Unit 5:- Application of Partial Differentiation.

* Question Bank *.

Q.1) If $x = uv$, $y = \frac{u+v}{u-v}$ then find $\frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)}$

Soln Given, $x = uv$, $y = \frac{u+v}{u-v}$

consider, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$x_u = v$, $x_v = u$.

$y_u = \frac{(u-v)(1) - (u+v)(1)}{(u-v)^2} = \frac{u-v-u-v}{(u-v)^2} = \frac{-2v}{(u-v)^2}$ → 1M

$y_v = \frac{(u-v)(1) - (u+v)(-1)}{(u-v)^2} = \frac{u-v+u+v}{(u-v)^2} = \frac{2u}{(u-v)^2}$ → 1M

$\therefore J = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix}$

$J = \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}$ → 2M

But $J' = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{J} = \frac{(u-v)^2}{4uv}$. → 1M

Q.2) Discuss maxima and minima of

$$f(x,y) = x^2 + y^2 + 6x + 12.$$

Soln Given :- $f(x,y) = x^2 + y^2 + 6x + 12$

$f_x = 0 \rightarrow 2x + 6 = 0$

$2x = -6 \quad \boxed{x = -3}$

$f_y = 0 \rightarrow 2y = 0 \quad \boxed{y = 0}$

Stationary points is $(x,y) = (-3,0)$ → 2M

$r = f_{xx} = 2$

$s = f_{xy} = f_{yx} = 0$

$t = f_{yy} = 2.$ → 1M

$\therefore rt - s^2 = (2)(2) - 0 = 4 > 0$

& $r = 2 > 0$

$\boxed{r > 0}$

f is minimum at $(-3,0)$ → 1M

$f_{min} = (-3)^2 + (0) + 6(-3) + 12$

$= 9 - 18 + 12$

$\boxed{f_{min} = 3}$ → 1M

Q.3) In calculating the volume of a right circular cone, errors of 2% and 1% are made in measuring the height and radius of base respectively &

Soln

Volume of Right Circular cone $(V) = \frac{1}{3}\pi r^2 h$ → 1M

take log on both side

$$\log V = \log\left(\frac{1}{3}\pi r^2 h\right)$$

$$\log V = \log\left(\frac{\pi}{3}\right) + \log(r^2) + \log h$$ → 1M

$$\log V = \log\left(\frac{\pi}{3}\right) + 2\log r + \log h$$

on differention.

$$\frac{1}{V}dV = 0 + 2\frac{1}{r}dr + \frac{1}{h}dh$$ → 1M

$$100\frac{dV}{V} = 2\left(100\frac{dr}{r}\right) + 100\frac{dh}{h}$$ → 1M

$$= 2(1) + 2$$

$$\boxed{\%V = 4}$$ → 1M

Q.4) Examine whether $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$. If dependent, if so, find the relation between them.

Sol^n $\left. u = \frac{x+y}{1-xy}, \; v = \tan^{-1}x + \tan^{-1}y \right\}$ Given.

consider, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}$

$$U_x = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2} \rightarrow \underline{\underline{1M}}$$

$$U_y = \frac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2} \rightarrow \underline{\underline{1M}}$$

Now,

$$V_x = \frac{1}{1+x^2} \qquad V_y = \frac{1}{1+y^2}$$

consider,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = 0 \rightarrow \underline{\underline{2M}}$$

$\therefore$ u & v are functionally dependent.

