Sec. 2.3 Basic Solutions

Ex. (<u>geometry</u> of LP) Solve this LP.

$$\min z = -50x_1 - 80x_2 \quad \text{--- objective function (cost)}$$

$$\text{s.t.} \quad x_1 + 2x_2 \overset{(1)}{\le} 32,\ 3x_1 + 4x_2 \overset{(2)}{\le} 84,\ x_2 \overset{(3)}{\le} 12,\ x_i \ge 0,\ i=1,2. \quad \text{--- constraints}$$

Sol. Idea: <u>geometry</u> of LP.

Step 1. Graph the feasible set $K = \{(x_1, x_2) \mid (1), (2), (3), x \ge 0\}$.

$K = \text{polygon}\{O, A, D, F, G\}$

Step 2. Consider $-50x_1 - 80x_2 \overset{\text{Let}}{=} \text{constant}$

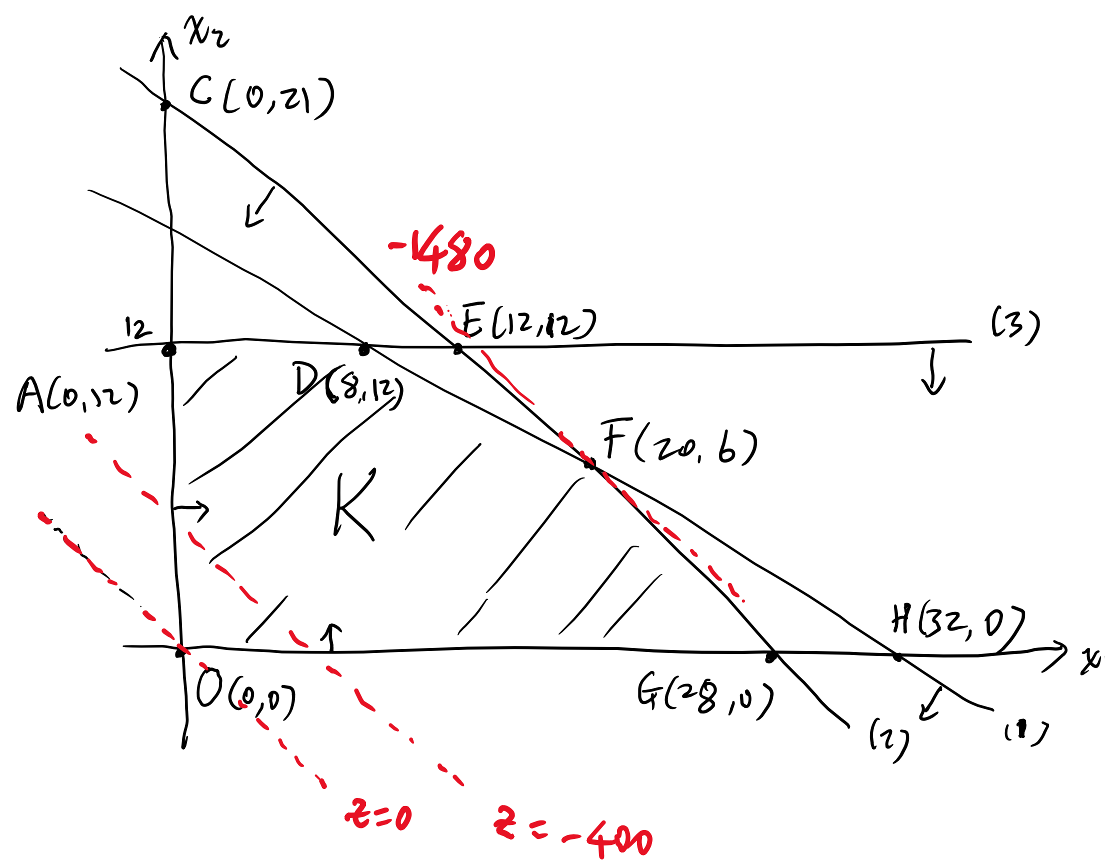
"constant cost lines"

$z = 0.$

$\vdots$

$z = -1480 = z^*$

$x^* = (20, 6)$



Observations:

std LP: $\min z = c^T x$, s.t. $Ax = b, x \geq 0$

1). K is convex.

2). Polygon, corners (vertices)

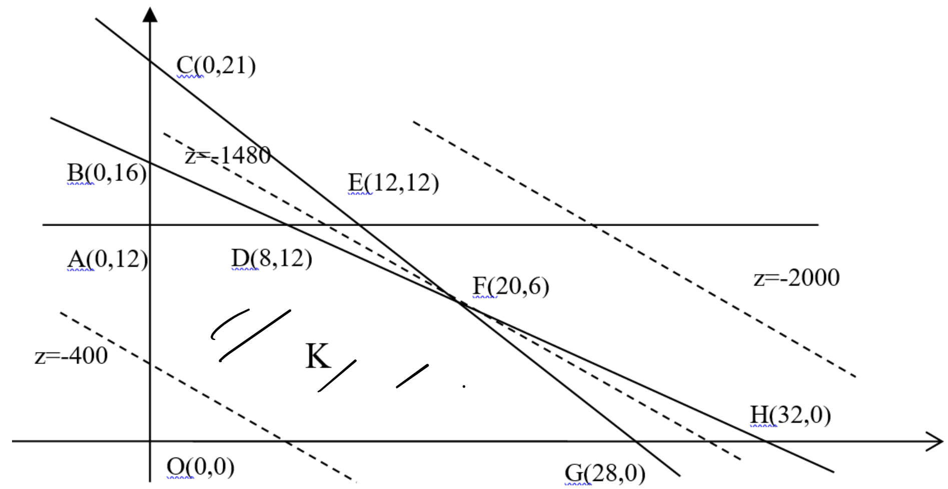
3). # of pts in $K = \infty$

4). #(vertices) $< \infty$

5). min. over K is equivalent to min over its vertices.

