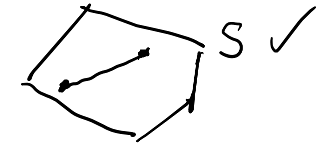
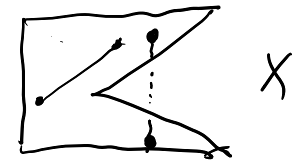
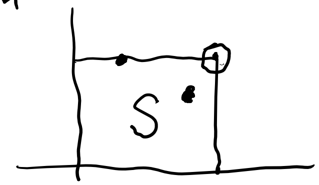
Sec. 2.5 Relations to Convexity (cf. Appendex B)



Def. A set S is a convex set if $\forall$ (for any) $x, y \in S$, $0 \le \alpha \le 1$, $\underbrace{\alpha x + (1-\alpha) y}_{\text{convex combination}} \in S$.



Def. $x \in S$ is an extreme point (or vertex) of a convex S if it cannot be strictly between two distinct pts. of S.



Ex. $K = \{x \mid Ax = b,\ x \ge 0\}$, the feasibility set of std LP, is convex.

proof. $\forall$ $x \in K$: $Ax = b$, $x \ge 0$ $\leftarrow \alpha$

$y \in K$: $Ay = b$, $y \ge 0$ $\leftarrow (1-\alpha)$ } (+) $\Rightarrow$ $A\alpha x + A(1-\alpha)y = \alpha b + (1-\alpha) b$, i.e. $A(z) = b$

$z \triangleq \alpha x + (1-\alpha) y \ge 0$

$\alpha \in [0,1]$ $Az = b$, $z \ge 0$ $\Rightarrow z \in K$. $\square$

FTLP. $x^* =$ a b.f.s. optimal

Ex. in §2.3. consider b.f.s / vertices / extreme pts.

Theorem. extreme points of K <--> b.f.s.

(geometry) (algebra)

Pf. (<===) x is a b.f.s. Say $x=(x_B, 0)$, $Ax=b$, $x\geq 0$ ✓ Want to show x is an extreme pt.

Suppose (not) x is not an extreme pt, we would write $x = ty + (1-t)z$ for $y \neq z$ in K, $0<t<1$

$y\geq 0$, $z\geq 0$.

$(x_B, 0) = t(y_B, y_D) + (1-t)(z_B, z_D)$

$0 = t y_D + (1-t) z_D \Rightarrow y_D = 0, z_D = 0$

$Ax=b \Leftrightarrow Bx_B = b$, B^{-1}.

$\downarrow$

$x_B = B^{-1}b$

$Ay=b \Leftrightarrow By_B = b$ $y_B = B^{-1}b$

$Az=b \Leftrightarrow Bz_B = b$ $z_B = B^{-1}b$

$\Downarrow$

$y = z$ $(=x)$

$\Rightarrow$ x is an extreme pt.

$(\Longrightarrow)$. x is an extreme pt. of K. Want to show x is a b.f.s.

After reordering, we get $x = (x_B, 0)$ with $x_B > 0$. $\underbrace{Bx_B + \overset{x_D=0}{0} = b}$.

Case 1. Cols of B are linearly indep. $\xrightarrow{LA}$ Expand B with additional linearly indep. cols. to get

$\bar{B}x_{\bar{B}} + 0 = b$, $\bar{B}$ with $\bar{B}^{-1}$ ②. $x_{\bar{B}} = (x_B, 0)$, $x \geq 0$ ③

$x = (x_{\bar{B}}, 0)$ ①

$\Rightarrow$ x is a b.f.s. (satisfying 3 conditions ①, ②, ③) case 2. Cols of B are linearly dep. Let $k = \#(\text{cols. of } B)$. Want to show case 2 is impossible (x is not an extreme pt)

$\Rightarrow \exists$ (there is) $p \neq 0$ s.t. $p_1 B_1 + p_2 B_2 + \cdots + p_k B_k = 0$

$x_1 B_1 + x_2 B_2 + \cdots + x_k B_k = b, \quad x_i > 0$

t

$(x_1 + t p_1) B_1 + \cdots + (x_k + t p_k) B_k = b$

$\underbrace{\begin{matrix} + t p_1 \\ - t p_1 \end{matrix}}_{\geq 0}$ $\qquad$ $\underbrace{\begin{matrix} + t p_k \\ - t p_k \end{matrix}}_{\geq 0}$

$\underbrace{t \approx 0 \qquad\qquad t \approx 0}_{\text{smallest } t \approx 0}$

x is strictly between 2 sols.

$$\left.\begin{array}{ll} x_i \pm t p_i \geq 0, & i = 1, \cdots, k. \\ \text{zero-extension: } p_i = 0, & i = k+1, \cdots, n \end{array}\right\} \Rightarrow x \pm t p \geq 0$$

$y \hat{=} x + tp \in K$, $z \hat{=} x - tp \in K$. $\quad x + tp \neq x - tp$ since $p \neq 0$.

