Chapter 2. Basic Properties of Linear Programming

Sec. 2.1 Introduction

Optimization:

Optimize $z = f(x)$ the **objective** function

subject to $h(x) = 0$ the **equality** constraints

$g(x) \leq 0$ the **inequality** constraints

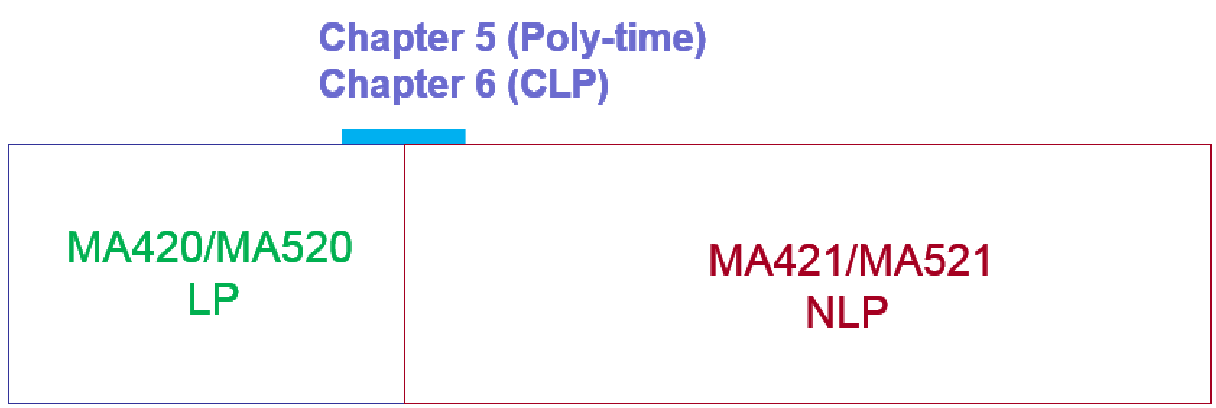
$x \in S$ in R^n. the **set** constraint (may or may not be included in h and g)

MA420/520 Linear Programming:

(LP): $f(x)$, $h(x)$, $g(x)$ **all linear** functions; straight boundary of S;

MA421/521 Nonlinear Programming:

(NLP): At least one of $f(x)$, $h(x)$, and $g(x)$ **is nonlinear**; or curved boundary of S;



Examples of nonlinear models:

<> the Cobb-Douglas production function that relates output (Y) to labor (L) and capital (K) can be written as $Y = aL^bK^r$.

<> free falling $h(t) = 0.5gt^2$.

<> exponential growth (compound interest) $p(t) = p_0e^{at}$.

<> stock price in time (no analytic formula).

<> shapes of physical objects.

<> the shortest and equal time curves (inverted cycloid).

<> least squares fitting of data. (AI)

<> probability density functions.

<> …

.

Linear Programming (Linear Optimization)

Ex1 Ex2

Linear -------- functions $f(x) = 2x + 3$, $x+4$

equations: $2x + 3 = 1$, $x+4=3$

inequalities: $2x + 3 \leq 1$, $x+4 \leq 5$

Optimization --- minimization/maximization, both minimization&maximization (game)

say minimize above linear functions → Answer: (optimal f-value) $f^* = -\infty$

solve equation: $2x + 3 = 1$ → solution $x=-1$

solve inequality: $2x + 3 \leq 1$ → solutions $x \leq -1$

⇒ Algebra

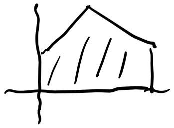
Not interesting in LP.

To make the LP more meaningful, we add (on top of f(x)):

more variables. min $f(x,y) = 2x + 3y + 3$. → $f^* = -\infty$

(linear) constraints: min $f(x,y) = 2x + 3y + 3$ subject to (s.t.) $x \geq 0$ and $y \geq 0$. → $f^* = f(0,0) = 4$, $x^* = y^* = 0$.

Def. Standard LP: min $z=c^Tx$ s.t. "$Ax=b,\ x\geq 0$", $x\in R^n$, $b\in R^m$, $\boxed{A_{m\times n}}$. $c\in R^n$ $\longrightarrow$ $\boxed{A, b, c}$



§2.5
feasible sol. x

Def. x^* is an optimal sol. if (1) $Ax^*=b$, $x^*\geq 0$. (feasibility) (2). For any feasible x, $c^Tx^*\leq c^Tx$ (smallest obj-f value)

"Solve LP" — Find x^* and $z^*=f^*=f(x^*)$. $\longrightarrow$ Your answer to any LP should be $\begin{cases} x^*,\ z^* \\ \text{not feasible.}\end{cases}$

Def. (LP$_1$) is equivalent to (LP$_2$) if any opt. sol. of one LP induces an opt. sol. of the other LP.

