to every point $x \in X$. For more details on tangent bundles, vector fields, and manifolds, I recommend the book of Lee [3]. You do not need these concepts for this exercise. Equivalent (but more informal) notations for the time derivative of the flow at $t = 0$ are $\frac{d}{dt}\phi^t(x)$, $\frac{d}{dt}x$, and $\dot{x}$. Parameters of a dynamical system change its behavior (i.e., the map ϕ) in rather arbitrary, mostly smooth ways. Such parameters can be indicated as a subscript to the evolution operator: the symbol ϕ_α indicates that the operator ϕ depends on a certain number of parameters $\alpha \in \mathbb{R}^k$. A bifurcation analysis of a dynamical system is concerned with qualitative changes of the system when the values of the parameters change. Qualitative change is formalized through the notion of topological equivalence, i.e. a system is *qualitatively the same* as another system, if it is *topologically equivalent*:

Definition 1. ***Topological equivalence.*** *A dynamical system (I, X, ϕ) is topologically equivalent to another dynamical system (I, Y, ψ) if there is a homeomorphism $h : X \to Y$ mapping orbits of the first system onto orbits of the second system, preserving the direction of time.*

Note that this definition does not take into account parameters. There is a separate definition for parameterized systems that includes a second map between the parameter spaces in the book of Kuznetsov [2]. You should read up on it to understand this additional notion, too! You can download the book through your TUM account at Springer[1], but you can also find it in a protected folder on Moodle.

Definition 2. ***Steady states / equilibrium points / fixed points.*** *A point $x_0 \in X$ is called an equilibrium (fixed point, steady state) if $\phi(t, x_0) = x_0$ for all $t \in I$.*

To visualize a given vector field v, phase portraits are very powerful. Essentially, they show suitably chosen sample vectors from v and orbits around qualitatively interesting parts of state space. In python, the `matplotlib` package offers a convenient way to visualize vector fields as phase portraits, through the method `streamplot`[2].

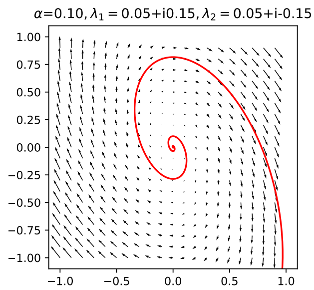

Figure 1: Phase portrait of a dynamical system with state $x \in \mathbb{R}^2$ and parametrized vector field $v_\alpha(x) = A_\alpha x = (\alpha x_1 + \alpha x_2, -0.25x_1)$, with $\alpha = 0.1$. A trajectory is shown in red, and the eigenvalues of the matrix A_α are shown in the title.

Definition 3. ***Bifurcation, see [2, p.57].*** *The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.*

Numerical solution to ordinary differential equations

Equation (2) defines the local derivative of the flow map at every point in the state space. There are thousands of books and hundreds of years of work in mathematics on the solution to the following problem: Which flow map (which function ϕ) satisfies equation (2), i.e. has local derivatives $v(x)$ at every point x? A sub-problem asks for individual trajectories: Given a point x, how does the set $\{\phi(t, x) | t \in I\} \subset X$ look like? This set is called *orbit*. The basic idea to find an orbit numerically is encoded in Euler's algorithm:

1. Start with a point $x_0 \in X$, and set the current iteration number to $n = 1$.
2. To generate a new point x_n on the orbit through x_0, define a small time step $\Delta t \in \mathbb{R}$, and compute
$$x_n = x_{n-1} + \Delta t \cdot v(x_{n-1}). \tag{3}$$
3. Increase n by one, and iterate. If Δt is small enough and v well-behaved, this will create an approximation to a part of the orbit through x_0.

For some systems, it is possible to use a negative value for Δt, and "integrate backwards in time", thus creating the orbit in the other direction. Note that you should avoid using Euler's method as much as possible, as it is not very accurate and you do not have any control over the error to the true orbit. Many robust and accurate solvers exist, in python you should look at `scipy.integrate.solve_ivp` (solves initial value problems).
Note: the number of points per exercise is a rough estimate of how much time you should spend on each task.

