Sec. 2.4 The Fundamental Theorem of Linear Programming

Recall: Standard LP: min $z = c^T x$, s.t. $Ax = b$, $x \geq 0$. (Assume full rank of A) $\text{rank}(A) = m$

Theorem (FTLP) (1). feasible solution → There is a **b**.f.s. B^{-1}, $x_D = 0$

(2). finite optimal sol → There is an optimal **b**.f.s. ➔ Key word basic. $[B \quad D] \begin{bmatrix} x_B \\ x_D \end{bmatrix} = b$

Pf. Notation. Let A = $[a_1 \ldots a_n]_{m \times n}$ in n col vectors.

Show (1). Let x be a f.s. $Ax = b$, $x \geq 0$

$$[a_1\ a_2 \cdots a_n]\, x = b, \quad x \geq 0$$

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b, \quad x \geq 0$$

Let p = #(positive components of x) $\leq n$

Assume w.o.l.o.g., 1^{st} p components of x are > 0. $\Rightarrow x_1 > 0, \cdots, x_p > 0, x_{p+1} = 0, \cdots, x_n = 0$

Case 1. $a_1, \cdots, a_p$ are linearly indep. $p \leq n$; $p \leq m$

Claim: The given x is already basic.

$p = m$, $B \hat{=} [a_1 \cdots a_p]$. $x_j = 0$, $j = p+1, \cdots, n$ (nonbasic)

$p < m$, $B = [a_1 \cdots a_p$ all $m-p$ cols$]$, $x_j = 0$, j = nonbasic $\Rightarrow x$ is basic, degenerate.

Case 2. $a_1, \ldots, a_p$ are linearly dep.

Idea: Remove redundancy $\underbrace{\rightarrow \cdots \rightarrow}_{\text{Keep feasibility}}$ indep. $\Rightarrow$ case 1.

$\rightarrow y_1 a_1 + \cdots + y_p a_p + 0 + \cdots + 0 = 0$ with at least one $y_{i_0} > 0$. $(y_{p+1} = 0, \cdots, y_n = 0)$ ($*(-1)$ if necessary)

$*\varepsilon$

$x_1 a_1 + \cdots + x_p a_p + 0 + \cdots + 0 = b$ (-1)

$*$ $(x_1 - \varepsilon y_1) a_1 + \cdots + (x_p - \varepsilon y_p) a_p + 0 + \cdots + 0 = b$ $\Leftrightarrow$ $A(x - \varepsilon y) = b$

$\leftarrow$ new coeff. vector $x - \varepsilon y$.

$\varepsilon = ?$ $x_j - \varepsilon y_j \geq 0$ ✓, $j = 1, \cdots, p$. $\Rightarrow$ $\varepsilon^* = \min \left\{ \frac{x_j}{y_j} : y_j > 0 \right\}$ $\Rightarrow$ minimizing j_0

not empty (i_0).

$x_{j_0} - \varepsilon y_{j_0} = 0$. $\Rightarrow$ # of zero components of $x - \varepsilon y$ $< p$

Repeat this process until no redundancy $\Rightarrow$ Apply case 1.

(2). opt. sol. $\Rightarrow$ opt. b.f.s. --- almost the same proof.

Just need to check $x - \varepsilon y$ opt. $\leftarrow$ x opt.

$$c^T x - \varepsilon c^T y \qquad\qquad c^T x$$

$$\downarrow$$

show $c^T y$ is 0.

Suppose not. $c^T y \neq 0$.

$$c^T(x - \varepsilon y) = \overset{\text{better}}{c^T x - \varepsilon c^T y} < \overset{\text{opt.}}{c^T x} \quad \text{for some } \varepsilon.$$

$x - \varepsilon y$

$\uparrow$

$\varepsilon < 0$

⊕ $0 < \varepsilon << 1$, $x - \varepsilon y \geq 0 \Rightarrow$ ⨳

⊖ $-1 << \varepsilon < 0$, $x - \varepsilon y \geq 0 \Rightarrow$ ⨳.

$x > 0$.