$\underline{\underline{4\,1M}}$

Relation :- $v = \tan^{-1}x + \tan^{-1}y$

$$v = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$v = \tan^{-1}(u)$$

$$\boxed{v = \tan^{-1}u} \text{ or } \boxed{\tan v = u} \longrightarrow \underline{2M}$$

Q5) Find the Extreme values of $x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$

Sol^n Given :- $F(x,y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$ — ①

$$\frac{\partial F}{\partial x} = 0 \rightarrow 2x + 0 + 2\left(\frac{-1}{x^2}\right) = 0$$

$$x - \frac{1}{x^2} = 0$$

$[a^3 - b^3 = (a-b)(a^2+ab+b^2)]$

$$x^3 - 1 = 0$$

$$(x-1)(x^2+x+1) = 0$$

$$\boxed{x = 1}$$

$$\frac{\partial F}{\partial y} = 0 \rightarrow 2y + 2\left(\frac{-1}{y^2}\right) = 0$$

$$y^3 - 1 = 0$$

$$\boxed{y = 1}$$

Stationary point in $(x,y) = (1,1)$ $\longrightarrow \underline{2M}$

[II] Find r, s, t

$r = F_{xx} = 2 + \frac{4}{x^3}$

$s = F_{xy} = F_{yx} = 0$

$t = F_{yy} = 2 + \frac{4}{y^3}$

at $(1,1)$

$r = 6$, $s = 0$, $t = 6$ → 1M

[III] $rt - s^2 = 6(6) - 0 = 36 > 0$

& $r = 6 > 0$

→ F is minimum at $(1,1)$ is → 1M

$f_{min.} = (1)^2 + (1)^2 + \frac{2}{1} + \frac{2}{1}$

$\therefore \boxed{F_{min} = 6}$ → 1M

Q6. If $u = x+y^2$, $v = y+z^2$, $w = z+x^2$, using Jacobians find $\frac{\partial x}{\partial u}$.

Sol^n let

$f = x+y^2-u=0$

$g = y+z^2-v=0$

$h = z+x^2-w=0$

$$\therefore \frac{\partial x}{\partial u} = -\frac{\frac{\partial(f,g,h)}{\partial(u,y,z)}}{\frac{\partial(f,g,h)}{\partial(x,y,z)}} = -\frac{N}{D} \quad \text{--- (1)} \rightarrow \underline{1M}$$

$$N = \frac{\partial(f,g,h)}{\partial(u,y,z)} = \begin{vmatrix} f_u & f_y & f_z \\ g_u & g_y & g_z \\ h_u & h_y & h_z \end{vmatrix} = \begin{vmatrix} -1 & 2y & 0 \\ 0 & 1 & 2z \\ 0 & 0 & 1 \end{vmatrix}$$

$$N = -1(1) - 2y(0) + 0$$

$$\boxed{N=-1} \quad \text{--- (2)} \qquad \rightarrow \underline{1M}$$

$$D = \frac{\partial(f,g,h)}{\partial(x,y,z)} = \begin{vmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{vmatrix}$$

$$D = \begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix} = 1(1) - 2y(-4xz) + 0$$

$$= 1 + 8xyz \quad \text{—} \ (3) \qquad \rightarrow 2M$$

put in Eqⁿ (1) values of Eqⁿ (2) & (3)

$$\frac{-N}{D} = \frac{-(-1)}{1+8xyz} = \boxed{\frac{1}{1+8xyz}} \qquad \rightarrow 1M$$

Q.7) A power dissipited in a resistor is given by $p = \frac{E^2}{R}$. If Errors of 3% and 2% are found in E & R Respectively, Find the percentage error in P.

Solⁿ $p = \frac{E^2}{R}$ — given.

take log on both sides

$$\log p = \log\left(\frac{E^2}{R}\right)$$

$$\log p = \log E^2 - \log R$$

$$\log p = 2\log E - \log R \quad \text{—} \ (1) \qquad \rightarrow 2M$$

on differentiation

$$\frac{1}{p}\,dp = 2\,\frac{1}{E}\,dE - \frac{1}{R}\,dR$$

$$100\frac{dp}{p} = 2\left(100\frac{dE}{E}\right) - \left(100\,\frac{dR}{R}\right) \qquad \rightarrow 2M$$

$$100\,\frac{dp}{p} = 2(3) - 2 = 4 \qquad \rightarrow 1M$$

$$\boxed{\%p = 4\%} \qquad \rightarrow$$

Q.8) Using Lagrange's method find extreme value of xyz if $x+y+z=a$.