[1] `https://link.springer.com/book/10.1007%2F978-1-4757-3978-7`
[2] `https://matplotlib.org/3.1.1/gallery/images_contours_and_fields/plot_streamplot.html`

Task 1/5: Vector fields, orbits, and visualization **Points: 10/100**

A good way to visualize a dynamical system with a one- or two-dimensional state space is through its *phase portrait*. Consider the following linear dynamical system, with state space $X = \mathbb{R}^2$, $I = \mathbb{R}$, parameter $\alpha \in \mathbb{R}$, and flow ϕ_α defined by

$$\left.\frac{d\phi_\alpha(t,x)}{dt}\right|_{t=0} = A_\alpha x, \tag{4}$$

where $A_\alpha \in \mathbb{R}^{2\times 2}$ is a parametrized matrix. To produce the phase portraits in figure 2, A_α has the form

$$A_\alpha = \begin{bmatrix} \alpha & \alpha \\ -\frac{1}{4} & 0 \end{bmatrix}. \tag{5}$$

With the linear system defined in equation (4), construct a figure similar to Fig. 2.5 in [2, p.49] for hyperbolic

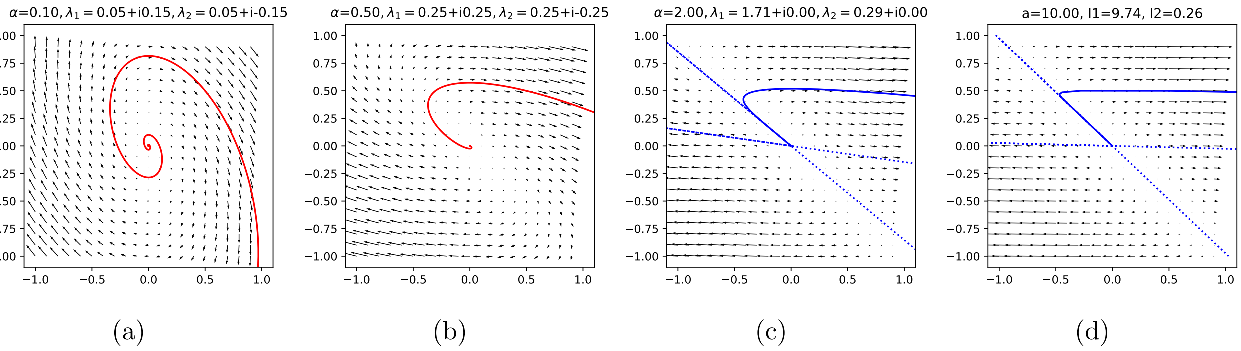

Figure 2: Phase portraits of system (4) with different values of α. The eigenvalues $\lambda_{1,2}$ of A_α are shown in the title of each portrait. Trajectories in red color indicate complex eigenvalues.

equilibria on the plane. Your phase portraits may look slightly different than in the book, and you have to use a different, parametrized matrix than A_α above to produce all the necessary phase portraits. You are also allowed to use more than one parametrized matrix if you cannot think of one that covers all necessary phase portraits. Specify the value of the parameter for each of your phase portraits. Are the systems in your figure topologically equivalent? Why, or why not? Have a look at [2, p.43ff] and repeat the arguments in your own words (no proof necessary!).

Checklist:

- ☐ Construct a figure similar to Fig. 2.5 in the book of Kuznetsov.
- ☐ Specify the value of the parameter for each of your phase portraits (ideally in the figure itself).
- ☐ Are these systems topologically equivalent? Why, or why not (no formal proof necessary)?
- ☐ Short description of the setup in the report? Which parametrized matrix did you choose?
- ☐ Verbose discussion of the results in the report?
- ☐ **Code:** modular, concise, well documented?