6). vertex —— LA. (system of linear eqns.)

7). higher dim cf. §2.5 (later)



Assume (w.o.l.o.g.): std LP with $b \geq 0$, rank(A) = m (full row rank), m < n.

std LP: min $z = c^T x$, s.t. $Ax = b$, $x \geq 0$

Def. x is a sol. if $Ax=b$

---- feasible sol if $Ax=b$, $x \geq 0$.

---- basic sol. (b.s.) if $Ax=b$, $A = [B \ \ D]$, $x = (x_B, x_D)$. $\Rightarrow [B \ D] \begin{bmatrix} x_B \\ x_D \end{bmatrix} = b$, $Bx_B + Dx_D = b$

$m \ \ n$, B: $m \times m$, B^{-1} (basic), $x_B = B^{-1}b$, $x_D = 0$, $Dx_D = 0$

---- degenerate b.s. ---- b.s. with at least basic var $= 0$.

---- basic feasible sol. (b.f.s.) ---- b.s. + feasible

Recall: x^* is an opt. sol. if $\begin{cases} 1. & x^* \text{ is feasible.} \\ 2. & z(x^*) \text{ is mini.} \end{cases}$ ---- $z(x^*) \leq z(x)$ for any feasible x.

Note: $\#(\text{b.s.}) \leq \binom{n}{m} = \dfrac{n!}{(n-m)!\, m!} < \infty$

$\#(\text{b.f.s}) \leq$ -- $< \infty$

Ex (again, but now with algebraic approach)

now add 3 slack variables x_3, x_4, x_5

(LP) $\min z = -50x_1 - 80x_2$

s.t. $x_1 + 2x_2 \le 32$, $3x_1 + 4x_2 \le 84$, $x_2 \le 12$, $x_i \ge 0$, $i=1,2$.

Std LP: $\min z = -50x_1 - 80x_2 + 0x_3 + 0x_4 + 0x_5$

s.t. $x_1 + 2x_2 + 1x_3 + 0x_4 + 0x_5 = 32$

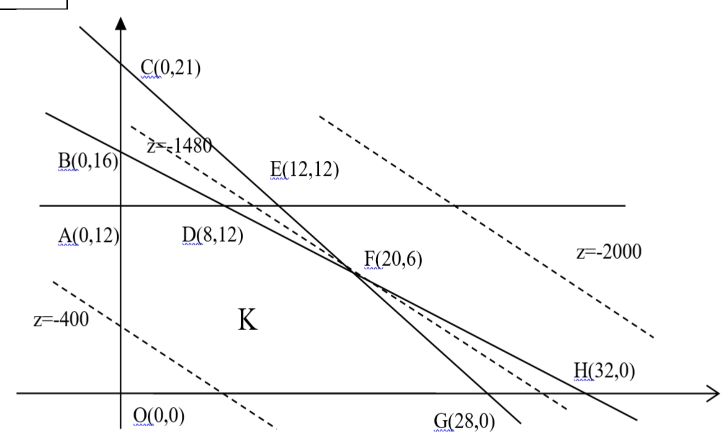
$+x_4 \quad = 84$

$+x_5 \quad = 12$

$x \geq 0$

(equivalent) Std LP (dim-5)		Original LP (dim=2)	Cost z
nonbasic variables	basic solution x	vertex (*if feasible)	$c^T x$
1,2	(0,0,32,84,12)	*O(0, 0)	0
1,3	(0,16,0,20,-4)	B(0, 16)	
3,4	(20,6,0,0,6)	*F(20, 6)	-1480(best)
4,5	(12,12,-4,0,0)	E(12, 12)	
2,5	n/a	n/a	
1,4	(0,21,-10,0,-9)	C(0, 21)	
1,5	(0,12,8,36,0)	*A(0,12)	-960
2,3	(32,0,0,-12,12)	H(32, 0)	
2,4	(28,0,4,0,12)	*G(28, 0)	-1400
3,5	(8,12,0,12,0)	*D(8, 12)	-1360

new



Ex. Given A= $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$, b=(1,4,0), B= $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Find the b.s.

Sol. $\text{rref}([B \mid b]) \to$

b.s. = $x = (\underline{1}, \underline{2}, \underline{1}, \ , 0)$

nondegenerate, feasible.

Ex. Given A= $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & -2 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ b = $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$. Find the b.s.

Sol. $\text{rref}(\qquad\qquad) \to$

$x = (0, 1, 1, 0)$

degenerate, b.f.s.