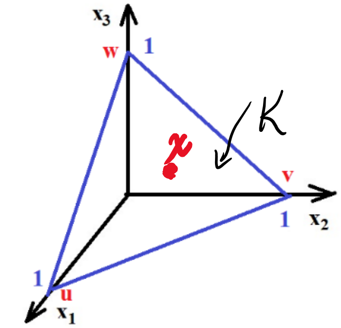
$x = \frac{1}{2} y + \frac{1}{2} z \Rightarrow x$ is NOT an extreme pt $\quad \cancel{\times}$

(x is strictly between y and z)

Ex1. K={(x_1,x_2,x_3): $x_1+x_2+x_3 = 1$, $x_1\geq 0$, $x_2\geq 0$, $x_3\geq 0$}

$Ax = b$ $\quad\quad x \geq 0$

claim: When K = bdd., any pt x in K is a convex combination of its vertices



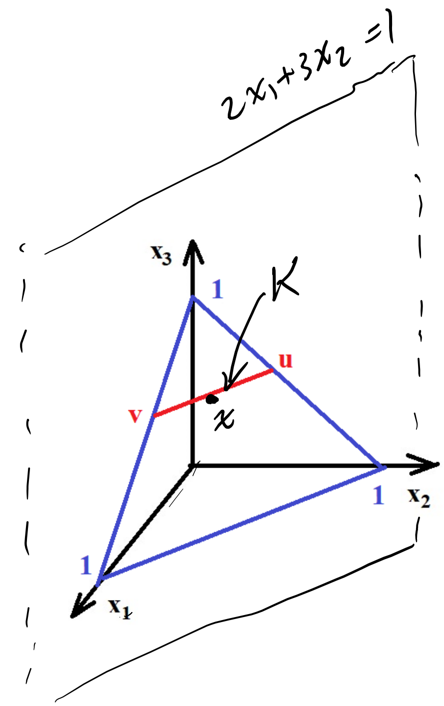
$\alpha_i \geq 0$, $\sum \alpha_i = 1$.

x = (2/3, 1/6, 1/6) = $\underset{2/3}{\underline{\alpha_1}}$ u + $\underset{1/6}{\underline{\alpha_2}}$ v + $\underset{1/6}{\underline{\alpha_3}}$ w. — by LA.

Ex2. K={(x_1,x_2,x_3): $x_1+x_2+x_3 = 1$, $\underline{2x_1+3x_2=1}$, $x_1\geq 0$, $x_2\geq 0$, $x_3\geq 0$}

$x_1=0$: $x_2, x_3 \Rightarrow u$

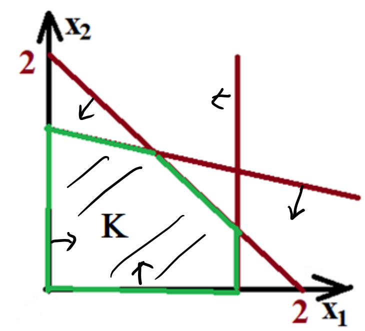
$x_2=0$: $x_1, x_3 \Rightarrow v$



x = (1/3, 1/9, 5/9) = $\overset{1/3}{\underline{s}}$ u + $\overset{2/3}{\underline{t}}$ v, where u = (0, 1/3, 2/3), v=(1/2,0,1/2).

$$\begin{bmatrix} 1/3 \\ 1/9 \\ 5/9 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/3 & 0 \\ 2/3 & 1/2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \iff Ay=b. \quad \text{rref}([A \mid b]) = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Ex3. (similar to ex of Sec. 2.3, also HW2.14)



$K = \{(x_1, x_2)\colon\ x_1 + 8/3x_2 \le 4,\ x_1 + x_2 \le 2,\ 2x_1 \le 3,\ x_i \ge 0,\ i=1,2\}$.

Find all b.s.'s and b.f.s.'s.

Sol.

$$x_1 + \frac{8}{3}x_2 + x_3 = 4$$

$$x_1 + x_2 + x_4 = 2$$

$$2x_1 + x_5 = 3$$

$$x \geq 0$$

Std LP (dim-5)		Original LP (dim=2)
nonbasic variables	basic solution x	vertex
1,2	(0,0,4,2,3)	(0, 0)
1,3	(0,3/2,0,1/2,3)	(0, 3/2)
1,4	(0,2,-4/3,0,3)	(0, 2)
1,5	(n/a)	(n/a)
2,3	(4,0,0,-2,-5)	(4, 0)
2,4	(2,0,2,0,-1)	(2, 0)
2,5	(3/2,0,5/2,1/2,0)	(3/2, 0)
3,4	(4/5,6/5,0,0,7/5)	(4/5, 6/5)
3,5	(3/2,15/16,0,-7/16,0)	(3/2, 15/16)
4,5	(3/2,1/2,7/6,0,0)	(3/2, 1/2)