Task 2/5: Common bifurcations in nonlinear systems **Points: 20/100**

Consider a dynamical system on the real line $X = \mathbb{R}$, time $I = \mathbb{R}$, with the evolution described by

$$\dot{x} = \alpha - x^2. \tag{6}$$

For $\alpha > 0$, this system has two steady states at $x_0 = \pm\sqrt{\alpha}$, and for $\alpha < 0$ there are no steady states. What type of bifurcation happens at $\alpha = 0$? Plot the bifurcation diagram of the system for values of α in a suitable range (for example: $(-1, 1)$), visually indicating the stability of the steady states (for example, with color or different line types). Then, do the same for the following system:

$$\dot{x} = \alpha - 2x^2 - 3. \tag{7}$$

Are the systems (6) and (7) topologically equivalent at $\alpha = 1$? Why, or why not? What about the systems at $\alpha = -1$? Argue why the systems (6) and (7) have the same normal form (i.e., independent of the parameter value of α, they can be mapped to the same system). You do not need to formally prove your hypotheses, short arguments suffice. It is very useful for this task to think about the following question: if a system has zero (or one) steady states, can it be topologically equivalent to a system with two steady states? Why (not)? You do not need to discuss this separately in the report, but it is helpful to answer the questions above.

Checklist:

- [x] What type of bifurcation happens at $\alpha = 0$ for system (6)?
- [x] Plot the bifurcation diagram of the system for suitable values of α.
- [x] Visually indicate the stability of the steady states.
- [x] Do the same for the other system (7).
- [x] Are the systems (6) and (7) at $\alpha = 1$ topologically equivalent? Why, or why not?
- [x] What about the systems at $\alpha = -1$?
- [ ] Argue why the systems have the same normal form.
- [ ] Short description of the setup in the report?
- [ ] Verbose discussion of the results in the report?
- [ ] **Code:** modular, concise, well documented?

Task 3/5: Bifurcations in higher dimensions **Points: 10/100**

Bifurcations can happen for dynamical systems with state spaces of arbitrary dimension, and also in more than one parameter. Some bifurcations do not occur if the state space is one-dimensional (and the system is continuous). An important bifurcation for systems with one parameter exists for two-dimensional state spaces: the Andronov-Hopf bifurcation [2, p.57], with the vector field in normal form

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \tag{8}$$

1. Visualize the bifurcation of the system by plotting three phase diagrams at representative values of α.
2. For $\alpha = 1$, numerically compute and visualize two orbits of the system forward in time, starting at the point $(2, 0)$ and at $(0.5, 0)$. You can use Euler's method with a very small time step, or any numerical solver, but you have to descibe how you obtain the results.

Another important bifurcation occurs already in one state space dimension $X = \mathbb{R}$, but with two parameters $\alpha \in \mathbb{R}^2$: the cusp bifurcation, with normal form

$$\dot{x} = \alpha_1 + \alpha_2 x - x^3. \tag{9}$$

Visualize the bifurcation surface (i.e. all points (x, α_1, α_2) where $\dot{x} = 0$) of the cusp bifurcation in a 3D plot, with α_1, α_2 on the bottom plane and x in the third direction. An easy way to plot this surface is to sample points (x, α_2) uniformly and then plot the surface as a function α_1 of (x, α_2). Why is it called *cusp* bifurcation (where can one see the cusp)?

Checklist:

- [x] Visualize the bifurcation of the system by plotting three phase diagrams.
- [x] For $\alpha = 1$, numerically compute and visualize two orbits of the system (8).
- [ ] Visualize the bifurcation surface of the cusp bifurcation (9) in a 3D plot.
- [ ] Why is it called cusp bifurcation?
- [ ] Short description of the setup in the report?
- [ ] Verbose discussion of the results in the report?
- [ ] **Code:** modular, concise, well documented?

Task 4/5: Chaotic dynamics **Points: 30/100**

Dynamical systems can behave in very irregular ways, and changes in their parameters can lead to very drastic changes in their behavior.

Part 1: Consider the discrete map

$$x_{n+1} = r x_n (1 - x_n),\ n \in \mathbb{N}, \tag{10}$$

with the parameter $r \in (0, 4]$. Perform the following bifurcation analyses separately:

1. Vary r from 0 to 2. Which bifurcations occur? At which numerical values do you find steady states of the system?

2. Now vary r from 2 to 4. What happens? Describe the behavior (no formal proofs or exact statements necessary).

3. Plot a bifurcation diagram for r between 0 and 4 (horizontal axis), x between 0 and 1 (vertical axis), roughly indicating the positions of steady states and limit cycles.