Soln

(1) let $u = xyz$ — (1)

$\phi = x+y+z-a=0$ — (2)

(2) consider

$F = u + \lambda\phi$

$F = xyz + \lambda(x+y+z-a)$ — (3)

$\frac{\partial F}{\partial x} = 0 \rightarrow yz + \lambda(1) = 0$

$\boxed{-\lambda = yz}$ — (4)

$\frac{\partial F}{\partial y} = 0 \rightarrow xz + \lambda(1) = 0$

$\boxed{-\lambda = xz}$ — (5)

$\frac{\partial F}{\partial z} = 0 \rightarrow xy + \lambda(1) = 0$

$\boxed{-\lambda = xy}$ — (6) $\rightarrow$ 2M

(III) from eqn (4), (5) & (6)

$yz = xz = xy$

divide by xyz

$$\frac{yz}{xyz} = \frac{xz}{xyz} = \frac{xy}{xyz}$$

$$\frac{1}{x} = \frac{1}{y} = \frac{1}{z} = \frac{1}{k} \text{ (say)}$$

$\rightarrow x = k, \ y = k, \ z = k$ $\longrightarrow$ 1M

Eqn (2) becomes

$x + y + z = a$

$k + k + k = a$

$3k = a$

$$\boxed{k = \frac{a}{3}}$$

$\therefore x = \frac{a}{3}, \ y = \frac{a}{3}, \ z = \frac{a}{3}$ $\longrightarrow$ 1M

Stationary points are

$$(x, y, z) = \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$$

$\therefore$ Extreme value ~~are~~ is, $U_{ex} = \frac{a}{3} \cdot \frac{a}{3} \cdot \frac{a}{3} = \frac{a^3}{27}$

$\hookrightarrow$ 1M

Q.9) If $x = u + v^2$, $y = v + w^2$, $z = w + u^2$, using Jacobian find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Sol^n Given : $x = u + v^2$, $y = v + w^2$, $z = w + u^2$

consider,

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$J = \begin{vmatrix} 1 & 2v & 0 \\ 0 & 1 & 2w \\ 2u & 0 & 1 \end{vmatrix} = 1(1) - 2v(-4uw) + 0$$

$$\boxed{J = 1 + 8uvw} \rightarrow \underline{4M}$$

Now,

$$J' = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$\boxed{\frac{1}{J} = \frac{1}{1 + 8uvw}} \rightarrow \underline{1M}$$

Q10) If $x = u+v$, $y = v^2+w^2$, $z = u^3+w^3$ then find $\frac{\partial u}{\partial x}$.

Sol^n let:- $f = u+v-x$, $g = v^2+w^2-y$, $h = u^3+w^3-z$

$$\therefore \frac{\partial u}{\partial x} = -\frac{\frac{\partial(f,g,h)}{\partial(x,v,w)}}{\frac{\partial(f,g,h)}{\partial(u,v,w)}} = \frac{-N}{D} \quad \text{—— (1)} \rightarrow \underline{\underline{1M}}$$

$$N = \frac{\partial(f,g,h)}{\partial(x,v,w)} = \begin{vmatrix} f_x & f_v & f_w \\ g_x & g_v & g_w \\ h_x & h_v & h_w \end{vmatrix} = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 2v & 2w \\ 0 & 0 & 3w^2 \end{vmatrix}$$

$$N = 6vw^2 \qquad \rightarrow \underline{\underline{1M}}$$

After solving determinant we get

$$D = 6w^2v + 6u^2w. \qquad \rightarrow \underline{\underline{2M}}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{-(-6vw^2)}{6vw^2+6u^2w}$$

$$\boxed{\therefore \frac{\partial u}{\partial x} = \frac{vw}{vw+u^2}} \qquad \rightarrow \underline{\underline{1M}}$$

Q10) If $x = u+v$, $y = v^2+w^2$, $z = u^3+w^3$ then find $\frac{\partial u}{\partial x}$.