Claim: Any LP $\Leftrightarrow$ std (LP) (A, b, c)

Ex. max $w=c^Tx$ s.t. $Ax=b$, $x\geq 0$. $\Leftrightarrow$ std LP. min $z=-c^Tx$ s.t. $Ax=b$, $x\geq 0$. Equivalence $\begin{cases} \text{same } x^* \\ w^*=-z^*\end{cases}$

Ex1. LP$_1$: min $z=c^Tx$ s.t. $\underline{Ax\leq b}$, $x\geq 0$. $\Leftrightarrow$ std LP$_2$.

Sol. LP$_1$ $\Leftrightarrow$ min $z=c^Tx$ s.t. $Ax+s=b$, $x\geq 0$, $s\geq 0$. s is called a slack var.

$x'=\begin{bmatrix}x\\ s\end{bmatrix}$ $\Leftrightarrow$ min $w=c'^Tx'$ s.t. $\begin{bmatrix} & \end{bmatrix}x'=b'$, $x'\geq 0$

$c'=\begin{bmatrix}c\\ 0\end{bmatrix}$, $\begin{bmatrix}A & I\end{bmatrix}$, $b'=b$.

std LP with c^T, b'; $A'=[A\ I]$.

Equivalence. same x^*, $s^*=b-Ax^*$, $z^*=w^*$. $\left(w=\begin{bmatrix}c\\ 0\end{bmatrix}'^{T}\cdot\begin{bmatrix}x\\ s\end{bmatrix}'=c^Tx+0^Ts=c^Tx=z\right)$

Ex2. min $z = C^T x$ s.t. $\underline{Ax \geq b}$, $x \geq 0$.

$\downarrow$

$Ax - s = b$, $s \geq 0$ called surplus var.

Ex. LP_1: min $z = C^T x$ s.t. $Ax = b$, $0 \leq x \leq d$. (bounded var.)

Sol. - - - - - - - - - - - - - - - , $x + s = d$, $s \geq 0$. ---- std LP. (LP_2).

$\Leftrightarrow$ LP_2: min $w = \begin{bmatrix} c \\ 0 \end{bmatrix}^T \begin{bmatrix} x \\ s \end{bmatrix}$ s.t. $\begin{bmatrix} A & 0 \\ I & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$, $x' = \begin{bmatrix} x \\ s \end{bmatrix} \geq 0$.

Equivalence. same x^*, $s^* = d - x^*$, $z^* = w^*$

Ex. (free var.) 3-4.

min $z = 2x_1 - 3x_2$ s.t. $-3x_1 + x_2 = 5$, $x_2 \geq 0$. $\Leftrightarrow$ std LP ?

Sol. (1). replacement of free x_1 by two nonnegative vars. $x_1 = u - v$, $u \geq 0$, $v \geq 0$

std LP. min $z = 2(u-v) - 3x_2$ s.t. $-3(u-v) + x_2 = 5$, $x_2 \geq 0$, $u \geq 0$, $v \geq 0$.

$A' =$ $b' =$ $c' =$

(2). Eliminate x_1, $x_1 = \dfrac{5 - x_2}{-3} = -\dfrac{5}{3} + \dfrac{1}{3}x_2$

LP: $2\left(\dfrac{-5}{3} + \dfrac{1}{3}x_2\right) - 3x_2$ s.t. $x_2 \geq 0$ $\Leftrightarrow$ min $\dfrac{-10}{3} + \dfrac{2}{3}x_2 - 3x_2$ s.t. $x_2 \geq 0$

$\Leftrightarrow$ std LP. min $\overset{w=}{\dfrac{-7}{3}x_2}$ s.t. $x_2 \geq 0$ $c = -\dfrac{7}{3}$, $A = 0$, $b = 0$.

Equivalence, same x_2^*, $x_1^* = -\dfrac{5}{3} + \dfrac{1}{3}x_2^*$, $z^* = w^* - \dfrac{10}{3}$