The system described by equation (10) is well studied. It is called the "logistic map", and is a good example of chaos in discrete maps on one-dimensional spaces.

Part 2: Dynamical systems in continuous time cannot have smooth evolution operators that produce chaotic dynamics if the dimension of the state space is smaller than three. The Lorenz attractor [4] is a famous example for a system in three-dimensional space that forms a *strange attractor*, a fractal set on which the dynamics are chaotic.[3] Visualize a single trajectory of the Lorenz system starting at $x_0 = (10, 10, 10)$, until you reach the end time of $T_{\text{end}} = 1000$ (note that T_{end} is not the iteration count, but the simulated time!), at the parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. What does the attractor look like? The chaotic nature of the system implies that small perturbations in the initial condition will grow larger at an exponential rate, until the error is as large as the diameter of the attractor. Test this by plotting another trajectory from $\hat{x}_0 = (10 + 10^{-8}, 10, 10)$ in 3D, and separately plot the difference between the two trajectories over time (plot $\|(x(t) - \hat{x}(t)\|^2$ against t, where $x(t), \hat{x}(t) \in \mathbb{R}^3$ are the two trajectories for the two initial conditions). At what time is the difference between the points on the trajectory larger than 1?

Now, change the parameter ρ to the value 0.5 and again compute and plot the two trajectories. What is the difference in terms of the sensitivity to the initial conditions? Is there a bifurcation (or multiple ones) between the value 0.5 and 28? Why, or why not? Again, short arguments suffice.

[3] You can find a vector field with parameters for the chaotic regime on Wikipedia, `https://en.wikipedia.org/wiki/Lorenz_system`.

Checklist:

- [x] In system (10), vary r from 0 to 2. Which bifurcations occur?
- [x] At which numerical values do you find steady states and limit cycles of the system?
- [x] Now vary r from 2 to 4. What happens?
- [x] Plot and discuss a bifurcation diagram for r between 0 and 4.
- [x] Visualize a single trajectory of the Lorenz system starting at $x_0 = (10, 10, 10)$, discuss the results.
- [ ] Test initial condition dependence by plotting another trajectory.
- [ ] At what time is the difference between the points on the trajectory larger than 1?
- [x] Change to $\rho = 0.5$ and again compute and plot the two trajectories. Is there a bifurcation?
- [ ] Short description of the setup in the report?
- [ ] Verbose discussion of the results in the report?
- [ ] **Code:** modular, concise, well documented?

Task 5/5: Bifurcations in crowd dynamics **Points: 30/100**

In this task, you have to apply your knowledge about bifurcation theory to analyze and describe a given SIR model (from [5]). As described in their paper, for the number (or part of the whole population) of susceptible (S), infective (I), and removed (R), the differential equation based model is the following:

$$\frac{dS}{dt} = A - \delta S - \frac{\beta SI}{S+I+R} \quad (11)$$
$$\frac{dI}{dt} = -(\delta+\nu)I - \mu(b,I)I + \frac{\beta SI}{S+I+R} \quad (12)$$
$$\frac{dR}{dt} = \mu(b,I)I - \delta R, \quad (13)$$

where A is the "recruitment rate" (or birth rate) of susceptible population, $\delta > 0$ is the per capita natural death rate, $\nu > 0$ is the per capita disease-induced death rate, $\beta > 0$ is the average number of adequate contacts per unit time with infectious individuals, and μ is the per capita recovery rate of infectious individuals that depends on the number of infective persons I and the number of beds per 10,000 persons b. The function μ has the following form:

$$\mu(b,I) = \mu_0 + (\mu_1 - \mu_0)\frac{b}{b+I}, \quad (14)$$

with μ_0, μ_1 the minimum and maximum recovery rates based on the number of available beds. Note that we have slightly changed the notation from the parameter d (in [5]) to δ, to make the distinction from $\frac{dS}{dt}$ (derivative of S by t) and δS (multiplication of δ and S) clear. The parameters are set to the following values:

$$A = 20,\ \delta = 0.1,\ \nu = 1,\ \mu_0 = 10,\ \mu_1 = 10.45,\ \beta = 11.5,\ b = 0.01. \quad (15)$$

To complete the task, you have to do the following. You are allowed to use another programming language or visualization, but then you have to re-implement everything yourself and make sure the numerical integration method is correct (Euler's method will fail here!).