Sol^n let:- $f = u+v-x$, $g = v^2+w^2-y$, $h = u^3+w^3-z$

$$\therefore \frac{\partial u}{\partial x} = -\frac{\frac{\partial(f,g,h)}{\partial(x,v,w)}}{\frac{\partial(f,g,h)}{\partial(u,v,w)}} = \frac{-N}{D} \quad \text{--- (1)} \rightarrow \underline{1M}$$

$$N = \frac{\partial(f,g,h)}{\partial(x,v,w)} = \begin{vmatrix} f_x & f_v & f_w \\ g_x & g_v & g_w \\ h_x & h_v & h_w \end{vmatrix} = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 2v & 2w \\ 0 & 0 & 3w^2 \end{vmatrix}$$

$$N = 6vw^2 \qquad \rightarrow \underline{1M}$$

After solving determinant we get

$$D = 6w^2v + 6u^2w \qquad \rightarrow \underline{2M}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{-(-6vw^2)}{6vw^2+6u^2w}$$

$$\boxed{\therefore \frac{\partial u}{\partial x} = \frac{vw}{vw+u^2}} \qquad \rightarrow \underline{1M}$$

Q11. Discuss maxima and minima of $f(x,y) = x^2+y^2+xy+x-4y+5$.

Sol^n Given:-

(I) $f(x,y) = x^2+y^2+xy+x-4y+5$ —— (1)

$$\frac{\partial f}{\partial x} = 2x+0+y+1-0+0$$

$$\frac{\partial f}{\partial x} = 2x+y+1=0 \quad \text{—— (2)}$$

$$\frac{\partial f}{\partial y} = 2y+x-4=0 \quad \text{—— (3)}$$

After solving (2) & (3) we get

$\boxed{x=-2}$ & $\boxed{y=3}$

$\therefore$ Stationary point is $(x,y) = (-2,3) \rightarrow 2M$

(II) Find r, s, t

$r = f_{xx} = 2$,

$s = f_{xy} = f_{yx} = 1$

$t = f_{yy} = 2$ $\rightarrow 1M$

(III) $rt - s^2 = (2)(2) - 1 = 3 > 0$

& $r > 0$

$\Rightarrow f$ is minimum at $(-2,3)$. $\rightarrow 1M$

$\therefore f_{min}$ at $(-2,3)$

$f_{min} = 4+9-6-2-12+15 = -2$

$\boxed{f_{min} = -2}$ $\rightarrow 1M$

Q.12) Examine whether $u = \frac{x-y}{1+xy}$, $v = \tan^{-1}x - \tan^{-1}y$ dependent, if so, find the relation between them.

Sol^n Given :- $u = \frac{x-y}{1+xy}$, $v = \tan^{-1}x - \tan^{-1}y$.

consider

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}$$

$$\therefore U_x = \frac{1+xy(1) - (x-y)(y)}{(1+xy)^2} = \frac{1+y^2}{(1+xy)^2}$$

$$U_y = \frac{(1+xy)(-1) - (x-y)(x)}{(1+xy)^2} = \frac{-1-x^2}{(1+xy)^2}$$

$$V_x = \frac{1}{1+x^2} \qquad V_y = \frac{-1}{1+y^2}.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1+xy)^2} & \frac{-1-x^2}{(1+xy)^2} \\ \frac{1}{1+x^2} & \frac{-1}{1+y^2} \end{vmatrix}$$

$\therefore \frac{\partial(u,v)}{\partial(x,y)} = 0$ $\therefore$ u & v are functionally dependent → 3M

$\therefore$ Relation :- $v = \tan^{-1}x - \tan^{-1}y$

$$v = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \quad \boxed{v = \tan^{-1}u} \text{ or } \boxed{\tan v = u}$$

→ 2M