1. Download the example Jupyter notebook from Moodle and try to run it. It implements parts of the model, and also has the proper setup for its numerical integration. The model is not complete yet, and the notebook is not very well documented–you need to fix this as well, using the methods in the separate `sir_model.py` file.

2. You may have to fix some imports that are missing (it should only need `numpy`, `scipy`, and `matplotlib`), and you have to actually implement the SIR model equations that are missing in the `model` method.
3. The parameters are already set. However, the parameter b can be changed for a special bifurcation to occur! Try changing it from 0.01 to 0.03 in very small increments (e.g. 0.001) and observe what happens from the starting points $(S_0, I_0, R_0) = (195.3, 0.052, 4.4)$ (very slow behaviour, needs long integration time!), $(195.7, 0.03, 3.92)$, and $(193, 0.08, 6.21)$. Illustrate what happens by plotting either the 3D picture, or just the 2D projection in the (S, I) plane. Include at most nine plots in the report for this subtask (all on one page), the ones that illustrate best what happens when you change the parameter.
4. What kind of bifurcation happens between $b = 0.02$ and $b = 0.03$? What is the normal form of it? At what parameter value of b does it happen exactly (to three decimal places)?
5. In the paper [5], the authors define the reproduction rate $\mathbb{R}_0$ in equation (3.1) based on another paper [6]. Describe which variables are used to compute the reproduction rate, and what it means for the number of infective persons if the parameter β increases or decreases. You don't need to read [6] to answer this, but it is informative.
6. In [5], a part of theorem 3.2 is that "For the system (2.2) [the SIR model], the disease free equilibrium $E_0 = (A/d, 0, 0)$ at $\mathbb{R}_0 < 1$ is an attracting node". What does that mean? What happens for values of (S, I, R) close to E_0?
7. Bonus (10 points): discuss and illustrate (i.e. change the parameters and show trajectories) another type of bifurcation of the model, based on the analysis in [5]. There are many more, so it should be easy to find one—but be careful, it is not so easy to visualize them correctly! It is a rather complex model, so there may be different bifurcations in different parts of the state space. You get all points if you correctly identify a bifurcation, implement it by changing the parameters and visualize the correct region, and then discuss what happens and why.

Checklist:

- [ ] Implement the SIR model equations that are missing in the model method.
- [ ] Change b from 0.01 to 0.03 in small increments. What happens (from three starting points)?
- [ ] What bifurcation is that?
- [ ] Describe which variables are used to compute the reproduction rate...
- [ ] ... and what it means for the number of infective persons.
- [ ] What does E_0 being an attractive node mean?
- [ ] What happens for values of (S, I, R) close to E_0?
- [ ] Short description of the setup in the report?
- [ ] Verbose discussion of the results in the report?
- [ ] **Code:** modular, concise, well documented?
- [ ] **Bonus:** discuss another type of bifurcation of the model, including illustrations / visualizations.

References

[1] Felix Dietrich, Thomas N. Thiem, and Ioannis G. Kevrekidis. On the Koopman Operator of Algorithms. *SIAM Journal on Applied Dynamical Systems*, 19(2):860–885, January 2020.

[2] Yuri A. Kuznetsov. *Elements of Applied Bifurcation Theory*. Springer New York, 2004.

[3] John M. Lee. *Introduction to Smooth Manifolds*. Springer New York, 2012.

[4] Edward N. Lorenz. Deterministic Nonperiodic Flow. *Journal of the Atmospheric Sciences*, 20(2):130–141, March 1963.

[5] Chunhua Shan and Huaiping Zhu. Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds. *Journal of Differential Equations*, 257(5):1662–1688, September 2014.

[6] P. van den Driessche and James Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1-2):29–48, November 2002.